Chapter 5
Higher-Order ODEs Polynomial Form

5.1 Introduction

If a function $f(t, x, \dot{x}, \ldots, x^{(n)})$ is defined and is continuous in a subset of $\mathbb{R}^{n+2} (n \geq 1)$, then the equation

$$f(t, x, \dot{x}, \ldots, x^{(n)}) = 0 \tag{5.1}$$

is said to be *ordinary differential equation of nth-order*.

Fortunately, it happens very often that (5.1) can be transformed to the form

$$x^{(n)} = f_1(t, x, \dot{x}, \ldots, x^{(n-1)}). \tag{5.2}$$

The Cauchy problem for Eq. (5.2) is that of finding a solution $x(t)$ satisfying the series of the following initial conditions

$$x(t_0) = x_0, \quad \dot{x}(t_0) = \dot{x}_0, \ldots, x^{(n-1)}(t_0) = x_0^{(n-1)}. \tag{5.3}$$

Theorem 5.1 (Peano). *If the function f_1 is continuous in certain open subset, then for an arbitrary point $(t_0, x_0, \dot{x}_0, \ldots, x_0^{(n-1)})$ belonging to this subset there is a solution to Eq. (5.2) defined in a neighbourhood of t_0, which satisfies (5.3).*

Theorem 5.2 (Cauchy–Picard). *If the function f_1 satisfies both the conditions of Theorem 5.1 and the Lipschitz conditions with regard to the variables $x, \dot{x}, \ldots, x^{(n-1)}$, then for arbitrary initial conditions (5.3) there is only one solution (uniqueness) to Eq. (5.2).*

J. Awrejcewicz, *Ordinary Differential Equations and Mechanical Systems*,
DOI 10.1007/978-3-319-07659-1_5

Recall that the Lipschitz constant L for the first-order case can be found from the inequality

$$\left|\frac{f(t,x_1)-f(t,x_2)}{x_1-x_2}\right| = \left|\frac{\partial f}{\partial x}(x)\right| \le L. \tag{5.4}$$

In words the Lipschitz constant can be defined by an upper bound of $\frac{\partial f}{\partial x}$. In the nth-order case, when $f = f(t, x_1, \ldots, x_n)$, the Lipschitz constant L can be defined by the inequality

$$\text{Max}\left\{\left|\frac{\partial f}{\partial x_1}\right|, \ldots, \left|\frac{\partial f}{\partial x_n}\right|\right\} \le L. \tag{5.5}$$

To show a uniqueness of a solution let us recall the example of the equation $\frac{\partial f}{\partial x} = 1+x^2$ given in the book [191]. In this case $f(t,x_1) = 1+x_1^2$ and $f(t,x_2) = 1+x_2^2$. Hence $|f(t,x_1) - f(t,x_2)| = \left|x_1^2 - x_2^2\right| = |x_1 - x_2|\,|x_1 + x_2| \le 2\,|x_1 - x_2|$ in the rectangle $|t| < 1, |x| < 1$. Theorem 5.2 states that there exists only one solution passing through (0,0). In fact, this solution has the following analytical form: $x(t) = \text{tg}(t)$. Let D be a domain composed of points, where any point corresponds to only one solution of the Cauchy problem. The function

$$x = \phi(t, C_1, \ldots, C_n) \tag{5.6}$$

is said to be a *general solution* to Eq. (5.2), if the following assumptions are satisfied:

(i) the function ϕ has the nth-order derivative with respect to t;
(ii) for any point defined by the series (5.3) the following equations

$$\begin{aligned} x_0 &= \phi(t, C_1, \ldots, C_n),\\ \dot{x}_0 &= \dot{\phi}(t, C_1, \ldots, C_n),\\ &\vdots\\ x_0^{(n-1)} &= \phi^{(n-1)}(t, C_1, \ldots, C_n), \end{aligned} \tag{5.7}$$

have the unique solutions with regard to the constants $C_1^0, \ldots, C_n^0$;
(iii) the function $\phi(t, C_1^0, \ldots, C_n^0)$ is the solution to (5.2) for arbitrary constants $C_1^0, \ldots, C_n^0$, which are the solutions to (5.7).

If a general solution is given in the implicit way

$$\phi_0(t, x, C_1, \ldots C_n) = 0, \tag{5.8}$$

then Eq. (5.8) defines a general solution to Eq. (5.2).

Any arbitrary solution (5.6) for the specified values of constants $C_1, \ldots, C_n$ is said to be the *particular solution* of Eq. (5.2). The following steps are required in order to find a particular solution knowing a general one:

(i) the algebraic set of Eq. (5.7) should be derived from either (5.6) or (5.8), which yields the constants $C_1, \ldots, C_n$;
(ii) the found specified values $C_1^0, \ldots, C_n^0$ are substituted to (5.6) or (5.8), which are now solutions to a Cauchy problem.

Note that sometimes a general solution can be represented in a parametric form

$$\begin{aligned} t &= t(p, C_1, \ldots, C_n), \\ x &= x(p, C_1, \ldots, C_n), \end{aligned} \tag{5.9}$$

where p is a parameter. In the most general case governed by Eq. (5.1), the following theorem satisfies an existence and uniqueness of a Cauchy problem.

Theorem 5.3. *Assume that the function defined by (5.1) is continuous and posses-ses the continuous derivatives with regard to $x, \dot{x}, \ldots, x^{(n)}$. Hence, for an arbitrary point $(t_0, x_0, \dot{x}_0, \ldots, x_0^{(n)})$ such that*

$$f(t_0, x_0, \dot{x}_0, \ldots, x_0^{(n)}) = 0, \quad \frac{\partial f}{\partial x^{(n)}}(t_0, x_0, \dot{x}_0, \ldots, x_0^{(n)}) \neq 0, \tag{5.10}$$

there is exactly only one solution to Eq. (5.1) defined in the neighbourhood of t_0 and satisfying the initial conditions (5.3).

A general solution constitutes of a family of integral curves in the plane (t, x) with n parameters $C_1, \ldots, C_n$.

5.2 Linear Homogeneous Differential Equations

Consider the following homogeneous nth-order differential equation

$$L_n(y) \equiv y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0, \tag{5.11}$$

where L_n is called the nth-order linear differential operator (here $y^{(n)} = \frac{d^n y}{dt^n}$). Recall that in general, a linear differential operator (a function) has the following properties:

(i) $L(Cy) = CL(y)$,
(ii) $L(y_1 + y_2) = L(y_1) + L(y_2)$,
(iii) $L(C_1y_1 + \cdots + C_my_m) = C_1L(y_1) + \cdots + C_mL(y_m)$, for any y_i and C_i.

Theorem 5.4. *If $y_1, \dots, y_n$ are the solutions to homogeneous equation (5.11), then $y = C_1 y_1 + \cdots + C_n y_n$ is also a solution to $L_n(y) = 0$ for arbitrary constant numbers $C_1, \dots, C_n$.*

Remark 5.1. The set of functions $\{y_1(t), \dots, y_n(t)\}$ is said to be a *fundamental set of solutions* of $L_n(y) = 0$.

Remark 5.2. The function $y(t) = C_1 y_1(t) + \cdots + C_n y_n(t)$ is called the *complementary* function of $L_n(y) = 0$

Definition 5.1 (Linearly Independent Functions). Let $u_1(t), \dots, u_m(t)$ are the functions defined in the interval $a \le t \le b$. If there are numbers $\alpha_1, \dots, \alpha_m$, not all of them equal to zero, and the following equation is satisfied

$$z = \alpha_1 u_1 + \cdots + \alpha_m u_m \equiv 0, \qquad a \le t \le b, \tag{5.12}$$

then we say that the functions $u_1(t), \dots, u_m(t)$ are linearly dependent in the considered interval.

