Equivalent 2D Sequential and Parallel Thinning Algorithms

Kálmán Palágyi

Department of Image Processing and Computer Graphics,
University of Szeged, Hungary
palagyi@inf.u-szeged.hu

Abstract. Thinning is a frequently applied skeletonization technique: border points that satisfy certain topological and geometric constraints are deleted in iteration steps. Sequential thinning algorithms may alter just one point at a time, while parallel algorithms can delete a set of border points simultaneously. Two thinning algorithms are said to be equivalent if they can produce the same result for each input binary picture. This work shows that the existing 2D fully parallel thinning algorithm proposed by Manzanera et al. is equivalent to a topology-preserving sequential thinning algorithm with the same deletion rule.

Keywords: Discrete Geometry, Digital Topology, Thinning, Equivalent Thinning Algorithms.

1 Introduction

Thinning is an iterative layer-by-layer erosion until only some skeleton-like shape features are left [6,12]. Thinning algorithms use *reduction operator*s that transform binary pictures (i.e., images containing only black and white points) only by changing some black points to white ones, which is referred to as deletion.

Parallel thinning algorithms are comprised of *reduction*s that can delete a set of border points simultaneously [2,7,13], while sequential thinning algorithms traverse the boundary of objects and may remove just one point at a time [7,13]. In the parallel case the initial set of black points is considered when the deletion rule is evaluated. On the contrary, the set of black points is dynamically altered during a sequential reduction.

Thinning algorithms generally classify the set of black points into two (disjoint) subsets: the deletion rule of an algorithm is evaluated for the elements of its set of *interesting* points, and black points in its *constraint set* are not taken into consideration. Constraint sets comprise the set of interior points (i.e., black points that are not border points) [6] and they may contain some types of border points in subiteration-based (or directional) parallel algorithms [2] or border points that are not in the active subfield in the case of subfield-based parallel algorithms [2]. In addition, endpoints (i.e., some border points that provide important geometrical information relative to the shape of the objects [2])

R.P. Barneva, V.E. Brimkov, and J. Šlapal (Eds.): IWCIA 2014, LNCS 8466, pp. 91–100, 2014.

or isthmuses (i.e., generalization of curve and surface interior points [1]) can also be accumulated in the constraint sets.

Two reductions (i.e., thinning phases) are said to be *equivalent* if they produce the same result for each input binary picture [9]. A deletion rule is called *equivalent* if it yields a pair of equivalent parallel and sequential reductions [9]. As far as we know, no one showed that there exists a pair of equivalent parallel and sequential thinning algorithms.

The sequential approach suffers from the drawback that different visiting order of interesting points may yield various results. Order-independent sequential reductions can produce the same result for any visiting order of the elements in the actual set of interesting points [3,10]. It is obvious that only an order-independent sequential reduction can be equivalent to a parallel one.

In [9] the author gave some sufficient conditions for equivalent deletion rules. This paper shows that the deletion rule of the known 2D fully parallel thinning algorithm proposed by Manzanera et al. [8] is equivalent. Hence an example of a pair of equivalent parallel an sequential thinning algorithms is presented. In addition the topological correctness of these algorithms is also proved.

The rest of this paper is organized as follows. Section 2 gives an outline from basic notions and results from digital topology, topology preservation, and equivalent reductions. Then in Section 3 we rephrase the known parallel thinning algorithm proposed by Manzanera et al. [8]. In Section 4 we show that the considered parallel algorithm is equivalent to a topology-preserving sequential thinning algorithm. Hence the topological correctness of an existing parallel thinning algorithm is also verified. Finally, we round off the paper with some concluding remarks.

2 Basic Notions and Results

We use the fundamental concepts of digital topology as reviewed by Kong and Rosenfeld [6].

Let p be a point in the 2-dimensional digital space $\mathbb{Z}^2$. Let us denote $N_m(p)$ the set of points that are *m-adjacent* ($m = 4, 8$), see Fig. 1. Note that, throughout this paper, all figures are depicted on the square grid that is dual to $\mathbb{Z}^2$.

$\lozenge$	$\blacklozenge$	$\lozenge$
$\blacklozenge$	p	$\blacklozenge$
$\lozenge$	$\blacklozenge$	$\lozenge$

Fig. 1. The considered adjacency relations in $\mathbb{Z}^2$. The set $N_4(p)$ contains point p and the four points marked "$\blacklozenge$". The set $N_8(p)$ contains $N_4(p)$ and the four points marked "$\lozenge$".

