Chapter 2
First-Order ODEs

Modelling of various problems in engineering, physics, chemistry, biology and economics allows formulating of differential equations, where a being searched function is expressed via its time changes (velocities). One of the simplest example is that given by a first-order ODE of the form

$$\frac{dy}{dt} = F(y), \tag{2.1}$$

where $F(t)$ is a known function, and we are looking for $y(t)$. Here by t we denote time. In general, any given differential equation has infinitely many solutions. In order to choose from infinite solutions those corresponding to a studied real process, one should attach initial conditions of the form $y(t_0) = y_0$.

In general, there is no direct rule/recipe for construction of an ODE. Let $y = y(t)$ be a dependence between t and y of the investigated process. We are going to monitor the difference $y(t + \Delta t) - y(t)$ caused by the disturbance Δt. Then, if we take

$$\dot{y} \equiv \frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{y(t + \Delta t) - y(t)}{\Delta t},$$

we obtain a differential equation, i.e. dependence of the process velocity in the point t governed by the function $F(y)$.

There are also cases where a function $y(t)$ appears under an integral and the obtained equation is called *the integral equation*, which in simple cases can be transformed to a differential equation.

J. Awrejcewicz, *Ordinary Differential Equations and Mechanical Systems*,
DOI 10.1007/978-3-319-07659-1_2

2.1 General Introduction

A differential equation of the form

$$f\left(t, y, \frac{dy}{dt}\right) = 0 \tag{2.2}$$

is called the first-order ordinary differential equation, where t is the independent variable (here referred to time, but in general it can be taken as a space variable x), and $y(t)$ is the unknown function to be determined. Observe that Eq. (2.2) is not solved with respect to its derivative dy/dt. In many cases, however, one deals with the following differential equation

$$\frac{dy}{dt} = f(t, y), \tag{2.3}$$

which is called the first-order ODE solved with respect to the derivative. Alternatively, one may deal often with the following form of first-order ODE

$$P(t, y)dt + Q(t, y)dy = 0, \tag{2.4}$$

where P, Q are given functions.

We say that $y = \phi(t)$ is a solution to either (2.2) or (2.3) in an interval J, if

$$f\left(t, \phi(t), \frac{d\phi(t)}{dt}\right) \equiv 0, \tag{2.5}$$

or

$$\frac{d\phi(t)}{dt} = f(t, \phi(t)), \tag{2.6}$$

for all $t \in J$.

One may also find a solution to Eq. (2.2) in the *implicit form* $\varphi(t, \phi(t))$, where $\phi(t) = y$ is a solution to Eq. (2.2). Solution in the form of $\varphi(t, \phi(t))$ is also referred to as *the integral of Eq. (2.2)*.

A graph of solution $y = \phi(t)$ of Eq. (2.2) is called *the integral curve* of the studied differential equation. Projection of the solution graph onto the plane (t, y) is called *the phase curve* (or *trajectory*) of the investigated first-order ODE.

A problem related to finding a solution $y = \phi(t)$ satisfying the initial condition $y(t_0) = y_0$ is called the Cauchy problem.

If we take a point (t, y) for $t \in J$, then a tangent line passing through this point creates with the axis t an angle α, then $\tan\alpha = f(t, y)$. A family of all tangent lines defines a *direction field* for the studied differential equation. If we draw a short line segment possessing the slope $f(t, y)$ through each of representative collection of points (t, y), then all line segments constitute *a slope field* for the investigated ODE.

A curve constituting of points with the same slope field is called the *isocline*. In other words all integral curves passing through an isocline intersect the axis t with the same angle.

Example 2.1. Prove that the function $y = \phi(t)$ given in the parametric form $t = xe^x$, $y = e^{-x}$ satisfies the following differential equation

$$(1+ty)\frac{dy}{dt} + y^2 = 0.$$

We have

$$\begin{aligned}(1+ty)\frac{dy}{dt} + y^2 &= (1 + xe^x e^{-x})\frac{dy}{dx}\frac{dx}{dt} + e^{-2x} \\ &= -(1+x)e^{-x} \cdot \frac{1}{(1+x)e^x} + e^{-2x} = 0,\end{aligned}$$

which proves that $\phi(t)$ satisfies the studied equation. □

Example 2.2. Construct a differential equation of a family of ellipses of the following canonical form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $0 < b < a$.

Acting by d/dx on both sides of this algebraic equation yields

$$\frac{x}{a^2} + \frac{y\frac{dy}{dx}}{b^2} = 0.$$

Solving both equations we get

$$\sqrt{a^2 - x^2}\frac{dy}{dx} + \frac{b}{a}x = 0.$$

□

Example 2.3. Construct a differential equation of the force lines of a dipole constituted by two electric charges $(+q, -q)$ located on the distance $2a$, where the force lines satisfy the Coulomb algebraic equation of the form

$$\frac{x+a}{r_1} - \frac{x-a}{r_2} = C,$$

where: $r_1^2 = (x+a)^2 + y^2, \quad r_2^2 = (x-a)^2 + y^2.$

A differentiation of the algebraic equation yields

$$\frac{r_1 - (x+a)\frac{dr_1}{dx}}{r_1^2} - \frac{r_2 - (x-a)\frac{dr_2}{dx}}{r_2^2} = 0,$$

and also

$$\frac{dr_1}{dx} = \frac{x + a + y\frac{dy}{dx}}{r_1}, \quad \frac{dr_2}{dx} = \frac{x - a + y\frac{dy}{dx}}{r_2}.$$

Finally, after a few of transformations we get

$$\left(\frac{x-a}{r_2^3} - \frac{x-a}{r_1^3}\right)\frac{dy}{dx} + \left(\frac{1}{r_2^3} - \frac{1}{r_1^3}\right) y = 0.$$

□

Example 2.4. How many solutions of the equation $(x-1)\frac{dy}{dx} + y = 0$ defines the relation

$$y(x-1) = C,$$

for each fixed $C \in \mathbb{R}$. Find the solutions associated with the initial conditions $y(0) = 0$, $y(0) = -1$, $y(2) = 1$. Define intervals of solution existence as well as the corresponding integral and phase curves.

First we verify that $\varphi(x) = \frac{C}{x-1}$ satisfies the given differential equation. We have $\varphi_1(x) = \frac{C}{x-1}$ with $x \in (C, +\infty)$ and $\varphi_2(x) = \frac{C}{x-1}$ with $x \in (1, +\infty)$.

The initial condition $y(0) = 0$ is satisfied by the solution $y = 0$. Its integral curve corresponds to the axis of abscissa, whereas its phase corresponds to a projection of the integral curve into the axis of ordinates, i.e. the point $y = 0$.

In the case of $y(0) = -1$ we find that $C = 1$. It means that the integral curve of this solution corresponds the hyperbola branch $y(x-1) = 1$ for $x \in (-\infty, 1)$. The phase curve of this solution is the ray $y < 0$.

Finally, in the case $y(2) = 1$ we obtain $C = 1$. Integral curve of the solution $y = \frac{1}{1-x}$ is the hyperbola $y(x-1) = 1$ branch, where $x \in (1, +\infty)$ phase curve is the ray $y > 0$. □

2.2 Separable Equation

The first-order differential equation of the form

$$\frac{dy}{dx} = f(x)g(y) \tag{2.7}$$

is called a *separable differential equation*.