The vector z is called a *linear combination* of the members of the set $\{u_1, \dots, u_m\}$, whereas the set $\{\alpha_1, \dots, \alpha_m\}$ is called a *set of coefficients* of the linear combination.

Recall that:

(i) the set of all linear combinations of members of $\{y_1, \dots, y_m\}$ is called the *linear span* (or *simply span*) of $\{y_1, \dots, y_m\}$;

(ii) a linear space U is a set that satisfies the following properties:

  (a) if $u, v \in U$ then $u + v \in U$;
  (b) if $u \in U$ and $c \in R$, then $Cu \in U$;

(iii) the span defined in (i) forms a linear space.

Assuming that Eq. (5.12) is satisfied if and only if $\alpha_1 = \cdots = \alpha_m \equiv 0$, then the functions $u_1, \dots, u_m$ are called *linearly independent*. A linearly independent spanning set for a linear space $U = \{u_1, \dots, u_m\}$ is called *a basis for U*. A dimension of U is defined by the number of vectors in a basis of vector space U. A necessary condition for linear dependence of the functions $u_1, \dots, u_n$ can be obtained in the following way. Let the given functions $u_1, \dots, u_n$ to be are linearly dependent. It means that they satisfy (5.12). Differentiating $n-1$ times Eq. (5.12) one gets

$$\alpha_1 u_1^{(k)} + \cdots + \alpha_n u_n^{(k)} = 0, \quad k = 1, \dots, n-1. \tag{5.13}$$

Of course, we have assumed that the functions $\{u_1, \dots, u_n\}$ have $n-1$ continuous derivatives on some interval (they are C^{n-1} smooth). It is known from algebra that the system of Eqs. (5.12) and (5.13) has non-trivial solution of $\alpha_1, \dots, \alpha_n$, when the following determinant function (called Wronskian) is equal to zero:

$$W(u_1,\ldots,u_n)=\begin{vmatrix} u_1 & \cdots & u_n \\ u_1' & \cdots & u_n' \\ \vdots & & \\ u_1^{(n-1)} & \cdots & u_n^{(n-1)} \end{vmatrix}=0,\quad a\le t\le b. \tag{5.14}$$

Theorem 5.5. *If the functions $y_1,\ldots,y_n$ are the solutions of $L_n(y)=0$ on $a\le t\le b$, then $W(y_1(t),\ldots,y_n(t))$ is either zero for every t from [a,b] or is never zero for any $t\in[a,b]$.*

Proof. We take $n=2$ and follow the proof given in the book [191]. Here we omit a generalization for any n. Consider the following linear operator with variable coefficients

$$L_2(y)\equiv y''+p_1(t)y'+p_2(t)y=0.$$

The Wronskian

$$W(y_1,y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=y_1y_2'-y_2y_1'.$$

On the other hand

$$\begin{aligned} W' &= (y_1y_2'-y_2y_1')'=y_1y''_2-y''_1y_2=y_1(-p_1y_2'-p_2y_2)-y_2(-p_1y_1'-p_2y_1)\\ &=-p_1(y_1y_2'-y_2y_1')=-p_1W, \end{aligned}$$

and hence

$$\ln W=-\int p_1dt+\ln C.$$

Finally, one gets

$$W(y_1,y_2)=Ce^{-\int p_1(t)dt},$$

which for $C=0$ is identically zero, or for $C\neq 0$ is never zero. □

Theorem 5.6. *If $y_1,\ldots,y_n$ are solutions of the equation $L_n(y)=0$ and $W(y_1,\ldots,y_n)|_{t=t_0}=0$, then $W(y_1,\ldots,y_n)\equiv 0$ for $a\le t\le b$, and $y_1,\ldots,y_n$ are linearly dependent solutions in the interval $t\in[a,b]$.*

Theorem 5.7. *A necessary and sufficient condition for solutions $z_1,\ldots,z_n$ of $L_n(z)=0$ to be linearly independent in an arbitrary point is $W(z_1,\ldots,z_n)|_{t=t_0}\neq 0$. If $W(t_0)\neq 0$, then also $W(t)\neq 0$ for $a\le t\le b$, and if $W(t_0)=0$, then also $W(t)=0$.*

Theorem 5.8. *If $y_1, \dots, y_n$ are linearly independent solutions of the equation $L_n(y) = 0$, then $y = C_1 y_1 + \dots + C_n y_n$ is a general solution for $a \le t \le b$, $-\infty < y_k < +\infty$, $k = 0, 1, \dots, n-1$.*

In the case of a homogeneous linear nth-order equation $L_n(y) = 0$, there are n linearly independent solutions $y_1, \dots, y_n$ forming the *fundamental set of solutions* (proof is omitted here). It is interesting to note that the Wronskian can be used to define a differential equation, when a fundamental set is known.

Recall that a set of all solutions of the linear equation $L(y) = 0$ is called a *null space* or *kernel* of the operator L.

Theorem 5.9. *A homogeneous linear differential equation $L_n(y) = 0$ has n linearly independent solutions that form a basis for the set of all solutions. The dimension of the kernel of an* n*th-order linear differential operator is n.*

Example 5.1. Define a linear differential equation possessing the fundamental solution: $y_1(t) = \sin t$, $y_2(t) = e^t$.

We define the Wronskian

$$
\begin{aligned}
W(\sin t, e^t, y) &= \begin{vmatrix} \sin t & e^t & y \\ \cos t & e^t & y' \\ -\sin t & e^t & y'' \end{vmatrix} e^t \begin{vmatrix} \sin t & 1 & y \\ \cos t & 1 & y' \\ -\sin t & 1\&y'' & \end{vmatrix} \\
&= e^t \left(\sin t \begin{vmatrix} 1 & y' \\ 1 & y'' \end{vmatrix} - 1 \begin{vmatrix} \cos t & y' \\ -\sin t & y'' \end{vmatrix} + y \begin{vmatrix} \cos t & 1 \\ -\sin t & 1 \end{vmatrix} \right) \\
&= e^t [\sin t (y'' - y') - (y'' \cos t + y' \sin t) + y(\cos t + \sin t)] \\
&= e^t [(\sin t - \cos t) y'' - 2 \sin t + y' + (\sin t + \cos t) y] = 0,
\end{aligned}
$$

and the being sought differential equation is defined immediately as

$$L_2(y) = (\sin t - \cos t) y'' - 2 \sin t y' + (\sin t + \cos t) y = 0.$$

In order to verify the obtained results we check:

$$
\begin{gathered}
L_2(\sin t) = -(\sin t - \cos t) \sin t - 2 \sin t \cos t + (\sin t + \cos t) \sin t = 0, \\
L_2(e^t) = e^t L_2(\sin t) = 0.
\end{gathered}
$$

□

5.3 Differential Equations with Constant Coefficients

Since characteristic equation corresponding to an nth-order homogeneous differential equation cannot be solved using radicals already for $n \ge 5$, very often numerical or approximate analytical methods are used. The most important observation is that the nth-order differential equation is reduced to the nth-order polynomial (algebraic) equation. The solutions differ from each other qualitatively for distinct and repeated (multiple) roots of a characteristic equation, which we are going to discuss.

