Misspecified Dependency Modelling: What Does It Mean for Risk Measurement?

Theo Berger

Abstract Forecasting portfolio risk requires both, estimation of marginal return distributions for individual assets and dependence structure of returns as well. Due to the fact, that the marginal return distribution represents the main impact factor on portfolio volatility, the impact of dependency modeling which is required for instance in the field of Credit Pricing, Portfolio Sensitivity Analysis or Correlation Trading is rarely investigated that far. In this paper, we explicitly focus on the impact of decoupled dependency modeling in the context of risk measurement. We do so, by setting up an extensive simulation analysis which enables us to analyze competing copula approaches (Clayton, Frank, Gauss, Gumbel and t copula) under the assumption that the "true" marginal distribution is known. By simulating return series with different realistic dependency schemes accounting for time varying dependency as well as tail dependence, we show that the choice of copula becomes crucial for VaR, especially in volatile dependency schemes. Albeit the Gauss copula approach does neither account for time variance nor for tail dependence, it represents a solid tool throughout all investigated dependency schemes.

1 Introduction

Interdependencies between individual assets need to be captured to measure diversification effects and to precisely measure a single asset risk contribution on an aggregated portfolio level. Albeit, as Fantazzini [3] points out, the impact of misspecified marginals offsets the bias in dependency modeling on a portfolio level, precise dependency measurement represents a crucial information. For instance, a risk manager needs to know the effect of a hedged risk position on the overall portfolio risk. As well, correlation trading, the modeling of derivatives and measuring risk

T. Berger (✉)
University of Bremen, Wilhelm-Herbst-Str. 5, 28359 Bremen, Germany
e-mail: theoberger@uni-bremen.de

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diversification heavily depends on the information which are exclusively captured by dependency measurement.

So far there are only a few analysis which explicitly address the impact of dependency modeling on the measurement of Portfolio risk. Ane and Kharoubi [1] analyse the choice copula in the context of VaR forecasts, and show that inadequate dependency modeling explains up to 18 % of a VaR misspecification.[1] However, given the dominant impact stemming from the marginal return distributions, the separated impact of dependency modeling on aggregated portfolio risk in the absence of misspecified margins has not been explicitly investigated so far.

Thus, we add to the literature and set up an extensive simulation analysis accounting for realistic dependency scenarios such as time varying dependency and tail dependency. Both phenomena are discussed in a realistic as well as disproportionated environment. Further we investigate the dependency bias of modern dependency approaches on portfolio risk, in the absence of any bias caused by the modeling of marginal return distributions. More concrete, we generate samples with predefined margins, characterized by different dependency schemes and apply competing dependency models to forecast portfolio risk. By doing so, we are able to explicitly compare the forecasting bias caused by the applied dependency approaches in an applied risk measurement environment via out of sample analysis. Specifically, the simulation exercise should answer the question whether the choice of copula does affect the VaR performance when the data generating process is described by time varying conditional correlations or tail dependence.

The remainder is structured in the following way: Sect. 2 gives a brief overview about the relevant dependency approaches and Sect. 3 describes the setup of the simulation analysis. Section 4 gives the empirical results and Sect. 5 summarizes the results of this paper.

2 Methodology

2.1 Copulas

The copula approach is based on Sklar‘s Theorem [7]:
Let $X_1, \ldots, X_n$ be random variables, $F_1, \ldots, F_n$ the corresponding marginal distributions and H the joint distribution, then there exists a copula C: $[0, 1]^n \rightarrow [0, 1]$ such that:

$$H(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)). \tag{1}$$

Conversely if C is a copula and $F_1, \ldots, F_n$ are distribution functions, then H (as defined above) is a joint distribution with margins $F_1, \ldots, F_n$.

[1] The analysis is based on applied loss functions in an empirical setup.

The Gaussian and t copula belong to the family of elliptical copulas and are derived from the multivariate normal and t distribution respectively.