If $g(C_0) = 0$ in the point $y = C_0$, then the function $y = C_0$ is the solution to Eq. (2.7). If $g(y) \neq 0$, then the following relation is obtained

$$\int \frac{dy}{g(y)} - \int f(x)dx = C. \tag{2.8}$$

Theorem 2.1. *Let the function $f(x)$ and $g(x)$ are continuously differentiable in the vicinity of points $x = x_0$, $y = y_0$ respectively, where $g(y_0) \neq 0$. Therefore, there is a unique solution $y = \phi(x)$ of Eq. (2.7) with the attached initial condition $\phi(x_0) = y_0$ in the vicinity of the point $x = x_0$, satisfying the relationship*

$$\int_{y_0}^{\phi(x)} \frac{dy}{g(y)} = \int_{x_0}^{x} f(x)dx.$$

If we have the equation

$$\frac{dy}{dx} = f(ax + by + c), \tag{2.9}$$

then introducing a new variable

$$z = ax + by + c, \tag{2.10}$$

we get

$$\frac{dz}{dx} = bf(z) + a, \tag{2.11}$$

i.e. the problem is reduced to Eq. (2.7).

One may use the following physical interpretation of the differential equation

$$\frac{dy}{dx} = f(y). \tag{2.12}$$

Let us attach to each point y a vector of the length $|f(y)|$, which direction is defined by the axis $0y$ providing that $f(y) > 0$. Therefore, a set of all vectors defines a vector field. The points $f(y) = 0$ are called *singular points* of the vector field (or *its equilibrium positions* in the case when we deal with time). Having drawn the vector field of the given Eq. (2.12) one may draw schematically the integral curves.

Example 2.5. Find a solution of the following differential equation

$$x(1 + y^2) + y(1 + x^2)\frac{dy}{dx} = 0.$$

We transform the studied equation to the form

$$\int \frac{x dx}{1+x^2} + \int \frac{y dy}{1+y^2} = 2 \ln C$$

and hence after integration we get

$$\ln(1+x^2) + \ln(1+y^2) = \ln C,$$

which means that

$$(1+x^2)(1+y^2) = C.$$

□

Example 2.6. Solve the following ODE

$$\frac{dy}{dx} + y = 2x + 1.$$

In order to transform the given ODE into that of separable variables we introduce the following new variable

$$y - 2x - 1 = z,$$

and hence

$$\frac{dz}{dx} + z + 2 = 0.$$

Separating variables and integrating we get

$$\int \frac{dz}{z+2} + \int dx = 0,$$

which means that

$$\ln|z+2| + x = \ln C_0, \quad |z+2| = C_0 e^{-x}, \quad C_0 > 0.$$

Observe that $z = -2$ satisfies the studied equation directly, and therefore, all its solutions are given by the following formula

$$z = -2 + Ce^{-x}, \quad C \in \mathbb{R},$$

and finally we get

$$y = 2x - 1 + Ce^{-x}.$$

□

In what follows we proceed with a few examples of real-world applications.

Example 2.7. A particle of mass m is subjected to action of a constant force, and it moves with the constant acceleration a. The viscous damping of the surrounding medium is c. Find the particle velocity providing that $v(0) = 0$.

The second Newton law gives

$$\frac{dv(t)}{dt} = \frac{ma - cv(t)}{m},$$

or equivalently

$$\frac{dv}{dt} = -\frac{c}{m}v + a.$$

The trivial (time independent solution) is

$$v(t) = \frac{m}{c}a,$$

and hence all solutions are given by the formula

$$v(t) = \frac{m}{c}a + Ce^{-\frac{c}{m}t}$$

The initial condition allows to find $C = -\frac{m}{c}a$, and finally

$$v(t) = \frac{m}{c}a\left(1 - e^{-\frac{c}{m}t}\right),$$

which means also that

$$\lim_{t\to\infty} v(t) = \frac{m}{c}a.$$

□

Example 2.8. A meteorite of mass M starts to move from its rest position into the Earth centre linearly from the height h (Fig. 2.1). Determine the meteorite velocity, when it touches the Earth surface assuming the Earth radius R.

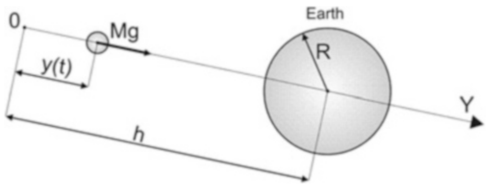

Fig. 2.1 Meteorite movement towards Earth centre

We denote by $y = y(t)$ the meteorite distance from its movement beginning point $y(0) = 0$, and by $h - y(t)$ we denote the meteorite distance from the Earth centre in time instant t. The meteorite is subjected to action of two forces: Ma and Mg. Owing to the Newton principle we have

$$\frac{Ma}{R^2} = \frac{Mg}{(h-y)^2},$$

and hence

$$a = \frac{gR^2}{(h-y)^2}.$$

Therefore,

$$a = \frac{dv}{dt} = \frac{dv}{dy} \cdot \frac{dy}{dt} = \frac{dv}{dy} v,$$

and the following governing ODE is obtained

$$v \frac{dv}{dy} = \frac{gR^2}{(h-y)^2},$$

or equivalently

$$\frac{1}{2} \frac{d(v)^2}{dy} = \frac{gR^2}{(h-y)^2}.$$

Integration of the obtained equation yields

$$v^2 = \frac{2gR^2}{h-y} + C.$$

Taking into account $y(0) = 0$, we get $C = -\frac{2gR^2}{h}$, and finally

$$v^2 = \frac{2gR^2 y}{h(h-y)}.$$

On the Earth surface $y = h - R$, and we get

$$v = \sqrt{2gR\left(1 - \frac{R}{h}\right)}$$

Taking into account that $h \to \infty$, the last formula yields

$$v = \sqrt{2gR}.$$

□

Example 2.9. Two substances A and B undergo a chemical reaction yielding a substance C. We assume amount of the C substance by $y(t)$ in the time instant t after the reaction, and we denote by α and β the amount of substance A and B, in the beginning of reaction, respectively. Find $\frac{dy}{dt}$ assuming that the reaction velocity is proportional to the product of reacting masses.

The governing equation is

$$\frac{dy}{dt} = p(\alpha - y)(\beta - y),\ p > 0,$$

and p is the proportionality coefficient. Separation of the variables yields

$$\frac{dy}{y - \alpha} - \frac{dy}{y - \beta} = -p(\beta - \alpha)dt.$$

After integration one gets

$$\frac{y - \alpha}{y - \beta} = Ce^{-p(\beta - \alpha)t}.$$

Taking into account the initial condition $y(0) = 0$ we obtain the constant $C = \alpha/\beta$, i.e.

$$\frac{y - \alpha}{y - \beta} = \frac{\alpha}{\beta} e^{-p(\beta - \alpha)t},$$

or equivalently

$$y(t) = \alpha\beta \frac{1 - e^{-p(\beta - \alpha)t}}{\beta - \alpha e^{-p(\beta - \alpha)t}}.$$

Observe that for $\beta > \alpha$ we have

$$\lim_{t \to \infty} y(t) = \alpha,$$

whereas for $\beta < \alpha$ we obtain

$$\lim_{t \to \infty} y(t) = \lim \alpha\beta \frac{e^{p(\beta - \alpha)t} - 1}{\beta e^{p(\beta - \alpha)t} - \alpha} = \beta.$$

In the case when $\alpha = \beta$ the governing equation is

$$\frac{dy}{dt} = p(\alpha - y)^2.$$

Separation of the variables of this equation and the integration allows to find the following dependence

$$\frac{1}{\alpha - y} = pt + C.$$

Since $y(0) = 0$, therefore $C = 1/\alpha$. In this case the reaction B governed by the equation

$$y(t) = \alpha \left(1 - \frac{1}{1 + \alpha pt}\right),$$

which for $t \to \infty$ yield

$$\lim_{t \to \infty} y(t) = \alpha.$$

2.3 Homogenous Equations

A function $F(x, y)$ is called homogenous of order k, if for all $\sigma > 0$ the following property holds [208]

$$F(\sigma x, \sigma y) = \sigma^k F(x, y) \tag{2.13}$$

For instance the functions

$$\frac{x + y}{x - y}, \quad \frac{x^2 + xy}{y - x}, \quad x^2 + y^2 + 2xy \tag{2.14}$$

are homogenous of order $k = 0, 1, 2,$, respectively.

