Chapter 9
Phase Plane and Phase Space

9.1 Introduction

A dynamical state of an autonomous system is completely determined by the generalized coordinates $y_i(t)$ and the generalized velocities $\dot{y}_i(t)$ ($i = 1, 2, \ldots, n$, where n is the number of degrees of freedom). Treating time t as a parameter, a point of the coordinates $(y_i, \dot{y}_i)$ will be a point of $2n$-dimensional phase space. Motion of this point describes a phase trajectory as time increases. In the case of $n = 1$ a vibrating system has one degree-of-freedom and the phase space reduces to the phase plane. Then, a phase trajectory is a curve lying in the plane, and a set of all phase trajectories, corresponding to distinct initial conditions, form a phase portrait.

If the motion of one degree-of-freedom autonomous system (or two-dimensional system because it is governed by two first-order differential equations) is governed by the equation

$$\ddot{y} = F(y, \dot{y}), \tag{9.1}$$

then phase plane is said to be a plane with the rectangular coordinate system $(y, \dot{y} = v)$.

Equation (9.1) is transformed into a system of two first-order differential equations

$$\begin{aligned} \dot{y} &= v, \\ \dot{v} &= F(y, v). \end{aligned} \tag{9.2}$$

Equation (9.2) describes motion of a point $A(y, v)$ in the phase plane. Eliminating the time, we obtain an integral curve (phase trajectory) formula of the form

$$C(y, v) = 0. \tag{9.3}$$

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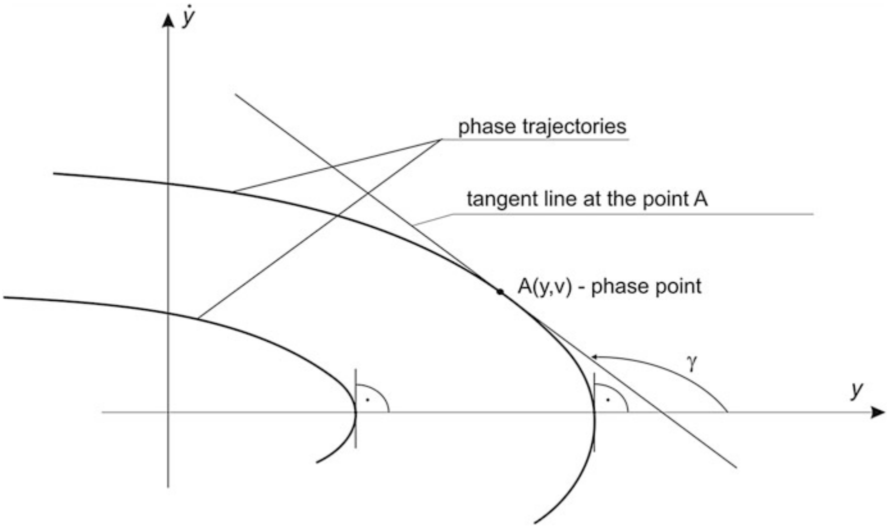


Fig. 9.1 Phase plane and phase trajectories

Dividing both sides of the system of Eq. (9.2) by themselves, we obtain

$$\frac{dv}{dy} = \frac{F(y, v)}{v} = \tan\gamma, \tag{9.4}$$

where γ is an angle between a phase trajectory and positive direction of the y-axis. There are a few phase trajectories depicted in Fig. 9.1 with the marked phase point and the angle γ.

Phase points, at which a tangent line is determined, will be called *ordinary* or *regular points*. Phase points, at which a tangent line is not determined, will be called *singular points*. The latter are equilibrium positions, determined from the equation

$$F(y, 0) = 0. \tag{9.5}$$

One can see in Fig. 9.1 that the phase trajectories intersect the y-axis at right angles. It turns out that each phase trajectory must pass through a regular point lying on the y-axis at right angle, since

$$\lim_{v\to 0} \frac{dv}{dy} = \lim_{v\to 0} \frac{F(y, v)}{v} = \infty, \tag{9.6}$$

and hence the value of γ at these points is $\pi/2$. A characteristic feature of nonlinear system follows from Eq. (9.5). These systems can possess one or several equilibrium positions depending on the character of the function $F(y, 0)$. Phase trajectories have some general properties, given below.

1. Direction of motion of the phase point $A(y, v)$ along the phase trajectory is such that the positive velocity v is in correspondence with increment of the displacement y according to positive direction of the y-axis, and the negative velocity v is in correspondence with the increment, which is opposite to the positive direction of the y-axis.
2. A phase trajectory cannot have a tangent line parallel to the v-axis at regular points, which do not lie on the y-axis. The phase trajectory cannot have a tangent line parallel to the v-axes at points, which do not lie in the v-axis.
3. If any continuous phase trajectory intersects the y-axis at two successive points, then there is at least one singular point between them.
4. In time interval, in which a continuous phase trajectory does not intersect the y-axis, the trajectory can intersect, at most, once any straight line parallel to the v-axis.
5. Closed curves in a phase plane correspond to periodic motions.

9.2 Phase Plain and Singular Points

A broader class of physical systems can be described by first-order differential equations of the form:

$$\frac{dy}{dt} = Q(y, v), \quad \frac{dv}{dt} = P(y, v). \tag{9.7}$$

Equation (9.4) is a particular case of Eq. (9.7). In what follows we analyse the linearized equation (9.7):

$$\frac{dv}{dy} = \frac{ay + bv}{cy + dv}. \tag{9.8}$$

In the dynamical system described by Eq. (9.8) there can be three types of phase trajectories, namely: a point, a closed (corresponds to a periodic solution) and open (corresponds to a non-periodic solution) curve. The aim of qualitative examination of the dynamical systems (9.8) is to determine a phase portrait and its topological structure. By a notion of topological structure we mean such properties of a phase portrait that remain unchanged under topological (i.e. unique and mutually continuous) mapping of a plane into itself. In order to perform such a qualitative analysis of the dynamical system (9.8), in most cases one can confine oneself to determining equilibrium positions, periodic trajectories and limit cycles, and phase semi-trajectories, which are curves separating qualitatively different phase trajectories in a neighbourhood of equilibrium position.

A limit cycle is said to be a closed phase curve, surrounded by a region completely filled with trajectories tending to the curve as $t \to +\infty$ or $t \to -\infty$. After Taylor expanding the functions $P(y, v)$ and $Q(y, v)$ about the analysed singular point we obtain

$$\frac{dv}{dy} = \frac{ay + bv + P'(y, v)}{cy + dv + Q'(y, v)}. \tag{9.9}$$

Ignoring the nonlinear terms, Eq. (9.9) takes the form

$$\begin{bmatrix} \dot{v} \\ \dot{y} \end{bmatrix} = \mathbf{A} \begin{bmatrix} v \\ y \end{bmatrix}, \tag{9.10}$$

$$\mathbf{A} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}. \tag{9.11}$$

The point $y = v = 0$ is a *critical point* (or a trivial solution) of Eq. (9.10) and if $\det \mathbf{A} \neq 0$, then the system (9.10) is called simple [191]. Eigenvalues of the matrix $\mathbf{A}$ allow to determine the canonical basis in $\mathbf{R}^2$, where the matrix $\mathbf{A}$ takes a canonical form. A characteristic equation leading to determination of eigenvalues can be obtained by standard procedure, namely by assuming solutions of the form

$$\begin{aligned} v &= C_1 e^{\lambda t}, \\ y &= C_2 e^{\lambda t}, \end{aligned} \tag{9.12}$$

where C_1 and C_2 are constants. Substituting (9.12) into (9.10) we obtain

$$\begin{vmatrix} b - \lambda & a \\ d & c - \lambda \end{vmatrix} = 0, \tag{9.13}$$

and after expanding

$$\lambda^2 - (b + c)\lambda + bc - ad = 0. \tag{9.14}$$

By the above equation we find the discriminant

$$\Delta = (b - c)^2 + 4ad. \tag{9.15}$$

The above equation possesses the following roots

$$\lambda_{1,2} = \frac{1}{2}\left[(b + c) \pm \sqrt{(b - c)^2 + 4ad}\right]. \tag{9.16}$$

Considerations based on the phase plane (x, y) are transferred into the plane (ξ, η) and correspond to the canonical form of the matrix $\mathbf{A}$. After the transformation, corresponding curves in both planes are rotated and deformed but their qualitative features remain unchanged, e.g. a circle corresponds to an ellipse but both curves are closed. The character of the curves depends on the ratio λ_1/λ_2 and a constant C (see (9.20)).

We will consider the following cases:

1. Both roots are real and distinct, and of the same sign. We have such a situation when $\Delta > 0$, $ad - bc < 0$. Then, the matrix $\mathbf{A}$ in the canonical basis has the form

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \tag{9.17}$$

where $\lambda_{1,2} \in \mathbf{R}$, and Eq. (9.10) takes the canonical form

$$\begin{aligned} \dot{\eta} &= \lambda_1 \eta, \\ \dot{\xi} &= \lambda_2 \xi. \end{aligned} \tag{9.18}$$

Equation (9.18) can be easily solved by separating the variables. Consequently, we obtain

$$\begin{aligned} \eta(t) &= C_1 e^{\lambda_1 t}, \\ \xi(t) &= C_2 e^{\lambda_2 t}. \end{aligned} \tag{9.19}$$

Next, we have

$$\ln \frac{\eta}{C_1} = \frac{\lambda_1}{\lambda_2} \ln \frac{\xi}{C_2},$$

thus

$$\ln \frac{\eta}{C_1} = \ln \left(\frac{\xi}{C_2} \right)^{\frac{\lambda_1}{\lambda_2}},$$

hence

$$\eta = C |\xi|^{\frac{\lambda_1}{\lambda_2}}, \tag{9.20}$$

where $C = (C_1/C_2)^{\frac{\lambda_1}{\lambda_2}}$. The singular point $(0, 0)$ is called a stable (unstable) node.

