Analytic Approximations for Linear Differential Equations with Periodic or Quasi-periodic Coefficients

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Abstract A perturbative procedure is proposed to compute analytic approximations to the fundamental matrix of linear differential equations with periodic or quasi-periodic coefficients. The algorithm allows one to construct high-order analytic approximations to the characteristic exponents and thus analyze the stability of the system. In addition, the approximate matrix solutions preserve by construction qualitative properties of the exact solution.

1 Introduction

The linear system of differential equations

$$\dot{Y} \equiv \frac{dY}{dt} = A(t)Y, \qquad Y(0) = I, \tag{1}$$

with $A(t)$ a T-periodic matrix, is an example of a reducible system: by means of the transformation $Y = P(t)Z$, with $P(t)$ a non singular periodic matrix, a new system $\dot{Z} = KZ$ is obtained, where now the coefficient matrix $K = P^{-1}(t)A(t)P(t) - P^{-1}(t)\dot{P}(t)$ is constant. This is the so-called Lyapunov transformation [1]. As a consequence, the solution of the original system can be written globally as $Y(t) = P(t)\exp(tK)$. This is just a rephrasing of the well known Floquet theorem for linear periodic differential equations [9].

From this result it is clear that the stability conditions of the solution $Y(t)$ only depend on the matrix K, specifically on its eigenvalues (the characteristic exponents of the system), whose real parts are uniquely determined. Thus, the trivial solution of (1) is asymptotically stable if and only if the real part of the characteristic exponents is negative, and it is stable if and only if all the characteristic exponents have non positive real part, with the vanishing or purely imaginary characteristic exponents being simple elementary divisors of the matrix $K - \lambda I$, $\lambda \in \mathbb{C}$ [9]. From

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these properties, it is clear that computing the matrix K or the monodromy matrix $Y(T) = \exp(TK)$ is extremely useful. Unfortunately, although the Floquet theorem gives us information about the structure of solution of the system (1), it does not provide any practical method to get K and/or the transformation matrix $P(t)$.

Here we propose an algorithmic procedure to get approximations to both K and $P(t)$, and therefore to the solution $Y(t)$ in the form prescribed by the Floquet theorem when $A(t) = A_0 + \varepsilon A_1(t) + \varepsilon^2 A_2(t) + \cdots$ in terms of the parameter $\varepsilon > 0$. The algorithm is recursive and determines the periodic transformation $P(t)$ as the exponential of a certain matrix $\Omega(t)$. This property guarantees by construction that the approximations preserve certain qualitative properties of the exact solution. In addition the algorithm can be easily implemented with a symbolic algebra package.

If, on the other hand, the coefficient matrix $A(t)$ is quasi-periodic, the problem of reducing (1) to a system with constant coefficients is far more difficult. When the terms $A_1(t), A_2(t), \ldots$ are sufficiently small, Shtokalo [8] constructed asymptotic expansions for the solution which allowed him to examine the stability of the system. It turns out that the procedure we have developed for periodic systems can also be generalized to this setting with only minor modifications.

2 Algorithm

Let us consider the $d \times d$ system

$$\frac{\partial}{\partial t} Y(t, \varepsilon) = A(t, \varepsilon)\, Y(t, \varepsilon), \qquad Y(t_0 = 0, \varepsilon) = I \tag{2}$$

with

$$A(t, \varepsilon) = A_0 + \sum_{j \geq 1} \varepsilon^j A_j(t) = A_0 + \varepsilon A_1(t) + \varepsilon^2 A_2(t) + \cdots \tag{3}$$

and $A_j(t + T) = A_j(t)$, $j \geq 1$, for a certain $T > 0$. The goal is then to construct a transformation $P(t, \varepsilon)$ with inverse

$$Y(t, \varepsilon) \xrightarrow{P(t,\varepsilon)} Z(t, \varepsilon) = P^{-1}(t, \varepsilon)\, Y(t, \varepsilon)\, P(0, \varepsilon) \tag{4}$$

such that for the system in the new coordinates one has

$$\frac{\partial}{\partial t} Z(t, \varepsilon) = K(\varepsilon)\, Z(t, \varepsilon), \qquad Z(0, \varepsilon) = I, \tag{5}$$

with a constant coefficient matrix given by

$$K(\varepsilon) = P^{-1}(t, \varepsilon) A(t, \varepsilon) P(t, \varepsilon) + \frac{\partial P^{-1}(t, \varepsilon)}{\partial t} P(t, \varepsilon). \tag{6}$$

