Subdirect Decomposition of Concept Lattices[*,**]

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Dedicated to Garrett Birkhoff on the occasion of his seventieth birthday

1. Introduction

In [1], G. Birkhoff exhibited the subdirect product of algebraic structures as a universal tool, which since has been extensively used in the study of algebraic theories. Although a subdirect product is not uniquely determined by its factors, there are useful construction methods based on subdirect products (cf. Wille [8], [9], [10]). The aim of this paper is to make these methods available for handling the "Determination Problem" of concept lattices as it is exposed in Wille [11]. In particular, a useful method for determining concept lattices via its scaffoldings will be developed under some finiteness condition.

2. Concept lattices

First we recall some notions from Wille [11]. A *context* is defined as a triple (G, M, I) where G and M are sets, and I is a binary relation between G and M; the elements of G and M are called *objects* and *attributes*, respectively. If gIm for $g \in G$ and $m \in M$ we say: the object g has the attribute m. The relation I establishes a *Galois connection* between the power sets of G and M (cf. Birkhoff [2]) which is expressed by the definition

$$A' := \{m \in M \mid gIm \quad \text{for all} \quad g \in A\} \quad \text{for} \quad A \subseteq G,$$
$$B' := \{g \in G \mid gIm \quad \text{for all} \quad m \in B\} \quad \text{for} \quad B \subseteq M.$$

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Following traditional philosophy, a *concept* of a context (G, M, I) is defined as a pair (A, B) where $A \subseteq G$, $B \subseteq M$, $A' = B$, and $B' = A$; A and B are called the *extent* and the *intent* of the concept (A, B), respectively. The relation "subconcept–superconcept" is captured by the definition

$$(A_1, B_1) \leq (A_2, B_2) :\Leftrightarrow A_1 \subseteq A_2 (\Leftrightarrow B_1 \supseteq B_2)$$

for concepts (A_1, B_1) and (A_2, B_2) of (G, M, I). $\mathfrak{B}(G, M, I)$ denotes the set of all concepts of the context (G, M, I) and $\underline{\mathfrak{B}}(G, M, I) := (\mathfrak{B}(G, M, I), \leq)$. A subset D of a complete lattice L is called *infimum-dense* (*supremum-dense*) if $L = \{\bigwedge X \mid X \subseteq D\}$ $(L = \{\bigvee X \mid X \subseteq D\})$. Now, we are able to formulate the basic theorem of "concept lattices" (cf. Wille [11]).

THEOREM 1. *Let (G, M, I) be a context. Then $\underline{\mathfrak{B}}(G, M, I)$ is a complete lattice, called the* concept lattice *of (G, M, I), in which infima and suprema can be described as follows*:

$$\bigwedge_{j \in J} (A_j, B_j) = \left(\bigcap_{j \in J} A_j, \left(\bigcap_{j \in J} A_j \right)' \right),$$

$$\bigvee_{j \in J} (A_j, B_j) = \left(\left(\bigcap_{j \in J} B_j \right)', \bigcap_{j \in J} B_j \right).$$

Conversely, if L is a complete lattice then $L \cong \underline{\mathfrak{B}}(G, M, I)$ if and only if there are mappings $\gamma : G \to L$ and $\mu : M \to L$ such that γG is supremum-dense in L, μM is infimum-dense in L, and gIm is equivalent to $\gamma g \leq \mu m$ for all $g \in G$ and $m \in M$; in particular, $L \cong \underline{\mathfrak{B}}(L, L, \leq)$.

For the complete lattice $L := \underline{\mathfrak{B}}(G, M, I)$ the mappings $\gamma : G \to L$ and $\mu : M \to L$ in Theorem 1 are naturally defined by

$$\gamma g := (\{g\}'', \{g\}') \quad \text{for} \quad g \in G,$$
$$\mu m := (\{m\}', \{m\}'') \quad \text{for} \quad m \in M.$$

An important problem is: How can one determine the concept lattice of a given context? One way to approach this problem is based on the idea to construct the concept lattice of a context by the concept lattices of some suitable subcontexts. Here a *subcontext* of a context (G, M, I) is understood as a triple $(H, N, I \cap (H \times N))$ with $H \subseteq G$ and $N \subseteq M$; we often write (H, N) instead of $(H, N, I \cap (H \times N))$. In Wille [11], it has been shown that for a partition $\{N_j \mid j \in J\}$ of M an isomorphism of the $\bigvee$-semilattice $\underline{\mathfrak{B}}(G, M, I)$ onto a subdirect product of the $\bigvee$-semilattices $\underline{\mathfrak{B}}(H, N_j, I \cap (G \times N_j))(j \in J)$ is given by $(A, B) \mapsto ((B \cap N_j)', B \cap N_j)_{j \in J}$. Since the construction methods are more powerful if they are based on subdirect products of complete lattices instead of subdirect products of $\vee$-semilattices, we analyse in the following subdirect decompositions of $\underline{\mathfrak{B}}(G, M, I)$ as a complete lattice. A main purpose is to obtain for contexts satisfying the "chain condition" a method for determining the scaffolding of $\underline{\mathfrak{B}}(G, M, I)$ directly from the context (G, M, I). Then the concept lattice can be constructed as an isomorphic copy of the ideal lattice of its scaffolding (see Wille [9]).

