Computing an Upper Bound for the Longest Edge in an Optimal TSP-Solution

Hans Achatz and Peter Kleinschmidt

Abstract A solution of the traveling salesman problem (TSP) with n nodes consists of n edges which form a shortest tour. In our approach we compute an upper bound u for the longest edge which could be in an optimal solution. This means that every edge longer than this bound cannot be in an optimal solution. The quantity u can be computed in polynomial time. We have applied our approach to different problems of the TSPLIB (library of sample instances for the TSP). Our bound does not necessarily improve the fastest TSP-algorithms. However, the reduction of the number of edges might be useful for certain instances.

1 Introduction

The traveling salesman problem (TSP) is one of the most studied problems in combinatorial optimization and has got applications in many different areas. The TSP consists of finding a shortest tour in a complete graph whose edges (i,j) have cost (distance) c_{ij}. A comprehensive treatment of the traveling salesman problem can be found in [3].

In this paper we do not assume that the cost matrix is symmetric. However, our figures will refer to symmetric instances. We consider a dual relaxation of the original problem—the assignment problem A based on the same cost matrix. The result is a dual relaxation, possibly with subtours, as shown in Fig. 1. This problem can also be solved by any code for the assignment problem, e.g. [1].

H. Achatz (✉) · P. Kleinschmidt
University of Passau, Innstr. 39, 94036 Passau, Germany
e-mail: hans.achatz@uni-passau.de

P. Kleinschmidt
e-mail: pkleinschmidt@t-online.de

D. Huisman et al. (eds.), *Operations Research Proceedings 2013*,
Operations Research Proceedings, DOI: 10.1007/978-3-319-07001-8_1,

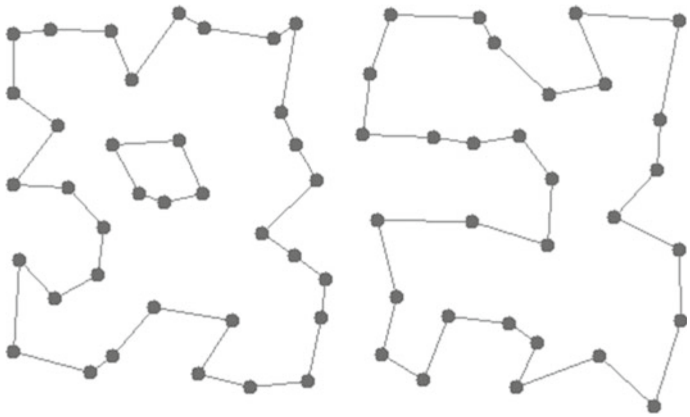

Fig. 1 Dual relaxation with subtours

$$A: \qquad \min \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \tag{1}$$

s.t.

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad 1 \leq j \leq n \tag{2}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad 1 \leq i \leq n \tag{3}$$

$$0 \leq x_{ij} \leq 1 \qquad 1 \leq i, j \leq n \tag{4}$$

This optimal solution can be transformed into a tour as shown in the figure below. The value of the objective function of the solution in this example is 2,744 and it is an upper bound for the optimal solution (Fig. 2). By using the Lin-Kernighan heuristic [4] we can obtain an even better upper bound 2,726. The optimal value of the dual heuristic is 2,426. Hence, the length of an optimal tour is between these two values.

2 Computing an Upper Bound

In this paper we introduce a new relaxation A' of the TSP. Due to inequality 6 and 7 every node must have at least one adjacent edge and at most two adjacent edges. Equation 8 assures that there are exactly $n - 1$ edges.