The equivalence classes relative to the *m-connectivity* relation (i.e., the transitive closure of the reflexive and symmetric m-adjacency relations) are the *m-components* of a set of points $X \subseteq \mathbb{Z}^2$.

A $(8,4)$ *digital picture* $\mathcal{P}$ is a quadruple $(\mathbb{Z}^2, 8, 4, B)$. Each element of $\mathbb{Z}^2$ is said to be a *point* of $\mathcal{P}$. Each point in $B \subseteq \mathbb{Z}^2$ is called a *black point*. Each point in $\mathbb{Z}^2 \setminus B$ is said to be a *white point*. A *black component* is an 8–component of B, while a *white component* is a 4–component of $\mathbb{Z}^2 \setminus B$.

A black point is called a *border point* in a $(8,4)$ picture if it is 4–adjacent to at least one white point. A black point in a picture is said to be an *interior point* if it is not a border point.

A reduction (on a 2D picture) is *topology-preserving* if each black component (as a set of points) in the original picture contains exactly one black component of the produced picture, and each white component in the output picture contains exactly one white component of the input picture [6].

A black point is *simple* in a picture if and only if its deletion is a topology-preserving reduction [6]. We mention now the following characterization of simple points:

Theorem 1. [6] *Black point p is simple in a picture $(\mathbb{Z}^2, 8, 4, B)$ if and only if all of the following conditions hold:*

1. *$N_8(p) \setminus \{p\}$ contains exactly one black component.*
2. *p is a border point.*

Recall that a deletion rule is equivalent if it yields a pair of equivalent parallel and (order-independent) sequential reductions. The author gave the following sufficient conditions for equivalent deletion rules:

Theorem 2. [9] *Let R be a deletion rule. Let $(\mathbb{Z}^2, 8, 4, B)$ be an arbitrary picture, and let $q \in B$ be any point that is deleted from that picture by R. Deletion rule R is equivalent if the following two conditions hold for any $p \in B \setminus \{q\}$:*

1. *If p can be deleted from picture $(\mathbb{Z}^2, 8, 4, B)$ by R, then p can be deleted from picture $(\mathbb{Z}^2, 8, 4, B \setminus \{q\})$ by R.*
2. *If p cannot be deleted from picture $(\mathbb{Z}^2, 8, 4, B)$ by R, then p cannot be deleted from picture $(\mathbb{Z}^2, 8, 4, B \setminus \{q\})$ by R.*

Reductions associated with parallel thinning phases may delete a set of black points and not just a single simple point. Hence we need to consider what is meant by topology preservation when a number of black points are deleted simultaneously. Various authors proposed sufficient conditions for reductions to preserve topology [4,5,11]. The author established the following ones:

Theorem 3. [9] *A (parallel) reduction with deletion rule R is topology-preserving if the following conditions hold:*

1. *R is equivalent.*
2. *R deletes only simple points.*

3 An Existing Fully Parallel 2D Thinning Algorithm

In this section we recall the 2D parallel thinning algorithm proposed by Manzanera et al. [8]. That existing algorithm falls into the category of fully parallel thinning [2] since it uses the same reduction in each thinning phase (i.e., iteration step). The set of interesting points associated with the reduction of the algorithm contains the set of border points in the actual picture. Hence its constraint set comprises all interior points in the input picture of the actual iteration step.

Manzanera et al. [8] gave the deletion rule of their algorithm by three classes of patterns. The base patterns α_1, α_2, and β are depicted in Fig. 2. All their rotated versions are patterns as well, where the rotation angles are 90°, 180°, and 270°. All α_1- and α_2-type of elements are *removing patterns*, while β and its rotated versions are the *preserving patterns*. A black point is designated to be deleted if at least one removing pattern matches it, but it is not matched by any preserving pattern.

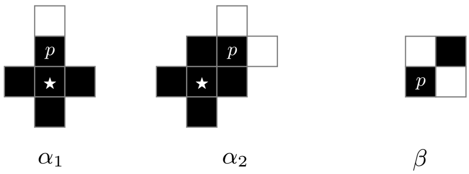

Fig. 2. The three base patterns associated with the 2D fully parallel algorithm proposed by Manzanera et al. Notations: each black position is a black point; each white element is a white point; black positions marked "p" are the central positions of the patterns. (Note that each position marked "★" is an interior point.)

