Robust Design of Accelerated Life Testing and Reliability Optimization: Response Surface Methodology Approach

Taha-Hossein Hejazi, Mirmehdi Seyyed-Esfahani, and Iman Soleiman-Meigooni

Abstract Due to cost and time savings and improving reliability, accelerated life tests are commonly used; in which some external stresses are conducted on items at higher levels than normal. Estimation and optimization of the reliability measure in the presence of several controllable and uncontrollable factors becomes more difficult especially when the stresses interact. The main idea of this chapter is employing different phases of response surface methodology to obtain a robust design of accelerated life testing. Since uncontrollable variables are an important part of accelerated life tests, stochastic covariates are involved in the model. By doing so, a precise estimation of reliability measure can be obtained. Considering the covariates as well as response surface methodology simultaneously are not addressed in the literature of accelerated life test. This methodology can be used on the conditions that a broad spectrum of variables is involved in the accelerated life test and the failed units have a massive cost for producers. Though considering covariates in the experiments, the optimization of reliability can generate more realistic results in comparison with noncovariates model. For the first step of this study, experimental points using D-optimal approach are designed to decrease the number of experiments as well as the prediction variance. The reliability measure is estimated under right censoring scheme by Maximum likelihood estimator (MLE) assuming that lifetime data have an exponential distribution with parameter, λ, depending on the design and stress variables as well as covariates. In order to find the best factor setting that leads to the most reliable design, response surface methodology is applied to construct the final mathematical program. Finally, a numerical example is analyzed by the proposed approach and sensitivity analyses are performed on variables.

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1 Introduction

In the recent years, various industries are encountered with a tremendous growth in competition. In this regard, producers have paid a worthy attention to develop and design new products that satisfy the consumers' expectations. Producers should consider different characteristics in their products to increase their share in the market. Reliability is one of the most important quality dimensions and plays a pivotal role in increasing lifetime of products, reducing warranty costs, and achieving expected product's functions. Therefore, manufacturers have focused their attention to design products with high quality and reliability to decrease the number of failures in the warranty period. For this purpose, various tests should be applied and many constraints would be considered in this field such as the high cost of products that fail tests, test duration, and material limitations. Accelerated life testing (ALT) is a well-grounded method to save cost and achieve lifetime data in a shorter time period. Units in ALT are subjected to higher stress levels, and the output data of ALT are used for the estimation of product life at normal conditions. ALT can generate many opportunities for manufacturers, such as a proper maintenance scheduling for products to be kept in an acceptable level of reliability and determining a reasonable warranty period to reduce the warranty costs (Smith 1983).

Reliability can be defined as a probability of expected performance of a unit in a normal and operating condition as well as a predefined time interval. Reliability improvement is a vital part of product quality development. Although quality can be illustrated in quite a few ways, one statement which always is acceptable among the definitions is: a not reliable product is not a high quality one. Reliability function is defined as Eq. (1)

$$R(t) = 1 - F(t) \tag{1}$$

where $F(t)$ is the cumulative distribution function of lifetime. Several methods have been applied to predict and evaluate the various aspects of product's reliability. The mentioned methods collect different data and use them to estimate and analyze reliability measurements.

One of the most important issues in reliability analysis is the use of historical or experimental data to estimate product's lifetime. Due to the mentioned constraints such as long time needed for running the test and high cost of failed units, accelerated life test has been introduced to overcome the problems. Accelerated life testing is based on the principle that products have the same behavior in both conditions of high stress in short time and low stress in longer time. The aim of such testing is to quickly obtain failure data which, properly modeled and analyzed, yield desired information on product life or performance under normal use (Nelson 2009). It can be inferred from the definition of ALT that in the first phase examiner specifies the lifetime of units at the high stress condition and in the second phase estimates the lifetime of units at the normal condition. It is clear that the key point

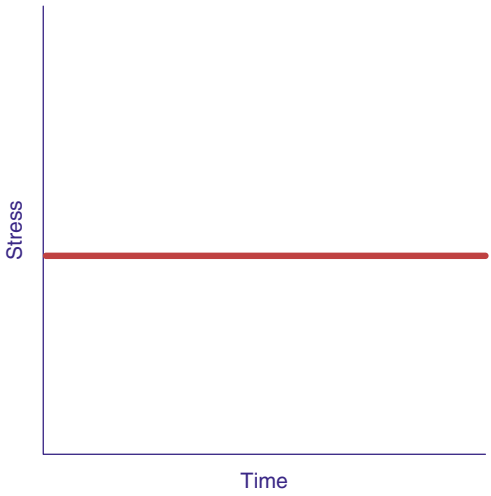

Fig. 1 Constant stress accelerated life test

in designing an optimal ALT is to construct a robust and appropriate relationship model between estimated life time in tests and the normal conditions. With all these taken into account, in order to design an adequate ALT considering some prerequisites, namely types of stress loading, relationships between life and stress, types of variables, censored data, and types of estimation methods are absolutely essential. In this regard, in the next subsections a brief review on the mentioned topics is presented.

1.1 Stress Loading

To begin with, in order to construct an ALT a broad spectrum of loads can be considered. The different types of stress loading can be categorized into two main classes with respect to dependency of stresses to time factor. The first class includes those types of stresses loading which are independent from time, in more precisely definition, in Constant Stress ALT (CSALT), items are tested at a specified constant stress level which does not vary during the experiments. Since these tests are simple to perform and have many merits such as available models, CSALT is widely used in reliability tests. Tang et al. (2002) considered two alternative ways of planning CSALT with three stress levels which optimize both stress levels and sample allocations. In Fig. 1 a constant stress accelerated life test is shown.

In second class, the stresses on the units are dependent on time and increase in a discrete way. In other words, at the initial time of test, units are subjected to a low stress level and when a specific time period is passed, the stress level is increased to a higher value. This process can be continued based on practical constraints of experiments. This type of ALT is called Step Stress ALT (SSALT). Firstly, Geol in 1971 introduced the implication of step stress partially accelerated life test.

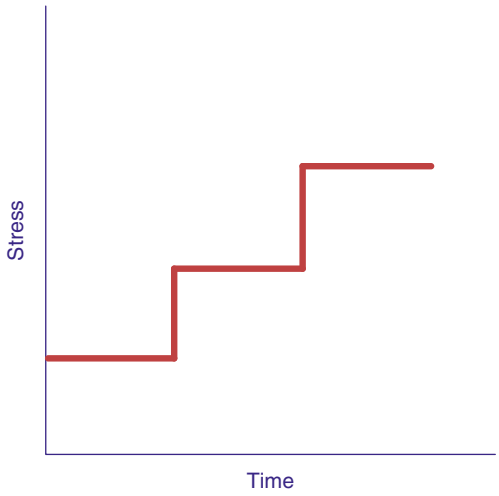

Fig. 2 Step stress accelerated life test

Miller and Nelson (1983) presented the optimal design of simple SSALT with considering some assumptions, namely, one stress variable, complete data, and exponential lifetime distribution. Khamis (1997) proposed optimal design of accelerated life test by considering K stress variables and M levels. It is assumed that lifetime has an exponential distribution and a complete information about life–stress relationship has been existed. Li and Fard (2007) proposed an optimum step-stress accelerated life test for two stress variables which include censored data. Fard and Li (2009) derived a simple Step-stress ALT (SSALT) model to determine the optimal hold time at which the stress levels changed; they assumed a Weibull distribution for failure time. In Fig. 2 a SSALT is demonstrated.

The last class of stress loading is Progressive Stress ALT (PSALT). In PSALT units are exposed under a stress which is a nondecreasing continuous function of time. In Fig. 3 a PSALT is shown.

1.2 Relationship Between Stress and Life

After determining the type of stress loading, experimenters should make a very important decision about the relationship between stress and life. In fact, in this phase of designing an accelerated life test and to achieve an accurate estimation of life time, practical conditions of problem should be considered in the model. Hence, selecting a proper relationship model between stress and life is the core of ALT design. Therefore, verification of other computations would be dependent on the validation of this model. Two main steps should be completed to construct such a relationship model. In the first step an appropriate lifetime distribution should

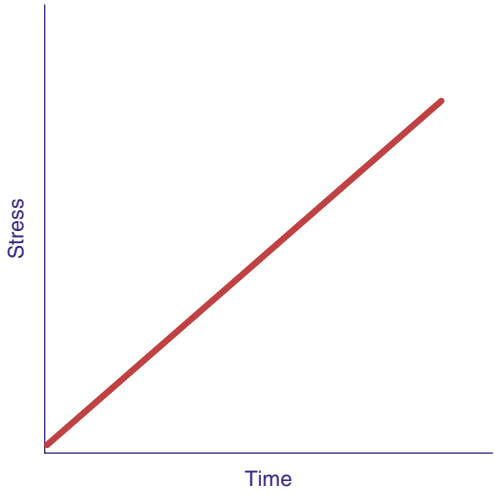

Fig. 3 Progressive accelerated life test

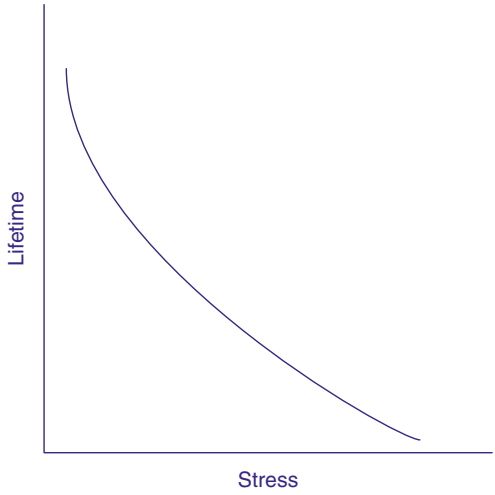

Fig. 4 Life–stress relationship

be determined for units based on the practical conditions and experts' opinions. At the second step the appropriate relationship between stress variables and the parameter(s) of lifetime distribution is derived. It should be considered that the parameter(s) of lifetime is a function of stress and not the own lifetime. In Fig. 4 a relationship between lifetime and stress is demonstrated. According to this figure, by increasing the stress the lifetime will decrease. Through a change to logarithmic scale, Fig. 5 is obtained.

