A Variant of Pure Two-Dimensional Context-Free Grammars Generating Picture Languages

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Abstract. Considering a large variety of approaches in generating picture languages, the notion of pure two-dimensional context-free grammar ($P2DCFG$) represents a simple yet expressive non-isometric language generator of picture arrays. In the present paper, we introduce a new variant of $P2DCFG$s that generates picture arrays in a leftmost way. We concentrate our attention on determining their generative power by comparing it with the power of other picture generators. We also examine the power of these generators that regulate rewriting by control languages.

Keywords: Two-dimensional arrays, Array grammars, Pure grammars, Context-free grammars.

1 Introduction

Recently, several two-dimensional (2D) picture generating grammars [4,10,11,16,17] have been introduced and investigated. The introduction of these grammars has been motivated by problem areas ranging from tiling patterns through certain floor designs up to geometric shapes. These 2D grammars have been mainly developed based on the concepts and techniques of string grammar theory. In essence, there exist two basic variants–(i) isometric array grammars in which geometric shape of the rewritten portion of the array is preserved, and (ii) non-isometric array grammars that can alter the geometric shape. In the present paper, we discuss *pure 2D context-free grammar* ($P2DCFG$), which is related to (ii) (see [15]). In essence, the notion of $P2DCFG$ involves only terminal symbols as in any pure grammar [5] and tables of context-free (CF) rules. In this grammar, all the symbols in a column or a row of a rectangular picture array are rewritten by CF rules with all symbols being replaced in parallel by

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strings of equal length, thus maintaining the rectangular form of the array. In [1,2,14], various properties of this 2D grammar model are studied.

In string grammars, leftmost derivations (see, for example, [3,6,8]) have been extensively studied. Recall that in the case of context-free grammars, corresponding to an ordinary derivation, there is an equivalent leftmost derivation that rewrites only the leftmost nonterminal in a sentential form (see [7]). In this paper, we discuss leftmost rewriting in terms of $P2DCFG$. In other words, while a $P2DCFG$ allows rewriting any column or any row of a picture array by the rules of an applicable column rule table or row rule table respectively, in the variant under the investigation in the present paper, only the leftmost column or the uppermost row of an array is rewritten. We refer to the $P2DCFG$ working under this derivation mode as $(l/u)P2DCFG$ and the corresponding family of picture languages generated by them as $(l/u)P2DCFL$. We demonstrate that $(l/u)P2DCFL$ and the family of picture languages generated by $P2DCFG$s are incomparable, and that $(l/u)P2DCFL$ is not closed under union and intersection. The effect of regulated rewriting in $(l/u)P2DCFG$s by control languages is also examined, and it is demonstrated that this regulation results into an increase in the generative power.

2 Preliminaries

For notions related to formal language theory we refer to [7,12,13] and for array grammars and two-dimensional languages we refer to [4].

A word or a string $w = a_1 a_2 \dots a_n \quad (n \geq 1)$ over a finite alphabet Σ is a sequence of symbols from Σ. The length of a word w is denoted by $|w|$. The set of all words over Σ, including the empty word λ with no symbols, is denoted by Σ^*. For any word $w = a_1 a_2 \dots a_n$, we denote by ${}^t w$ the word w written vertically, with ^{t} having lower precedence than concatenation, so that ${}^t w = {}^t(w)$. For example, if $w = abb$ over $\{a, b\}$, then ${}^t w$ is $\begin{matrix} a \\ b \\ b \end{matrix}$. A two-dimensional array (also called picture array or picture) is a rectangular $m \times n$ array p over Σ of the form

$$p = \begin{matrix} p(1,1) & \cdots & p(1,n) \\ \vdots & \ddots & \vdots \\ p(m,1) & \cdots & p(m,n) \end{matrix}$$

where each $p(i,j) \in \Sigma, 1 \leq i \leq m,\ 1 \leq j \leq n$. A pixel is an element $p(i,j)$ of p. $|p|_{row}$ and $|p|_{col}$ denote the number of rows of p and the number of columns of p, respectively. The size of p is the pair $(|p|_{row}, |p|_{col})$. The set of all rectangular arrays over Σ is denoted by Σ^{**}, which includes the empty array λ. $\Sigma^{++} = \Sigma^{**} - \{\lambda\}$. A picture language is a subset of Σ^{**}.

