# Berge-Zhukovskii Optimal Nash Equilibria

Noémi Gaskó, Mihai Suciu, Rodica Ioana Lung, and Dan Dumitrescu

Babes-Bolyai University of Cluj-Napoca, Romania

**Abstract.** The Berge-Zhukovskii optimal Nash equilibrium combines the properties of the popular Nash equilibrium with the ones of the less known Berge-Zhukovskii by proposing yet another Nash equilibrium refinement. Moreover, a computational approach for the detection of these newly proposed equilibria is presented with examples for two auction games.

## 1 Introduction

The most used equilibrium concept in non-cooperative game theory is the Nash equilibrium [16], which assumes rational players that care only about themselves and make rational choices to achieve the best possible payoffs. There are many criticisms brought to Nash equilibria: it does not ensure the highest payoff for players (e.g. trust games, social dilemmas); in some cases it may not exist in pure form and in other cases a game may present an infinite number of equilibria [11].

One way to solve the problem of multiple equilibria is to consider some refinements of the Nash equilibrium (e.g. strong Nash [2], coalition proof Nash [4], etc.). Another way to approach the problem is to propose alternatives, such as the Berge-Zhukovskii (BZ) equilibrium [21]. While Nash equilibrium is stable against unilateral deviations of players, BZ is stable against deviations of all other players. Berge-Zhukovskii equilibrium can be interpreted as an other-regarding, altruistic equilibrium [5].

The Berge-Zhukovskii optimal Nash (BZON) equilibrium as a new refinement for Nash equilibrium is introduced that presents BZ properties, which are characterized by the use of a generative relation.

A Differential evolution (DE) [6] algorithm based on the aforementioned generative relation is used to compute BZON equilibria for first-price and second-price auction games illustrating the approach.

The article is organized as follows: the next section presents some basic game theory notions and concepts. The DE is described in Section 3. The fourth section presents numerical experiments. The paper ends with Conclusions.

## 2 Game Equilibria

A finite strategic non-cooperative game is defined by the set of players involved in the game, a set of possible actions associated with each player and their corresponding payoffs. Formally, a game is a system $G = ((N, S_i, u_i), i = 1, ..., n)$, where:

A.-A. Tantar et al. (eds.), *EVOLVE - A Bridge between Probability, Set Oriented Numerics, and Evolutionary Computation V*, Advances in Intelligent Systems and Computing 288,
DOI: 10.1007/978-3-319-07494-8_4, 

- $N$ represents the set of players, $n$ is the number of players;
- $S_i$ is the set of actions available to player $i \in N$, and $S = S_1 \times S_2 \times ... \times S_n$ is the set of all possible strategies of the game, $s = (s_1, ...s_n) \in S$ is a strategy (or strategy profile) of the game;
- for each player $i \in N$, $u_i : S \to R$ represents the payoff function for player $i$.

The most popular and used equilibrium concept is the Nash equilibrium [16]. When playing in Nash sense no player can improve its payoff by deviating from its strategy only by himself.

Let us denote by $(s_i, s^*_{-i})$ the strategy profile obtained from $s^*$ by replacing the strategy of player $i$ with $s_i$ : $(s_i, s^*_{-i}) = (s^*_1, ..., s_i, ...., s^*_n)$.

**Definition 1 (Nash equilibrium).** *A strategy profile $s^* \in S$ is a Nash equilibrium if*

$$u_i(s_i, s^*_{-i}) \leq u_i(s^*),$$

*holds* $\forall i = 1, ..., n, \forall s_i \in S_i$.

We will denote by

$$S_{-i} = S_1 \times ... \times S_{i-1} \times S_{i+1} \times ... \times S_n,$$

with

$$s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_n)$$

and

$$(s^*_i, s_{-i}) = (s_1, s_2, ..., s^*_i, ..., s_n).$$

A more general equilibrium concept is the Berge equilibrium [3].

**Definition 2 (Berge equilibrium).** *Let $M$ be a finite set of indices. Denote by $P = \{P_t\}, t \in M$ a partition of $N$ and $R = \{R_t\}, t \in M$ be a set of subsets of $N$. A strategy profile $s^* \in S$ is an equilibrium strategy for the partition $P$ with respect to the set $R$, or simply a Berge equilibrium strategy, if and only if the condition*

$$u_{p_m}(s^*) \geq u_{p_m}(s_{R_m}, s^*_{N-R_m})$$

*holds for each given $m \in M$, any $p_m \in P_m$ and $s_{R_m} \in S_{R_m}$.*

If we consider that each class $P_i$ of partition $P$ consists of one player $i$ and each set $R_i$ is the set of players $N$ except $i$, we obtain the Berge-Zhukovskii equilibrium [21]. We have $M = N$, $P_i = \{i\}$ and $R_i = N - i, \forall i \in N$, .