Some of the most common models for the relationship between stress and life are summarized in Table 1.

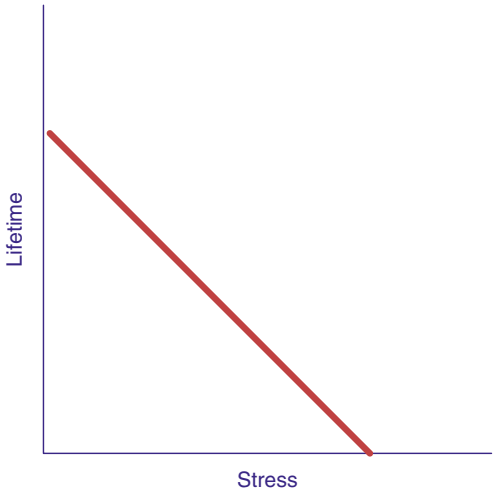

Fig. 5 Linear life–stress relationship

1.3 Types of Variables

In order to find an appropriate relationship model between stress and life, variables of model should be chosen with respect to experts' ideas and historical data. Classification of variables can be viewed from different aspects. In a point of view, some of the variables are input variables and some of them are output variables (responses) which are dependent on the inputs. However, this classification should be studied more precisely. In the one hand, some variables are known, controllable, and are of interest so should be set at ideal levels to optimize response variables. On the other hand, some variables are not of interest as a factor but exist in the model, which are called nuisance variables. These variables are very common in practical conditions and ignoring them may generate a noticeable inaccuracy in the model. Montgomery (2005) has categorized such variables into the following three groups.

1. In the first group, variables are known and controllable but are not of interest as a factor in model. Different techniques are available to handle these kinds of variables such as blocking method. These statistical techniques can eliminate effects of such variables on the results of experiments. The variation generated through changing the shifts of workers is an example of such variables.
2. In the second group, variables are known and uncontrollable. This type of variables is usually called covariates, which have a significant effect on the results of model. It should be noted that the value of such variables is measured during the experiments. Considering covariates and whose interactions with other variables have a profound impact on improving the modeling of responses. The chemical and physical properties of raw materials are examples for such variables.

Table 1 Life–stress models

Relationship	Use conditions	Model name
$\ln(L(V)) = \ln(C) + \frac{B}{V}$	When stress is thermal (for example, temperature)	Arrhenius
$L(V) = \frac{1}{V} \times e^{-(A - \frac{B}{V})}$	When stress is thermal or humidity	Eyring
$\ln(L) = -\ln(K) - n\ln(V)$	When stress is nonthermal (for example, voltage)	Inverse power law
$\ln(L(V, U)) = \ln(A) + \frac{\varnothing}{V} + \frac{b}{U}$	When stresses are temperature and humidity	Temperature–humidity
$\ln(L(U, V)) = \ln(C) - n\ln(U) + \frac{B}{V}$	When a temperature stress and a nonthermal stress are conducted on units	Temperature–nonthermal
$\ln(L) = \alpha_0 + \alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_n X_n$	When experiment is involved of multiple stresses	General log-linear
$\lambda\,(t, X) = \lambda_0(t) \cdot e^{\alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_m X_m}$	Widely used in medical fields and has new applications in reliability engineering. Cox considers a great number of stresses in model	Proportional hazards (Cox)

3. In the last group, variables not only are unknown but also are uncontrollable. This type of variables is not detectable and their levels randomly change during the experiment. The randomization technique is the main approach used to decrease the effects of such variables on the outputs. Humidity and climate variables are examples for this group of nuisance variables.

1.4 Censored Data

Through considering three last implications, a parametric relationship model between stress and life can be constructed. However, the parameters of this basic model are to be estimated. In an ideal condition, models' parameters (coefficients) are estimated using complete life data. The term of complete indicates that the failure data for all of test units are available. However, due to quite a few practical constraints such as time and cost limitation often a few number of failure data are missed. These missed data are called censored data (Chenhua 2009). Censored data play a crucial role in reliability testing, in general, and ALT, in particular. Therefore, removing censored data from the analysis will decrease the validity of estimations and accuracy of the results. Considering the mentioned reasons, it is clear that censored data should be considered in a model to reach a better design and optimized accelerated life test. Through considering the censored data a proper modification can be performed on estimation methods, which are introduced in next subsection, based on the types of censored data. These modifications significantly affect the accuracy of estimation in a model. In this regard, some important types of censored data will be presented as follows (Nelson 2009; Lewis 1987).

1. *Left censoring*: A failure time is lower than a point of inspection so it cannot be determined.
2. *Right censoring*: Some units are not failed in the experiment period. Right censoring is most common censoring scheme in reliability testing. The reasons for right censoring are as follows: (a) units exit from experiment for any reasons before that failure occurs. (b) In the end of test some units work correctly. (c) Units exit from experiment due to failure modes out of the purpose of the study.
3. *Interval censoring*: Time is missed because the units fail between two specific inspection times.

Ling et al. (2011) presented a SSLAT for two stress variables to find optimal times for to change stress levels based on type-censoring plan. In the type-censoring the censoring scheme has a predetermined termination time. They supposed that lifetime has an exponential distribution and used MLE as the estimator method. Ng et al. (2012) studied the estimation of three-parameter Weibull distribution under progressively type II censored samples. In the type censoring the experiment will be terminated after occurrence of a specific number of failures. To obtain reliable initial estimates for iterative procedures they also proposed application of censored

estimation technique with one-step bias correction. Wang et al. (2012) presented a step-stress partially accelerated life tests to estimate the parameters of Weibull distribution which considered multiply censored data.

1.5 Methods of Estimation

In the last step, the approximate relationship model between responses and independent variables should be constructed. An adequate regression model such as first and second order based on the aforementioned criteria should be selected and the unknown parameters are estimated. The following three methods are prevalent for obtaining the point estimator: (a) method of moments, (b) Bayes method, and (c) method of maximum likelihood (Ramachandran and Tsokos 2009). The most important and common method of point estimation is maximum likelihood estimation (MLE). MLE has many suitable properties and can be applied for a vast area of models and data. MLE such as other estimation methods has a simple and effective theorem behind their often massive computations. In MLE method the likelihood function of the occurred events is maximized. Although MLE has a specific algorithm to obtain estimations, a great number of techniques and methods can be used in its different phases. The process of MLE can be divided into two main phases. In the first phase, the likelihood function is demonstrated by the occurrences' probability of observed events and in the second phase the obtained function is maximized with respect to unknown parameters of the distribution function. The maximization process can be done through a wide range of techniques. In the following section, the mentioned process will be illustrated more precisely.

2 Problem Definition

2.1 Purposes of Study

The main purpose of this study is to obtain a precise estimation of reliability measurements considering stochastic covariates. To explain in more details, covariates are mostly ignored in ALT models, and this problem can drastically decline the quality of models' estimations. With respect to the mentioned reason, it can be inferred that covariates should be considered in ALT models with an equal importance as other variables. In addition, quite a few number of issues are involved in constructing a robust accelerated life test such as censored data, interaction between variables, high cost of failed units, and too name but a few. To satisfy these objectives quite a few of methods and techniques are used in the presented methodology and most of practical conditions of ALT are considered in the model. In order to cope with the mentioned problems in this study, Response

Surface Methodology is applied to design ALT models. In normal situations units perform whose functions in presence of more than one stress variable. However, the mentioned operational condition is ignored due to difficulty of computations in estimation process usually in researches. As a result, considering more than one stress variable in the model can develop the design of ALT to practical conditions. In addition, in the proposed approach ALT is designed with considering the censored data. By accepting this as a true that due to the lengthy time of many experiments, censored data is not avoidable, paying attention to censored data has a significant effect on accuracy of predictions. Through performing different phases of RSM a significant improvement in results of estimation and a noticeable decrease in number of tests will be achieved. Moreover, an optimal region that minimizes the prediction variance of estimations will be obtained using optimal designs in the first phase of RSM. In Sect. 3, the steps of RSM are explained and some examples are shown for more illustration.