We construct $P(t,\varepsilon)$ as a near-identity transformation, i.e., $P(t,\varepsilon) = I + \mathscr{O}(\varepsilon)$, in such a way that it satisfies an equation similar to (2) but now with respect to ε. More specifically, in view of Eq. (4), we impose

$$\frac{\partial}{\partial \varepsilon} P^{-1}(t,\varepsilon) = L(t,\varepsilon) P^{-1}(t,\varepsilon), \qquad P^{-1}(t,0) = I \tag{7}$$

in terms of a (still unknown) generator $L(t,\varepsilon)$. Alternatively,

$$\frac{\partial}{\partial \varepsilon} P(t,\varepsilon) = -P(t,\varepsilon) L(t,\varepsilon), \qquad P(t,0) = I. \tag{8}$$

Once $L(t,\varepsilon)$ has been determined, it is possible to obtain $P(t,\varepsilon)$ by formally applying the Magnus expansion [3,6] to the linear equation (7), so that

$$P^{-1}(t,\varepsilon) = \exp \Omega(t,\varepsilon), \tag{9}$$

where $\Omega(t,\varepsilon)$ is an infinite series depending $L(t,\varepsilon)$ and its nested commutators.

To determine the generator $L(t,\varepsilon)$, we differentiate Eq. (6) with respect to ε and use (7)–(8) to get

$$\frac{\partial K}{\partial \varepsilon} = [L, K] + P^{-1} \frac{\partial A}{\partial \varepsilon} P + \frac{\partial L}{\partial t}, \tag{10}$$

that is,

$$\frac{\partial K}{\partial \varepsilon} = [L, K] + \mathrm{e}^{\mathrm{ad}_{\Omega}} \frac{\partial A}{\partial \varepsilon} + \frac{\partial L}{\partial t}, \tag{11}$$

with

$$\mathrm{e}^{\Omega} \frac{\partial A}{\partial \varepsilon} \mathrm{e}^{-\Omega} = \mathrm{e}^{\mathrm{ad}_{\Omega}} \frac{\partial A}{\partial \varepsilon} = \sum_{n \geq 0} \frac{1}{n!} \mathrm{ad}_{\Omega}^{n} \frac{\partial A}{\partial \varepsilon} \tag{12}$$

in terms of the adjoint operator ad: $\mathrm{ad}_{\Omega} B \equiv [\Omega, B] = \Omega\, B - B\, \Omega$ and $\mathrm{ad}_{\Omega}^{n} B \equiv [\Omega, \mathrm{ad}_{\Omega}^{n-1} B]$.

Since $A(t,\varepsilon)$ is given as a series in powers of ε, (see Eq. (3)), we determine both the generator $L(t,\varepsilon)$ and the new coefficient matrix $K(\varepsilon)$ also as formal series in ε:

$$K(\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^{n} K_{n}, \qquad L(t,\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^{n} L_{n+1}(t). \tag{13}$$

The successive terms K_n, $L_n(t)$ in (13) can be obtained from Eq. (11) by applying the following procedure:

1. Insert the series $L(t,\varepsilon)$ into Eq. (7) and compute the Magnus expansion of $\Omega(t,\varepsilon)$,

$$\Omega(t,\varepsilon) = \sum_{n=1}^{\infty} \varepsilon^n v_n(t), \tag{14}$$

in terms of $L_k(t)$. This step has been thoroughly analyzed in [4], where in particular a recursive algorithm for the computation of $v_n(t)$ is given. The first terms in the series (14) read

$$\begin{aligned} v_1 &= L_1, \\ v_2 &= \frac{1}{2} L_2, \\ v_3 &= \frac{1}{3} L_3 - \frac{1}{12}[L_1, L_2] \\ v_4 &= \frac{1}{4} L_4 - \frac{1}{12}[L_1, L_3]. \end{aligned} \tag{15}$$

