Students' Non-realistic Mathematical Modeling as a Drawback of Teachers' Beliefs About and Approaches to Word Problem Solving

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Abstract Over the past decades numerous scholars have become aware of many compelling observations of students in mathematics classes abandoning their sense-making capabilities when doing word problems, and, in particular, carrying out arithmetic calculations that do not make sense in relation to the situations described. This led us, together with several other scholars, to embark upon an extended investigation of the phenomenon, the results of which are reported, among others, in two books (Verschaffel L, Greer B, De Corte E, Making sense of word problems. Swets & Zeitlinger, Lisse, 2000; Verschaffel L, Greer B, Van Dooren W, Mukhopadhyay S, Words and worlds: modelling verbal descriptions of situations. Sense Publishers, Rotterdam, 2009). The goal of the present chapter is to bring together and critically review the theoretical analyses and empirical studies that have focused on major aspects of teachers' instructional practices that affect – directly or indirectly – students' non-realistic approaches to and beliefs about word problem solving. Special attention will be given to the problems that appear in students' mathematical textbooks as well as to teachers' beliefs about word problems and what are appropriate ways to solve them, and to their instructional behavior, and how these factors affect students' beliefs about and approaches to word problems. While the focus is on research that has been done in our own center, we also integrate relevant studies by others.

Keywords Word problem solving • Mathematical modeling • Students' beliefs • Instructional approaches • Teachers' beliefs

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Word Problems as Exercises in Mathematical Modeling

Word problems have been assigned a central role in the mathematics curriculum in the elementary school (see e.g., National Council of Teachers of Mathematics 2010), not only because of their potential for motivating students and for the meaningful development of new mathematics concepts and skills, but also – and from a historical perspective even principally – to develop in students the skills of knowing *when* and *how* to apply their mathematics effectively in situations encountered in everyday life and at work (Boaler 1993; Hiebert et al. 1996; Verschaffel et al. 2000). Word problems are typically defined as essentially verbal descriptions of problem situations in which one or more questions are raised for which the answer(s) can be obtained by the application of one or more mathematical operations to the numerical data available in the problem statement (Verschaffel et al. 2000, p. ix). As they are composed of a mathematics structure embedded in a more or less realistic context, word problems can ideally serve as tools for mathematical modeling, which may be viewed according to Greer (1997) "as the link between the 'two faces' of mathematics, namely its grounding in aspects of reality, and the development of abstract formal structures" (p. 300).

Applying mathematics to solve problem situations in the real world can be usefully thought of as a complex process involving a number of phases. There are many different descriptions of this modeling process (e.g., Blum and Niss 1991; Burkhardt 1994; Mason 2001; Verschaffel et al. 2000), but, in essence, they all involve the following components (which do not necessarily follow a strictly linear order): (1) understanding and defining the problem situation leading to a situation model; (2) constructing a mathematical model of the relevant elements, relations, and conditions involved in the situation model; (3) working through the mathematical model using disciplinary methods to derive some mathematical results; (4) interpreting the outcome of the computational work in relation to the original problem situation; (5) evaluating the modeling process by checking if the interpreted mathematical outcome is appropriate and reasonable for its purpose; and (6) communicating the obtained solution of the original real-world problems.

For a long time, many teachers, textbook writers, and researchers in mathematics education assumed an unproblematic relationship between the situation and the mathematical model: Solving a word problem was considered as a direct translation process from the word problem text to mathematical symbols. However, more and more scholars have pointed to the difficulty of assuming a one-to-one relationship between mathematical models and real-world phenomena (Gerofsky 1997; Nesher 1980). This bridging problem became even clearer as empirical studies revealed that, after several years of schooling, many students have developed an approach to problem solving, whereby they ignore essential aspects of reality and whereby the mathematical actions they perform are based on a superficial analysis of the numbers and keywords provided in the problem text (Schoenfeld 1991). In this respect, we refer to the famous example of the captain problem: "There are 26 sheep and 10 goats on a ship. How old is the captain?". Confronted with this problem, many students were prepared to offer an answer to this absurd problem by combining the numbers given in the problem (e.g., 26 + 10) to produce an answer (i.e., 36) without showing any awareness of the meaninglessness of the problem and their solution (Baruk 1985). Inspired by this and some other striking examples of this phenomenon of "suspension of sense-making" (Schoenfeld 1991) when doing school word problems, Greer (1993) and Verschaffel et al. (1994) carried out two parallel studies in Northern Ireland and Belgium (Flanders). Paper-and-pencil tests were administered to upper elementary and lower secondary school students involving problems such as "Steve bought 4 planks of 2.5 m each. How many planks of 1 m can he saw out of these planks?" (=planks item), "John's best time to run 100 m is 17 s. How long will it take him to run 1 km" (=runner item). These authors termed each of these items "problematic" in the sense that they require the application of judgment based on real-world knowledge and assumptions rather than the straightforward application of one or more simple arithmetical operations. In both studies, students demonstrated a very strong tendency to exclude realistic considerations when confronted with these problematic items. For a more detailed overview of the design and the results of these two studies we refer to Greer et al. (2002).