Theorem 5.10. *Given the* n*th-order linear differential equation* $L_n(y) = y^{(n)} + p_1 y^{(n-1)} + \cdots + p_{n-1} y' + p_n y = 0$. *Let the characteristic polynomial corresponding to* $L_n(y) = 0$ *possess:*

(i) *n distinct real roots r_i, $i = 1, \ldots, n$. Then the n functions $y_i(t) = e^{r_i t}$ form a basis for the kernel of $L_n(y) = 0$. A general solution has the form*

$$y(t) = \sum_{i=1}^{n} C_i e^{r_i t};$$

(ii) *Single root repeated n times. Then the n functions $y_1(t) = e^{rt}, y_2(t) = te^{rt}, \ldots, y_n(t) = t^{n-1}e^{rt}$ form a basis for the kernel of $L_n(y) = 0$. A general solution can be written as*

$$y(t) = (C_1 + c_2 t + \cdots + C_n t^{n-1})e^{rt};$$

(iii) *Single root repeated m times and there are $n - m$ distinct roots $r_{m+1}, r_{m+2}, \ldots, r_n$. Then the n functions*

$$\begin{aligned} y_1(t) &= e^{rt}, \\ y_2(t) &= te^{rt}, \\ &\vdots \\ y_m(t) &= t^{m-1}e^{rt}, \\ y_{m+1}(t) &= e^{r_{m+1}t}, \\ &\vdots \\ y_n(t) &= e^{r_n t} \end{aligned}$$

form the kernel of $L_n(y) = 0$. A general solution has the form

$$y(t) = (C_1 + C_2 + \cdots + C_m t^{m-1})e^{rt} + C_{m+1}e^{r_{m+1}t} + \cdots + C_n e^{r_n t}.$$

Example 5.2. Consider the following third-order differential equation

$$L_3(y) = y''' - 8y'' + 19y' - 12y = 0.$$

Introducing $y = e^{rt}$ one obtains the characteristic polynomial $r^3 - 8r^2 + 19r - 12 = 0$, with three distinct roots: $r_1 = 1, r_2 = 3, r_3 = 4$. The linearly independent solutions are: $y_1(t) = e^t, y_2(t) = e^{3t}, y_3(t) = e^{4t}$, and every solution is written as $y(t) = C_1 e^t + C_2 e^{3t} + C_3 e^{4t}$. □

Example 5.3. Consider the fifth-order differential equation $L_5(y) = y^{(5)} - 12y^{(4)} + 56y''' - 126y'' + 135y' - 54 = 0$.

The characteristic equation corresponding to $L_5(y)$ has the following roots: $r_1 = 1$, $r_2 = 2$ and $r_3 = r_4 = r_5 = 3$ (multiplicity 3). The following set forms a basis $\{e^t,\ e^{2t},\ e^{3t},\ te^{3t},\ t^2e^{3t}\}$. Every solution in the kernel has the form

$$y(t) = C_1e^t + C_2e^{2t} + C_3e^{3t} + C_4te^{3t} + C_5t^2e^{3t}.$$

In order to investigate their linear dependence one calculates

$$W(e^t, e^{2t}, e^{3t}, te^{3t}, t^2e^{3t})$$

$$= \begin{vmatrix} e^t & e^{2t} & e^{3t} & te^{3t} & t^2e^{3t} \\ e^t & 2e^{2t} & 3e^{3t} & e^{3t} + 3te^{3t} & 2te^{3t} + 3t^2e^{3t} \\ e^t & 4e^{2t} & 9e^{3t} & 6e^{3t} + 9te^{3t} & 2e^{3t} + 12te^{3t} + 9t^2e^{3t} \\ e^t & 8e^{2t} & 27e^{3t} & 27e^{3t} + 27te^{3t} & 16e^{3t} + 54te^{3t} + 27t^2e^{3t} \\ e^t & 16e^{2t} & 81e^{3t} & 108e^{3t} + 81te^{3t} & 108e^{3t} + 216te^{3t} + 81t^2e^{3t} \end{vmatrix}$$

$$= e^te^{2t}e^{3t}e^{3t}e^{3t} \begin{vmatrix} 1 & 1 & 1 & t & t^2 \\ 1 & 2 & 3 & 1+3t & 2t+3t^2 \\ 1 & 4 & 9 & 6+9t & 2+12t+9t^2 \\ 1 & 8 & 27 & 27+27t & 16+54t+27t^2 \\ 1 & 16 & 81 & 108+81t & 108+216t+81t^2 \end{vmatrix} \neq 0.$$

Theorem 5.11. *Given the* n*th-order linear differential equation* $L_n(y) = y^{(n)} + p_1y^{(n-1)} + \cdots + p_{n-1}y' + p_ny = 0$. *Let the characteristic equation corresponding to* $L_n(y) = 0$ *to have:*

(i) A pair of complex roots $r = a \pm bi$, $i^2 = -1$. *Then the functions* $y_1(t) = e^{at}\cos bt$ *and* $y_2(t) = e^{at}\sin bt$ *are linearly independent solution of* $L_n(y) = 0$.

(ii) A pair of complex conjugate roots repeated m times. Then the 2m functions

$$y_1(t) = e^{at}\cos bt, \quad y_2(t) = te^{at}\cos bt \quad ,\ldots, \quad y_m(t) = t^{m-1}e^{at}\cos bt,$$

$$y_{m+1}(t)=e^{at}\sin bt, \quad y_{m+2}(t)=te^{at}\sin bt \quad ,\ldots, \quad y_{2m}(t) = t^{m-1}e^{at}\sin bt$$

are linearly independent solutions of $L_n(y) = 0$. *In addition, every function* $y(t) = (C_1 + C_2t + \cdots + C_mt^{m-1})e^{at}\cos bt + (C_{m+1} + C_{m+2}t + \cdots + C_{2m}t^{m-1})e^{at}\sin bt$ *is a solution to the studied equation.*

In order to get a better imagination of understanding of repeated roots of a characteristic equation we review some properties of polynomials.

(i) A polynomial $w(t)$ has a root r if $w(t) = (t - r)w_0(t)$.
(ii) A polynomial $w(t)$ has a double root if $w(t) = (t - r)^2 w_1(t)$. In this case $w(t)$ and $w'(t) = 2(t - r)w_1(t)$ have a common root r.
(iii) In general, a polynomial $w(t)$ has a root r of multiplicity k if $w(t) = (t-r)^k w_{k-1}(t)$. In this case, $w(t), w'(t), \dots, w^{k-1}(t)$ have a common root r.

Now we focus our attention on a second polynomial corresponding to differential operator $L_2(y) = y'' - 2ry' + r^2 y$. The corresponding quadratic equation is $w(t) = t^2 - 2rt + r^2 = (t - r)^2$. According to Theorem 5.10 we have the solutions: $y_1(t) = e^{rt}, y_2(t) = te^{rt}$. It is true, because $L_2(y_1) = e^{rt}(r^2 - 2r^2 + r^2) = 0$, and $L_2(te^{rt}) = (te^{rt})'' - 2r(te^{rt})' + r^2 te^{rt} = (e^{rt} + rte^{rt})' - 2r(e^{rt} + rte^{rt}) + r^2 te^{rt} = 2re^{rt} + r^2 te^{rt} - 2re^{rt} - 2r^2 te^{rt} + r^2 te^{rt} = 0$. The above example leads to more general observation. Namely, have got $L_2(e^{rt}) = e^{rt} w(r)$. Observe that $\frac{\partial}{\partial r} L_2(e^{rt}) = \frac{\partial}{\partial r}[e^{rt} w(r)] = te^{rt} w(r) + w'(r)e^{rt}$. But recall that r is the double root of the characteristic equation $w(r) = 0$ and hence $w(r) = w'(r) = 0$, i.e. $\frac{\partial}{\partial r} L_2(e^{rt}) = 0$. On the other hand $\frac{\partial}{\partial r} L_2(e^{rt}) = L_2(\frac{\partial}{\partial r} e^{rt}) = L_2(te^{rt}) = 0$, which shows that te^{rt} is a solution of the differential operator L_2. Finally, let us check a linear dependence of both solutions. The Wronskian determinant gives

$$w(e^{rt}, te^{rt}) = \begin{vmatrix} e^{rt} & te^{rt} \\ e^{rt} & e^{rt} + te^{rt} \end{vmatrix} = e^{2rt} \neq 0,$$

which means that they are independent. This observation can be generalized by the following theorem.

