On Values of Games

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Abstract. In this paper, we present some recent results of infinite games played on a finite graph. We mainly work with generalized reachability games and Büchi games. These games are two-player concurrent games in which each player chooses simultaneously their moves at each step. We concern here with a description of winning strategies and payoff functions over infinite plays. Each play and the outcome of a game are completely determined by strategies of the players. We classify strategies regarding their use of history. Our goal is to give simple expressions of values for each game. Moreover, we are interested in the question of what type of optimal (ϵ-optimal) strategy exists for both players depending on the type of games.

Keywords: reachability games, Büchi games, optimal strategy.

1 Introduction

We consider two-player simultaneous games played on finite graphs. For each round of the game, Player I and Player II choose their actions simultaneously and then the next state is determined. A finite or infinite sequence of state obtained are the result of the play. We investigate on generalized reachability games whose payoff functions can be described as a label function on the set of states over the non-negative real numbers. We mainly focus on Büchi games where the Player I want to visit target states infinitely many times and the Player II want to prevent from reaching the target states infinitely often. These are zero-sum games, and the reachability objective is one of the most basic objectives among the Borel hierarchy. Since there are two players on whose decisions the probability depends, we talk about the highest probability that the Player I can achieve against any opponent's strategy. Similarly, we also discuss the lowest probability that the Player II can achieve against any strategy of Player I. If these two quantities are equal, we call them the value of the game and say the game is determined. An optimal strategy for Player I is a strategy that guarantees the value of the game from each position. The way to determine the winner is called a winning objective. It is a set of infinite plays that we define as a winning for the Player I. In this study, we give answers to the following questions. Are the games determined or can we derive the value of game? Is it possible to show optimal and ϵ-optimal strategies in some way?

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Although Martin's theorem showed that every Borel game is determined, our results provide a specific proof for these types of games and may give more insight into this area, especially games on graphs.

1.1 Related Works and Motivation

In 1953, Gale and Stewart [10] introduced the general theory of infinite games, called Gale-Stewart games, which are two-player infinite games with perfect information. The theory of Gale-Stewart games has been investigated by many mathematicians and logicians, and until now it is one of the interesting topics in game theory and mathematical logic. This game is an infinite zero-sum game with perfect information because one of the players always wins and the other losses and the game is played in turn. The determinacy results for turn-based games with Borel objective was established by a deep result of Martin [14]. He proved that under some fairly general assumptions, one player has a winning strategy. On the other hand, the determinacy for one-round simultaneous games was proven by von Neumann [15] using his famous minmax theorem. Infinite versions of von Neumann's games were introduced by David Blackwell [1]. The determinacy for such games with Borel objective was established by Martin [13]. The proof of Martin's theorem is the culmination of a long series of results proving the determinacy of games of increasing Borel hierarchy.

Recently, Jan Krcǎl in [11] studied a determinacy of stochastic turn-based games focused on some winning objectives. Turn-based stochastic games are infinitely long sequential games with perfect information played by two players and a random player. He mainly discussed the reachability games and showed that the games are determined whether the games are finite or infinite. He also proved the existence of an optimal memoryless and deterministic strategy in the finite Büchi games. There are still many challenging open problems in the area of turn-based stochastic games. The existing results about infinite-state games usually concern on Markov Decision Process (MDPs) [12]. Moreover, many of the fundamental results are still waiting to be discovered in the infinite games with imperfect information, especially their use of payoff functions. Over the games on graph, the typical and most studied payoff functions are the limit-average (also called mean-payoff) and the discounted sum of the rewards along the path. To know the definition of mean payoff functions for example see [9], [8], and [16]; discounted payoff for details see also [9] and [6]. Besides their simple definitions, these two payoff functions enjoy the property that memoryless optimal strategies always exist, especially in turn-based stochastic games. In [2], they introduced a multi-mean payoff on turn-based stochastic parity games. This work can be seen as an extension of [3] where mean-payoff parity games have been studied. While Chatterjee et al. [4] defined another simple payoff functions which contain both the limit-average and the discounted sum functions in two-player turn-based games on a graph. In our study, a labelling function defined in generalized reachability games can be seen as a weighted reachability payoff function assigns to every infinite play either 0 if the game does not visit a target state, or the reward (positive real number) of the first target state visited by the player.

This paper is organized as follows. We first introduce the terminology of games, strategies and values. We then study a generalized reachability game and described its values as limits of finite-step games. Our main contribution is showing the existence of memoryless and randomized ε-optimal strategy for Player I, that is, the strategy in which depend only on the current state, satisfies the objective with probability within an ε difference of the value of the game. We then turn to games with Büchi objectives and use the results of generalized rachability games to show the value of Büchi games can be approximated in some way.

2 Games

This section gives preliminaries that is necessary for understanding the argument presented in subsequent sections.

Definition 1. *A (two-player simultaneous infinite) game is a quadruple* $\mathbb{G} = (S, A_{\mathrm{I}}, A_{\mathrm{II}}, \delta)$*, where* S*,* A_{I} *and* A_{II} *are nonempty finite sets and* δ *is a function from* $S \times A_{\mathrm{I}} \times A_{\mathrm{II}}$ *into* S*. Elements of* S *are called states. Elements of* A_{I} *are called actions or moves of Player I. Similarly, elements of* A_{II} *are called actions or moves of Player II.* δ *is called a transition function.*

Definition 2. *A path or a play of a game* $\mathbb{G} = (S, A_{\mathrm{I}}, A_{\mathrm{II}}, \delta)$ *is a finite or infinite sequence* $s_0 s_1 s_2 ...$ *of states in* S *such that for all* $n \in \mathbb{N}$*, there exist* $a_n \in A_{\mathrm{I}}$ *and* $b_n \in A_{\mathrm{II}}$ *where* $\delta(s_n, a_n, b_n) = s_{n+1}$*. Infinite paths of* $\mathbb{G}$ *are sometimes called runs. We write* $\Omega(\mathbb{G})$ *for the set of all infinite plays; and* $\Omega^{\mathrm{fin}}(\mathbb{G})$ *for the set of all finite plays of non-zero length. Sometimes we write* Ω *or* Ω^{fin} *instead of* $\Omega(\mathbb{G})$ *or* $\Omega^{\mathrm{fin}}(\mathbb{G})$ *when* $\mathbb{G}$ *is clear from the context.*

