Along Paths Inspired by Ludwig Streit: Stochastic Equations for Quantum Fields and Related Systems

Sergio Albeverio

Abstract The interaction between quantum mechanics, quantum field theory, stochastic partial differential equations and infinite dimensional analysis is briefly surveyed, referring in particular to models and techniques to which L. Streit has given outstanding contributions.

Keywords Functional integration • Feynman path integrals • Quantum field theory • Stochastic quantization • Stochastic partial differential equations • Infinite dimensional integrals

1 Introduction

It is a great pleasure to present a contribution to this volume dedicated to Ludwig Streit, on the occasion of his 75th birthday. I have written before in [3] about Ludwig's work and the strong influence it had on my own development. In the present paper I will concentrate on some aspects relating quantum theory with stochastic analysis and infinite dimensional analysis, stressing interactions between these areas and mentioning some open problems.

In Sect. 2 I shall briefly describe the relations between oscillatory and probabilistic integrals.

In Sect. 3 I shall describe canonical quantum mechanics from this point of view.

In Sect. 4 I shall discuss the complex relations between canonical quantum fields and S(P)DEs, and also relate them with Euclidean (and relativistic) quantum fields.

In Sect. 5 I shall provide some remarks on related systems.

S. Albeverio (✉)
Institut für Angewandte Mathematik, Universität Bonn, Endenicher Allee 60, 53115, Bonn, Germany

HCM, Bonn, Germany

BIBOS, Bielefeld, Bonn, Germany

IZKS, Bonn, Germany

CERFIM, Locarno, Switzerland
e-mail: albeverio@iam.uni-bonn.de

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2 Oscillatory and Probabilistic Integrals

Since the work of Feynman (preceded in part by Wentzell and Dirac), see, e.g., [2, 107–110], a very fruitful approach to quantum mechanics and quantum field theory is based on the consideration of heuristic complex-valued measures of the form

$$\mu_F(\mathrm{d}\gamma) = \text{“}Z^{-1} e^{iS(\gamma)}\, \mathrm{d}\gamma\text{”} \tag{1}$$

on a “state space” Γ, hence $\gamma \in \Gamma$, with an “action functional” $S(\gamma) : \Gamma \to \mathbb{R}$ associated with an underlying “classical system”, $\mathrm{d}\gamma$ a “flat measure on Γ”, i the imaginary unit and Z a normalization constant.

Physical quantities of interest are then computed as “averages” (linear functionals, integrals)

$$\int_\Gamma f(\gamma)\mu_F(\mathrm{d}\gamma) \tag{2}$$

of suitable “observable functionals” $f : \Gamma \to \mathbb{C}$.

For example, to solve the Schrödinger equation for a non relativistic quantum mechanical particle moving for a time from the interval $[0, t]$, $t > 0$ on a D-dimensional manifold M one takes as Γ a space of paths from $[0, t]$ into M.

For a quantum (scalar) field, vibrating for a time from the interval $[0, t]$ and in a spatial $(d - 1)$-dimensional domain $\Lambda \subset \mathbb{R}^{d-1}$ (d being then the space-time-dimension) one takes Γ as a space of paths from $[0, t]$ into a space of real-valued (generalized) functions depending on a space variable $x \in \Lambda$.

Analogous choices are made, e.g., for quantum gauge fields (with Γ a space of connection 1-forms in a principal fibre bundle over a Lorentzian manifold) and for quantum gravity (with Γ a space of locally Lorentzian space-time metrics).

In the case of scalar quantum fields, f could be a product of $n \in \mathbb{N}$ field operators $\prod_{i=1}^n \gamma(t_i, x_i)$, $t_i \in \mathbb{R}$, $x_i \in \mathbb{R}^{d-1}$, in which case above integrals would yield “correlation functions” (Wightman functions) associated with the quantum field $\gamma(t, x) \in \mathbb{R}$, $t \in \mathbb{R}$, $x \in \mathbb{R}^{d-1}$. Similar choices of relevant observable functions can be given in the other cases alluded above. See, e.g., [1, 2, 9, 33, 48, 52, 78, 80, 107, 109, 110], and references therein.

The choice of the action functional S depends on the problem at hand. In above examples, S has a typical Lagrangian form, e.g., in the case of a non relativistic particle (of unit mass) moving during the time interval $[0, t]$ in the D-dimensional Euclidean space $\mathbb{R}^D$ under the force given by a (continuous) potential (function) $v : \mathbb{R}^D \to \mathbb{R}$ one has:

$$S(\gamma) = S_{QM}(\gamma) = \frac{1}{2}\int_0^t |\dot{\gamma}(s)|^2 \,\mathrm{d}s - \int_0^t v(\gamma(s))\,\mathrm{d}s, \tag{3}$$

and for scalar quantum fields moving at arbitrary times $s \in \mathbb{R}$ over the space $\mathbb{R}^{d-1}$ one has

$$S(\gamma) = S_{QF}(\gamma) = \frac{1}{2}\int_{\mathbb{R}^d} \dot{\gamma}(s,\vec{x})^2 \,\mathrm{d}s\,\mathrm{d}\vec{x} - \frac{1}{2}\int_{\mathbb{R}^d} \left|\nabla\gamma(s,\vec{x})\right|^2 \,\mathrm{d}s\,\mathrm{d}\vec{x} \\ - \frac{m^2}{2}\int_{\mathbb{R}^d} \left|\gamma(s,\vec{x})\right|^2 \,\mathrm{d}s\,\mathrm{d}\vec{x} - \int_{\mathbb{R}^d} v(\gamma(s,\vec{x}))\,\mathrm{d}s\,\mathrm{d}\vec{x}, \tag{4}$$

$(s,\vec{x}) \in \mathbb{R}\times\mathbb{R}^{d-1}$, $m \geq 0$.

From the quantum description given in terms of (1), (2), (3), and (4) one obtains heuristically the classical (particle resp. field) mechanical one by inserting the (reduced) Planck constant $\hbar$ at the right place, i.e. replacing S by $\frac{1}{\hbar}S$ and making corresponding changes in the observable f in (2), and considering $\hbar$ to be small.

Heuristically then from (2) with such replacements one should be able to derive from Feynman's heuristic quantum description (1) and (2) the corresponding classical description in the limit $\hbar \downarrow 0$, with its corresponding "small $\hbar$-corrections". In the case of relativistic models, the velocity of light c appears as a parameter. Then one should be able to recover the non relativistic limit of the relativistic quantum description (1), (2), (3), and (4) by letting the velocity of light c go to $+\infty$ (c occurs implicitly in S and f). In principle one could proceed in similar ways to describe the limiting dependence of the quantum averages (2) on other parameters occurring in the action functionals and observables occurring in the examples mentioned above.

There are well known difficulties in trying to transform this heuristic approach into a rigorous one. E.g., even in the case $v \equiv 0$ (the "free field" case), in the computation of (2), the covariance of the "complex Gaussian measure" given by the covariance $S_{QF}(\gamma)|_{v=0}$ appears. It is, for $m^2 = 1$, the fundamental solution $(\Box - 1)^{-1}(s,\vec{x};s';.\vec{x}')$, which has a singularity on the submanifold $(s-s')^2-(\vec{x}-\vec{x}')^2 = 1$, indicating that it should be interpreted as a generalized function, which then means that the relevant γ's in (2) also are generalized functions. This creates difficulties in giving mathematical meaning to the complex-valued non linear function $f(\gamma)$ appearing in (2). In the linear case where $v \equiv 0$, taking e.g. f to be of the Wick product type, a sense can easily be given to (2). However, for $v \not\equiv 0$ and neither linear nor quadratic, $v(\gamma)$ is ill defined for γ which are typical relative to the complex Gaussian measure μ_F for the case $v \equiv 0$.

