Networking Theories to Understand Beliefs and Their Crucial Role in Mathematics Education

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Abstract Many publications present research on teacher beliefs, whether concretized for pre-service or in- service teachers. Most of them have in common that they highlight the crucial role that beliefs play in the classroom. In this chapter, we explore more deeply what those crucial aspects are, what they consist of, and how they interact with other variables. Different theoretical lenses will be brought together to underline different perspectives and to gain explanatory power going beyond the single approaches. For the case of practising teachers, we will discuss some thoughts on the classical contributions by Shulman (Educ Res 15(2):4–14, 1986) and Schoenfeld (Issues Educ 4(1):1–94, 1998). On the one hand, we extend the knowledge categorization provided by Shulman to the fields of beliefs and goals. On the other hand, we elaborate on Schoenfeld's theory of Teaching-In-Context. For the case of pre-service teachers, we combine the classification of mathematical beliefs based on the work of Ernest with ideas of conceptual change originally conceived in the context of knowledge (cf. Ernest, J Educ Teach 15:13–33, 1989; Vosniadou and Verschaffel, Learn Instr 14(5):445–451, 2004).

Keywords Pedagogical content beliefs • Conceptual change • Networking of theories

Introduction

In recent years the networking of theories has received much attention in mathematics education, as can be seen in the overview articles by Artigue et al. (2006), Bikner-Ahsbahs and Prediger (2006) and Arzarello et al. (2007). Some authors particularly focus on using theoretical diversity to strengthen theory development, and make suggestions on strategies and methods for networking (cf.

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Bikner-Ahsbahs and Prediger 2006), while others discuss whether theoretical plurality and diversity hinder moving forward as a field (Dahl 2006; Lerman 2006). In particular, at CERME 4 and 5 a group of researchers dealt with challenges induced by theoretical diversity (Arzarello et al. 2007) and they remind us that "researchers with different theoretical perspectives consider empirical phenomena […] from different perspectives and hence come to very different results in their empirical studies" (p. 1618). The authors continue by asking, "How can the results from different studies be integrated or at least understood in their difference?" (p. 1618). That is, results of empirical studies might be incompatible or even contradictory, a fact that, in the long run, can impede progress in the field of mathematics education. In this regard, Artigue et al. (2006) consider it as an important task of the community to pay attention to the different theoretical frameworks in terms of networking:

> If we can develop and maintain a certain degree of networking between some of the advocates of the different theoretical stances that are currently evident within mathematics education, this will constitute an important step on the path towards establishing mathematics education as a scientific discipline. (p. 1242)

As an essential endeavour, the authors identify integrating or synthesizing theoretical approaches into a new framework. Connecting theoretical approaches can then follow a bottom-up development while using a concrete empirical phenomenon as starting point; or a top-down development while using different theories from the beginning and focusing on the relationship of theories (cf. Arzarello et al. 2007). In this chapter, we elaborate on the latter aspect and pursue a deductive approach to networking theories. We will explore how different theories serve to analyse similar data to gain a more comprehensive understanding of relationships and interdependencies of the underlying frameworks to increase their explanatory power. In particular, regarding beliefs research, we point out how research directions were determined while developments took place in two different research fields, mathematics education and psychology, and which, of course, have influenced methodological approaches and choices. To underline our thoughts we present two examples of networking theories that come from those two research areas.

First, we will discuss some thoughts on the classical contributions by Shulman (1986, 2005a, b) and Schoenfeld (1998). On the one hand, we elaborate on Schoenfeld's theory of Teaching-in-Context[1] which explains teacher behaviour from a local view as a function of knowledge, goals and beliefs, while extending the knowledge categorization provided by Shulman (1986) to the fields of beliefs and goals. On the other hand, we use Shulman's (2005a, b) overarching theory of signature pedagogies to additionally understand the significant role of beliefs from a global view. Second, we develop further ideas of conceptual change originally conceived in the context of knowledge (Ernest 1989; Vosniadou and Verschaffel 2004; Appleton 1997) with regard to belief change and, in addition, use the classification