Intuitively, given a game $\mathbb{G} = (S, A_{\mathrm{I}}, A_{\mathrm{II}}, \delta)$, a function $F : \Omega(\mathbb{G}) \to [0, 1]$ and a state $s \in S$, we imagine the following infinite game $\mathbb{G}_s(F)$: at stage $n \in \mathbb{N} \setminus \{0\}$, we have the finite play $w \upharpoonright n$ with $w(0) = s$, and each player selects their actions $a_{\mathrm{I}} \in A_{\mathrm{I}}$ and $a_{\mathrm{II}} \in A_{\mathrm{II}}$, simultaneously, and, then, the next state $w(n) = \delta(w(n-1), a_{\mathrm{I}}, a_{\mathrm{II}})$ is determined. In this case the value of the play w is $F(w)$. We assume that Player I wants to maximize the value, whereas Player II wants to minimize. For a subset X of $\Omega(\mathbb{G})$, the infinite game $\mathbb{G}_s(X)$ is defined in the same way considering X as its characteristic function. Thus, in the case of a set X instead of a function F, Player I wants to put w into X, whereas Player II wants to avoid it.

The notion of strategies for infinite games plays an important role. Informally, a strategy for a player in the game is a rule that specifies the next move of the player for a given finite play.

For a set A, a *probability distribution* on A is a function $\mu : A \to [0, 1]$ with $\sum_{a \in A} \mu(a) = 1$. We use $\mathcal{D}(A)$ for the set of all probability distributions on A.

Definition 3. *Let* $\mathbb{G} = (S, A_{\mathrm{I}}, A_{\mathrm{II}}, \delta)$ *be a game. A (randomized) strategy of Player I in* $\mathbb{G}$ *is any function* $\sigma : \Omega^{\mathrm{fin}}(\mathbb{G}) \to \mathcal{D}(A_{\mathrm{I}})$*. We write* $\Sigma_{\mathrm{I}}^{\mathbb{G}}$ *or* Σ_{I} *for the set of all strategies of Player I. Similary, a (randomized) strategy of Player II in* $\mathbb{G}$ *is any function* $\tau : \Omega^{\mathrm{fin}}(\mathbb{G}) \to \mathcal{D}(A_{\mathrm{II}})$*, and we write* $\Sigma_{\mathrm{II}}^{\mathbb{G}}$ *or* Σ_{II} *for the set of all strategies of Player II.*

Definition 4. *Let $\mathbb{G} = (S, A_{\mathrm{I}}, A_{\mathrm{II}}, \delta)$ be a game. A strategy σ of Player I is called memoryless if $\sigma(p) = \sigma(q)$ holds whenever $p, q \in \Omega^{\mathrm{fin}}(\mathbb{G})$ satisfy $p(|p|-1) = q(|q|-1)$. A memoryless strategy of Player II is defined similarly. We write $\Sigma_{\mathrm{I}}^{\mathrm{M}}$ and $\Sigma_{\mathrm{II}}^{\mathrm{M}}$ for the set of all memoryless strategies of Player I and Player II, respectively.*

Intuitively, for a given finite play, memoryless strategies give the next action depending on the current state rather than depending on the finite play.

Clearly, given a memoryless strategy $\sigma \in \Sigma_{\mathrm{I}}^{\mathrm{M}}$, there exists *the* function $\sigma' : S \to \mathcal{D}(A_{\mathrm{I}})$ such that $\sigma(ps) = \sigma'(s)$ holds for any $ps \in \Omega^{\mathrm{fin}}(\mathbb{G})$ with $s \in S$. We sometimes identify σ with σ'. Similar identification will be used for Player II.

A pair $(\sigma, \tau) \in \Sigma_{\mathrm{I}} \times \Sigma_{\mathrm{II}}$ and a state $s \in S$ determine a probability measure $P_s^{\sigma,\tau}$ on $\Omega_s = \{w \in \Omega : w(0) = s\}$ as follows.

Definition 5. *Let $G = (S, A_{\mathrm{I}}, A_{\mathrm{II}}, \delta)$ be a game. For a pair $(\sigma, \tau) \in \Sigma_{\mathrm{I}}^{\mathbb{G}} \times \Sigma_{\mathrm{II}}^{\mathbb{G}}$ of strategies and a state $s \in S$, $P_s^{\sigma,\tau}$ denotes the probability measure on Ω_s determined by*

$$P_s^{\sigma,\tau}([p]) = \prod_{n \in \{1,\cdots,|\rho|-1\}} \sum \{\sigma(p \restriction n)(a)\tau(p \restriction n)(b) : (p(n-1), a, b) \in \delta^{-1}(p(n))\}$$

for any $p \in \Omega_s^{\mathrm{fin}} = \{q \in \Omega^{\mathrm{fin}} : q(0) = s\}$, where $[p] = \{w \in \Omega : p \subset w\}$.

Intuitively, for a function $F : \Omega \to [0,1]$ with $P_s^{\sigma,\tau}(F) = \int_{\Omega_s} F dP_s^{\sigma,\tau}$ exists, $P_s^{\sigma,\tau}(F)$ means the expectated value of an infinite game $\mathbb{G}_s(F)$ when Player I and Player II use the strategy σ and τ, respectively. In the case of a subset X of Ω instead of F, $P_s^{\sigma,\tau}(X)$ means the probability that the infinite play in Ω_s belongs to X when Player I and Player II use the corresponding strategies.

Let $\mathbb{G} = (S, A_{\mathrm{I}}, A_{\mathrm{II}}, \delta)$ be a game, and let $F : \Omega(\mathbb{G}) \to [0,1]$ satisfy that $P_s^{\sigma,\tau}(F)$ exists for any $\sigma \in \Sigma_{\mathrm{I}}^{\mathbb{G}}$, $\tau \in \Sigma_{\mathrm{II}}^{\mathbb{G}}$ and $s \in S$. We call such a function F a *payoff function* of $\mathbb{G}$. (In the game $\mathbb{G}(X)$, the set X with such a property is called a *winning set* of $\mathbb{G}$.) The value of Player I in a game $\mathbb{G}_s(F)$ for a state s is the supremum of expected value which Player I can ensure. Formally, it is $\sup_{\sigma \in \Sigma_{\mathrm{I}}^{\mathbb{G}}} \inf_{\tau \in \Sigma_{\mathrm{II}}^{\mathbb{G}}} P_s^{\sigma,\tau}(F)$. Let $\neg F$ be a function defined by $\neg F(w) = 1 - F(w)$. The value of Player II is defined as $\sup_{\sigma \in \Sigma_{\mathrm{I}}^{\mathbb{G}}} \inf_{\tau \in \Sigma_{\mathrm{II}}^{\mathbb{G}}} P_s^{\sigma,\tau}(\neg F)$. This value is equal to $1 - \inf_{\tau \in \Sigma_{\mathrm{II}}^{\mathbb{G}}} \sup_{\sigma \in \Sigma_{\mathrm{I}}^{\mathbb{G}}} P_s^{\sigma,\tau}(F)$. We say that the game $\mathbb{G}(F)$ is *determinate* if