The setup of the Gaussian copula is given by:

$$C^{Ga}(x_1, \ldots, x_n) = \Phi_\rho(\Phi^{-1}(x_1), \ldots, \Phi^{-1}(x_n)), \tag{2}$$

$$= \int_{-\infty}^{\Phi^{-1}(x_1)} \cdots \int_{-\infty}^{\Phi^{-1}(x_n)} \frac{1}{2(\pi)^{\frac{n}{2}} |\rho|^{\frac{1}{2}}} \exp\left(-\frac{1}{2} z^T \rho^{-1} z\right) dz_1 \ldots dz_n \tag{3}$$

whereas Φ_ρ stands for the multivariate normal distribution with correlation matrix ρ and Φ^{-1} symbolizes the inverse of univariate normal distribution.

Along the lines of the Gaussian copula, the t copula is given by:

$$C^t(x_1, \ldots, x_n) = t_{\rho,v}(t_v^{-1}(x_1), \ldots, t_v^{-1}(x_n)), \tag{4}$$

$$= \int_{-\infty}^{t^{-1}(x_1)} \cdots \int_{-\infty}^{t^{-1}(x_n)} \frac{\Gamma\left(\frac{v+n}{2}\right)}{\Gamma\left(\frac{v}{2}\right) (v\pi)^{\frac{n}{2}} |\rho|^{\frac{1}{2}}} \left(1 + \frac{1}{v} z^T \rho^{-1} z\right)^{-\frac{v+n}{2}} dz_1 \ldots dz_n, \tag{5}$$

in this setup $t_{\rho,v}$ stands for the multivariate t distribution with correlation matrix ρ and v degrees of freedom (d.o.f.). t_v^{-1} stands for the inverse of the univariate t distribution and v influences tail dependency. For $v \rightarrow \infty$ the t distribution approximates a Gaussian.

In contradiction to the elliptical copulas, the Clayton copula belongs to the group of Archimedean copulas and is given by:

$$C^{Clayton}(x_1, x_2) = \left(\max\{x_1^\theta + x_2^\theta - 1, 0\}\right)^{\frac{1}{\theta}}, \tag{6}$$

with $\theta \in [-1, \infty)\backslash\{0\}$. Note that the Clayton copula describes stronger dependence in the negative tail than in the positive, for $\theta \rightarrow \infty$ the Clayton copula describes comonoticity, and for $\theta \rightarrow 0$ independence.

Another popular Archimedean copula is represented by the Gumbel copula which, in contradiction to the Clayton copula, exhibits higher dependence in the positive tail than in the negative. The copula is given by:

$$C^{Gumbel}(x_1, x_2) = \exp\left(-[(-ln\ x_1)^\delta + (-ln\ x_2)^\delta]^{\frac{1}{\delta}}\right), \tag{7}$$

with $\delta \in [1, \infty)$. Analogue to the Gumbel copula, we get comonoticity for $\theta \rightarrow \infty$ and independence for $\theta \rightarrow 0$.

As well we introduce the Frank copula as defined by Nelson (1999) which is given by:

$$C^{Frank}(x_1, x_2) = -\frac{1}{\theta} ln \left(1 + \frac{(e^{-\theta x_1} - 1)(e^{-\theta x_2} - 1)}{e^{-\theta} - 1}\right), \quad (8)$$

for $\theta \in \mathbb{R} \backslash \{0\}$.

Due to the fact that estimating parameters for higher order copulas might be computationally cumbersome, all parameters are estimated in a two step maximum likelihood method given by Joe [5]. This approach is also known as inference for the margins (IFM). The two steps divide the log likelihood into one term incorporating all parameters concerning univariate margins and into one term involving the parameters of the chosen copula. Thus, this method enables us to explicitly isolate the dependency modeling from fitting the univariate marginals.