We now recall a pure 2D context-free grammar introduced in [14,15].

Definition 1. *A pure 2D context-free grammar (P2DCFG) is a 4-tuple*

$$G = (\Sigma, P_1, P_2, \mathcal{M}_0)$$

where

i) Σ *is a finite alphabet of symbols;*

ii) $P_1 = \{c_i | \ 1 \leq i \leq s_c\}$, *where* c_i *is called a column rule table and* s_c *is some positive integer; each* c_i *is a finite set of context-free rules of the form* $a \to \alpha, a \in \Sigma, \alpha \in \Sigma^*$ *such that for any two rules* $a \to \alpha, b \to \beta$ *in* c_i, *we have* $|\alpha| = |\beta|$ *i.e.* α *and* β *have equal length;*

iii) $P_2 = \{r_j | \ 1 \leq j \leq s_r\}$, *where* r_j, *is called a row rule table and* s_r *is some positive integer; each* r_j *is a finite set of rules of the form* $c \to {}^t\gamma, c \in \Sigma, \gamma \in \Sigma^*$ *such that for any two rules* $c \to {}^t\gamma, d \to {}^t\delta$ *in* r_j, *we have* $|\gamma| = |\delta|$*;*

iv) $\mathcal{M}_0 \subseteq \Sigma^{**} - \{\lambda\}$ *is a finite set of axiom arrays.*

A derivation in a $P2DCFG$ G is defined as follows: Let $p, q \in \Sigma^{**}$. The picture q is derived from picture p in G, denoted by $p \Rightarrow q$, if q is obtained from p either *i)* by rewriting in parallel all the symbols in a column of p, each symbol by a rule in some column rule table or *ii)* rewriting in parallel all the symbols in a row of p, each symbol by a rule in some row rule table. All the rules used to rewrite a column (or row) have to belong to the same table.

The picture language generated by G is the set of picture arrays $L(G) = \{M \in \Sigma^{**} | \ M_0 \Rightarrow^* M$ for some $M_0 \in \mathcal{M}_0\}$. The family of picture languages generated by $P2DCFGs$ is denoted by $P2DCFL$.

Example 1. Consider the $P2DCFG$ $G_1 = (\Sigma, P_1, P_2, \{M_0\})$ where $\Sigma = \{a, b, e\}$, $P_1 = \{c\}, P_2 = \{r\}$, where $c = \{a \to bab, e \to aea\}$, $r = \left\{ e \to \begin{matrix} e \\ a \end{matrix}, a \to \begin{matrix} a \\ b \end{matrix} \right\}$, and $M_0 = \begin{matrix} a & e & a \\ b & a & b \end{matrix}$.

G_1 generates a picture language L_1 consisting of picture arrays p of size $(m, 2n+1)$, $m \geq 2$, $n \geq 1$ with $p(1, j) = p(1, j+n+1) = a$, for $1 \leq j \leq n$; $p(1, n+1) = e$; $p(i, n+1) = a$, for $2 \leq i \leq n$; $p(i, j) = b$, otherwise. A member of L_1 is shown in Figure 1.

$$\begin{matrix} a & a & a & e & a & a & a \\ b & b & b & a & b & b & b \\ b & b & b & a & b & b & b \\ b & b & b & a & b & b & b \\ b & b & b & a & b & b & b \end{matrix}$$

Fig. 1. A picture in the language L_1

We note that the rows in the generated picture arrays of L_1 do not maintain any proportion to the columns since the application of the column rule table c can take place independent of the row rule table r. But the picture array will have an equal number of columns to the left and right of the middle column ${}^t(ea \ldots a)$.

We now recall a $P2DCFG$ with a control language on the labels of the column rule and row rule tables in the $P2DCFG$, which is introduced in [14,15].