$$\sup_{\sigma \in \Sigma_{\mathrm{I}}^{\mathbb{G}}} \inf_{\tau \in \Sigma_{\mathrm{II}}^{\mathbb{G}}} P_s^{\sigma,\tau}(F) + \sup_{\tau \in \Sigma_{\mathrm{II}}^{\mathbb{G}}} \inf_{\sigma \in \Sigma_{\mathrm{I}}^{\mathbb{G}}} P_s^{\tau,\sigma}(\neg F) = 1$$

holds for any $s \in S$. Or equivalently, the game $\mathbb{G}(F)$ is determinate if and only if

$$\sup_{\sigma \in \Sigma_{\mathrm{I}}^{\mathbb{G}}} \inf_{\tau \in \Sigma_{\mathrm{II}}^{\mathbb{G}}} P_s^{\sigma,\tau}(F) = \inf_{\tau \in \Sigma_{\mathrm{II}}^{\mathbb{G}}} \sup_{\sigma \in \Sigma_{\mathrm{I}}^{\mathbb{G}}} P_s^{\sigma,\tau}(F)$$

holds for any $s \in S$. In this case, we write $\mathrm{val}_s^{\mathbb{G}}(F)$ or $\mathrm{val}_s(F)$ instead of $\sup_{\sigma \in \Sigma_{\mathrm{I}}^{\mathbb{G}}} \inf_{\tau \in \Sigma_{\mathrm{II}}^{\mathbb{G}}} P_s^{\sigma,\tau}(F)$, and call it the *value* at s in the game $\mathbb{G}(F)$.

The following is well-known theorem obtained by Martin.

Theorem 1 (Martin [13]). *Let $\mathbb{G}$ be a game and let $F : \Omega(\mathbb{G}) \to [0,1]$ a Borel measurable function. Then the game $\mathbb{G}(F)$ is determinate.* □

Actually, the function F is Borel measurable function for any game $\mathbb{G}(F)$ studied in this paper later. Thus any game studied in this paper is determinate.

Definition 6. *Let $\mathbb{G} = (S, A_{\rm I}, A_{\rm II}, \delta)$, $F : \Omega \to [0,1]$ and $\epsilon \in [0,1]$. Suppose that $\mathbb{G}(F)$ is determinate. A strategy $\sigma \in \Sigma_{\rm I}$ of Player I is ϵ-optimal if $\inf_{\tau \in \Sigma_{\rm II}^{\mathbb{G}}} P_s^{\sigma,\tau}(F) \geq \mathrm{val}_s(F) - \epsilon$ holds for any $s \in S$. Similarly, a strategy $\tau \in \Sigma_{\rm II}$ of Player II is ϵ-optimal if $\sup_{\sigma \in \Sigma_{\rm I}^{\mathbb{G}}} P_s^{\sigma,\tau}(F) \leq \mathrm{val}_s(F) + \epsilon$ holds for any $s \in S$. Optimal strategies are 0-optimal strategies.*

By the definitions a strategy $\sigma \in \Sigma_{\rm I}$ of Player I is optimal if and only if $\inf_{\tau \in \Sigma_{\rm II}} P_s^{\sigma,\tau}(F) = \mathrm{val}_s(F)$ holds for all $s \in S$, and $\tau \in \Sigma_{\rm II}$ is optimal if and only if $\sup_{\sigma \in \Sigma_{\rm I}} P_s^{\sigma,\tau}(F) = \mathrm{val}_s(F)$ holds for all $s \in S$.

When $\mathbb{G}(F)$ is determinate and ε is a positive real number, then ε-optimal strategies of Player I and Player II always exist by the definition. However, there are some cases that Player I or Player II has no optimal strategy.

Let $\mathbb{G} = (X, A_{\rm I}, A_{\rm II}, \delta)$ be a game and let $V : S \to [0,1]$. We define $F_V : \Omega(\mathbb{G}) \to [0,1]$ by $F_V(w) = V(w(1))$. Games of the form $\mathbb{G}(F_V)$ are called *one-step games*. We write $\mathbb{G}(V)$ meaning $\mathbb{G}(F_V)$, and we write $\mathrm{val}_s(V)$ for $s \in S$ instead of $\mathrm{val}_s(F_V)$. In one-step games optimal strategies always exist for each player. This theorem is well-known as von Neumann's minmax theorem.

Theorem 2 (von Neumann [15]). *In any one-step game, both players have their optimal strategies.* □

3 Generalized Reachability Games

Reachability games are in some respect the simplest infinite games. We will prove some basic facts on a generalized version of reachability games, called generalized reachability games. We will describe the value of reachability games as a limit value of finite-step games. We will see that Player II has a memoryless optimal strategy, and Player I has a memoryless ε-optimal strategy in any generalized reachability games for any positive real number ε. Nevertheless, in general it is known that, even in a reachability game, Player I may not have an optimal strategy.

Definition 7. *Let $\mathbb{G} = (S, A_{\rm I}, A_{\rm II}, \delta)$ be a game. A function ℓ is called a label on S if $\mathrm{dom}(\ell) \subset S$ and $\ell(s) \in [0,1]$ for any $s \in \mathrm{dom}(\ell)$. We define $\mathcal{R}^{\mathbb{G},\ell} : \Omega(\mathbb{G}) \to [0,1]$ by*

$$\mathcal{R}^{\mathbb{G},\ell}(w) = \begin{cases} \ell(w(N_w)) & \text{if } (\exists N \in \mathbb{N})[w(N) \in \mathrm{dom}(\ell)], \\ 0 & \text{otherwise}, \end{cases}$$

where N_w is the least natural number N such that $w(N) \in \mathrm{dom}(\ell)$. A game of the form $\mathbb{G}(\mathcal{R}^{\mathbb{G},\ell})$ is called a generalized reachability game.