2. If $bc - ad = 0$ and $b + c < 0$, then by (9.14) we get

$$\lambda[\lambda - (b + c)] = 0, \tag{9.21}$$

and this implies $\lambda_2 = 0$ and $\lambda_1 = b + c < 0$. In this case, the analysed system is not simple. The matrix $\mathbf{A}$ has the following canonical form

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{9.22}$$

and its rank equals 1. Such a singular point is called a stable centre. It is noteworthy that when the rank of the matrix **A** equals 0, then the matrix is a zero matrix and each point of the phase plane is critical.

Critical points, called nodes, also occur when the discriminant of (9.14) $\Delta = 0$. Then $\lambda_1 = \lambda_2 = \lambda_0$ (double root) and if two linearly independent vectors are associated with a double eigenvalue, then canonical form of the matrix **A** reads:

$$\mathbf{A} = \begin{bmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{bmatrix}. \tag{9.23}$$

A critical point corresponding to this matrix is called a star-shaped node, which is stable, if $\lambda_0 < 0$ (and conversely). Only one eigenvector can be associated with a double eigenvalue. Then, a canonical form of the matrix **A** takes the form of the following Jordan block

$$\mathbf{A} = \begin{bmatrix} \lambda_0 & 0 \\ 1 & \lambda_0 \end{bmatrix}. \tag{9.24}$$

The differential equations (9.10) take the form

$$\begin{aligned} \dot{v} &= \lambda_0 v, \\ \dot{y} &= v + \lambda_0 y, \end{aligned} \tag{9.25}$$

and their solutions follow

$$\begin{aligned} v &= C_1 e^{\lambda_0 t}, \\ y &= (C_2 + C_1 t) e^{\lambda_0 t}. \end{aligned} \tag{9.26}$$

In this case, the critical point $(0, 0)$ is a *degenerate node*, which is stable for $\lambda_0 < 0$ and unstable for $\lambda_0 > 0$.

3. In this case both roots are real and have opposite signs. The orbits surround a singular point, which is called a *saddle*. Two orbits approach and move away from this point—these are axes of a coordinate system.
4. If the discriminant of (9.14) $\Delta < 0$ and $b + c \neq 0$, then the roots λ_1 and λ_2 are complex conjugate. Then, the critical point is a stable $b + c < 0$ or unstable $b + c > 0$ focus. Assume that $\lambda_{1,2} = \alpha \pm i\omega$, while $\alpha \neq 0$ and $\omega \neq 0$ (farther we will assume $\omega > 0$). In this case the canonical matrix has the form

$$\mathbf{A} = \begin{bmatrix} \alpha & -\omega \\ \omega & \alpha \end{bmatrix}, \tag{9.27}$$

and we will consider the following system of equations

$$\begin{aligned} \dot{v} &= \alpha v - \omega y, \\ \dot{y} &= \omega v + \alpha y. \end{aligned} \tag{9.28}$$

Parametric equations of orbits of the above system (its general solutions) are:

$$\begin{aligned} v(t) &= Ce^{\alpha t}\cos(\omega t + \varphi), \\ y(t) &= Ce^{\alpha t}\sin(\omega t + \varphi), \end{aligned} \tag{9.29}$$

where C and φ are any constants.

Orbits in neighbourhood of a focus can be also presented in the polar coordinates (ϱ, θ). Let us make a change of the variables

$$\begin{aligned} v &= \rho\cos\theta, \\ y &= \rho\sin\theta. \end{aligned} \tag{9.30}$$

By Eqs. (9.28) and (9.30) we get

$$\begin{aligned} \dot{\rho}\cos\theta - \rho\dot{\theta}\sin\theta &= \alpha\rho\cos\theta - \omega\rho\sin\theta, \\ \dot{\rho}\sin\theta + \rho\dot{\theta}\cos\theta &= \alpha\rho\cos\theta + \omega\rho\sin\theta. \end{aligned} \tag{9.31}$$

Multiplying (9.31) respectively by $\cos\theta$ and $\sin\theta$ (and by $\sin\theta$ and by $-\cos\theta$), and adding the equations we get

$$\begin{aligned} \dot{\rho} &= \alpha\rho, \\ \dot{\theta} &= \omega. \end{aligned} \tag{9.32}$$

The solution in the polar coordinates takes the form

$$\begin{aligned} \rho &= \rho_0 e^{\alpha t}, \\ \theta &= \omega t + \theta_0, \end{aligned} \tag{9.33}$$

where ρ_0 and θ_0 define any initial conditions. The solutions (9.33) have simple physical interpretation. The argument θ grows linearly in time, while a ray originating from the focus and passing through the point $(y(t), v(t))$ rotate anticlockwise at angular velocity ω [rad/s].

By Eq. (9.33) after eliminating the time we obtain

$$\rho = \rho_0 e^{-\frac{\alpha}{\omega}\theta_0} e^{\frac{\alpha}{\omega}\theta}. \tag{9.34}$$

In this case the orbit is represented by a curve called a logarithmic spiral in the coordinates (ρ, θ). It is worth emphasizing that in the case of an unstable focus $\lim_{t \to +\infty} \rho(t) = +\infty$, and the shape of logarithmic spirals depends on the ratio $\frac{\alpha}{\omega}$. In the case when $b + c = 0$, then $\lambda_{1,2} = \pm i\omega$ $(\alpha = 0)$. Then Eq. (9.32) we get

$$\begin{aligned} \rho &= \rho_0 \equiv \text{const.}, \\ \theta &= \omega t + \theta_0. \end{aligned} \tag{9.35}$$

The above formulas represent a circle of radius ρ_0 in the polar coordinates (ρ, θ). While, by Eq. (9.28) we get

$$\begin{aligned} \dot{v} &= -\omega y, \\ \dot{y} &= \omega v, \end{aligned} \tag{9.36}$$

and eliminating the time we obtain

$$\frac{dy}{dv} = -\frac{v}{y}, \tag{9.37}$$

and hence $v^2 + y^2 = C^2$.

A critical point, in this case, is called a centre. The centre is a stable point but not asymptotically stable in Lyapunov's sense.

There is only one particular case left to discuss, namely the case of vanishing discriminant $\Delta = 0$, when zero is a double root, and the matrix $\mathbf{A}$ has the following canonical form

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{9.38}$$

A normal form of a system of differential equations takes the form

$$\begin{aligned} \dot{v} &= 0, \\ \dot{y} &= v, \end{aligned} \tag{9.39}$$

and their solutions are the following functions

$$\begin{aligned} v &= C_1, \\ y &= C_1 t + C_2. \end{aligned} \tag{9.40}$$

In Table 9.1 one classified phase portraits associated with critical points in two-dimensional space $\mathbf{R}^2$. Using linear transformation

$$\xi = \alpha y + \beta v, \quad \eta = \gamma y + \delta v. \tag{9.41}$$

Table 9.1 Phase portraits classification

Eigenvalues λ_1, λ_2	Eigenvectors (comment)	Name of a critical point
$0 < \lambda_1 < \lambda_2$		Unstable node
$\lambda_1 < \lambda_2 < 0$		Stable node
$\lambda_1 < 0 < \lambda_2$		Saddle
$0 = \lambda_1 < \lambda_2$		Unstable centre
$\lambda_1 < \lambda_2 = 0$		Stable centre
$\lambda_1 = \lambda_2 < 0$	Two eigenvectors	Stable star-shaped node
$\lambda_1 = \lambda_2 > 0$	Two eigenvectors	Unstable star-shaped node
$\lambda_1 = \lambda_2 < 0$	One eigenvector	Stable degenerate node
$\lambda_1 = \lambda_2 > 0$	One eigenvector	Unstable nondegenerate node
$\lambda_1 = \lambda_2 = 0$	One eigenvector	Degenerate centre
$\lambda_{1,2} = \alpha \pm i\omega$	$\alpha > 0, \omega \neq 0$	Unstable focus
$\lambda_{1,2} = \alpha \pm i\omega$	$\alpha < 0, \omega \neq 0$	Stable focus
$\lambda_{1,2} = \alpha \pm i\omega$	$\alpha = 0, \omega \neq 0$	Stable centre

one can transform Eq. (9.8) into the form of separated variables (see 9.18)

$$\frac{d\eta}{d\xi} = \frac{\lambda_1 \eta}{\lambda_2 \xi}. \tag{9.42}$$

Equation (9.41) yields

$$d\xi = \alpha dy + \beta dv, \quad d\eta = \gamma dy + \delta dv. \tag{9.43}$$

Inserting the nominator and denominator of the formula (9.8) instead of dy and dv, we obtain

$$\frac{d\eta}{d\xi} = \frac{\gamma(cy + dv) + \delta(ay + bv)}{\alpha(cy + dv) + \beta(ay + bv)}. \tag{9.44}$$

Comparing nominators and denominators of the above equation and of the formula (9.42), and using the linear transformation (9.41) we obtain the following system of equations

$$\begin{aligned} \gamma(cy + dv) + \delta(ay + bv) &= \lambda_1 \eta = \lambda_1(\gamma y + \delta v), \\ \alpha(cy + dv) + \beta(ay + bv) &= \lambda_2 \xi = \lambda_2(\alpha y + \beta v). \end{aligned} \tag{9.45}$$

In order to determine the constants γ and δ, for the first of the formulas (9.45), we equate the terms occurring by y and v. We obtain two algebraic equations of the form

$$\begin{aligned} \gamma(c-\lambda_1)+\delta a &= 0, \\ \gamma d+\delta(b-\lambda_1) &= 0. \end{aligned} \tag{9.46}$$

The algebraic equations for the second equation (9.45) have very similar structure and allow to determine the coefficients α and β

$$\begin{aligned} \alpha(c-\lambda_2)+\beta a &= 0, \\ \alpha d+\beta(b-\lambda_2) &= 0. \end{aligned} \tag{9.47}$$

This implies that λ_1 and λ_2 are roots of the same characteristic equation, which is formed by equating the characteristic determinant of the system of Eqs. (9.46) and (9.47) to zero, i.e.

$$\begin{vmatrix} c-\lambda & a \\ d & b-\lambda \end{vmatrix} = 0. \tag{9.48}$$

9.3 Analysis of Singular Points

Nowadays there are many softwares allowing to solve the differential equation (9.10) analytically and numerically. The obtained results are automatically plotted in a plane in the coordinates (y, v). The character of a singular point under consideration depends only on the coefficients a, b, c, d. The obtained phase trajectories are slightly deformed but it is possible to rotate them by solving the differential equation (9.10). This equation allows to obtain the rectified trajectories in the coordinate system (ξ, η). The shape of these graphs depends only on a ratio of the roots λ_1 and λ_2 of the characteristic equation (9.10).