Playing in Berge-Zhukovskii sense can be interpreted as each player maximizing the payoff of the other players. More formally:

**Definition 3 (Berge-Zhukovskii equilibrium).** *A strategy profile $s^* \in S$ is a Berge-Zhukovskii equilibrium if the inequality*

$$u_i(s^*) \geq u_i(s^*_i, s_{-i})$$

*holds for each player $i = 1, ..., n$, and all $s_{-i} \in S_{-i}$.*

The strategy $s^*$ is a Berge-Zhukovskii equilibrium, if the payoff of each player $i$ does not decrease considering any deviation of the other $N - \{i\}$ players.

Let $Q \subset S$ and $s^* \in S$ a strategy profile. Then $s^*$ is a *Berge-Zhukovskii equilibrium with respect to* $Q$ if the inequality

$$u_i(s^*) \geq u_i(s_i^*, q_{-i})$$

holds for each player $i = 1, ..., n$, and all $q_{-i} \in Q_{-i}$.

**Definition 4 (Berge-Zhukovskii Optimal Nash).** *A Nash equilibrium profile that is also a Berge-Zhukovskii with respect to the entire set of Nash equilibria is called a Berge-Zhukovskii optimal Nash equilibrium of the game.*

A Berge-Nash equilibrium is introduced in [1] as a Berge equilibrium which is also a Nash equilibrium.

Figure 1 illustrates the connection between all these equilibria.

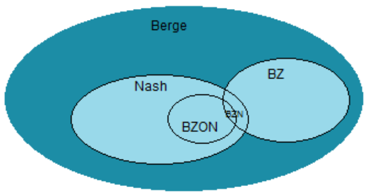


**Fig. 1.** Connection between Berge, Berge-Zhukovskii, Nash and BZON equilibria

## 3 BZON Detection

Due to all the similarities between multiobjective optimization problems [8] and non-cooperative games - the most important one being that both of them aim to optimize several payoff/objective functions in the same time - it is natural to assume that multiobjective optimization algorithms can also be used for game solving. Pareto based evolutionary approaches particularly can be suitable as they rely on the Pareto domination relation which can be changed with other relations redirecting the search towards different types of solutions/equilibria. However, the challenge is to find the appropriate relation for each equilibrium type.

Such a relation has been defined for Nash equilibria in [14] by using a quality measure $k(s, q)$ denoting the number of players that benefit from unilaterally switching their choices from $s$ to $q$:

$$k(s, q) = card\{i \in N, u_i(q_i, s_{-i}) \geq u_i(s), q_i \neq s_i\},$$

where $card\{M\}$ denotes the cardinality of the set $M$.

**Definition 5.** *Let $q, s \in S$. We say the strategy $q$ is better than strategy $s$ with respect to Nash equilibrium ($q$ Nash ascends $s$, and we write $q \prec_N s$, if the following inequality holds:*

$$k(q, s) < k(s, q).$$

**Definition 6.** *The strategy profile $q \in S$ is called Nash non-dominated, if and only if there is no strategy $s \in S, s \neq q$ such that*

$$s \prec_N q.$$

The relation $\prec_N$ is a generative relation for Nash equilibrium in the sense that the set of non-dominated strategies with respect to $\prec_N$ is equal to the set of Nash equilibria [14].

A similar quality measure exists also for the Berge-Zhukovskii equilibrium [9], counting how many players would benefit when all the others change their strategies from $s$ to $q$ :

$$b(s, q) = card\{i \in N, u_i(s) < u_i(s_i, q_{-i}), s_{-i} \neq q_{-i}\},$$

**Definition 7.** *Let $s, q \in S$. We say the strategy $s$ is better than strategy $q$ with respect to Berge-Zhukovskii equilibrium ($s$ BZ-dominates $q$), and we write $s \prec_B q$, if and only if the inequality*

$$b(s, q) < b(q, s)$$

*holds.*

**Definition 8.** *The strategy profile $s^* \in S$ is a Berge-Zhukovskii non-dominated strategy (BZ nondominated), if and only if there is no strategy $s \in S, s \neq s^*$ such that $s$ dominates $s^*$ with respect to $\prec_B$ i.e.*

$$\nexists s \in S : s \prec_B s^*.$$

Relation $\prec_B$ is a generative relation of the Berge-Zhukovskii equilibrium.