[1] Our ideas are based on Roesken and Rolka (2011).

of mathematical beliefs based on the work by Dionne (1984) and Ernest (1989) to illustrate belief change in more detail.[2]

Setting the Frame of Networking Theories: Beliefs Research in Psychology and Mathematics Education

From a historical viewpoint, the mathematics-related research in the field of beliefs has its roots in the "failure of the problem-solving based reforms" of the mathematics curriculum in the United States in the late 1980s (Schoenfeld 2007). More concretely, Roesken et al. (2011) emphasize that "numerous studies detected that one reason for that 'failure' were the 'inappropriate' beliefs of teachers concerning mathematics in general, the process of problem solving and characteristics of doing mathematics in particular, in addition to strong teacher convictions concerning students' apparent lack of ability […]" (p. 452). From that time on, the discussion on the role and significance of beliefs became more elaborated and led to the seminal papers by Pajares (1992) and Thompson (1992) in the 1990s which indicated a starting point for specific and targeted research on beliefs that has entailed numerous studies (cf. Philipp 2007) and encouraged substantial discussion. The reader will find almost no studies on beliefs that do not refer at least to one of these papers.

While comparing the work by Pajares (1992) and Thompson (1992) more closely, it is evident that the two researchers pursue different approaches since they come from different fields of educational research: educational psychology and mathematics education. These different roots entailed particular emphases which are briefly sketched in the following.

Among other aspects, the psychologists' viewpoint emphasizes the epistemological character of beliefs as those "play a key role on knowledge interpretation and cognitive monitoring" (Pajares 1992, p. 324); an issue that has dominated successive research in educational psychology. Moreover, Pajares (1992) refers to the work by Schommer (1990) who "argued that the study of epistemological beliefs may prove more valuable for understanding comprehension than either metacognition or schema theory, neither of which is able to explain students' failure to integrate information or monitor comprehension" (p. 328). As an overarching concept Pajares (1992) suggests *educational beliefs* as a construct that "is itself broad and encompassing" (p. 316). However, he reminds us that the concept's wide scope is difficult to operationalize and thus educational beliefs need to be specified by using the label *educational beliefs about*:

> Therefore, as with more general beliefs, *educational beliefs about* are required – beliefs about confidence to affect students' performance (teacher efficacy), about the nature of knowledge (epistemological beliefs), about causes of teachers' or students' performance (attributions, locus of control, motivation, writing apprehension, math anxiety), about

[2] These ideas are based on Liljedahl et al. (2007a, b) and Rolka et al. (2006).

> perceptions of self and feelings of self-worth (self-concept, self-esteem), about confidence to perform specific tasks (self-efficacy). There are also educational beliefs about specific subjects or disciplines (reading instruction, the nature of reading, whole language). (p. 316)

Subsequent research accentuated different directions in terms of personal epistemology (Hofer and Pintrich 1997; Hofer 2000), epistemic positions (Perry 1968), epistemic cognition (Kitchener 1983), epistemological beliefs (Schommer 1990), epistemological world views (Schraw and Olafson 2002) and epistemological understanding (Hofer 2004). In sum, research on beliefs in the field of pedagogy or psychology has developed strongly towards investigating epistemological aspects.

In comparison, the focus of Thompson (1992) is domain-specific as she explores explicitly teachers' conceptions of mathematics:

> A teacher's conception of the nature of mathematics may be viewed as that teacher's conscious or subconscious beliefs, concepts, meanings, rules, mental images, and preferences concerning the discipline of mathematics. (p. 132)