9.3.1 Unstable Node

The first singular point $(0, 0)$ will be a node. In this case the roots λ_1 and λ_2 of the characteristic equation must be real and distinct, and have the same signs. These conditions will be satisfied when

$$(b-c)^2+4ad>0 \quad \text{and} \quad bc-ad>0. \tag{9.49}$$

The solution curves in the plane (ξ, η) will be parabolas passing through the point $(0, 0)$. If $b+c>0$, then a critical point is an unstable node, a phase point moves away from the origin as time increases. These conditions are satisfied for e.g.: $a=0, b=2, c=1, d=1$. This situation is presented in Figs. 9.2 and 9.3.

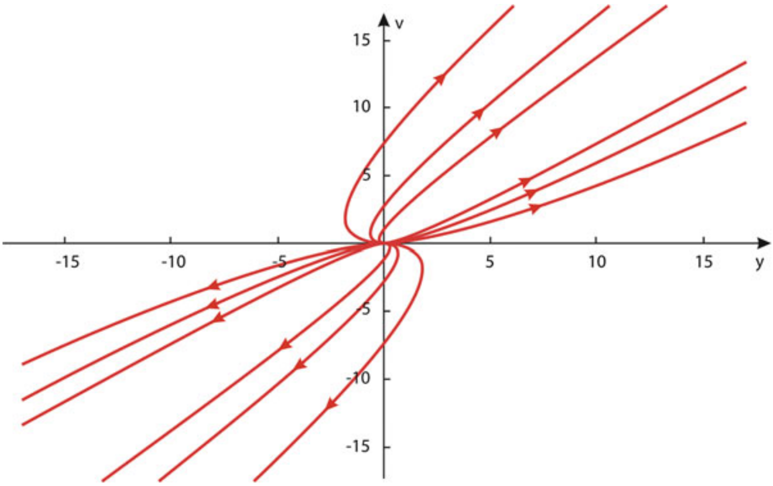


Fig. 9.2 The phase trajectories passing through the unstable node in the coordinates (y, v)

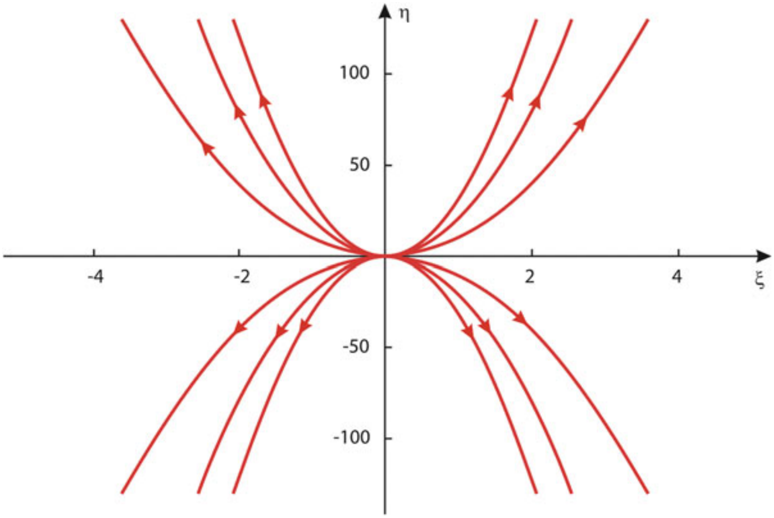


Fig. 9.3 Phase trajectories passing through the unstable node in the coordinates (ξ, η)

By the above graphs one can see that all the trajectories pass through the singular point $(0, 0)$, which is an unstable node because phase point move away from the node as time increases.

In Fig. 9.4 one can see the trajectory obtained numerically. This verifies the earlier obtained analytical solutions.

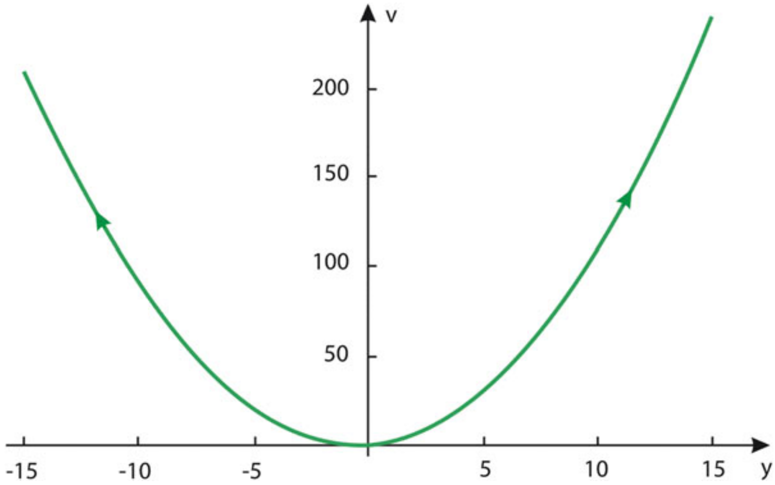


Fig. 9.4 Numerical solution for an unstable node in the coordinates (y, v)

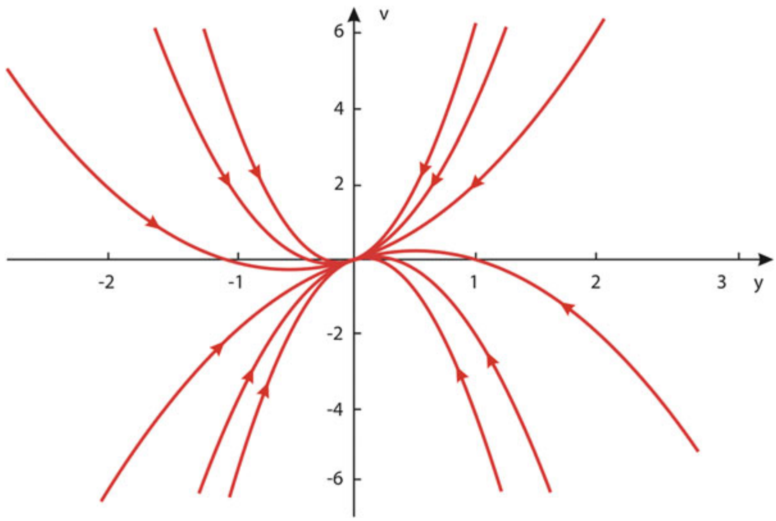


Fig. 9.5 Phase trajectories passing through the stable node in the coordinates (y, v)

9.3.2 Stable Node

If $b + c < 0$ then a phase point approaches to the singular point $(0, 0)$ as time grows. For instance, it takes place for $a = 1$, $b = -2$, $c = -1$, $d = 0$. These conditions are demonstrated in Figs. 9.5 and 9.6.

The phase trajectories are parabolas passing through the origin $(0, 0)$ of the coordinate system but the origin, which is a singular point is called a stable node,

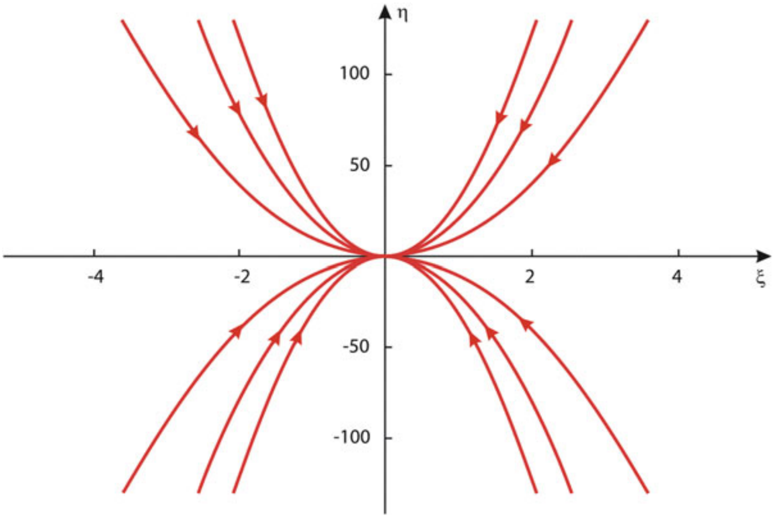


Fig. 9.6 Phase trajectories passing through the stable node in the coordinates (ξ, η)

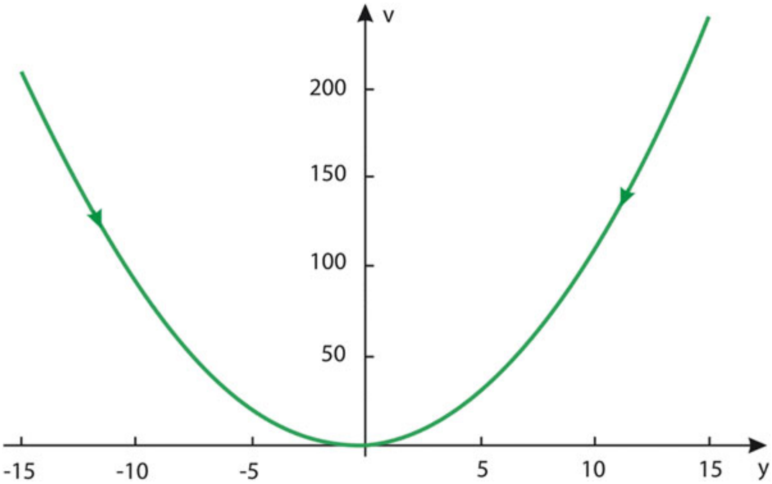


Fig. 9.7 Numerical solution for a stable node in the coordinates (y, v)

since phase points approach the point $(0, 0)$. Below in Fig. 9.7 one can see the numerical verification of the analytical solution.