3. Complete congruence relations

Throughout this section (G, M, I) will be a context and (H, N) a subcontext of (G, M, I). (H, N) is said to be *compatible* if $(A' \cap N)' \cap H \subseteq A''$ for all $A \subseteq G$ and $(B' \cap H)' \cap N \subseteq B''$ for all $B \subseteq M$. By $\pi(H, N)(A, B) := (A \cap H, B \cap N)$ we define a map $\pi(H, N)$ from $\underline{\mathfrak{B}}(G, M, I)$ into $\mathcal{P}(H) \times \mathcal{P}(N)$ where, in general, $\mathcal{P}(S)$ is the complete lattice of all subsets of a set S.

PROPOSITION 2. *(H, N) is compatible if and only if $\pi(H, N)$ is a complete lattice homomorphism from $\underline{\mathfrak{B}}(G, M, I)$ onto $\underline{\mathfrak{B}}(H, N, I \cap (H \times N))$.*

Proof. Let (H, N) be compatible. If $(A, B) \in \mathfrak{B}(G, M, I)$ then $(A \cap H)' \cap N = (B' \cap H)' \cap N = B'' \cap N = B \cap N$ and $(B \cap N)' \cap H = (A' \cap N)' \cap H = A'' \cap H = A \cap H$ wherefore $(A \cap H, B \cap H) \in \mathfrak{B}(H, N, I \cap (H \times N))$. If $(C, D) \in \mathfrak{B}(H, N, I \cap (H \times N))$ then $C'' \cap H = (C' \cap N)' \cap H = D' \cap H = C$ and $C' \cap N = D$. Hence $\pi(H, N)$ is a surjective map from $\mathfrak{B}(G, M, I)$ onto $\mathfrak{B}(H, N, I \cap (H \times N))$. By Theorem 1, $\pi(H, N)$ preserves arbitrary infima and suprema. Conversely, let $\pi(H, N)$ be a complete lattice homomorphism from $\underline{\mathfrak{B}}(G, M, I)$ onto $\underline{\mathfrak{B}}(H, N, I \cap (H \times N))$. Then, in particular, $(A'' \cap H, A' \cap N) \in \mathfrak{B}(H, N, I \cap (H \times N))$ for $A \subseteq G$; hence $(A' \cap N)' \cap H = A'' \cap H \subseteq A''$. This shows together with the dual argument that (H, N) is compatible.

The kernel of $\pi(H, N)$ is denoted by $\underline{\Theta}(H, N)$, if (H, N) is compatible Proposition 2 yields that $\underline{\Theta}(H, N)$ is complete congruence relation of $\underline{\mathfrak{B}}(G, M, I)$ and that $\underline{\mathfrak{B}}(H, N, I \cap (H \times N)) \cong \underline{\mathfrak{B}}(G, M, I)/\underline{\Theta}(H, N)$ (we recall that an equivalence relation Θ on a complete lattice L is a *complete congruence relation* if $x_j \Theta y_j (j \in J)$ always imply

$$\left(\bigwedge_{j \in J} x_j\right) \Theta \left(\bigwedge_{j \in J} y_j\right) \quad \text{and} \quad \left(\bigvee_{j \in J} x_j\right) \Theta \left(\bigvee_{j \in J} y_j\right)\Big).$$

For the rest of this section we assume that the context (G, M, I) satisfies the following *chain condition*: there is no infinite descending chain $\{g_1\}'' \supset \{g_2\}'' \supset \{g_3\}'' \supset \cdots$ with $g_1, g_2, g_3, \ldots$ in G and there is no infinite ascending chain $\{m_1\}' \subset \{m_2\}' \subset \{m_3\}' \subset \cdots$ with $m_1, m_2, m_3, \ldots$ in M.

For a complete congruence relation Θ of $\underline{\mathfrak{B}}(G, M, I)$ we define

$$G(\Theta) := \{g \in G \mid \gamma g \text{ is the smallest element of a } \Theta\text{-}class\}$$
$$M(\Theta) := \{m \in M \mid \mu m \text{ is the greatest element of a } \Theta\text{-}class\}.$$

PROPOSITION 3. *If Θ is a complete congruence relation of $\underline{\mathfrak{B}}(G, M, I)$ then $(G(\Theta), M(\Theta))$ is a compatible subcontext of (G, M, I) and $\Theta = \underline{\Theta}(G(\Theta), M(\Theta))$.*