In the focus are conceptions about the nature of mathematics indicating a philosophical viewpoint that is also pursued in the work by Ernest (1991, 1994) and Lerman (1983). In the beginning of beliefs research in mathematics education, some researchers applied the term mathematical *world views*; these included Schoenfeld (1985), and later Grigutsch (1996) and Grigutsch et al. (1998). A few papers explicitly address epistemology in mathematics education (Sierpinska and Lerman 1996; Steinbring 1998) while mostly such issues are only implicitly included in discussions with roots going back to philosophical positions on mathematics (Hersh 1991, 1997). That is, those articles address beliefs about the origin and acquisition of knowledge, and how such attitudes affect teaching in general and students' learning of mathematics specifically. Some studies explore epistemological beliefs held by mathematics teacher educators and investigate how those shape and influence their prospective teachers' beliefs and even actions in the classroom (e.g. Carter and Norwood 1997; Schraw and Olafson 2002). Roesken and Törner (2007), in this regard, yielded seven dimensions structuring university professors' beliefs about mathematics that encompass factors including characteristics of mathematics, main features of mathematical learning, philosophical aspects and sophisticated views on mathematics.

A subsequent step in binding together various approaches of beliefs and their influences (see also Ernest 1989) was the issue of *ZDM* in 1996 edited by Pehkonen and Törner. The book edited by Leder et al. (2002) reflected many of the views presented in the Special Issue, and beliefs were referred to as 'hidden variables', with reference to a famous paper by Bauersfeld (1980). Meanwhile, much research has been conducted in the field of mathematics-related beliefs.

In sum, the history of beliefs research particularly indicates that it is worthwhile to explore theoretical diversity as two different research strands, which originated in psychology and mathematics education, have developed almost independently of each other and have influenced subsequent research in both fields substantially. In what follows, we present two examples of how to capture the crucial role of beliefs by networking different theoretical lenses that have their roots either in *psychology* or *mathematics education*.

Example 1: Understanding Beliefs by Combining a Global and a Local View on Teaching

As regards the teaching of mathematics, many researchers have looked for underlying variables in order to understand and explain teacher behaviour in the classroom (cf. Baumert et al. 2010). Examples for such variables are teacher knowledge, teacher beliefs and teacher instructional goals – typically, researchers focus on one of these variables without considering their relationships or potential overlaps (Rösken et al. 2008). Schoenfeld's (1998) merit lies in providing a theory that accounts for a local view on teaching by modelling teacher behaviour as a function of a teacher's knowledge, goals and beliefs. His theory of Teaching-in-Context does not simply go beyond knowledge and beliefs by assigning an essential role to goals, but also emphasizes strongly that the three variables are pieces of a puzzle and the challenge is to explore how these fit together. In sum, the theory explains developments in teaching from a multi-faceted perspective and allows the didactical analysis for focusing on understanding, and explaining rich and complex teaching coherences. A teacher's spontaneous decision-making is characterized in terms of available knowledge, high priority goals, and beliefs. In his latest book, Schoenfeld (2010) modifies his initial theory as follows:

> The main claim in the book is that what people do is a function of their resources (their knowledge, in the context of available material and other resources), goals (the conscious or unconscious aims they are trying to achieve) and orientations (their beliefs, values, biases, dispositions, etc.). (p. viii)

What is new? While attention is still given to goals, Schoenfeld introduces the broader concepts of resources to refer to the category of teacher knowledge and of orientations to encompass the fields of beliefs, values, biases and dispositions. Regarding the former-used category of beliefs, Schoenfeld (2010) explains:

> Beliefs play much the same focal role that they did in my earlier work. Just as students' beliefs about themselves and about mathematics shape what they do while working on mathematics problems, teachers' beliefs about themselves, about mathematics, about teaching, and about their students shape what they do in the classroom. (p. 26)

Still he assigns a major role to beliefs and he gives the following explanation for his shift in terminology:

> The term "beliefs" worked well in characterizing problem solving and teaching (and it fit comfortably with the literature's use of the term), but it seemed less apt when I applied the theoretical ideas to other domains. In cooking, tastes and life style preferences are consequential; in other arenas (e.g., health care) one's values play a major role. For that reason I chose orientations as an all-encompassing term, to play the same role in general as beliefs do in discussions of mathematical and pedagogical behavior. (p. 27)

What is not new? Schoenfeld still aims at explaining comprehensive teaching behaviour:

> I argue that if enough is known, in detail, about a person's orientations, goals, and resources, that person's actions can be explained at both macro and micro levels. That is, they can be explained not only in broad terms, but also on a moment-by-moment basis. (p. viii)

Although Schoenfeld acknowledges a global level for analysing teachers' decision-making processes, we will reveal later in this section that Shulman's theoretical framework of signature pedagogies provides an additional source to understand teacher actions from a more global perspective. But first we elaborate on Shulman's seminal work on teacher knowledge, and his notion of pedagogical content knowledge which "goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge *for teaching*" (Shulman 1986, p. 7). Interestingly, Shulman's starting point for introducing a new category as an additional aspect of teacher knowledge is rooted in his observation that research in teacher cognition so far was either on teacher's subject matter knowledge or teacher's pedagogical knowledge. In a convincing manner, he explains pedagogical content knowledge as an essential link between the two:

> Within the category of pedagogical content knowledge I include, for the most regularly taught topics in one's subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that make it comprehensible to others. (p. 7)

Much research followed and led to advances in understanding the knowledge category, but only a few publications applied the typology to explore beliefs in more detail. Törner (2002), for instance, draws on global beliefs as mentioned by Thompson (1992) and mentions subject matter beliefs as those beliefs that relate to aspects of a teacher's subject matter knowledge. Kuntze (2011) brings forward those ideas and investigates local and global components of pedagogical content beliefs. However, he does not distinguish knowledge and beliefs but chooses the pragmatic solution that "beliefs and instruction-related convictions are […] understood to be contained in the notion of professional knowledge" (Kuntze 2011, p. 2). In what follows, we take up the idea of networking theories in first combining Schoenfeld's theory on teacher behaviour and an adaption of Shulman's categorization that was initially developed for teacher knowledge to understand teachers' actions in the classroom from a local view. That is, we adopt the classification of subject matter and pedagogical content knowledge for the constructs of beliefs and goals, which allows for a more fine-grained analysis of teaching incidents.

Second, we go back to Schoenfeld's theory of Teaching-in-Context, and have a closer look at his aim to explain a teacher's behaviour in the classroom, and the choices that he or she makes in any moment. According to Schoenfeld, teaching can be studied on a fine-grain level and the analysis focuses on the decision-making process, as Schoenfeld (2010) points out in the following:

> Decision making and resource access are largely automatic when people are engaged in well practiced behavior. Mechanisms for routine access to cognitive resources have been extensively studied and have various names in the literature. Depending on the tradition, they may be called *scripts, frames, routines,* or *schemata*. The core idea is the same: when people perceive a situation as being of a familiar type, they have a "default" set of expectations that guide their perceptions and/or actions. (p. 16)

On a more global level, going beyond the single classroom actions, we can find some similar ideas in Shulman's work (2005a, b) on signature pedagogies. While drawing on studies in law, engineering, the clergy, medicine, nursing and teaching, Shulman (2005a, b) describes the signature pedagogy of an entire field. In particular, the construct catches the characteristics of different professions and how those can be described and analysed:

> What I mean by "signature pedagogy" is a mode of teaching that has become inextricably identified with preparing people for a particular profession. This means it has three characteristics: One, it's distinctive in that profession. So you wouldn't expect clinical rounds in a law school. And even though it might be very effective, you wouldn't expect a case dialogue or case method teaching of this sort in a medical school. Second, it is pervasive within the curriculum. So that students learn that as they go from course to course, there are certain continuities that thread through the program that are part of what it means to learn to "think like a lawyer," or "think like a physician," or "think like a priest." There are certain kinds of thinking that are called for in the rules of engagement of each course, even as you go from subject to subject. The third feature is another aspect of pervasiveness, which cuts across institutions and not only courses. Signature pedagogies have become essential to general pedagogy of an entire profession, as elements of instruction and of socialization.