If the roots λ_1 and λ_2 differ from each other significantly, then the phase trajectories change the direction more rapidly. Moreover, if one of the roots equals zero, then the curves are transformed into vertical straight lines.

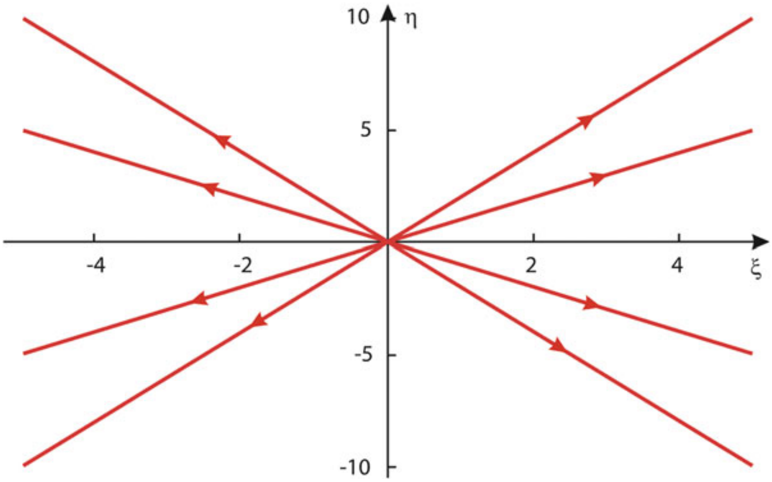


Fig. 9.8 Phase trajectories passing through the critical node in the coordinates (ξ, η)

9.3.3 *Critical Node*

When roots of the characteristic equation are real and $\lambda_1 = \lambda_2$, then a node is called a critical node. In this case we have

$$(b - c)^2 + 4ad = 0 \quad \text{and} \quad a = d = 0. \tag{9.50}$$

Let the coefficients be: $a = 0$, $b = 2$, $c = 2$, $d = 0$. If $b = c > 0$, then a phase point moves away from the origin as time increases. This situation is depicted in Fig. 9.8.

While $b = c < 0$, then the coefficients can be: $a = 0$, $b = -2$, $c = -2$, $d = 0$. Then phase points approach the point $(0, 0)$ as time increases. This situation is depicted in Fig. 9.9.

Figures 9.8 and 9.9 imply that the trajectories form a bunch of lines, on which a phase point approaches or moves away from the node $(0, 0)$, which is now called critical. Verification of the analytical solution is the numerical one depicted in Fig. 9.10.

9.3.4 *Degenerate Node*

We deal with a degenerate node if roots of the characteristic equation are equal and no special case occurs e.g.: $a = 2$, $b = 5$, $c = 1$, $d = -2$. Now, we have

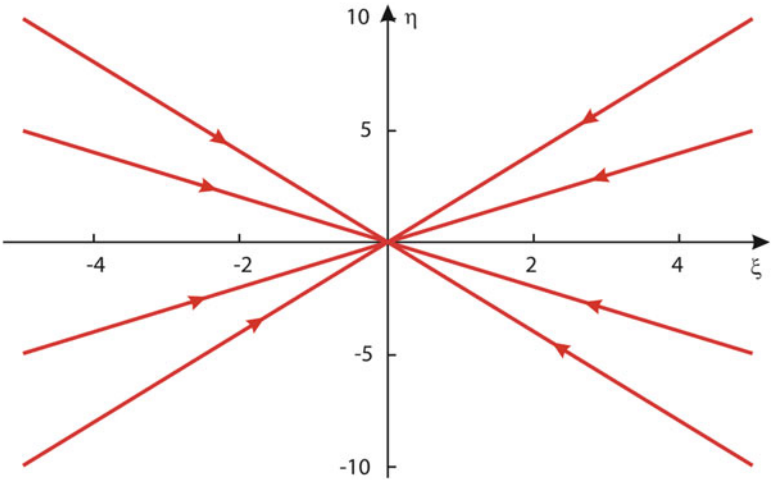


Fig. 9.9 Phase trajectories passing through the critical node in the coordinates (ξ, η)

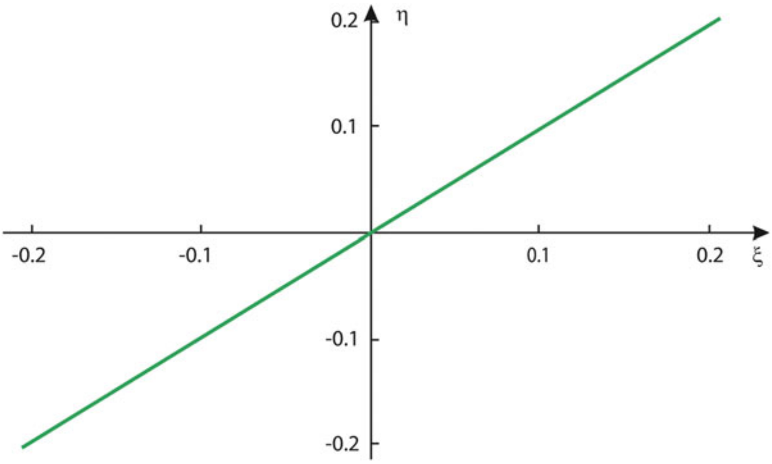


Fig. 9.10 Numerical solution for a critical node in the coordinates (ξ, η)

$b + c > 0$, and the singular point $(0, 0)$ is called a degenerate node and a phase point moves away from the origin of the coordinate system (Fig. 9.11).

When $a = 1$, $b = -2$, $c = -4$, $d = -1$, then $b + c < 0$, and a phase point approaches the origin and we also have to do with a degenerate node illustrated in Fig. 9.12. The numerical solution for a degenerate node is presented in Fig. 9.13.

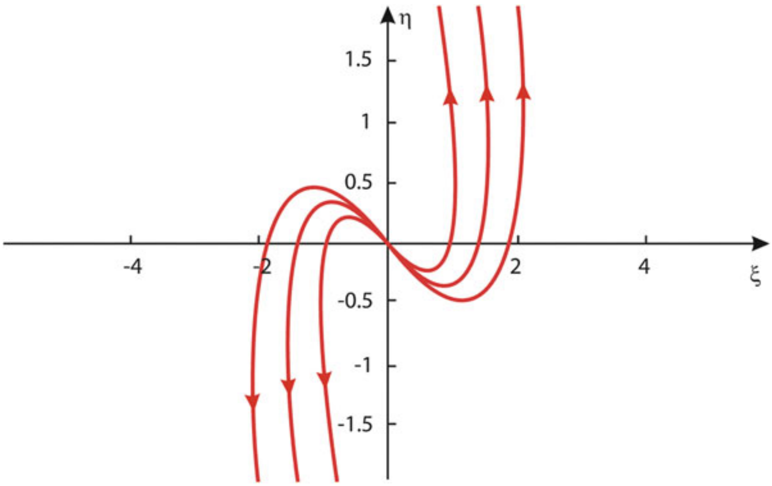


Fig. 9.11 Phase trajectories passing through the degenerated node in the coordinates (ξ, η)

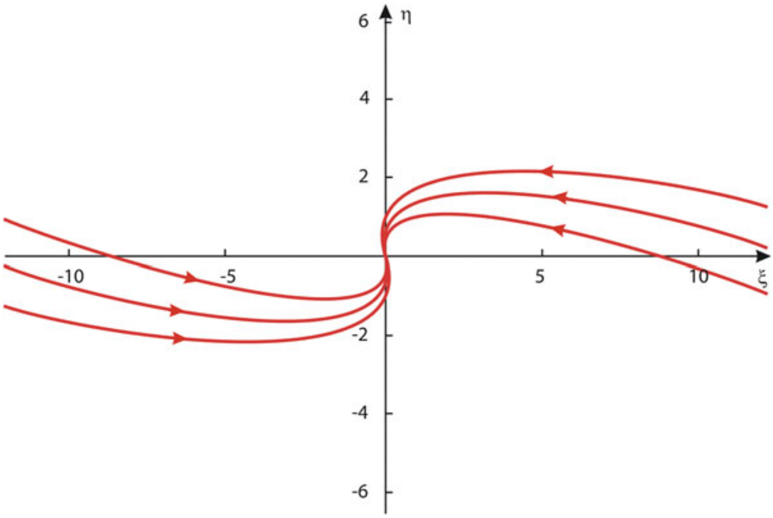


Fig. 9.12 Phase trajectories passing through the degenerated node in the coordinates (ξ, η)

9.3.5 Saddle

The second critical point is a saddle point, which is always unstable. In this case, the roots λ_1 and λ_2 are also real and distinct but they must be of opposite signs. This case occurs when the following conditions are satisfied:

$$(b - c)^2 = 4ad > 0 \quad \text{and} \quad bc - ad < 0. \tag{9.51}$$

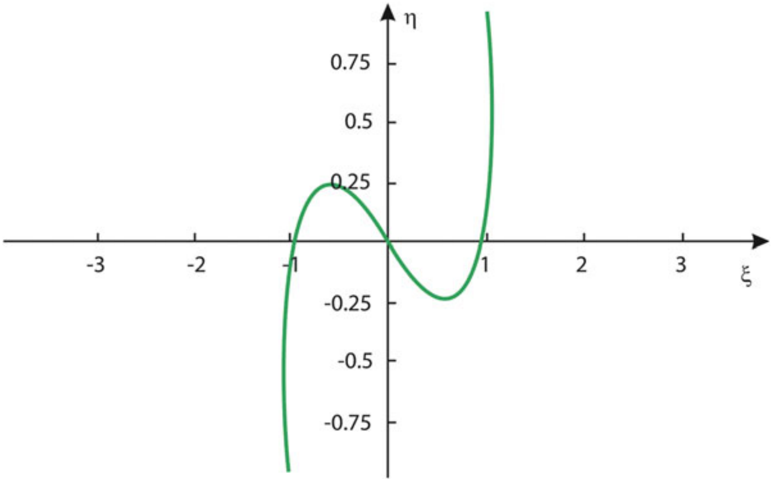


Fig. 9.13 A phase trajectory presenting a degenerated node obtained numerically in the coordinates (ξ, η)