The BZON equilibria can be computed by combining the above mentioned relations in the following manner:

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1. The Nash ascendancy relation is checked first.
2. The BZ domination second.
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Thus a new generative relation for BZON is obtained.

**Crowding Based Differential Evolution.** The Crowding based Differential evolutionary algorithm (CrDE) [20] can be easily adapted to compute different types of equilibria for static and dynamic games [10], [15], [19]. In a similar manner it can be used to compute BZON equilibria, by simply using the proposed generative relation when comparing two individuals.

# 4 Case Study: Auction Games

## 4.1 Prerequisites

Auction theory is important in economic transactions where an auction is a well defined micro-economic environment (for a survey see [12] or [18]).

The four main categories of auctions with complete information [7] are:

– first-price sealed bid auction - each bidder submits her/his own bid without seeing others bids, and the object is sold to the highest bidder at her/his bid;
– second-price sealed bid auction (Vickrey auctions)- each bidder submits her/his own bid, the object is sold to the highest bidder, who needs to pay only the second highest price for the object;
– open ascending-bid auctions (English auctions) - each bidder offers increasingly higher bids, the auction stops when no bidder wants to make a higher bid. The winner with the highest bid wins the object and needs to pay her/his own bid;
– open descending-bid auctions (Dutch auctions) - can be considered the inverse of the English auction, the initial price of the object is set by the auctioneer, the bidders lower the price, until there is no new bid. The winner pay her/his own bid.

Besides the mentioned categories other auction types are: all-pay auctions, Amsterdam auctions, unique bid-auctions, etc.

## 4.2 Game Theoretic Model of Auctions

Auction theory can be approached from different views: from a game theoretical perspective [13], from a market microstructure view, etc.

From a game theoretical view the auction has the following elements:

– players - the $n$ bidders, $n \geq 2$;
– actions - the set of possible bids ($b_i$ for the $i$th player );
– payoff function - depending on the type of auction, the player with the maximum bid gains the difference between the value of the object and maximum bid (or the difference between the value and the second highest bid - in second-price sealed bid auction);

We analyze some class of auctions from a game theoretical perspective. The first-price and second-price sealed bid auctions have several Nash equilibria. The aim is to show that it is possible to evolutionary compute the BZON equilibria.

## 4.3 Numerical Experiments

CrDE was run by using parameters presented in Table 1. For each experiment ten different runs were conducted.

**Table 1.** Parameter settings for CrDE used for the numerical experiments

| Parameter | |
|---|---|
| Population size | 50 |
| Max no evaluations | $2 \times 10^7$ |
| CF | 50 |
| F | 0.5 |
| Crossover rate | 0.9 |

**First-Price Sealed Bid Auction.** In the first-price sealed bid auction two players cast their bid independently. The value of the bidding objects is $v_1$ for the first player and $v_2$ for the second one. The winner is the highest bidder, who needs to pay his own bid ($b_i$). We can specify a simple agreement: if both have equal bids, the winner is the first bidder (another variant is to randomly choose a winner).

The payoff functions are the following [17]:

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_1, & \text{if } b_1 \geq b_2, \\ 0, & \text{otherwise.} \end{cases}$$

$$u_2(b_1, b_2) = \begin{cases} v_2 - b_2, & \text{if } b_2 > b_1, \\ 0, & \text{otherwise.} \end{cases}$$

The game has several Nash equilibria as any $v_1 \leq b_1^* = b_2^* \leq v_2$ is a Nash equilibrium of the game. The BZON equilibrium of the game is a single strategy profile $(b_1, b_2) = (v_1, v_1)$.

For numerical experiments we consider the object values $v_1 = 5$, $v_2 = 3$, naturally this means that the maximal bid is less than $5$ (nobody will cast more than the value of the object).

Numerical experiments over ten independent runs detect correctly the strategy profile $(b_1, b_2) = (v_2, v_2) = (3, 3)$. (with standard deviation $0.0$).

Figure 2 illustrates the strategies space by using 50000 randomly generated strategies; the Nash equilibrium, Berge-Zhukovskii equilibrium and the Berge-Zhukovskii optimal Nash equilibrium of the game are presented. In Figure 3 the corresponding payoffs are depicted.

Let us consider the $n$-person version of the game, the payoff functions can be described as follows:

$$u_i(b_1, ..., b_n) = \begin{cases} v_i - b_i, & \text{if } b_i = max\{b_1, b_2, ..., b_n\} \\ 0, & \text{otherwise.} \end{cases}$$

For the three player version of the game with the object values $v_1 = 5$, $v_2 = 4$, $v_3 = 3$ we also have infinite set of Nash equilibria but only one (correctly detected) BZON equilibria $(b_1, b_2, b_3) = (4, 4, 0)$. It is an advantage that the bidder does not need to play the entire value of the bidding object.

