Selected Theoretical Methods in Solid Earth Physics: Contribution from the Institute of Geophysics PAS

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Abstract Solid Earth Physics, including Seismology, Physics of the Earth, Earth Magnetism to name a few more topical disciplines, strongly relies on mathematical and numerical possibilities of modeling very complex physical processes ongoing in the Earth interior. Tremendous progress in geophysical instrumentation and still increasing quality and quantity of observational data also prompts for advanced processing methods in order to get more reliable interpretations. The goal of this chapter is to review some contributions from the Institute of Geophysics, Polish Academy of Sciences (IGF PAS) to physical and mathematical concepts used in Solid Earth Physics. We have selected some topics which are general enough to be interesting for a wide range of readers, leaving many topical issues uncovered in this review.

Keywords Computational geophysics · Inverse theory · Asymmetric continuum · Flow in porous media

1 Introduction

The distinguished position of geophysics among other earth sciences comes from the fact that geophysics attempts to describe the earth system using generally physical methodologies. It comprises observational techniques of physical parameters of the earth systems, often based on the very advanced technologies like satellite telemetry, very sensitive seismometric observations, or technologically very advanced deep and fast deep crust drilling to name a few techniques concerning solid earth observations. Moreover, most of observational sites have

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recently been networked together and infrastructures like international data centers have been created. Thus, nowadays we can talk about the Earth's *global observational system* providing enormous quantity of information on the Earth system and comparable only to the most advanced physical experimental facilities. Another particular feature of geophysics is using the physical approach to understand the natural and anthropogenic-caused processes occurring in the Earth system starting from the very basic physical principles like energy or momentum conservation, second thermodynamic law, etc. However, geophysics not only uses the concepts and methodologies developed by other physical disciplines. It also significantly contributes to development of various physical observational technologies, theories and computational techniques. This development was quite often inspired by pragmatic industrial and/or social requests and needs. For example, development of the very advanced numerical data processing techniques was inspired and founded by the prospecting industry, very much interested in more detailed and efficient mapping of underground structures. On the other hand, an advanced analysis of earthquake hazard, physics of earthquake sources or volcanic processes is of the greatest importance for contemporary societies. This particular demand of a "practice" of geophysical research has formed the third distinguished feature of geophysics—its pragmatism in formulating and solving the real scientific tasks.

At the early stage of the development of geophysical research, the most important research tasks were accumulating an evidence of some natural processes and linking them as much as possible to known physical principles, models and ideas. This stage of geophysical qualitative-type research was very soon finished by formulated demands of a quantitative analysis of gathered evidence and searching for possible physical description of observed phenomena. This refocusing of geophysical thinking from qualitative evidence accumulation to quantitative description has opened a way for introducing mathematical methods to generally understood Earth sciences. Many of the analyzed problems have been found out to be extremely complex so specific scientific methods needed to be developed. Let us concentrate on some of them.

Geophysical data used to infer the properties of the Earth's interior and/or provide information on processes undergoing in its depth are gathered practically on surface of the Earth or from the space. In such a case, an inference about the thought properties of the Earth or process in hand become a nontrivial inverse task. In consequence special data interpretation methods called inverse theory have been developed by geophysicists. The need for a quantitative estimation of accuracy of observational data analysis has lead to probabilistic formulation of the inverse theory.

Another astonishing example of development of general physical theory inspired by solid Earth problem analysis is the formulation of Asymmetric Continuum Media theory. This theory appears from very simple observations that damages of some constructions during earthquakes pointed out on relative rotational movements of different parts of constructions. However, the classical continuum elastic mechanics, which is the corner stone of contemporary seismology, predicts no rotational effects. Analysis of this apparent contradiction has lead to an idea of extending the classical theory of continuum elastic media.

In this chapter we concentrate on the mathematical and data analysis aspects of the geophysical research connected to the above tasks. This particular choice is by no means arbitrary. It simply follows from the fact of a significant contribution of IGF PAS to development of these particular types of geophysical research and personal authors' attachments.

2 Probabilistic Inverse Theory

2.1 General Concept

In geophysical investigations, when interpreting observational data we deal with the forward problem (also called forward modeling), and inverse problems. The forward problem aims at mathematical, or more often numerical simulations of given physical processes like energy release from seismic source, propagation of elastic waves, generation of electromagnetic signals at earthquake preparation stage, to name a few. On the contrary, the inverse tasks aim at quantitative description of process in hand. In this section we present the probabilistic approach to inverse problems, actively developed and promoted by IGF PAS.

