# Chapter 7
# Theory and Criteria of Similarity

For many years, theory of similarity has been used either in aware or unaware manner in many branches of science. It turns out that it is impossible to realize the examining real phenomena and processes, or objects due to their complexity and costs. For example, the examining water flow in rivers or behaviour of manned and unmanned flying objects during the flight. In such cases, one usually builds a smaller object (or larger) in such a way that one could carry out measurements in labs by means of modern apparatus. The obtained results should be reliable, i.e. there should exist the possibility of either direct or indirect transition to the real object. Theory of similarity gives an answer to the essential question: what range and under what conditions does a model represent a real object? In practice, a researcher fixes his attention on the object he examines, i.e. on the *similar* object. Sometimes, it is difficult for an individual to examine the proper object or the object is out of reach.

It may happen that engineer's intuition allows building a device that could not have been built on the basis of mathematical models due to the lack of sufficient development in a particular scientific discipline. The primary object (real or imagined) is called *original* and the object similar to the original is called *model*.

Let us notice that theory of similarity is based on dimensional analysis. The theory determines relationships between physical parameters, which influence the phenomenon under consideration. This approach is often used in nonlinear issues, especially in fluid mechanics and aerodynamics, or hydrodynamics.

It should be emphasized that *theory of similarity* is mainly supported by *measurement analysis* and defines dependencies between physical parameters having influence on a being investigated phenomenon. Such approach has been used often in nonlinear problems especially matched with fluid mechanics, aero-mechanics or hydromechanics. Obtained results due to theory of similarity can be transformed from laboratory devices to industry and allow for minimization of measurement numbers and costs.



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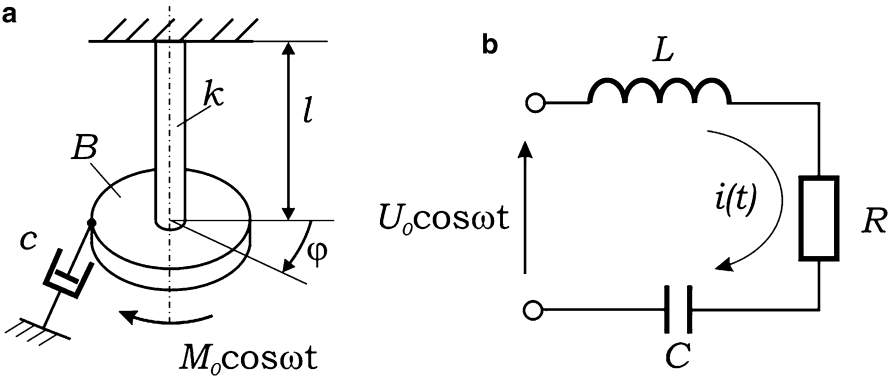


**Fig. 7.1** The original (mechanical system) and the model (electrical system)

Traditionally the following partition of *similarity* has been introduced.

(i) *Geometrical*—shapes and dimensions of two objects are similar, e.g. a ratio of two characteristic dimensions;
(ii) *Kinematical*—distributions of physical fields lines are similar, e.g. velocity, pressure or acceleration;
(iii) *Dynamical*—scale of similarity of different characteristic quantities is the same. In the case of incompressible fluids, Reynolds number *Re* is such number and in the case of compressible fluids one uses Mach, Strouhal or Prandtl numbers. This description requires introducing concepts of force and torque, or tension.

Let us remind that Reynolds number $Re = \frac{\rho l v}{\mu}$ where $l$ is characteristic dimension (e.g. diameter of a pipe), $v$ is characteristic velocity of fluid and $\mu$ is kinematical viscosity, and $\rho$ is density. For $Re < 2{,}300$ we have a laminar flow, for $2{,}300 < Re < 10{,}000$ transition flow and for $Re > 10{,}000$ turbulent flow appears.

Similarity numbers allow for qualitative evaluation of the examined phenomena. They are often used in applications. They follow: Abbe, Archimedes, Arrhenius, Biot, Euler, Fourier, Rayleigh or Weber numbers. Note that relationships original-model can concern both material systems and processes (e.g. flow), i.e. phenomena in general.

As an example, we consider a vibration phenomenon described by a second-order differential equation in the mechanical and electrical systems (Fig. 7.1).