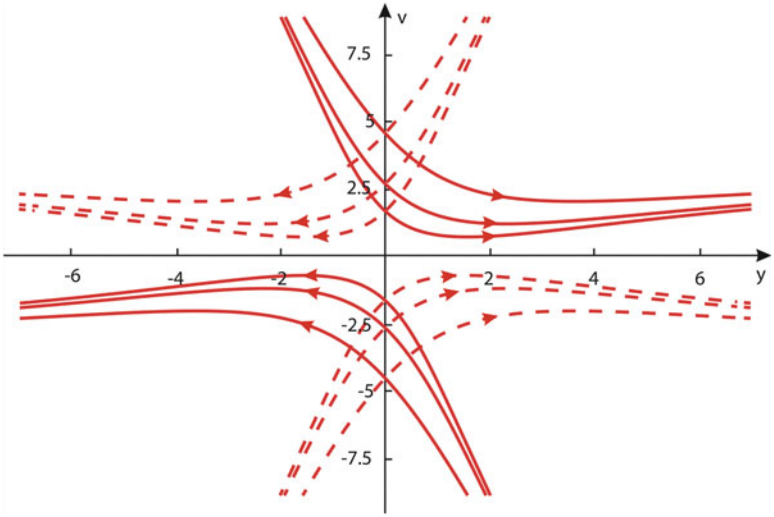


Fig. 9.14 Phase trajectories around the saddle in the coordinates (y, v)

The solution curves in the plane (ξ, η) are hyperbolae, which do not pass through a singular point. One of the roots (a positive one) is associated with the value growth of the solution as the time t increases, while the second solution tends to zero. In the plane (y, v) the curves will be deformed. Figure 9.14 illustrates this situation, where the coefficients: $a = 1$, $b = 2$, $c = -2$, $d = 1$ and then $\lambda_1 > \lambda_2$.

In order to "rectify" the phase trajectories, we transfer the solutions into the plane (ξ, η). This situation is illustrated in Fig. 9.15.

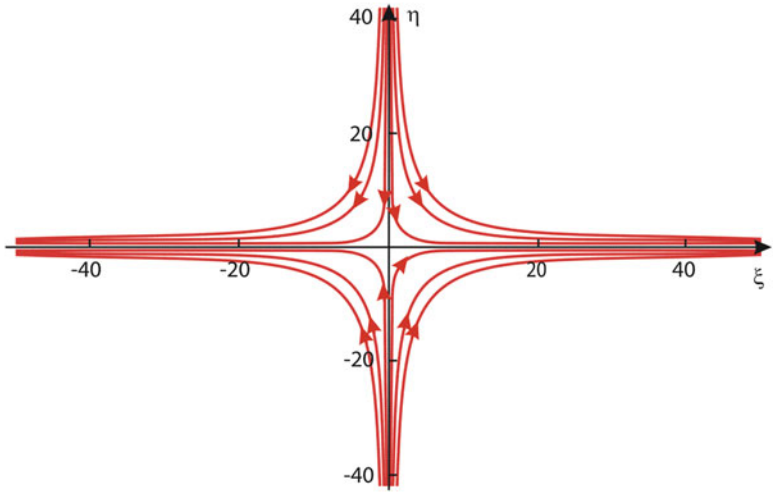


Fig. 9.15 Phase trajectories around the saddle in the coordinates (ξ, η)

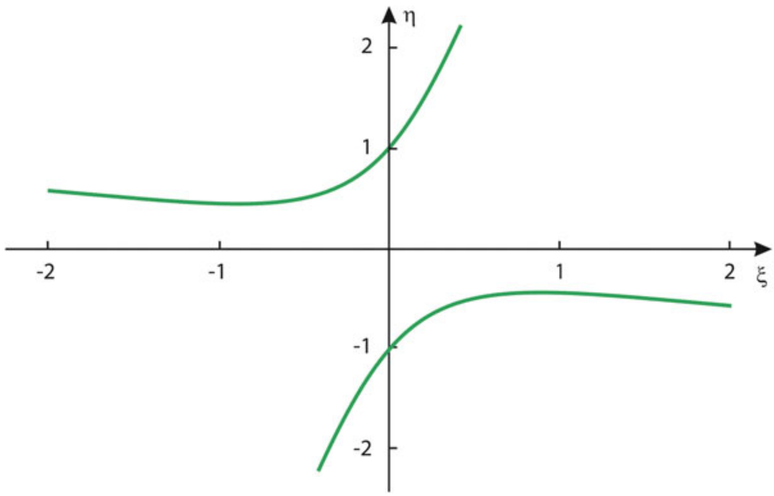


Fig. 9.16 Numerical solution in the case of a saddle in the coordinates (ξ, η)

Verification of the analytical solution is illustrated in Fig. 9.16, where there is numerically obtained singularity of saddle type.

All the analysed dynamical systems possessed real roots, which were solutions of the characteristic equation (9.14). This means that we did not have to do with any types of vibrations. Below, we characterize dynamical systems, whose roots of a characteristic equation are not real any more, i.e. there are no vibrations in these systems. For a stable and unstable focus damped oscillations appear, while in the case of a centre undamped oscillations appear.

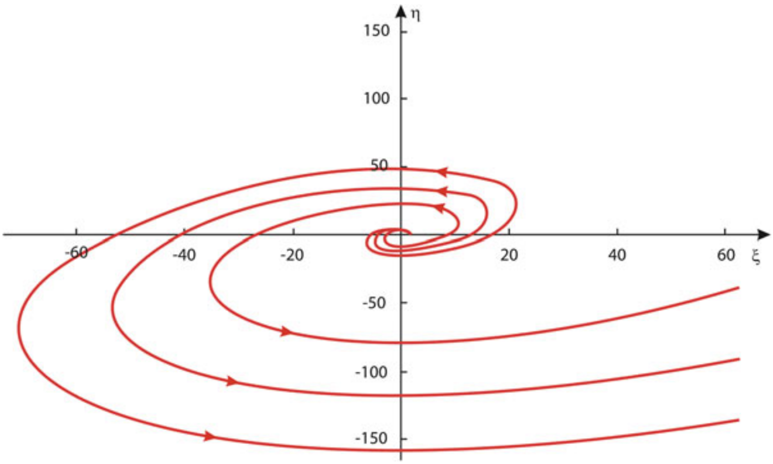


Fig. 9.17 Phase trajectories around the unstable focus in the coordinates (ξ, η)—numerical computations

9.3.6 Unstable Focus

The next analysed singular point appears, when

$$(b - c)^2 + 4ad < 0 \quad \text{and} \quad b + c \neq 0. \tag{9.52}$$

Then, roots of the characteristic equation (9.14) are complex conjugate, but any of the roots is neither real nor purely imaginary. It is possible for e.g. the following coefficients $a = 2, b = 0, c = 1, d = -1$ then $b + c > 0$, and a singularity of this type is called a non-stable focus, from which phase trajectories move away.

Figures 9.17 and 9.18 illustrate this situation.

9.3.7 Stable Focus

When $b + c < 0$ and the coefficients equal e.g. $a = -2, b = 0, c = -2, d = 2$ then we have a stable focus. Then, the phase trajectories approaches the origin of the coordinate system (Fig. 9.19).

We have also added the numerical solution of this problem (Fig. 9.20).

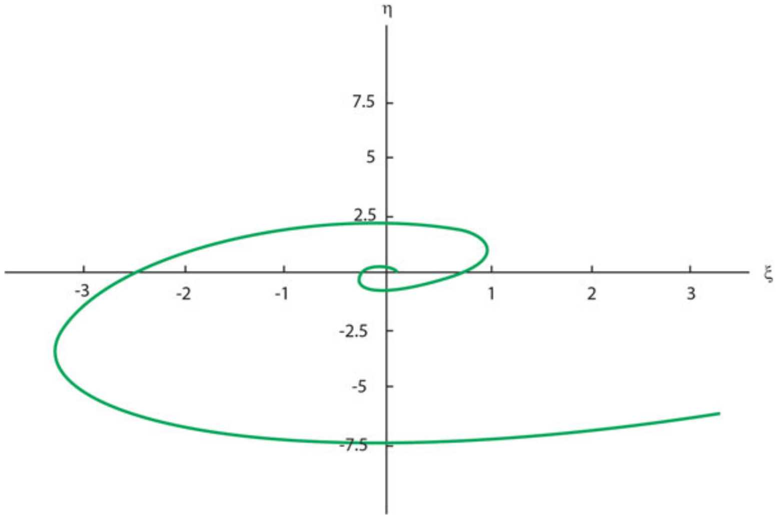


Fig. 9.18 Phase trajectories around the unstable focus in the coordinates (ξ, η)—numerical computations

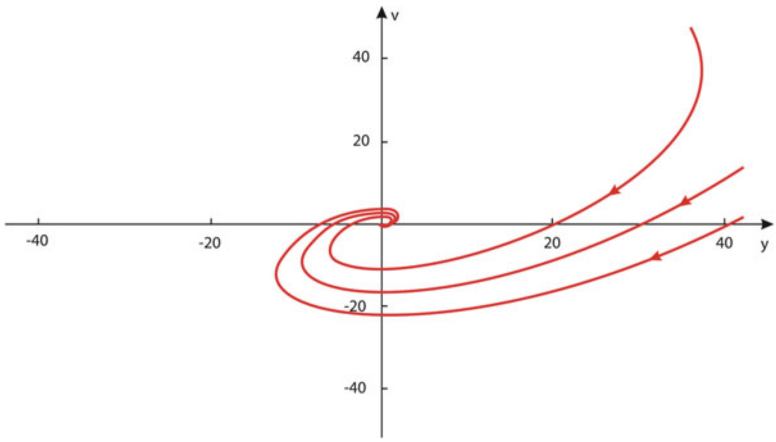


Fig. 9.19 Phase trajectories around the stable focus in the coordinates (y, v)