While the forward problems intend to explain the nature of phenomena at hand, the inverse questions concern its precise and quantitative description. The simplest answer to such tasks would be a direct measurement of a quantity we are interested in. However, the capacity to carry out direct measurements is very limited in geophysics (Tarantola 2005). In cases where we cannot directly observe given parameters $\mathbf{m}$ we are interested in, we need to carry out an "indirect" measurement. We measure another parameter $\mathbf{d}$ and using the forward modeling relation (Eq. 1) try to infer information about the sought $\mathbf{m}$.

Contrary to the forward problem, inverse tasks are quite often non-unique, which happens frequently in the case of nonlinear multi-parameter inverse problems. Non-uniqueness means that there may be many different sets of parameters $\mathbf{m}$ which will predict the same observational effect (Tarantola 2005). It can happen, for example, when the observational data contain no information about the sought parameters or forward modeling results do not depend on the particular parameters. In any case, some sought parameters remain unresolved by experimental data. In such a situation, to obtain any solution we need to use an additional piece of information, called a priori information. It allows to choose the desired solution from the set of equivalent (from the observational point of view) models.

Let us consider the most often met inverse problem: a task of estimating unknown parameters. To achieve our goal we need:

- observational data
- a model (theory) which provides a theoretical prediction for any set of model parameters
- a priori information (expectations) about the sought parameters

All three elements: the data, model and a priori expectations provide us with some knowledge (information) about the problem. Experimental data tell us what the "reality" is. The model (theory) is a kind of theoretical knowledge (information) which allows us to predict the possible outcome of a given experiment. Finally, the a priori expectations come from subjective experience, previous experiments, knowledge accumulated during similar research and so forth. Thus, solving the parameter estimation task we actually use the above-mentioned three kinds of information and combine them into final a posteriori knowledge. Thus, inversion can be regarded not just as a mathematical method of fitting parameters to data but rather as a process of handling, accumulation and inference of pieces of information. This generalization which is the corner-stone of the probabilistic inverse theory (Tarantola 2005) allows to treat a variety of inverse problems like parameter estimation, error analysis, discrimination among different theories (models), planning new experiments and so forth in a homogeneous way.

Currently, the inverse theory faces now a new challenge in its development. In many applications, the classical solution leading to an optimum "best data fitting" model according to a selected optimization criterion is no longer sufficient. We need to know how plausible the obtained model is or, in other words, how large the uncertainties are in the final solutions (Malinverno 2002; Debski 2010). Actually, the necessity of estimating the inversion uncertainties within the parameter estimation class of inverse problems is one of the most important requirements imposed on any modern inverse theory. It can only partially be fulfilled within the classical approaches. For example, assuming Gaussian-type inversion errors, inversion uncertainty analysis can, in principle, be performed for linear inverse problems (see, e.g., Zhdanov 2002), although in the case of large inverse tasks like seismic tomography this can be quite difficult (Nolet et al. 1999; Yao et al. 1999). On the other hand, in the case of non-linear tasks a comprehensive evaluation of the inversion errors is usually impossible. In such a case, only a linearization of the inverse problem around the optimum model allows the inversion errors to be estimated, provided that the original nonlinearity does not lead to multiple solutions, null space, etc. (see, e.g., Debski 2004). The probabilistic technique based on the information inferring principle offers a very general, flexible and unified approach outperforms any classical inversion technique in such applications.

2.2 Inverse Problem: A Probabilistic Point of View

As already mentioned, solving the inverse problem can be regarded as an inference process in which available information is combined into the final, a posteriori knowledge of the system. The strict mathematical formulation of this inference process has been proposed by Tarantola (2005) in terms of the probabilistic language. The main points are the following.

Following the Bayesian interpretation of mathematical notion of probability, all available information, including theoretical predictions, priori knowledge, and posteriori one, can be represented by probability distributions (Tarantola 2005). According to this reasoning, the solution of the inverse problem is not a single, optimum model, but rather the a posteriori probability distribution. This makes the most important difference with respect to the classical approach which casts the inverse tasks into the optimization, parameter-fitting type problem. The crucial point of this approach is the proper construction of the a posteriori probability providing we have all necessary a priori observational and theoretical information.