A $P2DCFG$ with a regular control is $G^c = (G, \Gamma, \mathcal{C})$ where $G = (\Sigma, P_1, P_2, \mathcal{M}_0)$ is a $P2DCFG$, Γ is a set of labels of the tables of G, given by $\Gamma = P_1 \cup P_2$ and $\mathcal{C} \subseteq \Gamma^*$ is a regular (string) language. The words in Γ^* are called control words of G. Derivations $M_1 \Rightarrow_w M_2$ in G^c are done as in G except that if $w \in \Gamma^*$ and $w = l_1 l_2 \ldots l_m$, then the tables of rules with labels l_1, l_2, ..., and l_m are successively applied starting with the picture array M_1 to finally yield the picture array M_2. The picture array language generated by G^c consists of all picture arrays obtained from axiom arrays of G with the derivations controlled as described above. We denote the family of picture languages generated by $P2DCFG$s with regular control by $(R)P2DCFL$.

3 Pure 2D Context-Free Grammar with (l/u) Mode of Derivations

We now consider a variant in the rewriting process of a $P2DCFG$. The concept of leftmost derivation in a context-free grammar in string language theory, is well-known [12,13], especially in the context of LL parsers. In fact, in [9], the leftmost derivation concept is generalized to obtain derivation trees for context-sensitive grammars. On the other hand, leftmost derivations have been considered in other string grammars as well. For example, Meduna and Zemek [8] have studied the generative power of one-sided random context grammars working in the leftmost way. These studies, especially the study in [8], motivate to consider a corresponding notion of "leftmost kind" of derivation in pure 2D context-free grammars with a view to compare the resulting picture generative power with the $P2DCFG$ [14] as well as to examine other kinds of results such as closure properties. The idea is to rewrite the leftmost column of a picture array by a column rule table or the uppermost row by a row rule table unlike the unrestricted way of rewriting any column or any row (if a column rule or row rule table is applicable) in a $P2DCFG$. This kind of a restriction on rewriting results in a picture language family which neither contains nor is contained in $P2DCFL$.

Definition 2. *Let $G = (\Sigma, P_1, P_2, \mathcal{M}_0)$ be a $P2DCFG$ with the components as in Definition 1. An (l/u) mode of derivation of a picture array M_2 from M_1 in G, denoted by $\Rightarrow_{(l/u)}$, is a derivation in G such that only the leftmost column or the uppermost row of M_1 is rewritten using respectively, the column rule tables or the row rule tables, to yield M_2. The generated picture language is defined as in the case of a $P2DCFG$ but with $\Rightarrow_{(l/u)}$ derivations. The family of picture languages generated by $P2DCFG$s under $\Rightarrow_{(l/u)}$ derivations is denoted by $(l/u)P2DCFL$. For convenience, we write $(l/u)P2DCFG$ to refer to $P2DCFG$ with $\Rightarrow_{(l/u)}$ derivations.*

We illustrate with an example.

Example 2. Consider an $(l/u)P2DCFG$ $G_2 = (\Sigma, P_1, P_2, \{M_0\})$ where $\Sigma = \{a, b\}$, $P_1 = \{c\}$, $P_2 = \{r\}$ with $c = \{a \to ab, b \to ba\}$, $r = \left\{a \to \begin{smallmatrix} a \\ b \end{smallmatrix}, b \to \begin{smallmatrix} b \\ a \end{smallmatrix}\right\}$, and $M_0 = \begin{smallmatrix} b & a \\ a & b \end{smallmatrix}$.

G_2 generates a picture language L_2 consisting of arrays p of size (m, n), $m \geq 2$, $n \geq 2$ with $p(1,1) = b$; $p(1,j) = a$, for $2 \leq j \leq n$; $p(i,1) = a$, for $2 \leq i \leq m$; $p(i,j) = b$, otherwise. A member of L_2 is shown in Figure 2. A sample derivation

$$\begin{matrix} b & a & a & a & a & a \\ a & b & b & b & b & b \\ a & b & b & b & b & b \\ a & b & b & b & b & b \\ a & b & b & b & b & b \end{matrix}$$