A differential equation

$$\frac{dy}{dx} = F(x, y) \tag{2.15}$$

is called *homogenous*, if the function $F(x, y)$ is of order zero.

Equation

$$F_1(x, y)dx + F_2(x, y)dy = 0 \tag{2.16}$$

is called homogeneous, if the function F_1, F_2 are homogeneous of the same order.

In the case of a homogeneous equation the introduction of a new variable $y = zx$ allows to get en equation with separable variables. One may use also polar coordinates (ϱ, φ) and by substitution $x = \varrho \cos \varphi$, $y = \varrho \sin \varphi$ again an equation with separable variables is obtained.

It should be mentioned that the equation

$$\frac{dy}{dx} = F\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right) \tag{2.17}$$

can also be transformed to a homogeneous equation through the following linear transformation

$$x = x_0 + X, \ y = y_0 + Y, \tag{2.18}$$

where (x_0, y_0) is the point of intersection of straight lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$. If the lines do not intersect then $a_1/b_1 = a_2/b_2$, and in this case Eq. (2.17) is transformed to that with separable variables using

$$a_1x + b_2y + c_1 = X. \tag{2.19}$$

The function $G(x, y)$ is called quasi-homogenous of order k, if for certain α and β the following relation holds

$$G(\sigma^\alpha x, \sigma^\beta y) = \sigma^k G(x, y), \tag{2.20}$$

for all $k > 0$.

Exponents α, β are called *weights*. We say that $x(y)$ has weight $\alpha(\beta)$, and for instance $7x^2y^5$ has the weight $2\alpha + 5\beta$.

Differential equation (2.15) is called quasi-homogeneous if the associated function $F(x, y)$ is quasi-homogeneous with weights α and β of order $\beta - \alpha$, i.e. $F(\sigma^\alpha x, \sigma^\beta y) = \sigma^{\beta-\alpha} F(x, y)$.

A quasi-homogeneous differential equation can be reduced to a homogeneous one. However, in many practical cases one may use the direct variables change $y = zx^{\frac{\beta}{\alpha}}$ allowing to get an equation with separable variables.

Example 2.10. Find a solution of the following ODE

$$\frac{dy}{dx} = \frac{xy + y^2 e^{-\frac{x}{y}}}{x^2}.$$

We introduce the new variable $y = zx$, and obtain

$$x\frac{dz}{dx} + z = z + z^2 e^{-\frac{1}{z}},$$

or equivalently

$$\frac{e^{\frac{1}{z}}}{z^2} dz = \frac{dx}{x}.$$

Integration of the last equation yields

$$-e^{\frac{1}{z}} = \ln|x| - C,$$

or equivalently

$$e^{\frac{x}{y}} + \ln|x| = C.$$

□

Example 2.11. Solve the following equation

$$\frac{dy}{dx} = 2\left(\frac{y+1}{x+y-2}\right)^2.$$

We introduce the following variables

$$y + 1 = Y,\ x - 3 = X,$$

and we get

$$\frac{dY}{dX} = 2\frac{Y^2}{(X+Y)^2}.$$

Now we introduce the following new variable

$$Y = uX,$$

and the following ODE is obtained

$$X\frac{du}{dX} + u = \frac{2u^2}{(1+u)^2},$$

or equivalently

$$\ln|u| + 2\text{arctan}u + \ln|X| = \ln C,$$

which means that

$$uX = C\exp(-2\text{arctan}u).$$

In the original variable the solution is

$$(y+1)\exp\left(2\text{arctan}\frac{y+1}{x-3}\right) = C.$$

□

Example 2.12. Prove that integral curves of the equation

$$[2x(x^2-axy+y^2)-y^2\sqrt{x^2+y^2}]dx+y[2(x^2-axy+y^2)+x\sqrt{x^2+y^2}]dy=0$$

are closed curves surrounding the coordinates origin for $|a|<2$.

Since the studied equation is homogenous, then we introduce polar coordinates to get

$$\begin{aligned}&\varrho^3[2(1-a\sin\varphi\cos\varphi)\cos\varphi-\sin^2\varphi](\cos\varphi d\varrho-\varrho\sin\varphi d\varphi)\\ &+\varrho^3[2\sin\varphi(1-a\sin\varphi\cos\varphi)+\cos^2\varphi](\sin\varphi d\varrho+\varrho\cos\varphi d\varphi)=0\end{aligned}$$

or equivalently

$$2(1-a\sin\varphi\cos\varphi)d\varrho+\varrho\sin\varphi d\varphi=0.$$

Separating the variables we obtain

$$\frac{d\varrho}{\varrho}+\frac{\sin\varphi}{2-a\sin 2\varphi}d\varphi=0,$$

and after integration we get

$$\ln\varrho+\int_0^{\varphi}\frac{\sin u du}{2-a\sin 2u}=\ln\varrho_0,\quad \varrho_0=\varrho(0),$$

or equivalently

$$\varrho=\varrho_0\exp\left(\int_0^{\varphi}\frac{\sin u}{2-a\sin 2u}du\right).$$

If we prove that the function $\int_0^{\varphi}\frac{\sin u du}{2-a\sin 2u}$ is periodic regarding φ with the period 2π, then $\varrho=\varrho(\varphi)$ for arbitrary $\varrho_0>0$ is the 2π periodic function and its integral curve is closed. We have

$$\begin{aligned}\int_0^{\varphi+2\pi}\frac{\sin u du}{2-a\sin 2u}&=\int_0^{2\pi}\frac{\sin u du}{2-a\sin 2u}+\int_{2\pi}^{\varphi+2\pi}\frac{\sin u du}{2-a\sin 2u}\\ &=\int_0^{\pi}\frac{\sin u du}{2-a\sin 2u}-\int_{\pi}^{2\pi}\frac{\sin u du}{2-a\sin 2u}+\int_0^{\varphi}\frac{\sin(2\pi+u)du}{2-a\sin 2(2\pi+u)}\end{aligned}$$

$$= \int_0^{\pi} \frac{\sin du}{2 - a \sin 2u} - \int_0^{\pi} \frac{\sin u du}{2 - a \sin 2u} + \int_0^{\varphi} \frac{\sin u du}{2 - a \sin 2u} = \int_0^{\varphi} \frac{\sin u du}{2 - a \sin 2u}$$

which proves that $\varrho(\varphi) = \varrho(\varphi + 2\pi)$. □

Example 2.13. Solve the following differential equation

$$\frac{dy}{dx} = \frac{4x^6 - y^4}{2x^4 y}.$$

Let $x(y)$ assign the weight $\alpha(\beta)$. Then

$$F(x, y) = \frac{4\sigma^{6\alpha} x^6 - \sigma^{4\beta} y^4}{2\sigma^{4\alpha+\beta} x^4 y} = \sigma^{\beta-\alpha} \frac{4x^6 - y^4}{2x^4 y}.$$