The studies of Greer (1993) and Verschaffel et al. (1994) were replicated in several other countries, using a similar methodology and, to a considerable extent, the same items. The findings were strikingly consistent across many countries: Almost none of the problematic items was answered in a realistic fashion by more than a small percentage of students. The mean percentage of realistic answers on the problematic items across the studies varied from 12 % in Hidalgo's (1997) study to 30 % in Caldwell's (1995) study. Realistic reactions were typically higher on the division with remainder problems. The obtained results strongly surprised some of these other researcher(s) who had anticipated that the "disastrous" picture of the Irish and Flemish pupils would not apply to their students (for an extensive overview of these replication studies, see Verschaffel et al. 2000).

Beliefs and Word Problem Solving

In search for an explanation of the students' non-realistic responses to word problems, Schoenfeld (1991) suggested that it is not a cognitive deficit as such that causes students' general and strong abstention from sense-making when solving mathematical word problems in a typical school setting. Rather, students seemed to be engaged in sense-making of a different kind: "such behavior is sense making of the deepest kind. In the context of schooling, such behavior represents the construction of a set of beliefs and behaviors that result in praise for good performance, minimal conflict, fitting in socially, etc. What could be more sensible than that? The problem, then, is that the same behavior that is sensible in one context (schooling as an institution) may violate the protocols of sense-making in another (the culture of mathematics and mathematicians)" (Schoenfeld 1991, p. 340). In other words, students' tendency to neglect real-world knowledge and realistic considerations when confronted with problematic word problems is assumed to be due to their beliefs about word problems and how to solve them, which they have gradually, implicitly, and tacitly developed in accordance with the "word problem game" (Verschaffel et al. 2000), or, as others would call it, the "didactical contract" (Brousseau 1998), or the "sociomathematical norms and practices" (Yackel and Cobb 1996) within "the culture of the mathematics classroom" (Seeger et al. 1998). Apart from some anecdotal indications collected in individual interviews, direct empirical evidence for the existence of these assumptions and beliefs is scarce. One exception is a study by Reusser and Stebler (1997) that provided some evidence for their existence based on interviews with students who gave explanations for their non-realistic behavior on problematic word problems. Reusser and Stebler (1997, pp. 324–325) identified the following assumptions that students typically develop through being immersed in the culture and practices of school mathematics:

- Assume that every problem presented by a teacher or in a textbook makes sense.
- Do not question the correctness or completeness of problems.
- Assume that there is only one "correct" answer to every problem.
- Give an answer to every problem presented to you.
- Use all numbers that are part of the problem in order to calculate the solution.
- If a problem is perceived to be indeterminate, equivocal, or unsolvable, go for an obvious interpretation given the information in the problem text and your knowledge of mathematical operations.
- If you do not understand a problem, look at key words, or at previously solved problems, in order to determine a mathematical operation.

In addition, indirect evidence for the existence of the previously described assumptions was obtained in a series of studies by Jiménez and colleagues. Jiménez and Ramos (2011) investigated the impact of four of these specific beliefs about word problems that develop in students as a result of traditional schooling: (1) every word problem is solvable, (2) there is only one numerical and precise correct answer to every word problem, (3) it is necessary to do calculations to solve a word problem, and (4) all numbers that are part of the word problem should be used in order to calculate the solution. Specifically, 22 second and 22 third graders were asked in the context of an individual interview to solve four word problems that each violate one of these four beliefs, i.e., (1) an unsolvable word problem, (2) a word problem with multiple solutions, (3) a word problem containing the solution in the problem statement, and (4) a word problem including irrelevant data. For instance, the word problem including irrelevant data was "Laura buys a box with 12 crayons for the Plastic arts class. Her friend Silvia gives her another box containing 3 pens and 9 crayons. How many crayons does Laura have now?". Results revealed, first, that only one third of all students responded correctly to the four problem types. Second, the percentage correct answers was higher for solutions in the statement and irrelevant data problems (resp. 45.5 % and 43.2 %) than for unsolvable and multiple solution problems (resp. 20.5 % and 23.9 %). Third, no differences were found between second and third graders. Fourth, the vast majority of the errors originated from doing one or more arithmetic operations on all given numbers in the problem.

Finally, many verbal explanations of erroneous responses contained spontaneous expressions of the above-mentioned beliefs about word problems. In a cross-sectional study, Jiménez and Verschaffel (2014) investigated the development of these beliefs from first to sixth grade. Using individual interviews they administered to students the same four problems as in the previous study, that respectively violate the belief that (1) every word problem has a solution, (2) there is only one numerical and precise answer to a word problem, (3) it is necessary to do calculations to solve a word problem, and (4) that all numbers mentioned in a word problem are relevant to its solution. The amount of correct responses on the distinct problem types was respectively 18 %, 30 %, 46 %, and 57 %. These results indicate, first, that accuracy scores were relatively low for all problem types. Second, the percentages correct answers suggest that some beliefs about arithmetic word problems were more established in students' thinking (e.g., every word problem has a solution) than others (e.g., all numbers mentioned in the word problem are needed for its solution). Third, this difference in performance across the distinct problem types was observed in all grades. Fourth, there was an increase in correct responses from grade 1 (15.5 %) to grade 6 (56 %), however, this increase was small in the upper grades (49.5 % in grade 4 and 55.5 % in grade 5). In general, the results of Jiménez and Verschaffel (2014) paralleled the findings of Jiménez and Ramos (2011). Overall students were weak at solving word problems that violate less appropriate beliefs – at least from a modeling perspective – about word problem solving. Moreover, the same pattern of differences in accuracy to solve the distinct problem types was observed in both studies, suggesting that students' belief that all numbers in a word problem are relevant to its solution is more prevalent than the belief that every word problem has a solution.