In order to prove that the considered parallel algorithm is equivalent to a sequential thinning algorithm, we rephrase its deletion rule $\mathcal{MBPL}$ by eliminating the preserving patterns. The rephrased rule is given by the set of 32 removing patterns $\mathcal{P} = \{P_1, \ldots, P_{32}\}$, see Fig. 3.

It can be readily seen that the 16 patterns $\{P_1, \ldots, P_{16}\}$ are associated with the removing patterns of type α_1 with respect to the preserving patterns of type β. Similarly, the remaining 16 patterns $\{P_{17}, \ldots, P_{32}\}$ are assigned to the removing patterns of type α_2 with respect to the preserving patterns of type β.

One iteration step of the considered 2D fully parallel algorithm is sketched by Algorithm 1. The constraint set C contains all interior points of the actual binary image, hence deletion rule $\mathcal{MBPL}$ is evaluated for the set of border points $Y = X \setminus C$. A border point $p \in Y$ is deletable (i.e., $\mathcal{MBPL}(p, X) = \textbf{true}$) if at least one pattern depicted in Fig. 3 matches it. Deletable points (i.e., elements of the set D) are removed simultaneously. The entire reduction (i.e., iteration step) is repeated until no points are deleted (i.e., $D = \emptyset$).

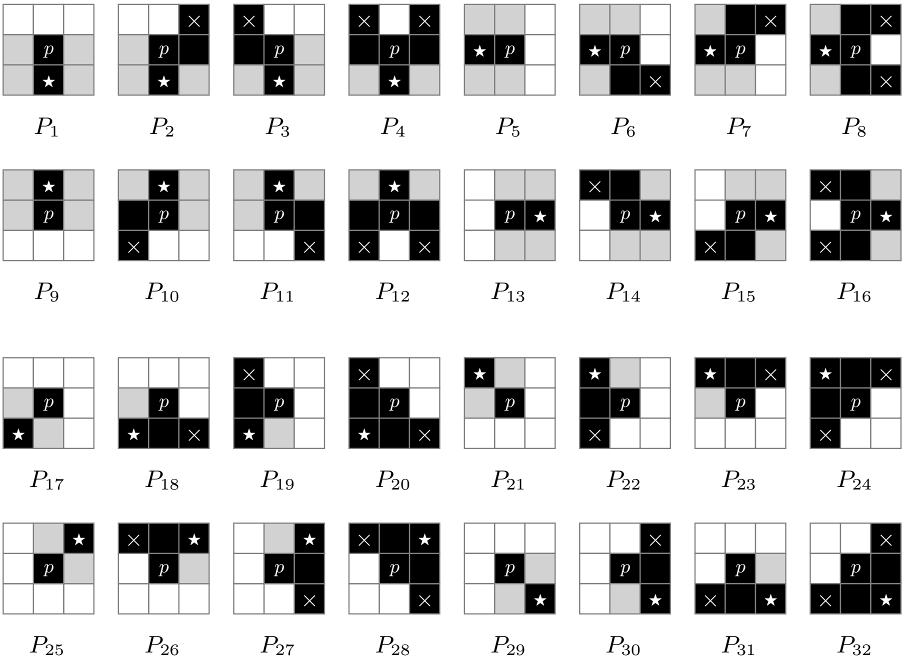

Fig. 3. The set of 32 patterns $\mathcal{P} = \{P_1, \ldots, P_{32}\}$ associated with the deletion rule $\mathcal{MBPL}$ (see Algorithm 1). Notations: each black position is a black point; each white element is a white point; each (don't care) position depicted in grey matches either a black or a white point; "p" indicates the central point of a pattern; each position marked "★" is an interior point; each position marked "×" is a border point (i.e., it is not an interior point).

4 A Sequential 2D Thinning Algorithm Being Equivalent to a Parallel One

We propose a sequential thinning algorithm that uses the deletion rule $\mathcal{MBPL}$ (see Algorithm 1). One iteration step of the derived algorithm is given by Algorithm 2. It (i.e., a phase of the thinning process) is repeated until stability is reached.

The derived sequential algorithm removes the actually visited border point $p \in Y$ right away if it is deletable in the actual image (i.e., $\mathcal{MBPL}(p, X) =$ **true**), therefore the set of black points X is dynamically altered within an iteration step.

We show that the new sequential algorithm is equivalent to the considered parallel thinning algorithm (i.e., Algorithms 1 and 2 are equivalent). In order to prove it, let us state some properties of the deletion rule $\mathcal{MBPL}$ (see Fig. 3). For the sake of brevity, a black point is said to be *deletable*, if it can be deleted by $\mathcal{MBPL}$ (i.e., at least one pattern in $\mathcal{P}$ matches it).