Proof. Suppose $g \in (A' \cap M(\Theta))' \cap G(\Theta)$ but $g \notin A''$ for some $A \subseteq G$. Then there is an $m \in A'$ such that $\{m\}'$ is maximal in $\{\{n\}' \mid n \in M \text{ and } (g, n) \notin I\}$. Since $\gamma g \leq \bigwedge \{\mu n \mid n \in M \text{ and } \mu m < \mu n\}$, μm is $\wedge$-irreducible and $\gamma g \vee \mu m$ covers μm. Hence $g \in G(\Theta)$ implies that μm is a greatest element of a Θ-class wherefore $m \in A' \cap M(\Theta)$. This contradicts $g \in (A' \cap M(\Theta))'$ and $(g, m) \notin I$. Hence $(A' \cap M(\Theta))' \cap G(\Theta) \subseteq A''$ for all $A \subseteq G$ and dually $(B' \cap G(\Theta))' \cap M(\Theta) \subseteq B''$ for all $B \subseteq M'$, i.e. $(G(\Theta), M(\Theta))$ is compatible. Let $(A, B) \in \mathfrak{B}(G, M, I)$, and let $(\underline{A}, \underline{B})$ and $(\bar{A}, \bar{B})$ be the smallest and the greatest concept in the Θ-class containing (A, B). Then $\gamma g \leq (A, B)$ for $g \in G(\Theta)$ implies $\gamma g \leq (A, B)$. Therefore $A \cap G(\Theta) = \underline{A} \cap G(\Theta)$ and dually $B \cap M(\Theta) = \bar{B} \cap M(\Theta)$. Hence $(A, B)\Theta(C, D)$ implies $A \cap G(\Theta) = C \cap G(\Theta)$ and $B \cap M(\Theta) = D \cap M(\Theta)$, i.e. $(A, B)\underline{\Theta}(G(\Theta), M(\Theta))(C, D)$. The converse implication follows from $(\underline{A}, \underline{B}) = \bigvee \gamma(A \cap G(\Theta))$ and $(\bar{A}, \bar{B}) = \bigwedge \mu(B \cap M(\Theta))$. This proves $\Theta = \underline{\Theta}(G(\Theta), M(\Theta))$.

For a characterization of the subcontexts $(G(\Theta), M(\Theta))$ we introduce the following notion: (H, N) is said to be *saturated* if $\{g\}' = (\{g\}'' \cap H)'$ implies $g \in H$ for all $g \in G$ and if $\{m\}' = (\{m\}'' \cap N)'$ implies $m \in N$ for all $m \in M$. Since $\bigvee \gamma A$ is the smallest element of a Θ-class for each $A \subseteq G(\Theta)$ and since $\bigwedge \mu B$ is the greatest element of a Θ-class for each $A \subseteq G(\Theta)$ and since $\bigwedge \mu B$ is the greatest element of a Θ-class for each $B \subseteq M(\Theta)$, the subcontext $(G(\Theta), M(\Theta))$ is saturated for all complete congruence relations Θ of $\underline{\mathfrak{B}}(G, M, I)$.

PROPOSITION 4. If (H, N) is saturated and compatible then $H = G(\underline{\Theta}(H, N))$ and $N = M(\underline{\Theta}(H, N))$.

Proof. For $h \in H$ and $(A, B) \in \mathfrak{B}(G, M, I)$, $\{h\}'' \cap H = A \cap H$ implies $\{h\}'' \subseteq A$; hence $\gamma h \in G(\underline{\Theta}(H, N))$. If γg is the smallest element of a $\underline{\Theta}(H, N)$-class for $g \in G$ then $\{g\}'' = (\{g\}'' \cap H)''$ wherefore $g \in H$ as (H, N) is saturated. This and the dual argument proves the assertion.

Proposition 3 and 4 yield a one-to-one correspondence between the complete congruence relations of $\underline{\mathfrak{B}}(G, M, I)$ and the saturated compatible subcontexts of (G, M, I). This correspondence is closely related to the duality elaborated in Urquhart [7].

4. Weak perspectivity

For the determination of subdirect decompositions of a concrete concept lattice it is commendable to reduce (if possible) the given context (G, M, I) to a minimal compatible subcontext (H, N) for which $\pi(H, N)$ is an isomorphism. Let us call a context *reduced* if $\pi(H, N)$ is not injective for each of its proper compatible subcontexts (H, N). Throughout this section we assume that (G, M, I) is a reduced context satisfying the chain condition (in Section 3). With Theorem 1 we conclude that γ is a bijective map from G onto the set of all $\bigvee$-irreducible elements of $\underline{\mathfrak{B}}(G, M, I)$ and μ is a bijective map from M onto the set of all $\bigwedge$-irreducible elements of $\underline{\mathfrak{B}}(G, M, I)$.

Congruence relations of lattices are successfully studied via the notions of weak perspectivity and weak projectivity (cf. Crawley and Dilworth [4], Grätzer [5]). These notions can be carried over to contexts. For $g \in G$ and $m \in M$, g is *weakly perspective* to m, in symbols $g \nearrow m$, if $\{m\}'$ is maximal in $\{\{n\}' \mid n \in M$ and $(g, n) \notin I\}$; dually, m is *weakly perspective* to g, in symbols $m \searrow g$, if $\{g\}''$ is minimal in $\{\{h\}'' \mid h \in G$ and $(h, m) \notin I\}$. If $g \nearrow m$ and $m \searrow g$, we call g and m *perspective* and write $g \sim m$ or $m \sim g$. In $G \cup M$ an element x is *weakly projective* to an element y, in symbols $x \approx_w y$, if $x = y$ or if there are elements $x = x_0, x_1, \ldots, x_k = y$ in $G \cup M$ such that x_{i-1} is weakly perspective to x_i for $i = 1, \ldots, k$. If $x \approx_w y$ and $y \approx_w x$, we call x and y *projective* and write $x \approx y$. For $X \subseteq G \cup M$ we define the *weakly projective closure* by $\langle X \rangle := \{y \in G \cup M \mid x \approx_w y$ for some $x \in X\}$.