There is a well known trick (going back to Nagumo, Schwinger, and Symanzik) of analytically continuing μ_F to a corresponding "Euclidean" probability measure (or functional) μ_E heuristically written as

$$\mu_E(\mathrm{d}\gamma) = \text{“}Z^{-1}e^{-S_E(\gamma)}\mathrm{d}\gamma\text{”} \tag{5}$$

with $S_E(\gamma)$ the "Euclidean" action (in contrast to the above relativistic or "Minkowskian" action (4))

$$S_E(\gamma) = \frac{1}{2} \iint \left[\left|\nabla_{s,\vec{x}}\gamma(s,\vec{x})\right|^2 + \frac{m^2}{2} \iint \left|\gamma(s,\vec{x})\right|^2 \right] \mathrm{d}s\,\mathrm{d}\vec{x} + \iint v(\gamma(s,\vec{x}))\,\mathrm{d}s\,\mathrm{d}\vec{x}, \tag{6}$$

in which case the typical γ's in (5) become still heuristically generalized random fields but at least μ_E would look as a "positive measure". This however, only really solves the problem for $v \equiv 0$ and arbitrary d, or special $v \not\equiv 0$ when $d = 2, 3$. In this case one interprets μ_E as a probabilistic (σ-additive) Gaussian measure, e.g., with support on the Schwartz tempered distributions space $S'(\mathbb{R}^d)$. For the mentioned special cases of $v \not\equiv 0$ and $d = 2, 3$, one manages, in fact, to "renormalize" the non-sensical $v(\gamma(s,\vec{x}))$ to give it a sense as a genuine generalized random field, obtaining non trivial quantum averages (2) (heuristically $\mu_F \neq \mu_F|_{v=0}$, that is, in particular, states which are typically with respect to one of the measures are atypical with respect to the other measure). Moreover from such averages one recovers by "analytic continuation" in the t variable a solution of the original problem (1) and (2).

This was achieved in connection with the programme of constructive quantum field theory [9, 71, 76, 78, 98, 106], to which Ludwig Streit contributed substantially and in various ways, see, e.g., [16–18, 65, 78, 79, 81, 101, 102, 107]. Independently of the proper methods of constructive quantum field theory, the original programme of giving directly a meaning to (1) and (2) has also been solved in some cases. E.g., μ_F and μ_E can be defined for various classes of (regularized) v in low space-time dimension d. The integral $\int f(\gamma)\,\mathrm{d}\mu_F(\gamma)$ is then realized as

$$_{(S)}\langle\langle f, \Phi_{\mu_F}\rangle\rangle_{(S)'} \tag{7}$$

in the framework of the infinite dimensional distributional setting

$$(S) \subset L^2(\mu_G) \subset (S)', \tag{8}$$

with pairing $\langle\langle\ ,\ \rangle\rangle$ and (S), $(S)'$ infinite dimensional analogues of the test functions space $S(\mathbb{R}^d)$ resp. tempered distributions space $S'(\mathbb{R}^d)$. μ_G is the Gaussian white noise measure on $\mathcal{S}'(\mathbb{R}^d)$, defined by its Fourier transform

$$\hat{\mu}_G(g) := \int_{\mathcal{S}'(\mathbb{R}^d)} e^{i\langle g,\gamma\rangle}\,\mathrm{d}\mu_G(\gamma) = e^{-\frac{1}{2}\|g\|^2_{L^2(\mathbb{R}^d)}}, \tag{9}$$

with $\langle g, \gamma\rangle$ the distributional pairing between $g \in \mathcal{S}(\mathbb{R}^d)$ and $\gamma \in \mathcal{S}'(\mathbb{R}^d)$.

Heuristically thus

$$\mu_G(\mathrm{d}\gamma) = \text{“}Z^{-1} e^{-\frac{1}{2}\int_{\mathbb{R}^d} |\gamma(x)|^2\,\mathrm{d}x}\,\mathrm{d}\gamma\text{”}, \tag{10}$$

and, e.g.,

$$\Phi_{\mu_F}(\gamma) = e^{iS(\gamma)} e^{\frac{1}{2}\int_{\mathbb{R}^d} |\gamma(x)|^2 \, \mathrm{d}x} \mu_G(\mathrm{d}\gamma). \tag{11}$$

For $d = 1$, i.e. for quantum mechanics (anharmonic oscillator for $m > 0$), the functional Φ_{μ_F} in (7) and (11) is well defined for a large class of continuous v and works for solving e.g. the problem of the Schrödinger equation with potential v and with smooth resp. δ-initial conditions (Green's function) by work by L. Streit and coworkers, see e.g. [74, 75] and, for a related approach [94].

Another approach to the rigorous construction of (1) and (2) (for $d = 1$, and also for $d \geq 2$, but the latter with regularized $v(\gamma)$-term in (4) resp. (6)) is presented in [26].

In the latter reference also work on the detailed expansion of (2) in fractional powers of $\hbar$ around $\hbar = 0$ by a rigorous method of stationary phase is described (for the case $d = 1$).

The relations between the white noise and other approaches are discussed, e.g., in [26, 39, 93, 94].

The analogues of Wightman functions in the Euclidean setting, called Schwinger functions (they are the moment functions to μ_E), for $d = 1$, i.e. related to quantum mechanics, have also been described in terms of white noise functionals, see [74, 75, 94]. Corresponding results for the case $v \equiv 0$ and all d, or for some $v \not\equiv 0$ and $d = 2, 3$ will be discussed in the next Sects. 3 and 4.

For characterization of interesting probability measures as white noise functionals (like μ_E and related measures) see, e.g., [88, 101, 102].

3 Canonical Quantum Mechanics

In the canonical approach to quantum mechanics (and resp. the theory of quantum fields) one takes the "Schrödinger representation" of the Hilbert space of states and operators acting in $L^2(\mathbb{R}^D, \mu)$ resp. $L^2(S'(\mathbb{R}^d), \mu)$, for some probability measures μ.

By work of Heisenberg, Coester, Haag, Araki from the early 1960s one uses, e.g., for the D-dimensional harmonic oscillators, that $L^2(\mathbb{R}^D)$ is isomorphic to $L^2(\mathbb{R}^D, \mu)$, where $\mu(\mathrm{d}x) = N(0; \mathbb{1})(\mathrm{d}x) = (\sqrt{2\pi})^{-D} e^{-\frac{1}{2}|x|^2} \, \mathrm{d}x$ is the canonical Gaussian measure on $\mathbb{R}^D$. E.g., for $D = 1$ the Hamiltonian H given on smooth functions by $H = -\frac{1}{2}\Delta + \frac{1}{2}x^2 - \frac{1}{2}$ acting in $L^2(\mathbb{R})$ is then unitarily equivalent to the operator $H_\mu = -L_\mu$ given on smooth functions by $H_\mu = -\frac{1}{2}\Delta + x \cdot \nabla$ in $L^2(\mathbb{R}, \mu)$.