2. Insert the series (14) into Eq. (12) to express $e^{\mathrm{ad}_\Omega} \frac{\partial A}{\partial \varepsilon}$ as a power series in ε,

$$e^{\mathrm{ad}_\Omega} \frac{\partial A}{\partial \varepsilon} = \sum_{n=0}^{\infty} \varepsilon^n w_n(t). \tag{16}$$

In particular,

$$\begin{aligned} w_0 &= A_1, \\ w_1 &= 2A_2 + [L_1, A_1], \\ w_2 &= 3A_3 + 2[L_1, A_2] + \frac{1}{2}[L_2, A_1] + \frac{1}{2}[L_1, [L_1, A_1]]. \end{aligned} \tag{17}$$

Again, a recursive procedure for the computation of $w_n(t)$ in (16) can be found in [4]. In general, w_n ($n \geq 1$) depends on A_k and L_m, with $1 \leq k \leq n+1$, $1 \leq m \leq n$.

3. Finally, insert the series (13) and (16) into Eq. (11), and equate terms of the same power in ε. In this way we arrive at

$$\begin{aligned} K_0 &= A_0 \\ \frac{dL_n}{dt} + [L_n, A_0] &= nK_n - F_n, \qquad n \geq 1 \end{aligned} \tag{18}$$

with

$$F_1 \equiv w_0 = A_1 \tag{19}$$

$$F_n \equiv \sum_{j=1}^{n-1} [L_{n-j}, K_j] + w_{n-1}, \qquad n > 1. \tag{20}$$

For the first terms we have explicitly

$$\frac{dL_1}{dt} + [L_1, A_0] = K_1 - A_1$$

$$\frac{dL_2}{dt} + [L_2, A_0] = 2K_2 - 2A_2 - [L_1, K_1 + A_1]$$

$$\frac{dL_3}{dt} + [L_3, A_0] = 3K_3 - 3A_3 - [L_2, K_1 + \frac{1}{2}A_1] - [L_1, K_2 + 2A_2 + \frac{1}{2}[L_1, A_1]].$$

These equations allow us to get K_n and $L_n(t)$ recursively once K_m and $L_m(t)$ with $m = 1, \ldots, n-1$ have been previously determined.

For later use, we notice that Eq. (18) can also be written as

$$\frac{dL_n}{dt} = \mathrm{ad}_{A_0} L_n + nK_n - F_n \tag{21}$$

in terms of the linear operator ad_{A_0}.

3 The Lyapunov Transformation in Periodic Systems

Since our goal is to construct approximations to the solution of (2) according with the Floquet theorem, we choose $K(\varepsilon)$ as a constant matrix and obtain the successive terms $L_n(t)$ as periodic matrices in t: $L_n(t+T) = L_n(t)$ for all $n \geq 1$. In this way, $\Omega(t+T,\varepsilon) = \Omega(t,\varepsilon)$ and $Z(t,\varepsilon) = \exp(tK(\varepsilon))$.

To begin with, we integrate Eq. (18) over the period and divide by T:

$$\frac{L_n(T) - L_n(0)}{T} = [A_0, \frac{1}{T}\int_0^T L_n(t)dt] + nK_n - \frac{1}{T}\int_0^T F_n(t)dt. \tag{22}$$

Since L is periodic, then $L_n(T) - L_n(0) = 0$, so that

$$nK_n = \langle F_n \rangle - [A_0, \langle L_n \rangle], \tag{23}$$

where $\langle F_n \rangle$ and $\langle L_n \rangle$ denote the average of F_n and L_n over the interval $[0, T]$, respectively:

$$\langle F_n \rangle \equiv \frac{1}{T} \int_0^T F_n(t)dt, \qquad \langle L_n \rangle \equiv \frac{1}{T} \int_0^T L_n(t)dt. \tag{24}$$