Aspects of Teachers' Instructional Practices That Influence Students' Non-realistic Behavior

In an attempt to explain how these beliefs about and tactics for the solution of school word problems develop in students, it is assumed that mainly three aspects of the instructional practice and culture of traditional school mathematics are responsible, namely (1) the stereotyped and unrealistic nature of the problems used in classrooms, (2) the way in which teachers conceive word problems, and (3) the way in which teachers treat word problems in their daily practice (Mason and Scrivani 2004; Verschaffel et al. 1999). Even though it is generally accepted that the culture and practice in regular mathematics classrooms is responsible for the beliefs that students develop about word problems and for their non-realistic word problem-solving tactics, only rarely has attention been paid to whether, when, and how students are exposed to realistic modeling experiences in their daily mathematics classroom (Verschaffel et al. 2010). In what follows, we will give an overview of the studies that yield empirical evidence on these aspects of the instructional environment that may affect – directly or indirectly – students' non-realistic approaches to and beliefs about word problem solving. First, we will focus on the nature of the word problems that appear in mathematical textbooks and that teachers use in their instructional practice. Second, we will report studies on teachers' beliefs about problematic word problems and how they evaluate students' non-realistic approaches. Third, an overview will be given of the way in which teachers deal with word problems in their regular classroom practice.

The Nature of Word Problems in Traditional School Mathematics

The research literature suggests two related criticisms regarding the nature of the problems to which students are exposed in regular mathematics classrooms. First, most problems can be solved by a simple and straightforward application of one or a combination of the four basic arithmetic operations (Davis-Dorsey et al. 1991; Gravemeijer 1997). Second, but related to the previous issue, problems that are closely related to students' experiential worlds, are rare (Gerofsky 1997; Palm 2002). In an attempt to give empirical grounding to this second criticism, Depaepe et al. (2009) investigated the nature of word problems in the most frequently used sixth-grade mathematics textbook in Flanders as well as the nature of the word problems actually selected and used by two typical teachers who used this textbook. We relied on Palm's (2002) conceptual framework for analyzing the realistic nature of word problems. The founding idea of his framework lies in the notion *simulation*: A word problem is considered to be realistic if its important aspects are taken under conditions representative for an out-of-school situation. The operationalization of the framework included 11 aspects that play an important role in the extent to which students may engage in similar mathematical activities in a school task as in an out-of-school situation: event, question, purpose in the figurative context, existence of data, realism of data, specificity of data, language use, availability of solution strategies, external tools, guidance, and solution requirements. Two classification levels were distinguished for all but one aspect. The two levels relate to whether a task was judged as simulating the aspects of a corresponding out-of-school situation to a reasonable degree (1) or not (0). For the aspect specificity of data three levels were distinguished. The operationalization of the different aspects of the framework is presented in Table 1.

The way in which we classified mathematical problems according to the aspects mentioned in Table 1 is illustrated in Fig. 1.

Overall, we found that the tasks from the textbook and those that were created by the teachers themselves were similar. The word problems seemed to simulate relatively well some aspects that are assumed to be important in designing realistic tasks according to Palm's coding scheme (e.g., event, language use), but failed to include others (e.g., specificity of data, purpose in the figurative context). Another important finding (that was however not revealed by Palm's coding scheme) was that almost all word problems could be solved straightforwardly by applying one or more arithmetic operation(s) with all numbers mentioned in the task.

