Analysis of Micro–Macro Transformations of Railway Networks

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Abstract A common technique in the solution of large or complex optimization problems is the use of *micro–macro* transformations. In this paper, we carry out a theoretical analysis of such transformations for the track allocation problem in railway networks. We prove that the *cumulative rounding* technique of Schlechte et al. satisfies two of three natural optimality criteria and that this performance cannot be improved. We also show that under extreme circumstances, this technique can perform inconveniently by underestimating the global optimal value.

1 Introduction

It is often the case in discrete optimization problems coming from applications that the data is too complex to be tractable by an efficient algorithm. However, much of the information in this precise (also called *microscopic*) model is not necessary to obtain a very good feasible solution. A common technique is to derive a simplified *macroscopic* model by aggregating the structures of the microscopic model, find a good solution to the macroscopic model, and retranslate it to the original problem. This idea has been used in diverse settings. In [1], an algorithm for solving linear programs exactly solves a sequence of increasingly detailed LPs until the desired degree of precision is reached. In [2], an algorithm for solving a dynamic progam over a large state space is described. A sequence of coarse DPs is solved, and the complexity/level of detail increases gradually. Reference [3] surveys aggregation and disaggregation techniques for optimization problems. This research was mostly influenced by Schlechte et al. [5], where a micro–macro transformation is used for

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solving the track allocation problem for railway networks (see [4, 5] for a precise definition), which is the problem considered in this paper. One of the main difficulties in developing an efficient micro–macro algorithm for this problem is choosing a reasonable time discretization. That is, given a time unit δ in the microscopic model, we seek to find a larger unit Δ for the macroscopic model and then determine the input times of the macroscopic model in multiples of Δ. It is on this last step that we will focus next. Given a microscopic running time of some route on a macroscopic track, the most natural choice is to round it to a close multiple of Δ. Rounding down can lead to infeasibilities, while rounding up all running times leads to an unnecessary increase in the optimal value. Therefore, a combination of both seems to be the best strategy. In this context, we consider the *cumulative rounding* method introduced by Schlechte et al. in [5]. This method consists of rounding up the running times along each route in order of traversal, until the total "lost" time accumulated is at least the time corresponding to the track currently considered, at which point we round down this running time and iterate.

While it is possible to give upper bounds on the overestimation error of the total time needed to traverse each route, the impact of this rounding on the originating network optimization problem as a whole has not been studied. The paper is structured as follows. In Sect. 2 we describe the general problem, the motivation and the goals of micro–macro transformations. In Sect. 3 we define three optimality criteria for a rounding strategy for the track allocation problem. We prove that the cumulative rounding strategy is optimal with respect to two of these criteria and that no strategy satisfies all three of them. Finally, in Sect. 4 we show an instance in which cumulative rounding yields a macroscopic value that is smaller than the microscopic optimum and whose solution is impossible to translate back to the original model without losing a significant factor. This shows the difficulty of acheiving global optimality or near-optimality.

2 Our Setting

We consider a general minimization[1] problem P_δ based on a time discretization δ with $k\delta = \Delta$, $k \in \mathbb{Z}$, $k > 0$. The problem P_Δ results from rounding all times of P_δ to multiples of Δ with respect to alternate rounding strategies. Let us consider the trivial rounding down ($\lfloor\,\rfloor$) and up ($\lceil\,\rceil$). Then for the optimal values v, we have:

$$v(P_\Delta^{\lfloor\rfloor}) \le v(P_\delta) \le v(P_\Delta^{\lceil\rceil})$$

On the one hand the solution of $P_\Delta^{\lceil\rceil}$ can be re-transformed, i.e., we maintain the orders of the trains and retranslate the departure and arrivals w.r.t. δ, to a feasible solution of P_δ retaining the same objective value or obtaining a better one. On

[1] In case of the track allocation problem we want to schedule a fixed number of trains on a network within a minimum time horizon.

the other hand $v(P_{\Delta}^{\lfloor\rfloor})$ only provides in general a valid lower bound. Thus, we can guarantee some solution quality provided by the lower bound $v(P_{\Delta}^{\lfloor\rfloor})$.

