Exact and Compact Formulation of the Fixed-Destination Travelling Salesman Problem by Cycle Imposement Through Node Currents

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Abstract The Travelling Salesman Problem (TSP) has been studied extensively for over half a century, but due to its property of being at the basis of many scheduling and routing problems it still attracts the attention of many research. One variation of the standard TSP is the multi-depot travelling salesman problem (MTSP) where the salesmen can start from and return to several distinct locations. This article focusses on the MTSP with the extra restriction that each salesman should return to his home depot, known as the fixed-destination MTSP. This problem (and its variations such as the multi-depot vehicle routing problem) is usually formulated using three-index binary variables, making the problem computationally expensive to solve. Here an alternative formulation is presented using two-index binary variables through the introduction of a limited amount of continuous variables to ensure the return of the salesmen to their home depots.

1 Introduction

The TSP has been a topic of research for over six decades [1], but it still attracts the attention of researchers due to its challenges and wide applicability. Many variations of the TSP have been introduced to model a real-world problem, such as the vehicle routing problem [12] and its many variations [8].

In this article we focus on the formulation of the TSP with multiple depots, were each salesman should return to his home depot. When considering scheduling and routing problems with multiple depots where each entity (e.g. a salesman or vehicle) should return to the home depot, we talk about fixed-destination problems [2]. Such problems are often formulated as a mixed-integer linear program (MILP) using a three-index formulation of binary variables

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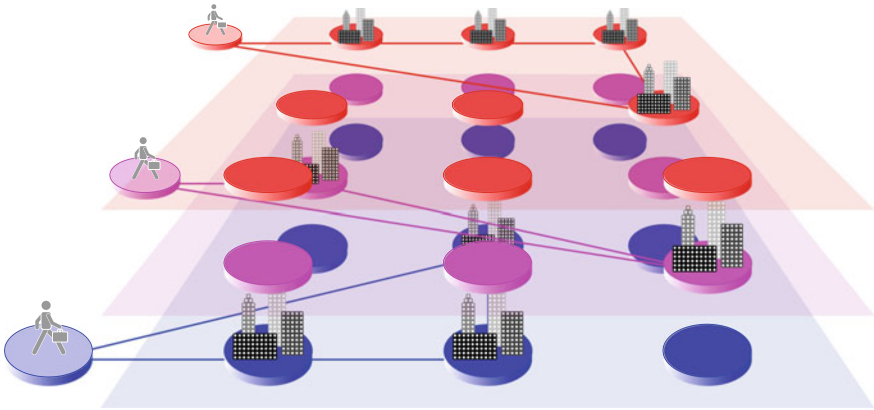

Fig. 1 A fixed-destination solution using a 3-index formulation for 3 depots and 9 cities

$$x_{ijk} = \begin{cases} 1 & \text{if location } i \text{ precedes location } j \text{ directly in a tour started at depot } k \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

resulting in $\mathsf{C}^2\mathsf{D}$ binary variables for a problem with C cities and D depots. This problem can be depicted by a layered graph as shown in Fig. 1, where each depot has a copy of all city nodes in a separate layer. When solving a MILP using standard solvers, the computation time is largely dependent on the number of integer (binary) variables that are used to represent the problem. Therefore, it is beneficial to try to reduce the number of binary variables.

Recently an alternative formulation using two-index binary variables has been presented in [3] using a multi-commodity formulation. The set of depot nodes is cloned to create sink and source nodes for the commodity flows. Using D continuous variables (representing the commodities) at each of the $\mathsf{C} + 2\mathsf{D}$ locations (cities plus depots) it is ensured that a flow of commodities starting at a (source) depot will end at the associated copied (sink) depot, thereby ensuring fixed-destination solutions.

In this article an alternative two-index formulation is presented that requires a little less binary variables and significantly less continuous variables as compared to the multi-commodity formulation. There is no need to copy the depot nodes, and only one additional continuous variable per location is needed, resulting in an increase of $\mathsf{C} + \mathsf{D}$ continuous variables compared to the (non-fixed destination) TSP. These continuous variables can be seen as node currents, inspired by the subtour elimination constraints using node potentials that were introduced by Miller et al. [10]. This formulation has been used for solving scheduling problems for micro-ferries [5] and harvesters [6]. Here we will discus the method in detail for the basic MTSP to make readers aware of the possibility to use this formulation as the basis of other multi-depot scheduling and routing problems.