This equation is satisfied when

$$6\alpha - 4\alpha - \beta = 4\beta - 4\alpha - \beta = \beta - \alpha$$

which means that $\beta/\alpha = 3/2$. Therefore, it has been proved that the studied equation is homogeneous. In order to separate its values, the following variable is introduced $y = zx^{\frac{3}{2}}$. We obtain

$$\frac{dz}{dx} x^{\frac{3}{2}} + \frac{3}{2} x^{\frac{1}{2}} z = \frac{4x^6 - z^4 x^6}{2x^4 z x^{\frac{3}{2}}},$$

and multiplying both sides by $x^{-\frac{1}{2}}$ we get

$$x \frac{dz}{dx} + \frac{3}{2} z = \frac{4 - z^4}{2z}$$

or equivalently

$$\frac{2z dz}{(z^2 + 4)(z^2 - 1)} + \frac{dx}{x} = 0, \quad z \neq \pm 1.$$

Direct integration yields

$$\ln \frac{|z^2 - 1|}{z^2 + 4} + 5 \ln |x| = \ln C. \tag{$*$}$$

In order to verify the obtained result we use the following differentiation formulas:

$$(\ln |y|)' = \frac{y'}{y}, \quad y = \frac{u}{v}, \quad u = z^2 - 1, \quad v = z^2 + 4.$$

Since

$$y' = \left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v} = \frac{2z(z^2+4) - 2z(z^2-1)}{(z^2+4)} = \frac{10z}{(z^2+4)^2},$$

hence

$$\frac{y'}{y} = \frac{10z}{(z^2+4)(z^2-1)}.$$

Full differentiation of (∗) yields

$$\frac{10z dz}{(z^2+4)(z^2-1)} + \frac{5}{x}dx = 0.$$

Formula (∗) yields

$$\frac{z^2-1}{z^2+4}x^5 = C.$$

Since $z^2 = y^2/x^3$, therefore

$$\frac{y^2-x^3}{y^2+4x^3}x^5 = C.$$

2.4 Linear Equations

Linear first-order ODE has the following form

$$\frac{dy}{dx} + a(x)y = f(x). \tag{2.21}$$

There exist three different methods yielding a solution of Eq. (2.21)

(i) *The Lagrange method.* This method is based on a constant variation. We consider first a homogeneous equation associated with (2.21) of the form

$$\frac{dy}{dx} + a(x)y = 0, \tag{2.22}$$

and its solution is

$$y = C \exp\left[-\int a(x)dx\right]. \tag{2.23}$$

We are looking for solution to Eq. (2.21) by variation of the constant $C = C(x)$, namely

$$y = C(x) \exp\left[-\int a(x)dx\right]. \tag{2.24}$$

Substituting (2.24) to (2.21) we obtain

$$\frac{dC(x)}{dx} = f(x) \exp\left[\int a(x)dx\right], \tag{2.25}$$

and hence

$$C(x) = C + \int \left[f(x) \exp\left[\int a(x)dx\right]\right] dx, \tag{2.26}$$

where C is the arbitrary constant.

Finally, substitution of (2.26) into (2.24) yields

$$y = \exp\left[-\int a(x)dx\right] \left\{C + \int \left[f(x) \exp\left(\int a(x)dx\right)\right]\right\}. \tag{2.27}$$

Any solution passing through the point (x_0, y_0) can be written in the following form

$$y = \exp\left[-\int_{x_0}^{x} a(z)dz\right] \left\{y_0 + \int_{x_0}^{x} f(u) \left[\exp\left(\int_{x_0}^{u} a(x)dx\right) du\right]\right\}. \tag{2.28}$$

(ii) *The Bernoulli method.* We are looking for a solution of (2.21) in the following form

$$y = u(x)v(x). \tag{2.29}$$

Substitution of (2.29) to (2.21) gives

$$\frac{du}{dx}v + u\frac{dv}{dx} + a(x)uv = f(x). \tag{2.30}$$

If we take $u(x)$ as the solution of equation

$$\frac{du}{dx} + a(x)u = 0, \tag{2.31}$$

then

$$u(x) = \exp\left[-\int a(x)dx\right]. \tag{2.32}$$

Substituting (2.32) into (2.30) gives

$$\exp\left[-\int a(x)dx\right]\frac{dv}{dx} = f(x), \tag{2.33}$$

and therefore

$$v(x) = C + \int f(x)\exp\left[\int a(x)dx\right]dx, \tag{2.34}$$

where C is a constant.

(iii) *The method of an integrating multiplier.* We multiply both parts of Eq. (2.21) by $\exp\left(\int a(x)dx\right)$, and we get

$$\frac{d}{dx}\left[y\exp\left(\int a(x)dx\right)\right] = f(x)\exp\left(\int a(x)dx\right) \tag{2.35}$$

or equivalently

$$y = \exp\left(-\int a(x)dx\right)\left[C + \int f(x)\exp\left(\int a(x)dx\right)dx\right] \tag{2.36}$$

Equation of the form

$$A(y) + [B(y)x - C(y)]\frac{dy}{dx} = 0 \tag{2.37}$$

can be transformed to the form (2.21). We multiply both sides by $\frac{1}{A}\frac{dx}{dy}$ and we get

$$\frac{dx}{dy} + \alpha(y)x = \beta(y) \tag{2.38}$$

where

$$\alpha(y) = \frac{B}{A}, \quad \beta(y) = \frac{C}{A}. \tag{2.39}$$

It should be emphasized that equations of the form

$$F'(y)\frac{dy}{dx} + F(y)a(x) = b(x), \tag{2.40}$$

where $'$ denotes $\frac{d}{dy}$ can be transformed to the linear equation by introduction of the relation $u = f(y)$.

A particular role in theory of first-order differential equations play *the Bernoulli and Riccati equations*. The equation

$$\frac{dy}{dx} + a(x)y = b(x)y^n,\ n \neq 0, 1 \tag{2.41}$$

is called *the Bernoulli equation*. It is transformed to the following form

$$y^{-n}\frac{dy}{dx} + a(x)y^{1-n} = b(x),\ y \neq 0, \tag{2.42}$$

and it is reduced to a linear equation via the variable change $u = y^{1-n}$. This approach will be illustrated through examples. One may also apply here *the Bernoulli method.*

The equation

$$\frac{dy}{dx} + a(x)y + b(x)y^2 = C(x) \tag{2.43}$$

is called a *Riccati equation*. In general it cannot be solved in quadratures. However, if one of its particular solutions is known, say $y_1(x)$ then the transformation $y = y_1 + u$ allows reduction of the problem to that of finding solution to the Bernoulli equation.

Example 2.14. A current in the electrical network with the resistance R, induction L and excitation voltage $u(t) = u_0 \sin \omega t$ is governed by the following equation

$$L\frac{di}{dt} + Ri = u_0 \sin \omega t, \quad i(0) = 0.$$

Find $i = i(t)$.

We have

$$\frac{di}{dt} + \alpha i = \beta \sin \omega t,$$

where $\alpha = \frac{R}{L}$, $\beta = \frac{u_0}{L}$. We apply here *the Bernoulli method*, i.e. we assume

$$i(t) = u(t)v(t).$$

Substitution of $i(t)$ into the governing equation yields

$$\frac{du}{dt}v + u\frac{dv}{dt} + \alpha uv = \beta \sin \omega t. \tag{$*$}$$

We consider a solution of the homogeneous equation

$$\frac{du}{dt} + \alpha u = 0$$

of the form

$$u = \exp(-\alpha t).$$

We substitute it to (∗) and we obtain $\frac{dv}{dt} = \beta e^{\alpha t} \sin \omega t$, what means that

$$v(t) = \beta \left[\int e^{\alpha t} \sin \omega t dt + C \right].$$

We successively compute

$$\begin{aligned} V(t) = \int e^{\alpha t} \sin \omega t dt &= -e^{\alpha t} \frac{1}{\omega} \cos \omega t + \frac{\alpha}{\omega} \int e^{\alpha t} \cos \omega t dt \\ &= -\frac{1}{\omega} e^{\alpha t} \cos \omega t + \frac{\alpha}{\omega^2} e^{\alpha t} \sin \omega t - \frac{\alpha^2}{\omega^2} \int e^{\alpha t} \sin \omega t dt \\ &= -\frac{1}{\omega} e^{\alpha t} \cos \omega t + \frac{\alpha}{\omega^2} e^{\alpha t} \sin \omega t - \frac{\alpha^2}{\omega^2} V(t), \end{aligned}$$

and therefore

$$V(t) = \frac{e^{\alpha t}(-\omega \cos \omega t + \alpha \sin \omega t)}{\omega^2 + \alpha^2}.$$