PROPOSITION 5. *A subcontext* (H, N) *of* (G, M, I) *is compatible if and only if* $\langle H \cup N \rangle = H \cup N$.

Proof. Let (H, N) be compatible. Suppose $g \in H$, $g \nearrow m$, but $m \notin N$. Then $g \in (\{m\}'' \cap N)' \cap H = \{m\}''' \cap H = \{m\}' \cap H$ what contradicts $(g, m) \notin I$. Therefore $g \in H$ and $g \nearrow m$ imply $m \in N$ and dually $m \in N$ and $m \searrow g$ imply $g \in H$. Hence $\langle H \cup N \rangle = H \cup N$. Conversely, let us assume $\langle H \cup N \rangle = H \cup N$. Suppose $g \in (A' \cap N)' \cap H$ but $g \notin A''$ for some $A \subseteq G$. Then there is an $m \in A'$ such that $\{m\}'$ is maximal in $\{\{n\}' \mid n \in M$ and $(g, n) \notin I\}$. It follows that $g \nearrow m$ and hence $m \in A \cap N$ what contradicts $g \in (A' \cap N)'$ and $(g, m) \notin I$. Therefore $(A' \cap N)' \cap H \subseteq A''$ for all $A \subseteq G$ and dually $(B' \cap H)' \cap N \subseteq B''$ for all $B \subseteq M$, i.e. (H, N) is compatible.

COROLLARY. *The compatible subcontexts of* (G, M, I) *form a complete sublattice* $\mathfrak{D}(G, M, I)$ *of the complete lattice* $\mathcal{P}(G) \times \mathcal{P}(M)$.

THEOREM 6. Let (G, M, I) be a reduced context satisfying the chain condition. Then $\Theta \mapsto (G(\Theta), M(\Theta))$ describes an antiisomorphism from the complete lattice of all complete congruence relations of $\underline{\mathfrak{B}}(G, M, I)$ onto the complete sublattice of $\mathcal{P}(G) \times \mathcal{P}(M)$ consisting of all compatible subcontexts of (G, M, I).

Proof. For $g \in G$ and $H \subseteq G$, $\{g\}' = (\{g\}'' \cap H)'$ implies $\gamma g = \gamma(\{g\}'' \cap H)$ and hence $g \in H$ as γ is a bijection from G onto the set of all $\bigvee$-irreducible elements of $\underline{\mathfrak{B}}(G, M, I)$. This shows together with the dual argument that every subcontext of (G, M, I) is saturated. Therefore, the described mapping is a bijection from the set of all complete congruence relation of $\underline{\mathfrak{B}}(G, M, I)$ onto $\mathfrak{D}(G, M, I)$ by Proposition 3 and 4. The preceding corollary states that $\mathfrak{D}(G, M, I)$ is a complete sublattice of $\mathcal{P}(G) \times \mathcal{P}(M)$. Obviously, $\Theta_1 \subseteq \Theta_2$ is equivalent to $G(\Theta_1) \supseteq G(\Theta_2)$ and $M(\Theta_1) \supseteq M(\Theta_2)$ for complete congruence relations of $\underline{\mathfrak{B}}(G, M, I)$. Hence the described mapping is an antiisomorphism.

COROLLARY. *Let* $\leq_w$ *be the (partial) order induced by* $\approx_w$ *on* $G \cup M/\approx$. *Then* $(G \cup M/\approx, \leq_w)$ *is isomorphic to the ordered set of all* $\bigwedge$*-irreducible complete congruence relations of* $\underline{\mathfrak{B}}(G, M, I)$, *and* $\mathfrak{D}(G, M, I)$ *is isomorphic to the complete lattice of all order filters of* $(G \cup M/\approx, \leq_w)$.

The results of this section show how we may study complete congruence relations of $\underline{\mathfrak{B}}(G, M, I)$ via the digraph $(G \cup M, \nearrow \cup \searrow)$ which can be easily derived from the context (G, M, I). The digraph $(G \cup M, \nearrow \cup \searrow)$ and the ordered set $(G \cup M/\approx, \leq_w)$ are closely related to the double digraph considered in Urquhart [7]. The connection to the digraph $(J(L), C)$ in Jónsson, Nation [6] should also be mentioned.

5. Subdirect product constructions

In this section we elaborate for concept lattices the construction methods developed for subdirect products of complete lattices in Wille [9]. By Theorem 6, the subdirect decompositions of a reduced context satisfying the chain condition are in one-to-one correspondence to the families of compatible subcontexts of which the join is the whole context. In general, for compatible subcontexts $(H_j, N_j)(j \in J)$ of a context (G, M, I), $(A, B) \mapsto (A \cap H_j, B \cap N_j)_{j \in J}$ is an isomorphism from $\underline{\mathfrak{B}}(G, M, I)$ onto a subdirect product of the $\underline{\mathfrak{B}}(H_j, N_j, I \cap (H_j \times N_j))$ if and only if $\underline{\Theta}(\bigcup_{j\in J} H_j, \bigcup_{j\in J} N_j)$ is the identity on $\underline{\mathfrak{B}}(G, M, I)$; the subcontext $(\bigcup_{j\in J} H_j, \bigcup_{j\in J} N_j)$ is again compatible as $g \in (A' \cap \bigcup_{j\in J} N_j)' \cap \bigcup_{j\in J} H_j$ for $A \subseteq G$ implies $g \in (A' \cap N_k)' \cap H_k \subseteq A''$ for some $k \in J$ and hence $(A' \cap \bigcup_{j\in J} N_j)' \cap \bigcup_{j\in J} H_j \subseteq A''$.