Shulman (2005a, b) also underlines the decisive role of teacher knowledge bundled in routines, but emphasizes additionally that the cultures and characteristics of the professions are transported as signature pedagogies. Signature pedagogy is an emerging concept in teacher education which catches the salient and pervasive teaching practices that characterize an entire field and thus allows for analysing practices from a global viewpoint. The Carnegie Foundation for the Advancement of Teaching has undertaken many studies (cf. Shulman 2005a, b) to describe the signature pedagogy of the different fields. Those studies of education in the professions share that they link the following aspects of a professional role:

> [. . .] professional education is a synthesis of three apprenticeships – a cognitive apprenticeship wherein one learns to think like a professional, a practical apprenticeship where one learns to perform like a professional, and a moral apprenticeship where one learns to think and act in a responsible and ethical manner that integrates across all three domains. (Shulman 2005b)

Building on knowledge as basis, Shulman (2005a) assigns a fundamental role to signature pedagogies since those "are designed to transform knowledge attained to knowledge-in-use […]". Moreover, Shulman (2005b) reverts to his knowledge categories and explains that "these forms of knowledge are foundational, necessary but not sufficient". In order to understand more deeply teachers' actions in the classroom, Shulman (2005a) thus refers to the crucial role of signature pedagogies:

> Signature pedagogies are important precisely because they are pervasive. They implicitly define what counts as knowledge in a field and how things become known. They define how knowledge is analyzed, criticized, accepted, or discarded. (p. 54)

Signature pedagogies possess a structure by which a discipline's pedagogies can be examined, elaborated and compared (Shulman 2005a). Following Shulman, we have to distinguish three dimensions: *surface structure*, *deep structure* and *implicit structure*. The surface structure "consists of concrete, operational acts of teaching and learning, of showing and demonstrating, of questioning and answering, of interacting and withholding, of approaching and withdrawing" (Shulman 2005b, p. 54). That is, the surface structure covers overtly social acts associated with teaching and learning the subject. According to Shulman (2005b), "any signature pedagogy also has a deep structure, a set of assumptions about how best to impart a certain body of knowledge and know-how" (p. 55). Thus, the deep structure transports assumptions about the teaching and learning within the field. Finally, Shulman (2005a) points out that the implicit structure addresses a moral dimension comprising a set of beliefs about professional attitudes, values, norms and dispositions.

The dimension especially interesting in our case is the surface structure since it maintains the relevance of the knowledge body of mathematics, a domain with a proud long and international history and many traditions. Roesken and Törner (2010) point out that this surface structure encompasses a specific language and semiotics, as well as particularities regarding the teaching style and the teacher–student relationship. Teaching mathematics has its characteristics that depend strongly on the underlying subject of mathematics and how it is taught at the universities and during teacher education. For instance, the style of speech in the lectures often shows possessive set phrases such as using the plural "we" or an authoritative wording like "let be", so that no one feels invited to say something against. Another interesting example is the notion of w.l.o.g. (without loss of generality), a well-known saying of mathematicians. Who wants to show any weakness by claiming that it is not trivial for him or her? One can imagine that such an education leaves its marks and affects a teacher's later behaviour in the classroom essentially. In the analysis that we present later, we will elaborate on these ideas and show the influence on teacher behaviour in the classroom.

So far, we have briefly sketched the contributions by Schoenfeld and Shulman. In what follows, we elaborate on those theoretical strands and offer some further ideas on Schoenfeld's theory of Teaching-In-Context and Shulman's work on teacher knowledge and signature pedagogies (cf. Rosken and Rolka 2011) by combining both theoretical approaches. In particular, we enrich Schoenfeld's theoretical lens of analysing teacher behaviour locally in terms of knowledge goals, and beliefs by extending Shulman's knowledge categorization to beliefs and goals, and use the notion of signature pedagogies to elaborate on the role of beliefs from a global viewpoint. The ideas emerged throughout our work on a paper that was dedicated to analysing a video-taped school lesson through the lenses of Schoenfeld's approach (Törner et al 2010). We now explicate how the networking approach contributes to a better understanding of the data.