2 Fixed-Destination Travelling Salesman Problems

We will discuss the TSP with multiple depots, where each salesman should return to its home depot at the end of his tour. A novel formulation for this fixed-destination MTSP using two-index decision variables will be presented next.

2.1 Node Potentials and Currents

The inspiration of this approach comes from the *subtour elimination constraints* of Miller et al. [10] using *node potentials*. To avoid cycles in (a part of) a graph one can assign continuous variables to the nodes representing a potential in an electric circuit, and add constraints on their values to avoid subtours. We reckoned that if there are node potentials in a network, and the nodes are connected by arcs, there should also be arc currents flowing between the nodes. Since for a solution to the MTSP each node has exactly one incoming and one outgoing arc (see Fig. 2) this current can be seen as a property of the nodes (instead of the arcs). We will present a methodology that can be seen as the dual to the MTZ subtour elimination constraints; *cycle imposement constraints* using *node currents*.

2.2 Description of the Problem

Consider a problem with D depots and C cities with sets $\mathscr{D}$ and $\mathscr{C}$ defined as

$$\mathscr{D} = \{1, \dots, \mathsf{D}\}, \quad \mathscr{C} = \{\mathsf{D}+1, \dots, \mathsf{N}\}, \quad \mathscr{N} = \mathscr{D} \cup \mathscr{C}, \tag{2}$$

where $\mathsf{N} = \mathsf{D} + \mathsf{C}$ denotes the total number of locations represented by the set $\mathscr{N}$. This problem can be depicted by a graph with N nodes, where associated with each possible directed arc (i, j) we define a decision variable

$$x_{ij} = \begin{cases} 1 & \text{if there is a connection from node } i \text{ to node } j, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

resulting in a total of C^2 binary variables in the MILP formulation.

As shown in Fig. 2 the graph can be split into two subgraphs: the nodes in $\mathscr{D}$ are associated with the depots and the nodes in $\mathscr{C}$ are associated with the cities. From each of the depots we want one salesman[1] to travel towards a city (represented by an

[1] It is possible to formulate this problem with multiple salesmen per depot as well. To avoid distraction from the main purpose of this section the problem is kept as simple as possible.

arc from $\mathscr{D}$ to $\mathscr{C}$) and returning to his home depot at the end of the tour (represented by an arc from $\mathscr{C}$ to $\mathscr{D}$).

Although cycles in the set $\mathscr{C}$ must be avoided to obtain a correct solution, within the set $\mathscr{N}$ we want *exactly* D cycles; one associated with each of the depots in $\mathscr{D}$ (see Fig. 2). To obtain such a solution we introduce N continuous variables k_i that can be seen as the dual to the node potentials u_i; they can be considered node currents. To *impose* the existence of D cycles in the graph we give each depot node an unique value and propagate it along the path.

2.3 Formulation of the Problem

The fixed-destination MTSP can be formulated as the mixed-integer linear program[2]

$$\text{minimise} \quad \sum_{i \in \mathscr{N}} \sum_{j \in \mathscr{N}} \mathsf{c}_{ij} x_{ij} \tag{4a}$$

$$\text{subject to} \quad \sum_{j \in \mathscr{C}} x_{hj} = 1, \quad \sum_{i \in \mathscr{C}} x_{ih} = 1 \qquad \forall\, h \in \mathscr{N} \tag{4b}$$

$$u_i - u_j + \mathsf{N} x_{ij} \leq \mathsf{N} - 1 \qquad \forall\, i, j \in \mathscr{C} \tag{4c}$$

$$k_d = d \qquad \forall\, d \in \mathscr{D} \tag{4d}$$

$$k_i - k_j + (\mathsf{D} - 1) x_{ij} \leq \mathsf{D} - 1 \quad \forall\, i, j \in \mathscr{N} \tag{4e}$$

$$x_{ij} \in \{0, 1\} \qquad \forall\, i, j \in \mathscr{N} \tag{4f}$$

where (4a) is the objective function representing the total travel distance, (4b) are the assignment constraints ensuring that each location is visited once and only once, and (4c) are the subtour elimination constraints. Using (4d) each variable k_d of the depot nodes is given a unique value, and (4e) propagate the value $k_i = d$ along cities i in the path of depot d. Note the strong resemblance to (4c); constraints (4c) might appear to be weaker versions of the subtour elimination constraints, but they actually impose the existence of D cycles (one for each depot) in the set $\mathscr{N}$ as explained next.

2.4 Node Current Propagation in Detail

In order to show that inequalities (4e) indeed enforce fixed-destination solutions (in combination with the assignment constraints (4b) and the subtour elimination constraints (4c)), we start by analysing the inequalities.