Theorem 5.12. *If r_0 is a root of m-multiplicity of the characteristic equation*

$$w(r) = r^n + a_1 r^{n-1} + a_2 r^{n-2} + \cdots + a_{n-1} r + a_n = 0,$$

where $1 \leq m \leq n$, then the functions $e^{r_0 t}, te^{r_0 t}, \dots, r^{m-1} e^{r_0 t}$ are the linearly independent solutions of the characteristic equation $w(r)$.

Proof. If r_0 is the root of $w(r)$ of multiplicity m, then

$$w(r_0) = \frac{dw}{dr}(r_0) = \frac{d^2 w}{dr^2}(r_0) = \cdots = \frac{d^{k-1} w(r_0)}{dr^{k-1}} = 0.$$

Consider now the differential operator

$$L(y) \equiv y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y.$$

It is easy to check that $L(e^{rt}) = w(r)e^{rt}$. Consider now $L(t^i e^{rt}) = L\left(\frac{\partial^i e^{rt}}{\partial r^i}\right) = \frac{\partial^i}{\partial r^i} L(e^{rt}) = \frac{\partial^i}{\partial r^i}(w(r)e^{rt})$. We have

$$\begin{aligned} i = 0 : &\quad w(r)e^{rt} = 0; \\ i = 1 : &\quad (w' + tw)e^{rt} = 0; \\ i = 2 : &\quad (w'' + 2tw' + w(1 + t^2))e^{rt} = 0; \\ &\vdots \end{aligned}$$

Since for $i = k-1$ we obtain $w^{k-1} = \cdots = w'' = w' = w = 0$, hence we have also $L(t^i e^{r_0 t}) = 0$. It means that $t^i e^{r_0 t}$ are solutions to $L(y)$. In addition, they are linearly independent, because the polynomials of different orders are linearly independent. □

In the case of complex conjugate roots (Theorem 5.12) we recall some fundamental properties known from algebra. According to the Euler's formula we have

$$e^{i\beta} = \cos\beta + i\sin\beta,$$

where: $\mathrm{Re}(e^{i\beta}) = \cos\beta$, $\mathrm{Im}(e^{i\beta}) = \sin\beta$ and $e^{a+ib} = e^a(\cos b + i\sin b)$.

Now let the characteristic equation representing the differential equation $L(y) = 0$ to have conjugate complex roots $a \pm ib$, and we obtain the following solutions to the differential equation: $e^{(a+ib)t} = e^{at}(\cos bt + i\sin bt)$ and $e^{(a-ib)t} = e^{at}(\cos bt - i\sin bt)$. Observe that instead of taking this solution one can take $y_1 = e^{at}\cos bt$ and $y_2 = e^{at}\sin bt$. Both of them are real-valued independent solutions (check), and they span the same real-valued solutions as the complex-valued solutions $e^{(a\pm ib)t}$. This observation can be generalized by the following theorem.

Theorem 5.13. *If u and v are in the domain composed of real elements of a real-valued function L, and we take $y = u + iv$, then $L(y) = L(u) + iL(v)$. In addition, if $L(y) = 0$, then $L(u) = L(v) = 0$.*

Remark 5.3. Theorem 5.12 shows that the domain of L can be extended to include complex-valued component.

Our considerations can be briefly outlined in the following way [191]. The solutions to homogeneous linear nth-order ordinary differential equation (5.11) are sought in forms of exponential functions

$$y = e^{rt}, \tag{5.15}$$

where r is constant. We have

$$y^{(k)} = \frac{\partial^k y}{\partial t^k} = r^k e^{rt}, \quad k = 1, \ldots, n. \tag{5.16}$$

Substituting the assumed solution (5.15) to (5.16) we get

$$L(e^{rt}) = e^{rt} P_n(r) = 0, \tag{5.17}$$

where the polynomial

$$P_n(r) = r^n + p_1 r^{n-1} + \cdots + p_{n-1} r + p_n = 0 \tag{5.18}$$

is called the *characteristic equation* (or *the characteristic polynomial*). The roots of the characteristic equation (5.18) are called the *characteristic roots*. If the roots $r_1, \dots, r_n$ are real and different, then we have n independent solutions of the form

$$y_1 = e^{r_1 t}, \dots, y_n = e^{r_n t}. \tag{5.19}$$

The Wronsky determinant has the form

$$\begin{aligned}
w(e^{r_1 t}, \dots, e^{r_n t}) &= \begin{vmatrix} e^{r_1 t} & \dots & e^{r_n t} \\ r_1 e^{r_1 t} & \dots & r_n e^{r_n t} \\ \vdots & \vdots & \vdots \\ r_1^{n-1} e^{r_1 t} & \dots & r_n^{n-1} e^{r_n t} \end{vmatrix} \\
&= e^{(r_1 + r_2 + \dots + r_{n-1})t} \begin{vmatrix} 1 & 1 & \dots & 1 \\ r_1 & r_2 & \dots & r_n \\ \vdots & \vdots & \vdots & \vdots \\ r_1^{n-1} & r_2^{n-1} & \dots & r_n^{n-1} \end{vmatrix} \\
&= e^{(r_1 + r_2 + \dots + r_{n-1})t} (r_n - r_1)(r_n - r_2) \dots (r_n - r_{n-1})(r_{n-2} - r_1) \dots \\
&\quad \dots (r_{n-1} - r_{n-2}) \dots (r_2 - r_1) \neq 0.
\end{aligned} \tag{5.20}$$

Observe that $w(t) \to 0$, if $t \to +\infty$, when $r_k < 0, k = 1, \dots, n$. A general (or every) solution to (5.11) has the form

$$y = C_1 e^{r_1 t} + \dots + C_n e^{r_n t}. \tag{5.21}$$

If $r_k < 0, k = 1, \dots, n$, then every solution y governed by (5.21) $y \to 0$ for $t \to +\infty$.

Consider the case, when $r_1, \dots, r_n$ are different, but some of them are complex conjugate. Let $r_1 = a + ib$. Since the coefficients of the differential equation (5.11) are real, then also $r_2 = \bar{r}_1 = a - ib$ exists. A general solution has the form $y = u(t) + iv(t)$, where $u(t)$ and $v(t)$ are real-valued functions. However, if $y(t_0) = u(t_0) + iv(t_0) = y_0,\ y^{(k)}(t_0) = u^{(k)}(t_0) + iv^{(k)}(t_0) = y_0^{(k)},\ k = 1, \dots, n-1$, where y_0 and $y_0^{(k)}$ are real-valued, then $v(t_0) = v'(t_0) = \dots = v^{(n-1)}(t_0) = 0$, and hence $v(t) \equiv 0$.