2.2 VaR

In order to make the results of the competing copula approaches comparable, we translate the figures into a VaR universe, so that we are able to evaluate the properties of different copulas within a realistic risk measurement framework. Generally, VaR is defined as the quantile at level α of the distribution of portfolio returns:

$$VaR_\alpha = F^{-1}(\alpha) = \int_{-\infty}^{VaR_\alpha} f(r)dr = P(r \leq VaR_\alpha). \quad (9)$$

So that, the respective quantiles are direct functions of the variances, which enables us to directly translate the quantiles of the estimated portfolio variances into VaR figures. Let α be the quantile, H_t the covariance matrix and w the vector of portfolio weights, then VaR at time t is given by: $VaR_t = -\alpha\sqrt{w' H_t w}$ for both normal and t distributions. For instance the 99 % VaR of PF return y_t represents the empirical 1 % quantile of the variance.

3 Simulation Design

The aim of the simulation exercise is to analyze the impact of dependency modeling apart from the choice of optimal marginal distribution. Further, we explicitly address the isolated impact stemming from dependency modeling on quantile forecasts from two angles:

Three different dependency scenarios (weak/medium/strong) are investigated for each copula (Table 1).

Once the sample covering 1001 observations is generated, we use the introduced copula approaches to forecast Value-at-Risk. At this, the forecast is based on 1000

Table 1 Archimedean copulas: simulated scenarios

Copula	Low dependency	Medium dependency	High dependency
Clayton	$\theta = 0, 5$	$\theta = 1, 5$	$\theta = 2, 5$
Gumbel	$\theta = 1$	$\theta = 3$	$\theta = 5$
Frank	$\theta = 10$	$\theta = 20$	$\theta = 30$

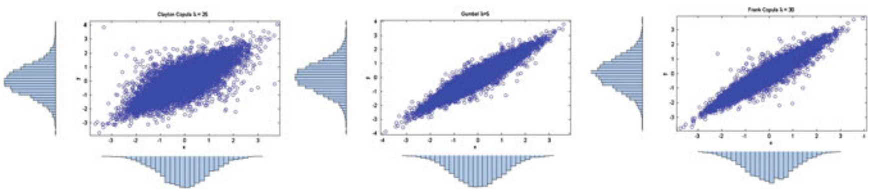

Fig. 1 1001 normally distributed return observations generated by Clayton copula ($\theta = 25$), Gumbel copula ($\theta = 5$) and Frank Copula ($\theta = 30$)

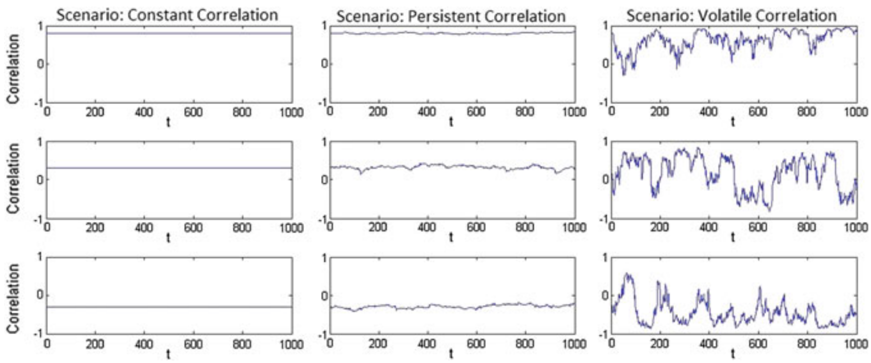

Fig. 2 Correlation scenarios

observations and can be evaluated by the 1001th one. This approach enables us to investigate the VaR performance and thus the compatibility of the competing copula approaches when the samples are not in line with the underlying assumptions. Figure 1 illustrates examples of the investigated scenarios. 10.000 scenarios for each dependency modification are simulated and evaluated. Secondly, to take account for financial time series specific properties, we generate normally distributed return series characterized by time varying conditional correlations (see Fig. 2) to investigate the consequences of time varying dependency schemes on the applied copula approaches. Again, for each scenario 10.000 return series covering 1001 observations are generated and the competing copula models are evaluated by the VaR forecasting performance regarding the 1001th observation.