Table 1 Framework for analyzing the realism of word problems

Aspect	Description
Event	1 = The event in the school task could be encountered in real life outside school.
	0 = The school task is about an imaginary event; the event includes objects from the real world, but is still a fictious event; or the school task is a pure mathematical task which is not embedded in a context.
Question	1 = The question in the school task has been asked, or might be asked, in the stimulated event. The answer to the question is of practical value or of interest for others than just the people very interested in mathematics.
	0 = The question in the school task is judged not to have been asked, and neither would be asked, in the event described in the task.
Purpose in the figurative context	1 = The purpose of solving the task is explicitly mentioned in the school task and in concordance with the purpose of solving the task in the stimulated situation.
	0 = The purpose of solving the task in the stimulated situation is unclear. The school context could be generally described, not pointing to a specific situation, resulting in many possible situations and purposes of the task solving. In other tasks the situation described in the task is more specific but still open for more than one purpose.
Existence of data	1 = The relevant data that are important for the solution in the simulated situation coincide with the accessible data in the school task.
	0 = The data that are important for the solution in the simulated situation are not the same as the accessible data in the school situation and/or this information is accessible only by applying other competencies that are different from those required in the simulated situation.
Realism of data	1 = Numbers and values given are identical to or very close to the corresponding numbers and values in the simulated situation.
	0 = Numbers and values given are not realistic.
Specificity of data	2 = The text of the task describes a specific situation in which the subjects, objects, and places in the school context are specific. If graphs are used, the source is mentioned.
	1 = The situation in the school task is not specific, but at a minimum the objects that are the foci of mathematical treatment are specific.
	0 = The situation in the school context is a general situation in which the subjects and objects are not specified.
Language use	1 = The task is linguistically similar to the corresponding simulated situation. Specific mathematical concepts which are not used in daily language are avoided.
	0 = The terminology, sentence structure or amount of text in the school task is judged to affect more than an insignificant proportion of students in such a way that the possibility to use the same mathematics in the school task and in the simulated situation is greatly impaired.
Availability of solution strategies	1 = The students' available solution strategies allow them to solve the task in the same way as the taken character in the simulated situation would have done. The textbook is not directing the student in a specific direction to solve the problem.
	0 = The students' available solution strategies to solve the task are different than in the simulated situation. The textbook is directing the students into a specific solution strategy, which the problem solver would not necessary have used while solving a similar problem in real life.

(continued)

Table 1 (continued)

Aspect	Description
External tools	1 = The availability of external tools (i.e., concrete tools outside the mind: calculator, map, ruler…), important for the solution of a task, in the school task is similar to the simulated real situation.
	0 = There is a discrepancy between the tools in the two corresponding situations.
Guidance	1 = The same guidance is provided in the school task and in the corresponding out-of-school situation.
	0 = The task does not match in the guidance given between the school task and the corresponding out-of-school situation.
Solution requirements	1 = The explicit or implicit requirements on the solution to a task are considered to be similar to the corresponding situation in real life.
	0 = The explicit or implicit requirements on the solution to a school task are not considered to be similar to the corresponding real-life situation.

Problem

A jeweler makes a golden ring of 11 gram pure gold (12 euro each gram) and 4 gram silver (2.50 euro each gram).

Q [a] How much does the alloy cost?

C S

Gold g →

Silver g → OK

Scoring

Event	Quest.	Purp.	Exist.	Real.[b]	Specif.	Lang.	Sol. strat.	Ext. tools	Guid.	Sol. req.
1	1	0	1	1	1	1	0	0	0	1

[a] *The textbook writers explicitly refer to (parts of) the problem-solving process by means of letters which symbolize certain heuristics (S = scheme) and phases (Q = question; C = computation; OK = checking and interpreting the outcome). Since such hints do normally not occur in real life situations, a "0" was given for the aspect guidance.*

[b] *At the time of the analysis the value of the unit-price of gold and silver was realistic, therefore a code "1" was given for realism of data. However, it should be admitted that prices of objects involved in the task often automatically expires, which is a problem that is inherently connected to textbook tasks which are still used in a classroom years after their design.*

Fig. 1 Illustration of a textbook word problem and its classification according to the framework for analyzing the realism of word problems

Very recently, Gkoris et al. (2013) analyzed the nature of the word problems of the compulsory national mathematics textbook for fifth grade in Greece, both before and after a major educational reform in 2003 that aimed among others at promoting critical reasoning in problem solving (Pedagogic Institute 2003). They also relied on Palm's framework for analyzing the authentic nature of word problems (Palm

2002), but additionally they investigated the problematic nature of the word problems in the old and the new textbook. The clearest difference between the old and the new textbook was observed for the aspect event. Whereas most of the new textbook problems related to students' personal interests and experiences, the majority of the problems in the old textbook provided only minimal contextual information. But also with respect to other aspects of Palm's framework the new textbook problems were more authentic than those from the old textbook: The purpose of solving the school problem was more in alignment with the purpose of solving the task in the simulated real-life situation, the use of external tools similar to situations in real life – such as the calculator – was allowed, and the information provided in the new textbook problems was more specific than in the old ones. However, like in Depaepe et al.'s analysis, both textbooks scored low on the aspect problematicity, meaning that Greek students were and are hardly confronted with problems that stimulate the use of real-life reasoning skills. Moreover, the few tasks in which the relation between the situation model (the problem context) and the mathematical model (the required mathematical operations) was neither straightforward nor simple were division with remainder problems, in which real-life considerations should be taken into account when interpreting the obtained results. However, for these type of problems, previous studies reported better realistic modeling results when compared to the other types of problematic tasks from Verschaffel et al.'s (1994) study (Hidalgo 1997; Reusser and Stebler 1997; Yoshida et al. 1997).

Similarly, Vicente et al. (2011) analyzed all word problems of two Spanish elementary textbooks (grade 1–6). The analysis focused on (1) the level of authenticity of the word problems (strongly relying on Palm's framework) and (2) the proportion of challenging problems (e.g., problems with irrelevant information or missing information that problem solvers must infer from their prior knowledge, problem posing activities). The results indicated that the Spanish word problems simulated well most aspects of Palm's framework such as language use, external tools, solution requirements, realism of data, question, event. Only the aspect purpose in the figurative context was only in 6 % of the word problems well simulated. However, their analysis of the problematic nature of word problems revealed that 95 % of the word problems were stereotyped, easy, and non-challenging.