9.3.8 Centre

The last possible singularity, occurring in the origin is a centre point. The roots λ_1 and λ_2 are then complex conjugate and purely imaginary $\lambda_{1,2} = i\omega$, when

$$(b-c)^2 + 4ad < 0 \quad \text{and} \quad b + c = 0. \tag{9.53}$$

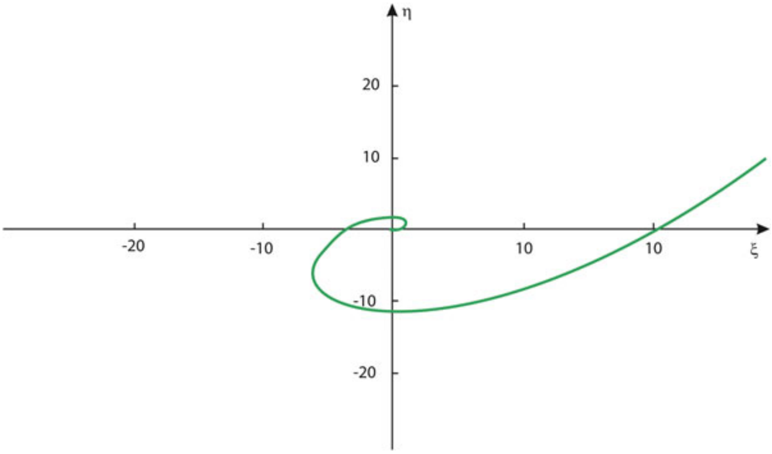


Fig. 9.20 The stable focus obtained numerically in the coordinates (ξ, η)

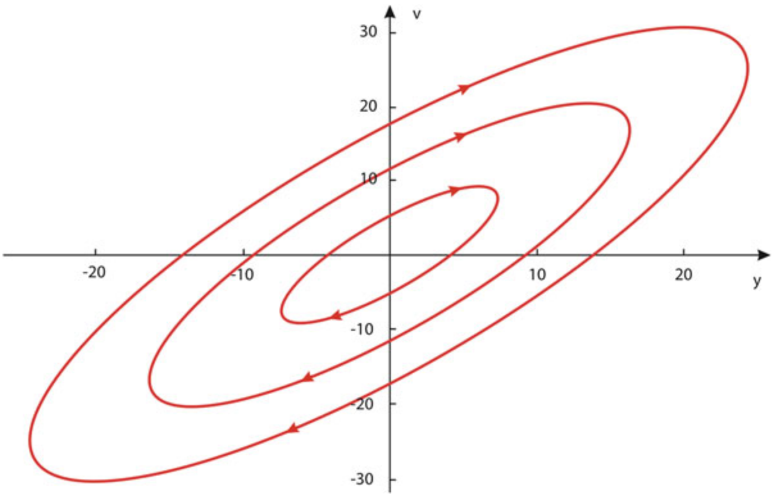


Fig. 9.21 Phase trajectories around the centre in the coordinates (y, v)

In the plane (y, v) the phase trajectories are deformed, but their character is left unchanged, thus they are closed curves surrounding the origin of the coordinate system. The coefficients can be selected in the following way: $a = -3$, $b = 2$, $c = -2$, $d = 2$. This is illustrated in Fig. 9.21.

In this case, the normal form of the equations differs from the previous one since we have to do with the case described by Eq. (9.37).

Figure 9.22 presents the phase trajectories around the origin of the coordinates (ξ, η), whereas the numerically obtained solution is shown in Fig. 9.23.

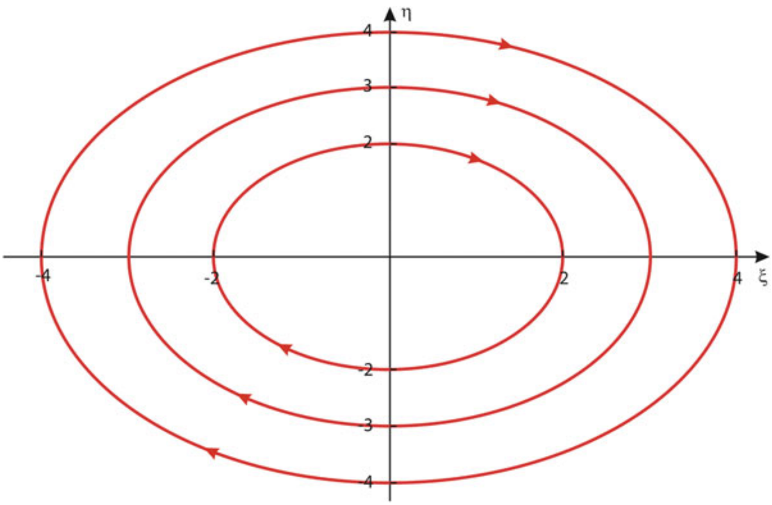


Fig. 9.22 Phase trajectories around the centre in the coordinates (ξ, η)

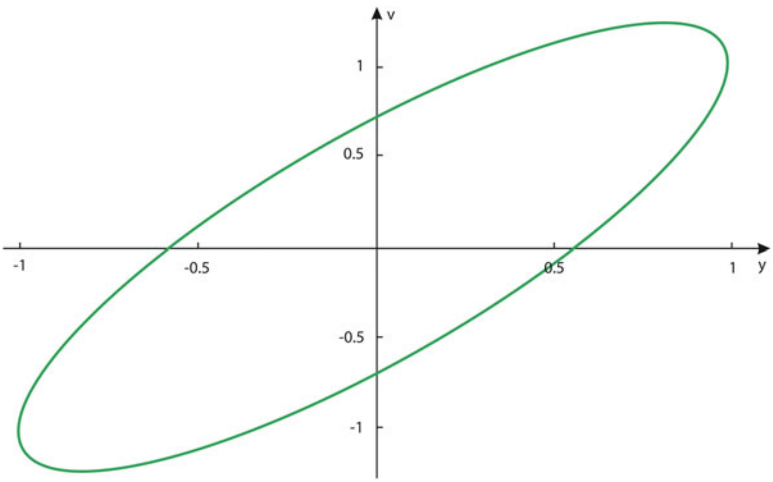


Fig. 9.23 The phase trajectory obtained numerically for a singularity of centre type in the coordinates (y, v)

It follows from the analysis performed in this subsection that the character of an equilibrium position and the shape of phase trajectories near the position depends only on the coefficients a, b, c, d. They have significant influence on the structure of the characteristic equation (9.14).

9.4 Analysis of Singular Points Governed by Three Differential Equations of First Order

In this section we deal with dynamical systems which are governed by three differential equations of first order. The obtained solutions will be presented by means of three-dimensional graphs of phase trajectories. Both analytical and numerical solutions will be plotted for properly selected values of all three constants C_1, C_2, C_3. Selecting in a proper way the values occurring in the equations, we will obtain singularities of special types. In Sect. 9.4.1, we will present the analysed system of equations and its characteristic equation, which will serve for determining proper matrices.

A given matrix will be characteristic for a specific type of a considered singularity. While, in Sect. 9.4.2, graphs of solutions of the corresponding system obtained numerically and analytically will be presented. These are solutions of a system of three first-order differential equations.

9.4.1 Theory Concerning the Solving a System of Differential Equations and Method for Determining Roots of a Polynomial of Third Degree

Considerations will be based on a system of three first-order differential equations. The analysed system of differential equations written in a form of rectangular coordinate system can be presented in the following way:

$$\begin{aligned}\frac{dx}{dt} &= ax + by + cz,\\ \frac{dy}{dt} &= dx + ey + fz,\\ \frac{dz}{dt} &= gx + hy + iz.\end{aligned} \tag{9.54}$$

In this system, the coefficients (characterizing the equations), i.e. a, b, c, d, e, f, g, h, i, can take on real as well as complex values. A solution of this system of equations, we will seek in the form

$$\begin{aligned}x &= C_1 \exp(\lambda t),\\ y &= C_2 \exp(\lambda t),\\ z &= C_3 \exp(\lambda t).\end{aligned} \tag{9.55}$$

The characteristic equation can be written in a matrix form

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0. \tag{9.56}$$

This equation has a trivial solution when we equate the determinant below to zero

$$\begin{vmatrix} a-\lambda & b & c \\ d & e-\lambda & f \\ g & h & i-\lambda \end{vmatrix} = 0. \tag{9.57}$$

Expanding the determinant we get the following characteristic equation

$$\lambda^3-(a+e+i)\lambda^2+(ai+ei+ae-cg-fh-bd)\lambda+bdi+afh+ceg-gbf-dhc-aei = 0. \tag{9.58}$$

The above properly selected coefficients allow to obtain singularities, we are interested in, in a three-dimensional space. The coefficients are responsible for the character of curves plotted after solving the system of differential equations (9.54).

9.4.2 Analysis of Singular Points Described by Three First-Order Differential Equations

Below, we consider and analyse different dynamical systems, in which we select and change the values, which we will write in the matrix form:

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}. \tag{9.59}$$

9.4.2.1 Unstable Node

An unstable node will be the first analysed type of equilibrium in a three-dimensional phase space. We meet this type of singularity, when components of the matrix **A** are following

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

Then the characteristic equation (9.58) possesses three roots. All of them are positive and real. They are: $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$.