2.2 Model Assumptions

Quite a few of assumptions are considered in the proposed approach to design an optimal ALT. Some of which are principal assumptions that the validation of model is dependent on these. One of the most important problems in designing an ALT is extrapolation errors result from the difference between the operational stress and accelerated stress. In order to overcome this error, some notions should be considered before designing ALTs. To begin with, the relationship between stress and life should be modeled with respect to experts' opinions as well as physical properties of units. In this regard, various information about the failures mechanism of units should be collected and analyzed by the experts. Furthermore, some essential assumptions should be considered to design ALTs. First, the life time distribution is known and remains unchanged for any level of stresses. Second, the obtained model for failure data in accelerated levels has the same behavior in operational level. As a result, the extrapolation for data in operational levels has an acceptable validation for calculated parameters. The other aspect that should be noticed to design ALTs would be the dependency of lifetimes' parameter to time. To explain in more details, units or products have a life cycle based on their failure rate. The cycle is divided into three periods. In "burn-in" period, the failure rate will be decreased with passage of time. In the "useful life" period, units have a constant failure rate. In fact, the failure rate in the useful life period is independent of time. In the last period, which is called wear out, with passage of time the failure rate will be increased.

The duration of useful life period is much greater than burn-in and wear-out periods. In addition, the failures that occur in burn-in period are due to initial problems and mistakes in design of product. Usually by a proper redesign process the mentioned failures can be removed after passing a short time. The wear-out period includes units approaching to the end of life. In this regard, designing

reliability tests in burn-in and wear-out periods has less importance than in useful-life period. Hence, mostly the designs of ALT models are performed in useful life period. Therefore, accepting the assumption that failure rate is independent of time is a reasonable hypothesis.

In addition to the mentioned principal assumptions, this study includes additional assumptions on the conditions of experiments. To begin with, it is assumed that stresses are constant during running of the experiment. In other words, the model is a constant stress accelerated life test, in which stresses do not vary with passage of time. A right censoring plan is assumed for data regarding the censoring scheme. In addition, it is assumed that the lifetime parameters are a function of design and stress variables, as well as covariates.

The proposed approach has the advantage that is not dependent on the lifetime distribution and according to condition of problem different distributions can be chosen for lifetime data.

3 Methodology

With respect to mentioned issues in the last section, an effective approach to design an accelerated life test is proposed in this section. In the presented approach to design an optimized ALT, three main steps should be performed. The steps can be explained as follows: firstly an appropriate approximate relationship among the response (parameters of lifetime distribution) and input variables (design and stress variables as well as covariates) should be modeled. Secondly, the unknown parameters in the approximate relationship should be estimated. In the last step, an optimization process should be conducted on the obtained relationship with respect to input variables. In this regard, the RSM is an absolutely well-designed method for covering and handling the mentioned steps. RSM is a collection of mathematical and statistical techniques which are applied to construct a proper functional relationship between response and input variables (Myers et al. 2004). Moreover, RSM is a powerful tool for improving and optimizing the response variables. Initial works in the field of RSM were proposed in 1950. At the mentioned years RSM was widely applied in the chemical industries. However, in recent years the applications of RSM have a tremendous growth in a broad spectrum of systems and processes (Myers et al. 2011). This methodology consists of three major phases that are consistent with ALT design phases. At first step, a design of experiment should be selected to run the tests. The aim of this phase is to select design points where response variable should be evaluated. Through applying the first phase two major advantages come up to design ALT. Firstly by choosing a statistical design to conduct the experiment, the numbers of test unit which might be failed during the test would be significantly decreased. Secondly, by considering the best set of design points to run the experiments, the prediction error of unknown parameters would be reduced. As a result, the accuracy of estimations would be meaningfully increased. At second step, the unknown parameters in the designed relationship between response and input

variables would be estimated by using an appropriate method. By doing so, lifetime of the units can be predicted using the obtained relationship model. In the last step of the methodology, an optimization method would be applied with respect to different objectives such as improving the reliability of units and involved constraints in the practical condition of experiments. The main three steps are illustrated by simple examples in the following subsections.

3.1 Design of Experiments

The basic model to design a relationship between response(s) and input variables can be defined as Eq. (2).

$$y = f'(x)\beta + \epsilon \tag{2}$$

where $= (x_1, x_2, \ldots, x_k)'$, y is the response variable, and $f(x)$ is a P-elements vector function, which contains powers and Cartesian products of $x_1, x_2, \ldots, x_k$. In addition, β is a p-element vector of unknown coefficients related to input variables and ϵ is the experimental random error, which is assumed to have a zero mean. With respect to presented model, it is clear that the value of $f'(x)\beta$ is the expected value of response variable (y). In other words, $f'(x)\beta$ is the mean of response variable and can be shown as $\mu\,(x)$. Two useful and important equations that are commonly used in response surface methodology are first order and second order equations. These equations are shown in Eqs. (3) and (4).

$$y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \epsilon \tag{3}$$

$$y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum\sum_{i<j} \beta_{ij} x_i x_j + \sum_{i=1}^{k} \beta_{ii} x_i^2 + \epsilon \tag{4}$$

In order to achieve the mentioned goals in the last section for first phase of response surface methodology, initially a number of experiments should be conduct on units. By doing so, the values of input variables and its response variable can be determined. The input variables can be presented in a design matrix as follows.

$$\mathcal{D} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1k} \\ x_{22} & x_{22} & \cdots & x_{2k} \\ . & . & \cdots & \\ . & . & \cdots & \\ . & . & \cdots & \\ x_{n1} & x_{n2} & \cdots & x_{nk} \end{bmatrix}$$

where X_{ui} is the Uth design set of input variable X_i ($i = 1, 2, \ldots, k$; $u = 1, 2, \ldots, n$). Each row of this matrix presents a design point. If y be a response that is obtained by applying Uth design point, it can be calculated through Eq. (5)

$$y = f'(x_u)\beta + \epsilon_u, \quad u = 1, 2, \ldots, n \tag{5}$$

where ϵ_u presents the error of Uth design point. Equation (5) can be rewritten as Eq. (6).

$$y = X\beta + \epsilon \tag{6}$$

where $y_u = (y_1, y_2, \ldots, y_n)'$, X is a $n \times p$ matrix in which Uth row is $f'(x_u)$, and $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)'$. The elements of the first column are valued by 1 s to enable fitting with nonzero intercept. This is assumed that the mean of ϵ is zero and its variance is $\sigma^2 I_n$ (Rencher and Schaalje 2008). In this regard, OLS can provide a linear estimator of unknown parameters as shown in Eq. (7)

$$\widehat{\beta} = \left(X'X\right)^{-1} X'y \tag{7}$$

with the following variance–covariance matrix.

$$\mathrm{Var}(\widehat{\beta}) = \sigma^2 \left(X'X\right)^{-1} \tag{8}$$

Accordingly, Eq. (9) estimates the response mean at the design point u.

$$\widehat{\mu}(x_u) = f'(x_u)\widehat{\beta}, \quad u = 1, 2, \ldots, n \tag{9}$$

In addition, by considering the mentioned point at the previous section, this is clear that $f'(x_u)\widehat{\beta}$ is the value of response prediction. On the whole, in each design point in region of experiment, the prediction of response can be achieved as Eq. (10).

$$\widehat{y}(x) = f'(x)\widehat{\beta}, \quad x \in R \tag{10}$$

where R is the feasible region of experimental factors.

Since $\widehat{\beta}$ is a unbiased estimator for B, $\widehat{y}(x)$ is a unbiased estimator for $f'(x)\beta$. As a result, the variance for $\widehat{y}(x)$ can be obtained by applying Eq. (11) (Díaz-García et al. 2005).

$$\mathrm{Var}\left[\widehat{y}(x)\right] = \sigma^2 f'(x)\left(X'X\right)^{-1} f(x) \tag{11}$$

As shown earlier, we can only decide on $X'X$ since other elements are supposed to be fixed, constant, or predetermined. Therefore, values in the design matrix have

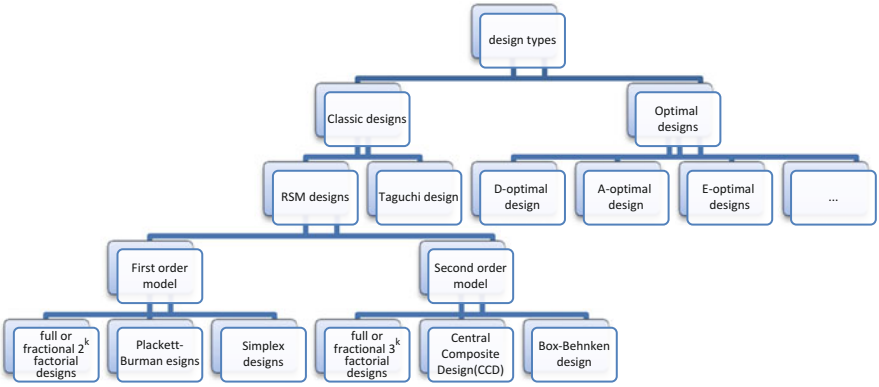

Fig. 6 Classification of design methods

a meaningful effect on the variance of prediction. The vital point of first phase of response surface methodology is to construct the proper design matrix. With respect to Eq. (11), it can be inferred that by choosing a proper design matrix that satisfy some criteria, a relevant decrease in prediction variance would be obtained, and as a result a significant improvement in accuracy of estimations would be provided.

In order to reach the mentioned purposes, the design matrix should have some features, among which orthogonality and rotatability have a more importance in comparison to others. If the matrix $X'X$ be a diagonal, then the design matrix is orthogonal. The great merit of orthogonal matrix is that the elements of $\widehat{\beta}$ will be uncorrelated. The ability of rotation can be defined in terms of the constant prediction variance for each design point. In simple words, each design point in a matrix with the ability of rotation has a same distance from the design center. The main merit for such a matrix is that the prediction variance remains unchanged under any rotation of axes. Since the prediction variance is constant and different responses can only be compared based on their mean values, the process of comparison among them can become easier than normal condition (Ai and Mukhopadhyay 2010).