Let us denote the data and model spaces by ($\mathcal{D}$) and ($\mathcal{M}$), respectively. These are space of all possible values of the measurements (data space) and possible values of the model parameters (model space) (Tarantola 2005). Let the forward (modelling) task be described by the G operator acting between these two spaces (Eq. 1), allowing calculation of observational effects (d) for a given model m.

$$\mathrm{d} = \mathrm{G}(\mathrm{m}) \tag{1}$$

Based on the Bayesian paradigm, each of the pieces of information we have in hand about d, m, and G can be described by an appropriate probability density functions (Tarantola 2005). We can join them using, for example, the Bayesian theorem to get the a posteriori probability $p(\mathrm{m}|\mathrm{d})$ expressing our belief that the true value of the thought parameters is m provided that we have measured the data d and the relation between m and q given in Eq. 1 is subjected to some modeling errors described by a conditional probability $p(\mathrm{d}|\mathrm{m})$. It reads (Tarantola 2005)

$$p(\mathrm{m}|\mathrm{d}) = k\, p_{apr}(\mathrm{m})\, p(\mathrm{d}|\mathrm{m}) \tag{2}$$

where k is the constant (normalization factor independent of m) and $p_{apr}(\mathrm{m})$ is the probability distribution describing the a priori information. The so-called likelihood function $p(\mathrm{d}|\mathrm{m})$ being the conditional probability of predicting the data d provided the model parameters are m is often taken in the form (Tarantola 2005; Debski 2010)

$$p(\mathrm{d}|\mathrm{m}) = \exp\left(-||\mathrm{d}^{obs} - \mathrm{G}(\mathrm{m})||\right) \tag{3}$$

where d^{obs} are data measured in the experiment.

Having defined the a posteriori distribution, the question is how to inspect it to extract the required information. The point is that in most of practical cases the a posteriori *PDF* is a complicated, multi-parameter function. Basically, there are two different strategies to explore the a posteriori probability density function, either by the evaluation of some point estimators or by the calculation of the marginal a posteriori *PDF* distributions.

The first approach relies on calculation of some integrals, among which the most popular are the lowest-order moments of the a posteriori *PDF* (Tarantola 2005; Debski 2010):

1. The maximum likelihood model

$$\mathrm{m}^{mll} = \max_{m \in \mathcal{M}} p(\mathrm{d}|\mathrm{m}), \tag{4}$$

2. The average model

$$\mathrm{m}^{avr} = \int_{\mathcal{M}} \mathrm{m}\, p(\mathrm{d}|\mathrm{m})\, \mathrm{dm}, \tag{5}$$

3. The covariance matrix

$$C_{ij}^{PO} = \int_{\mathcal{M}} (\mathrm{m}_i - \mathrm{m}_i^{avr})(\mathrm{m}_j - \mathrm{m}_j^{avr}) p(\mathrm{d}|\mathrm{m})\, \mathrm{dm}. \tag{6}$$

If a more comprehensive description of $p(\mathrm{d}|\mathrm{m})$ is required, higher-order moments can also be calculated (Jeffreys 1983).

The exhaustive description of the probabilistic inverse theory can be found, for example, in an excellent book by Tarantola (2005).

2.3 Practical Applications: Examples

For a long time the probabilistic inverse theory was treated as an interesting proposition of managing inverse problems but limited computational resources prevented using it in real geophysical applications. Situation started to change with a wide availability of high speed computers, and development of the very efficient numerical techniques of sampling in multidimensional spaces (see, e.g., Debski 2010; Sambridge 1999). Here we describe a few examples of such applications.

One of the first real seismological task which has been addressed by probabilistic inverse theory is location of seismic sources (Lomax et al. 2000; Gibowicz and Kijko 1994). In this problem we try to estimate the coordinates of hypocenter and rupture time of a seismic source on the basis of the recorded seismic waves. What probabilistic approach is bringing to the problem is an exhaustive and a reliable evaluation of location uncertainties (see, e.g., Lomax et al. 2000; Rudzinski and Debski 2012). It allows also a comprehensive analysis of efficiency and accuracy of different location techniques (Rudzinski and Debski 2012) and seismic network sensitivity (Gibowicz and Kijko 1994). While in first applications of the Bayesian technique the source location problem was solved by using observed time onsets of given seismic waves, now it is possible to use the full seismograms to enhance the location reliability or to locate events like volcanic tremors, ice-quake tremors etc. (see, e.g., Larmat et al. 2008). The idea behind this approach relies in two elements: using the time-reversal symmetry of the wave equation to perform the back-propagation of the recorded signal (the so-called time-reversal mirroring; (Fink 1997) and performing implicit sampling simultaneously with back-propagation of the recorded seismograms.