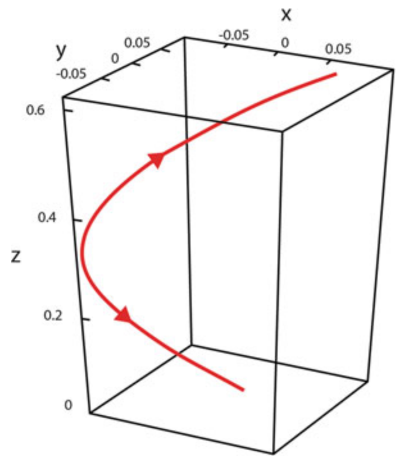


Fig. 9.24 The analytical solution, when an unstable node is an equilibrium position in the three-dimensional space

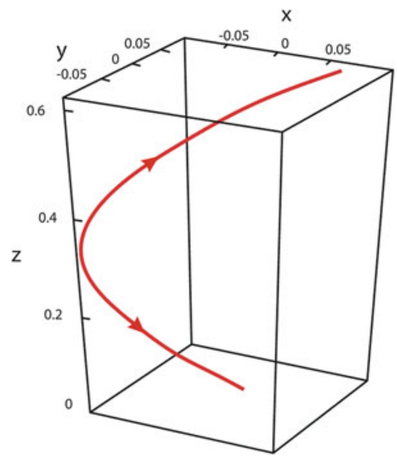


Fig. 9.25 The numerical solution, when an unstable node is an equilibrium position in the three-dimensional space

In Fig. 9.24, one can see that the solution is a parabola. One could have expected this, since we have obtained a similar graph during the analysis of equilibrium positions in the phase plane. Verification of this solution is a numerically obtained graph depicted in Fig. 9.25.

Another example of the matrix **A** (this matrix enabled to obtain equilibrium position of unstable node type), whose elements are

$$\mathbf{A} = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.06 & 0 \\ 0 & 0 & 0.01 \end{bmatrix}.$$

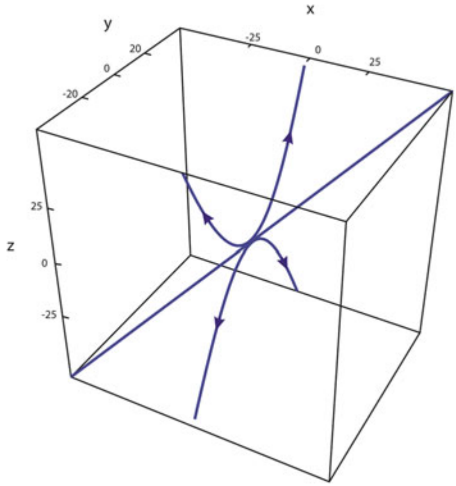


Fig. 9.26 The analytical solution, when an unstable node is an equilibrium position in three-dimensional space

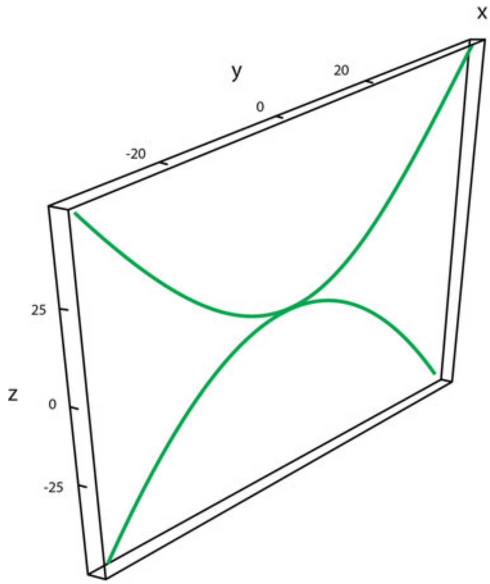


Fig. 9.27 The numerical solution, when an unstable node is an equilibrium position in three-dimensional space

Then roots of the characteristic equation are positive and real, but one of the roots is a double root. They are: $\lambda_1 = 0.06$, $\lambda_2 = 0.1$, $\lambda_3 = 0.1$. Below (Fig. 9.26), one can see the analytical solution graph (Fig. 9.27).

The graph presented in Fig. 9.26 was obtained with the use of symmetry principles and selecting the constants, which appear as a result of solving the system of differential equations (9.54).

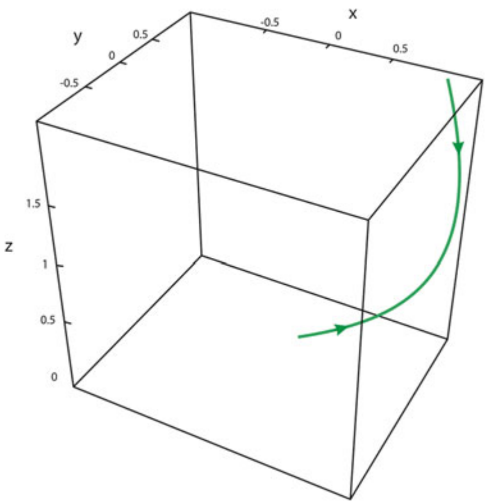


Fig. 9.28 The analytical solution, when a stable node is an equilibrium position in three-dimensional space

9.4.2.2 Stable Node

The second equilibrium—a stable node occurs, when the matrix **A** has the following components

$$\mathbf{A} = \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & -1 & 3 \end{bmatrix}.$$

Then, the characteristic equations (9.58) possesses also three real roots but all of them are negative. They are: $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -3$. Both analytical and numerical solutions of this example coincide and they are reported in Fig. 9.28.

9.4.2.3 Saddle

A next equilibrium position is a saddle point, which is always unstable. The matrix **A** has the following form then

$$\mathbf{A} = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.09 & 0 \\ 0 & 0 & -0.09 \end{bmatrix}.$$

The characteristic equation (9.58) possesses then three real roots, but they are of opposite signs, i.e. two of them are positive and the last one is negative. They are: $\lambda_1 = -0.09$, $\lambda_2 = 0, 09$, $\lambda_3 = 0.1$. The analytical solution is depicted in Fig. 9.29.

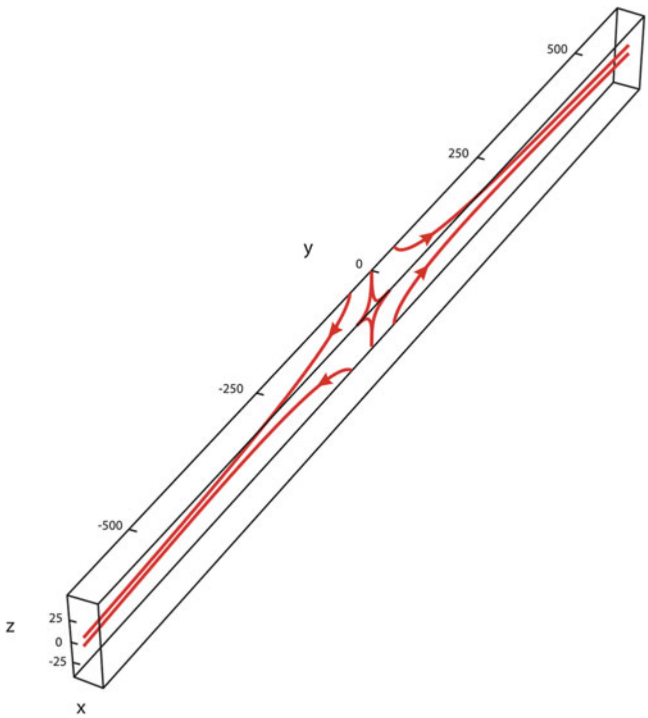


Fig. 9.29 The analytical solution, when a saddle is an equilibrium position in three-dimensional space

The solution is hyperbola, just like in the case of a two-dimensional saddle. In this case, when two roots of the characteristic equation are positive and one is negative, then the solution approaches the equilibrium position.

When a solution of Eq. (9.58) is three real roots of opposite signs, but two of them are negative and one is positive, then we also have to do with equilibrium position of saddle type. The matrix **A** has the following elements:

$$\mathbf{A} = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & -0.07 & 0 \\ 0 & 0 & -0.1 \end{bmatrix}.$$

Roots, as one can predict, are following: $\lambda_1 = 0.1$, $\lambda_2 = -0.07$, $\lambda_3 = -0.1$, and the analytical solution with the use of symmetry principles is depicted in Fig. 9.30.

In Fig. 9.30, similarly to Fig. 5.30, a hyperbola is a solution. This result differs from the previous one, since the phase trajectories move away from the equilibrium position. Roots of the characteristic equation have influence on this situation, since both of them are negative and previously were positive.

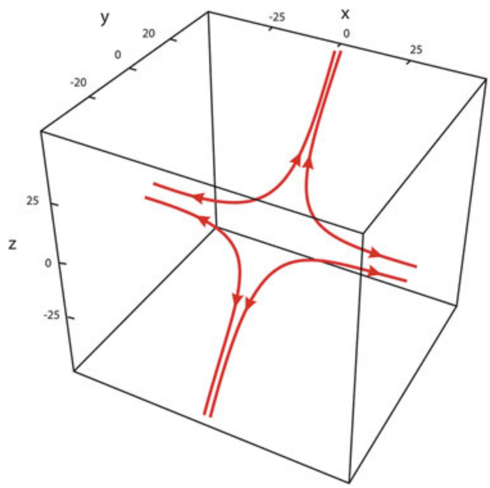


Fig. 9.30 The analytical solution, when a saddle is an equilibrium position in three-dimensional space

9.4.2.4 Unstable Focus

Now, we will consider the cases, when only one root of the characteristic equation (9.58) is real, while two remaining roots are complex conjugate.

When the matrix **A** has the following components

$$\mathbf{A} = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.1 & -1 \\ 0 & 1 & 0.1 \end{bmatrix},$$

then roots of Eq. (9.58) equal: $\lambda_1 = 0.2$, $\lambda_2 = 0.1 + i$, $\lambda_3 = 0.1 - i$. Then we have to do with equilibrium position of unstable focus type. This situation occurs since the real root as well as the real parts of complex roots are positive. The obtained result is depicted in Fig. 9.31.

As a result, we obtained spirals stretching along the x-axis. As one can see a radius of these spirals grows and moves away from the equilibrium position. This type of singularity is called an unstable focus. Verification of this solution is a numerically obtained graph depicted in Fig. 9.32.