Illustrating the Networking of the Two Theories with Empirical Data

In the following, we further support our ideas on networking the above- mentioned theories by presenting evidence that we found in an empirical study (Törner et al. 2010). In particular, we show how the networking of theories that tackle aspects of teacher knowledge and beliefs can help to analyse local and global beliefs relevant in the classroom, and how beliefs interfere with teacher goals. This empirical study emerged from a bi-national in-service teacher training[3] that aimed at working out cultural differences and/or similarities in teaching styles. For this purpose, a Dutch and a German lesson on linear functions were videoed, forming the basis of discussions within the teacher training. The German teacher who taught the lesson on linear functions possesses 30 years of professional experiences. She has attended numerous in-service teacher training courses, in particular on using computer algebra systems and open tasks in mathematics teaching. In the lesson, linear relationships as motivation for the treatment of linear functions were embedded in various tasks. Students had to work in small groups of two or three on one of the tasks by using the computer, in particular the software Excel.

The teacher engaged very eagerly to implement newly imparted issues into her teaching on linear functions, a topic that she has taught in rather traditional ways several times previously. Although the teacher planned the lesson thoroughly, its course developed unexpectedly. At the beginning of the lesson, the teacher pursued a rather open and problem-oriented approach where students worked in small groups using the computer. However, as the lesson developed and time seemed to run out, the teacher suddenly changed her teaching style in favour of a more traditional approach. That is, she shifted back to her solid and approved methods in terms of a monologue on definitions in a formalized structure.

Another important data source is an interview that took place several days after the lesson. After watching the video and immediately recognizing the turning point in the course of the lesson, we wanted to find out more about the teacher's goals and beliefs underlying the planning and teaching of that lesson.

On the one hand, we resort to Schoenfeld's theory and identify the teacher's knowledge, goals and beliefs that were observable in the lesson but also expressed by her in the interview. On the other hand, we draw on the work by Shulman (1986) and adapt his categorization for the domain of knowledge to the one of beliefs, and we differentiate between pedagogical content goals and beliefs, and subject matter goals and beliefs. Basically, the knowledge categories can be directly adapted to beliefs. That is, the pedagogical content goals and beliefs concentrate on how to teach the subject of mathematics while the subject matter goals and beliefs are derived from the subject itself. We illustrate the categories by some examples.

In the interview with the teacher after the lesson, we identified statements that can be interpreted as both pedagogical content goals and beliefs. To be more concrete, the expressed goals were strongly rooted in beliefs and the beliefs influenced the goals to be fixed. The conclusion of the duality of the two constructs was even strengthened by the teacher justifying her goals and hence revealing subjective convictions that can be understood as beliefs. For the teacher, the use of the computer plays a central role in her teaching in general, but also in the specific lesson and the interview. She formulates as a goal:

> Whenever possible, I employ the computer in mathematics lessons (pedagogical content goal).

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[3] Funded by the Robert-Bosch Foundation.

This statement can be interpreted as belief in the sense that employing the computer whenever possible is rooted in the conviction that there is an additional value compared with the abdication of the computer. She complements this goal by a belief that is related to the topic of the lesson:

> The theme linear functions can be mediated by the computer (pedagogical content belief).

This pedagogical content belief was also realized as the teacher actually employed the computer when introducing linear functions. A reformulation of this belief in terms of a goal could have been "Students shall use the computer when dealing with linear functions". Hence, this expressed belief corresponds to an implicit goal.

Although the pedagogical content goals and beliefs were highly relevant during the first 29 min of teaching, the teacher suddenly shifted to her approved and traditional style while the computer lost its central role. Besides articulating frustration about the use of open tasks, she provided some subject matter goals and beliefs that explain the shift in the teaching trajectory from her point of view:

> Linear functions are defined by their slopes. The slope of a linear function is its most important characteristic (subject matter belief).
>
> Functions are important for Calculus in grade 12 (subject matter belief). The central term to be mediated in the context of linear functions is the concept of slope, which prepares students for the concept of derivative (subject matter belief).

From this results the following specific mathematical goal, which can also be identified as a kind of output directive for the lesson:

> The term slope must be mentioned in this lesson (subject matter goal).