For a subset T of S, let $\mathcal{R}^{\mathbb{G},T} = \mathcal{R}^{\mathbb{G},\ell_T}$, where $\ell_T : T \to \{1\}$. Games of the form $\mathbb{G}(\mathcal{R}^{\mathbb{G},T})$ are called *reachability games*.

Definition 8. *Let $\mathbb{G} = (S, A_{\mathrm{I}}, A_{\mathrm{II}}, \delta)$ be a game and let ℓ a label on S. For every state $s \in S$ and $n \in \mathbb{N}$, we define $V_n^{\mathbb{G},\ell} : S \to [0,1]$ inductively by*

$$V_0^{\mathbb{G},\ell}(s) = \begin{cases} \ell(s) & \text{if } s \in \mathrm{dom}(\ell), \\ 0 & \text{otherwise}, \end{cases} \qquad V_{n+1}^{\mathbb{G},\ell}(s) = \begin{cases} \ell(s) & \text{if } s \in \mathrm{dom}(\ell), \\ \mathrm{val}_s(V_n^{\mathbb{G},\ell}) & \text{otherwise}. \end{cases}$$

We let $V^{\mathbb{G},\ell}(s) = \lim_{n\to\infty} V_n^{\mathbb{G},\ell}(s)$ for any state s, and we call it the limit value at s.

For a label ℓ on S and $n \in \mathbb{N}$, we define $\mathcal{R}_n^{\mathbb{G},\ell} : \Omega(\mathbb{G}) \to [0,1]$ by $\mathcal{R}_n^{\mathbb{G},\ell}(w) = s_w$ if there exists $m \leq n$ with $w(m) \in \mathrm{dom}(\ell)$ and $\mathcal{R}_n^{\mathbb{G},\ell}(w) = 0$ otherwise.

Theorem 3. *For any $n \in \mathbb{N}$, both players have their optimal strategy in the game $\mathbb{G}(\mathcal{R}_n^{\mathbb{G},\ell})$, and the equality $V_n^{\mathbb{G},\ell}(s) = \mathrm{val}_s(\mathcal{R}_n^{\mathbb{G},\ell})$ holds for all $s \in S$.*

Proof. We define σ_n^* and τ_n^* inductively. Let σ_0^* and τ_0^* be any strategies. Now suppose that we have constructed σ_n^* and τ_n^*. Choose σ and τ as optimal strategies of Player I and II respectively in the one-step game $\mathbb{G}(V_n^{\mathbb{G},\ell})$. Define σ_{n+1}^* by $\sigma_{n+1}^*(s) = \sigma(s)$ and $\sigma_{n+1}^*(s\rho) = \sigma_n^*(\rho)$ for any $s \in S$ and any $\rho \neq \emptyset$ with $s\rho \in \Omega^{\mathrm{fin}}$. Similarly, define τ_{n+1}^* by $\tau_{n+1}^*(s) = \tau(s)$ and $\tau_{n+1}^*(s\rho) = \tau_n^*(\rho)$ for any $s \in S$ and any $\rho \neq \emptyset$ with $s\rho \in \Omega^{\mathrm{fin}}$. It is easy to see by induction on n that σ_n^* and τ_n^* satisfy the equalities $V_n^{\mathbb{G},\ell}(s) = \inf_{\tau\in\Sigma_{\mathrm{II}}} P_s^{\sigma_n^*,\tau}(\mathcal{R}_n^{\mathbb{G},\ell}) = \sup_{\sigma\in\Sigma_{\mathrm{I}}} P_s^{\sigma,\tau_n^*}(\mathcal{R}_n^{\mathbb{G},\ell})$. This equalities imply that the σ_n^* and τ_n^* are optimal strategies in the game $\mathbb{G}(\mathcal{R}_n^{\mathbb{G},\ell})$ and $V_n^{\mathbb{G},\ell}(s) = \mathrm{val}_s(\mathcal{R}_n^{\mathbb{G},\ell})$ holds. □

Now we verify the value $\mathrm{val}_s(\mathcal{R}^{\mathbb{G},\ell})$ is equivalent to the limit value $V^{\mathbb{G},\ell}(s)$.

Theorem 4. *For any state $s \in S$, the equation $V^{\mathbb{G},\ell}(s) = \mathrm{val}_s(\mathcal{R}^{\mathbb{G},\ell})$ holds.*

Proof. It is enough to show that the following inequalities:

$$\inf_{\tau\in\Sigma_{\mathrm{II}}} \sup_{\sigma\in\Sigma_{\mathrm{I}}} P_s^{\sigma,\tau}(\mathcal{R}^{\mathbb{G},\ell}) \leq V^{\mathbb{G},\ell}(s) \leq \sup_{\sigma\in\Sigma_{\mathrm{I}}} \inf_{\tau\in\Sigma_{\mathrm{II}}} P_s^{\sigma,\tau}(\mathcal{R}^{\mathbb{G},\ell}).$$

To show the first inequality, choose an optimal strategy τ^* of Player II in the one-step game $\mathbb{G}(V^{\mathbb{G},\ell})$. We may see τ^* as a memoryless strategy of Player II in the generalized reachability game $\mathbb{G}(\mathcal{R}^{\mathbb{G},\ell})$. We show that τ^* satisfies the inequality $\sup_{\sigma\in\Sigma_{\mathrm{I}}} P_s^{\sigma,\tau^*}(\mathcal{R}^{\mathbb{G},\ell}) \leq V^{\mathbb{G},\ell}(s)$ for any $s \in S$. (Thus, if we prove the second inequality, then we can say this τ^* is, in fact, an optimal strategy of Player II in the game $\mathbb{G}(\mathcal{R}^{\mathbb{G},\ell})$.) It is enough to show that $\sup_\sigma P_s^{\sigma,\tau^*}(\mathcal{R}_n^{\mathbb{G},\ell}) \leq V^{\mathbb{G},\ell}(s)$ for any $s \in S$ and $n \in \mathbb{N}$. We show this by induction on n. The case $n = 0$ is clear. Suppose that $\sup_\sigma P_s^{\sigma,\tau^*}(\mathcal{R}_n^{\mathbb{G},\ell}) \leq V^{\mathbb{G},\ell}(s)$ holds for any $s \in S$ as an induction hypothesis. Fix $s \in S$. If $s \in \mathrm{dom}(\ell)$, then it is obvious that the inequality holds for s. Otherwise, we have the equality $P_s^{\sigma,\tau^*}(\mathcal{R}_{n+1}^{\mathbb{G},\ell}) = \sum_{s'\in S} P_s^{\sigma,\tau^*}([ss'])P_{s'}^{\sigma,\tau^*}(\mathcal{R}_n^{\mathbb{G},\ell})$ for any $\sigma \in \Sigma_{\mathrm{I}}$. By the induction hypothesis, we know that $P_s^{\sigma,\tau^*}(\mathcal{R}_{n+1}^{\mathbb{G},\ell}) \leq \sum_{s'\in S} P_s^{\sigma,\tau^*}([ss'])V^{\mathbb{G},\ell}(s')$. Hence the equalities