Algorithm 1. one iteration step of the fully parallel thinning algorithm

Input: set of black points X and set of interior points $C \subset X$
Output: set of black points X
$Y = X \setminus C$
$D = \{ \ p \mid p \in Y$ and $\mathcal{MBPL}\,(p, X) = \textbf{true} \ \}$
$X \ = \ X \setminus D$

Algorithm 2. one iteration step of the sequential thinning algorithm

Input: set of black points X and set of interior points $C \subset X$
Output: set of black points X
$Y = X \setminus C$
foreach $p \in Y$ **do**
  if $\mathcal{MBPL}\,(p, X) = \textbf{true}$ **then**
    $X = X \setminus \{p\}$
end

Proposition 1. *Each deletable point is 8-adjacent to at least one interior point.*

Notice that each 3×3 pattern in $\mathcal{P}$ contains a position marked "$\star$" (i.e., an element that is an interior point), see Fig. 3.

Proposition 2. *In each pattern in $\mathcal{P}$, the "opposite" positions of the "$\star$" elements are white points.*

Figure 4 illustrates the "opposite" white points that are associated with an interior point (marked "$\star$"). Since each deletable point is 8-adjacent to at least one interior point by Proposition 1, Proposition 2 holds.

Proposition 3. *All deletable points are simple points.*

It is really apparent that both conditions of Theorem 1 hold from a careful examination of the patterns in $\mathcal{P}$.

Proposition 4. *Deletable points are not interior points.*

Condition 2 of Theorem 1 is not satisfied for interior points. Hence they are not deletable points by Proposition 3.

Proposition 5. *All the 26 patterns in $\{P_3, P_5, \ldots, P_{16}, P_{19}, P_{21}, \ldots, P_{32}\}$ can be derived from the six base patterns in $\{P_1, P_2, P_4, P_{17}, P_{18}, P_{20}\}$.*

Pattern P_3 is a reflected version of P_2; P_{19} is a reflected version of P_{18}; P_5, P_9, and P_{13} are the rotated versions of P_1; P_6, P_{10}, and P_{14} are the rotated versions of P_2; P_7, P_{11}, and P_{15} are the rotated versions of P_3; P_8, P_{12}, and P_{16} are the rotated versions of P_4; P_{21}, P_{25}, and P_{29} are the rotated versions of P_{17}; P_{22}, P_{26}, and P_{30} are the rotated versions of P_{18}; P_{23}, P_{27}, and P_{31} are the rotated

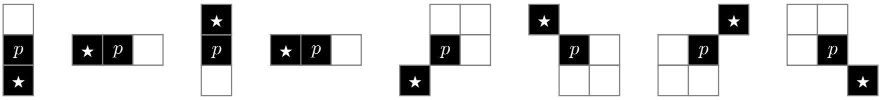

Fig. 4. Interior points marked "★" that are 8-adjacent to the central point "p". The depicted white points are the "opposite" positions associated with the corresponding interior point.

versions of P_{19}; and P_{24}, P_{28}, and P_{32} are the rotated versions of P_{20}. Hence Proposition 5 holds.

Now we are ready to state the key theorem.

Theorem 4. *Deletion rule $\mathcal{MBPL}$ is equivalent.*

Proof. Let $(\mathbb{Z}^2, 8, 4, B)$ be an arbitrary picture. To prove this theorem we must show that both conditions of Theorem 2 are satisfied. Let $q \in B$ be an arbitrary deletable point. Then the following two points are to be proved for any point $p \in B \setminus \{q\}$:

1. If p is a deletable point in picture $(\mathbb{Z}^2, 8, 4, B)$, then p is a deletable point in picture $(\mathbb{Z}^2, 8, 4, B \setminus \{q\})$.
2. If p is not a deletable point in picture $(\mathbb{Z}^2, 8, 4, B)$, then p is not a deletable point in picture $(\mathbb{Z}^2, 8, 4, B \setminus \{q\})$.

Since deletion rule $\mathcal{MBPL}$ is given by 3×3 patterns, there is nothing to prove if $q \notin N_8(p)$. Hence it is sufficient to show that if we alter just one element in any pattern $P \in \mathcal{P}$, then

- we get another pattern $P' \in \mathcal{P}$ (written as $P \Rightarrow P'$),
- we do not get a pattern in $\mathcal{P}$ (i.e., $P \Rightarrow P" \notin \mathcal{P}$), and the considered black position (or the altered white pattern element in question) is not a deletable point, or
- point p is in the constraint set (hence p is not a deletable point before/after the deletion of q).