This episode underlines that the subject matter beliefs on the relevance of linear functions can be understood as a key prerequisite, which in the last instance characterize unavoidably the subject matter goal that the teacher tried to obtain desperately in the lesson. That is, the moment the teacher realized that she could not achieve her central subject matter goal of introducing the term slope, she let the students simply switch off the computer. From this point onwards, global subject matter goals dominated the lesson activities to reach the one goal: the term slope must be mentioned. In other words, all pedagogical content goals and beliefs lost their rather positive value and stepped aside to make room for subject matter goals and beliefs.

Regarding this teaching episode, the questions arising for us are the following. Why are goals and beliefs so closely connected and attached to the same idea? Does this observation depend on the subject of mathematics and its specific structure? In addition to our analysis, we found some answers on a meta level while drawing on Shulman's work on signature pedagogies that we will discuss in the following.

In a talk at a conference in Germany, Shulman gave some examples for signature pedagogies in the domain of mathematics. For instance, he characterized the domain of teaching mathematics at university as a kind of dorsal teaching while showing a picture of a mathematics lecturer in front of a blackboard, turning his back to the classroom and writing down formulas while the students tried eagerly to copy the text on the boards. That is, all mathematics lectures are given in a specific style and thus elements of a signature pedagogy even permeate the field of teacher education.

Teacher students are confronted with a specific culture that is related to the subject they are studying.

Going deeper into the construct of signature pedagogy, we identify as surface structure influencing the domain of mathematics teaching in school the stable network provided by the discipline in terms of definitions, theorems and examples. In the teaching incident that we observed, the subject structure served as a kind of safety net for the teacher. That is, the subject matter goals and beliefs are rooted robustly in mathematics and dominate the pedagogical content goals and beliefs. The possibility of abandoning the term slope does not occur for the teacher either during the lesson or in the aftermath of the lesson while reflecting on the teaching.

Such a signature, obviously a powerful frame, maybe blurs the differences between goals and beliefs and serves as an overarching theme so that both constructs appear as two sides of the same coin.

Example 2: Networking a Beliefs Classification and Conceptual Change Approaches to Understand Changes in Beliefs

Whereas conceptual change theories initially focused on knowledge systems, Pintrich et al. (1993) called for also taking into consideration the affective domain. Beliefs are part of the affective domain (McLeod 1992), and can be used to explain why learners who possess the cognitive resources to succeed at mathematical tasks still fail (Di Martino and Zan 2001). In this context, beliefs transport what learners assume to be true about mathematics. Beliefs about mathematics are often based on an individual's own experiences as learner of mathematics. For example, beliefs that mathematics is 'difficult', 'all about one answer', or 'all about memorizing formulas' stem from classroom experiences where these ideas were implicitly conveyed and constantly reinforced. Research has shown that such beliefs are slow to form but, once established, are resistant to change even in the face of intervention (Op't Eynde et al. 2001; Schommer-Aikins 2004). In the context of teaching mathematics, beliefs have been used to explain the discordance between teachers' knowledge of mathematics and their teaching practices. This research has revealed that beliefs about *teaching* mathematics also arise from teachers' experiences as *learners* of mathematics (Calderhead and Robson 1991; Chapman 2002; Feiman-Nemser and Featherstone 1992; Feiman-Nemser et al. 1987; Fosnot 1989; Liljedahl 2006; Lortie 1975; Millsaps 2000; Skott 2001; Uusimaki and Nason 2004). So, a belief that teaching mathematics is 'all about telling how to do it' may come from a belief that learning mathematics is 'all about being told how to do it', which in turn may have come from personal experience as a learner of mathematics. Or it may not have! Implication and causality is difficult to determine in the context of beliefs.