Another seismological analysis where probabilistic approach has been proved to be very important is an analysis of kinematics of seismic sources including such detailed tasks like energy release, rupture duration, static and dynamic stress drop to name a few (Debski 2008; Kwiatek 2008). This, very fundamental seismological task requires a very careful analysis of inversion uncertainties, because the obtained results are being currently the only available observational information on rupture physics. Thus, their interpretation in terms of physical models of material failure is critically dependent on the inversion errors. Again, the probabilistic approach provides tools for the reliable estimation of the final uncertainties in the solutions found. Debski (2008) has managed to obtain the relative source time function for mining induced events with magnitude between 2.4 and 3.0. He has demonstrated that while the obtained source time functions suggested some complexity of the rupture process and two modal energy releases, the analysis of the a posteriori errors showed that such an interpretation has no justification. Kwiatek (2008) performed a similar analysis to check weather the complexity of calculated source time functions was only a numerical artefact or a reliable signature of the complexity of the rupture processes.

The maturity of the probabilistic inversion technique allows to apply the method to large scale inversion problems like tomographic imaging of the velocity heterogeneities at least in the regional or local scales (Debski 2013). Some first attempts of the full seismic waveform inversion have also been recently undertaken (Bodin and Sambridge 2009). In tomographic applications the possibility of evaluation of the reliability of obtained results is again the biggest advantage of the method. The fully nonlinear estimation of the a posteriori errors and their spatial distribution possible in the framework of the approach significantly outperforms other semi-quantitative methods based on various types of resolution tests (Debski 2013). Moreover, using the advanced techniques of exploring the a posteriori distribution based on trans-dimensional sampling techniques (Green 1995; Bodin and Sambridge 2009) allows to optimally adjust achieved spatial resolution to data at hand.

The applications of the probabilistic inversion, as discussed above, were essentially the classical parameter estimation tasks. What really brings the probabilistic technique to this problems is the possibility of an exhaustive error analysis. However, the power of the method is most apparently visible when non-parametric inversion tasks are concerned. One of such problems is discussed by (Debski and

Tarantola 1995) in the context of the AVO (Amplitude Vs. Offset) prospecting techniques. The question which authors posed and answered was whether the AVO data can distinguish between different sets of physical parameters used equivalently to describe the elastic medium. If so, which parameters are the best resolved by seismic data? This is an example of the non-parametric inverse problem, where we are interested not in an (accurate) estimation of numerical values of some physical parameters, but we are comparing different relations between physical parameters. The answer was, that in a wide angle refraction experiment the P-wave impedance contrast and the Poisson's ratio are best resolved by the data, while the density remains unresolved. A similar problem has recently been addressed in the context of verification of assumptions about characteristics of the a posteriori probability functions constructed for time-reversal based location algorithm when statistic of observational or/and modeling errors are unknown. None of these tasks can be easily treated with the classical approach.

3 Rotational Waves and Asymmetric Theory of Elasticity

Classical theory of elasticity (Aki and Richards 1985) which describes the behavior of the elastic material under a small external perturbation is the symmetric theory in the sense that the strain and stress tensors remain symmetric. Let us denote by u a displacement field in the body and let x_i be a set of Cartesian coordinates $x_i = (x, yz)$. Than, the strain tensor for infinitesimally small deformation u can be written down in the differential form (Aki and Richards 1985)

$$E^c_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{7}$$

According to the constitutive relation, the deformation of the body described by the strain tensor E^c_{ij} is related to stresses acting on/within the body: any deformation rises internal stresses and vice versa any external load causes deformation of the body. In the case of perfectly elastic, isotropic and homogeneous medium, this constitutive relation takes the form of Hook's law

$$\sigma_{ij} = \lambda \delta_{ij} E^c_{kk} + 2\mu E^c_{jk} \tag{8}$$

where σ_{ij} is the stress tensor, δ_{ij} stands for the Kronecker's symbol, summation over repeated indexes is assumed and λ and μ are two Lame's parameters. This relation is symmetric with respect to interchange of i and j indexes.