Let us see the six base patterns of $\mathcal{P}$, see Fig. 5. It is easy to see that we do not need to take "don't care" positions (depicted in grey) into account. Note that pattern elements marked "★" (i.e., interior points) cannot be altered by Proposition 4.

Let us consider the remaining black and white elements of the six base patterns (see Fig. 5).

- Let white point a (in base patterns P_1, P_2, P_{17}, and P_{18}) be the deleted point q.
  - If the point marked "$\diamond$" (a "don't care" position in that patterns that is 8-adjacent to the point a in question) is white, then we do not get a pattern in $\mathcal{P}$, and $a = q$ is not a deletable point, since no deletable points are matched by a β-type preserving pattern (see Fig. 2).

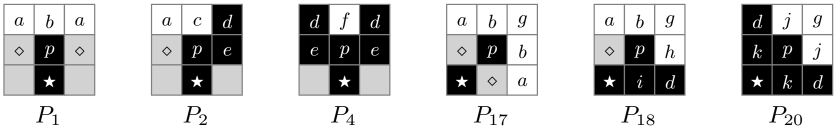

Fig. 5. The six base patterns in $\mathcal{P}$. Note that positions marked "★" are interior points, and black points marked "*d*" are border points (i.e., they are not interior points).

  - If the point marked "◇" is black, then we get another pattern in $\mathcal{P}$: $P_1 \Rightarrow P_2$ or P_3; $P_2 \Rightarrow P_4$; $P_{17} \Rightarrow P_{18}$ or P_{19}; $P_{18} \Rightarrow P_{20}$.
- Let white point b (in base patterns P_1, P_{17}, and P_{18}) be the deleted point q. Then we do not get a pattern in $\mathcal{P}$, and $b = q$ is not a deletable point by Propositions 1 and 2.
- Let white point c (in base pattern P_2) be the deleted point q. Since $c = q$ is a deleted point, c is 8-adjacent to an interior point by Proposition 1. It can be readily seen that black point e may be the only interior point that is 8-adjacent to c.
  - If the point marked "◇" (a "don't care" position in P_2) is black, then p is an interior point before deletion of q. Hence p is in the constraint set, and it is also not a deletable point after the deletion of q.
  - If the point marked "◇" is white, then we get another pattern in $\mathcal{P}$: $P_2 \Rightarrow P_{13}$ or P_{15}.
- Let black point d (in base patterns P_2, P_4, P_{18}, and P_{20}) be the deletable point q. If $d = q$ is deleted, then we get another pattern in $\mathcal{P}$: $P_2 \Rightarrow P_1$; $P_4 \Rightarrow P_2$ or P_3; $P_{18} \Rightarrow P_{17}$; $P_{20} \Rightarrow P_{18}$ or P_{19}.
- Let black point e (in base patterns P_2 and P_4) be deletable point q. Then we do not get a pattern in $\mathcal{P}$, and $e = q$ is not a deletable point by Propositions 1 and 2.
- Let white point f (in base pattern P_4) be the deleted point q. Then we do not get a pattern in $\mathcal{P}$, and $f = q$ is not a deletable point by Propositions 1 and 2. (Elsewise, p is an interior point before deletion of q. Hence p is in the constraint set, and it is also not a deletable point after the deletion of q.)
- Let white point g (in base pattern P_{18}, P_{19}, and P_{20}) be the deleted point q. Then we do not get a pattern in $\mathcal{P}$, and $g = q$ is not a deletable point, since no deletable points are matched by a β-type preserving pattern (see Fig. 2).
- Let white point h (in base pattern P_{18}) be the deleted point q. Since $h = q$ is a deleted point, it is 8-adjacent to an interior point by Proposition 1. It can be readily seen that black point i may be the only interior point that is 8-adjacent to h. Then we get another pattern in $\mathcal{P}$: $P_{18} \Rightarrow P_1$.
- Let black point i (in base patterns P_{18}) be the deletable point q. Then we do not get a pattern in $\mathcal{P}$, and $i = q$ is not a deletable point by Propositions 1 and 2.
- Let white point j (in base pattern P_{20}) be the deleted point q. Since $j = q$ is a deleted point, it is 8-adjacent to an interior point by Proposition 1. It can be readily seen that a black point k may be the only interior point that is 8-adjacent to c. In this case we get another pattern in $\mathcal{P}$: $P_{20} \Rightarrow P_3$ or P_6.