L_μ can be interpreted probabilistically as the restriction to smooth functions of the Ornstein-Uhlenbeck operator associated with the classical Dirichlet form $\frac{1}{2}\int \nabla u \cdot \nabla v \, \mathrm{d}\mu$, for u, v in a suitable dense subset of $L^2(\mathbb{R}, \mu)$. For classical Dirichlet forms (on $\mathbb{R}^D$ and also on infinite dimensional spaces) see, e.g., [4, 7, 8, 10, 28, 34, 37, 49, 57, 59, 66, 69, 72, 92]. Through the replacement of $N(0; \mathbb{1})$ by a more general, not necessarily Gaussian probability measure μ, one can set up a corresponding transformation for the generator L_μ (called Dirichlet operator) associated with the classical Dirichlet form $\frac{1}{2}\int_{\mathbb{R}^D} \nabla u \cdot \nabla v \, d\mu$ in $L^2(\mathrm{R}^D, \mu)$ to a generator $-H$, $H \geq 0$ in $L^2(\mathbb{R}^D)$.

This "ground state transformation" was first studied in details in [27], where a general and powerful approach to singular interactions in quantum mechanics on $\mathbb{R}^D$ via Dirichlet forms has been initiated. E.g., 3 nucleons interacting via 2-particle δ-functions in $\mathbb{R}^3$ can be rigorously defined and discussed in this approach. This Dirichlet form approach yields, more generally, a unified picture of quantum mechanics suited both for singular interactions and infinite dimensional extensions (the analogues of above classical Dirichlet forms being naturally also defined on infinite dimensional spaces; the latter forms describe namely quantum fields, see Sect. 4).

Remark 3.1 The probabilistic technique of subordination permits to treat quantum systems with relativistic kinematics in a way similar to those with nonrelativistic kinematics, see [6, 44–46, 50].

To general classical Dirichlet forms, there are (properly) associated good Markov processes $X = (X_t)$, $t \geq 0$, see [69]. Intuitively these can be seen as Brownian motions distorted by a drift term given by the logarithmic derivative of μ, μ itself constitutes then an invariant measure for X_t. In particular X_t can be extended to a μ-symmetric process for all $t \in \mathbb{R}$.

In the quantum mechanical situation corresponding to (4) with $d = 1$, $n = 1$, and $D = 1$, the path space measure μ_E for X_t, $t \in \mathbb{R}$ (giving the joint distribution of $(X_{t_1}, \ldots, X_{t_n})$, $t_i \in \mathbb{R}$, $i = 1, \ldots, n$) is heuristically given by μ_E as specified in (5) and (6) with a suitable v. If we write $S^0(\gamma) := \frac{1}{2}\int \left[|\dot{\gamma}|^2(s) + |\gamma(s)|^2 \right] ds$, then

$$\mu_E(d\gamma) = Z^{-1} e^{-S^0(\gamma)} e^{-\int v(\gamma(s))\, ds} \, d\gamma. \tag{12}$$

This can be rigorously defined for suitable v, e.g. as a probability measure on $\mathcal{S}'(\mathbb{R}^d)$. An analytic continuation in time of the (time ordered) moment functions given by μ_E ("Schwinger functions") permits to go over to the corresponding quantities in terms of μ_F ("Wightman functions"), e.g. for v resp. a polynomial or exponential function. This yields the full information on the dynamical quantum mechanical "$P(\varphi)_1$" resp. "$\exp(\varphi)_1$"-"models". Such models can also be expressed conveniently in terms of the white noise functionals mentioned in Section 2 ([74]).

One proves that both quantum mechanical Schwinger and Wightman functions can be reformulated as limits of quantities defined in the canonical Hilbert space $L^2(\mathbb{R}, N(0,1))$, since for $v = 0$ the restriction of μ_E to the σ-algebra generated by X_0 is just $\mu = N(0,1)$ (X_t acts just in the same L^2-space $L^2(\mathbb{R}, N(0,1))$ where also H_μ and the CCR are realized) and the inclusion of the v-term can be handled by "perturbation and passage to the limit", see, e.g., [22, 23, 27, 55, 111].

Let us summarize the situation in quantum mechanics for a particle moving in $\mathbb{R}^D$. As a zero step ("Level 0") we have a (fixed time) Hilbert space, say $L^2(\mathbb{R}^D)$, an Hamiltonian H, as well as position and momentum operators, with the relative canonical commutation relations (CCR), and a fixed time distribution μ on $\mathbb{R}^D$ (that exists if the lower end of the spectrum of H is an isolated simple eigenvalue) which yields the natural Hilbert space $L^2(\mathbb{R}^D, \mu)$. At a first step ("Level 1") one has a Markov process X_t (diffusion) with values in $\mathbb{R}^D$, t being the time parameter, its path space measure μ_E (on, say, $\mathcal{S}'(\mathbb{R}^D)$), and its moments functions (Schwinger functions). This process has μ as invariant measure. In the case $v \equiv 0$, $m = 1$, the path measure is $\mu_E = N\left(0; C^{-1}\right)$, with C the operator $\left(-\frac{d^2}{dt^2} + 1\right) 1\!\!1_D$ (on $\mathcal{S}'(\mathbb{R}, \mathbb{R}^D)$), X_t is the $\mathbb{R}^D$-valued Ornstein-Uhlenbeck process satisfying $\mathrm{d}X_t = -X_t\,\mathrm{d}t + \mathrm{d}W_t$, with W_t a D-dimensional Brownian motion. It is associated with the classical Dirichlet form $\frac{1}{2}\int_{\mathbb{R}^D} \nabla u \cdot \nabla v\, \mathrm{d}\mu$. For $D = 1$ this coincides with the measure μ_E given by (5) and (6), for $d = 1$ and $m = 1$. For $v \equiv 0$ one has $\mu_E = N(0; (-\Delta + 1)^{-1})$ with Δ the Laplacian on $\mathbb{R}$.

At "Level 2" one has another Markov process (diffusion) Y_τ, $\tau \geq 0$. For $D = 1$ this has μ_E (as just described at level 1 given by the $D = 1$, $m = 1$ version of (5) and (6)), as invariant measure. The Markov process is given by the stationary solution of the SDE $\mathrm{d}Y_\tau(t) = \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} - 1\right) Y_\tau(t)\,\mathrm{d}\tau - v'(Y_\tau(t))\,\mathrm{d}\tau + \mathrm{d}W_\tau(t)$, $\tau \geq 0$, with $\mathrm{d}W_\tau(t)$ a (τ, t)-Gaussian white noise, $\tau \in \mathbb{R}_+$, $t \in \mathbb{R}$.

τ can be interpreted as a "computer time" (t is the "time" in the Euclidean version of quantum mechanics; it will be replaced by a space-time variable in $\mathbb{R}^d$, $d \geq 2$, in a quantum field interpretation, see Sect. 4, the quantum (particle) mechanics corresponding thus to the case $d = 1$. The equation for Y_τ is called "stochastic quantization equation". It is a particular case of Parisi-Wu's approach [100] for computing invariant measures of the type of μ_E.

Note that Y_τ can be looked upon (as a modification of) the standard diffusion process associated with the classical Dirichlet form $\frac{1}{2}\int_{S'(\mathbb{R}^D)} \nabla u \nabla v\, \mathrm{d}\mu_E = \frac{1}{2}\, {}_{(S)}\langle\langle \nabla u \nabla v, \mu_E \rangle\rangle_{(S)'}$ given by μ_E.

The determination of the associated generator on smooth cylinder functions has been given in [31, 32, 87] (based on previous results in [84, 85]).

We shall now briefly go over to describe similar Level 0–2 structures for quantum fields, i.e. associated with (5) and (6) (with non trivial dependence on the space variable $\vec{x} \in \mathbb{R}^{d-1}$, $d \geq 2$).