Fig. 2. A picture array in the language L_2

in $(l/u)P2DCFG$ G_2 starting from M_0 and using the tables c, r, c, c in this order is shown in Figure 3. We note that in this derivation (unlike in a derivation in a $P2DCFG$), the application of the column rule table c rewrites all symbols in the leftmost column in parallel and likewise, the application of the row rule table r rewrites all symbols in the uppermost row. We now compare the generative

$$M_0 = \begin{matrix} b & a \\ a & b \end{matrix} \Rightarrow_{(l/u)} \begin{matrix} b & a & a \\ a & b & b \end{matrix} \Rightarrow_{(l/u)} \begin{matrix} b & a & a \\ a & b & b \\ a & b & b \end{matrix} \Rightarrow_{(l/u)} \begin{matrix} b & a & a & a \\ a & b & b & b \\ a & b & b & b \end{matrix} \Rightarrow_{(l/u)} \begin{matrix} b & a & a & a & a \\ a & b & b & b & b \\ a & b & b & b & b \end{matrix}$$

Fig. 3. A sample derivation under (l/u) mode

power of $(l/u)P2DCFL$ with $P2DCFL$.

Theorem 1. *The families of $P2DCFL$ and $(l/u)P2DCFL$ are incomparable but not disjoint, when the alphabet contains at least two symbols.*

Proof. It is clear that the families are not disjoint since the non-trivial picture language of all rectangular picture arrays over $\{a, b\}$ belongs to both of them. In fact the corresponding grammar needs to have only two tables

$$c = \{a \to aa, a \to ab, b \to ba, b \to bb\}, r = \left\{a \to \begin{matrix} a \\ a \end{matrix}, a \to \begin{matrix} a \\ b \end{matrix}, b \to \begin{matrix} b \\ a \end{matrix}, b \to \begin{matrix} b \\ b \end{matrix}\right\}$$

and axiom pictures a, b.

The picture language L_2 in Example 2 belongs to $(l/u)P2DCFL$ but it cannot be generated by any $P2DCFG$. In fact every column (including the leftmost column) in the picture arrays of L_2 involves the two symbols a, b and only these two. So to generate the picture arrays of L_2 starting from an axiom array, we have to specify column rules for both a, b. The leftmost column will require a column rule that will rewrite b into $ba \cdots a$ and a into $ab \cdots b$ but then the table with these rules can be applied to any other column in a $P2DCFG$. This will result in picture arrays not in the language L_2.

On the other hand the picture language L_1 in Example 1 belongs to $P2DCFL$ but it cannot be generated by any $(l/u)P2DCFG$. In fact there is an unique middle column in every picture array of L_1. Also to the left and right of this middle column there are an equal number of identical columns. Since only the leftmost column can be rewritten in an $(l/u)P2DCFG$, it is not possible to maintain this feature of "equal number of identical columns" if leftmost column rewriting is done. □

Remark 1. The families $P2DCFL$ and $(l/u)P2DCFL$ coincide if we restrict to only a unary alphabet. Since there is a single symbol and the column rules and the row rules can use only one symbol, rewriting any column is equivalent to rewriting the leftmost column of a picture array.

We now exhibit non-closure of the family $(l/u)P2DCFL$ under the Boolean operations of union and intersection.

Theorem 2. *The family $(l/u)P2DCFL$ is not closed under union.*

Proof. Let $L_1 \subseteq \{a, b, d\}^{**}$ be a picture language such that each $p \in L_1$ of size $(m, n), m \geq 2, n \geq 2$ has the following properties: $p(1, 1) = b$; $p(1, j) = a$, for $2 \leq j \leq n$; $p(i, 1) = a$, for $2 \leq i \leq m$; $p(i, j) = d$, otherwise. Let $L_2 \subseteq \{a, b, e\}^{**}$ be a picture language such that each $p \in L_2$ of size $(r, s), r \geq 2, s \geq 2$ has the following properties: $p(1, 1) = b$; $p(1, j) = a$, for $2 \leq j \leq s$; $p(i, 1) = a$, for $2 \leq i \leq r$; $p(i, j) = e$, otherwise. The languages L_1 and L_2 are generated by $(l/u)P2DCFGs$ G_1 and G_2, respectively. We mention here only the tables of rules and axiom arrays of these grammars. The other components are understood from the tables of rules. The column rule table of G_1 is

$$c_1 = \{b \to ba, a \to ad\}$$

while the row rule table is

$$r_1 = \left\{ b \to \begin{matrix} b \\ a \end{matrix}, a \to \begin{matrix} a \\ d \end{matrix} \right\}.$$