Finally, we find

$$v(t) = \beta \left[\frac{e^{\alpha t}(-\omega \cos \omega t + \alpha \sin \omega t)}{\omega^2 + \alpha^2} + C \right],$$

and

$$i(t) = u(t)v(t) = \beta \left(\frac{-\omega \cos \omega t + \alpha \sin \omega t}{\omega^2 + \alpha^2} + Ce^{-\alpha t} \right).$$

Since $i(0) = 0$, $C = \frac{\omega}{\omega^2 + \alpha^2}$, and therefore

$$i(t) = \frac{\beta}{\omega^2 + \alpha^2} (-\omega \cos \omega t + \alpha \sin \omega t + \omega e^{-\alpha t}).$$

Observe that

$$\lim_{t \to \infty} i(t) = \frac{u_0}{L(\omega^2 + \alpha^2)} (-\omega \cos \omega t + \alpha \sin \omega t) = \frac{u_0}{\sqrt{(L\omega)^2 + R^2}} \sin(\omega t - \varphi),$$

where $\tan \varphi = \frac{\omega}{\alpha}$ denotes the initial current phase. □

Example 2.15. Show that equation

$$\frac{dy}{dx} + \alpha y = f(x), \quad \alpha > 0,$$

possesses only one bounded solution assuming that $f(x)$ is bounded for all $x \in \mathbb{R}$. Find this solution, and show that if $f(x + x_0) = f(x)$, then $y(x) = y(x + x_0)$, where x_0 is a period.

First we find a solution to the homogeneous equation

$$\frac{dy}{dx} + \alpha y = 0.$$

After variables separation we get

$$\frac{dy}{y} = -\alpha dx,$$

and hence

$$\ln |y| + \alpha x = \ln C,$$

which means that

$$y = Ce^{-\alpha x}$$

assuming that $y \neq 0$.

We apply here *the Lagrange's method*. Namely, we have

$$y(x) = C(x)e^{-\alpha x},$$

and substitution of $y(x)$ into the governing equation gives

$$\frac{dC}{dx} = e^{\alpha x} f(x).$$

It means that

$$C(x) = C(x_0) + \int_{x_0}^{x} e^{\alpha z} f(z) dz.$$

The sought solution has the following form

$$y(x) = C(x_0)e^{-\alpha x} + \int_{x_0}^{x} e^{-\alpha(x-z)} f(z) dz. \qquad (*)$$

Assuming $y(x_0) = y_0$ we obtain

$$y_0 = C(x_0)e^{-\alpha x_0},$$

or equivalently

$$C(x_0) = y_0 e^{\alpha x_0}.$$

Therefore, solution (∗) takes the following form

$$y(x) = e^{-\alpha(x-x_0)} y_0 + \int_{x_0}^{x} e^{-\alpha(x-z)} f(z)dz.$$

We multiply both sides of the last equation by $e^{\alpha(x-x_0)}$ to get

$$e^{\alpha(x-x_0)} y(x) = y_0 + \int_{x_0}^{x} e^{\alpha(z-x_0)} f(z)dz.$$

We consider the case $x \to -\infty$ (the case of $x \to +\infty$ can be studied in the similar way). We have

$$\lim_{x\to-\infty} e^{\alpha(x-x_0)} y(x) = y_0 + \int_{x_0}^{-\infty} e^{\alpha(z-x_0)} f(z)dz,$$

and hence

$$y_0 = \int_{-\infty}^{x_0} e^{\alpha(z-x_0)} f(z)dz,$$

because for a bounded solution $\lim_{x\to-\infty} e^{\alpha(x-x_0)} y(x) = 0$. It means that

$$\lim_{x\to-\infty} y(x) = Y(x) = \int_{-\infty}^{x} \exp(-\alpha(x-z)) f(z)dz \qquad (**)$$

is bounded, assuming that $f(z)$ is bounded.

In what follows we show that $Y(x)$ is the only bounded solution of the studied equation. Let us assume that there exists one more bounded solution denoted by $Y_*(x)$. It means that the difference

$$\Delta Y = Y(x) - Y_*(x)$$

is bounded. We also have

$$\frac{dY}{dx} + \alpha Y = f(x),$$
$$\frac{dY_*}{dx} + \alpha Y_* = f(x),$$

which means that

$$\frac{d(\Delta Y)}{\Delta Y} + \alpha(\Delta Y) = 0,$$

and hence

$$\Delta Y(x) = Ce^{\alpha x}.$$

Owing to our introduced assumption $\Delta Y(x)$ is bounded for all $x \in \mathbb{R}$, which means that $Ce^{\alpha x}$ must be bounded. This is true only if $C = 0$, which yields $Y(x) = Y_*(x)$.

Let us now show that if $f(x + x_0) = f(x)$ than $Y(x) = Y(x + x_0)$. It follows from (∗∗) that

$$Y(x + x_0) = \int_{-\infty}^{x+x_0} e^{-\alpha(x+x_0-z)} f(z)dz = \int_{-\infty}^{x} e^{-\alpha(x-\tau)} f(\tau + x_0)d\tau$$
$$= \int_{-\infty}^{x} e^{-\alpha(x-\tau)} f(\tau)d\tau = Y(x).$$

□

Example 2.16. Solve the following Bernoulli equation

$$x\frac{dy}{dx} + y = y^2 \ln x.$$

We use the Bernoulli method, and we look for a solution of the form

$$y = u(x)v(x).$$

Substitution of $y(x)$ into the studied equation yields

$$xu\frac{dv}{dx} + xv\frac{du}{dx} + uv = u^2v^2 \ln x \qquad (*)$$

We take

$$x\frac{du}{dx} + u = 0,$$

and hence

$$u = \frac{1}{x}.$$

Substituting this result into (∗) we get

$$x^2 \frac{dv}{dx} = v^2 \ln x,$$

and hence

$$\frac{1}{v^2} dv = \frac{1}{x^2} \ln x dx.$$

Integration of the last obtained equation gives

$$-\frac{1}{v} = \int \frac{\ln x}{x^2} dx = -\frac{\ln x}{x} - \frac{1}{x} - C,$$

which means that

$$v(x) = \frac{x}{1 + Cx + \ln x},$$

and finally

$$y(x) = u(x)v(x) = \frac{1}{1 + Cx + \ln x}.$$

□

Example 2.17. Solve the following Bernoulli equation

$$(1 + x^2)\frac{dy}{dx} - 2xy = 4\sqrt{y(1 + x^2)}\arctan x.$$

Assuming

$$y(x) = u(x)v(x)$$

we get

$$(1 + x^2)\left(\frac{du}{dx}v + u\frac{dv}{dx}\right) - 2xuv = 4\sqrt{uv(1 + x^2)}\arctan x$$

or equivalently

$$(1 + x^2)\frac{du}{dx}v + (1 + x^2)\left(\frac{dv}{dx} - \frac{2x}{1 + x^2}v\right)u = 4\sqrt{uv(1 + x^2)}\arctan x.$$

We take an arbitrary solution to the equation

$$\frac{dv}{dx} - \frac{2x}{1+x^2}v = 0,$$

i.e. for example the following one

$$v(x) = 1 + x^2.$$

Therefore, we get

$$(1+x^2)^2\frac{du}{dx} = 4(1+x^2)\sqrt{u}\arctan x.$$

One of the solution is $u = 0$, and the other solutions are found through the successive transformations

$$\frac{du}{dx} = \frac{4\arctan x}{1+x^2}\sqrt{u},$$
$$\frac{du}{2\sqrt{u}} = \frac{2\arctan x}{1+x^2}dx,$$
$$\sqrt{u} = \arctan^2 x + C.$$