4 Simulation Results

To sum it up, the empirical VaR backtesting performance for all investigated scenarios is given in Table 2. Obviously, given that dependency is modeled via time varying linear correlation coefficient, the elliptical copulas outperform the Archimedean copulas in terms of empirical VaR performance. The Clayton copula underestimates risk when the data generating process is not a Clayton copula whereas mixed evidence can be reported for both Gumbel and Frank copula. However, both approaches, are not able to adequately capture tail dependence generated via Clayton copula. By comparing both Gauss and t copula, the overall performance of the t copula is slightly more precise when it comes to forecast VaR.[2] According to the return series generated by Archimedean copulas, mixed evidence can be reported. For 95 % VaR forecasts, the Gumbel and Frank copula applied to returns generated via Clayton copula result in an inappropriately high number of misspecifications[3] and thus both dependency approaches would be rejected by standard statistical VaR backtesting.[4] Interestingly, the Gumbel and Frank copula lack in capturing tail dependence generated by Clayton copula and vice versa. However, given that the empirical backtesting performance of all the other investigated models are "statistically" acceptable, leads us to conclude that misspecified dependency modeling has an impact on the rejection of a VaR model. Thus, given that the rejection rate is impacted by the choice of the applied copula approach, our findings are twofold:

- From a pregulatory perspective, the classical gauss and t copula seem to be an appropriate choice for all investigated dependency scenarios. According to our results, it is the Gauss and t copula which mainly result in the "second best" solution[5] when the returns are generated by different copulas. Thus, the higher parametrisation of the competing copula approaches does not lead to more precise dependency measurement and hence more accurate VaR failure rate. Further, due to the higher parameterisation, the Archimedean copulas lack in terms of preciseness when the underlying sample does not exhibit the characteristics of the applied copulas. Moreover, having in mind that Gauss copulas do neither account for time varying dependency structure nor for tail dependence, we show that the parsimonious approach leads to acceptable VaR figures throughout all investigated scenarios.
- However, from an institutional point of view, it is not only the rejection rate which is relevant but also the absolute size of VaR. If we analyse both, the rejection rate as well as the absolute amount of VaR forecast, we favour the model which results in the lowest amount of VaR forecast, given that the empirical failure rate is in line with the expectations. For time varying linear dependency, again, the elliptical

[2] We applied the CPA test proposed by Giacomini and White (2006) to prove this fact. Results are available upon request.

[3] The empirical backtesting performance would get rejected by statistical backtesting criteria, "conditional coverage", by Christoffersen (1998).

[4] Results are available upon request.

[5] The "first best" solution is always the original model.

Table 2 Empirical misspecification performance, 95 % and VaR forecasts

Scenario	G Cop (%)	t Cop (%)	Clayton (%)	Gumbel (%)	Frank (%)
95 % VaR					
Elliptical	4,97	4,99	4,03	4,68	5,04
Clayton	6,10	6,09	5,16	7,05	6,37
Gumbel	4,71	4,70	4,60	4,82	4,77
Frank	5,07	5,07	4,89	5,44	4,92

copulas do outperform the Archimedean approaches, since they result in the lowest VaR values and show an acceptable empirical failure rate. Along the lines of the linear dependency scenarios, the elliptical copulas also represent the (second-) best choice for VaR forecasts for samples which are generated by Archimedean copulas. The Archimedean copulas heavily depend on the assumptions of the underlying samples, so that Frank copulas adequately capture tail dependency generated by Gumbel copula (and vice versa) whereas both Frank and Gumbel fail to capture characterstics generated by Clayton copula.

5 Conclusion

Albeit the main impact on multivariate portfolio VaR stems from the choice of marginal return distributions, the adequate modeling of dependency needs to be considered in order to achieve an appropriate VaR performance. So that, when it comes to the impact of dependency modeling on VaR forecasts, the choice of copula is crucial.

Based on the given extensive simulation analysis covering different dependency scenarios and triggered by the comparison of competing copula approaches, we conclude that the investigated elliptical copulas do outperform the Archimedean copulas due to more precise VaR forecasts. Thus, having in mind that both the Gauss and t copula are straightforward to apply to multivariate asset portfolios comprising three or more assets, we strongly recommend the application of elliptical copulas in the context of VaR forecasts.

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