Now, let (G, M, I) be an arbitrary context and let $(H_j, N_j)(j \in J)$ be compatible subcontexts of (G, M, I) such that $\underline{\Theta}(\bigcup_{j\in J} H_j, \bigcup_{j\in J} N_j)$ is the identity on $\underline{\mathfrak{B}}(G, M, I)$. We define a map

$$\alpha_{jk} : \underline{\mathfrak{B}}(H_k, N_k, I \cap (H_k \times N_k)) \to \underline{\mathfrak{B}}(H_j, N_j, I \cap (H_j \times N_j)) \quad (j, k \in J)$$

by

$$\alpha_{jk}(A_k, B_k) := (A_k'' \cap H_j, A_k' \cap N_j) \quad \text{for all} \quad (A_k, B_k) \in \mathfrak{B}(H_k, N_k, I \cap (H_k \times N_k)).$$

PROPOSITION 7. *α_{jk} is the greatest of the $\bigvee$-preserving maps*

$$\alpha : \underline{\mathfrak{B}}(H_k, N_k, I \cap (H_k \times N_k)) \to \underline{\mathfrak{B}}(H_j, N_j, I \cap (H_j \times N_j))$$

satisfying

$$\alpha(\{g\}'' \cap H_k, \{g\}' \cap N_k) \leq (\{g\}'' \cap H_j, \{g\}' \cap N_j) \text{ for all } g \in G.$$

Proof. By Proposition 2, α_{jk} is a map into $\mathfrak{B}(H_j, N_j, I \cap (H_j \times N_j))$. Using the general formulas $\bigcap_{t\in T} X_t' = (\bigcap_{t\in T} X_t')'' = (\bigcup_{t\in T} X_t)'$ and $(X'' \cap (X \cup Y))' = X'$ and the assumption that (H_k, N_k) is compatible, for $(A_t, B_t) \in \mathfrak{B}(H_k, N_k, I \cap (H_k \times N_k))$ $(t \in T)$ we obtain

$$\begin{aligned}\left(\left(\bigcap_{t\in T} B_t\right)' \cap H\right)' &= \left(\left(\bigcap_{t\in T} (A_t' \cap N)\right)' \cap H\right)' = \left(\left(\bigcap_{t\in T} A_t'\right)'' \cap N\right)' \cap H' \\ &= \left(\left(\bigcap_{t\in T} A_t'\right)' \cap H\right)' = \left(\left(\bigcup_{t\in T} A_t\right)'' \cap H\right)' = \left(\bigcup_{t\in T} A_t\right)' = \bigcap_{t\in T} A_t'.\end{aligned}$$

Therefore

$$\begin{aligned}\alpha_{jk} \bigvee_{t\in T} (A_t, B_t) &= \alpha_{jk}\left(\left(\bigcap_{t\in T} B_t\right)' \cap H, \bigcap_{t\in T} B_t\right) \\ &= \left(\left(\left(\bigcap_{t\in T} B_t\right)' \cap H\right)'' \cap H_j, \left(\left(\bigcap_{t\in T} B_t\right)' \cap H\right)' \cap N_j\right) \\ &= \left(\left(\bigcap_{t\in T} A_t'\right)' \cap H_j, \bigcap_{t\in T} A_t' \cap N_j\right) = \bigvee_{t\in T} \alpha_{jk}(A_t, B_t);\end{aligned}$$

hence α_{jk} is $\bigvee$-preserving. For an arbitrary $\bigvee$-preserving map α specified in Proposition 7 we have

$$\alpha(A_k, B_k) = \alpha \bigvee_{g \in A_k} (\{g\}'' \cap H_k, \{g\}' \cap N_k)$$

$$= \bigvee_{g \in A_k} \alpha(\{g\}'' \cap H_k, \{g\}' \cap N_k) \leq \bigvee_{g \in A_k} (\{g\}'' \cap H_j, \{g\}' \cap N_j)$$

$$= ((A_k' \cap N_j)', A_k' \cap N_j) = \alpha_{jk}(A_k, B_k);$$

hence $\alpha \leq \alpha_{jk}$.

Now, we are ready to apply Konstruktion I in Wille [9]. This leads to the following theorem.