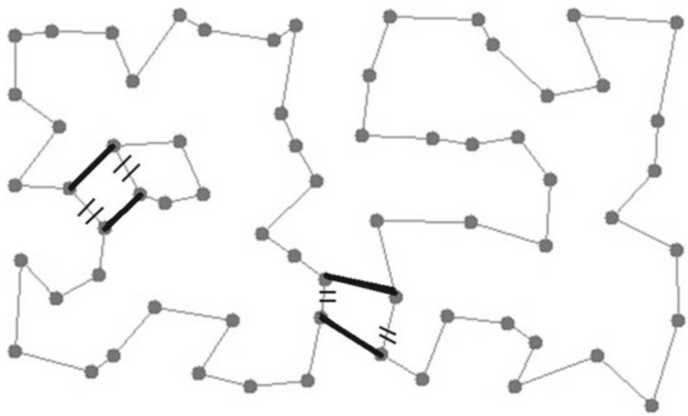

Fig. 2 Transformation into a primal feasible solution

$$A': \qquad \min \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \tag{5}$$

$$s.t.$$

$$\sum_{i=1}^{n} x_{ij} + \sum_{i=1}^{n} x_{ji} \leq 2 \qquad 1 \leq j \leq n \tag{6}$$

$$\sum_{i=1}^{n} x_{ij} + \sum_{i=1}^{n} x_{ji} \geq 1 \qquad 1 \leq j \leq n \tag{7}$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} = n - 1 \tag{8}$$

$$M, \text{ a set of valid TSP constraints} \tag{9}$$

$$0 \leq x_{ij} \leq 1 \qquad 1 \leq i, j \leq n \tag{10}$$

The set M may consist of some valid TSP constraints which do not contradict constraint 8. For example, M could be chosen as a set of subtour elimination constraints. We have tested our approach with $M = \{x : x_{ij} + x_{ji} \leq 1, 1 \leq i, j \leq n\}$ to avoid 2-cycles. An optimal solution (objective value 2,624) for this problem is shown in Fig. 3. If we delete the constraints of type 9 then the resulting problem A^* is comparable to an assignment problem where only $n - 1$ nodes are assigned. In [2] the first author analyzed the bipartite weighted matching problem with respect to slightly changed problems of the original problem. In one type of problem two nodes are deleted in the bipartite graph (one at each partition). The solution is of course a complete matching (an assignment) with $n - 1$ edges and therefore also a solution for A^* which can be computed in $O(n^3)$.

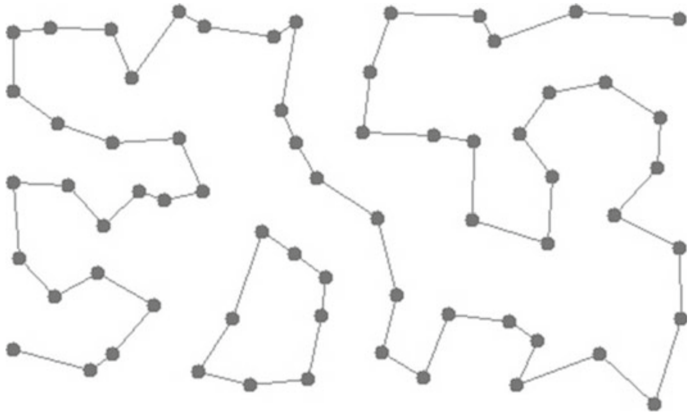

Fig. 3 Relaxation with $n - 1$ edges

Let $f(A')$ be the objective value of the above problem A'. OPT denotes the optimal solution of the original TSP and P' is any primal feasible solution. Then, of course we have

$$f(A') \leq OPT \leq f(P') \tag{11}$$

Theorem 1 *$f(P') - f(A')$ is an upper bound for the longest edge in an optimal solution of the TSP.*

Proof For any primal feasible solution P with objective value $f(P) \leq f(P')$ we claim:

If (i, j) is the longest edge in P then $c_{ij} \leq f(P') - f(A')$.