The wave equation describing the propagation of small disturbances can now be obtained be equaling the total internal forces $\partial \sigma_{ij}/\,\partial x_i$ to the derivative of momentum for infinitesimally small volume of body and reads (Aki and Richards 1985)

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \sum_j \frac{\partial \sigma_{ij}}{\partial x_j} + F_i(\mathrm{x}, t) \tag{9}$$

where a source term $F_i(x, t)$ describing external forces acting on the medium has been introduced. The solution of this equation consists of two types of waves called P and S waves propagating with constant velocities (Aki and Richards 1985; Gibowicz and Kijko 1994)

$$\begin{aligned} v_p &= \sqrt{(\lambda + 2\mu)/\rho} \\ v_s &= \sqrt{\mu/\rho}. \end{aligned} \tag{10}$$

These two types of waves differ essentially in the sense of movements of medium's particle through which waves are passing. In case of P waves, the particles are moving parallel to wave propagation direction. In case of the S-waves, the particle's displacement is perpendicular to the wave propagation direction and the waves can obviously have two perpendicular polarizations. From mathematical point of view. the P-wave corresponds to the solution with null rotation, while the S-wave is the solution with the null divergence. In this classical elastodynamics any mechanical wave in an infinite elastic body can be described as a superposition of these two waves.

However, as it has already been mentioned, the observations of damages caused by earthquakes provide evidence on rotational movements connected with seismic waves (Trifunac 2009; Zembaty 2009). Being inspired by this observation, Teisseyre and Gorski (2009) proposed another way of extending the classical elastodynamics towards the theory including rotational effects. The basic idea is to split the total (observable) rotations and displacements into elastic and "internal" or self-field components connected with existing internal micro-structure of the medium. This basic idea is implemented in terms of the shearing and rotational stress tensors rather than the displacements like in the classical approach (Teisseyre and Gorski 2007) The idea of using stresses as the "elementary fields" instead of displacements (deformations) comes from observations that in the solid continuum point transports and point rotations are actually non-important (rotations can even be hardly defined at the continuum level) but an important role is played by the deformations between neighborhood points, that is, the rotational and shear stresses. We shall come back to this point latter on.

The main points of the theory are following.

Let us consider waves propagation in solids. The wave motion equations follow directly from the derivatives of the classic Newton formula. Balancing all acting forces within the body we obtain the equation of motion similar to those in the classical elasticity theory. However, the deformation tensor E is now assumed to have both symmetric and antisymmetric parts and in general can be decomposed as follows

$$E_{ij} = \frac{\partial u_i}{\partial x_j}; \quad E_{ij} = E_{(ij)} + E_{[ij]} = \delta_{ij}\bar{E} + \hat{E}_{(ij)} + \breve{E}_{[ij]} \tag{11}$$

where u is the displacement field, symbols (ij) and $[ij]$ stands for symmetric and antisymmetric parts of the appropriate tensors, $\bar{E} = \frac{1}{3}\sum_i E_{ii}$ describes the elastic deformation part while $\hat{E} = E_{ij} - \delta_{ij}\bar{E}$ and $\breve{E}$ are symmetric and antisymmetric self-field deformations due to an internal micro-structure of the body. Similar decomposition expressions for stress tensor can be easily obtained assuming linear Hook's law (Teisseyre and Gorski 2007). The resulting set of equation describing propagation of seismic waves in the body reads

$$\begin{aligned}
&(\lambda + 2\mu)\sum_s \frac{\partial^2 E}{\partial x_s \partial x_s} - \rho\frac{\partial^2 E}{\partial t^2} = 0 \\
&\mu\sum_s \frac{\partial^2 \hat{E}_{(ij)}}{\partial x_s \partial x_s} - \rho\frac{\partial^2 \hat{E}_{(ij)}}{\partial t^2} + (\lambda + \mu)\left(3\frac{\partial^2 \bar{E}}{\partial x_i \partial x_j} - \delta_{ij}\sum_s \frac{\partial^2 \bar{E}}{\partial x_s \partial x_s}\right) = 0 \\
&\mu\sum_s \frac{\partial^2 \breve{E}_{[ij]}}{\partial x_s \partial x_s} - \rho\frac{\partial^2 \breve{E}_{[ij]}}{\partial t^2} = 0
\end{aligned} \tag{12}$$

where the external force term was omitted. When considering only shear and rotation strains (at constant pressure) we will obtain:

$$\begin{aligned}
&\mu\sum_s \frac{\partial^2 \hat{E}_{(ij)}}{\partial x_s \partial x_s} - \rho\frac{\partial^2 \hat{E}_{(ij)}}{\partial t^2} = 0 \\
&\mu\sum_s \frac{\partial^2 \breve{E}_{[ij]}}{\partial x_s \partial x_s} - \rho\frac{\partial^2 \breve{E}_{[ij]}}{\partial t^2} = 0
\end{aligned} \tag{13}$$

The interpretation of the above relation is that during an earthquake, or other fracture process, a few mechanisms are activated. First seismic waves are generated following the release-rebound mechanism (see, e.g., Teisseyre 2011): molecular bonds are breaking in the source area which leads to a release of rotation and shear fields. They are subsequently converting each other: rotation generates a shear field and vice versa, release of shears are generating rotation strains. Without such a mutually combined system it is difficult to understand the related propagation motions at least at meter scales (seismological) much larger than typical scales for molecular dynamics.