Various types of design can be categorized into two main classes: classic and optimal designs. Figure 6 shows these two main groups and their subgroups.

To begin with, in classic RSM design various methods for first order designs are available, among which 2^k factorial, Plackett–Burman, and simplex design are the most prevalent ones. In a 2^k factorial design each input variable is set at two levels, coded by $+1$ and -1. By doing so, a design matrix will be constructed including all possible combinations of input variables. Thus, this design requires to run $n = 2^k$. For example, suppose the lifetime of a lamp depends on two factors, namely, voltage and temperature. The condition of experiments is as follows: The voltage is set at 120 and 220 V and the temperature falls into 30 or 40 °C. By scaling process the

mentioned values can be converted to 1 and −1. The required number of runs for this example equals to $2^2 = 4$. The design matrix for this example has been shown below.

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ -1 & -1 \end{bmatrix}$$

The 2^k factorial designs result in an orthogonal matrix. However, by increasing the number of input variables the number of required design points grows drastically. Hence, a significant increase in cost will be generated. By considering this matter of fact that in a first order model only $k+1$ unknown parameters should be estimated, to overcome the mentioned problem the 2^k factorial design can be conducted by a fraction of design such as half or one-fourth fraction design which is greater than $k+1$. The last design is called fractional factorial design and is very common in design of experiments. The other method in this subgroup is Plackett–Burman designs which have a vast application in design of experiments. Plackett–Burman designs such as 2^k factorial designs assume two levels for each input variables but require fewer number of design points (precisely $n = k+1$) compared to 2^k factorial design. Therefore, Plackett–Burman designs can reduce cost of experiments. The last design is simplex in which $k+1$ design points are required. Although the simplex designs based on aforementioned note are economical, because of its difficult computations usually are not of interest. All the mentioned designs are orthogonal and it is crystal clear that orthogonal design matrix provides best results for estimation of unknown parameters.

The most prevalent designs for second order models are, namely, the 3^k factorial designs, the Central Composite Design (CCD), and Box–Behnken designs. The 3^k factorial designs have a same procedure like 2^k factorial designs but set variables in three levels, namely, 1, 0, −1. Because of the mentioned reasons in the last section fractional factorial designs are considered in these designs. The minimum number of design points which are required in a second order model like Eq. (4) is $h = 1 + 2k + \frac{1}{2}k\,(k-1)$. Probably, the most applicable design for second order models is CCD. CCD has well-organized steps to construct design matrix. In the first part CCD, 2^k design points are selected based on first order model. Hence, the first part is complete by using first order model information. In the second part of CCD, more details about the second order model are added to design matrix. Finally, Box–Behnken designs need a small part of a 3^k factorial design. Therefore, this design has a great cost benefit and is adequate for real industrial problems (Ai and Mukhopadhyay 2010).

The other subgroup for classic designs is Taguchi design that mainly deals with nuisance factors or dynamic environments. Due to a simple procedure, this group of design is very applicable for practitioners. According to Taguchi's viewpoint of robustness and minimum variations in quality issues, these designs are constructed with a special focus on orthogonality measure (Taguchi et al. 2005).

Optimal designs are another important class of design matrix. This type of design is based on some optimality criteria. The various criteria have a same goal which is the closeness of the estimated response to the mean response (Atkinson et al. 2007). In other words, such criteria consider the confidence region of unknown coefficient, B, and try to minimize this region in different indices. The confidence region is a general form of a confidence interval in a multidimensional space. Commonly, this region has an ellipsoid shape near the estimated point. Each point in the confidence region has a probability to occur. Therefore, by minimizing the area of this region the accuracy of estimation would be improved. All the different optimality criteria using different methods try to make this region as small as possible. By doing the last step, the response prediction variance would be minimized and a proper design matrix will be obtained. To explain in more details, the variance–covariance matrix of predicted unknown parameters vector, $\widehat{\beta}$, is presented as follows.

$$\sum \widehat{\beta} = \begin{pmatrix} \mathrm{Var}\left(\widehat{B}_0\right) & \cdots & \mathrm{Cov}\left(\widehat{B}_0, \widehat{B}_k\right) \\ \vdots & \ddots & \vdots \\ \mathrm{Cov}\left(\widehat{B}_0, \widehat{B}_k\right) & \cdots & \mathrm{Var}\left(\widehat{B}_0\right) \end{pmatrix}$$

As can be seen from the variance–covariance matrix of $\widehat{\beta}$, the main diagonal elements are the variances of estimated coefficients. Hence, the minimizing process should be performed on these elements. With respect to Eq. (8) if it supposed that the value of σ^2 is one, then for minimizing the predicted variance of model coefficient only the $(X'X)^{-1}$ must be considered. The inverse of variance–covariance matrix is called information matrix (Goos and Jones 2011). Information matrix presents the available information of the model. It is crystal clear that by more information the accuracy of estimation will increase. If a criterion is defined to minimize the predicted variance of factors, it can have their application to maximize the information in the information matrix. Therefore, it can be claimed that all optimality criteria can be defined in two different ways that have same target.

Now that the aim of optimal designs is illuminated, different optimality criteria will be introduced. Probably, the most popular optimality criterion is D-optimality. The D is a clue of determinant. In a D-optimal design the determinant of $(X'X)^{-1}$ would be minimized. In other words, the determinant of information matrix should be maximized. The D-optimality criterion has a logical assumption by which, minimization of the determinant of $(X'X)^{-1}$ leads to smaller variances in the prediction step. It should be considered that if $(X'X)^{-1}$ be a diagonal matrix, then the determinant of $(X'X)^{-1}$ is equal to product of diagonal elements of this matrix. This case makes the last mentioned assumption easy to understand. In addition, when $(X'X)^{-1}$ is orthogonal the unknown parameters can be estimated independently. However, the orthogonally assumption for $(X'X)^{-1}$ might be violated in real–world situations especially where some nuisances are to be included in the statistical model. In this regard, one of the most prominent advantages of optimal designs is that such designs are robust against not satisfying of orthogonally assumption.

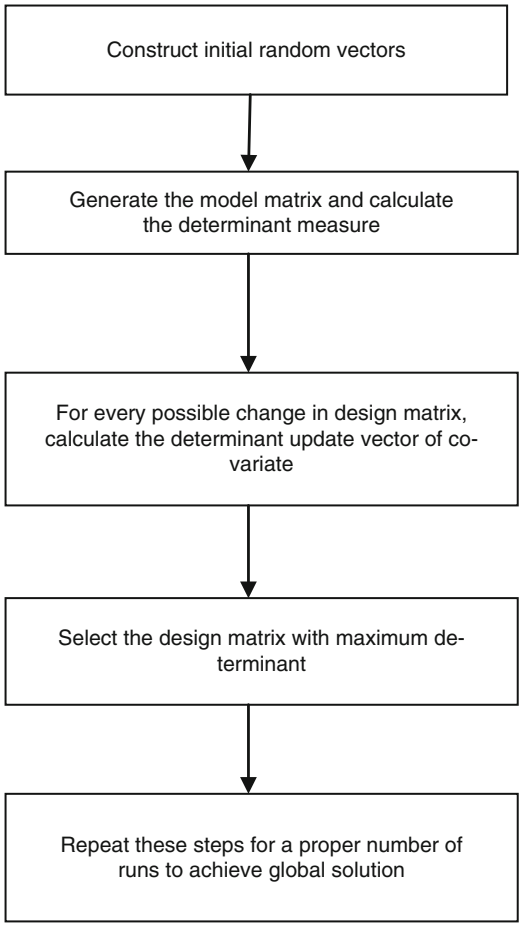

Fig. 7 Coordinate exchange algorithm steps

In fact, optimal designs will remain a proper method for nonorthogonal matrices. Although nonorthogonal matrix might generate inaccuracy in model by variance inflation as well as covariance among estimated parameters, optimal designs obtain best combinations of factors among available designs so that the violation would be small as possible. Compared to the factorial design, optimal design would consider the variance of prediction by assuming some relationships models between response and factors. Therefore, the design points are determined so that the variance of prediction is minimized.

A great number of algorithms are available to obtain *D*-optimal designs, one of which is coordinate exchange algorithm. Coordinate exchange algorithm is a useful algorithm with clear steps to minimize the determinant of the variance–covariance matrix and has a wide application in constructing of *D*-optimal designs. The steps of coordinate exchange algorithm are shown in Fig. 7.

For the mentioned reasons, in this study the coordinate exchange algorithm has been used for constructing a *D*-optimal design. The details of these steps are explained as follows.