Given the initial conditions one finds the being sought constants $C_1, \dots, C_n$ from the algebraic equations

$$\begin{aligned}
y_0 &= C_1 e^{r_1 t_0} + \dots + C_n e^{r_1 t_0}, \\
y_0^{(k)} &= C_1 r_1^k e^{r_1 t_0} + \dots + C_n r_n^k e^{r_n t_0}, \quad k = 1, \dots, n-1.
\end{aligned} \tag{5.22}$$

It can be shown that the solution (5.21), after substituting the constant $C_i i = 1, \dots, n$ from (5.22), is real-valued. Hence, since the solution is real-valued then instead of complex-valued form of the solution we can use the real-valued form of the solution. For our case $r_1 = a + ib$, $r_2 = a - ib$ we get

$$C_1 e^{r_1 t} + C_2 e^{r_2 t} = C_1 e^{at}(\cos bt + i \sin bt) + C_2 e^{at}(\cos bt - i \sin bt)$$
$$= (C_1 + C_2)e^{at} \cos bt + i(C_1 - C_2)e^{at} \sin bt = C_{10} e^{at} \cos bt + C_{20} \sin bt,$$

where: $C_{10} = C_1 + C_2$, $C_{20} = i(C_1 - C_2)$. It is easy to check that $C_{1,2} = C_{10} \pm C_{20}/\mathrm{i}$, i.e. they are complex conjugate, but $C_{i0}, i = 1, 2$ are real-valued. In other words

$$y = C_{10} e^{at} \cos bt + C_{20} e^{at} \sin bt + C_3 e^{r_3 t} + \dots + C_n e^{r_n t}. \tag{5.23}$$

For real-value initial conditions $y_0,\ y'_0, \dots,\ y^{(n-2)}$ the solution (5.23) is real-valued and also C_{10} and C_{20} should be real-valued. Since $e^{r_1 t} = e^{at}(\cos bt + i \sin bt)$ is the solution, then also $y_1 = e^{at} \cos bt$ and $y_2 = e^{at} \sin bt$ are the solutions generated by the roots $r_1 = a + ib$, $r_2 = a - bt$. Recall now the discussed case of the root $r_1 = \dots = r_m$ with mth-multiplicity. The kth-order derivative of (5.11) yields

$$\frac{d^k L(e^{rt})}{dr^k} = L\left[\frac{d^k e^{rt}}{dr^k}\right] = L(t^k e^{rt}), \tag{5.24}$$

and

$$L(t^k e^{rt}) = (t^k e^{rt})^{(n)} + p_1 (t^k e^{rt})^{n-1} + \dots + p_{n-1}(t^k e^{rt})' + p_n (t^k e^{rt}). \tag{5.25}$$

Owing to the Leibniz formula applied to dot product of two functions we have

$$\begin{aligned}(uv)^{(n)} &= uv^{(n)} + \binom{n}{1} u' v^{(n-1)} + \binom{n}{2} u'' v^{(n-2)} + \dots \\ &\dots + \binom{n}{n-1} u^{(n-1)} v' + u^{(n)} v = \sum_{k=0}^{n} u^{(k)} v^{(n-k)},\end{aligned} \tag{5.26}$$

where: $u^{(0)} = u$, $v^{(0)} = v$. Applying Leibniz formula (5.26) to (5.25) we obtain

$$\begin{aligned}L(t^k e^{rt}) &= t^k e^{rt} P_n(r) + C_1^{(k)} t^{k-1} e^{rt} P'_n(r) \\ &\quad + C_2^{(k)} t^{k-2} e^{rt} P''_n(r) + \dots + e^{rt} P_n^{(k)}(r),\end{aligned} \tag{5.27}$$

where the numbers $C_l^{(k)}$ depend on k and l. Since $r_1 = \dots = r_m$, hence the characteristic equation has the following property

$$P_n(r_1) = P'_n(r_1) = \dots = P_n^{m-1}(r_1) = 0. \tag{5.28}$$

It means that $L(t^k e^{rt}) = 0$ for $k = 1, 2, \ldots, m-1$ and $r = r_1$, where r_1 is the root of mth multiplicity. In other words, m times repeated root generates m solutions of the form $e^{r_1 t}, te^{r_1 t}, \ldots, t^{m-1}e^{r_1 t}$. If $r_1, \ldots, r_l$ are different roots of the corresponding multiplicity $m_1, \ldots, m_l$ and there are no other roots (i.e. $m_1 + \cdots + m_l = n$), then every solution to homogeneous linear differential equation (5.11) has the form

$$y = P_{m_1-1}(t)e^{r_1 t} + P_{m_2-1}(t)e^{r_2 t} + \cdots + P_{m_l-1}(t)e^{r_l t}, \tag{5.29}$$

where: $P_{m_1-1}(t), P_{m_2-1}(t), \ldots, P_{m_l-1}(t)$ are the polynomials of t of the corresponding orders: $m_1 - 1, m_2 - 1, \ldots, m_l - 1$. It can be checked that a set $\{e^{r_1 t}, re^{r_1 t}, \ldots, t^{m_1-1}e^{r_1 t}, \ldots, e^{r_l t}, te^{r_l t}, \ldots, t^{m_l-1}e^{r_l t}\}$ has members which are linearly independent.

Recall now our considerations related to repeated complex roots. If among the roots $r_1, \ldots, r_l$ a complex one appears, say $r_1 = a + ib$, then also exists $r_2 = a - ib$ with the same multiplicity. The used real initial conditions $y_0, y_0', \ldots, y_0^{(n-1)}$ allow to find a set of constants to define a real solution.

Assume that $r = a + ib$ and $\bar{r} = a - ib$ are m times repeated roots. Every solution includes the functions

$$\begin{aligned} &e^{rt}, te^{rt}, \ldots, t^{m-1}e^{rt}, \\ &e^{\bar{r}t}, te^{\bar{r}t}, \ldots, t^{m-1}e^{\bar{r}t}. \end{aligned} \tag{5.30}$$

Since $t^k e^{rt} = t^k e^{at} e^{ibt} = t^k e^{at}(\cos bt + i \sin bt)$ is the solution, then its real and imaginary part are also solutions. The same consideration holds for $\bar{r}$. In the case when a root is purely imaginary with mth multiplicity ($a = 0$), then every solution is composed of the functions $t^k \cos bt, t^k \sin bt, k = 0, 1, \ldots, m$.

Let us sum up our brief conclusions related to a repeated root. Any real-valued root r with mth multiplicity generates m solutions, whereas a complex-valued root r with mth multiplicity generates $2m$ solutions. Every solution corresponding to m-times repeated root is represented by

$$y = \sum_{k=0}^{m-1} C_k t^k e^{rt}. \tag{5.31}$$

Every solution corresponding to complex root with mth multiplicity has the form

$$y = \sum_{k=0}^{m-1} \left(A_k t^k e^{at} \cos bt + B_k t^k e^{at} \sin bt \right). \tag{5.32}$$

Finally, every solution corresponding to repeated imaginary roots $r = ib, \bar{r} = -ib$ has the form

$$y = \sum_{k=0}^{m-1} \left(A_k t^k \cos bt + B_k t^k \sin bt \right). \tag{5.33}$$

In both latter cases, A_k and B_k are defined by the initial conditions.