3 Optimality Criteria

While the ultimate objective in the track allocation problem is to find a microscopic solution of optimal or near-optimal value, it is in general not clear how to obtain a feasible microscopic solution from a macroscopic solution such that the objective value does not increase. For that reason, we will try to judge the quality of a transformation by comparing the values of the obtained macroscopic and microscopic solutions. There are several (often conflicting) possibilities of defining an "optimal" rounding algorithm, and it is not obvious which of them should be considered. Here we consider three very natural optimality criteria:

1. **Global optimality**: The total time is not underestimated and the corresponding (overestimating) error is minimal.
2. **Route-wise optimality**: The total time on each individual route is not underestimated and the corresponding (overestimating) error is minimal.
3. **Local optimality**: The overestimating error on any subroute $(j_m, j_{m+1}, \ldots, j_{m+n})$ of a route r is less than Δ.

The no-underestimating condition guarantees that we can obtain feasible solutions. The first two conditions are self-explaining and the third condition guarantees that the approximation is good on a local level, i.e., on intervals.

In this section we prove that the cumulative rounding technique satisfies the last two properties.

Theorem 1 *For the track allocation problem, a rounding strategy is route-wise optimal if and only if on every route j it rounds up the traversal times corresponding to exactly $\left\lceil \frac{\sum_{j\in D} \widehat{t}_j^r}{\Delta} \right\rceil$ tracks.*

Proof In the same setting as above, let r be a route. For every track j in the route, let t_j^r be the time (in units of δ) needed to traverse j, and let $\widehat{t}_j^r \equiv t_j^r \pmod{\Delta}$. If for this track we decide to round up, the (overestimating) error will be $\Delta - \widehat{t}_j^r$, while if we round down, the (underestimating) error is $\widehat{t}_j^r$. Let J be the set of tracks in route r, let $U \subset J$ (the set of tracks for which we round up) and $D = J \backslash U$ (the tracks for which we round down). Now, the total overestimating error is

$$\varepsilon^r = \sum_{j\in U} (\Delta - \widehat{t}_j^r) - \sum_{j\in D} \widehat{t}_j^r = |U|\Delta - \sum_{j\in J} \widehat{t}_j^r.$$

Since $\sum_{j\in J} \widehat{t}_j^r$ is independent of the choice of U and D, the total error depends only on the cardinality of U. By the non-overestimating property, we are looking for a set U of minimal cardinality such that ε^r is nonnegative and minimal. Clearly, this is achieved by choosing U with $|U| = \left\lceil \frac{\sum_{j \in D} \widehat{t}_j^{\,r}}{\Delta} \right\rceil$. □

Corollary 1 *The cumulative rounding strategy is route-wise optimal and locally optimal.*

Proof The authors of [5] have proven that on each route, the total error caused by cumulative rounding is in the interval $[0, \Delta)$. By the proof of the previous theorem, this is a minimizer and thus the strategy is route-wise optimal.

To prove local optimality, let us consider a subroute $r^1 = (j_m, \ldots, j_{m+n})$. We can picture this subroute as the difference between subroutes $r^2 = (j_1, j_2, \ldots, j_{m+n})$ and $r^3 = (j_1, j_2, \ldots, j_{m-1})$. As before, let us denote by ε^r the overestimating error of a subroute r. By the result in [5] we just mentioned above, we have $0 < \varepsilon^{r^1} < \Delta$ and $0 < \varepsilon^{r^2} < \Delta$. Suppose $\varepsilon^{r^3} > \Delta$. Then, we clearly have $\varepsilon^{r^2} = \varepsilon^{r^1} + \varepsilon^{r^3} > \Delta$, which is a contradiction. □

Theorem 2 *There exists no rounding strategy that satisfies all three described optimality criteria.*

Proof Let us consider the following network, with $\Delta = k\delta$ for some $k \geq 3$:

On this network, let us consider trains 1 and 2 traveling from A to D, and train 3 traveling from D to A. We are interested in minimizing the time until the last train arrives at its destination. We assume that for every track, the headway time corresponding to two trains in the same direction is Δ. Similarly, the headway time corresponding to two trains in opposite directions along track j is $t_j + \Delta$. Suppose trains can not stop at intermediate stations and there are no restrictions on the departure or arrival times. A feasible and in fact optimal solution is to let trains 1, 2 and 3 leave their initial stations at times 0, Δ and δ, respectively. As trains 1 and 2 go from B to C in one direction, train 3 goes from C to B in the opposite direction without violating the headway constraints. The time until the last train (train 2) arrives is $5\Delta + \delta$. Suppose we have a route-wise and locally optimal strategy. Let us consider r^1, the route corresponding to train 1. By route-wise optimality, we know that exactly one traversal time is rounded down. If this time corresponds to either track AB or track CD, we know that the remaining two tracks form a subroute with an overestimating error of Δ, which contradicts local optimality. Without loss of generality, the same reasoning applies to routes r^2 and r^3, so the resulting macroscopic network is given by the numbers below the arcs on Fig. 1.