1. To begin with a random design matrix should be constructed. Before starting to construct a random design matrix a recommended prerequisite which is factors scaling should be introduced. Although scaling of variables is not an absolute necessity for generating an optimal design, this process brings some merits to coordinate exchange algorithm. By doing so, the continuous variables are set between –1 and 1. In addition, categorical variables which have two levels can be set at –1 and +1. The other kinds of categorical variables can also be coded based on the condition of experiment. Scaling of variables has an advantage for comparing factors' effect. For instance, when two factors X_1 and X_2 have a same scale and the model coefficient of which are $B_1 = 3$ and $B_2 = 6$, it can be inferred that X_2 has a twice impact than X_1 on the response variable. By taking all these into account, to construct a random design matrix random values between -1 and $+1$ must be assigned to the controllable scaled variables. In addition to controllable variables, the design matrix is contained covariates which might have a stochastic behavior. Commonly in practice a specific distribution function such as normal distribution is defined for covariates. Hence, a random value with respect to the mentioned distribution function should be generated for covariates. By doing so, a random design matrix is constructed and the others steps would be performed on this matrix.
2. Since the optimality criterion is to minimize the determinant of variance–covariance matrix, firstly the information matrix should be calculated. In this regard, the model matrix X would be generated. The model matrix is obtained through using design matrix. In fact, each row of the model matrix represents the value of each term in the model for a specific design point. After obtaining the model matrix, the information matrix can be constructed through the mentioned method. This should be considered when the computation of determinant for a small information matrix is simple but for more complicated matrices these calculations become very time consuming. Therefore, this process must be performed by using computer programs. By obtaining the determinant of initial information matrix a start point for algorithm is created. The rest of the algorithm includes replicating steps designed to maximize the mentioned determinant.
3. In third step, every possible change in design matrix would be performed. For any unique design matrix the determinant of information matrix should be calculated and saved. To do so, an element-by-element process should be applied. In this regard, when a change occurs in an element the other elements should be remained constant to make possible the comparison and evaluation of each change in design matrix. This must be considered when covariates are not controllable variables. Thus, the effects of such variables should be removed from the determinant of information matrix. In this regard, for every change in design matrix a sufficient number of random values would be generated for each covariate. By considering the mentioned point, for each change in design

matrix a great number of design matrices with different covariate column would be obtained. Now the determinant measure for the obtained design matrix must be computed. The sample average of these determinants can be proposed as the determinant of the information matrix for every change. By doing so, the effect of covariates will be justified in the computations. Note that the updating process of covariates should be performed equal to a sufficient number for every change in design matrix. This process would be continued till all the possible changes in design matrix terminate. It is worthy to note that this procedure is a kind of Monte-Carlo approach. The design matrix associated to the maximum value of determinant of information matrix would be chosen as the best combination of factors for running the experiments.

4. The obtained solution from the last step is a local optimum solution. To explain in more details, this solution is the best one among the neighborhood of this point. However, it can be possible that by starting with a different initial random design matrix a better solution is obtained. In this regard, to achieve the global optimum solution the last three steps should be repeated for a sufficient number of runs. It is like a mountain climber who has climbed a hill, by looking around the hill he cannot come to the conclusion that it is the highest point in that mountain. However, to find the highest point in that mountain he should try climbing different hills. This action should be repeated until the highest hill in the mountain can be recognized. By doing so, the global optimum solution is likely to be a design matrix which will obtain the greatest determinant of information matrix among all the repeated loops in the algorithm. The final solution of algorithm is the best combination of factors for conducting the experiments.

3.2 Parameter Estimation

In order to achieve an appropriate relationship between response and factors in an accelerated life test model, an adequate method of estimation should be selected. More precisely, after obtaining the best design points for constructing the model, the unknown coefficients of model must be estimated through a point estimator. Since the process of estimation is performed in the obtained design from the last step, the predicted variance of response variable becomes as small as possible. With respect to aforementioned methods of point estimation, in this subsection four main methods of point estimation are introduced, namely, least square error (LSE), the method of moments, MLE, and Bayes method. To begin with LSE is a useful method to minimize the difference between the real value of response variable and the estimated value of it. This difference can be called as error, because the ideal estimated points are those of ones which are nearest to the real value of response variables.

The other main method of estimation is the method of moments. The method of moments is one of the oldest methods of estimation. This method contains some equivalency equations between the initial moments of sample and population.

The number of equations is equal to the number of unknown parameters that must be estimated in the model. For instance, the number of equations to estimate the mean and variance of a normal distribution is two. The method of moments has a broad application in statistical and other fields of science.

Bayesian estimators are applied when prior information about the unknown parameters is available. The risk of a bad estimation is defined by some loss function such as mean square/absolute errors. The estimation has a minimum expected loss function over the prior distribution of the parameters (El and Casella 1998).

The last but the most prominent method of estimation in reliability analysis is MLE. The MLE has a probabilistic basis for computations. In the MLE a likelihood function would be constructed. The likelihood function is a relationship that demonstrates the occurrence probability of real response variable values. An illustrative example is presented to explain the process of constructing the likelihood function. For example, in ALT parts which are conducted in stress levels have a Bernoulli distribution with P as the probability of failures. In this test five parts are conducted in an accelerated life test. In the inspection stations two first parts are failed. Hence, the probability of happening the mentioned situation (likelihood function) is as follows.

$$\begin{aligned} L &= p\left(x_1 = 0, x_2 = 0, x_3 = 1, x_4 = 1, x_5 = 1\right) \\ &= P\left(x_1 = 0\right) P\left(x_2 = 0\right) P\left(x_3 = 1\right) P\left(x_4 = 1\right) P\left(x_5 = 1\right) \end{aligned} \tag{12}$$

where if $x_{\mathrm{i}} = 0$ the part i is failed and reversely. Since the parts have a Bernoulli distribution the likelihood function can be rewritten as follows

$$L = (1-P)(1-P)(1-P)(P)(P) = (1-P)^2 P^3 \tag{13}$$

The obtained likelihood function demonstrates the occurrence probability of observed results in the sample. Therefore, the implication of likelihood function can be understood by the mentioned example.

The next step in the MLE method is to maximize the obtained likelihood function with respect to unknown parameters of distribution. By doing so, the estimated parameters achieve the best prediction for response variables. In other words, the value that maximizes the likelihood function estimates the unknown parameters of model based on observed results of experiments. Thus, the response variables estimated by these parameters are properly estimated. To explain in more details, the mentioned example is extended to the second step. In this step, the likelihood function should be maximized. Therefore, the obtained likelihood function should be derived with respect to P and the result would be equal to zero. In this regard the following equation is achieved.

$$\frac{\partial L}{\partial P} = P^2\left(5P^2 + 8P + 3\right) = 0 \tag{14}$$

By solving the above-mentioned equation, the P is estimated by 0.6. It can be claimed that if $P = 0.6$ then the occurrence probability of observed events is maximum. Therefore, the best possible estimation would be achieved.

Although two main steps of MLE are constant and clear, some details can be varied in the method with respect to the problem conditions. In the case of constructing an accelerated life test, experimenters are confronting with two major issues. First, commonly ALT involves censored data which have a profound impact on estimation of unknown parameters. Second, due to a great number of unknown parameters in the relationship between response variable and input variables, which are contained stress and design variables as well as covariates, the system of equations for maximization step in MLE become very large. In these regards, to make an adequate estimation for an ALT some additional steps should be considered in the two main steps of MLE.

By considering the type of censored data which are discussed in the previous section, a proper likelihood function should be constructed for available data as well as censored data. To achieve this objective, depending on the type of censored data the probability of each event is evaluated in the likelihood function. Right censored data can be seen in a broad spectrum of accelerated life tests. In this regard, in this study assumed that data are right censored. As mentioned in the previous section, the failure time of a right censored observation is not available for estimation of parameters. If censored data are ignored or analyzed similar to complete data a significant error in estimations will be generated. In order to cope up with this problem, the censored data should be considered in a model with a proper likelihood function. When a failure time is censored from right side, it is clear that the lifetime of the part is equal or greater than the censoring time. By accepting this fact as true the mentioned part is not failed before censoring time. Hence, the cumulative distribution function is a proper representative term for right censored data in likelihood function. More precisely, cumulative distribution function demonstrates the summation of probabilities before a specific value, which in the case of right censored data in ALT is equal to censoring time. If $f(t)$ presents the lifetime distribution function of the parts, then the lifetime cumulative distribution function, $F(t)$, for censoring time can be obtained through Eq. (15).

$$F\left(t = T\right) = p\left(t \leq T\right) = \int_0^T f(t)dt \tag{15}$$

where T is the censoring time. In view of these considerations, the likelihood function can be constructed for an accelerated life test model with the right censored data as Eq. (16).

$$L\left(\theta_1, \ldots, \theta_p, \ldots, \theta_k\right) = \prod_{i=1}^{n-m} f\left(t_i, \theta_1, \ldots, \theta_p, \ldots, \theta_k\right) \prod_{i=n-m+1}^{n} F\left(t_i = T, \theta_1, \ldots, \theta_p, \ldots, \theta_k\right) \tag{16}$$

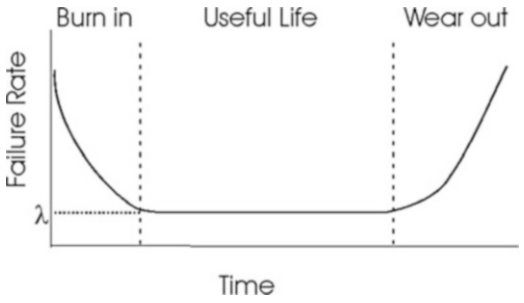

Fig. 8 Lifetime cycle

where $i = 1, \ldots, n$, n is the number of experiments, and m is the number of right censored observations. In addition, θ_p is the parameter of distribution functions and k is the number of these unknown parameters. It is clear that in this step the lifetime distribution function is required to be known. In the design of experiments step, the process can be performed by applying LSE which is independent of the lifetime distribution of parts. In this regard, a great number of choices are available, among which log-normal, Weibull, and exponential distribution have more applications in accelerated life testing and reliability analysis. Some notes should be considered before choosing the life time distribution. Firstly, the region of interest in accelerated life tests commonly located in useful time period. In the useful period that are shown in Fig. 8 the failure rate is approximately constant and this not dependent on time. Therefore, because of features of negative exponential distribution, it is a proper choice for lifetime of parts. In addition, this choice simplifies the future calculations for estimating process.