The mechanical system equations of motion, based on Newton's second law, can be written in the form

$$B\ddot{\varphi} + c\dot{\varphi} + k\varphi = M_0 \cos \omega t, \tag{7.1}$$

where $(\cdot) \equiv \frac{d}{dt}$, $B$ is moment of inertia of a shield suspended on a weightless rod of torsion rigidity $k = (GI_0)l^{-1}$, $G$ is shear modulus, $I_0$ is moment of inertia of cross-section and $l$ is length of the rod. $M_0$ and $\omega$ are amplitude and frequency of harmonic excitation respectively, $\varphi$ is generalized coordinate (angle measured in radian).

The electrical system equation of motion is based on Kirchhoff's law, i.e. we equate the sum of voltage drops on all elements of the circuit to the voltage on terminals. We obtain

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i dt = U_0 \cos \omega t, \tag{7.2}$$

where $L$ is self-induction coefficient, $R$—resistance, $C$—capacitance, $i$—current intensity, $U_0$ and $\omega$ are amplitude and frequency of external voltage (driving), respectively.

Since $i = \frac{dq}{dt}$, where $q$ is electric charge, then by Eq. (7.2) one gets

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = U_0 \cos \omega t. \tag{7.3}$$

It is easy to see that Eqs. (7.1) and (7.3) are similar. We have the following similar quantities: the rotation angle $\varphi$ and the charge $q$; the angular velocity $\dot{\varphi}$ and the current intensity $i$; the moment of inertia $B$ and the self-induction coefficient $L$; the viscous damping coefficient $c$ and the resistance $R$; torsion rigidity $k$ and reciprocal of the capacitance $C^{-1}$; the amplitude of the torque $M_0$ and the voltage $U_0$; frequency $\omega$ of torque and voltage.

We aim to determine the "period of damped vibrations in both systems". On this purpose, we transform the homogeneous equation, obtained from (7.1) and (7.3), into the form

$$\ddot{\varphi} + 2h\dot{\varphi} + \alpha^2\varphi = 0, \tag{7.4}$$

$$q'' + 2h'q' + \alpha'^2 q = 0, \tag{7.5}$$

where $\alpha^2 = \frac{k}{B}$, $2h = \frac{c}{B}$; $\alpha'^2 = (LC)^{-1}$, $2h' = \frac{R}{L}$, $' = \frac{d}{dt'}$, $\dot{} = \frac{d}{dt}$.

In order to determine relationships between the original and the model, we make use of *dimensional analysis*. Let us introduce the following values of the scales

$$k_\varphi = \frac{\varphi}{q}, \quad k_t = \frac{t}{t'}, \quad k_h = \frac{h}{h'}, \quad k_\alpha = \frac{\alpha}{\alpha'}, \tag{7.6}$$

where ($'$) refers to the model. Making use of (7.6) in (7.4) we get

$$\frac{k_\varphi}{k_t^2}q'' + \frac{k_\varphi k_h}{k_t}2h'q' + k_\varphi k_\alpha^2 \alpha'^2 q = 0. \tag{7.7}$$

The equations of the original and the model are identical when there are the following relationships between the scales

$$\frac{k_\varphi}{k_t^2} = \frac{k_\varphi k_h}{k_t} = k_\varphi k_\alpha^2. \tag{7.8}$$

The above relationships imply

$$k_t k_h = 1, \quad k_\alpha k_t = 1, \\ k_\alpha = k_h \tag{7.9}$$

which means

$$ht = h't' \quad \text{and} \quad \alpha t = \alpha' t',$$

or

$$\frac{h}{\alpha} = \frac{h'}{\alpha'}. \tag{7.10}$$

We have obtained the criterion of similarity (7.10). This means that some dimensionless combinations of the signal parameters must equal to the analogous combinations of the model parameters. By Eq. (7.10) we get as well

$$\frac{c}{\sqrt{kB}} = R\sqrt{\frac{C}{L}} \tag{7.11}$$

The above equation implies that two of the three parameters $R, C, L$ can be chosen freely but the third one is obtained from (7.11).