4 Canonical Quantum Fields and S(P)DEs

We shall only discuss, for simplicity, scalar fields.

At Level 0 we have as the analogue of the position operators resp. momentum operators in quantum mechanics, the time zero quantum field $\varphi(\vec{x})$, $\vec{x} \in \mathbb{R}^{d-1}$ resp. the time zero momentum field $\Pi(\vec{x})$, satisfying the canonical commutation relations (CCR)

$$[\Pi(\vec{y}), \varphi(\vec{x})] = \frac{1}{i}\delta(\vec{x} - \vec{y}), \quad \vec{x}, \vec{y} \in \mathbb{R}^{d-1}, \tag{13}$$

on some suitable dense domain in some Hilbert space.

Since the situation for the case $v \not\equiv 0$ is quite complicated, because of the well known problem of divergences, let us first consider the case where $v \equiv 0$, i.e. the case of free fields describing first Level 0–2 for this case. If $v \equiv 0$ in S_{QF} as given by (4), then the natural Hilbert space for the CCR representation is $L^2(\mathcal{S}'(\mathbb{R}^{d-1}), \mu_E^{0,0})$, with $\mu_E^{0,0} = N(0; (-\Delta_{d-1} + 1)^{-\frac{1}{2}})$, the free time zero field measure (of mass 1), which can be realized on $\mathcal{S}'(\mathbb{R}^{d-1})$. One shows that $\mu_E^{0,0}$ is the restriction to the σ-algebra generated by the time zero fields of the measure μ_E^0, described heuristically as $\mu_E^0 = Z^{-1} e^{-S_E^0(\gamma)}\, \mathrm{d}\gamma$ with

$$S^0(\gamma) := \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \left(\left|\dot{\gamma}(s, \vec{x})\right|^2 + \left|\nabla\gamma(s, \vec{x})\right|^2 + \left|\gamma(s, \vec{x})\right|^2 \right) \mathrm{d}s\, \mathrm{d}\vec{x}. \tag{14}$$

μ_E^0 is realized rigorously as $N(0; (-\Delta_d + 1)^{-1})$ on $\mathcal{S}'(\mathbb{R}^d)$. It is called the Euclidean free field measure, see [98].

The above $L^2(\mathcal{S}'(\mathbb{R}^{d-1}), \mu_E^{0,0})$ is naturally isomorphic to the Fock space ("second quantization"), see [78].

Consider the classical Dirichlet form $\mathcal{E}(u, v) = \frac{1}{2} \int \nabla u \nabla v \, \mathrm{d}\mu_E^{0,0}$, so that $\mathcal{E}(u, v) = \left(u, -L_{\mu_E^{0,0}} v\right)$ for u, v in the relevant domains, in $L^2\left(\mu_E^{0,0}\right) := L^2(\mathcal{S}'(\mathbb{R}^{d-1}), \mu_E^{0,0})$ with $L_{\mu_E^{0,0}}$ the associated Dirichlet operator. One has that $H^0 := -L_{\mu_E^{0,0}}$ is self-adjoint (on its natural domain) and is a realization of the Hamiltonian for the free relativistic quantum field. In an analogous way one can consider the generators of the whole Poincaré group on space-time $\mathbb{R} \times \mathbb{R}^{d-1}$.

The generators of the Lorentz group are not positive, but can be shown to be essentially self-adjoint on smooth cylinder functions (by using the isomorphism of $L^2(\mu_E^{0,0})$ with Fock space, i.e. second quantization).

Then one has a rigorous full implementation of canonical relativistic free quantum fields in $L^2(\mu_E^{0,0})$, see [48].

Note that $L_{\mu_E^{0,0}}$ generates a diffusion process $X_t(\vec{x})$, $t \geq 0$, $\vec{x} \in \mathbb{R}^{d-1}$ with state space $\mathcal{S}'(\mathbb{R}^{d-1})$ satisfying the stochastic differential equation

$$\mathrm{d}X_t(\vec{x}) = -\left[(-\Delta_{d-1} + 1)^{\frac{1}{2}} X_t(\vec{x})\right] \mathrm{d}t + \mathrm{d}w_t(\vec{x}), \quad t \geq 0, \vec{x} \in \mathbb{R}^{d-1}. \tag{15}$$

$\mathrm{d}w_t(\vec{x})$ denotes the Gaussian white noise in the t- and $\vec{x}$-variables.

This was first discussed in [23] and [81]. Note also that $\mu_E^{0,0}$ is the unique stationary measure for $X_t(\vec{x})$.

The path space distribution of $X_t(\vec{x})$ is the Euclidean free field measure $\mu_E^0 = N\left(0; (-\Delta_{\mathbb{R}^d} + 1)^{-1}\right)$ on $\mathcal{S}'(\mathbb{R}^d)$, as described above.

This has been discussed originally by K. Symanzik [112] and E. Nelson [98] (and, in connection with statistics, by L. Pitt, [103]).

That the restriction of μ_E^0 to the σ-algebra $\sigma(X_0)$ generated by the time zero fields $X_0(\,\cdot\,)$ can be identified with $\mu_E^{0,0}$ due to the global Markov property of the Euclidean free field [29, 38, 98, 104].

The Level 2 is described by the SDE (SPDE)

$$\mathrm{d}X_\tau(y) = (\Delta_d - 1)X_\tau(y) + \mathrm{d}W_\tau(y), \tag{16}$$

$y \in \mathbb{R}^d$, $\tau \geq 0$ (τ is a "computer time", y the space-time variable). $\mathrm{d}W_\tau(y)$ denotes the Gaussian white noise in the τ- and y-variables.

The invariant measure to $X_\tau(y)$ is μ_E^0. The SPDE (16) is called stochastic quantisation equation (SQE) associated with the Euclidean free field (over $\mathbb{R}^d$).

Already for $d = 2$, in the case $v \not\equiv 0$ the construction of Levels 0–2 is much more complicated.

Levels 0 and 1 have been achieved first in models with exponential interaction (for which v is of the form $v(u) = e^{\alpha u}$, $|\alpha| < \alpha_0$, for suitable $\alpha_0 > 0$, $u \in \mathbb{R}$) with renormalization, in [82] resp. [20] (see also [11, 21, 89]).

In the same period the case of v of polynomial type, with renormalization, was achieved, see [71, 106], and references therein.

The model with v of trigonometric type was first discussed in [19] (with regularization, see also [14]) and solved (with renormalizations) in [68] (see also [14, 67]).

All models are covered and unified by the white noise calculus approach [16–18].

Let us give a short summary of these constructions. At Level 0 one constructs a non Gaussian $\mu^{v,0}$ on $\mathcal{S}'(\mathbb{R}^{d-1})$. At Level 1 one constructs the analogue of (12) for quantum fields, i.e.

$$\mu_E^v(\mathrm{d}\gamma) = \text{“}Z^{-1} e^{-S(\gamma)} \mathrm{d}\gamma\text{''}, \tag{17}$$

with $S(\gamma) = S^0(\gamma) + \iint v(\gamma(s,\vec{x}))\,\mathrm{d}s\,\mathrm{d}\vec{x}$. (Thus μ_E^v coincides with the expression given by (5) and (6)). At Level 2 the measure μ_E^v appears as invariant measure for the solution of the stochastic quantization equation (also called, in other contexts, Allen-Cahn equation, see, e.g., [99], also related to the Ginzburg-Landau equation), associated to Euclidean fields with interaction given by v:

$$\mathrm{d}X_\tau(y) = (\Delta_d - 1)X_\tau(y)\,\mathrm{d}\tau - v'(X_\tau(y))\,\mathrm{d}\tau + \mathrm{d}w_\tau(y), \quad \tau \geq 0,\ y \in \mathbb{R}^d. \tag{18}$$

The associated classical Dirichlet forms given by the restriction $\mu_E^{v,0}$ of μ_E^v to the σ-algebra of the zero fields, which is thus a measure on $S'(\mathbb{R}^{d-1})$, (coinciding with the level 0 measure $\mu^{v,0}$) resp. by μ_E^v itself, yield in turn generators of Markov processes (diffusions) $X_t(\vec{x})$ resp. $X_\tau(y)$, $t, \tau \geq 0, \vec{x} \in \mathbb{R}^{d-1}$, $y \in \mathbb{R}^d$, the latter solving (18).