On the other hand, the formal solution of Eq. (21) reads

$$L_n(t) = \mathrm{e}^{t\,\mathrm{ad}_{A_0}} L_n(0) + \mathrm{e}^{t\,\mathrm{ad}_{A_0}} \int_0^t \mathrm{e}^{-s\,\mathrm{ad}_{A_0}} (nK_n - F_n(s))\, ds. \tag{25}$$

Now, inserting (23) into this expression we get

$$L_n(t) = \mathrm{e}^{t\,\mathrm{ad}_{A_0}} L_n(0) + (I - \mathrm{e}^{t\,\mathrm{ad}_{A_0}})\langle L_n \rangle + \mathrm{e}^{t\,\mathrm{ad}_{A_0}} \int_0^t \mathrm{e}^{-s\,\mathrm{ad}_{A_0}} (\langle F_n \rangle - F_n(s))\, ds,$$

where we have used the formal identity

$$\int_0^t \mathrm{e}^{-s\,\mathrm{ad}_{A_0}} (-\mathrm{ad}_{A_0} \langle L_n \rangle) = (\mathrm{e}^{-t\,\mathrm{ad}_{A_0}} - I)\langle L_n \rangle.$$

If we denote by $G_n(s)$ the antiderivative of $\mathrm{e}^{-s\,\mathrm{ad}_{A_0}}(\langle F_n \rangle - F_n(s))$, i.e., $G_n(t)$ is such that

$$\frac{dG_n(t)}{dt} = \mathrm{e}^{-t\,\mathrm{ad}_{A_0}} (\langle F_n \rangle - F_n(t)),$$

then clearly

$$L_n(t) = \mathrm{e}^{t\,\mathrm{ad}_{A_0}} L_n(0) + (I - \mathrm{e}^{t\,\mathrm{ad}_{A_0}})\langle L_n \rangle + \mathrm{e}^{t\,\mathrm{ad}_{A_0}} (G_n(t) - G_n(0)). \tag{26}$$

In summary, the new constant coefficient matrix and the generator of the transformation are given recursively by

$$\begin{aligned} nK_n &= \langle F_n \rangle - [A_0, \langle L_n \rangle] \\ L_n(t) &= \langle L_n \rangle + \mathrm{e}^{t\,\mathrm{ad}_{A_0}} (L_n(0) - \langle L_n \rangle + G_n(t) - G_n(0)), \end{aligned} \tag{27}$$

for $n \geq 1$, starting with $K_0 = A_0$. Notice that there are two undetermined parameters at each step in these expressions, both related with the generator: its initial value $L_n(0)$ and the average $\langle L_n \rangle$. To construct explicitly the transformation we have to fix these values. The problem then admits infinite solutions. Next we consider just two different possibilities:

1. We fix the initial condition $L_n(0) = 0$. Then, $L_n(T) = 0$ by periodicity and (26) evaluated at $t = T$ leads to

$$0 = (I - e^{T\,\mathrm{ad}_{A_0}})\langle L_n\rangle + e^{T\,\mathrm{ad}_{A_0}}(G_n(T) - G_n(0)). \tag{28}$$

In other words, we can choose $\langle L_n\rangle$ as an arbitrary solution of the matrix equation (28) or alternatively,

$$\int_0^T e^{-s\,\mathrm{ad}_{A_0}}[A_0, C_n]\,ds = G_n(T) - G_n(0) = \int_0^T e^{-s\,\mathrm{ad}_{A_0}}(\langle F_n\rangle - F_n(s))ds, \tag{29}$$

where C_n denotes the unknown matrix. In this way, the problem is solved if we take

$$\begin{aligned} nK_n &= \langle F_n\rangle - [A_0, C_n] \\ L_n(t) &= C_n + e^{t\,\mathrm{ad}_{A_0}}\left(G_n(T) - G_n(0) - C_n\right), \end{aligned} \tag{30}$$

with C_n any particular solution of Eq. (29). As a matter of fact, this is a non-homogeneous system of d^2 linear equations with d^2 unknowns (the elements of C_n) that has a unique solution C_n if and only if $\lambda_k - \lambda_l \neq 0 \bmod \frac{2\pi i}{T}$, $k \neq l$, where λ_k, λ_l are distinct eigenvalues of A_0. Otherwise, some preliminary transformations lead the matrix A_0 to this situation [7].