In conclusion, these recent studies that analyzed the nature of word problems (Depaepe et al. 2009; Gkoris et al. 2013; Vicente et al. 2011) reveal, on the one hand, that the negative image of the unrealistic nature of the set of tasks students are confronted with in the mathematics classroom – as expressed in previous publications (e.g., Verschaffel et al. 2000) – does currently not anymore count to the same degree for regular classroom practices. This positive development may partly be a result of the past 15 years of research on students' suspension of sense-making. But, most probably, it has also been impacted by the global reform movement towards more realistic mathematics education in which policy makers, textbook writers and teachers generally believe that students should be confronted with realistic word problems. On the other hand, these studies also demonstrate that most word problems are characterized by only a restricted problematic nature. If one really wants students to develop appropriate beliefs towards solving word problems and to become competent problem solvers in real life, one should integrate more problematic problems into the mathematics curriculum, since most real word problems which one encounters in life beyond school are modeling problems in which the translation of the situation model into a mathematical model is neither straightforward nor simple.

Teachers' Knowledge and Beliefs About Mathematical Modeling Problems

At least as important as the nature of the word problems is the way in which these problems are conceived and approached by the teacher. Hiebert et al. (1996, p. 16) argue: "given a different culture, even large-scale real-life situations can be drained of their problematic possibilities. Tasks are inherently neither problematic nor routine. Whether they become problematic depends on how teachers and students treat them". Accordingly, the teacher may play an important role in stimulating or discouraging students to take into account realistic considerations. In this section we will focus on teachers' knowledge (how do they solve problematic items themselves?) and beliefs (how do they value students' realistic considerations when solving mathematical word problems?) regarding realistic mathematical modeling.

Verschaffel et al. (1997) administered a paper-and-pencil test consisting of seven standard and seven problematic word problems to 332 Flemish prospective elementary teachers. The teachers were, first, asked to solve the word problems themselves. Afterwards, the test was given a second time to all prospective teachers and they were asked to score different answers from students to all word problems (including a typical non-realistic and a realistic answer). The results indicated that, similar to elementary and secondary students, prospective teachers demonstrated a strong overall tendency to exclude real-world knowledge and realistic considerations when confronted with the problematic word problems. Moreover, they valued students' non-realistic responses to these problematic items considerably more than realistic answers.

Similar results were obtained by replication studies in different countries. A study of Bonotto and Wilczewski (2007) with Italian prospective teachers revealed that their overall evaluations of the non-realistic answers were also considerably more positive than for the realistic ones, suggesting that these future teachers also seemed to believe that the activation of realistic context-based considerations should not be stimulated, rather, discouraged in elementary school mathematics. Xu (2005) asked 117 prospective and 72 in-service Chinese teachers to solve the seven problematic items from Verschaffel et al.'s (1997) study and to value students' realistic and non-realistic answers to these problematic items. The study indicated that Chinese (prospective) teachers, first, showed more realistic problem-solving behavior, and, second, evaluated students' realistic answers more positively than their Flemish and Italian peers. Another Chinese study (Chen et al. 2011) with 208 prospective teachers confirmed Chinese prospective teachers' more realistic disposition towards word problem solving, both in terms of their own problem-solving behavior and their valuation of students' responses to problematic items. This discrepancy between the performances of Chinese and Western teachers might be not so surprising, since studies revealed that Chinese teachers acquire more content and pedagogical content knowledge (e.g., Ma 1999; Zhou et al. 2006). Another factor that might have impacted the more realistic behavior of Chinese teachers compared to their Western peers is the increased emphasis on realistic mathematics education in the Chinese curriculum (Chen et al. 2011).

Inspired by the study of Verschaffel et al. (1997) Duan et al. (2011) asked 20 Chinese teachers of upper elementary school to value the educational suitability of six standard and six problematic word problems. Moreover, they were asked to justify their choices and to make suggestions to improve the word problems. First, the Chinese teachers favored – in alignment with the previously mentioned studies (Chen et al. 2011; Xu 2005) – realistic mathematical modeling approaches to the problematic items, even though their performance on these problematic items was considerably lower than on the standard items. Second, Chinese teachers evaluated the educational suitability of the problematic word problems clearly lower than of the standard word problems. Although some teachers acknowledged the possible additional value of the problematic items in word problem solving, most teachers expressed criticism on the ambiguous character of these problematic word problems which might mislead and confuse students (e.g., a teacher commented on the runner item mentioned in section "Word problems as exercises in mathematical modeling" "This is not realistic, because running 100 m is quite different from running 1,000 m"). In other words, these teachers assumed that the problematic word problems would have been better formulated in a clear, unambiguous way. This was also evident in their response to the question whether and how these problematic items could be improved: their typical reaction was to transform them into a standard format by eliminating all "problematic" aspects (e.g., for the runner item teachers suggested that it should be explicitly stated that the speed is fixed). Consequently, although the findings of Duan et al.'s (2011) study were in alignment with the previous Chinese studies (Chen et al. 2011; Xu 2005) indicating that some teachers performed very well on these problematic items and acknowledged that these items could help students to deal with complex and ambiguous problem situations, Duan et al.'s study additionally provided evidence that the same teachers seemed, in general, to attach little value to the opportunities that these word problems offer for students' realistic mathematical modeling.