The generator of $X_t(\vec{x})$ is identifiable (via a natural isomorphism) on smooth cylinder functions with the Hamiltonian acting in $L^2\left(\mu_E^{v,0}\right)$, of the interacting quantum fields, given the global Markov property of the Euclidean fields [29]. The generators of the Poincaré group are also realizable as self-adjoint operators on $L^2(\mu_E^{v,0})$ [21, 47].

Whether all generators are already determined on smooth cylinder functions is a hard problem, only solved for the case $X_\tau(y)$ in a bounded region when $d =$

2 [61, 90] (it has been solved in bounded and unbounded regions for $d = 1$ in [31, 32, 87]).

For recent work on Level 2 for $d = 2$ with exponential interaction see [30, 31].

Remark 4.1 The SQE has also been studied intensively with noise regularized in the y-variable and corresponding modified drift coefficient, so as to heuristically maintain the same invariant measure.

See [41–43, 56, 60, 61, 64, 70, 83–86, 95–97, 99, 100, 105], and also e.g. [58] for related problems.

For $d = 3$ partial results on Level 0–2 have been achieved for v which are of the form of a 4-th power (with renormalization), see, e.g. the references in [36].

Recently solutions of the SQE of Level 2 has been constructed by different methods in [77]. However the Markov character of the solutions (a property included if one uses processes associated with Dirichlet forms) has not yet been discussed.

The difficulties in constructing solutions is related to the expected singularity of μ_E^v with respect to μ_E^0 even in bounded regions (this has been proved rigorously in [35] only for the restriction of these measure to the σ-algebra generated by the time zero fields).

Other types of relativistic models however also exist for all d including $d = 4$, provided one relaxes the axioms to the ones which have been introduced (by Strocchi-Wightman and Morchio-Strocchi) for gauge fields, see the work [73] (in white noise analysis) and [12, 13] (and references therein).

For the construction of gauge fields for $d = 2$ see [1, 24, 25, 47, 48] (and references therein).

For $d = 3$ in the Euclidean case a model of gauge fields has been constructed using μ_F (Chern-Simons model) [15, 47, 91], again using methods from white noise analysis (to which Ludwig has fundamentally contributed).

5 Some Remarks on Related Systems

Methods related to the ones discussed in the preceding Sections can also be used in other areas.

E.g., in recent years equations for neuronal dynamics of the FitzHugh-Nagumo type have been discussed in [5, 6].

In the simplest version they describe a signal propagating along a single neuron under the influence of external noise and of certain salts concentrations. They are of the form of 2 coupled random variables, the components of a vector $X \in L^2([0, 1]) \times L^2([0, 1])$ satisfying a SDE of the form $\mathrm{d}X_t = AX_t\,\mathrm{d}t + F(X_t)\,\mathrm{d}t + \varepsilon\,\mathrm{d}w_t$, $t \geq 0$, with w_t space-time Gaussian white noise and $\varepsilon > 0$ a (small diffusion) parameter.

A is a diffusion operator, $F(x)$ is a non linear term, e.g., of the Fitz-Hugh-Nagumo type $F(x) = \begin{pmatrix} -x(x-1)(x-\theta), \quad 0 < \theta < 1 \\ 0 \end{pmatrix}$, $x \in \mathbb{R}$.

Using dissipativity, in [5, 54] the existence, uniqueness of solutions and the uniqueness of the invariant measure have been discussed, as well as the $\varepsilon \downarrow 0$ behaviour in terms of detailed asymptotic expansions, with L^p-control, $1 \le p < +\infty$, on remainders.

The invariant measure is exhibited, as being of Gibbsian type with a density e^{-G} with respect to μ_A, and $G' = -F$, where μ_A is the invariant Gaussian measure associated to the linear stochastic equation for $F \equiv 0$.

It is interesting that in the case of Gaussian white noise replaced by Lévy noise also an invariant measure has been found in [6]. Its character is presently under discussion.

In the finite dimensional case explicit invariant measures of related systems can be constructed via an analogue of the general ground state transformation discussed early in [27], see [6, 50]. Andrisani and Cufaro-Petroni [50] have interesting applications, e.g. to the study of halo formation in intense beams of charged particles in accelerators.

Stochastic pde's of the form of those we have discussed in connection with quantum field theory and neuronal dynamics also occur in many other areas, including, e.g., hydrodynamics and polymer physics, to which Ludwig Streit has given outstanding contributions. Let us mention, in particular, his work on the Burgers equation [51] and his work on polymer-type measures, a prototype for the latter being given by the Edwards' model described heuristically by the invariant measure

$$\mu(\mathrm{d}\gamma) = \text{“}Z^{-1} e^{-\lambda \int_0^t \int_0^t \delta(\omega(s)-\omega(s'))\,\mathrm{d}s\,\mathrm{d}s'} \mu_0(\mathrm{d}\omega)\text{”} \tag{19}$$

μ_0 being Wiener measure for Brownian motion on $\mathbb{R}^d$, δ the Dirac distribution, λ a (positive) constant, $t > 0$. The Gibbs factor inhibits self-intersection of paths. For such measures and related ones see, e.g., [9, 40, 62, 63].

Varadhan showed in [113] that setting $L_\varepsilon = \int_0^t \int_0^t \delta_\varepsilon(\omega(s) - \omega(s'))\,\mathrm{d}s\,\mathrm{d}s'$ with δ_ε a natural ε-regularization of δ one has that $L_\varepsilon - E(L_\varepsilon)$ converges as $\varepsilon \downarrow 0$ in $L^2(\mu_0)$ to a limit $\tilde{L}$, in such a way that

$$\| \tilde{L} - [L_\varepsilon - E(L_\varepsilon)] \|_{L^2(\mu_0)} \le C\varepsilon^\alpha \tag{20}$$

for some constant $C > 0$, for all $\alpha < \frac{1}{2}$.

This implies in particular the existence of the polymer measure μ, understood as weak limit of μ_ε as $\varepsilon \downarrow 0$ defined by (19) with

$$L := \int_0^t \int_0^t \delta(\omega(s) - \omega(s'))\,\mathrm{d}s\,\mathrm{d}s' \tag{21}$$

replaced by $L_\varepsilon - E(L_\varepsilon)$.

The above estimate (20) has been improved to $\alpha < 1$ in a recent publication by Ludwig and coworkers [53].

6 Conclusions

We have just been able to discuss a few of the many important developments inspired, initiated and rigorously pursued by Ludwig Streit. I am very grateful to Ludwig for having been a constant source of inspiration for me and many coworkers since many years. I wish him many more years of good health, happiness, “frohes Schaffen”.