In summary, if we impose the initial condition $L_n(0) = 0$ and periodicity for $L_n(t)$, then we can build explicitly the series $\Omega(t + T, \varepsilon) = \Omega(t, \varepsilon)$, with $\Omega(0, \varepsilon) = 0$, so that the solution is given by

$$Y(t,\varepsilon) = P(t,\varepsilon)\,e^{tK(\varepsilon)} = e^{-\Omega(t,\varepsilon)}\,e^{tK(\varepsilon)} = \exp\left(-\sum_{n\geq 1}\varepsilon^n v_n(t)\right)\exp\left(t\sum_{n\geq 0}\varepsilon^n K_n\right) \tag{31}$$

where $K_0 = A_0$ and K_n, $n \geq 1$, are constant matrices. In addition, the series obtained for $K(\varepsilon)$ and $P(t, \varepsilon)$ are convergent for sufficiently small values of ε [5].

2. As a second option, we construct L_n such that its average $\langle L_n\rangle = 0$. In that case, from (27),

$$K_n = \frac{1}{n}\langle F_n\rangle. \tag{32}$$

Then we determinate the value of $L_n(0)$ so that $L_n(t)$ in (27) is T-periodic, in particular $L_n(T) = L_n(0)$. From (27) we get

$$L_n(0) = e^{T\,\mathrm{ad}_{A_0}}(L_n(0) + G_n(T) - G_n(0))$$

or

$$\int_0^T \frac{d}{ds}\left(\mathrm{e}^{-s\,\mathrm{ad}_{A_0}} L_n(0)\right) ds = G_n(T) - G_n(0).$$

Since

$$\frac{d}{ds}\left(\mathrm{e}^{-s\,\mathrm{ad}_{A_0}} L_n(0)\right) = \frac{d}{ds}\left(\mathrm{e}^{-sA_0} L_n(0)\mathrm{e}^{sA_0}\right) = \mathrm{e}^{-sA_0}\left(L_n(0)A_0 - A_0 L_n(0)\right)\mathrm{e}^{sA_0},$$

it turns out that $L_n(0)$ has to satisfy Eq. (29). Therefore, the new coefficient matrix and the corresponding generator are given by

$$K_n = \frac{1}{n}\langle F_n\rangle \tag{33}$$
$$L_n(t) = \mathrm{e}^{t\,\mathrm{ad}A_0}(C_n + G_n(t) - G_n(0)),$$

where $C_n = L_n(0)$ is any solution of (29). In general $L_n(0) \neq 0$ and therefore $\Omega(0,\varepsilon) \neq 0$, so that the solution of (2) reads

$$Y(t,\varepsilon) = \mathrm{e}^{-\Omega(t,\varepsilon)}\,\mathrm{e}^{tK(\varepsilon)}\,\mathrm{e}^{\Omega(0,\varepsilon)}. \tag{34}$$

Here $\Omega(t+T,\varepsilon) = \Omega(t,\varepsilon)$ is computed with the generators L_n. In consequence

$$Y(t+T,\varepsilon) = Y(t,\varepsilon)\,\mathrm{e}^{-\Omega(0,\varepsilon)}\,\mathrm{e}^{TK(\varepsilon)}\,\mathrm{e}^{\Omega(0,\varepsilon)}.$$

We notice that, although the structure prescribed by Floquet's theorem is no longer reproduced, $M \equiv \mathrm{e}^{-\Omega(0,\varepsilon)}\,\mathrm{e}^{TK(\varepsilon)}\,\mathrm{e}^{\Omega(0,\varepsilon)}$ *is* a monodromy matrix, with the same eigenvalues as $\mathrm{e}^{TK(\varepsilon)}$. In other words, the eigenvalues of the new matrix $K(\varepsilon)$ given by (33) are also the characteristic exponents of the system.