Although the proposed methodology has not any limitations about the type of lifetime distributions, based on the mentioned reasons considering exponential distribution for parts in the next section is a logical choice. If the lifetime distribution has only one unknown parameter, then the likelihood function can be rewritten as Eq. (17).

$$L\left(t, \Theta\right) = \prod_{i=1}^{n-m} f_t\left(t_i; \Theta\right) \prod_{j=n-m+1}^{n} F_t\left(t_j; \Theta\right) \tag{17}$$

where F_t is the cumulative distribution function for exponential distribution. As mentioned in the previous sections, in an accelerated life test the unknown parameters of lifetime distribution are a function of input variables. In fact, although the unknown parameter of lifetime distribution is not dependent on time, it is dependent on stress and design variables as well as covariates.

$$\begin{aligned}\theta_i = B_0 + \sum_{j=1}^{s} B_{1j} S_{ji} + \sum_{t=1}^{x} B_{2t} X_{ti} + \sum_{d=1}^{c} B_{3d} C_{di} + \sum_{j=1}^{s}\sum_{t=1}^{x} B_{4jt} S_{ji} X_{ti} \\ + \sum_{j=1}^{s}\sum_{d=1}^{c} B_{5jd} S_{ji} C_{di} + \sum_{t=1}^{x}\sum_{d=1}^{c} B_{6td} X_{ti} C_{di} + E_i\end{aligned} \tag{18}$$

where S, X, and C represent the stress variables, design variables, and covariates, respectively. In addition, S, X, and C are the number of stress variables, design variables, and covariates. The vector of coefficients $B = (B_0, \ldots, B_w)$, where w is equal to summation of main effects and first order interactions, is the unknown parameters of model. Therefore, using the MLE method these unknown parameters should be estimated. Before constructing the likelihood function one point should be considered that the type of relationship between the parameters and input variables should be selected based on features which are described in previous section. Thus, the first order interaction model chosen for this study is an arbitrary choice which can be varied depending on the purpose of the research. However, in the proposed first order interaction model the interaction between covariates and other controllable variables is considered which can improve the accuracy of estimation, because some of deviations from the mean of response variable are based on these terms in the model. All in all, to construct the likelihood function for proposed problem the relationship for distribution parameter should be located in Eq. (14). By doing so, the likelihood function would be depending on the coefficients stress variables, design variables, covariates, and their interactions. By doing so, the likelihood function would be constructed and the second step of MLE method will be started.

The aim of this step is to maximize the value of likelihood function. The usual method in this step is to analyze the derivations of likelihood function with respect to vector B and find its roots. Broad spectrums of methods are provided to solve this system of equations. In the one hand, some methods such as Gradient methods and Simlex (Nelder–Mead) are methods which will obtain local optimum solutions. On the other hand, Simulated Annealing (SA) and Genetic algorithm are methods which will obtain global optimum solutions. In addition, the first group is not effective for large-scale models with a great number of parameters, while the second group contains methods that are appropriate for the mentioned models. More precisely, by increasing the number of unknown parameters in the likelihood function, the number of equations in the system of equations will increase. Hence, solving the system of equation becomes a major problem. In this situation using the second group of methods is more appropriate.

Hence, to overcome the mentioned problem about difficulty in calculation for maximization of likelihood function, the second group and particularly SA algorithm can be used in this study and an optimization model can be constructed. It is clear that the objective function of this model is to maximize the likelihood function. However, due to the simplicity in calculations a natural logarithm of likelihood function is usually used as the objective function. This should be considered that the optimization process should be performed in the region obtained in the design of experiments phase. Hence, the design variables, stress variables, and covariates set in their design points value for each run and the lifetime of parts will be calculated for every design point. The estimation process is performed based on these design points value and the related responses.

In order to solve this optimization model, a meta-heuristic algorithm which is called Simulated Annealing is applied in this study. Simulated Annealing (SA) is a local optimization method for solving hard optimization problems. Firstly ideas about this method were proposed by the work of Kirkpatrick et al. (1983) who applied some similarities between simulating the annealing process of solids in optimization problems. Afterwards, SA has been used to optimize variety of problems in areas such as locational analysis, image processing, molecular physics and chemistry, and job shop scheduling (Eglese 1990).

The main steps of SA in analyzing the optimization problem can be expressed as follows.

- Define set, S, of feasible solutions
- Calculate objective function $f(s)$ for each solution
- Define a neighborhood structure which can be obtained by making a move in the current solution, a move being the change in value of one or more variables.
- Do the following iterative steps
  - Starting with an initial solution s generated by other means
  - Repeatedly move to a neighboring solution that meet one of the following conditions:
    - Having a better objective function
    - Having a worse objective function but the neighborhood solution meets certain probability. The probability of accepting an uphill move is normally set to exp $(-\Delta/T)$ where T is a control parameter which corresponds to temperature in the analogy with physical annealing, and Δ is the change in the objective function value.
  - Update the parameters such as T.
- Stop algorithm when one of the following criteria are met
  - Specified number of iterations
  - Specified change in two successive objective function
  - Specified process time

After solving the optimization model by SA algorithm the most accurate possible estimations of coefficient of models vector B would be obtained. Hence, locating this vector in the failure rate relationship, the estimation of failure rate will be created. By doing so, in different times the reliability of parts can be estimated. Therefore, the first main goal of this study is satisfied. In this study the MATLAB optimization toolbox is used for solving the optimization model by SA algorithm. Hence, more information about the steps of this algorithm can be found in software. More detailed for this section is presented in numerical example.

3.3 Multiobjective Optimization

In this phase, a multiobjective optimization model has been proposed to maximize the expected value of the predicted response variable and to maximize the probability of the covariates. After obtaining the proper life–stress relationship, by a few converts the reliability function can be constructed. It is clear that the improvement of reliability is the most important aim of this study. Hence, firstly the proper model for this optimization problem should be designed. Secondly, the adequate method to solve the obtained optimization model will be selected. By considering this matter of fact as a true that the reliability is a function of lifetime, the reliability function simply will be generated by using the obtained lifetime model from the last step. The other objective of the proposed optimization model is to maximize the probability of covariates. Since the covariates are stochastic and uncontrollable variables, the second objective improves the region of optimization process. In other words, by applying the second objective, results would be given with more likely values for the covariates. It is accepted as a true that considering the real experiment conditions in optimization process will develop the quality of results. All in all, the goal of the proposed optimization model is to maximize the reliability, while it is tried to consider the most probable region of the stochastic covariates.

The reliability $R(t, \Theta)$ is depended to the time in lifetime distribution, which t is the time and Θ is the parameter of lifetime distribution for parts. In addition, based on the aforementioned discussions in the previous sections the lifetime is a function of stress and design variables as well as covariates. Thus, the reliability function can be rewritten as $R(t; S, X, C)$, which S, X, and C are representatives of variables and defined in the previous section. Consider the reliability function given below (Smith 1983):

$$R(t, \Theta) = \int_{t}^{\infty} f(t, \Theta)\, d(t) \tag{19}$$

where $f(t, \Theta)$ is the lifetime distribution function. One might be interested in analyzing the reliability by available information about the failure rate. In this case, failure rate is a function of stress variables, design variables, covariates, and time. By considering the implication of failure rate the reliability function can be rewritten as equation given below (Smith 1983).

$$R(t, s, x, c) = \mathrm{e}^{-\int_{0}^{t} \lambda(t, s, x, c)\, dt} \tag{20}$$

which should be maximized as the first objective function of the optimization model.

A decision about the distribution of covariates should be made to construct the second objective. In this regard, a broad spectrum of distributions can be assigned to the covariates. Hence, to select best distribution for such variables different

aspects of the covariates should be evaluated. With respect to the mentioned points about the covariates, the second objective can be constructed as maximization of probability distribution function of covariates (Hejazi et al. 2011; Salmasnia et al. 2013). In addition to objective functions, the model contains some constraints about the acceptable region and limitations for stress variables, design variables, and covariates. Therefore, the optimization model can be constructed as follows

$$\begin{gathered}\text{maximize } \mu\left(\widehat{R}\right)(t,s,x,c) = \mathrm{e}^{-\int_0^t \mu\left(\widehat{\lambda}(t,s,x,c)\right)dt} \\ \text{maximize } f(C_t) \quad \forall t = 1,2,\ldots,l \\ \text{Subject to :} \\ L < X < U \\ S \text{ fixed in specified use stress} \\ C \in \Omega\end{gathered} \tag{21}$$

where $\mu\left(\widehat{R}\right)$ and $\mu\left(\widehat{\lambda}(t,s,x,c)\right)$ are the means of estimated reliability and failure rate. In addition, L is the lower bound and U is the upper bound for design variable X and Ω is the feasible region for covariates.

To explain the constraints of model, the first constraint defines acceptable limits for the design variables. The design variables should be fallen between their lower and upper bounds. This interval is specified with respect to the operational condition of problem and the experimental design. The second constraint is about the stress levels. In an ALT stress variables are set at some accelerated levels. Therefore, to optimize the reliability function these variables should change to the operational or design condition. The last constraint refers to acceptable region of covariates. The mentioned region is determined with respect to domain of the covariates' distribution function.