Example 5.4. Using formula (5.25) and (5.26) for $k = 2$ find a corresponding number C_l^k when

$$L(y) = y''' + p_1 y'' + p_2 y' + p_3 y.$$

Formula (5.26) gives

$$(uv)''' = uv''' + \binom{3}{1} u'v'' + \binom{3}{2} u''v' + u'''v,$$

$$(uv)'' = uv'' + \binom{2}{1} u'v' + u''v',$$

$$(uv)' = u'v + v'y,$$

whereas from (5.25) we get

$$\begin{aligned} L(t^k e^{rt}) &= (t^k e^{rt})'' + p_1(t^k e^{rt})'' + p_2(t^k e^{rt})' + p_3(t^k e^{rt}) \\ &= (t^2r^3+6tr^2+6r)e^{rt}+p_1(t^2r^2+4tr+2)e^{rt}+p_2(t^2r+2t)e^{rt}+p_3t^2e^{rt} \\ &= e^{rt}[t^2(r^3 + p_1r^2 + p_2r + p_3) + t(6r^2 + 4rp_1 + 2p_2) + 6r + 2p_1]. \end{aligned}$$

Since

$$\begin{aligned} P_3(r) &= r^3 + p_1r^2 + p_2r + p_3, \\ P'_3(r) &= 3r^2 + 2p_1r + p_2, \\ P''_3(r) &= 6r + 2p_1, \end{aligned}$$

therefore

$$L(t^2e^{rt}) = t^2e^{rt}P_3(r) + C_1^2 P_3'(r)e^{rt} + e^{rt}P_3''(r),$$

and we have $C_1^2 = 2$.

5.4 Linear Non-homogeneous Differential Equations with Constant Coefficients

In the previous section we have considered linear homogeneous operator defined by Eq. (5.11). Here we are going to consider non-homogeneous problems.

Theorem 5.14. *If the fundamental solutions* $y_1(t), \ldots, y_n(t)$ *to Eq. (5.11) are known, then a general solution to the following non-homogeneous equation*

$$L_n(y) = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = f(t) \tag{5.34}$$

has the form

$$y = y_p + C_1 y_1 + C_2 y_2 + \cdots + C_n y_n, \tag{5.35}$$

where $C_1, C_2, \ldots, C_n$ *are arbitrary numbers, and* y_p *is an arbitrary particular solution of (5.34).*

There are two general methods to solve a non-homogeneous problem, i.e. the method of *undetermined coefficients* and the method of *variations of parameters*. The method of undetermined coefficients relies on observation that a being sought solution is somehow similar to the right-hand side function $f(t)$.

Example 5.5. Find a general solution to the non-homogeneous differential equation

$$L(y) = y'' - 3y' + 2y = 3t^2 - 2t + 4.$$

First we consider the corresponding homogeneous equation $L(y) = y'' - 3y' + 2y = 0$. The corresponding characteristic equation is $r^2 - 3r + 2 = 0$ with the roots $r_1 = 1$ and $r_2 = 2$, and hence the general solution of the homogeneous equation $y = C_1 e^t + C_2 e^{2t}$. Let us look for a particular solution of the form $y_p = at^2 + bt + c$. The simple calculus yields

$$\begin{aligned} L(y_p) &= 2at - 3(2at + b) + 2(at^2 + bt + c) \\ &= 2at^2 + t(2b - 6a) + 2c + 2a - 3b. \end{aligned}$$

Equating the coefficients standing by the same powers of t we obtain:

$$\begin{aligned} t^0 &: 2c + 2a - 3b = 4, \\ t^1 &: \quad 2b - 6a = -2, \\ t^2 &: \qquad 2a = 3. \end{aligned}$$

The above linear algebraic equations give: $a = 1.5$, $b = 3.5$, $c = 5.75$. The general solution of the non-homogeneous problem is $y = C_1 e^t + C_2 e^{2t} + 1.5t^2 + 3.5t + 5.75$. □

Example 5.6. Find a general solution of the non-homogeneous equation

$$L(y) = y'' + 2hy' + \alpha_0^2 y = q \cos \omega t.$$

This equation governs oscillation of a one-degree-of-freedom mechanical system with mass m and with a viscous positive damping, a linear stiffness and a harmonic excitation. The parameters have the following physical meaning: c—damping ($2h = c/m$); k—stiffness ($\alpha_0^2 = k/m$); P_0—amplitude of exciting force ($q = p_0/m$) and ω is frequency of excitation. The characteristic equation is

$$r^2 + 2hr + \alpha_0^2 = 0,$$

which gives the roots $r_{1,2} = -h \pm \sqrt{h^2 - \alpha_0^2}$. If $h > \alpha_0$, then we have two real roots and oscillations do not appear. If $h = \alpha_0$ we have so-called critical damping $c_{cr} = 2\sqrt{km}$ and the root is double. Here we consider the case $\alpha_0^2 > h^2$ which corresponds to oscillations and $r_{1,2} = -h \pm i\lambda$, where $\lambda = \sqrt{\alpha_0^2 - h^2}$. Since $e^{(-h+i\lambda)t} = e^{-ht}(\cos\lambda t + i\sin\lambda t)$ and according to our earlier discussion we can take the following real-valued general solution corresponding to homogeneous equation: $y = C_1 e^{-ht}\cos\lambda t + C_2 e^{-ht}\sin\lambda t$. As it has been mentioned, the form of the particular solution to the non-homogeneous equation is similar to the right-hand side excitation and is assumed to be: $y_p = A\cos\omega t + B\sin\omega t$. Since

$$y' = -A\omega\sin\omega t + B\omega\cos\omega t,$$
$$y'' = -A\omega^2\cos\omega t - B\omega^2\sin\omega t,$$

hence

$$L(y_p) = -A\omega^2\cos\omega t - B\omega^2\sin\omega t - 2hA\omega\sin\omega t + 2hB\omega\cos\omega t + \alpha_0^2 A\cos\omega t + \alpha_0^2 B\sin\omega t = q\cos\omega t.$$

Equating the terms standing by $\sin\omega t$ and $\cos\omega t$ yields

$$A(\alpha_0^2 - \omega^2) + 2h\omega B = q,$$
$$-2h\omega A + (\alpha_0^2 - \omega^2)B = 0,$$

and therefore

$$A = \frac{W_A}{W} = \frac{\begin{vmatrix} q & 2h\omega \\ 0 & \alpha_0^2 - \omega^2 \end{vmatrix}}{\begin{vmatrix} \alpha_0^2 - \omega^2 & 2h\omega \\ -2h\omega & \alpha_0^2 - \omega^2 \end{vmatrix}} = \frac{q(\alpha_0^2 - \omega^2)}{(\alpha_0^2 - \omega^2)^2 + 4h^2\omega^2},$$

$$B = \frac{W_B}{W} = \frac{\begin{vmatrix} \alpha_0^2 - \omega^2 & q \\ -2h\omega & 0 \end{vmatrix}}{\begin{vmatrix} \alpha_0^2 - \omega^2 & 2h\omega \\ -2h\omega & \alpha_0^2 - \omega^2 \end{vmatrix}} = \frac{2h\omega q}{(\alpha_0^2 - \omega^2)^2 + 4h^2\omega^2}.$$

Finally, the general solution to homogeneous equation has the form

$$y = C_1 e^{-ht} \cos \lambda t + C_2 e^{-ht} \sin \lambda t + A \cos \omega t + B \sin \omega t.$$

□

The obtained solution possesses clear physical interpretation. It is composed of oscillation of autonomous damped system (this part vanishes, when $t \to +\infty$) and a particular solution generated by the harmonic excitation. A steady state oscillation is represented by the solution

$$\lim_{t \to \infty} y(t) = A \cos \omega t + B \sin \omega t = a \cos(\omega t + \varphi).$$

Observe that

$$a \cos(\omega t + \varphi) = a \cos \varphi \cos \omega t - a \sin \varphi \sin \omega t = A \cos \omega t + B \sin \omega t.$$

Equating terms standing by $\cos \omega t$ and $\sin \omega t$ we obtain

$$A = a \cos \varphi, \quad B = -a \sin \varphi,$$

and hence

$$a = \sqrt{A^2 + B^2} = \frac{q}{\sqrt{(\alpha_0^2 - \omega^2)^2 + 4h^2\omega^2}},$$

$$\varphi = -\arctan \frac{B}{A} = -\arctan \frac{2h\omega}{\alpha_0^2 - \omega^2}.$$

This result indicates that the oscillations are harmonic with the amplitude a and are delayed by the phase shift ϕ in comparison to excitation. The latter example allows to solve more general problem, which possesses many applications in engineering, and particularly in mechanics.