Finally, the solutions are

$$y = 0,$$
$$y = (1+x^2)(\arctan^2 x + C)^2.$$

□

Example 2.18. Solve the following Riccati equation

$$\frac{dy}{dx} + y^2 = \frac{2}{x^2}.$$

Let us look for a particular solution of the form

$$y_1 = \frac{A}{x}.$$

Substituting y_1 into the studied equation yields

$$-\frac{A}{x^2} + \frac{A^2}{x^2} = \frac{2}{x^2}.$$

The second-order algebraic equation yields two roots $A_1 = -1$, $A_2 = 2$. Let us introduce a new variable z of the form

$$y = z - \frac{1}{x},$$

and therefore

$$\frac{dz}{dx} + \frac{1}{x^2} + z^2 - \frac{2z}{x} + \frac{1}{x^2} = \frac{2}{x^2},$$

or equivalently

$$\frac{dz}{dx} - \frac{2}{x}z = -z^2.$$

We multiply both sides of the obtained equation by x^2 to get

$$x\frac{d}{dx}(zx) = 3zx - (zx)^2.$$

We take

$$zx = u,$$

and integrate the following equation

$$x\frac{du}{dx} = u(3-u).$$

Separation of the variables yields

$$\frac{du}{u(3-u)} = \frac{dx}{x}.$$

Since

$$\frac{1}{u(3-u)} = \frac{1}{3u} + \frac{1}{3(3-u)},$$

therefore

$$\frac{1}{3}\int\frac{du}{u} + \frac{1}{3}\int\frac{du}{3-u} = \int\frac{dx}{x},$$

and consequently

$$\frac{1}{3}\left[\ln|u| - \ln|3-u|\right] = \ln|x| + \ln C_1, \qquad C_1 > 0.$$

Finally, we find

$$\ln\left|\frac{u}{3-u}\right| = 3\ln|C_1 x|,$$

or

$$\ln\left|\frac{u}{3-u}\right| = \ln|(C_1 x)^3|.$$

We consider two cases:

(i)

$$\frac{u}{3-u} \geq 0.$$

In this case we have

$$\frac{u}{3-u} = Cx^3,$$

which means that

$$\frac{zx}{3-zx} = Cx^3,$$

and hence

$$z = \frac{3Cx^2}{Cx^3+1}.$$

We finally get

$$y = z - \frac{1}{x} = \frac{2}{x}$$

and

$$y = \frac{3Cx^2}{Cx^3+1} - \frac{1}{x} = \frac{2Cx^3-1}{x(1+Cx^3)}.$$

(ii)

$$\frac{u}{3-u} < 0.$$

In this case we have

$$\frac{u}{u-3} = Cx^3,$$

which means that

$$zx = C(zx - 3)x^3,$$

and hence

$$z = \frac{3Cx^2}{Cx^3 - 1}.$$

We finally obtain

$$y = \frac{2}{x}$$

and

$$y = \frac{2Cx^3 + 1}{x(Cx^3 - 1)}$$

□

2.5 Exact Differential Equations

The differential equation

$$M(x, y)dx + N(x, y)dy = 0 \tag{2.44}$$

is called an *exact differential equation* if its left-hand side is the full differential of a certain function $V(x, y)$ such that

$$dV(x, y) \equiv \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy = M(x, y)dx + N(x, y)dy = 0. \tag{2.45}$$

A necessary condition that Eq. (2.44) is exact one follows

$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}. \tag{2.46}$$

If $V(x, y)$ is known than all solutions of (2.44) satisfy the condition

$$V(x, y) = C, \tag{2.47}$$

where C is an arbitrary constant.

We show how we can find the function $V(x, y)$. Since

$$\frac{\partial V}{\partial x} = M(x, y), \quad \frac{\partial V}{\partial y} = N(x, y), \tag{2.48}$$

then

$$V(x, y) = \int M(x, y)dx = \psi(x, y) + \psi(y). \quad (2.49)$$

We differentiate (2.49) to get

$$\frac{\partial\psi(x, y)}{\partial y} + \frac{\partial\psi(y)}{\partial y} = N(x, y). \quad (2.50)$$

In some cases the general form given by (2.44) can be transformed to an exact differential equation by introduction of a so-called integrating multiplier $m(x, y)$ [208]. In Eq. (2.46) we introduce $m(x, y)$, and we obtain the following exact differential equation

$$\frac{\partial}{\partial y}(mM) = \frac{\partial}{\partial x}(mN), \quad (2.51)$$

which means that m should satisfy the following equation

$$m\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\frac{\partial m}{\partial x} - M\frac{\partial m}{\partial y}. \quad (2.52)$$

The obtained general form (2.52) can be simplified in the following cases

(i) If $m(x, y) = m(x)$ then

$$\frac{1}{m}\frac{dm}{dx} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}. \quad (2.53)$$

(ii) If $m(x, y) = m(y)$ then

$$-\frac{1}{m}\frac{dm}{dx} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M}. \quad (2.54)$$

(iii) If $m(x, y) = m(r(x, y))$, where $r(x, y)$ is a known function then

$$\frac{1}{m}\frac{dm}{dr} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N\frac{\partial r}{\partial x} - M\frac{\partial r}{\partial y}}. \quad (2.55)$$

Example 2.19. Solve the differential equation

$$(2xy + 3y^2)dx + (x^2 + 6xy - 3y^2)dy = 0.$$

We have

$$M(x, y) = 2xy + 3y^2,\ N(x, y) = x^2 + 6xy - 3y^2,$$

and hence

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2x + 6y.$$

It means that the left-hand side of the differential equation is a full differential of a certain function $V(x, y)$.

We have

$$\frac{\partial V}{\partial x} = 2xy + 3y^2, \quad \frac{\partial V}{\partial y} = x^2 + 6xy - 3y^2.$$

First equation of the above yields

$$V(x, y) = x^2y + 3xy^2 + \psi(y).$$

We differentiate the last equation with respect to y and thus

$$\frac{\partial V}{\partial y} = x^2 + 6xy + \frac{\partial \psi(y)}{\partial y} = x^2 + 6xy - 3y^2.$$

It means that

$$\psi(y) = -y^3 + C.$$

Hence

$$V(x, y) = x^2y + 3xy^2 - y^3 + C,$$

and a general solution to the studied ODE is defined implicitly by the equation

$$x^2y + 3xy^2 - y^3 = C.$$

□

Example 2.20. Solve the differential equation

$$2x\left(1 + \sqrt{x^2 - y}\right)dx - \sqrt{x^2 - y}dy = 0.$$

Observe that

$$\frac{\partial}{\partial y}[2x(1 + \sqrt{x^2 - y})] = \frac{\partial}{\partial x}(-\sqrt{x^2 - y}) = -\frac{x}{\sqrt{x^2 - y}},$$

and hence we deal with the exact differential equation. We have

$$\frac{\partial V}{\partial x} = 2x(1 + \sqrt{x^2 - y}), \quad \frac{\partial V}{\partial y} = -\sqrt{x^2 - y}, \tag{$*$}$$

and integration of the first equation yields

$$V(x,y) = \int (2x + 2x\sqrt{x^2 - y})dx = x^2 + \frac{2}{3}(x^2 - y)^{\frac{3}{2}} + \psi(y).$$

Substitution of $V(x,y)$ into the second equation of (∗) gives

$$\frac{\partial}{\partial y}[x^2 + \frac{2}{3}(x^2 - y)^{\frac{3}{2}} + \psi(y)] = -\sqrt{x^2 - y},$$

or equivalently

$$-\sqrt{x^2 - y} + \frac{d\psi}{dy} = -\sqrt{x^2 - y},$$

which means that

$$\psi(y) = C.$$

Finally, we have

$$V(x,y) = x^2 + \frac{2}{3}(x^2 - y)^{\frac{3}{2}},$$

and a general solution to the studied differential equation is

$$x^2 + \frac{2}{3}(x^2 - y)^{\frac{3}{2}} = C$$

or

$$y = x^2 - [\frac{3}{2}(C - x^2)]^{\frac{2}{3}}.$$

□

Example 2.21. A mirror reflects solar radiation in a way that a light ray coming from a source 0 after the reflection is parallel to a given direction $0X$, which is the rotation axis. Figure 2.2 shows a scheme of the light ray $0A$ coming from the light source 0, and the rectangular coordinates $0XY$. Derive the mirror shape analytically.