A difference between the Classic and Asymmetric theories shortly sketched above follows from the assumption that we have a simultaneous appearance of a number of fields independently released; we admit an independent release of some physical fields, e.g., shear strains or rotation strains, in an earthquake source. Formally, these released fields might be expressed again by some displacements, but such displacements do not exist in a continuum—we prefer to treat them as the potential fields only. In fact the recording of the seismic waves, even very long, means that we record only the deformation which becomes integrated during an adequate time by a seismometer to reveal the displacement motion.

The assumption about symmetricity of the observed tensors leads to the constraints on the symmetric and antisymmetric parts of the total stress and strain tensors. Simplifying the problem one can say that the symmetric parts of tensors can now be connected mainly to the translational deformations while the antisymmetric ones to the rotations of nuclei in the body. From seismological point of view, the most important conclusion of the theory is that an existence of grains (or other nuclei) within the crust (more generally some micro-structure) allows propagation of rotational waves—an additional type of elastic waves not predicted by the classical theory of elasticity.

The proposed antisymmetric elastodynamics is mathematically a very elegant theory and it is already supported by seismological observations. However, its importance goes much beyond seismology and is actually related to a very basic principles underlying the continuum mechanics. Let us shortly discuss this point. Any macroscopic, rigid body (when plastic effects, creeping, and similar non-elastic effects are excluded) has six degrees of freedom: three translations and three rotation. They are canonically connected to three components of the momentum vector and moment of momentum, respectively. The continuum mechanics is built under the assumption that a given medium can be described as a set of infinitesimally small elements for which all forces can be balanced and this way the equation of dynamic evolution is constructed. The limit of the infinitesimally small elements is necessary to describe the medium dynamics in terms of differential equation. However, a way of approaching this limit is by no means obvious. The classical approach leads to disregarding the rotation and moment of momentum of such infinitesimally small elements. In consequence, the strain and stress tensors becomes symmetric. The other approach leads to micromorphic continuum mechanics (Eringen 1999). The asymmetric theory discussed here is another attempt of dealing with this "limit problem" which could be applied at the seismological scales (1 m–1 km) without going to microscopic scale of, for example, the micromorphic model of Eringen (1999).

4 Flow of Incompressible Viscous Fluids in Porous Media

The modeling of porous media and fluid flows through the media has a significant role in the Earth sciences, i.e., geophysics, seismology, exploration of underground resources or CO_2 sequestration to name a few. The properties of permeability of porous media and physics of volatile transport through the medium play a crucial role in building of appropriate mathematical models to be used in practical applications. For example, in seismology the pore fluids are known to be factor very important in earthquake triggering. Secondly, exploration of natural resources, i.e., oil, gases, or geothermal heated water is related to flow through a porous strata. Contamination transport is another example of an application of poromechanics. Last but not least, the underground acoustics describes wave propagating in the uppermost part of the strongly water-saturated, porous, ocean-bottom sediment layers.

In continuum mechanics, there are at least two approaches of the description and modeling of porous media. One approach is the phenomenological one (Coussy 2004) whereas the other is based on a contemporary continuum micromechanics and mathematical methods, based especially on the theory of asymptotic homogenization (see, e.g., Mikelic 2000). We use the second approach which exploits micro-structure of the medium.

The main task of the description of the porous media is to determine the filtration law, the classic example of which is the so-called Darcy's law which states that

$$\mathrm{v} = -\mathrm{K}(\nabla p - f), \tag{14}$$

where v is the vector of fluid velocity. K is positive permeability coefficient (matrix for anisotropic flows), p is a pressure and f are forces (gravitational, for example).

The homogenization method is particularly convenient to address this issue, because it does not require any a priori knowledge of an infiltration mechanisms. It follows from the suitable assumptions on the micro-structure of a porous skeleton. Let us shortly discuss this task important for geophysical problems.

In seismology, it is important to study properties of one-phase fluid flow through a porous medium, linearly deformable, i.e., liable to a small elastic deformation. The presence of fluid is of importance here because it influences such characteristics of the medium like seismic wave speed, intrinsic attenuation or seismic wave anisotropy (see, e.g., Carcione 2001).