$$\sup_\sigma P_s^{\sigma,\tau^*}(\mathcal{R}_{n+1}^{\mathbb{G},\ell}) \leq \sup_\sigma \sum_{s'\in S} P_s^{\sigma,\tau^*}([ss'])V^{\mathbb{G},\ell}(s') = V^{\mathbb{G},\ell}(s)$$

hold by the optimality of τ^* in the one-step game. Let us now show the second inequality. We have $P_s^{\sigma,\tau}(\mathcal{R}_n^{\mathbb{G},\ell}) \le P_s^{\sigma,\tau}(\mathcal{R}^{\mathbb{G},\ell})$ since $\mathcal{R}_n^{\mathbb{G},\ell}(w) \le \mathcal{R}^{\mathbb{G},\ell}(w)$ for any $w \in \Omega$. Hence $\sup_\sigma \inf_\tau P_s^{\sigma,\tau}(\mathcal{R}_n^{\mathbb{G},\ell}) \le \sup_\sigma \inf_\tau P_s^{\sigma,\tau}(\mathcal{R}^{\mathbb{G},\ell})$ holds. By Theorem 3, $V_n^{\mathbb{G},\ell}(s) = \mathrm{val}_s(\mathcal{R}_n^{\mathbb{G},\ell}) = \sup_\sigma \inf_\tau P_s^{\sigma,\tau}(\mathcal{R}_n^{\mathbb{G},\ell})$ holds. Thus the second inequality holds. □

Corollary 1. *Player II has a memoryless and randomized optimal strategy in any generalized reachability game.* □

Contrary to the case of Player II, Player I has no even optimal strategy in some reachability games. We give such an example below.

Example 1. Consider the following simultaneous reachability game as shown in Figure 1. Let $S = \{s_0, s_1, s_2\}$, $A_{\mathrm{I}} = \{x_1, x_2\}$ and $A_{\mathrm{II}} = \{y_1, y_2\}$. Define a transition function δ by $\delta(s_0, x_1, y_1) = s_0$, $\delta(s_0, x_2, y_2) = s_2$, $\delta(s_0, x_1, y_2) = \delta(s_0, x_2, y_1) = s_1$ and $\delta(s_i, x, y) = s_i$ for any $i \in \{1, 2\}$ and $(x, y) \in A_{\mathrm{I}} \times A_{\mathrm{II}}$. Let $\mathbb{G} = (S, A_{\mathrm{I}}, A_{\mathrm{II}}, \delta)$. Now consider the reachability game $\mathbb{G}(\mathcal{R}_{\{s_1\}})$.

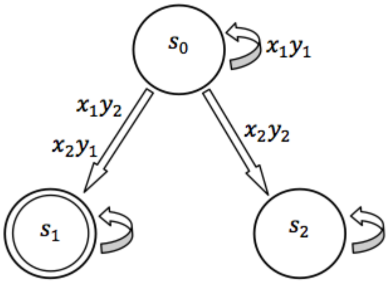

Fig. 1. An illustration of reachability game

One can prove that $\mathrm{val}_{s_0}(\mathcal{R}_{\{s_1\}}) = 1$. We show that Player I has no optimal strategy in the reachability game $\mathbb{G}(\mathcal{R}_{\{s_1\}})$.

Proof. Fix a strategy $\sigma \in \Sigma_{\mathrm{I}}$. We construct $\tau \in \Sigma_{\mathrm{II}}$ such that $P_{s_0}^{\sigma,\tau}(\mathcal{R}_{\{s_1\}}) < 1$. For $\rho \in \Omega^{\mathrm{fin}}(\mathbb{G})$, define $\tau(\rho)(y_1) = 1$ if $\sigma(\rho)(x_1) = 1$, and define $\tau(\rho)(y_2) = 1$ otherwise. It is clear that $P_{s_0}^{\sigma,\tau}(\mathcal{R}_{\{s_1\}}) < 1$ by the definitions of $\mathbb{G}$ and τ. □

The next theorem says that, given a generalized reachability game, Player I always has a memoryless ε-optimal strategy in this game for any positive real number ε. In fact, this result for reachability games was shown by Chatterjee et al. [5] in a slightly different setting. We essentially use their method to prove our theorem.

Theorem 5. *In every generalized reachability game $\mathbb{G}(\mathcal{R}^{\mathbb{G},\ell})$, there exist an ε-optimal memoryless strategy of Player I for any $\varepsilon > 0$.*

Proof. Let $\mathbb{G} = (S, A_{\mathrm{I}}, A_{\mathrm{II}}, \delta)$ be a game and let ℓ a label on S. Without loss of generality, we may assume that if $s \in \mathrm{dom}(\ell)$ or $\mathrm{val}_s(\mathcal{R}^{\mathbb{G},\ell}) = 0$, then $\delta(s, x, y) = s$ holds for any $(x, y) \in A_{\mathrm{I}} \times A_{\mathrm{II}}$.

Fix a positive real $\varepsilon > 0$. Choose $n \in \mathbb{N}$ such that for any $s \in S$, the inequality $V_{n-1}^{\mathbb{G},\ell}(s) \geq \mathrm{val}_s(\mathcal{R}^{\mathbb{G},\ell}) - \varepsilon$ holds, and $\mathrm{val}_s(\mathcal{R}^{\mathbb{G},\ell}) > 0$ implies $V_{n-1}^{\mathbb{G},\ell}(s) > 0$. For $m \leq n$, choose $\sigma_m \in \Sigma_{\mathrm{I}}^{\mathrm{M}}$ such that σ_m is an optimal strategy of Player I in the one-step game $\mathbb{G}(V_{m-1}^{\mathbb{G},\ell})$. We define a strategy $\sigma^* \in \Sigma_{\mathrm{I}}^{\mathrm{M}}$ by $\sigma^*(s) = \sigma_{m_s}(s)$ for any $s \in S$, where m_s is the least number $m \leq n$ such that $V_m^{\mathbb{G},\ell}(s) = V_n^{\mathbb{G},\ell}(s)$. By the definition, $V_{m_s}^{\mathbb{G},\ell}(s) = \inf_{\tau \in \Sigma_{\mathrm{II}}^{\mathrm{M}}} P_s^{\sigma^*,\tau}(V_{m_s-1}^{\mathbb{G},\ell})$ holds for any $s \in S \setminus \mathrm{dom}(\ell)$. Now choose a strategy $\tau^* \in \Sigma_{\mathrm{II}}^{\mathrm{M}}$ such that $P_s^{\sigma^*,\tau^*}(\mathcal{R}^{\mathbb{G},\ell}) = \inf_\tau P_s^{\sigma^*,\tau}(\mathcal{R}^{\mathbb{G},\ell})$ for all $s \in S$.