Since A belongs to the mirror surface, the marked angles φ before and after reflection are equal, and $n(t)$ denotes a normal (tangent) to the curve being intersection of the mirror and surface $0XY$.

Owing to the reflection principle (the angle of incidence is equal to the reflection angle) $0A = 0B$, and hence

$$\tan\varphi = \frac{AA'}{B0 + 0A'} = \frac{AA'}{\sqrt{(0A')^2 + (A'A)^2} + 0A'},$$

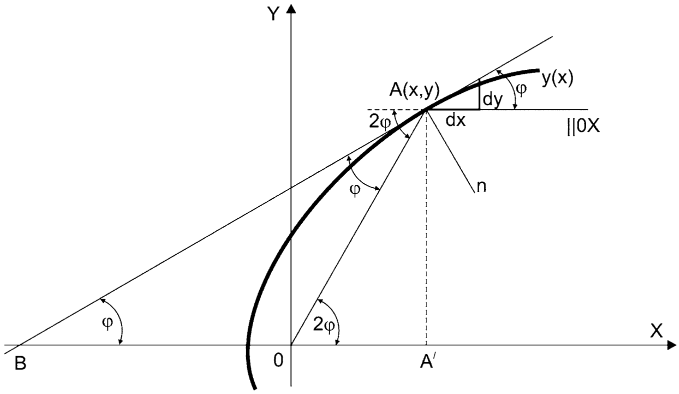

Fig. 2.2 The mirror shape $y(x)$ and light rays

or equivalently

$$\frac{dy}{dx} = \frac{y}{x + \sqrt{x^2 + y^2}}.$$

We may rewrite the latter equation in the following way

$$xdx + ydy = \sqrt{x^2 + y^2}dx,$$

because

$$dy = \frac{y(x - \sqrt{x^2 + y^2})}{(x + \sqrt{x^2 + y^2})(x - \sqrt{x^2 + y^2})}dx = \frac{x - \sqrt{x^2 + y^2}}{-y}dx.$$

Applying the integrating multiplier

$$m(x, y) = \frac{1}{\sqrt{x^2 + y^2}},$$

we get

$$\frac{xdx + ydy}{\sqrt{x^2 + y^2}} - dx = 0$$

or equivalently

$$\frac{d(x^2 + y^2)}{2\sqrt{x^2 + y^2}} - dx = 0.$$

It means that

$$\sqrt{x^2 + y^2} = x + C,$$

which allows to find the mirror surface as a paraboloid that intersects with the surface $0XY$ yielding a parabola governed by the equation

$$y^2 = 2Cx + C^2.$$

□

Example 2.22. Solve the differential equation

$$ydx - (x + x^2 + y^2)dy = 0$$

assuming the integrating multiplier $m = m(r(x, y))$, where $r(x, y) = x^2 + y^2$.

We apply formula (2.55) directly, and we get

$$\frac{1}{m}\frac{dm}{dr} = \frac{1 + 1 + 2x}{-2(x + x^2 + y^2)x - 2y^2} = \frac{2(1 + x)}{-2(1 + x)(x^2 + y^2)} = -\frac{1}{r}.$$

Therefore, the following differential equation is obtained

$$\frac{dm}{m} + \frac{dr}{r} = 0,$$

which yields

$$m(x, y) = \frac{1}{r(x, y)} = \frac{1}{x^2 + y^2}.$$

Now, we multiply by m the studied differential equation to get

$$\frac{ydx}{x^2 + y^2} - \left(\frac{x}{x^2 + y^2} + 1\right)dy = 0,$$

which is an exact differential equation, i.e.

$$\frac{\partial V}{\partial x} = \frac{y}{x^2 + y^2}, \quad \frac{\partial V}{\partial y} = -\left(\frac{x}{x^2 + y^2} + 1\right).$$

Integration of the first equation in the above gives

$$V(x, y) = \int \frac{y}{x^2 + y^2}dx = \arctan\frac{x}{y} + \psi(y),$$

and hence

$$\frac{\partial V}{\partial y} = \frac{\partial}{\partial y}\left(\arctan\frac{x}{y} + \psi(y)\right) = -\frac{x}{x^2 + y^2} - 1.$$

It means that

$$-\frac{1}{1+\frac{x^2}{y^2}}\frac{x}{y^2} + \frac{d\psi}{dy} = -\frac{x}{x^2 + y^2} - 1,$$

and finally

$$\frac{d\psi}{dy} = -1, \quad \psi(y) = -y + C_1,$$

and

$$V(x, y) = \arctan\frac{x}{y} - y + C_1.$$

We have the following solutions: one given explicitly $y = 0$, and other given implicitly

$$\arctan\frac{x}{y} - y = C.$$

2.6 Implicit Differential Equations Not Solved with Respect to a Derivative

We consider here the differential equation (2.2), which cannot be solved with respect to $\frac{dy}{dt}$, i.e. we cannot reduce the problem to that of Eq. (2.3). It may happen, however, that Eq. (2.2) can be solved with respect to either x or y. In what follows we describe briefly the method of the parameter introduction yielding a solution in the latter case. Let

$$y = f(x, y'), \quad y' \equiv \frac{dy}{dx} = p, \tag{2.56}$$

where p is *the introduced parameter*. The full differential of $y = f(x, y')$ follows

$$p dx = \frac{\partial f}{\partial x} dx + \frac{\partial(x, p)}{\partial p} dp. \tag{2.57}$$

It means that we have got the exact differential equation form (2.44), where

$$M(x, y) = \frac{\partial f}{\partial x}, \quad N(x, y) = \frac{\partial(x, p)}{\partial p}. \tag{2.58}$$

In the previous section supplemented by many examples we have described various methods yielding solutions to Eq. (2.56). Namely, we can take

$$x = \psi(p, c), \quad y = f(x, p), \tag{2.59}$$

where $x = \psi(p, c)$ is the implicit form of solution governed by Eq. (2.56).

Theorem 2.2. *Suppose that the function $f(x, y, y')$ in a neighbourhood of the point (x_0, y_0, y_0'), where y_0' is one of the roots of the equation $f(x_0, y_0, y_0') = 0$, is continuous regarding x and it is continuously differentiable with respect to y, y', and $\frac{\partial f}{\partial y'}(x_0, y_0, y_0') \neq 0$. Then there exists a unique solution $y = \psi'(x)$ of the Cauchy problem $f(x, y, y') = 0$, $y(x_0) = y_0$ defined in a satisfactorily close neighbourhood of the point x_0, where $\psi'(x_0) = y_0'$.*

Recall that the uniqueness of problem of Eq. (2.2) means that the point (x_0, y_0) is a point of the solution uniqueness, i.e. there are no other integral curves of (2.2) which pass through the point (x_0, y_0) and have the same slope in this point. Otherwise, the solution uniqueness is violated.