The column rule table of G_2 is

$$c_2 = \{b \to ba, a \to ae\}$$

while the row rule table is

$$r_2 = \left\{ b \to \begin{matrix} b \\ a \end{matrix}, a \to \begin{matrix} a \\ e \end{matrix} \right\}.$$

The axiom pictures of G_1 and G_2 are $\begin{matrix} b & a \\ a & d \end{matrix}$ and $\begin{matrix} b & a \\ a & e \end{matrix}$, respectively. Now the union picture language $L_1 \cup L_2$ cannot be generated by any $(l/u)P2DCFG$. In fact, the smallest pictures in $L_1 \cup L_2$ are $\begin{matrix} b & a \\ a & d \end{matrix}$ and $\begin{matrix} b & a \\ a & e \end{matrix}$. Both of these will be the axiom arrays in any $(l/u)P2DCFG$ that could be formed to generate $L_1 \cup L_2$. Also in order to generate the pictures of L_1, column rules of the form $a \to ad \cdots d$ will be required while to generate the pictures of L_2 column rules of the form $a \to ae \cdots e$ will be needed. Likewise for row rules. But then there is no restriction on the application of the tables of rules which will therefore generate pictures not in $L_1 \cup L_2$. □

Theorem 3. *The family $(l/u)P2DCFL$ is not closed under intersection.*

Proof. Let L_s be a picture language consisting of square sized arrays p of the language L_2 in Example 2 i.e. pictures p of size $(n, n), n \geq 2$ with $p(1, 1) = b$; $p(1, j) = a$, for $2 \leq j \leq n$; $p(i, 1) = a$, for $2 \leq i \leq n$; $p(i, j) = b$, otherwise. We denote here by L_r the picture language L_2 in Example 2 noting that the picture arrays of L_2 are rectangular arrays.

We consider a language L containing of the following three sets of picture arrays:
i) Square arrays with the uppermost row in each array being of the form $xd \cdots d$, the leftmost column of the form ${}^t xe \cdots e$ and with b in all other positions
ii) Rectangular arrays with the uppermost row in each array being of the form $yd \cdots d$, the leftmost column of the form ${}^t ye \cdots e$ and with b in all other positions
iii) the picture arrays of L_s.

The picture language L_r is generated by the $(l/u)P2DCFG$ of Example 2 while L is generated by an $(l/u)P2DCFG$ G, for which we mention here only the column rule and row rule tables and the axiom array. The column rule tables are

$$c_1 = \{x \to yd,\, e \to eb\},\ c_2 = \{x \to b,\, e \to a\}.$$

The row rule tables are

$$r_1 = \left\{ y \to \begin{matrix} x \\ e \end{matrix},\, d \to \begin{matrix} d \\ b \end{matrix} \right\},\ r_2 = \{b \to b,\, d \to a\}.$$

The axiom array is $\begin{matrix} x & d \\ e & b \end{matrix}$. We note that an application of the column rule table c_1 will increase the number of columns by one, after which only the row rule table r_1 can be applied which will then increase the number of rows by one, thereby yielding a square sized array. The application of the tables of rules c_2, r_2 produce the picture arrays in L_s. It is clear that $L_s \subset L$ and $L_s = L_r \cap L$. It can be seen that L_s cannot be generated by any $(l/u)P2DCFG$ (using the alphabet $\{a, b\}$), since the application of the column rule and row rule tables are independent and hence cannot ensure square size of the pictures generated. □

Analogous to $(R)P2DCFG$, we can define a controlled $(l/u)P2DCFG$.