When we have a model (the electrical system) we can measure the period of damping vibrations $T'$, and then determine the period of vibrations in the mechanical system (the original) in the following way

$$T = k_t T' \tag{7.12}$$

where

$$k_t = \frac{1}{k_\alpha} = \frac{\alpha'}{\alpha} = \frac{\sqrt{\frac{1}{LC}}}{\sqrt{\frac{k}{B}}} = \sqrt{\frac{B}{kLC}}. \tag{7.13}$$

There is another often applied method of obtaining similarity criteria, namely the transforming equations of both the original and the model into the same dimensionless form, and the imposing of identity conditions on the form of the equations and their coefficients. The vibration phenomenon will be described by the only one dimensionless differential equation of second order. We introduce to the original and to the model, the following dimensionless values

$$\varphi_1 = \frac{\varphi}{\varphi_0}, \quad t_1 = \frac{t}{T}, \quad q_1 = \frac{q}{q_0}, \quad t_1' = \frac{t'}{T'}, \tag{7.14}$$

where $\varphi_0$, $q_0$, $T'$ and $T$ are values of the angle and the charge, and the time (period).

Inserting (7.14) into Eqs. (7.1) and (7.3) we get

$$\frac{B\ddot{\varphi}_1}{kT^2} + \frac{c}{kT}\dot{\varphi}_1 + \varphi_1 = \frac{M_0}{k\varphi_0} \cdot \cos 2\pi t_1, \tag{7.15}$$

$$\frac{LC\ddot{q}_1}{T'^2} + \frac{RC}{T'}\dot{q}_1 + q_1 = \frac{U_0 C}{q_0} \cdot \cos 2\pi t_1, \tag{7.16}$$

since the cosine input values are assumed to be the same, namely $\omega T = 2\pi$, $\omega' T' = 2\pi, \omega' t' = \omega t$.

One can assume that $\frac{\omega}{\omega'} = \frac{t'}{t} = \frac{T'}{T} = 1$, which implies $t_1 = t_1{}'$.

Equations (7.15) and (7.16) possess the same independent variable $t_1$. The quantities $\varphi_1$ and $q_1$ are dimensionless, so the original and the model can be represented by the second-order differential equation

$$(T_1^2 p^2 + T_0 p + 1)x = K \cos 2\pi t_1 \tag{7.17}$$

where

$$\begin{aligned} p &= \frac{d}{dt_1}, \\ x &= \varphi_1 = q_1, \\ T_1 &= \frac{B}{kT^2} = \frac{LC}{T'^2}, \\ T_0 &= \frac{c}{kT} = \frac{RC}{T'}, \\ K &= \frac{M_0}{k\varphi_0} = \frac{U_0 C}{q_0}. \end{aligned} \tag{7.18}$$

One can say that the obtained relationships are equivalent to (7.10), (7.11).

Now, let us come to the third way of determining similarity criteria between the original and the model, using Eqs. (7.4) and (7.5). Each of the equation terms (7.4) or (7.5) possesses the same dimension. The terms can be written in the following way

$$\left[\frac{\varphi}{t^2}\right] = \left[\frac{h\varphi}{t}\right] = [\alpha^2 \varphi], \tag{7.19}$$

dividing the above equality by $\left[\frac{\varphi}{t^2}\right]$ we get

$$[1] = [ht] = [\alpha^2 t^2]. \tag{7.20}$$

This implies the products $ht$ and $\alpha t$ are dimensionless combinations of dimensional quantities for the mechanical system. Similarly, with regard to Eq. (7.5) we get that the products $h't'$ and $\alpha't'$ are dimensionless combinations of dimensional quantities for the electric system.

Newton formulated the following law of similarity of two phenomena.

> We say that two phenomena are similar to each other when the quantities occurring in the equations which describe the phenomena form dimensionless combinations are of equal values.

In regard to the analysed example, the similarity criteria are the following

$$ht = h't' \quad \text{and} \quad \alpha t = \alpha' t' \tag{7.21}$$

which lead to (7.10).

The example was based on the knowledge of equations describing the phenomena in the original and the model.

Now, we will try to determine similarity criteria only through the analysis of physical quantities characterizing the phenomenon (the equations describing the phenomena are unknown), basing ourselves on dimensional analysis.

Every dimensional quantity can be presented in the following way

$$Q = q x_1^{a_1} x_2^{a_2} \ldots x_m^{a_m}, \tag{7.22}$$

where $x_1, \ldots, x_m$ are dimensional units, $q$ is dimensionless, and $a_1, \ldots, a_m$ are real numbers. As an example, let us put $1\,W = 1\,\mathrm{N\,ms^{-1}}$ or $g = 9.81\,\mathrm{ms^{-2}}$.