Theorem 2.2 yields sufficient conditions of a solution existence and uniqueness for Eq. (2.2).

Assuming that the function $f(x, y, y')$ is continuous with respect to x and continuously differentiable with respect to y and y', then a possible set of singular points is defined via the following system of algebraic equations

$$\begin{aligned} f(x, y, y') &= 0, \\ \frac{\partial f}{\partial y'}(x, y, y') &= 0. \end{aligned} \tag{2.60}$$

It is required, while solving Eq. (2.2) to find singular solution, i.e. we remove y' from Eq. (2.60) and we get a so-called discriminant-type curve. Each branch of this curve should be verified if it is a solution to Eq. (2.2). Assuming a positive reply, our next step consists of checking if its points correspond to the solution non-uniqueness.

The method of parameter introduction can be directly applied either to the so-called Claurait equation

$$y = xy' + \psi(y'), \tag{2.61}$$

or to the so-called Lagrange equation

$$y = x\varphi(y') + \psi(y'). \tag{2.62}$$

Example 2.23. Solve the following Claurait equation

$$\sqrt{(y')^2 + 1} + xy' - y = 0.$$

We introduce $p = y'$ to get

$$y = xp + \sqrt{1 + p^2}.$$

Differentiation of the last equation with respect to x yields

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + \frac{p\frac{dp}{dx}}{\sqrt{1 + p^2}},$$

and hence

$$\left(x + \frac{p}{\sqrt{1 + p^2}}\right)\frac{dp}{dx} = 0.$$

It means that either

$$x = -\frac{p}{\sqrt{1 + p^2}}.$$

or

$$p = C.$$

A solution to the problem is as follows:

$$y = Cx + \sqrt{1 + C^2}$$

or equivalently

$$x = -\frac{p}{\sqrt{1 + p^2}},$$
$$y = px + \sqrt{1 + p^2}.$$

□

Example 2.24. Solve the following Lagrange equation

$$y' + y = x(y')^2.$$

It is easily solved with respect to y, i.e.

$$y = x(y')^2 - y'$$

or equivalently

$$y = xp^2 - p,$$

where $p = y'$. Differentiation of this algebraic equation yields

$$p \equiv \frac{dy}{dx} = p^2 + 2px\frac{dp}{dx} - \frac{dp}{dx},$$

or equivalently

$$p(p-1)\frac{dx}{dp} = (1-2px),$$
$$\frac{dx}{dp} + \frac{2x}{p-1} = \frac{1}{p(p-1)}.$$

In other words, the problem has been reduced to a linear differential equation with the following solution

$$x = \frac{p - \ln p + C}{(p-1)^2}.$$

□

Example 2.25. Derive an equation governing a family of equipotential curves of the electric field generated by a dipole. Recall that the equipotential curves are orthogonal to force curves of the electric field (see Example 2.3).

As it has been shown previously in Example 2.3, we have

$$\left(\frac{x-a}{r_2^3} - \frac{x+a}{r_1^3}\right)\frac{dy}{dx} - \left(\frac{1}{r_2^3} - \frac{1}{r_1^3}\right) = 0,$$

where

$$r_1^2 = (x+a)^2 + y^2, \quad r_2^2 = (x-a)^2 + y^2. \tag{$*$}$$

We may generalize the studied case in Example 2.3. Namely, we began with the algebraic problem governed by the following equation

$$F(x, y, a) = 0,$$

where

$$F(x, y, a) = \frac{x+a}{\sqrt{(x+a)^2 + y^2}} - \frac{x-a}{\sqrt{(x-a)^2 + y^2}} - C.$$

For a given C, we have a family of one parameter curves. In what follows we define another family of the *isogonal curves*, which interset the first family curves with the same angle φ, for $\varphi = \pi/2$ we say that both trajectories (curves) are orthogonal.

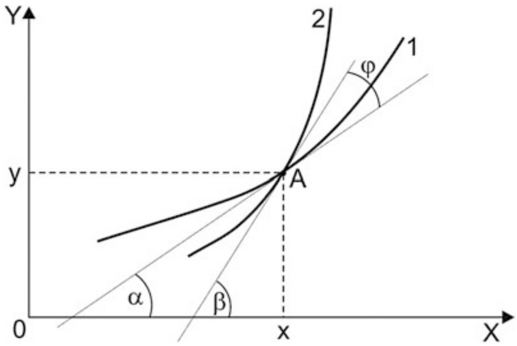

Fig. 2.3 Two curves 1 and 2 intersecting in point A

We differentiate the algebraic equation to get

$$\frac{dF}{dx} \equiv \frac{\partial F(x,y,a)}{\partial x} + \frac{\partial F(x,y,a)}{\partial y}\frac{dy}{dx} = 0.$$

We may also exclude the parameter a using the equation $F = 0$. In our case we have

$$\frac{\partial F}{\partial x} = \left(\frac{1}{r_2^3} - \frac{1}{r_1^3}\right)y, \quad \frac{\partial F}{\partial y} = \frac{x-a}{r_2^3} - \frac{x-a}{r_1^3}.$$

In Fig. 2.3 two curves belonging to both families are shown intersecting in the point $A = A(x,y)$.

The angle between two curves at point A is φ (known), which is given by the formula

$$\pi = \alpha + \varphi + \pi - \beta.$$

Therefore, we get

$$\tan\beta = \tan(\alpha + \varphi) = \frac{\tan\alpha + \tan\varphi}{1 - \tan\alpha\tan\varphi}.$$

We apply the following notation $\tan\alpha = y'$, $\tan\beta = y'_*$, $\tan\varphi = m$, and hence

$$y'_* = \frac{y' + m}{1 - my'}.$$

In a case of orthogonal trajectories we have $\varphi = \pi/2$, and therefore

$$\tan\beta = \frac{\tan\alpha + \tan\frac{\pi}{2}}{1 - \tan\alpha\tan\frac{\pi}{2}}$$

$$= \lim_{\varphi \to \frac{\pi}{2}} \frac{1 + \frac{\tan\alpha}{\tan\varphi}}{\frac{1}{\tan\varphi} - \tan\alpha} = -\frac{1}{\tan\alpha}$$

or equivalently

$$y'_* = -\frac{1}{y'}.$$

The so far consideration implies a simple recipe. In order to find a differential equations of the family of *isogonal trajectories* to the trajectories (curves) governed by equation $F(x, y, a) = 0$, we need to substitute the term $y' = \frac{dy}{dx}$ standing in equation $\partial F/\partial x + \partial F/\partial y y' = 0$, by the term y'_*. In a case for $\varphi = \frac{\pi}{2}$ (orthogonal trajectories) we substitute y' by $-\frac{1}{y'} = -\frac{dx}{dy}$.

In the studied case, using the so far described orthogonality property we obtain the following differential equation

$$\left(\frac{x-a}{r_2^3} - \frac{x+a}{r_1^3}\right)\left(-\frac{1}{\frac{dy}{dx}}\right) - \left(\frac{1}{r_2^3} - \frac{1}{r_1^3}\right) y = 0,$$

or equivalently

$$(x-a)r_1^3 - (x+a)r_2^3 + y(r_1^3 - r_2^3)\frac{dy}{dx} = 0.$$

From (∗) we get

$$r_1 dr_1 = (x+a)dx + ydy,$$
$$r_2 dr_2 = (x-a)dx + ydy,$$

therefore the problem is reduced to the following differential equation

$$r_1^3 r_2 dr_2 = r_2^3 r_1 dr_1,$$

which yields the following solution

$$\frac{1}{r_2} - \frac{1}{r_1} = C,$$

and hence

$$\frac{1}{\sqrt{(x-a)^2 + y^2}} - \frac{1}{\sqrt{(x+a)^2 + y^2}} = C.$$

□