THEOREM 8. *Let (G, M, I) be a context and let $(H_j, N_j)(j \in J)$ be compatible subcontexts of (G, M, I) such that $\underline{\Theta}(\bigcup_{j \in J} H_j, \bigcup_{j \in J} N_j)$ is the identity on $\underline{\mathfrak{B}}(G, M, I)$. Then $(A, B) \mapsto (A \cap H_j, B \cap N_j)_{j \in J}$ describes an isomorphism from $\underline{\mathfrak{B}}(G, M, I)$ onto a subdirect product of the $\underline{\mathfrak{B}}(H_j, N_j, I \cap (H_j \times N_j))(j \in J)$ which has $G(\alpha_{jk} \mid j, k \in J) := \{\alpha_{jk}(A_k, B_k)_{j \in J} \mid k \in J$ and $(A_k, B_k) \in \mathfrak{B}(H_k, N_k, I \cap (H_k \times N_k)) \backslash \{(N_k' \cap H_k, N_k)\}\}$ as a supremum-dense subset.*

If $G(\alpha_{jk} \mid j, k \in J)$ is considered as a partial $\bigvee$-semilattice (induced by the join in the direct product) then $\underline{\mathfrak{B}}(G, M, I)$ is isomorphic to the complete lattice of all complete ideals of $G(\alpha_{jk} \mid j, k \in J)$ (we recall that a subset A of a partial $\bigvee$-semilattice is a *complete ideal* if $x \leq a$ and $a \in A$ imply $x \in A$ and if $X \subseteq A$ implies $\bigvee X \in A$ whenever $\bigvee X$ exists). An isomorphic copy of the partial $\bigvee$-semilattice $G(\alpha_{jk} \mid j, k \in J)$ is its inverse image in $\underline{\mathfrak{B}}(G, M, I)$ under the isomorphism of Theorem 8 which can be described by

$$G((H_j, N_j) \mid j \in J) :=$$
$$\{\gamma A_j \mid j \in J \text{ and } (A_j, B_j) \in \mathfrak{B}(H_j, N_j, I \cap (H_j \times N_j)) \backslash \{(N_j' \cap H_j', N_j)\}\}$$

COROLLARY. *$\underline{\mathfrak{B}}(G, M, I)$ is isomorphic to the complete lattice of all complete ideals of $G((H_j, N_j) \mid j \in J)$.*

6. Scaffoldings

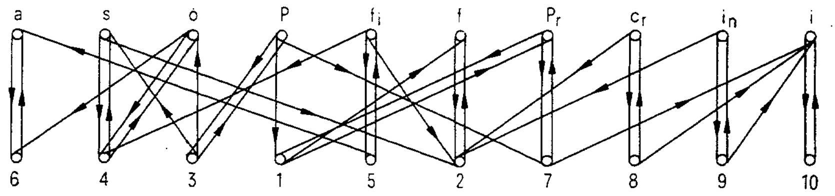
The construction of the concept lattice $\underline{\mathfrak{B}}(G, M, I)$ via $G((H_j, N_j) \mid j \in J)$ or $G(\alpha_{jk} \mid j, k \in J)$ is usually more economical if the compatible subcontexts $(H_j, N_j)(j \in J)$ are smaller. The extreme case is present if all $\underline{\Theta}(H_j, N_j)(j \in J)$ are $\bigwedge$-irreducible. If (G, M, I) is a reduced context satisfying the chain condition then, by Theorem 6, $\underline{\Theta}(H, N)$ is $\bigwedge$-irreducible for a compatible subcontext (H, N) of (G, M, I) if and only if $H \cup N = \langle\{g\}\rangle$ for some $g \in G$ (notice that for every $m \in M$ there is a $g \in G$ with $g \sim m$ and dually). This together with Theorem 8 yields the following theorem.

THEOREM 9. *Let* (G, M, I) *be a reduced context satisfying the chain condition. Then* $(A, B) \mapsto (A \cap \langle\bar{g}\rangle, B \cap \langle\bar{g}\rangle)_{\bar{g} \in G/\approx}$ *describes an isomorphism from* $\underline{\mathfrak{B}}(G, M, I)$ *onto a subdirect product of the completely subdirect irreducible concept lattices* $\underline{\mathfrak{B}}(\langle\bar{g}\rangle \cap G, \langle\bar{g}\rangle \cap M, I \cap \langle\bar{g}\rangle^2)$ $(\bar{g} \in G/\approx)$ *which has* $G(\alpha_{\bar{g}\bar{h}} \mid \bar{g}, \bar{h} \in G/\approx)$ *as a supremum-dense subset.*

The partial $\bigvee$-semilattice $G((\langle\bar{g}\rangle \cap G, \langle\bar{g}\rangle \cap M \mid \bar{g} \in G/\approx)$ isomorphic to $G(\alpha_{\bar{g}\bar{h}} \mid \bar{g}, \bar{h} \in G/\approx)$ is called the *scaffolding* of $\underline{\mathfrak{B}}(G, M, I)$ (see Wille [9], [10]) and denoted by $\mathscr{G}(G, M, I)$. We recall that $\underline{\mathfrak{B}}(G, M, I)$ is isomorphic to the complete lattice of all complete ideals of its scaffolding $\mathscr{G}(G, M, I)$. How a concept lattice may be constructed via its scaffolding shall be demonstrated by an example. We choose the following reduced finite context which occurs in the analysis of homomorphisms of partial algebras (see Burmeister and Wojdyło [3]).

	i	a	o	f	s	f_i	c_r	i_n	P_r	P
1	×	×	×							
2	×	×	×						×	×
3	×	×		×			×	×	×	
4	×	×		×			×	×	×	×
5	×			×	×		×	×	×	×
6	×			×	×	×	×	×	×	×
7		×	×	×	×	×				
8		×	×	×	×	×			×	×
9		×	×	×	×	×	×		×	×
10		×	×	×	×	×	×	×	×	×

The set $G := \{1, 2, \ldots 10\}$ of objects consists of names for concrete homomorphisms. The attributes in the set $M := \{i, a, o, f, s, f_i, c_r, i_n, P_r, P\}$ are explained by i: injective, a: almost onto, o: onto, f: full, s: strong, f_i: final, c_r: relatively closed, i_n: initial, P_r: relatively P-closed, P: P-closed. The crosses of the table indicate the relation I. First we determine the digraph $(G \cup M, \nearrow \cup \searrow)$.