The fluid flow is governed by the non-stationary Stokes equation. In typical geophysical applications the fluid can be assumed incompressible, the Reynolds number is low and for simplicity the periodic micro-structure of the porous skeleton can be assumed. In this case, the homogenization can be carried out in the following way.

Let us denote by $\mathrm{u} = \mathrm{u}(x, t)$ and $\mathrm{v} = \mathrm{v}(x, t)$ the displacement vector of solid phase and the velocity field of fluid phase, respectively; the tensor of elastic properties of the solid part is referred to as $\mathbf{C}$, and F^L and F^S are forces of liquid and solid part, respectively. Tensor e(u) is the symmetric gradient of the vector field u and reads

$$\mathrm{e}(\mathrm{u}) = \frac{1}{2}(\nabla \mathrm{u} + \nabla \mathrm{u}^T) \tag{15}$$

or in the index notation

$$e_{ij}(\mathrm{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{16}$$

The system of equations for fluid flow through an elastic skeleton now reads:

$$\varrho^S \ddot{\mathrm{u}}^\varepsilon = \nabla \cdot \mathrm{Ce}(\mathrm{u}^\varepsilon) + \mathrm{F}^S, \qquad x \in \Omega_\varepsilon^S \tag{17}$$

$$\varrho^L \dot{\mathrm{v}}^\varepsilon = \varepsilon^2 \eta \nabla \cdot \mathrm{e}(\mathrm{v}^\varepsilon) - \nabla p^\varepsilon + \mathrm{F}^L, \qquad x \in \Omega_\varepsilon^L \tag{18}$$

$$\nabla \cdot \mathrm{v}^{\varepsilon} = 0, \qquad x \in \Omega_{\varepsilon}^{L}. \tag{19}$$

where η is the constant viscosity of the fluid. Here ϱ^S and ϱ^L are mass densities of solid and fluid part, respectively. The small parameter $\varepsilon << 1$ is used to characterize the micro-structure of the reservoir Ω. In the process of homogenization, the small parameters tend to zero and the asymptotic solution is obtained.

The reservoir occupied by porous medium Ω is divided onto two parts, Ω_S^{ε} and Ω_L^{ε}. The subscripts S and L denote solid and liquid part of the reservoir Ω, respectively. Interface between Ω_S^{ε} and Ω_L^{ε} is denoted by Γ^{ε}. For the solid part, the equation of linear theory of elasticity is satisfied and in the liquid part the Stokes equation of incompressible fluid is valid. The appropriate boundary and initial conditions must be added according to a physical problem considered; for example, we can assume that $\mathrm{u}^{\varepsilon} = \mathrm{v}^{\varepsilon}$ on the interface Γ^{ε}. It means that in this case the so-called non-slip condition is posed.

Homogenization means passing with parameter ε to zero. The small parameter characterizes the ratio of diameter (d) of periodic cell Y and diameter of reservoir (D). Usually we set $\varepsilon = \frac{d}{D} < < 1$. This method requires at least two scales, say macroscopic denoted traditionally by $x \in \Omega$ and the microscopic denoted by $y = \frac{x}{\varepsilon} \in Y$, where Y is the so-called periodicity cell.

Through homogenization of the Stokes equations we can obtain the effective infiltration law and the dynamic equations for the poroelastic medium. The obtained equations describing the non-local in time filtration law are much more general than the Biot model providing important extensions and can be expressed in the form

$$\mathrm{v}_{rel}(x,t) = \frac{1}{\rho^L} \int_0^t \mathrm{A}(t-\tau)\big(F^L(\tau,x) - \varrho^L \ddot{u}(\tau,x) - \nabla p(\tau,x)\big) d\tau, \tag{20}$$

where A is the permeability tensor obtained from the solution of an auxiliary *local problem* on the periodicity cell Y described by a set of equations:

$$\varrho^L \dot{\mathrm{w}}^{(i)}(y,t) = \eta \Delta \mathrm{w}^{(i)}(y,t) - \nabla_y q^{(i)}(y,t) + \mathrm{e}^i \qquad \text{in } Y \tag{21}$$

$$\nabla \cdot \mathrm{w}^{(i)}(y,t) = 0 \qquad \text{in } Y \tag{22}$$

$$\mathrm{w}(y,t) = 0 \qquad \text{on } \partial Y. \tag{23}$$

Then, the permeability matrix $\mathrm{A}(t)$ given by

$$A_{ij}(t) = \frac{1}{|Y|} \int_{Y_L} \frac{\partial \mathrm{w}^{(i)}(t,y)}{\partial t} \cdot \mathrm{e}_j \, \mathrm{d}y \tag{24}$$

is symmetric and positive definite.