– Let black point k (in base pattern P_{20}) be the deletable point q. Then we do not get a pattern in $\mathcal{P}$, and $k = q$ is not a deletable point by Propositions 1 and 2.

Since the remaining 26 patterns in P_{20} can be derived from the six base patterns by Proposition 5, the proof can be carried out for all elements of the set of patterns $\mathcal{P}$ (that is associated with the deletion rule $\mathcal{MBPL}$). □

Theorem 4 means that the 2D fully parallel thinning algorithm proposed by Manzanera et al. [8] (see Algorithm 1) and the sequential algorithm with the same deletion rule $\mathcal{MBPL}$ (see Algorithm 2) are equivalent.

An easy consequence of Theorem 4 and Proposition 3 is that both algorithms (i.e., the original parallel and the derived sequential ones) are topology-preserving, hence we presented an alternative proof concerning the topological correctness of an existing parallel thinning algorithm.

5 Conclusions

In an earlier work the author laid a bridge between the parallel and the sequential reductions. Some sufficient conditions for equivalent parallel and order-independent sequential reductions with the same deletion rule were given.

This work shows that an existing 2D (fully parallel) thinning algorithm is equivalent to a topology-preserving sequential thinning algorithm. Hence an example is found that a useful parallel algorithm can be replaced by a sequential one.

Acknowledgements. This work was supported by the European Union and co-funded by the European Social Fund. Project title: "Telemedicine-focused research activities on the field of Mathematics, Informatics and Medical sciences." Project number: TÁMOP-4.2.2.A-11/1/KONV-2012-0073.

References

1. Bertrand, G., Couprie, M.: Transformations topologiques discrètes. In: Coeurjolly, D., Montanvert, A., Chassery, J. (eds.) Géométrie Discrète et Images Numériques, pp. 187–209. Hermès Science Publications (2007)
2. Hall, R.W.: Parallel connectivity-preserving thinning algorithms. In: Kong, T.Y., Rosenfeld, A. (eds.) Topological Algorithms for Digital Image Processing, pp. 145–179. Elsevier Science B.V. (1996)
3. Kardos, P., Palágyi, K.: Order-independent sequential thinning in arbitrary dimensions. In: Proc. Int. Conf. Signal and Image Processing and Applications, SIPA 2011, pp. 129–134 (2011)
4. Kardos, P., Palágyi, K.: On Topology Preservation in Triangular, Square, and Hexagonal Grids. In: Proc. 8th Int. Symposium on Image and Signal Processing and Analysis, ISPA 2013, pp. 782–787 (2013)

5. Kong, T.Y.: On topology preservation in 2–d and 3–d thinning. International Journal of Pattern Recognition and Artificial Intelligence 9, 813–844 (1995)
6. Kong, T.Y., Rosenfeld, A.: Digital topology: Introduction and survey. Computer Vision, Graphics, and Image Processing 48, 357–393 (1989)
7. Lam, L., Lee, S.-W., Suen, S.-W.: Thinning methodologies — A comprehensive survey. IEEE Trans. Pattern Analysis and Machine Intelligence 14, 869–885 (1992)
8. Manzanera, A., Bernard, T.M., Pretêux, F., Longuet, B.: n-dimensional skeletonization: a unified mathematical framework. Journal of Electronic Imaging 11, 25–37 (2002)
9. Palágyi, K.: Deletion Rules for Equivalent Sequential and Parallel Reductions. In: Ruiz-Shulcloper, J., Sanniti di Baja, G. (eds.) CIARP 2013, Part I. LNCS, vol. 8258, pp. 17–24. Springer, Heidelberg (2013)
10. Ranwez, V., Soille, P.: Order independent homotopic thinning for binary and grey tone anchored skeletons. Pattern Recognition Letters 23, 687–702 (2002)
11. Ronse, C.: Minimal test patterns for connectivity preservation in parallel thinning algorithms for binary digital images. Discrete Applied Mathematics 21, 67–79 (1988)
12. Siddiqi, K., Pizer, S. (eds.): Medial representations – Mathematics, algorithms and applications. Computational Imaging and Vision, vol. 37. Springer (2008)
13. Suen, C.Y., Wang, P.S.P. (eds.): Thinning methodologies for pattern recognition. Series in Machine Perception and Artificial Intelligence, vol. 8. World Scientific (1994)