4 Generalization to the Quasi-periodic Case

Let us consider now Eq. (3) in the quasi-periodic case, i.e., when the matrices $A_j(t)$, $j = 1, 2, \ldots$, in (3) are of the form

$$A_j(t) = \sum_{l=1}^{r} C_{j,l}\, e^{i\mu_l t}. \tag{35}$$

Here $C_{j,l}$ are constant matrices, and μ_l are real numbers, so that the elements of the matrices $A_j(t)$ are trigonometric polynomials with arbitrary frequencies μ_l. The algorithm proposed by Shtokalo [8] for analyzing the stability of the trivial solution of system (2) consists essentially in constructing a change of variables that transform Eq. (2) into (5),

$$\frac{\partial}{\partial t} Z(t,\varepsilon) = \left(A_0 + \sum_{j \geq 1} \varepsilon^j K_j \right) Z(t,\varepsilon), \qquad (36)$$

where K_j are constant matrices. In Shtokalo's procedure, the change of variables and the matrix $K(\varepsilon)$ are constructed perturbatively, as power series of ε, without paying much attention to the approximations of the solution of (3) and the preservation of the main qualitative properties if may possess [5, 8].

It turns out that the procedure developed in the previous sections for constructing the Lyapunov transformation for periodic linear systems can also be applied in this setting with only minor changes. To proceed, let us first recall that for a quasi-periodic function $f(t)$, there exists the limit

$$\lim_{T\to\infty} \frac{1}{T} \int_a^{a+T} f(t) dt = \langle f \rangle, \qquad (37)$$

uniformly with respect to a. The number $\langle f \rangle$ is called the *mean value* of the quasi-periodic function $f(t)$. In addition, this mean value defined for quasi-periodic functions coincides with the usual mean value over the period for periodic functions. Moreover, if $f(t)$ is a trigonometric polynomial,

$$f(t) = C_0 + \sum_{l=1}^{r} C_l \, e^{i\mu_l t},$$

where $\mu_l \neq 0$, $l = 1, \ldots, r$, the mean value $\langle f \rangle = C_0$.

Again, the starting point is Eq. (18). Integrating over the interval $t \in [0, T]$, for an arbitrary $T > 0$, and dividing by T, we get Eq. (22). Taking the limit $T \to \infty$ results in

$$\lim_{T\to\infty} \frac{L_n(T)}{T} = [A_0, \langle L_n \rangle] + nK_n - \langle F_n \rangle.$$

Since we aim to construct the terms of the generator as trigonometric polynomials we impose

$$\lim_{T\to\infty} \frac{L_n(T)}{T} = 0,$$

so that we recover in this setting the expressions (27) for K_n and L_n, where now $\langle \cdot \rangle$ denotes the mean value (37).

At this point, at least two alternatives are possible:

1. Choose $L_n(0) = 0$. Then, a trigonometric polynomial for $L_n(t)$ results as long as $\langle L_n \rangle = -G_n(0)$. In other words,

$$\begin{aligned} K_n &= \frac{1}{n}\langle F_n \rangle + \frac{1}{n}[A_0, G_n(0)] \\ L_n(t) &= -G_n(0) + \mathrm{e}^{t\,\mathrm{ad}_{A_0}} G_n(t). \end{aligned} \tag{38}$$

2. Determine L_n as a trigonometric polynomial with zero mean value, $\langle L_n \rangle = 0$. This can be achieved by taking $L_n(0) = G_n(0)$, and thus

$$\begin{aligned} K_n &= \frac{1}{n}\langle F_n \rangle \\ L_n(t) &= \mathrm{e}^{t\,\mathrm{ad}_{A_0}} G_n(t). \end{aligned} \tag{39}$$

A detailed treatment of this case will be the subject of subsequent work [2].

5 Illustrative Example

We next illustrate the algorithm on a simple periodic example. In particular, we consider the system

$$\begin{aligned} \dot{y}_1 &= \varepsilon(-1 + 2\sin t)y_1 + \varepsilon\, y_2 \\ \dot{y}_2 &= -y_2 + \varepsilon\, y_1 \end{aligned} \tag{40}$$

worked out by Malkin [1]. Here ε is a real parameter and the period $T = 2\pi$. Using the method of small parameters, he showed that the characteristic exponents of the system are negative at least for $\varepsilon < 1/9$, whereas in [9] the domain of values of ε that ensure asymptotic stability is extended up to $\varepsilon < 2/3$.