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References

1. Albeverio, S.: Loop groups, random gauge fields, Chern-Simons models, strings: some recent mathematical developments. In: Léandre, R., Paycha, S., Wurzbacher, T. (eds.) Espace des lacets, Proceedings of the Conference Loop Spaces ’94, Strasbourg, pp. 5–34 (1996) (Publ. Inst. R. Descartes)
2. Albeverio, S.: Wiener and Feynman-Path Integrals and Their Applications. Proceedings of Symposia in Applied Mathematics, vol. 52, pp. 163–194. AMS, Providence (1994)
3. Albeverio, S.: Some themes of the scientific work of Ludwig Streit. In: Albeverio, S., Blanchard, Ph., Ferreira, L., Hida, T., Kondratiev, Y., Vilela Mendes, R. (eds.) Mathematical Physics and Stochastic Analysis, Essays in Honour of Ludwig Streit, pp. 1–7. World Scientific, Singapore (2000).
4. Albeverio, S.: Theory of Dirichlet forms and applications. In: Bernard, P. (ed.) Lectures on Probability Theory and Statistics (Saint-Flour, 2000). Lecture Notes in Mathematics, vol. 1816, pp. 1–106. Springer, Berlin (2003)
5. Albeverio, S., di Persio, L., Mastrogiacomo, Z.: Small noise asymptotic expansions for stochastic PDE’s I. The case of a dissipative polynomially bounded nonlinearity. Tohôku Math. J. **63**, 877–898 (2011)
6. Albeverio, S., di Persio, L., Mastrogiacomo, E., Smii, B.: A class of Lévy driven SDEs and their explicit invariant measures. Potential Anal. (2016). doi:10.1007/s111118-016-9544-3
7. Albeverio, S., Fan, R., Herzberg, F.S.: Hyperfinite Dirichlet Forms and Stochastic Processes. Lecture Notes of the UMI. Springer, Heidelberg (2011)
8. Albeverio, S., Ferrario, B.: Some methods of infinite dimensional analysis in hydrodynamics: an introduction. In: SPDE in Hydrodynamic: Recent Progress and Prospects. Lecture Notes in Mathematics, vol. 1942, pp. 1–50. Springer, Berlin/London (2008)
9. Albeverio, S., Fenstad, J.E., Høegh-Krohn, R., Lindstrøm, T.: Nonstandard Methods in Stochastic Analysis and Mathematical Physics. Pure and Applied Mathematics, vol. 122. Academic, Orlando (1986). Reprint: Dover Publications, Mineola (2009)
10. Albeverio, S., Fukushima, M., Karwowski, W., Streit, L.: Capacity and quantum mechanical tunneling. Commun. Math. Phys. **80**, 301–342 (1981)
11. Albeverio, S., Gallavotti, G., Høegh-Krohn, R.: Some results for the exponential interaction in two or more dimensions. Commun. Math. Phys. **70**, 187–192 (1979)
12. Albeverio, S., Gottschalk, H., Wu, J.-L.: Models of local relativistic quantum fields with indefinite metric (in all dimensions). Commun. Math. Phys. **184**(3), 509–531 (1997)

13. Albeverio, S., Gottschalk, H., Yoshida, M.W.: Systems of classical particles in the grand canonical ensemble, scaling limits and quantum field theory. Rev. Math. Phys. **17**(2), 175–226 (2005)
14. Albeverio, S., Haba, Z., Russo, F.: A two-space dimensional semilinear heat equation perturbed by (Gaussian) white space. PTRF **121**, 319–366 (2001)
15. Albeverio, S., Hahn, A., Sengupta, A.: Chern-Simons theory, Hida distributions and state models. IDAQP **6**, 65–81 (2003). (Special Issues on Diff. Geom. Stoch. Anal. I).
16. Albeverio, S., Hida, T., Potthoff, J., Röckner, M., Streit, L.: Dirichlet forms in terms of white noise analysis I – Construction and QFT examples. Rev. Math. Phys. **1**, 291–312 (1990)
17. Albeverio, S., Hida, T., Potthoff, J., Röckner, M., Streit, L.: Dirichlet forms in terms of white noise analysis II – Closability and Diffusion Processes. Rev. Math. Phys. **1**, 313–323 (1990)
18. Albeverio, S., Hida, T., Potthoff, J., Streit, L.: The vacuum of the Høegh-Krohn model as a generalized white noise functional. Phys. Lett. **B 217**, 511–514 (1989)
19. Albeverio, S., Høegh-Krohn, R.: Uniqueness of the physical vacuum and the Wightman functions in the infinite volume limit for some non-polynomial interactions. Commun. Math. Phys. **30**, 171–200 (1973)
20. Albeverio, S., Høegh-Krohn, R.: The Wightman axioms and the mass gap for strong interactions of exponential type in two-dimensional space-time. J. Funct. Anal. **16**, 39–82 (1974)
21. Albeverio, S., Høegh-Krohn, R.: Canonical relativistic quantum fields in two space-time dimensions. Preprint Series, Institute of Mathematics, University of Oslo, No. 23 (1975)
22. Albeverio, S., Høegh-Krohn, R.: Quasi invariant measures, symmetric diffusion processes and quantum fields. In: Editions du CNRS, Proceedings of the International Colloquium on Mathematical Methods of Quantum Field Theory. Colloques Internationaux du Centre National de la Recherche Scientifique, No. 248, pp. 11–59 (1976)
23. Albeverio, S., Høegh-Krohn, R.: Dirichlet forms and diffusion processes on rigged Hilbert spaces. Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete **40**, 1–57 (1977)
24. Albeverio, S., Høegh-Krohn, R.: Diffusions, Quantum Fields and Groups of Mappings. Functional Analysis in Markov Processes. Lecture Notes in Mathematics, vol. 923, pp. 133–145 (1982)
25. Albeverio, S., Høegh-Krohn, R., Holden, H.: Stochastic multiplicative measures, generalized Markov semigroups and group valued stochastic processes and fields. J. Funct. Anal. **78**(1), 154–184 (1988)
26. Albeverio, S., Høegh-Krohn, R., Mazzucchi, S.: Mathematical Theory of Feynman Path Integrals, An Introduction, 2nd edn. Springer, Berlin (2008)
27. Albeverio, S., Høegh-Krohn, R., Streit, L.: Energy forms, Hamiltonians, and distorted Brownian paths. J. Math. Phys. **18**, 907–917 (1977)
28. Albeverio, S., Høegh-Krohn, R., Streit, L.: Regularization of Hamiltonians and processes. J. Math. Phys. **21**, 1636–1642 (1980)
29. Albeverio, S., Høegh-Krohn, R., Zegarlinski, B.: Uniqueness and global Markov property for Euclidean fields: the case of general polynomial interactions. Commun. Math. Phys. **123**, 377–424 (1989)
30. Albeverio, S., Kawabi, H., Mihalache, S.: Stochastic quantization of exponential interactions (in preparation)
31. Albeverio, S., Kawabi, H., Mihalache, S., Röckner, M.: Dirichlet form approach to stochastic quantization under exponential interaction in finite volume (in preparation)
32. Albeverio, S., Kawabi, H., Röckner, M.: Strong uniqueness for both Dirichlet operators and stochastic dynamics for Gibbs measures on a path space with exponential interactions. J. Funct. Anal. **262**(2), 602–638 (2012)
33. Albeverio, S., Kondratiev, Y., Kozitsky, Y., Röckner, M.: Statistical Mechanics of Quantum Lattice Systems: A Path Integral Approach. EMS Tracts in Mathematics, vol. 8. European Mathematical Society, Zürich (2009)
34. Albeverio, S., Kusuoka, S., Streit, L.: Convergence of Dirichlet forms and associated Schrödinger operators. J. Funct. Anal. **68**, 130–148 (1986)