The results of the study on porous media using homogenization theory, are included in the papers, for example, by Bielski et al. (2001); Bielski (2005); Bielski and Wojnar (2008); Telega and Bielski (2003).

Another important case is the two-phase fluid flow, i.e., dynamics of two immiscible fluids in a porous medium. In this case, the flow is significantly influenced by the presence of capillary forces at the interfaces between two fluids.

Modeling of hierarchical porous media appears to be of significance since over 20 % of the word oil reservoirs are found in rocks with double porosity properties. Some of the first models were introduced by Warren and Root (1963). The issue of description of the double-porosity properties can also be conveniently undertaken using the homogenization method. In this case we assume two scales of rock porosity. The results of this approach and modeling of double-porosity media may be found in the paper by Bielski and Wojnar (2008).

Porous media frequently exhibit a random micro-structure. Thus, it is crucial to derive a stochastic model of the media. The book by Torquato (2002) summarizes the results of finding effective quantities for the media with random micro-structures. In the paper Telega and Bielski (2003) various approaches to micro-macro passage for random porous media were reviewed. The results obtained by stochastic homogenization are very similar to those obtained for periodic micro-structures and the macroscopic filtration law (generalized Darcy's law) is expressed as

$$E(\chi_F(\mathrm{v}-\dot{\mathrm{u}})) = \frac{1}{\varrho^L}\int_0^t \mathrm{A}(t-s,\omega)\left[F^L(s,x)-\varrho^L\ddot{\mathrm{u}}(s,x,\omega)-\nabla_x p(s,x,\omega)\right]ds, \quad (25)$$

where $E()$ means the expected value over the probabilistic space and ω is an element of this space. The permeability matrix is defined by

$$A_{ij}(t,\omega) = E(\chi_F(\omega)\dot{\mathrm{w}}^{(i)}(t,\omega)\cdot \mathrm{e}_j), \quad i,j = 1,\ldots,n, \quad (26)$$

where e_j denotes the jth standard basis vector in $\mathbb{R}^n$. The matrix A_{ij} is symmetric and positive definite. The functions $\mathrm{w}^{(i)}, q^{(i)}$ are solutions to the following flow cell problem:

$$\varrho^L\frac{\partial \mathrm{w}^{(i)}}{\partial t} + \nabla_\omega q^{(i)} - \eta\nabla_\omega \mathrm{w}^{(i)} = \mathrm{e}_i \quad \text{on } (0,\tau)\times F, \quad (27)$$

$$\nabla_\omega\cdot \mathrm{w}^{(i)}(t,\omega) = 0 \quad \text{on } (0,\tau)\times F \quad (28)$$

$$\mathrm{w}^{(i)}(t,\omega) = 0 \quad \text{on } (0,\tau)\times \Gamma(\omega). \quad (29)$$

The details on this stochastic calculus and results obtained are discussed by Telega and Wojnar (2007). Finally, let us note that the study of porous media has become an interdisciplinary area and the range of scientific and engineering applications of the poro-mechanics is still increasing.

5 Conclusion

In this chapter we have presented some selected elements of theoretical, data processing, and mathematical tasks which appear in various aspects of the solid Earth physical analysis emphasizing contributions of authors to their formulation, development and applications. The selection is by no means complete and many important elements have been obviously omitted. We made it keeping in mind first of all our personal competence in their description and secondly trying to make the review logically consistent and concise. Some of the presented techniques are further discussed in more depth in other chapters of this book.

The selected topics illustrate the broad range of a scientific activity undertaken and continuing at the IGF PAS with respect to the solid Earth problems. They cover advanced mathematical methods, an attempt of building more realistic physical models of wave propagations, and finally the advanced technique of observational data analysis. This three elements form a kind of a "full chain" of methods used for solving problems at hand. Such an approach, always visible in IGF PAS activity, is motivated by two complementary factors. Firstly, by the fact that all research activity we undertook in the past and are also addressing today always refers to real problems met by seismology and physics of the Earth and all the elements mentioned above are usually necessary. The second factor is connected to different personal attitudes of researchers towards mathematics, theoretical physics, observational data analysis, and so on. The topics covered in this chapter are just illustration of these facts.

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