The fundamental matrix $Y(t, \varepsilon)$ corresponding to system (40) verifies Eq. (4) with

$$A(t, \varepsilon) = A_0 + \varepsilon A_1(t) \equiv \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} + \varepsilon \begin{pmatrix} -1 + 2\sin t & 1 \\ 1 & 0 \end{pmatrix}. \tag{41}$$

First we carry out the first procedure by fixing $L_n(0) = 0$, i.e., we determine K_n and L_n by Eq. (30), up to $n = 10$ and compute the solution matrix (31). In Fig. 1 we plot the difference between the Frobenius norm of our approximation, $Y(t, \varepsilon)$, and the exact result (as determined by numerical integration) when $n = 5$ and $n = 10$ terms are taken in the series.

Next we compute the eigenvalues of $K(\varepsilon)$ as a function of ε by applying the second alternative, i.e., by means of (33), and compare with the exact result (as

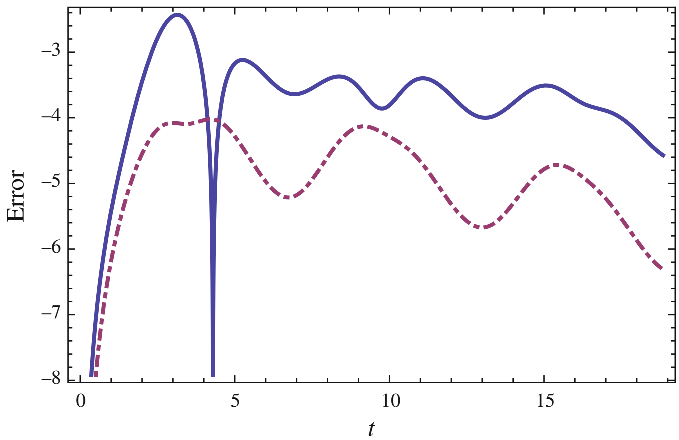

Fig. 1 Error in the approximation (in logarithmic scale) between the approximation of order ε^5 (*solid line*) and order ε^{10} (*dashed line*) with respect to the exact solution

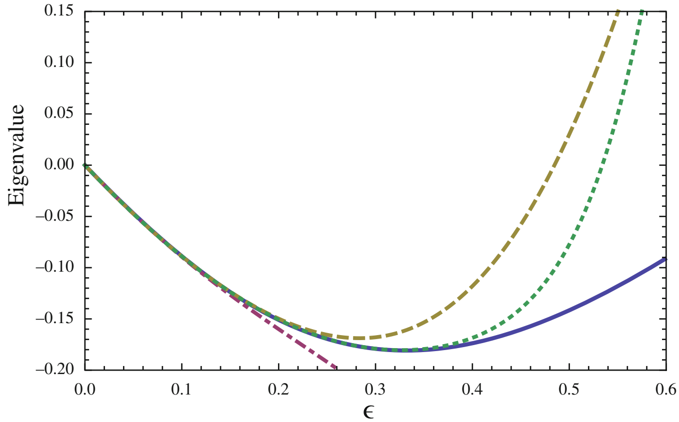

Fig. 2 One of the characteristic exponents of system (40), obtained by direct numerical integration (*solid line*), and by the perturbative algorithm of order ε^2 (*dot-dashed line*), ε^4 (*dashed line*) and ε^{10} (*dotted line*), as a function of ε

determined by the numerical integration of Eq. (40) with 25 digits of accuracy). One of the eigenvalues turns out to be always negative, whereas the second one is negative only for $\varepsilon < 0.745023$, so that it is this value which determines the stability region of the system.

In Fig. 2, we represent this exact eigenvalue (solid line) together with the results rendered by the perturbative algorithm of order ε^2 (dot-dashed line), ε^4 (dashed line) and ε^{10} (dotted line).

Notice that higher order approximations provide results that are indistinguishable from the exact value for increasingly larger values of the perturbation parameter ε.

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