Consider now the following non-homogeneous problem

$$L_n(y) = e^{at}[P(t) \cos bt + Q(t) \sin bt], \tag{5.36}$$

where $a, b \in R$ and $P(t)$ and $Q(t)$ are polynomials of orders p and q, respectively. Let the characteristic equation corresponding to the homogeneous equation has the form

$$r^n + p_1 r^{n-1} + \cdots + p_n = 0. \tag{5.37}$$

By s we denote the largest values among p and q (for $p = q$ we take $s = p = q$). If $a + ib$ is not a solution to the characteristic equation (5.38), then also $a - ib$ is not a solution. In this case the being sought particular solution has the form

$$y_p = e^{at}[R(t) \cos bt + S(t) \sin bt],$$

where $R(t)$ and $S(t)$ are polynomials of an order not higher than s. If $a + ib$ is repeated root of multiplicity k, then

$$y_p = t^k e^{at}[R(t)\cos bt + S(t)\sin bt], \tag{5.38}$$

where $R(t)$ and $S(t)$ are polynomials of an order not higher than s.

The steps of the method of undetermined coefficients can be even more detailed presented by introducing the following theorem (see [191]).

Theorem 5.15. *Apply the following steps to solve $L_n(y) = f(t)$ using the method of undetermined coefficients:*

(i) Factor the characteristic polynomial $w_{L_n}(r)$ of L_n into linear and irreducible quadratic factors and form a basis B_L for the kernel of L.
(ii) Separate f(t) into groups of terms that are annihilated by a single annihilator. Form $w_A(r)$, the characteristic polynomial for an annihilator of $f(t)$ factored into linear and irreducible quadratic factors. Recall that for a given function f, an annihilator of f is a linear operator A such that $A(f) = 0$. For instance, $y'' + 16y$ is an annihilator of $\sin\phi t$ and $\cos\phi t$, because the characteristic equation is $(r^2 + 16) = (r + 4i)(r - 4i)$.
(iii) The characteristic polynomial of A_L is defined by $w_A(r)w_L(r)$. Form the basis B_{AL}, and find B consisting of the functions in B_{AL} that are not in B_L.
(iv) Form $y_p(t)$ as a linear combination of the functions in B and equate coefficients of $L(y_p) = f(t)$.
(v) Determine values for the coefficients and define a particular solution.
(vi) Find $y(t) = y_g(t) + y_p(t)$.

Example 5.7. Solve the differential equation $L(y) = y''' - 2y'' + 9y' - 18y = e^t + 3\sin 2t$ using the method of undetermined coefficients.

(i) The characteristic polynomial $w_L(r) = r^3 - 2r^2 + 9r - 18 = (r-2)(r^2+9)$, and hence $\{B_L = e^{2t}, \sin 3t, \cos 3t\}$.
(ii) The function $f(t) = e^t + 3\sin 2t$ is annihilated by the characteristic polynomial $w_A = (r-1)(r^2+4)$, and the basis for kernel of A is $B_A = \{e^t, \sin 2t, \cos 2t\}$.
(iii) The characteristic polynomial of A_L is $w_A w_L = (r-2)(r^2+9)(r-1)(r^2+4)$, and the kernel of A_L is $B_{AL} = \{e^t, e^{2t}, \sin 2t, \cos 2t, \sin 3t, \cos 3t\}$. We have found also that $B = B_A = \{e^t, \sin 2t, \cos 2t\}$.
(iv) The particular solution $y_p = ae^t + b\sin 2t + c\cos 2t$. The successive differentiation gives

$$y'_p = ae^t + 2b\cos 2t - 2c\sin 2t,$$

$$y''_p = ae^t - 4b\sin 2t - 4c\cos 2t,$$

$$y'''_p = ae^t - 8b\sin 2t + 8c\cos 2t,$$

and

$$\begin{aligned} L(y_p) &= ae^t - 8b\sin 2t + 8c\cos 2t - 2(ae^t - 4b\sin 2t + 4c\cos 2t) \\ &\quad +9(ae^t + 2b\cos 2t - 2c\sin 2t) - 18(ae^t + b\sin 2t + c\cos 2t) \\ &= -10ae^t - 18(b+c)\sin 2t + 2(9b-c)\cos 2t. \end{aligned}$$

Equating the appropriate coefficients with those of $f(t)$ gives

$$\begin{aligned} e^t &: \quad -10a = 1, \\ \sin 2t &: -18(b+c) = 3, \\ \cos 2t &: \quad 9b - c = 0. \end{aligned}$$

(v) The solution of the latter linear algebraic equation is: $a = -10, b = -\frac{1}{60}$, $c = -\frac{3}{20}$, and hence the particular solution is

$$y_p(t) = -10e^t - \frac{1}{60}\sin 2t - \frac{3}{20}\cos 2t.$$

(vi) Finally, the general solution is

$$y = C_1e^{2t} + C_2\sin 3t + C_3\cos 3t - 10e^t - \frac{1}{60}\sin 2t - \frac{3}{20}\cos 2t.$$

□

Now we briefly describe the method of *variation of parameters*. Recall that even in the method of undetermined coefficients we have found the kernel of a homogeneous equation $L_n(y) = 0$ first.

Theorem 5.16. *A particular solution $y_p(t)$ of non-homogeneous equation $L_n(y) = f(t)$ can be found by quadratures if it will be sought in the form*

$$y_p(t) = C_1(t)y_1(t) + C_2(t)y_2(t) + \cdots + C_n(t)y_n(t), \tag{5.39}$$

where $\{y_1(t), y_2(t), \ldots, y_n(t)\}$ is the basis of the kernel of $L_n = 0$.

Theorem 5.17. *If the functions $C_1(t), \ldots, C_n(t)$ satisfy the system of equations*

$$\begin{gathered} C'_1y_1 + C'_2y_2 + \cdots + C'_ny_n \\ C'_1y'_1 + C'_2y'_2 + \cdots + C'_ny'_n \\ \vdots \\ C'_1y_1^{(n-1)} + C'_2y_2^{(n-1)} + \cdots + C'_ny_n^{(n-1)}, \end{gathered} \tag{5.40}$$

then the function y_p defined by (5.39) satisfies the non-homogeneous equation $L_n(y) = f(t)$.

Remark 5.4. The function $y_p(t)$ possesses the form of a general integral of $L_n(y) = 0$, where now the constant are the functions depending on t.

Remark 5.5. Observe that $C_1', C_2', \ldots, C_n'$ are uniquely defined, since the determinant of the system (5.41) is the Wronskian determinant, and by definition it never equals zero.

In order to get $C_1(t), \ldots, C_n(t)$ we integrate the functions $C_1'(t), \ldots, C_n'(t)$ without the constant of integrations, because they appear already in the part of n general solutions of the homogeneous equation.

Example 5.8. Find a general solution of the equation

$$L_2(y) = y'' + y = t^2.$$

The kernel of $L_2(y) = 0$ is $\{\sin t, \cos t\}$. The general solution of $L_2(y) = t^2$ is $y = y_p + C_1 \cos t + C_2 \sin t$.