Definition 3. *Let $G = (\Sigma, P_1, P_2, \{M_0\})$ be an $(l/u)P2DCFG$ G. Let $\Gamma = P_1 \cup P_2$ i.e. Γ is the set of labels of the column rule and row rule tables of G. Let $\mathcal{C} \subseteq \Gamma^*$, whose elements are called control words. The application of the tables in an l/u derivation in G is regulated by the control words of $\mathcal{C}$, called the control language. An $(l/u)P2DCFG$ with a regular and context-free control language is denoted by $(R)(l/u)P2DCFG$ and $(CF)(l/u)P2DCFG$, respectively. In addition, the family of picture languages generated by $(R)(l/u)P2DCFGs$ and $(CF)(l/u)P2DCFGs$ is denoted by $(R)(l/u)P2DCFL$ and $(CF)(l/u)P2DCFL$, respectively.*

It is known [15] that the family of $P2DCFL$ is properly contained in $(R)P2DCFL$. An analogical inclusion holds for the families $(l/u)P2DCFL$ and $(R)(l/u)P2DCFL$.

Theorem 4. $(l/u)P2DCFL \subset (R)(l/u)P2DCFL \subset (CF)(l/u)P2DCFL$.

Proof. The inclusions are straightforward since an $(l/u)P2DCFG$ is an $(R)(l/u)P2DCFG$ on taking the regular control language as Γ^* where Γ is the set of labels of the tables of the $(l/u)P2DCFG$. Also it is well-known [13] that the regular language family is included in the CF family.

The proper inclusion in $(l/u)P2DCFL \subset (R)(l/u)P2DCFL$ can be seen by considering a picture language L_3 consisting of square sized arrays p of the language L_s given in the proof of Theorem 3. This picture language can be generated by the $(l/u)P2DCFG$ G_2 in Example 2 with a regular control language $(cr)^*$. But it is clear that L_3 cannot be generated by an $(l/u)P2DCFG$, since the applications of the column rule and row rule tables are independent.

The proper inclusion of $(R)(l/u)P2DCFL$ in $(CF)(l/u)P2DCFL$ can be shown by considering a picture language L_4 consisting of picture arrays p as in Example 1 but of sizes $(k+1, 2k+1), k \geq 1$. The $(CF)(l/u)P2DCFG$ $G^c = (G_4, \Gamma, \mathcal{C})$ generates L_4, where $G_4 = (\Sigma, P_1, P_2, \{M_0\})$ where $\Sigma = \{a, b, e\}$, $P_1 = \{c_1, c_2, c_3\}, P_2 = \{r\}$ with

$$c_1 = \{e \to ea, a \to ab\}, c_2 = \{e \to ae, a \to ba\}, c_3 = \{a \to aa, b \to bb\},$$

$$r = \left\{ e \to \begin{matrix} e \\ a \end{matrix}, a \to \begin{matrix} a \\ b \end{matrix} \right\},$$

$M_0 = \begin{matrix} e & a \\ a & b \end{matrix}$ and the tables of rules c_1, c_2, c_3, r are themselves taken as the labels of the corresponding tables, constituting the set Γ. The CF control language is $\mathcal{C} = \{(c_1 r)^n c_2 c_3^n |\ n \geq 0\}$. In order to generate the picture arrays of L_4, the l/u derivations are done according to the control words of $\mathcal{C}$. Starting from the axiom array $M_0 = \begin{matrix} e & a \\ a & b \end{matrix}$ the leftmost column of M_0 is rewritten using the column rule table c_1 immediately followed by the row rule table r. This is repeated n times (for some $n \geq 0$) and then the column rule table c_2 is applied once, followed by the application of the column rule table c_3, the same number of times as c_1

followed by r was done, thus yielding a picture array in L_4. But L_4 cannot be generated by any $(l/u)P2DCFG$ with regular control. In fact in a generation of a picture array p in L_4 that makes use of a regular control, if the derivation is generating the part of p to the left of the middle column (made of one e as the first symbol and all other symbols in the column being a's), there will be no information available on the number of columns generated once the derivation "crosses" the middle column, so that the columns to the right of this middle column cannot be generated in equal number. □