Fix a $s \in S \setminus \mathrm{dom}(\ell)$ with $V_{m_s}^{\mathbb{G},\ell}(s) > 0$. Suppose that $V_n(s) \geq V_n(s')$ holds for any $s' \in S$ with $P_s^{\sigma^*,\tau^*}([ss']) > 0$. We have $V_{m_s}^{\mathbb{G},\ell}(s) = V_{m_s-1}^{\mathbb{G},\ell}(s') = V_n^{\mathbb{G},\ell}(s')$ for any $s' \in S$ with $P_s^{\sigma^*,\tau^*}([ss']) > 0$ since $V_n^{\mathbb{G},\ell}(s) = V_{m_s}^{\mathbb{G},\ell}(s)$, $V_{m_s-1}^{\mathbb{G},\ell}(s') \leq V_n^{\mathbb{G},\ell}(s')$ and $V_{m_s}^{\mathbb{G},\ell}(s) \leq P_s^{\sigma^*,\tau}(V_{m_s-1}^{\mathbb{G},\ell})$ hold. Therefore, if $s' \in S$ satisfies $P_s^{\sigma^*,\tau^*}([ss'])$, then $m_s > m_{s-1}$. As a result, we know that for any $s \in S \setminus \mathrm{dom}(\ell)$ there exists s' with $P_s^{\sigma^*,\tau^*}([ss']) > 0$ such that

$$V_n^{\mathbb{G},\ell}(s) < V_n^{\mathbb{G},\ell}(s') \text{ or } m_s > m_{s-1}.$$

Note that $\{V_n^{\mathbb{G},\ell}(s) : s \in S \setminus \mathrm{dom}(\ell)\}$ is finite, and $m_s = 0$ implies $s \in \mathrm{dom}(\ell)$ or $V_n^{\mathbb{G},\ell}(s) = 0$. Here $V_n^{\mathbb{G},\ell}(s) = 0$ implies $\mathrm{val}_s(\mathcal{R}^{\mathbb{G},\ell}) = 0$. Hence for any $s \in S$ there exists $\rho \in \Omega_s^{\mathrm{fin}}$ such that $P_s^{\sigma^*,\tau^*}([\rho]) > 0$ and

$$\rho(|\rho| - 1) \in \mathrm{dom}(\ell) \text{ or } \mathrm{val}_{\rho(|\rho|-1)}(\mathcal{R}^{\mathbb{G},\ell}) = 0.$$

As a conclusion, we have $P_s^{\sigma^*,\tau^*}(A) = 0$ for any $s \in S$, where $A = \{w \in \Omega : (\forall n \in \mathbb{N})[w(n) \in \mathrm{dom}(\ell) \ \& \ \mathrm{val}_{w(n)}(\mathcal{R}^{\mathbb{G},\ell}) > 0]\}$. Thus, the sum

$$\sum \left\{ V_n^{\mathbb{G},\ell}(\rho(|\rho| - 1)) P_s^{\sigma^*,\tau^*}([\rho]) : \rho \in \Omega_s^{\mathrm{fin}} \ \& \ |\rho| = k \right\}$$

tends to $P_s^{\sigma^*,\tau^*}(\mathcal{R}^{\mathbb{G},\ell})$ as k to ∞. It is easy to see by induction on $k \in \mathbb{N}$ that

$$\sum \left\{ V_n^{\mathbb{G},\ell}(\rho(|\rho| - 1)) P_s^{\sigma^*,\tau^*}([\rho]) : \rho \in \Omega_s^{\mathrm{fin}} \ \& \ |\rho| = k \right\} \geq V_{n-1}^{\mathbb{G},\ell}(s)$$

holds for any $k \in \mathbb{N}$. Hence we have $P_s^{\sigma^*,\tau^*}(\mathcal{R}^{\mathbb{G},\ell}) \geq V_{n-1}^{\mathbb{G},\ell}(s) \geq \mathrm{val}_s(\mathcal{R}^{\mathbb{G},\ell}) - \varepsilon$. □

4 Büchi Games

Our plan in this section is as follows. After giving the definition of Büchi games, we describe values of Büchi games as values of some generalized reachability games. The proof includes the information how we can construct ε-optimal strategies in a Büchi game for a given positive real number ε. But we see that, in general, Player I may not have a memoryless ε-optimal strategy in a Büchi game for some positive real ε.

Definition 9. *Let* $\mathbb{G} = (S, A_{\mathrm{I}}, A_{\mathrm{II}}, \delta)$ *be a game. For a run* $w \in \Omega(\mathbb{G})$*, we define* $\mathrm{Inf}(w)$ *as the set* $\{s \in S : (\forall n \in \mathbb{N})(\exists m \geq n)[w(m) = s]\}$*. For* $T \subset S$*, we set* $\mathcal{B}^{\mathbb{G},T} = \{w \in \Omega(\mathbb{G}) : \mathrm{Inf}(w) \cap T \neq \emptyset\}$*. Any game of the form* $\mathbb{G}(\mathcal{B}^{\mathbb{G},T})$ *is called a Büchi game.* T *is called the set of target states of the Büchi game* $\mathbb{G}(\mathcal{B}^{\mathbb{G},T})$*.*

In [7], they introduced quantitative game μ-calculus, and showed that the maximal probability of winning for Büchi game can be expressed as a fixpoint formulas. In particular, they characterized the optimality and the memory requirements of the winning strategies, that is, memoryless strategies suffice for winning games with reachability condition, and Büchi conditions require the use of strategies with infinite memory.