The scaffolding of $\mathfrak{B}(G, M, I)$ is already determined by the maximal subcontexts of the form $(\langle\{g\}\rangle \cap G, \langle\{g\}\rangle \cap M)$ with $g \in G$; these subcontexts are given by the following tables (the generator is encircled).

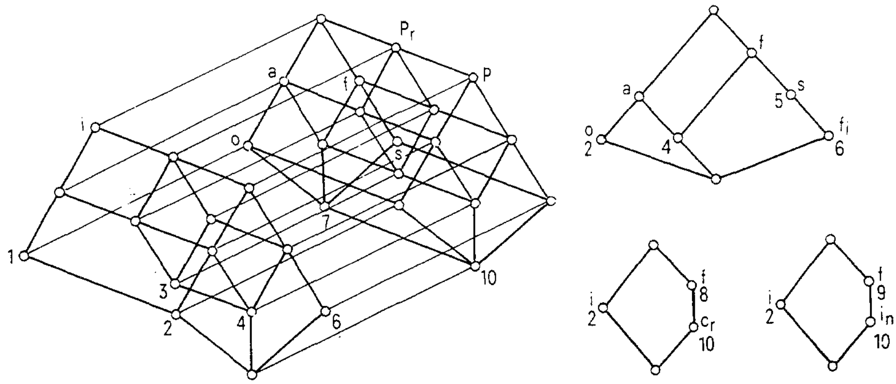
	i	a	o	f	s	P_r	P
1	×	×	×				
2	×	×	×			×	×
③	×	×		×		×	
4	×	×		×		×	×
6	×			×	×	×	×
7		×	×	×	×		
10		×	×	×	×	×	×

	a	o	f	s	f_i
2	×	×			
4	×		×		
⑤			×	×	
6			×	×	×

	i	f	c_r
2	×		
⑧		×	
10		×	×

	i	f	i_n
2	×		
⑨		×	
10		×	×

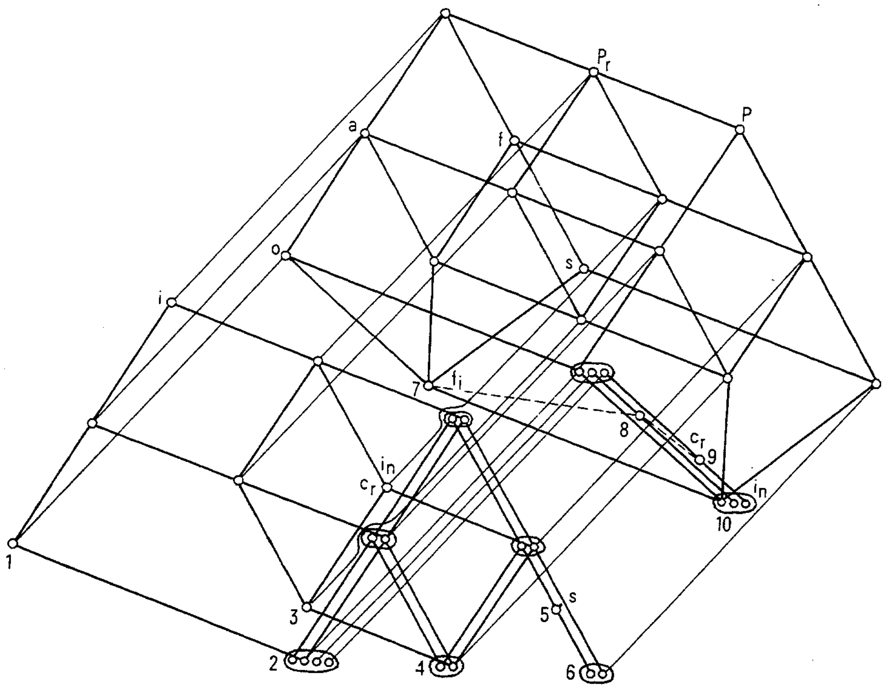
The concept lattices of the four subcontexts are described by the following Hasse diagrams. For the objects g (attributes m) labels indicate the smallest (greatest) concept of which the extent (intent) contains g (m).