Let t_j denote the microscopic time for each train on track j and T_j the corresponding macroscopic time. By choice of the microscopic headway times, the macroscopic headway times are still Δ and $T_j + \Delta$. Since now the tracks between B and C are of time Δ, the previous solution is no longer feasible. In fact, now the optimal solution is to let trains 1 and 2 go from A to D, and let train 3 depart only after the other two have arrived at D. This gives a total time of 12Δ, which is more than double the time needed in the microscopic instance.

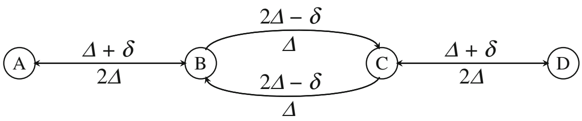

Fig. 1 The numbers *above* and *below* the *arcs* represent, respectively, the microscopic and macroscopic times for the corresponding tracks, assuming all trains have uniform speed

Table 1 Results for transformations of the optimization problem described in the proof of Theorem 2

Description	Discretization	Rounding technique	Optimal value
Original problem	δ	–	$5\Delta + \delta$
Approximation	Δ	$\langle\,\rangle$	12Δ
Feasible solution	δ	$\langle\,\rangle$	$10\Delta + 2\delta$
Feasible solution	Δ	$\lceil\,\rceil$	7Δ
Feasible solution	δ	$\lceil\,\rceil$	$5\Delta + \delta$
Lower bound	Δ	$\lfloor\,\rfloor$	4Δ

We use $\langle\,\rangle$ to denote any strategy that is route-wise- and locally optimal

Applying the conservative approach (rounding up all running times), we would get a total time of 7Δ as optimum, which is more than the microscopic optimal value but significantly smaller than 12Δ. Since the conservative rounding gives a smaller total time we can conclude that the considered strategy does not satisfy global optimality. □

While the previous proof shows that the conservative rounding strategy gives a better macroscopic total time, it is not immediately clear what the corresponding microscopic times are. If we take the solution given by cumulative rounding or a similar strategy and translate it back to the microscopic model, we obtain a total time of $10\Delta + 2\delta$, which is exactly double of the optimal time. We summarize these results in Table 1. Let us also note that we can easily make the macroscopic instance infeasible while keeping the original feasible. For example, we could require for all trains to arrive at their destinations at time 6Δ or before.

4 A Paradoxical Instance

In the previous section we saw some drawbacks to the cumulative rounding strategy, but we also proved that it is impossible to improve it to a globally optimal strategy while keeping both of its optimality properties. In this section, we will give an instance such that the macroscopic optimal value is much better than the microscopic optimal value. This shows that even if we relax the optimality requirement in the global optimality condition, the non-underestimating condition is not necessarily satisfied.

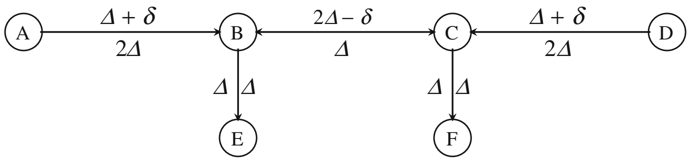

Fig. 2 The numbers *above* (*below*) and to the *left* (*right*) of the *arcs* represent the microscopic (macroscopic) times for the corresponding tracks

Furthermore, this hints that guaranteeing non-underestimation on a global level in general may be very hard. Consider the network with two trains in Fig. 2.

Here, train 1 has to go from A to F and train 2 from D to E. The only headway times of interest are those corresponding to track BC. They are defined as $t_{BC} + \Delta$. Trivially, an optimal solution is to let train 1 depart at time 0 and let train 2 depart when train 1 is about to reach C (to be precise, at time $3\Delta - \delta$). In this solution, train 2 arrives to its destination at time $7\Delta - \delta$.

As in the previous example, the macroscopic headway times of interest are now $T_{BC} + \Delta$. Letting train 1 depart at time 0 and train 2 at time 2Δ, the last train arrives at time 6Δ. Clearly, this objective value is impossible to attain in the microscopic problem.

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