For a label ℓ on S, define $\mathcal{R}_+^{\mathbb{G},\ell} : \Omega(\mathbb{G}) \to [0,1]$ by

$$\mathcal{R}_+^{\mathbb{G},\ell}(w) = \begin{cases} \ell(w(N_w)) & \text{if } (\exists N > 0)[w(N) \in \mathrm{dom}(\ell)], \\ 0 & \text{otherwise,} \end{cases}$$

where N_w is the least natural number $N > 0$ such that $w(N) \in \mathrm{dom}(\ell)$. Clearly, the results for generalized reachability games in the previous section hold even for games of the form $\mathbb{G}(\mathcal{R}_+^{\mathbb{G},\ell})$. We also call this kind of games generalized reachability games. To study Büchi games, these results for games of the form $\mathbb{G}(\mathcal{R}_+^{\mathbb{G},\ell})$ are useful.

Definition 10. *Let $\mathbb{G} = (S, A_{\mathrm{I}}, A_{\mathrm{II}}, \delta)$ be a game and let $T \subset S$. For any $n \in \mathbb{N}$, we define a label $\ell_n^{\mathbb{G},T}$ on S with the domain T inductively by*

$$\ell_0^{\mathbb{G},T}(s) = 1 \qquad\qquad \ell_{n+1}^{\mathbb{G},T}(s) = \mathrm{val}_s(\mathcal{R}_+^{\mathbb{G},\ell_n^{\mathbb{G},T}})$$

We set $\ell^{\mathbb{G},T}(s) = \lim_{n\to\infty} \ell_n^{\mathbb{G},T}(s)$.

For $T \subset S$ and $n \in \mathbb{N}$, we define $\mathcal{B}_n^{\mathbb{G},T} = \{w \in \Omega(\mathbb{G}) : (\exists^{\geq n} k > 0)[w(k) \in T]\}$. Here, read "$\exists^{\geq n} k > 0$" as "there exist at least n-many natural numbers $k > 0$".

Theorem 6. *The equality $\mathrm{val}_s^{\mathbb{G}}(\mathcal{B}_{n+1}^{\mathbb{G},T}) = \mathrm{val}_s^{\mathbb{G}}(\mathcal{R}_+^{\mathbb{G},\ell_n^{\mathbb{G},T}})$ holds for any $n \in \mathbb{N}$ and any $s \in S$.*

Proof. Let $\ell_n = \ell_n^{\mathbb{G},T}$ for any $n \in \mathbb{N}$. We show the equality holds by induction on n. Fix n. It is clear that the equation holds for any $s \in S$ when $n = 0$. Now, suppose that $\mathrm{val}_s(\mathcal{B}_{n+1}^{\mathbb{G},T}) = \mathrm{val}_s(\mathcal{R}_+^{\mathbb{G},\ell_n})$ holds for any $s \in S$. Then we have $\ell_{n+1}(s) = \mathrm{val}_s(\mathcal{R}_+^{\mathbb{G},\ell_n}) = \mathrm{val}_s(\mathcal{B}_{n+1}^{\mathbb{G},T})$ for any $s \in T = \mathrm{dom}(\ell_{n+1})$. Therefore, $\mathrm{val}_s(\mathcal{B}_{n+2}^{\mathbb{G},T}) = \mathrm{val}_s(\mathcal{R}_+^{\mathbb{G},\ell_{n+1}})$ holds for any $s \in S$. □

Theorem 7. *The equality $\mathrm{val}_s^{\mathbb{G}}(\mathcal{B}^{\mathbb{G},T}) = \mathrm{val}_s^{\mathbb{G}}(\mathcal{R}_+^{\mathbb{G},\ell^{\mathbb{G},T}})$ holds for any $s \in S$.*

Proof. Let $\ell = \ell^{\mathbb{G},T}$ and $\ell_n = \ell_n^{\mathbb{G},T}$ for any $n \in \mathbb{N}$. It is enough to show that the following inequalities:

$$\inf_{\tau \in \Sigma_{\mathrm{II}}} \sup_{\sigma \in \Sigma_{\mathrm{I}}} P_s^{\sigma,\tau}(\mathcal{B}^{\mathbb{G},T}) \leq \mathrm{val}_s(\mathcal{R}_+^{\mathbb{G},\ell}) \leq \sup_{\sigma \in \Sigma_{\mathrm{I}}} \inf_{\tau \in \Sigma_{\mathrm{II}}} P_s^{\sigma,\tau}(\mathcal{B}^{\mathbb{G},T}).$$

Note that $\mathcal{B}_n^{\mathbb{G},T} \supset \mathcal{B}^{\mathbb{G},T}$ holds for any $n \in \mathbb{N}$. Thus

$$\inf_{\tau} \sup_{\sigma} P_s^{\sigma,\tau}(\mathcal{B}^{\mathbb{G},T}) \leq \mathrm{val}_s(\mathcal{B}_{n+1}^{\mathbb{G},T}) = \mathrm{val}_s(\mathcal{R}_+^{\mathbb{G},\ell_n})$$

holds for any $n \in \mathbb{N}$. Since the righthand tends to $\mathrm{val}_s(\mathcal{R}_+^{\mathbb{G},\ell})$ as n to ∞, we know that the first inequality holds. Let us now show the second inequality also holds.

Fix a positive real ε. We construct a strategy $\sigma^* \in \Sigma_{\mathrm{I}}$ such that $\mathrm{val}_s(\mathcal{R}_+^{\mathbb{G},\ell}) - \varepsilon \leq \inf_\tau P_s^{\sigma^*,\tau}(\mathcal{B}^{\mathbb{G},T})$ holds for any $s \in S$. Choose sequences $\{\alpha_n\}_{n\in\mathbb{N}}$, $\{\beta_n\}_{n\in\mathbb{N}}$ of positive reals such that

$$\mathrm{val}_s(\mathcal{R}_+^{\mathbb{G},\ell}) - \varepsilon \leq \mathrm{val}_s(\mathcal{R}^{\mathbb{G},\ell})) \prod_{n\in\mathbb{N}}(1-\alpha_n), \quad \mathrm{val}_s(\mathcal{R}^{\mathbb{G},\ell})(1-\alpha_n) \leq \mathrm{val}_s(\mathcal{R}^{\mathbb{G},\ell}) - \beta_n$$