Suppose $f(P') - c_{ij} < f(A')$ then $P \setminus \{(i, j)\}$ is a feasible solution for problem A' with objective value $f(P) - c_{ij}$. Hence, $f(P) - c_{ij} \leq f(P') - c_{ij} < f(A')$ by our assumption. However, $f(A')$ was optimal and therefore we have a contradiction. This means that all edges longer than $f(P') - f(A')$ can not be in a better solution than P', in particular all these edges can not be in an optimal solution. □

In our example our best primal solution was 2,726 and the objective value of A' is 2,624. Therefore the difference 102 of these values is an upper bound for the longest edge in an optimal solution. This improves the value 2,352 computed via A^*. In Fig. 4 the edge (a, b) has length 104 and therefore this edge can not be in an optimal solution. All in all 3,542 edges (or 83 %) are longer than the computed bound and can be deleted.

Remark 1 There are TSP instances where the longest edge of the problem is in an optimal solution.

If all cities are on a semicircular then the longest edge (the diameter of the circular) is of course in the optimal solution. In this case our bound is useless.

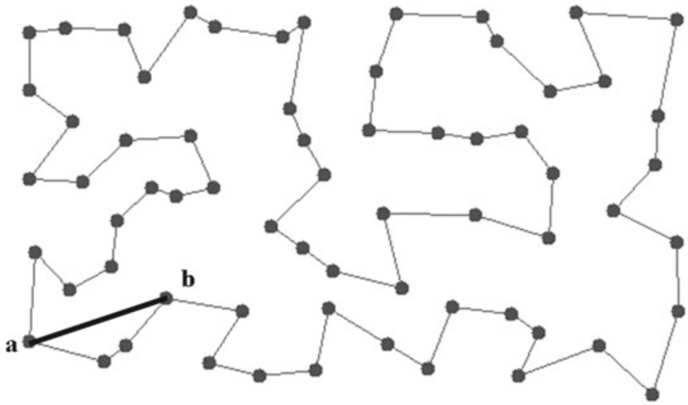

Fig. 4 Example for an edge to be deleted

Table 1 Upper bounds for longest edge

Instance	Cities	Length	Edges	Percentage
bay29	29	104	304	37
eil51	51	26	1,562	61
gr120	120	94	6,314	44
a280	280	130	32,788	42
att532	532	4,656	60,950	22

3 Computational Results

We have analyzed our approach with several instances in TSPLIB [5] where our set M was chosen to be $M = \{x : x_{ij} + x_{ji} \leq 1, 1 \leq i, j \leq n\}$. The first two columns denote the name and size of the problem.

The entries of the column "length" are the computed upper bounds for the respective instances. In the last two columns the number of edges longer than this bound and their percentage is given. This means for example for the drilling problem in instance a280 that 42 % of all edges are too long to be in an optimal solution. In all instances, computing the euclidian distances from the problem data takes more time than the computation of the LP-solution of A'. All primal feasible solutions were produced by the Lin-Kernighan heuristic [4].

Our computed bounds may be helpful computationally as they lead to potentially much sparser graphs to be considered in various algorithms.

References

1. Achatz, H., Kleinschmidt, P., & Paparrizos, K. (1991). A dual forest algorithm for the assignment problem. In P. Gritzmann, B. Sturmfels & V. Klee (Eds.), *Applied geometry and discrete mathematics. The Victor Klee festschrift* (pp. 1–10). Providence, R.I.: AMS (DIMACS'4).
2. Achatz, H. (1999). Sensitivity analysis of the bipartite weighted matching problem. In P. Kall (Ed.), *Operations Research Proceedings 1998. Selected Papers of the International Conference on Operations Research*, Zürich (pp. 135–141). Aug31–Sept 3, 1998. Berlin: Springer.
3. Applegate, D. L., Bixby, R. E., Chvátal, V., & Cook, W. J. (2007). *The traveling salesman problem. A computational story*. Princeton, NJ: Princeton University Press (Princeton series in applied mathematics).
4. Lin, S., & Kernighan, B. W. (1973). An effective heuristic algorithm for the traveling-salesman problem. *Operations Research*, *21*, 498–516.
5. Reinelt, G. (1991). TSPLIB—a traveling salesman problem library. *ORSA Journal on Computing*, *3*, 376–384.