[2] To see the relation between node potentials and node currents more clearly the original subtour elimination constraints from [10] are used in (4c). For actual implementation these constraints can be made tighter using the formulations presented in [7, 11].

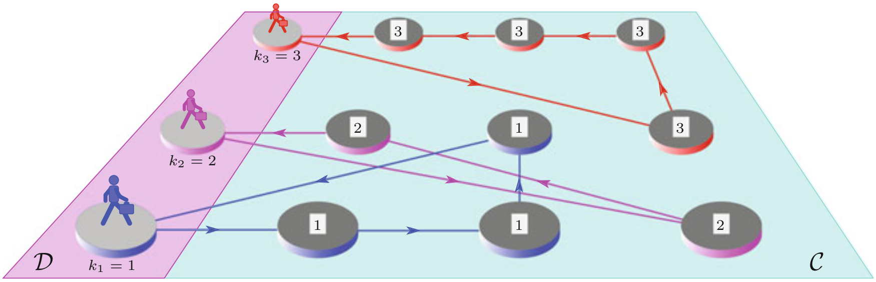

Fig. 2 Constraints (4e) ensure the existence of D cycles. This figure shows an example solution to the problem with $\mathsf{D} = 3$ depots and $\mathsf{C} = 9$ cities

When there is no direct path from location i to j we have $x_{ij} = 0$ and hence

$$k_i - k_j \leq \mathsf{D} - 1. \tag{5}$$

Since the cities will be associated to a depot with index number 1 to D, we expect the variable k_i to have a value in between 1 and D due to the equality constraints (4d). Therefore, inequality (5) is non-restrictive since $k_i \leq \mathsf{D}$ and $k_j \geq 1$. When $x_{ij} = 1$ it means that the path of a salesman goes from location i to j directly, and

$$k_i - k_j + \mathsf{D} - 1 \leq \mathsf{D} - 1 \quad \Leftrightarrow \quad k_i \leq k_j. \tag{6}$$

Hence, the value of k_j will be non-decreasing along the path. Since these inequalities should also hold for arcs (c, d) from a city c to depot d—and the value k_d of the depot is fixed by (4d)—a path that originates from depot d cannot return to a depot with a lower index number.

Now consider depot D. Since $k_i \leq k_j$ along each path we have $k_i \geq \mathsf{D}$ along the path originating from this depot. Due to the assignment constraints (4b) each node will have exactly one incoming arc and one outgoing arc, hence the path can only end in a depot node (otherwise there will be a city node with two incoming arcs). The only depot node d that can satisfy the constraint $k_d \geq k_c \geq \mathsf{D}$ is depot $d = \mathsf{D}$. Constraints (4d) and (4e) impose the existence of a cycle containing node D, and since $\mathsf{D} \leq k_c \leq k_d = \mathsf{D}$ we have $k_c = \mathsf{D}$ along the path of depot D.

Next consider depot node $\mathsf{D} - 1$. Along the path of this depot we have $k_c \geq \mathsf{D} - 1$, and since $k_d \geq k_c \geq \mathsf{D} - 1$ should hold when going from city c to depot d, the index number of the depot should be at least $\mathsf{D} - 1$. Since we know that depot D already has an incoming arc [and only one is allowed due to (4b)] the path started at depot $\mathsf{D} - 1$ can only return to depot $\mathsf{D} - 1$. Also $k_c = \mathsf{D} - 1$ along the path of depot $\mathsf{D} - 1$.

Continuing this reasoning one can see that each depot d has a path that returns to depot d, and $k_c = d$ along the path associated with depot d. Hence we have a solution with (at least) D cycles. Due to the subtour (cycle) elimination constraints

(4c) it is ensured that there are no cycles in $\mathscr{C}$; exactly D cycles exist in the graph, each associated with one of the depots. This resembles the solution to the fixed-destination MTSP, since each path returns to its home depot.

3 Conclusions

In this article we have demonstrated the use of node currents and cycle imposement constraints to formulate the fixed-destination travelling salesman problem as a mixed-integer linear program using two-index binary variables. The use of two-index formulations over three-index formulations results in shorter computation times and lower memory use when solving the problems using standard MILP solvers. Although specialised algorithms might outperform the standard MILP solvers, it is believed that the presented formulation might provide great benefits in solving variations of the multi-depot travelling salesman problem (such as the micro-ferry scheduling problem [5] and the multiple harvester routing problem [6]) for which specialised algorithms are (not yet) available.

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