holds for any $s \in S$ and $n \in \mathbb{N}$. Choose a sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ of strategies of Player I in the game $\mathbb{G}(\mathcal{R}^{\mathbb{G},\ell})$ such that σ_n is β_n-optimal for any $n \in \mathbb{N}$. For $k \in \mathbb{N}$, we define $\sigma_k^* \in \Sigma_{\mathrm{I}}$ as follows: for $\rho \in \Omega^{\mathrm{fin}}$, let $\sigma_k^*(\rho) = \sigma_{n+k}(\rho_{\mathrm{suf}})$, where $\rho_{\mathrm{suf}} \in \Omega^{\mathrm{fin}}(\mathbb{G})$ satisfies $\rho = (\rho \restriction m)\rho_{\mathrm{suf}}$ with $m = \max\{0, i-1 : \rho(i) \in T\}$, and $n = \#\{i \leq m : \rho(i) \in T\}$. We show by induction on n that the inequality

$$\mathrm{val}_s(\mathcal{R}_+^{\mathbb{G},\ell}) \prod_{i\leq n}(1-\alpha_{i+k}) \leq \inf_\tau P_s^{\sigma_k^*,\tau}(\mathcal{B}_n^{\mathbb{G},T}) \tag{1}$$

holds for any $s \in S$ and $k, n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Clearly (1) holds for any $s \in S$ and $k \in \mathbb{N}$ when $n = 0$. Now, suppose that n satisfies (1) for any $s \in S$ and $k \in \mathbb{N}$. For $s, s' \in S$, define $A_{s,s'}$ as the set $\{\rho \in \Omega_s^{\mathrm{fin}} : \rho(|\rho|-1) = s' \;\&\; (\forall i \in \{1, \cdots, |\rho|-2\})[\rho(i) \notin T]\}$. The inequalities

$$\begin{aligned}
&\inf_\tau P_s^{\sigma_k^*,\tau}(\mathcal{B}_{n+1}^{\mathbb{G},T}) \\
\geq &\inf_\tau \sum \left\{ P_s^{\sigma_k^*,\tau}([\rho]) \inf_{\tau'} P_{s'}^{\sigma_{k+1}^*,\tau'}(\mathcal{B}_n^{\mathbb{G},T}) : \rho \in A_{s,s'} \;\&\; s' \in T \right\} \\
\geq &\inf_\tau \sum \left\{ P_s^{\sigma_k^*,\tau}([\rho]) \mathrm{val}_{s'}(\mathcal{R}_+^{\mathbb{G},\ell}) \prod_{i\leq n}(1-\alpha_{i+k+1}) : \rho \in A_{s,s'} \;\&\; s' \in T \right\} \\
\geq &\mathrm{val}_s(\mathcal{R}_+^{\mathbb{G},\ell}) \prod_{i\leq n+1}(1-\alpha_{i+k})
\end{aligned}$$

holds for any $s \in S$ and $k \in \mathbb{N}$. Thus (1) holds for any $s \in S$ and $k, n \in \mathbb{N}$. Hence $\mathrm{val}_s(\mathcal{R}_+^{\mathbb{G},\ell}) - \varepsilon \leq \inf_\tau P_s^{\sigma_0^*,\tau}(\mathcal{B}_n^{\mathbb{G},T})$ holds for any $s \in S$ and $n \in \mathbb{N}$. Since $\mathcal{B}^{\mathbb{G},T} = \bigcap_{n\in\mathbb{N}} \mathcal{B}_n^{\mathbb{G},T}$ holds, we have $\mathrm{val}_s(\mathcal{R}_+^{\mathbb{G},\ell}) - \varepsilon \leq \inf_\tau P_s^{\sigma_0^*,\tau}(\mathcal{B}^{\mathbb{G},T})$ for any $s \in S$. □

The following example shows that there exist a positive real ε and a Büchi game in which Player I has no ε-optimal memoryless strategy.

Example 2. Let $\mathbb{G}(\mathcal{B}^{\mathbb{G},T})$ be the simultaneous Büchi game given as Figure 2.

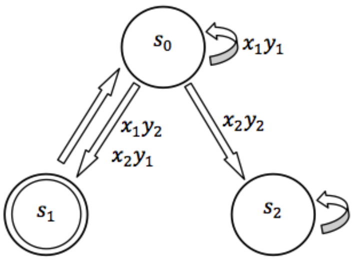

Fig. 2. An illustration of Büchi game

That is, let $\mathbb{G} = (S, A_{\mathrm{I}}, A_{\mathrm{II}}, \delta)$ be a game, where $S = \{s_0, s_1, s_2\}$, $A_{\mathrm{I}} = \{x_1, x_2\}$ and $A_{\mathrm{II}} = \{y_1, y_2\}$, and the transition function δ is given by $\delta(s_0, x_1, y_1) = s_0$, $\delta(s_0, x_2, y_2) = s_2$, $\delta(s_0, x_1, y_2) = \delta(s_0, x_2, y_1) = s_1$, $\delta(s_1, x, y) = s_0$ and $\delta(s_2, x, y) = s_2$ for all $(x, y) \in A_{\mathrm{I}} \times A_{\mathrm{II}}$. Define $T = \{s_1\}$. One can prove that $\mathrm{val}_{s_0}(\mathcal{B}^{\mathbb{G},T}) = 1$. We show that Player I has no ε-optimal memoryless strategy in the Büchi game $\mathbb{G}(\mathcal{B}^{\mathbb{G},T})$ for any positive real $\varepsilon < 1$.

Proof. Fix a positive real $\varepsilon < 1$ and a strategy $\sigma \in \Sigma_{\mathrm{I}}^{\mathrm{M}}$. We define a strategy $\tau \in \Sigma_{\mathrm{II}}$ by $\tau(\rho)(y_1) = 1$ if $\sigma(\rho)(x_1) = 1$, and $\tau(\rho)(y_2) = 1$ otherwise. In the case that $\sigma(s_0)(x_1) = 1$, we have $P_{s_0}^{\sigma,\tau}(\{s_0^{\mathbb{N}}\}) = 1$. Otherwise, we have $P_{s_0}^{\sigma,\tau}(\mathcal{R}^{\mathbb{G},\{s_2\}}) = 1$ since $\sigma(\rho) = \sigma(\rho')$ if $\rho(|\rho| - 1) = \rho'(|\rho'| - 1)$. Therefore, $\inf_{\tau'} P_{s_0}^{\sigma,\tau'}(\mathcal{B}^{\mathbb{G},\{s_1\}}) = 0 < 1 - \varepsilon = \mathrm{val}_{s_0}(\mathcal{B}^{\mathbb{G},T}) - \varepsilon$ holds. □

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