Analysis of Customers' Impatience in M/M/1 Queues with Server Subject to Random Breakdowns and Exponential Vacations

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Abstract In this paper we consider M/M/1queuing systems with server's vacations and random breakdowns. Customers are impatient, where customers' impatience is due to an absentee of the server upon arrival. This absentee is because either the server is on vacation or it is under repair. The mean number of customers in the system and the total waiting time the customer spends in the system have been derived in this work as a very common and important performance measures.

Keywords M/M/1 queue • Server vacation • Random breakdowns • Customers' impatience

1 Introduction

Queuing systems with server interruptions can be used in modeling numerous real-world queuing situations have arisen in systems such as manufacturing systems, communication systems, and production-inventory systems. There is now a growing interest in the analysis of queuing systems with impatient customers. This is due to the potential application of such systems in many related areas (cf. [1]).

Queuing systems with server vacations and/or random system breakdowns have been studied by numerous researchers, for instance, [2–4]. In their work [5–8] investigated a batch arrival queuing system with a Bernoulli scheduled vacation and random system breakdowns. In addition, [9] studied a queuing system with four different main server interruptions and a standby that ever replaces the main server during any potential stop.

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The impatience phenomenon has been studied under various assumptions by many authors. The pioneering work of [10, 11] seems to be the first to analyze queuing systems with impatient customers by considering the unlimited buffer M/M/c queue and assuming that each individual customer stays in the queue as long as his waiting time does not exceed an exponentially distributed impatience time. Furthermore, [12] analyzed queuing systems with server vacations, where each arriving customer who finds no servers on duty because it is on vacation activates an independent random impatience timer. If a server does not show up by the time the timer expires, the customer abandons the queue.

In this work we study the queuing system where the customers' impatience is due to the absence of servers upon arrival. This absence of the server is either because of the vacation or because of random breakdowns. If an arriving customer sees no server present in the system, he/she may abandon the queue if no server shows up within some time. Such a model, representing frequent behavior by waiting customers in service systems, has not been treated before in the literature.

2 The Model

The underlying process is a M/M/1 queue with multiple server ([12]). The customers arrive to the system according to a Poisson distribution with arrival rate λ. Service times are exponentially distributed with mean service rate. Also the vacation times are exponentially distributed with parameter γ. The system may breakdown at random, and breakdowns are assumed to occur according to a Poisson stream with mean breakdown rate $\alpha > 0$. Once the system breaks down, the required repairs start immediately. The duration of repairs follows an exponential distribution with repair rate $\theta > 0$.

We consider the impatience of a customer by noting that whenever a customer arrives to the system and realizes that the server is on vacation or under repair, he/she activates an "impatience timer" T, T which is exponentially distributed with parameter ξ. The impatience time T is independent of the queue size at that moment. If the server returns from his vacation before the time T expires (and starts providing service), the customer stays in the system until his/her service is completed. However, if T expires while the server is still on vacation or still under repairs, the customer leaves the queue, never to return.

3 Balance Equations

We start by letting L denote the total number of customers in the system, and we let J denote the number of working servers. When the server is on vacation or under repair, this means that $J = 0$, while $J = 1$ implies that the server is active. The pair (J, L) defines a continuous-time Markov process with transition-rate diagram

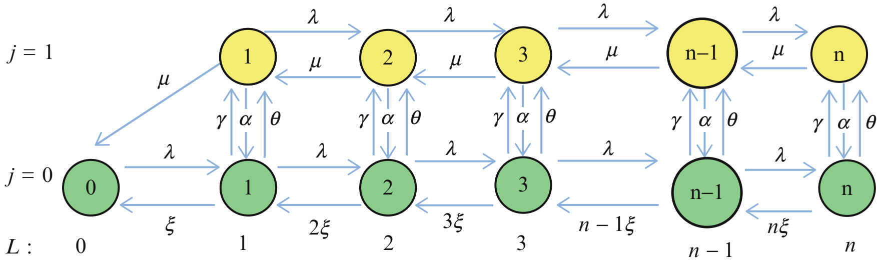

Fig. 1

as illustrated in Figure 1. Let $Pjn = P\{J = j, L = n\}(j = 0, 1; n = 0, 1, 2, \ldots)$ denote the (steady state) system state probabilities. The set of balance equations is given as

$$j = 0 \begin{cases} n = 0 & \lambda P_{00} = \mu P_{11} + \xi P_{01} \\ n \geqslant 1 & (\lambda + n\xi + \gamma + \theta)\, P_{0n} = (n+1)\,\xi P_{0n+1} + \lambda P_{0n-1} + \alpha P_{1n} \end{cases} \tag{1}$$

$$j = 1 \begin{cases} n = 1 & (\lambda + \mu + \alpha)\, P_{11} = \mu P_{12} + (\gamma + \theta)\, P_{01} \\ n \geqslant 2 & (\lambda + \mu + \alpha)\, P_{1n} = \mu P_{1n+1} + \lambda P_{1n-1} + (\gamma + \theta)\, P_{0n} \end{cases} \tag{2}$$