Lee (2012) investigated how prospective elementary teachers perceive real-life connections in mathematical word problem solving. In this study 71 US prospective teachers were first asked to formulate at least three criteria for exemplary story problems, to collect two story problems, and to pose two word problems that, in their opinion, best represent real-life connections. Based on these prospective teachers' responses Lee selected ten word problems. In a second assignment prospective teachers were asked to value these ten word problems on a 5-point scale in terms of the quality of real-life connections, and to comment on the strengths and weaknesses of each word problem. The results of the first assignment revealed that 42 % of teachers' criteria for realistic word problems were not directly related to real-life connectedness but, rather, to mathematics problems in general (e.g., age or grade level appropriate, involve high-order and critical thinking, utilize multiple modes of representation). Moreover, the word problems posed by the prospective teachers were typical standard problems that can be solved by a straightforward application of one or more arithmetic operations with the given numbers. Based on the second assignment, Lee concluded that, in general, prospective teachers demonstrated positive beliefs about real-life connections in word problem solving, albeit with insufficient specifics. The majority of the participants held an utilitarian view on realistic problem solving: They stated that reality in word problems is important for enhancing students' interest and motivation, for making mathematics more meaningful, and for enhancing the application of mathematics concepts in real life. Discrepancies were observed between their positive beliefs on reality and word problem solving and the way they valued word problems. For instance, word problems that contained many details in order to make the word problem more real were typically negatively valued by arguing that these details could and should be ignored in terms of mathematical problem solving.

In general, the previous studies on teachers' beliefs about word problems and what are appropriate ways to solve them provide evidence for their lack of disposition towards realistic mathematical modeling. These studies suggest that also prospective teachers themselves seem to share the belief that realistic considerations about the problem context should be neglected when solving problematic word problems. Although this observation counts for a lesser extent for Chinese (prospective) teachers, they also seem – similar to their Western peers – to depreciate word problems in which there is a complex, ambiguous relationship between the mathematical model and real-world phenomena. This lack of disposition towards realistic mathematical modeling, most probably, also impacts teachers' classroom practice.

Teachers' Approaches to Word Problem Solving

Arguably, also the way in which teachers actually treat word problems in their instructional practice can promote or inhibit students' realistic disposition towards word problem solving. As already mentioned in section "Word problems as exercises in mathematical modeling", we strongly recommend to conceive and use word problems as exercises in mathematical modeling, a non-trivial and non-linear complex process. This process aims at finding a proper balance between taking seriously into account the elements of the real world evoked by the problem statement, on the one hand, and finding an underlying mathematical structure that allows the use of the power of mathematics to efficiently understand and solve the problem, on the other hand. In modeling, not all aspects of reality can, nor should be modeled (Verschaffel 2002). In this regard, Ikeda and Stephens (2001) point to the pivotal task for the mathematical modeler of balancing appropriately between over-complication and over-simplification, taking into account the goals of the modeling task and one's personal and contextual constraints.

In this respect, Chapman (2006) made an interesting distinction – borrowed from Bruner (1985) – between two complementary modes of conceiving and treating word problems, namely a paradigmatic-oriented and a narrative-oriented mode. The paradigmatic approach towards a word problem is based on a focus on mathematical models and structures that are universal and context-free (e.g., fragmenting and translating the context into mathematical representations). The narrative mode, in contrast, deals with situational aspects of the word problem and, thus, focuses on context-sensitive explications (e.g., allowing students to resonate in the social context of the word problem to discuss specific aspects of it they were curious about or to critique it). Using audiotapes and field notes of two lessons related to word problems Chapman distilled different paradigmatic and narrative modes that emerged from the teaching of word problems of 14 experienced elementary, junior high, and senior high school teachers. Chapman's results revealed that the paradigmatic mode was more dominant, but also that it was combined with the narrative mode in different ways among the teachers. Based on her results, she made a plea for balancing between the paradigmatic and the narrative mode in order to realize realistic mathematical modeling.

Based on Chapman's (2006) distinction between a paradigmatic and narrative approach towards word problem solving we investigated whether two sixth-grade teachers used word problems as a vehicle for realistic mathematical modeling (Depaepe et al. 2010). Contrary to Chapman (2006) who studied the occurrence of the paradigmatic and narrative mode in the lessons of 14 teachers in general, we systematically analyzed to what degree the paradigmatic and narrative interventions were reflected in the teaching of each word problem in two classrooms over a period of 7 months. Inspired by Chapman's distinction in one of her narrative modes between the entry into the problem and the exit out of the problem, we distinguish in our analysis between the initial phases and the final phases of the problem-solving process. Our operationalization of teachers' paradigmatic and narrative interventions with regard to the entry and exit phase of the modeling process has resulted in a refinement of Chapman's scheme (see Table 2).