The functions $y_p(t)$ is sought in the form

$$y_p(t) = C_1(t) \cos t + C_2(t) \sin t.$$

From (5.41) we obtain

$$C'_1 \cos t + C'_2 \sin t = 0,$$
$$-C'_1 \cos t + C'_2 \sin t = t^2,$$

and hence

$$C_1' = -t^2 \sin t, \quad C_2' = t^2 \cos t.$$

Integration of C_1', C_2' yields

$$C_1(t) = (t^2 - 2) \cos t - 2t \sin t, \quad C_2(t) = (t^2 - 2) \sin t + 2t \cos t$$

and

$$y = t^2 - 2 + C_1 \cos t + C_2 \sin t.$$

5.5 Differential Equations with Variable Coefficients

We begin with a special class of higher order homogeneous differential equations, which can be reduced to the differential equations with constant coefficients. A solution to the differential equation

$$L_n(y) = p_n t^n \frac{d^n y}{dt^n} + p_{n-1} t^{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + p_1 t \frac{dy}{dt} + p_0 y, \qquad (5.41)$$

known as the Cauchy–Euler differential equation, can be sought in the form

$$y = t^r. \tag{5.42}$$

Substituting (5.42) to (5.41) gives

$$L_n(t^r) = w(r)t^r, \tag{5.43}$$

and hence $L_n = 0$, when $w(r) = 0$. The analysis is similar to previous one, but now instead of the function e^{rt}, we have t^r.

(i) Assume that there are $r_1, r_2, \ldots, r_m$ distinct real roots ($m \leq n$), then $t^{r_1}, t^{r_2}, \ldots, t^{r_m}$ are real independent solutions, and

$$y(t) = C_1 t^{r_1} + C_2^{r_2} + \cdots + C_m t^{r_m} \tag{5.44}$$

is a general solution (it defines an m-dimensional subspace of the kernel of (5.41)).

(ii) Assume that r_0 is a root with kth multiplicity, then its subkernel follows

$$\{t^{r_0}, t^{r_0}\ln t, \ldots, t^{r_0}(\ln t)^{k-1}\},$$

and a general solution

$$y(t) = C_1 t^{r_0} + C_2 t^{r_0}\ln t + \cdots + C_k t^{r_0}(\ln t)^{k-1} \tag{5.45}$$

defines a k-dimensional subspace.

(iii) Assume that $r = a \pm bi$, then $t^a\cos(b\ln t)$ and $t^a\sin(b\ln t)$ are real linearly independent solutions. In addition, if $a \pm bi$ has multiplicity k, then

$$\begin{array}{c} t^a\cos(b\ln t), t^a\ln t\cos(b\ln t), \ldots, t^a(\ln t)^{k-1}\cos(b\ln t), \\ t^a\sin(b\ln t), t^a\ln t\sin(b\ln t), \ldots, t^a(\ln t)^{k-1}\sin(b\ln t), \end{array} \tag{5.46}$$

are $2k$ real linearly independent solutions in the kernel of (5.41). Their linear combinations span a $2k$-dimensional subspace.

Now we show how to reduce the homogeneous and non-homogeneous Cauchy–Euler equation to an equation with constant coefficients.

Theorem 5.18. *Introducing a variable $x = \ln t$ in the non-homogeneous Cauchy–Euler equation*

$$L_n(y) = p_n t^n \frac{d^n y}{dt^n} + p_{n-1}t^{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_1 t\frac{dy}{dt} + p_0 y = f(t), \tag{5.47}$$

leads to conversion to the differential equation with constant coefficients

$$L[y(e^x)] = g(x), \tag{5.48}$$

where $g(x) = f(e^x)$.

Example 5.9. Transform the Cauchy–Euler differential equation

$$t^3 y''' - 4t^2 y'' + 3ty' + y = 0$$

to a differential equation with constant coefficients.

Let $t + e^x$, and use the chain rule to get

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}\frac{1}{t}, \quad \text{i.e.} \quad t\frac{dy}{dt} = \frac{dy}{dx};$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\frac{1}{t}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{1}{t} + \frac{dy}{dx}\frac{d}{dt}\left(\frac{1}{t}\right)$$

$$= \frac{1}{t^2}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right) \quad \text{i.e.} \quad t^2\frac{d^2y}{dt^2} = \frac{d^2y}{dx^2} - \frac{dy}{dx};$$

$$\frac{d^3y}{dt^3} = \frac{d}{dt}\left(\frac{d^2y}{dt^2}\right) = \frac{d}{dt}\left[\frac{1}{t^2}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right)\right]$$

$$= \frac{-2}{t^3}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right) + \frac{1}{t^2}\left[\frac{d}{dt}\left(\frac{d^2y}{dx^2}\right) - \frac{d}{dt}\left(\frac{dy}{dx}\right)\right]$$

$$= \frac{-2}{t^3}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right) + \frac{1}{t^3}\left(\frac{d^3y}{dx^3} - \frac{d^2y}{dx}\right)$$

$$= \frac{1}{t^3}\left(\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} - 2\frac{dy}{dx}\right), \quad \text{i.e.} \quad \frac{d^3y}{dt^3} = \frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} - 2\frac{dy}{dx}.$$

Our investigated equation becomes

$$\frac{d^3y}{dx^3} - 7\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + y = 0.$$

The characteristic polynomial is

$$r^3 - 7r^2 + 5r + 1 = (r-1)(r^2 - 6r - 1)$$

and possesses the characteristic roots

$$r_2 = 1, \quad r_1 = 3 - \sqrt{10}, \quad r_3 = 3 + \sqrt{10}.$$

Hence the solution is

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x} + C_3 e^{r_3 x} = C_1 e^{r_1 \ln t} + C_2 e^{r_2 \ln t} + C_3 e^{r_3 \ln t}.$$

□

A key role in mechanics plays second-order differential equation, since a second derivative corresponds to acceleration of a moving rigid body with one degree of freedom. It can happen that one of its solutions is known (for example, from theoretical considerations or from an experiment), and a question appears, how to find a second linearly independent solution?

Consider $L_2(y) = p_2(t)y'' + p_1(t)y' + p_0(t)y = 0$, and assume $y(t) = v(t)y_1(t)$, where $y_1(t)$ satisfies $L_2(y_1) = 0$. The assumed form of solution refers to the so-called Bernoulli method, which is often applied to solve first-order non-homogeneous differential equations. The successive differentiation gives

$$\begin{aligned} y' &= v'y_1 + vy'_1, \\ y'' &= v''y_1 + 2v'y'_1 + vy''_1, \end{aligned} \tag{5.49}$$

and hence

$$\begin{aligned} L_2(vy_1) &= p_2(t)(v''y_1 + 2v'y'_1 + vy''_1) + p_1(t)(v'y_1 + vy'_1) + p_0(t)vy_1 \\ &= v''(p_2y_1) + v'(2p_2y'_1 + p_1y_1) + v(p_2y''_1 + p_1y'_1 + p_0y_1) \\ &= p_2y_1v'' + (2p_2y'_1 + p_1y_1)v' = 0. \end{aligned} \tag{5.50}$$

Denoting $v' = w$ one can solve first-order linear differential equation with regard to w' to give

$$w(t) = y_1^{-2} e^{-\int p_1(t)/p_2(t)} dt. \tag{5.51}$$

Hence

$$v(t) = \int y_1^{-2} e^{-\int p_1(t)/p_2(t)dt} dt, \tag{5.52}$$

and the second solution is

$$y_2(t) = v(t)y_1(t) = y_1(t) \int y_1^{-2}(t) e^{-\int p_1(t)/p_2(t)dt} dt. \tag{5.53}$$