Stress variables are fixed at their use levels in the model. In addition the nature of covariates, which is stochastic, does not allow control of such variables. Therefore, optimization should be performed with respect to design variables. In the first objective the reliability function, which can be obtained in the last section by using the estimated lifetime, would be maximized with respect to design variables. In this regard, by using the obtained information from the accelerated life test and locating the stress variables in their use levels the reliability in operational condition can be optimized. The second objective causes that the stochastic condition consider in the problem. In simple words, by increasing the occurrence probability of covariates the problem will approach to their stochastic condition. To provide more analyses on the effects of the occurrence probability of covariates on the results, some constraints can be added so that the reliability optimization will be conducted as near as possible to the real condition of problem.

After constructing the optimization model a proper approach to solve the model should be selected. Since the optimization model contains two different objectives, the multiobjective optimization methods are adequate tools to solve the model.

The objectives in the presented model have conflict with each other. It is crystal clear that by increasing the occurrence probability of covariates the reliability will be decreased. Through increasing the occurrence probability of covariates the optimization region of the model becomes small and smaller. Therefore, the available points to optimize the reliability function would be limited and a decrease in reliability will be provided. Multiobjective methods are commonly used in such situations. These methods handle quite a few of conflicting objectives in an optimization model to achieve a satisfactory solution. The weighted p-norm, displaced ideal method, goal programming, global criterion, neutral compromise solution, weighted method, ϵ-constraint method, value function, loss function, and desirability are the methods that are frequently applied for multiobjective optimization (Ardakani and Wulff 2013).

Bounded objective method is one of the well-grounded approaches of multiobjective optimization which can be used to get more information and sensitivity analysis to the above-mentioned problem. In this method, the main objective function is maximized while the others considered in constraints with some satisfactory bounds. Since in the mentioned problem the main goal is maximization of reliability, the probability of covariate can be bounded by proper constraints. The key point in this approach is the evaluation of the bound associated to the covariates' occurrence probability. Thus, a sensitivity analysis should be performed on this factor. The optimization model for this approach is demonstrated in Eq. (22).

$$\begin{gathered} \text{Maximize} \quad Z = \mu\left(\widehat{R}\right)(t, s, x, c) \\ \text{Subject to :} \\ f(c) \geq \omega \\ L < X < U \\ S \ \text{ fixed in specified use stress} \\ C \in \Omega \end{gathered} \tag{22}$$

where ω is the lower bound for constraint of covariate probability. This bound could be expressed by the decision maker as a minimum acceptable confidence of results (Hejazi et al. 2011).

4 Numerical Example

In this section a hypothetical example is studied to illustrate the applications of the proposed approach. An exponential distribution with mean 120 h is assumed for the lifetime probability. With respect to the aforementioned reasons the exponential distribution is adequate ones for lifetime data (Eq. (23)).

$$f(t) = \lambda \mathrm{e}^{-\lambda t} \tag{23}$$

where t is the lifetime and λ is the parameter of lifetime distribution which is called failure rate. In this case the response variable λ is considered to be a function of one design variable, two stress variables, and one covariate. The covariate is also assumed to follow a standard normal probability distribution. In addition, stress and design variables are located between –1 and 1. It has also been assumed that the relationship between inputs and outputs is expressed by linear and interacts effects as follows:

$$\lambda_i = B_0 + B_1 S_{1i} + B_2 S_{2i} + B_3 X_i + B_4 C_i + B_5 S_{1i} S_{2i} + B_6 S_{1i} X_i + B_7 S_{1i} C_i + B_8 S_{2i} X_i + B_9 S_{2i} C_i + B_{10} X_i C_i + E_i \tag{24}$$

where i is the index of observations, S_1 and S_2 are stress variables, X is design variables, and C is the covariates. In addition, the vector $B = (B_0, B_1, \ldots, B_{10})$ is a vector of unknown coefficients. Finally, E is the modeling random error. According to the proposed approach, in the first step the optimum design points for conducting the experiments are obtained by modified exchange algorithm described in the previous sections. To do so, firstly an initial solution for the algorithm should be generated. Hence, all the stress and design variables are located in their lower bounds and covariates would be generated using the standard normal distribution. After obtaining the initial solution all the possible changes should be performed on the initial design matrix and the D-optimality criterion would be evaluated for each different design matrix. In this regard, element-by-element changes would be conducted on design matrix and best possible solution will be obtained. To reach the global solution this process should be repeated for a sufficient number of runs, for example, 1,000 runs. The design table and related response variable are presented in Table 2.

Assuming that the experiment was to be terminated after 120 h, observations that show lifetimes greater than 120 h have been censored.

In the next step likelihood function constructed as below:

$$L(\lambda) = \prod_{i=1}^{17} \lambda_i \mathrm{e}^{-\lambda_i t_i} \prod_{i=18}^{20} \mathrm{e}^{-\lambda_i t_i} \tag{25}$$

Because of the mentioned reasons in the last sections, a natural logarithm is performed on aforementioned equation. SA algorithm is a powerful method for mathematical optimization problems. In order to find the constraints for this optimization model some notes should be considered. Since the failure rate cannot get negative value a limitation will be imposed in optimization model. In order to avoid assigning a negative value to failure rate λ, natural logarithm is performed on the relationship between failure rate and input variables and the objective function is maximized with respect to it. The developed likelihood function is maximized using this algorithm in Matlab mathematical software package (Global Optimization Toolbox) with following parameters setting (Table 3).

Table 2 Design table and related responses for numerical example

Number of observations	Design variable, X	Stress variable, $S1$	Stress variable, $S2$	Covariate, C	Response, t
1	1.0000	1.0000	1.0000	–0.5688	599.248
2	1.0000	1.0000	–1.0000	0.3793	315.6477
3	0.4921	–1.0000	–1.0000	2.2987	127.7357
4	0.5531	–1.0000	–1.0000	0.2241	99.2245
5	0.5472	0.0006	0.5108	0.9217	89.8678
6	0.0319	–1.0000	0.0524	1.9916	98.8301
7	0.1317	0.0660	–1.0000	–1.2066	98.9784
8	–1.0000	–1.0000	–1.0000	0.1919	70.1959
9	0.4374	–1.0000	0.3851	0.6947	47.5488
10	0.0705	–1.0000	0.9022	–0.8877	36.0969
11	–1.0000	1.0000	–1.0000	–0.4162	38.9341
12	0.3135	0.7381	0.0404	0.5055	44.2734
13	0.3871	0.5501	0.2439	1.6344	32.5114
14	–1.0000	0.3541	1.0000	–1.0601	16.0658
15	0.0750	0.5073	0.4156	1.8357	22.4051
16	1.0000	0.1897	0.0920	0.2532	17.8965
17	0.6292	0.0224	1.0000	0.0706	29.177
18	0.0407	0.1193	1.0000	–0.3146	4.2173
19	–1.0000	0.9231	0.1504	2.0359	1.5249
20	–1.0000	0.8537	–1.0000	0.8313	0.7068

Table 3 Parameter setting of SA algorithm performed in MATLAB

Parameter	Function tolerance	Annealing function	Temperature updating	Initial temp.
Setting	$1e^{-6}$	Boltzmann function	Exponentially	Default: 100

In order to avoid assigning a negative value to failure rate λ, natural logarithm function is performed on Eq. (24) and likelihood function is maximized.

$$\begin{aligned}\mathrm{Ln}\left(\widehat{\lambda}_1\right) = {} & B_0' + B_1' S_{1i} + B_2' S_{2i} + B_3' X_i + B_4' C_i \\ & + B_5' S_{1i} S_{2i} + B_6' S_{1i} X_i + B_7' S_{1i} C_i \\ & + B_8' S_{2i} X_i + B_9' S_{2i} C_i + B_{10}' X_i C_i + E_i'\end{aligned} \tag{26}$$

The resulted MLE of B' is presented as Table 4.

After the statistical model was being estimated, values of stress variables related to the normal use condition should be utilized for freezing the variables S. Therefore, Eq. (26) would be only a function of the design variables and the covariate.

The goal of this problem is maximization of the reliability. Reliability can be obtained from formula that become in the next.

$$R(t) = \exp - \int_0^t \lambda(t) dt \tag{27}$$

Table 4 Result of maximization on likelihood function

$B_0^{'}$	$B_1^{'}$	$B_2^{'}$	$B_3^{'}$	$B_4^{'}$	$B_5^{'}$	$B_6^{'}$	$B_7^{'}$	$B_8^{'}$	$B_9^{'}$	$B_{10}^{'}$
–4.55428	0.712225	0.612446	1.118208	1.075125	0.438235	0.3018	0.657322	0.48	0.661788	–0.25823

Since in an exponential distribution, parameter lambda is constant, Eq. (27) could be rewritten as an exponential distribution like Eq. (29). Obviously, the minimum point of the lambda function corresponds to the maximum point of the reliability function.

$$P(t) = \lambda \exp(-\lambda t) \tag{28}$$

$$\exp\left(-\int_0^t \lambda dt\right) = \exp\left(-\lambda \int_0^t \lambda dt\right) = \exp(-\lambda t) \tag{29}$$

For the mentioned example, stress variables can be replaced by predetermined operational levels and consequently the relationship between natural logarithm of λ and design variable and covariate would be constructed. For this purpose, stress variables assigned to -2 and using Eq. (24) and Table 2, Eqs. (30) and (31) can be obtained

$$\mathrm{Ln}(\lambda_i) = -5.45068 - 0.44539X_i - 1.5631C_i + 0.25823X_iC_i + E_i \tag{30}$$

and consequently,

$$\begin{aligned} E\left(\mathrm{Ln}\left(\widehat{\lambda}_i\right)\right) &= -5.45068 - 0.44539X_i - 1.5631C_i + 0.25823X_iC_i \\ E\left(\widehat{R}(t)\right) &= \exp(-t\,(\exp(\mathrm{Ln}(-5.45068 - .44539X \\ &\quad -1.5631C + 0.25823XC))) \end{aligned} \tag{31}$$

Above equations represent the life parameter and the related reliability function at normal use condition. For mentioned situation, λ is a function of design variable and covariate. Figure 9 shows this relationship.