The results revealed, first, that both teachers adhered more to a paradigmatic than to a narrative approach towards word problem solving. Second, we observed that a strong focus on a paradigmatic approach does not exclude a strong narrative approach and vice versa. Indeed, notwithstanding the fact that the paradigmatic mode dominated in both classrooms, one teacher's approach towards problem solving reflected a substantially stronger combined paradigmatic *and* narrative focus than the other teacher. This finding reveals that a simultaneous emphasis on universal and context-free mathematical structures and models (a paradigmatic approach) and on contextual elements of the realistic situation to which the word problem refers (a narrative approach) is not only desirable (Chapman 2006), but also feasible. Third, it was observed that the relation between the mathematical model and the situation model (N4 and N7) was almost never addressed in both classrooms. Nevertheless, interventions that stress this relationship belong to the core of mathematical modeling and seem to be consistent with current perspectives on mathematics learning and teaching (e.g., Verschaffel et al. 2000, 2009).

Table 2 Paradigmatic and narrative interventions towards word problem solving distinguished in Depaepe et al.'s (2010) study

Approach	Phase	Intervention	Description
Paradigmatic	Entry	**P1:** Distinguishing relevant from irrelevant information	Differentiating between what does and does not "matter" for the problem solution and/or translating the "given" into mathematical terms.
		P2: Applying a prototypical scheme	Transforming the information given in the problem context into a representational or solution scheme which enables the modeler to solve the problem.
		P3: Addressing the underlying mathematical structure	Emphasizing the structural similarities of the problem with an analogous problem and/or labeling the problem in terms of a particular problem class.
	Exit	**P4:** Seeking confirmatory evidence for the solution being obtained	Checking whether no errors were made and/or whether all questions were answered.
		P5: Addressing the underlying structure	Emphasizing the structural similarities of the problem with an analogous problem, and/or labeling the problem in terms of a particular problem class, and/or reviewing how a type of problems can or should be solved in general.
Narrative	Entry	**N1:** Rewording the problem	Rewording the problem into your own words based on the information given in the problem.
		N2: Defining notions involved in the problem	Clarifying the meaning of objects, persons, occupations, and/or situations mentioned in the problem.
		N3: Building on students' real-life experiences and prior knowledge	Linking the problem to a personal experience, and/or referring to a related event that happened in the real world, and/or elaborating on students' experiences with regard to objects mentioned in the problem text, and/or building on students' prior knowledge.
		N4: Taking explicitly into account the realities of the problem context	Identifying the conditions and assumptions of the real-world context to which the modeler will attend as he or she mathematizes the situation. This may results in criticizing and/or reformulating the word problem as initially posed.
	Exit	**N5:** Interpreting the outcome	Interpreting the outcome with regard to the real-life situation and/or seeking for real-life explanations for the obtained solution.

(continued)

Table 2 (continued)

Approach	Phase	Intervention	Description
		N6: Thinking of corresponding real-life situations	Referring to corresponding real-world applications and/or indicating (practical) relevance for learning to solve a particular problem class.
		N7: Taking explicitly into account the realities of the problem context	Identifying the conditions and assumptions of the real-world context to which the modeler will attend as he or she mathematizes the situation. This may results in criticizing and/or reformulating the word problem as initially posed.

Also relying on the analytic distinction between a paradigmatic and narrative approach, Rosales et al. (2012) investigated the way in which 11 elementary school teachers (grade 3–5) approached two non-standard word problems that require additional mathematical and situational knowledge to solve them. An example of such a problem is the following: "A shepherd was taking care of a flock of sheep. The shepherd had a flock of 57 sheep. He wanted to increase the size of the flock because this year there was a good fodder. In order to do so the shepherd went to a market, where he decided to buy some more sheep. One evening the shepherd saw a pack of wolves in the area. The wolves were hungry and then they devoured 11 sheep and now there are 96 sheep left. How many sheep did the shepherd buy in the market?". Rosales et al. distinguished three different paradigmatic approaches: (1) data selection (contents devoted to selecting the data from the problem), (2) mathematical reasoning (contents related to a deep mathematical understanding of the problem, in terms of mathematical relations among the data involved), and (3) mathematical resolution (contents related to the selection and execution of mathematical algorithms). In addition, two narrative approaches were distinguished: (1) relevant situational knowledge (contents related to the intentions and goals of the characters, and the actions performed to reach the goals to link the situation to the mathematical model of the problem) and (2) irrelevant situational knowledge (contents not related to the causal chain generated by the character's goals, such as descriptions of characters, places, objects). The results can be summarized as follows. First, even with non-standard word problems in which additional context-based information should be taken into account, teachers only rarely relied on the narrative approaches. Second, teachers' paradigmatic approach to word problems was rather superficial involving selecting data and certain key words in the problem statement (data selection), followed by automatic triggering of the mathematical model and the execution of the calculations (mathematical resolution). Mathematical reasoning, typical for genuine processing of word problems, only rarely occurred in the observed lessons. Third, the results also suggest that teachers were willing to accept a mathematically correct, though situationally incorrect, problem-solving procedure (e.g., accepting the computation 96 – (57 – 11) to solve the above mentioned shepherd problem).