35. Albeverio, S., Liang, S.: A remark on the nonequivalence of the time-zero ϕ_3^4-measure with the free field measure. Markov Process. Rel. Fields **14**, 159–164 (2008)
36. Albeverio, S., Liang, S., Zegarliński, B.: Remark on the integration by parts formula for the ϕ_3^4-quantum field model. Infin. Dimens. Anal. Quantum Probab. Relat. Top. **9**, 149–154 (2006)
37. Albeverio, S., Ma, Z.M., Röckner, M.: Quasi-regular Dirichlet forms and the stochastic quantization problem, Festschrift Masatoshi Fukushima. In: Chen, Z.-Q., Jacob, N., Takeda, M., Uemura, T. (eds.) Honor of Masatoshi Fukushima's Sanju, pp. 27–58. World Scientific, Singapore (2015)
38. Albeverio, S., Mandrekar, V.: A remark on measure indexed Gaussian random fields (2015, submitted)
39. Albeverio, S., Mazzucchi, S.: Path integrals, mathematical aspects, tiblo-Scholarpedia (2015)
40. Albeverio, S., Oliveira, M.J., Streit, L.: Intersection local times of independent Brownian motions as generalized white noise functionals. Acta Appl. Math. **69**(3), 211–241 (2001)
41. Albeverio, S., Röckner, M.: Dirichlet forms, quantum fields and stochastic quantization. In: Elworthy, K.D., Zambrini, J.C. (eds.) Stochastic analysis, path integration and dynamics. Research Notes in Mathematics, vol. 200, pp. 1–21. Longman, Harlow (1989)
42. Albeverio, S., Röckner, M.: Classical Dirichlet forms on topological vector spaces – the construction of the associated diffusion process. Probab. Theory Relat. Fields **83**(3), 405–434 (1989)
43. Albeverio, S., Röckner, M.: Stochastic differential equations in infinite dimensions: solution via Dirichlet forms. Probab. Theory Relat. Fields **89**(3), 347–386 (1991)
44. Albeverio, S., Rüdiger, B.: Infinite dimensional Stochastic Differential Equations obtained by subordination and related Dirichlet forms. J. Funct. Anal. **204**(1), 122–156 (2003)
45. Albeverio, S., Rüdiger, B.: Subordination of symmetric quasi-regular Dirichlet forms. Random Oper. Stoch. Equ. **13**(1), 17–38 (2005)
46. Albeverio, S., Rüdiger, B., Wu, J.-L.: Analytic and probabilistic aspects of Lévy processes and fields in quantum theory. In: Barndorff-Nielsen, O., Mikoschi, T., Resnick, S. (eds.) Lévy Processes, Theory and Application, pp. 187–224. Birkhäuser, Basel (2001)
47. Albeverio, S., Sengupta, A.: A mathematical construction of the non-abelian Chern-Simons functional integral. Commun. Math. Phys. **186**, 563–579 (1997)
48. Albeverio, S., Sengupta, A.: From classical to quantum fields – a mathematical approach (in preparation)
49. Albeverio, S., Ugolini, S.: Complex Dirichlet forms: non symmetric diffusion processes and Schrödinger operators. Potential Anal. **12**, 403–417 (2000)
50. Andrisani, A., Cufaro-Petroni, N.: Markov processes and generalized Schrödinger operators. J. Math. Phys. **52**, 113509, 22pp (2011)
51. Benth, F., Streit, L.: The Burgers equation with a non-Gaussian random force. In: Decreusefond, L., Gjerde, J., Øksendal, B., Üstünel, A.S. (eds.) Stochastic Analysis and Related Topics. Stochastic Monographs vol. 6, pp. 187–210. Birkhäuser, Basel (1998)
52. Berezansky, Y.M., Kondratiev, Y.G.: Spectral Methods in Infinite-Dimensional Analysis, vol. 1, 2. Kluwer, Dordrecht (1995) (first issued in URSS. Naukova Dumka, Kiev, 1988)
53. Bock, W., Oliveira, M.J., da Silva, J.L., Streit, L.: Polymer measure: Varadhan's renormalization revisited. Rev. Math. Phys. **27**(3), 1550009 (2015).
54. Bonaccorsi, S., Mastrogiacomo, E.: Analysis of the stochastic Fitz-Hugh-Nagumo System. IDAQP **11**, 427–446 (2008).
55. Borasi, L.: Complex scaled time oscillatory infinite dimensional integrals and the Gell-Mann Low formula. Master Thesis, University of Pisa (2014)
56. Borkar, V.S., Chari, R.T., Mitter, S.K.: Stochastic quantization of field theory in finite and infinite volume. J. Funct. Anal. **81**(1), 184–206 (1988)
57. Bouleau, N., Hirsch, F.: Dirichlet Forms and Analysis on Wiener Space. Studies in Mathematics, vol. 14. deGruyter, Berlin (1991)
58. Bovier, A.: Metastability. In: Methods of Contemporary Mathematical Statistical Physics. Lecture Notes in Mathematics 1970, pp. 177–221. Springer, Berlin (2009)