Let $n$ dimensional quantities $Q_1, \ldots, Q_n$ be given in the description of the examined phenomenon, however only $m$ measurement units are required to complete the description, $n > m$. It turns out that the choice of these $m$ measuring units is free provided that the condition of dimension independence holds.

We say that the quantities $Q_1, \ldots, Q_m$ are dimension-independent if the equality

$$Q_1^{r_1} Q_2^{r_2} \ldots Q_n^{r_n} = \gamma,\ \gamma > 0,\ \gamma, r_1 \ldots r_m \in R \tag{7.23}$$

implies $r_1 = r_2 = \ldots = r_m = 0$ and $\gamma = 1$.

*Example 7.1.* Show that the displacement $s$, the acceleration $p$ and the force $F$ are dimension-independent.

According to (7.23) we have

$$s^{r_1} p^{r_2} F^{r_3} = \gamma,$$

and after taking into account the dimensions

$$m^{r_1} \left(ms^{-2}\right)^{r_2} \left(kgms^{-2}\right)^{r_3} = m^o kg^o s^o,$$

or equivalently

$$m^{r_1+r_2+r_3} kg^{r_3} s^{-2r_2-2r_3} = m^o kg^o s^o.$$

Equating the exponents of the same bases we get

$$\begin{aligned} r_1 + r_2 + r_3 &= 0, \\ r_3 &= 0, \\ -2\,(r_2 + r_3) &= 0, \end{aligned}$$

hence

$$r_1 = r_2 = r_3 = 0$$

which proves that $s$, $p$ and $F$ are dimension-independent.

One can do it in another way. On this purpose, one must determine the rank of the exponents matrix of each dimensional quantity. For the considered case from the example 7.1 we have

| | $m$ | $kg$ | $s$ |
|---|---|---|---|
| $s$ | 1 | 0 | 0 |
| $p$ | 1 | 0 | −2 |
| $F$ | 1 | 1 | −2 |

Three quantities $s$, $p$ and $F$ are linearly independent because the determinant of third order is nonzero, i.e.

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & -2 \\ 1 & 1 & -2 \end{vmatrix} = \begin{vmatrix} 0 & -2 \\ 1 & -2 \end{vmatrix} = 2$$

If there are $m$ of the $n$ dimension-independent quantities, then it is easy to determine $n - m$ non-dimensional combinations of these quantities referred to the form

$$\begin{aligned} Q_{m+1} Q_1^{r_{1,1}} Q_2^{r_{2,1}} \ldots Q_m^{r_{m,1}} &= \gamma_1, \\ Q_{m+2} Q_1^{r_{1,2}} Q_2^{r_{2,2}} \ldots Q_m^{r_{m,2}} &= \gamma_2, \\ &\vdots \\ Q_n Q_1^{r_{1,n-m}} Q_2^{r_{2,n-m}} \ldots Q_m^{r_{m,n-m}} &= \gamma_{n-m}, \end{aligned} \tag{7.24}$$

or of analogous form referred to the model (prime mark):

$$\begin{aligned} Q'_{m+1} Q_1'^{r_{1,1}} Q_2'^{r_{2,1}} \dots Q_m'^{r_{m,1}} &= \gamma'_1, \\ Q'_{m+2} Q_1'^{r_{1,2}} Q_2'^{r_{2,2}} \dots Q_m'^{r_{m,2}} &= \gamma'_2, \\ &\vdots \\ Q'_n Q_1'^{r_{1,n-m}} Q_2'^{r_{2,n-m}} \dots Q_m'^{r_{m,n-m}} &= \gamma'_{n-m}. \end{aligned} \tag{7.25}$$

The similarity criteria in number $n - m$ are obtained from the equations

$$\gamma_1 = \gamma'{}_1, \dots, \gamma_{n-m} = \gamma'{}_{n-m}. \tag{7.26}$$

Let us introduce the following scales of independent quantities

$$k_i = \tfrac{Q_i}{Q'_i},\ i = 1, \dots, m \tag{7.27}$$

and dependent quantities

$$k_{m+1} = \frac{Q_{m+1}}{Q'_{m+1}}, \tag{7.28}$$

After dividing both sides of Eqs. (7.24) and (7.25), and making use of (7.26)–(7.28) we obtain

$$k_{m+1} k_1^{r_{1,1}} k_2^{r_{2,1}} \dots k_m^{r_{m,1}} = 1, \tag{7.29}$$

hence

$$k_{m+1} = k_1^{-r_{1,1}} k_2^{-r_{2,1}} \dots k_m^{-r_{m,1}}. \tag{7.30}$$

Product of type (7.24) takes the form

$$\alpha t^a = \gamma_1, \quad h t^b = \gamma_2, \tag{7.31}$$

where $\alpha$ and $h$ are dimension-dependent.