The spirals stretching is better seen for a similar matrix **A**, which also characterizes an unstable focus, namely:

$$\mathbf{A} = \begin{bmatrix} 0.05 & 0 & 0 \\ 0 & 0.1 & -1.3 \\ 0 & 1.3 & 0.1 \end{bmatrix}.$$

The roots are following: $\lambda_1 = 0.05$, $\lambda_2 = 0.1 + 1.3i$, $\lambda_3 = 0.1 - 1.3i$, and the solution is depicted in Fig. 9.33.

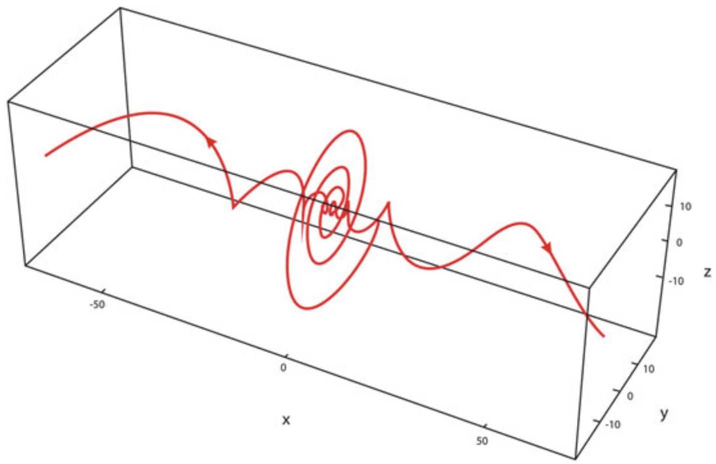
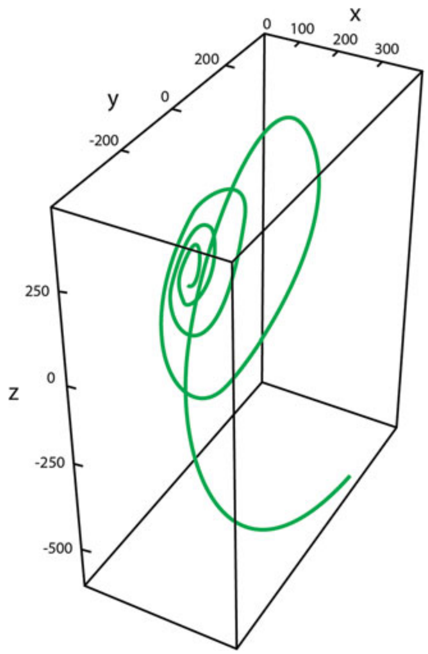


Fig. 9.31 The analytical solution, when an unstable focus is an equilibrium position in three-dimensional space

Fig. 9.32 The numerical solution, when an unstable focus is an equilibrium position in three-dimensional space



9.4.2.5 Stable Focus

Similar graphs, in which phase trajectories approach to the equilibrium position, occur in the case of a stable focus, which can be characterized by the following matrix

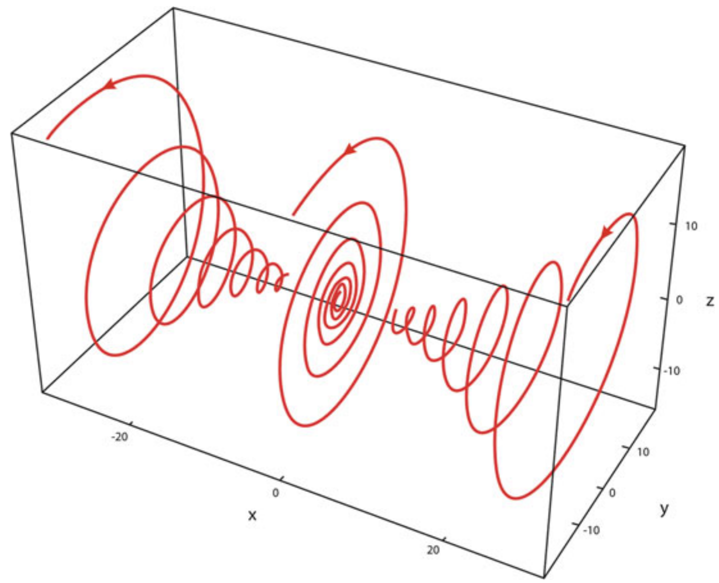


Fig. 9.33 The analytical solution, when an unstable focus is an equilibrium position in three-dimensional space

$$\mathbf{A} = \begin{bmatrix} -0.2 & 0 & 0 \\ 0 & -0.1 & -1 \\ 0 & 1 & -0.1 \end{bmatrix}.$$

In this case, a root of the characteristic equation (9.58) is negative. While complex conjugate roots have a negative real part: $\lambda_1 = -0.2$, $\lambda_2 = -0.1 + i$, $\lambda_3 = -0.1 - i$. Then, the solutions converge to the equilibrium position. This situation is illustrated in Fig. 9.34.

9.4.2.6 Saddle-Node

Similar graphs of phase trajectories can be obtained in the case of equilibrium position of saddle-node type. This singularity occurs when among three roots of Eq. (9.58), the real one is negative and real parts of the remaining complex conjugate roots are positive. It is possible when the matrix **A** has the following elements

$$\mathbf{A} = \begin{bmatrix} -0.1 & 0 & 0 \\ 0 & 0.09 & -0.5 \\ 0 & 0.5 & 0.09 \end{bmatrix}.$$

The roots are: $\lambda_1 = -0.1$, $\lambda_2 = 0.09{+}0.5i$, $\lambda_3 = 0.09{-}0.5i$, and the trajectories are presented in Fig. 9.35.

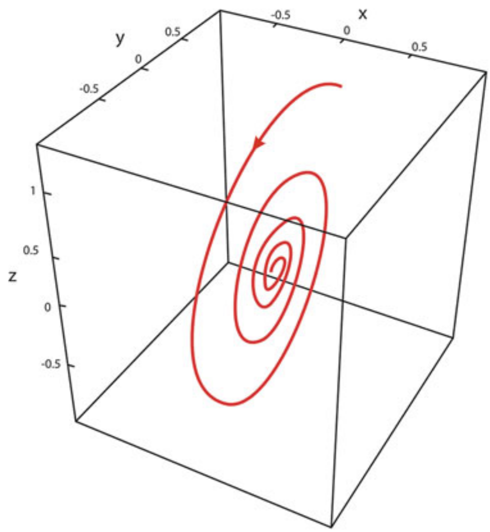


Fig. 9.34 The analytical solution, when a stable focus is an equilibrium position in three-dimensional space

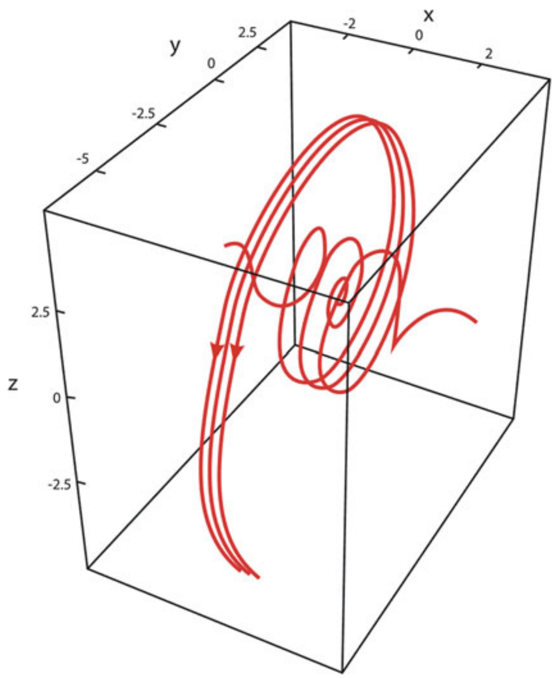


Fig. 9.35 The analytical solution, when a saddle-node is an equilibrium position in three-dimensional space

9.4.2.7 Saddle-Focus

An identical graph but flipped can be obtained if the matrix **A** has the form

$$\mathbf{A} = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & -0.1 & -0.5 \\ 0 & 0.5 & -0.1 \end{bmatrix}.$$

The roots are: $\lambda_1 = 0.1, \lambda_2 = -0.1+0.5i, \lambda_3 = -0.1-0.5i$. One can see that in this case a real root is positive and real parts of the complex roots are negative. The singularity of this type is called a saddle-focus, and the phase trajectories approach the equilibrium. This situation is illustrated in Fig. 9.36.

As one can see by the above considerations (just like in the case of phase plane), the graphs of three-dimensional phase trajectories corresponding to specific equilibrium points depend on coefficients occurring in the characteristic equation (9.58). Only they decide about the number of real roots of this equation and their values.

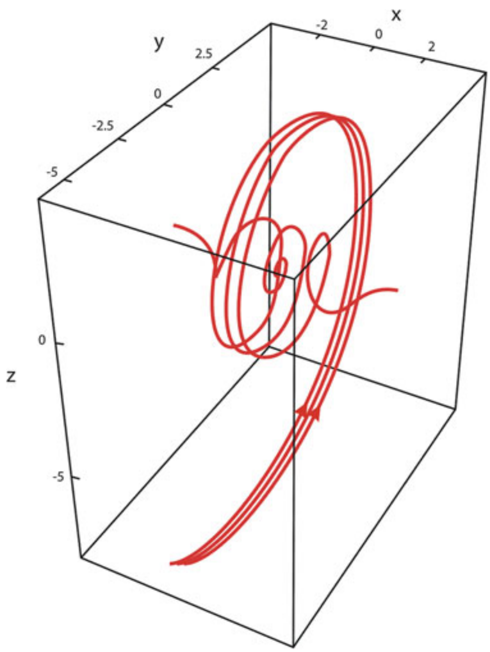


Fig. 9.36 The analytical solution, when a saddle-focus is an equilibrium position in three-dimensional space