As shown in Fig. 9, by increase of design variable to their upper bound, λ has been decreased. Thus, estimation of reliability in region of upper bound is reasonable. If X is fixed in 1, reliability is a function of covariate and time so can be obtained from Eq. (32).

$$R(t) = \exp(-t\,(\exp(\mathrm{Ln}(-5.89607 - 1.30487C)))) \tag{32}$$

In order to simplify the calculations, time unit is changed from hour to day. So Eq. (33) is obtained (Fig. 10).

$$R(t) = \exp(-t\,(24\exp(\mathrm{Ln}(-5.89607 - 1.30487C))) \tag{33}$$

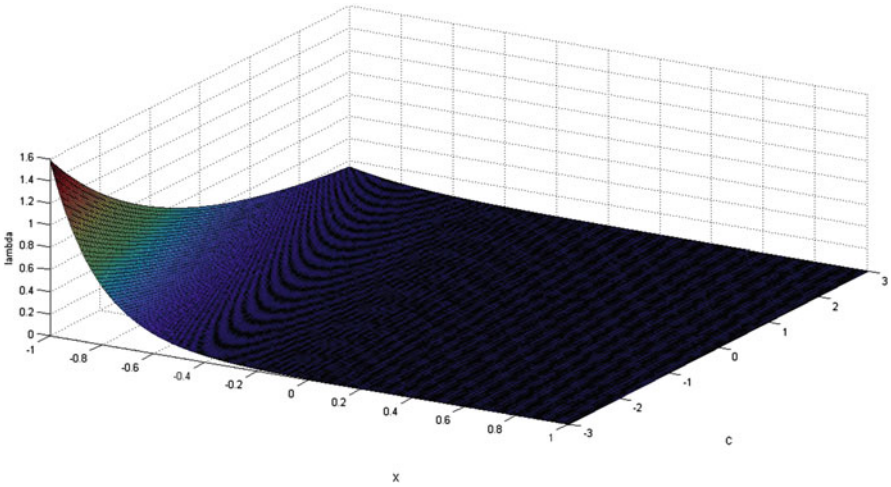

Fig. 9 Failure rate of exponential distribution as a function of the design factor and the covariate

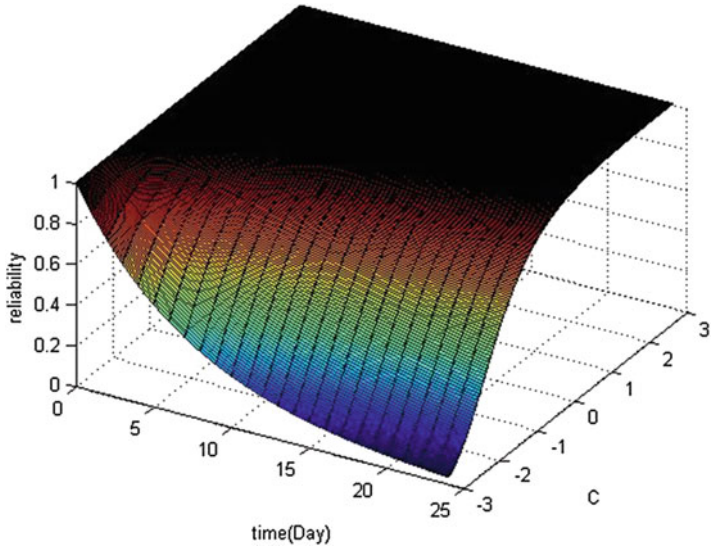

Fig. 10 A plot of reliability function with respect to time and covariate

Table 5 Probabilities of covariate and related mean of lifetimes

Minimum probability value of covariates	Maximum mean lifetime
0.26	585.58
0.3	577.15
0.35	537.55
0.39	454.34
0.4	Infeasible

In next step the mathematical model of the optimization problem can be formulated as follows.

$$\begin{gathered}\text{maximize} \quad \mu\left(\widehat{R}\right)(t,s,x,c) = \mathrm{e}^{-\int_0^t \mu\left(\widehat{\lambda}\right) t\,dt} \\ = \mathrm{e}^{-\int_0^t \frac{\left(\sum_{i=1}^{20} \widehat{B}_0+\widehat{B}_1 S_{1i}+\widehat{B}_2 S_{2i}+\widehat{B}_3 X_i+\widehat{B}_4 C_i+\widehat{B}_5 S_{1i} S_{2i}+\widehat{B}_6 S_{1i} X_i+\widehat{B}_7 S_{1i} C_i+\widehat{B}_8 S_{2i} X_i+\widehat{B}_9 S_{2i} C_i+\widehat{B}_{10} X_i C_i\right)}{20} t\,dt} \\ \text{maximize} \quad f(C) \\ \text{Subject to}: \\ -1 < X < 1 \\ S_1 = S_2 = -2 \\ C \in \Omega\end{gathered} \tag{34}$$

where $\left(\widehat{B}_0, \ldots, \widehat{B}_{10}\right)$ are estimated unknown parameters. The optimum results of this model are obtained using generalized reduced gradient method of optimization built in Microsoft Excel 2010 software. By solving this model the desired probability value for covariate is obtained at 0.26. In addition, according to the results the maximum mean of expected value for lifetime is 585.57 h and the design variable is equal to one that is upper bound of X. By using the obtained value for lower bound of covariate probability the later sensitivity analysis will be performed on this value and its effect on the results. In this regard, the second approach to solve the problem should be performed.

As it can be seen in Table 5 with an increase in the probability values of covariate the mean of lifetime decreases. In addition, when this value exceeds from 0.4 the problem does not have a feasible solution. Therefore, the cost of increase in probability of covariates is to reduce the reliability of products so manufacturer should make an important decision about this matter to hold the reliability in the desired goal whereas the probability of covariates locate in an acceptable range. The probability values of covariates and related mean of lifetimes are presented in Table 5.

Table 6 A comparison between optimization model considering and not considering occurrence probability of covariate

	Maximum mean of lifetime	Occurrence probability of covariate	Maximum mean of expected value for lifetime
Proposed model	2,252	0.26	585.58
Non covariate constraints model	15,358	0.0044	68.06

Now, a model in which covariates are not considered is constructed to make a comparison between an ALT with covariate probability constraint and ones without any covariate probability constraint. This model is shown in Eq. (35).

$$\begin{gathered} \text{Maximize} \quad \mu\left(\widehat{R}\right)(t,s,x,c) \\ \text{Subject to :} \\ -1 < X < 1 \\ -3 \leq C \leq +3 \end{gathered} \tag{35}$$

As it can be seen from the recent model, the only objective function is maximizing the expected value of reliability. In addition, there is no constraint for covariates; since the covariate has a standard normal distribution 99 % of data are located between -3 and 3. Hence, the covariates can assign any value between the mentioned values. In the recent model covariates can get values with a very low probability of occurrence. Therefore, it can be predicted that a significant increase in the value of reliability will be provided. However, by solving the recent model the covariate is set at 3 that has a very low occurrence probability. As a result, if the constraint of covariate probability is not considered, then the solutions would be located in regions which have a few chances to occur. Hence, the proposed approach obtains more logical and realistic solutions than the recent model. Table 6 presents the differences between two mentioned approaches.

The cases, which are claimed in Table 6, are based on the fact that by limiting the region of optimization process to the most probable region of covariates, a significant decrease in the maximum mean of lifetime will be provided logically. On the other hand, a noticeable increase in the maximum mean of the expected value for lifetime will be provided. Since in the noncovariate constraints model, covariates can get value, which have a very low probability of occurrence, it is clear that the obtained results have a logical guarantee.

5 Conclusion

Reliability functions and related measures in complex systems should be derived with respect to operational/technical factors as well as environmental variables. For this purpose, several statistical and computational algorithms have been suggested.

Among them, response surface methodology as a mathematical-statistical approach enables finding an optimal operational condition after it was estimated. In this chapter a constant accelerated life test has been developed to find the settings of variables that optimize the reliability of products when the stochastic covariate is affecting the performance. In this model interactions between covariate and other variables are considered. This model consists of censored data and MLE method used for parameters estimation. Afterwards, a mathematical program was constructed to maximize the reliability of the estimated life performance and to maximize the probability of covariate's occurrences. This approach would take the best settings of design variables and consider the reliability measure as well as the stochastic covariate as an unavoidable part of the experiments. The results show the superiority of the proposed approach either in process identification by proposing new design construction method or in reliability optimization by considering the stochastic covariates that affect the performance.

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