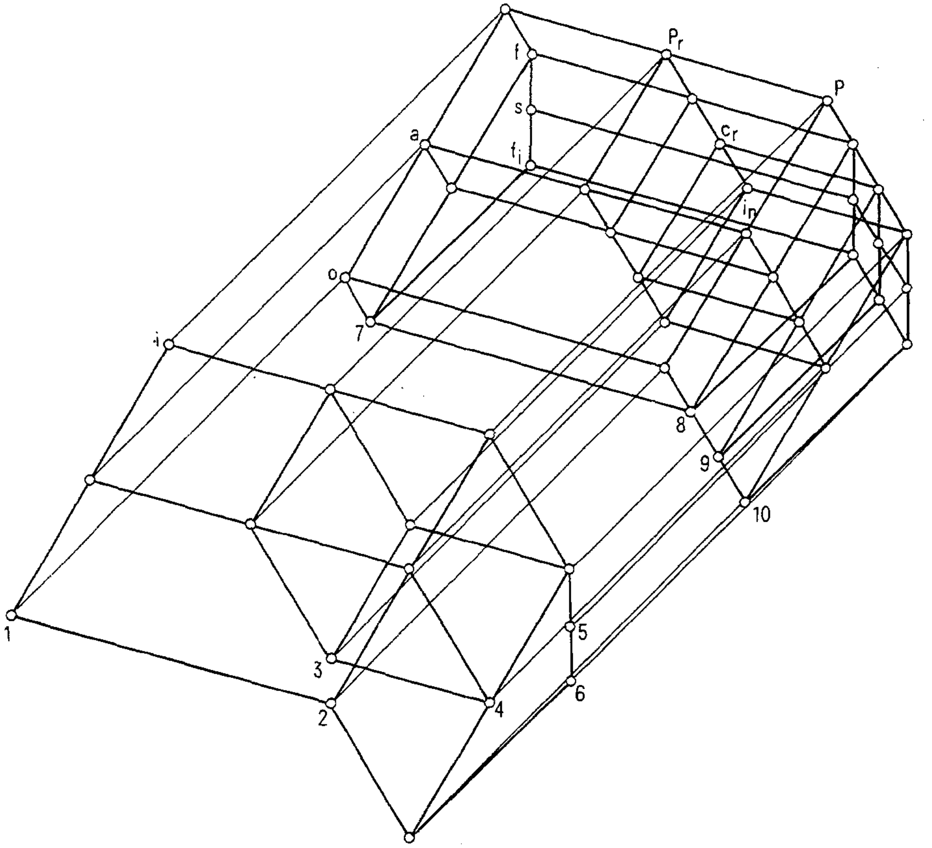
The $\bigvee$-preserving maps $\alpha_{\bar{g}\bar{h}}$ ($g, h \in \{3, 5, 8, 9\}$) are fixed by their images of the $\bigvee$-irreducible elements. Therefore, because of $\alpha_{\bar{g}\bar{h}}(\{k\}'' \cap \langle\bar{h}\rangle, \{k\}' \cap \langle\bar{h}\rangle) = \pi(\langle\bar{g}\rangle \cap G, \langle\bar{g}\rangle \cap M)(\{k\}'', \{k\}')$ for all $k \in \langle\bar{h}\rangle \cap G$, we can read the $\alpha_{\bar{g}\bar{h}}$ from the following table which describes the maps $\pi(\langle\bar{g}\rangle \cap G, \langle\bar{g}\rangle \cap M)$ restricted to γG.

G	1	2	3	4	5	6	7	8	9	10
$\bar{3}$	1	2	3	4	6	6	7	10	10	10
$\bar{5}$	2	2	4	4	5	6	0	0	0	0
$\bar{8}$	2	2	0	0	0	0	8	8	10	10
$\bar{9}$	2	2	0	0	0	0	9	9	10	10

By Theorem 9, an isomorphic copy of $\underline{\mathfrak{B}}(G, M, I)$ can be obtained by forming all joins in $\Pi(\mathscr{L}(\langle\bar{g}\rangle \cap G, \langle\bar{g}\rangle \cap M, I \cap \langle\bar{g}\rangle^2) \mid g \in \{3, 5, 8, 9\})$ of the ten elements described by the columns of the table. For another construction of $\underline{\mathfrak{B}}(G, M, I)$ we first determine the scaffolding $\mathscr{G}(G, M, I)$. For this we draw the disjoint union of the Hasse Diagrams above without the least elements. The drawing has to be completed to a diagram of the quasi-order Q defined by $(A_{\bar{g}}, B_{\bar{g}}) Q (A_{\bar{h}}, B_{\bar{h}}) :\Leftrightarrow \alpha_{\bar{g}\bar{h}}(A_{\bar{h}}, B_{\bar{h}}) \geq (A_{\bar{g}}, B_{\bar{g}})$; the equivalence relation $Q \cap Q^{-1}$ may be indicated by encircling. The resulting diagram describes the scaffolding $\mathscr{G}(G, M, I)$ because $(\bigvee \gamma A_1) \vee (\bigvee \gamma A_2)$ is in $\mathscr{G}(G, M, I)$ for $(A_i, B_i) \in \mathfrak{B}(\langle\bar{h}_i\rangle \cap G, \langle\bar{h}_i\rangle \cap M, I \cap \langle\bar{h}_i\rangle^2)$ $(i = 1, 2)$ if and only if there is a $g \in \{3, 5, 8, 9\}$ such that $(A_i, B_i) Q \alpha_{\bar{g}\bar{h}_1}(A_1, B_1) \vee \alpha_{\bar{g}\bar{h}2}(A_2, B_2)$ for $i = 1, 2$ (cf. Wille [10]. Construction II).



Forming the lattice of all (complete) ideals of $\mathscr{G}(G, M, I)$ leads to the following Hasse diagram of $\underline{\mathfrak{B}}(G, M, I)$.



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