**Second-Price Sealed Bid Auction.** In a second-price auction game the winner needs to pay the second highest bid, consider the two player version of the game [17]:

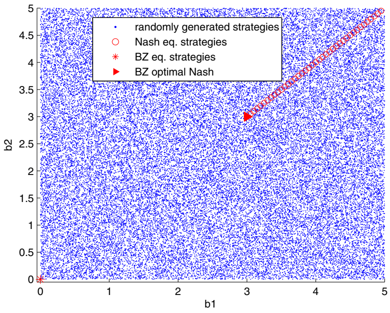


**Fig. 2.** Randomly generated strategies, Nash equilibria, Berge-Zhukovskii and Berge-Zhukovskii optimal Nash equilibrium strategies

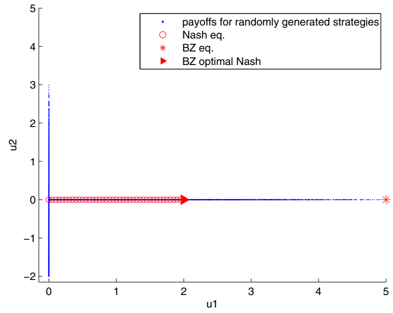


**Fig. 3.** Corresponding payoffs for randomly generated strategies, Nash equilibria, Berge-Zhukovskii and Berge-Zhukovskii optimal Nash equilibrium payoffs

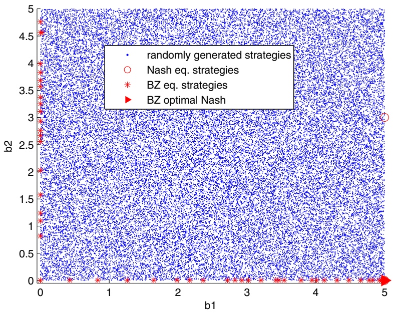


**Fig. 4.** Randomly generated strategies, Nash equilibria, Berge-Zhukovskii and Berge-Zhukovskii optimal Nash equilibrium strategies

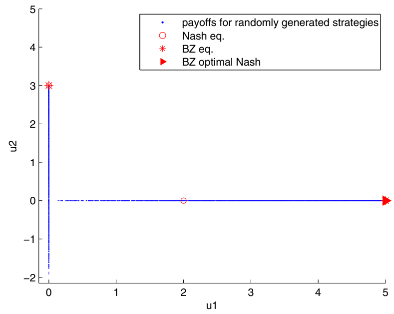


**Fig. 5.** Corresponding payoffs for randomly generated strategies, Nash equilibria, Berge-Zhukovskii and Berge-Zhukovskii optimal Nash equilibrium payoffs

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2, \text{ if } b_1 \geq b_2, \\ 0, \qquad\quad \text{otherwise.} \end{cases}$$

$$u_2(b_1, b_2) = \begin{cases} v_2 - b_1, \text{ if } b_2 > b_1, \\ 0, \qquad\quad \text{otherwise.} \end{cases}$$

In the second-price sealed bid auction also exist multiple Nash equilibria [17] every strategy profile $(b_1, b_2) = (v_1, v_2)$ is a Nash equilibrium of the game. Another equilibrium is $(b_1, b_2) = (v_1, 0)$, or $(b_1, b_2) = (v_2, v_1)$.

Let us consider $v_1 = 5$ and $v_2 = 3$. CrDE detects the strategy profile $(b_1, b_2) = (5, 0)$, also a single equilibrium point, which is the best possible outcome for the first player.

In Figure 4 the strategies space by using 50000 randomly generated strategies is illustrated; the Nash equilibrium, Berge-Zhukovskii equilibrium and the Berge-Zhukovskii optimal Nash equilibrium of the game are presented. In Figure 5 the corresponding payoffs are depicted.

## 5 Conclusion

This paper introduces the Berge-Zhukovskii optimal Nash equilibria as a refinment of Nash equilibria that are Berge-Zhukovskii with respect to the set of Nash equilibria. As Nash equilibria are stable against unilateral deviations and BZ are stable against the deviations of the others, the subset of Nash equilibria that present BZ properties among NEs is of interest both to rational and altruistic players.

Furthermore, the paper presents a simple way of computing the BZON by using a differential evolution algorithm. Numerical examples using auction games illustrate this approach.

**Acknowledgments.** The authors wish to thank the support of the OPEN-RES Academic Writing project 212/2012.

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