Define the probability generating functions (PGFs)

$$G_0(z) = \sum_{n=0}^{\infty} P_{0n} z^n, \qquad G_1(z) = \sum_{n=1}^{\infty} P_{1n} z^n.$$

Then by multiplying each equation for n in equation (2) by z^n , summing over n and rearranging terms, we get

$$G_1(z) = \frac{z(\gamma + \theta)G_0(z) - z((\gamma + \theta)P_{00} + \mu P_{11})}{((1 - z)(z\lambda - \mu) + z\alpha)} \tag{3}$$

In a similar manner from equation (1) we obtain

$$(1 - z)\xi G_0'(z) = (\lambda - \lambda z + \gamma + \theta)G_0(z) - ((\gamma + \theta)P_{00} + \mu P_{11}) - \alpha G_1(z), \tag{4}$$

where $G_0'(z) = \frac{d}{dz} G_0(z)$. In the next section we will derive the mean number of customers in the system when the server works and the mean number of costumers when the server does not work.

4 Derivation of $E(L_0)$ and $E(L_1)$

We begin by setting $A = (\gamma + \theta)P_{00} + \mu P_{11}$. The probability that the server is working is defined by

$$G_1(1) = \sum_{n=1}^{\infty} P_{1n} = P_1$$

From equation (3) we obtain

$$P_{1\cdot} = \frac{(\gamma + \theta)P_{0\cdot} - A}{\alpha} \tag{5}$$

Clearly the probability that the server does not work (on vacation or under repair) is

$$G_0(1) = \sum_{n=1}^{\infty} P_{0n} = P_{0\cdot}$$

Obviously $P_{0\cdot} + P_{1\cdot} = 1$, so from equation (5) we get

$$P_{1\cdot} = \frac{(\gamma + \theta) - A}{(\alpha + \gamma + \theta)} \tag{6}$$

Then

$$P_{0\cdot} = \frac{\alpha + A}{(\alpha + \gamma + \theta)}. \tag{7}$$

The mean number of customers in the system when the server does not work (on vacation or under repair) is given by $E(L_0) = G_0'(1) = \sum_{n=0}^{\infty} nP_{0n}$. So using L'Hospital's rule on equation (4), we get

$$E(L_0) = \lim_{z=1} G_0'(z) = \lim_{z=1} \frac{-\lambda G_0(z) + (\gamma + \theta)G_0'(z) - \alpha G_1'(z)}{-\xi}$$

Implying that

$$E(L_0) = \frac{\lambda P_{0\cdot} + \alpha E(L_1)}{(\xi + \gamma + \theta)} \tag{8}$$

To find the mean number of customers in the system when the server works $E(L_1) = G_0'(1)$, from equation (3), we get

$$E(L_1) = G_1'(1) = \frac{\alpha(\gamma+\theta)E(L_0) + (\lambda-\mu)((\gamma+\theta)P_{0\cdot} - A)}{\alpha^2}. \tag{9}$$

Substituting the value of $E(L_1)$ in equation (8) we get

$$E(L_0) = \frac{P_{0\cdot}(\alpha\lambda + (\lambda-\mu)(\gamma+\theta) - (\lambda-\mu)A}{\alpha\xi} \tag{10}$$

Then

$$E(L_1) = \frac{P_{0\cdot}(\alpha\lambda(\gamma+\theta) + (\xi+\gamma+\theta)(\lambda-\mu)(\gamma+\theta)) - (\xi+\gamma+\theta)(\lambda-\mu)A}{\alpha^2\xi} \tag{11}$$

If we define S to be the total sojourn time of customers in the system, measured from the moment of arrival until departure, either after completion of service or as a result of abandonment. By little's law

$$E(S) = \frac{E(L_0) + E(L_1)}{\lambda} \tag{12}$$

So using equations (10) and (11) we can get $E(S)$.

5 Conclusion

We have introduced and analyzed in this paper a new type of impatience behavior in which customers become impatient (and may leave the system) when the server goes on vacation or the server is broken down and under repair. This is in contrast with previously studied impatience behavior, which did not consider server breakdown and where customers may become impatient when the number of customers or the amount of workload queued in front of them is large. We derived explicit expressions for the PGF of the number of customers (conditioned on the server state) in the system. The closed forms to find the mean number of customers in the systems as well as the mean sojourn time are given also in this work.

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