The previous studies indicate that, in alignment with their beliefs about the relationship between mathematics and reality (see section "Teachers' knowledge and beliefs about mathematical modeling problems"), teachers only rarely include and use situational information when engaging in problem-solving or modeling activities with their whole class or with groups or individual students, even when approaching non-routine problems that require the application of judgment based on real-world knowledge and assumptions. Interventions that are typical and necessary for genuine mathematical modeling – i.e., aiming at a deep understanding of the situation and the mathematical model, and of the mathematical relations among the involved data – remain scarce. Although more research is needed, these studies strongly suggest that (elementary) students are only rarely confronted with genuine mathematical modeling in today's teaching of mathematics.

Conclusions and Discussion

Over the last decades numerous studies have revealed that, after several years of schooling, many students demonstrate a very strong tendency to exclude real-world knowledge and realistic considerations when confronted with problems that require – at least from the author's point of view – the application of judgment based on real-world knowledge and assumptions rather than the routine application of superficial solution strategies. It is assumed that this non-realistic problem-solving behavior is not "senseless" or "irrational". Indeed it is rather a result of students' beliefs about word problem solving that develop through being immersed in the culture and practices of traditional schooling. Especially three aspects of students' educational environment are assumed to directly or indirectly impact their beliefs about and approaches to word problem solving, namely (1) the nature of the word problems used in classrooms, (2) the nature of teachers' beliefs about word problems, and (3) the way in which teachers treat word problems in the classroom. This chapter has reviewed empirical research conducted at our own research center and by others, and focused on these three supposed causes of students' suspension of sense-making when solving mathematical word problems. Based on those studies we can, first, conclude that the caricature of unrealistic word problems in mathematics textbooks does not do justice to the reality of today's mathematics classrooms. But although many word problems are currently more closely connected to students' experiential worlds, most problems that students typically encounter in current mathematics lessons are still stereotyped in the sense that they require the routine application of simple arithmetical operations. Second, what concerns (prospective) teachers' beliefs towards realistic word problem solving, it was observed that they expressed positive beliefs regarding realistic connections in word problem solving. However, when they are confronted with students' answers to word problems they seem to value more non-realistic than realistic answers. Moreover, if they were asked to value word problems they are inclined to depreciate elements that make a word problem more realistic, such as a complicated relationship between real-world phenomena described in the problem and a mathematical model, and the addition of extraneous information from the story that is not necessarily needed to solve the problem. Third, the scarce studies that explicitly focused on the way in which teachers deal with word problems in regular classrooms revealed that students are offered only limited opportunities to exercise genuine mathematical modeling. Even when approaching non-routine word problems teachers rarely address situational information to provoke a deep understanding of the situation and the mathematical model. In conclusion, modifications on all three aspects mentioned above are needed to improve students' beliefs about and approaches to realistic mathematical modeling.

We acknowledge that the empirical evidence described in this chapter does not allow hard causal statements about the influence of the educational environment on students' non-realistic behavior, mainly due to a lack of an experimental component in the reviewed studies. To make such kind of causal statements further intervention research is needed which directly supports that modifications in the nature of the tasks, the beliefs of the teachers, and/or their instructional approach result in a change in students' beliefs and in more realistic problem-solving behavior. There are some intervention studies that indeed revealed positive effects of teachers' realistic modeling approaches in terms of students' performance, underlying processes, and motivational and affective aspects of learning (e.g., Mason and Scrivani 2004; Verschaffel et al. 1999). However, it should be admitted that these studies fall short in some aspects of either internal or external validity. After reviewing the available research evidence, Niss (2001, p. 8) concludes that "application and modeling capability can be learnt, and according to the above mentioned-findings has to be learnt, but at a cost, in terms of effort, complexity of task, time consumption, and reduction of syllabus in the traditional sense". Consequently, implementing these positive modifications in regular classroom practices is not an easy endeavor.

Pre-service and in-service teacher training can play an important role in preparing and equipping teachers to implement a realistic modeling approach towards word problem solving. Taking into account the findings of the reviewed studies, it is obvious that it is important to stress thereby, among others, the incorporation of modeling tasks in mathematics lessons. Moreover, it is necessary that pre-service as well as in-service teacher training address teachers' beliefs about the place and value of making realistic connections while solving word problems. Since these often hidden beliefs of the teacher are a major obstacle for change in school mathematics, only by explicitly addressing changing them, training will empower teachers to implement a genuine modeling approach to word problem solving (Ernest 1998). Meanwhile, changes are also needed at the meso- and macrolevel of the educational system that support a realistic modeling approach of school word problems. Textbooks should be revised in order to incorporate besides traditional word problems – that help students to master powerful schemes for identifying, understanding, and solving certain categories of problems (e.g., direct proportionality) – also genuine modeling problems – that may be used primarily as exercises in relating real-world situations to mathematical models and in reflecting upon that complex relationship between reality and mathematics. Finally, policy makers and school leaders may be supportive by creating working conditions that are helpful in teachers' implementation of a modeling approach.

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