In a $(R)P2DCFG$, the alphabet may contain some symbols called control symbols [2] which might not be ultimately involved in the picture arrays of the language generated. For example, the $(R)(l/u)P2DCFG$ with the $P2DCFG$ $(\{e, a, b\}, \{c_1, c_2\}, \{r\}, \{\begin{smallmatrix} e & a \\ a & b \end{smallmatrix}\})$ where

$$c_1 = \{e \to ea, a \to ab\}, c_2 = \{e \to a, a \to a\},$$

$$r = \left\{ e \to \begin{matrix} e \\ a \end{matrix}, a \to \begin{matrix} a \\ b \end{matrix} \right\},$$

and the control language $\{(c_1 r)^n c_2 |\ n \geq 0\}$ generates picture arrays p such that the uppermost row and the leftmost column of p involve only the symbol a while all other positions have the symbol b. But the alphabet contains a symbol e which ultimately does not appear in the picture arrays of the language. Such a symbol is referred to as a control symbol or a control character in the context of an $(R)(l/u)P2DCFG$. A picture language L_d is considered in [2] given by $L_d = \{p \in \{a, b\}^{++} |\ |p|_{col} = |p|_{row}, p(i, j) = b, \text{for } i = j, p(i, j) = a \text{ for } i \neq j\}$ and is shown to require at least two control symbols to generate it using a $P2DCFG$ and a regular control language.

Lemma 1. *[2] The language L_d cannot be defined by using less than two control characters and a $P2DCFG$ with a regular control language.*

We show in the following Lemma that in an $(R)(l/u)P2DCFG$ the picture language L_d can be generated with a single control character.

Lemma 2. *The language L_d can be defined by an $(R)(l/u)P2DCFG$ that uses a single control character. Moreover, L_d is not in $(l/u)P2DCFL$.*

Proof. The $(R)(l/u)P2DCFG$ with the $(l/u)P2DCFG$ given by $(\{0, 1, 2\}, \{c\}, \{r\}, \{\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\})$ where

$$c = \{1 \to 12, 0 \to 00\}, r = \left\{ 1 \to \begin{matrix} 1 \\ 0 \end{matrix}, 2 \to \begin{matrix} 0 \\ 1 \end{matrix}, 0 \to \begin{matrix} 0 \\ 0 \end{matrix} \right\},$$

and control language $(cr)^*$ generates L_d. Here, 2 is the only control character. It is clear that if there are only two symbols $0, 1$ in the alphabet, then, for example, there need to be two column rules $0 \to 01, 0 \to 00$ in a table to maintain the diagonal of 1's but this will yield pictures not in L_d. A similar reason holds for row rules. This shows L_d cannot be in $(l/u)P2DCFL$. □

Finally, we compare $(l/u)P2DCFL$ with the class LOC [4] of local picture languages whose pictures are defined by means of tiles i.e. square pictures of size $(2, 2)$.

Theorem 5. *The families $(l/u)P2DCFL$ and LOC are incomparable but their intersection is not empty.*

Proof. The picture language of all rectangular arrays over a one letter alphabet $\{a\}$ is clearly in $(l/u)P2DCFL$ and is also known [2] to be in LOC. But the language of rectangular pictures with an even number of rows and an even number of columns is not in LOC [2] but is in $P2DCFL$ [2] and hence in $(l/u)P2DCFL$, by Remark 1. On the other hand, the language L_d in Lemma 2 is in LOC [2] but again by Lemma 2, L_d is not in $(l/u)P2DCFL$. □

4 Conclusion

A variant of $P2DCFG$ [14,15] rewriting only the leftmost column or the uppermost row of a picture array is considered and properties of the resulting family $(l/u)P2DCFL$ of picture languages are obtained. Properties such as closure or non-closure under row or column concatenation of arrays or membership problem and others remain to be investigated. It will also be of interest to allow erasing rules of the form $a \to \lambda$ and examine the effect of using these rules in the derivations of the picture arrays.

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