59. Chen, Z.Q., Fukushima, M.: Symmetric Markov Processes, Time change, and Boundary Theory. Princeton University Press, Princeton (2012)
60. Da Prato, G., Debussche, A.: Strong solutions to the stochastic quantization equations. Ann. Probab. **31**(4), 1900–1916 (2003)
61. Da Prato, G., Tubaro, L.: Self-adjointness of some infinite dimensional elliptic operators and applications to stochastic quantization. Probab. Theory Relat. Fields **118**(1), 131–145 (2000)
62. de Faria, M., Hida, T., Watanabe, H., Streit, L.: Intersection local times as generalized white noise functionals. Acta Appl. Math. **46**, 351–362 (1997)
63. de Faria, M., Potthoff, J., Streit, L.: The Feynman integrand as a Hida distribution. J. Math. Phys. **32**, 2123–2127 (1991)
64. Doering, C.R.: Nonlinear parabolic stochastic differential equations with additive colored noise on $\mathbb{R}^d \times \mathbb{R}_+$: a regulated stochastic quantization. Commun. Math. Phys. **109**, 537–561 (1987)
65. Eachus, W.J., Streit, L.: Exact solution of the quadratic interaction Hamiltonian. Rep. Math. Phys. **4**, 161–182 (1973)
66. Eberle, A.: Uniqueness and Non-Uniqueness of Semigroups generated by Singular Diffusion Operators. Lecture Notes in Mathematics, vol. 1718. Springer, Berlin (1999)
67. Fröhlich, J., Park, T.M.: Remarks on exponential interactions and the quantum sine-Gordon equation in two space-time dimensions. Helv. Phys. Acta **50**, 315–329 (1977)
68. Fröhlich, J., Seiler, E.: The massive Thirring-Schwinger model (QED2): convergence of perturbation theory and particle structure. Helv. Phys. Acta **49**, 889–924 (1976)
69. Fukushima, M., Oshima, Y., Takeda, M.: Dirichlet Forms and Symmetric Markov Processes. 2nd revised and extended edn. de Gruyter Studies in Mathematics, vol. 19. Walter de Gruyter & Co., Berlin (2011)
70. Gatarek, D., Goldys, B.: Existence, uniqueness and ergodicity for the stochastic quantization equation. Studia Math. **119**, 179–193 (1996)
71. Glimm, J., Jaffe, A.: Quantum Physics. Springer, New York (1981). 2nd edn., (1986)
72. Gross, L.: Logarithmic Sobolev inequalities. Am. J. Math. **97**, 1061–1083 (1976)
73. Grothaus, M., Streit, L.: Construction of relativistic quantum fields in the framework of white noise analysis. J. Math. Phys. **40**, 5387–5405 (1999)
74. Grothaus, M., Streit, L., Vogel, A.: Feynman integrals as Hida distributions: the case of non-perturbative potentials. Astrisque **327**, 55–68 (2010)
75. Grothaus, M., Streit, L., Vogel, A.: The complex scaled Feynman-Kac formula for singular initial distributions. Stochastics **84**(2–3), 347–366 (2012)
76. Guerra, F., Rosen, J., Simon, B.: The $P(\Phi)_2$-Euclidean quantum field theory as classical statistical mechanics. Ann. Math. **101**, 111–259 (1975)
77. Hairer, M.: A theory of regularity structures. Invent. Math. **198**, 269–504 (2014)
78. Hida, T., Kuo, H.-H., Potthoff, J., Streit, L.: White Noise: An Infinite Dimensional Calculus. Mathematics and Its Applications, vol. 253. Kluwer Academic Publishers Group, Dordrecht (1993)
79. Hida, T., Potthoff, J., Streit, L.: Dirichlet forms and white noise analysis. CMP **116**, 235–245 (1988).
80. Hida, T., Streit, L.: Generalized Brownian functionals and the Feynman integral. Stoch. Process. Appl. **16**, 55–69 (1983)
81. Hida, T., Streit, L.: On quantum theory in terms of white noise. Nagrega Math. J. **68**, 21–34 (1977)
82. Høegh-Krohn, R.: A general class of quantum fields without cut-offs in two space-time dimensions. Commun. Math. Phys. **21**, 244–255 (1971)
83. Hu, Y., Kallianpur, G.: Exponential integrability and application to Stochastic quantization. Appl. Math. Optim. **37**, 295–353 (1998)
84. Iwata, K.: Reversible Measures of a $P(\phi)_1$-Time Evolution. Probabilistic Methods in Mathematical Physics, pp. 195–209. Academic Press, Orlando (1985)
85. Iwata, K.: An infinite dimensional stochastic differential equation with state space $C(\mathbb{R})$. Probab. Theory Relat. Fields **74**, 141–159 (1987)

86. Jona-Lasinio, G., Mitter, P.K.: On the stochastic quantization of field theory. Commun. Math. Phys. **101**, 409–436 (1985)
87. Kawabi, H., Röckner, M.: Essential self-adjointness of Dirichlet operators on a path space with Gibbs measures via an SPDE approach. J. Funct. Anal. **242**, 486–518 (2007)
88. Kondratiev, Y.G., Leukert, P., Potthoff, J., Streit, L., Westerkamp, W.: Generalized functionals in Gaussian spaces: the characterization theorem revisited. J. Funct. Anal. **141**(2), 301–318 (1996)
89. Kusuoka, S.: Høegh-Krohn's model of quantum fields and the absolute continuity of measures In: Albeverio, S., Fenstad, J.E., Holden, H., Lindstrøm, T. (eds.) Ideas and Methods in Mathematical Analysis, stochastics, and Applications, pp. 405–424. Cambridge University Press, Cambridge (1992)
90. Liskevich, V., Röckner, M.: Strong uniqueness for certain infinite dimensional Dirichlet operators and applications to stochastic quantization. Ann. Scuola Norm. Pisa **27**(1), 69–91 (1998)
91. Leukert, S., Schäfer, J.: A rigorous construction of Abelian Chern-Simons path integrals using white noise analysis. Rev. Math. Phys. **8**, 445–456 (1996)
92. Ma, Z.M., Röckner, M.: Introduction to the Theory of (Non-symmetric) Dirichlet Forms. Springer, Berlin (1992)
93. Mazzucchi, S.: Mathematical Feynman Path Integrals and Applications. World Scientific, Singapore (2009)
94. Mazzucchi, S.: Functional-integral solution for the Schrödinger equation with polynomial potential: a white noise approach. DAQP **14**, 675–688 (2011)
95. Mihalache, S.: Stochastische Quantisierung bei exponentieller Wechselwirkung. Diploma Thesis, Bonn (2006)
96. Mikulevicius, R., Rozovskii, B.: Martingale problems for stochastic PDE's. In: Carmona, R., Rozovskii, B. (eds.) Stochastic Partial Differential Equations: Six Perspectives, pp. 243–325. AMS, Cambridge (1999)
97. Mitter, S.: Markov random fields, stochastic quantization and image analysis. Math. Appl. **56**, 101–109 (1989)
98. Nelson, E.: The free Markov field. J. Funct. Anal. **12**, 211–227 (1973)
99. Otto, F., Weber, H., Westdickenberg, M.G.: Invariant measure of the stochastic Allen-Cahn equation: the regime of small noise and large system size. Electron. J. Probab. **19**, 76 (2014)
100. Parisi, G., Wu, Y.S.: Perturbation theory without gauge fixing. Sci. Sin. **24**, 483–496 (1981)
101. Potthoff, J., Streit, L.: A characterization of Hida distributions. J. Funct. Anal. **10**, 212–229 (1991)
102. Potthoff, J., Streit, L.: Invariant states on random and quantum fields: ϕ bounds and white noise analysis. J. Funct. Anal. **111**, 295–311 (1993)
103. Pitt, L.: A Markov property for Gaussian processes with a multidimensional time. Arch. Ration. Mech. Anal. **43**, 367–391 (1971)
104. Röckner, M.: Traces of harmonic functions and a new path space for the free quantum field. J. Funct. Anal. **79**, 211–249 (1988)
105. Röckner, M., Zhang, T.S.: Uniqueness of generalized Schrödinger operators and applications. J. Funct. Anal. **105**, 187–231 (1992)
106. Simon, B.: The $P(\Phi)_2$-Euclidean (Quantum) Field Theory. Princeton University Press, Princeton NJ, (1974)
107. Streit, L.: An introduction to theories of integration over function spaces. Acta Phys. Austriaca, Suppl. **II**, 2–21 (1965)
108. Streit, L.: Stochastic differential equations. A pedagogical random walk. In: Nonlinear Klein-Gordon and Schrödinger Systems: Theory and Applications (Madrid, 1995), pp. 87–109. World Scientific, River Edge (1996)
109. Streit, L.: Beyond Fock Space. An Introduction to Infinite Dimensional Analysis for Physicists. Methods and Applications of Infinite Dimensional Analysis, vol. 3–34. Central Visayan Institute Foundation, Jagna (2006)
110. Streit, L.: Feynman Integrals as Generalized Functions on Path Space: Things Done and Open Problems. Path Integrals, vol. 78–85. World Scientific, Hackensack (2008)

111. Streater, R.: Euclidean quantum mechanics and stochastic integrals. In: Stochastic Integrals. Lecture Notes in Mathematics, vol. 851, pp. 371–393. Springer, Berlin/New York (1981)
112. Symanzik, K.: Euclidean quantum field theory. In: Jost, R. (ed.) Local Quantum Theory. Academic, New York (1969)
113. Varadhan, S.R.S.: Appendix to “Euclidean quantum field theory” by K. Symanzik. In: Jost, R. (ed.) Local Quantum Theory. Academic Press, New York (1969)