From the first equation (7.31) we have

$$s^{-1} s^a = s^0, \tag{7.32}$$

hence $a = 1$, and the first invariant takes the form

$$\alpha t = \gamma_1. \tag{7.33}$$

The second equation (7.31) leads to determining the second invariant of the form

$$ht = \gamma_2. \tag{7.34}$$

Analogous considerations, carried out for the electrical system, lead to determining invariants of the form

$$\alpha' t' = \gamma'_1, \quad \text{and} \quad h' t' = \gamma'_2. \tag{7.35}$$

Since we get the following similarity criteria

$$\alpha t = \alpha' t', \quad \text{and} \quad ht = h' t', \tag{7.36}$$

dividing both sides we get the invariant

$$\frac{\alpha}{\alpha'}\frac{t}{t'} = \frac{\gamma_1}{\gamma_1'} \equiv 1, \quad \frac{h}{h'}\frac{t}{t'} = \frac{\gamma_2}{\gamma_2'} \equiv 1, \tag{7.37}$$

thus $k_\alpha k_t = 1, k_h k_t = 1$. □

At the end, we consider the procedure of transforming Duffing's equation into dimensionless form.

*Example 7.2.* Let us analyse an oscillator described by the second-order differential equation

$$\frac{d^2\bar{u}}{d\bar{t}^2} + f(\bar{u}) = 0.$$

We Maclaurin-expand the nonlinear function

$$f(\bar{u}) = f(0) + \bar{u}\frac{df}{d\bar{u}}(0) + \frac{1}{2}\bar{u}^2\frac{d^2 f}{d\bar{u}^2}(0) + \frac{1}{3!}\bar{u}^3\frac{d^3 f}{d\bar{u}^3}(0) + \cdots .$$

Let us take

$$f(0) = \frac{d^2 f}{d\bar{u}^2}(0) = 0 \quad \text{and} \quad \frac{df}{d\bar{u}}(0) > 0,$$

hence the resulting equation takes the form

$$\frac{d^2\bar{u}}{d\bar{t}^2} + \bar{u}\frac{df}{d\bar{u}}(0) + \frac{1}{6}\bar{u}^3\frac{d^3 f}{d\bar{u}^3}(0) = 0.$$

After introduction of the following non-dimensional quantities $u = \bar{u}l^{-1}$, $t = \bar{t}T^{-1}$ one gets

$$\frac{l}{T^2}\frac{d^2u}{dt^2} + ul\frac{df}{d\bar{u}}(0) + \frac{1}{6}u^3l^3\frac{d^3f}{d\bar{u}^3}(0) = 0,$$

which leads to the equation

$$\frac{d^2u}{dt^2} + uT^2\frac{df}{d\bar{u}}(0) + \frac{1}{6}u^3T^2l^2\frac{d^3f}{d\bar{u}^3}(0) = 0. \qquad (*)$$

Assume that

$$\left[T^2\frac{df}{d\bar{u}}(0)\right] = 1,$$

what follows from the often applied combination $[FT^2/ML] = 1$, where $L$ is the length, $T$ the time, $M$ the mass and $F$ the force. Note that $[f] = Nkg^{-1}$, $[\bar{u}] = m$, $\left[T^2\right] = s^2$, and hence

$$\left[T^2\frac{df}{d\bar{u}}(0)\right] = s^2(kgms^{-2})kg^{-1}m^{-1} = 1.$$

Equation (∗) takes the following dimensionless Duffing-type form

$$\frac{d^2u}{dt^2} + au + bu^3 = 0,$$

where:

$$a = \frac{df^*}{du}(0), \quad b = \frac{1}{6}\frac{d^3f^*}{du^3}(0), \quad f^* = \frac{fT^2}{l}.$$

Transformation of the dimensional equation into dimensionless one has a significant meaning. It enables to generalize the results, i.e. transition from one scientific discipline to another one, e.g. from mechanics to electrics. Moreover, transforming equations into dimensionless form sometimes allow the number of parameters to be reduced significantly. It has non-trivial meaning in analysis, especially during numerical computations. □