The above-mentioned examples of concrete beliefs about mathematics or its teaching and learning reveal a specific view on mathematics. In accordance with

Dionne (1984) and Ernest (1989, 1991), Törner and Grigutsch (1994) labelled such beliefs as belonging to the "toolbox aspect". Here, mathematics is seen as a set of rules, formulae, skills and procedures; while mathematical activity means calculating as well as using rules or formulae and mastering procedures. Besides this toolbox aspect, one finds in the literature at least two other components, sometimes with varying notions: the "system aspect" and the "process aspect" (Grigutsch et al. 1998). In the system aspect, mathematics is characterized by logic, rigorous proofs, exact definitions and a precise mathematical language, and doing mathematics consists of accurate proofs as well as of the use of a precise and rigorous language. In the process aspect, doing mathematics is considered as a constructivist process where relations between different notions and sentences play an important role. Here the mathematical activity involves creative steps, such as generating rules and formulae, thereby inventing or re- inventing mathematics. In addition to these three perspectives on beliefs about mathematics and its teaching and learning, a further component is the usefulness or utility of mathematics (Grigutsch et al. 1998). Since beliefs are often referred to as a "messy construct" (Furinghetti and Pehkonen 2002; Pajares 1992), considering them as consisting of different components enables some reduction in this "messiness". Besides, this classification allows for identifying changes in beliefs, as will be illustrated later.

However, using this classification in order to code beliefs about mathematics and its teaching and learning, and finally to trace changes in teachers' beliefs, does not explain how and why these changes are occurring. To be more clear, classifying mathematical beliefs as toolbox, system or process aspect at different points in time can show that changes have occurred, but does not give any information about the mechanisms behind this change. For a better understanding of the underlying conditions, two strands of conceptual change approaches will be briefly sketched in the following (Vosniadou and Verschaffel 2004; Appleton 1997).

Worth noting in this regard is that the conditions described for the conceptual change approach by Vosniadou and Verschaffel (2004) refer to the prerequisites that an individual needs to bring before undergoing any change. For conditions on the instructional level that may produce change see Rolka et al. (2006).

The conceptual change approach used in the field of learning and instruction, initially in the domain of science (Posner et al. 1982), is based on the philosophy and history of science (Kuhn 1970), and was afterwards adapted to mathematics learning as well (Vosniadou and Verschaffel 2004). In accordance with Kuhn (1970), Posner et al. (1982) suggest that three conditions must be fulfilled so that a conceptual change can take place: (a) students do not come to instruction as "tabulae rasae" but already possess knowledge about certain phenomena, and, in some cases, this stands in contrast to the accepted scientific theories that explain these phenomena; hence, it is important to note that these "misconceptions" are formed through lived experiences without formal instruction; (b) students must be dissatisfied or feel discomfort with existing conceptions or theories; and (c) there is a phenomenon of theory replacement, initiated by the mechanism of 'cognitive conflict' which basically refers to the assumption that before a new theory can be adopted the current theory needs to be rejected. In the best case, this model can be seen as partial understandings rather than incorrect understandings. The perfection of these models is achieved through further instruction based on constructivist theories of learning.

Although the theory of conceptual change focuses primarily on cognitive aspects of conceptual change, it is equally applicable to metaconceptual, motivational, affective and socio-cultural factors as well (Vosniadou and Verschaffel 2004; Liljedahl et al. 2007b). In the following, we briefly sketch how this approach can be used for describing changes in pre-service teachers' mathematical beliefs by outlining that each of the three criteria (a) to (c) presented above is equally applicable to tracing changes in teachers' mathematical beliefs. In the sense of the criteria "lived experiences", pre-service teachers also do not come to teacher education as "tabulae rasae", as Ball (1988) points out: "Long before they enroll in their first education course or math methods course, they have developed a web of interconnected ideas about mathematics, about teaching and learning mathematics, and about schools". During their time as students of mathematics they first formulated, and then concretized, deep-seated beliefs about mathematics and what it means to learn and teach mathematics. Unfortunately, these deep-seated beliefs often run counter to contemporary research on what constitutes good practice. As such, it is one of the roles of teacher education programmes to reshape these beliefs and extend insufficient beliefs that could impede effective teaching in mathematics (Green 1971). Certainly, one could raise the objection that the formation of pre-service teachers' mathematical beliefs cannot be viewed as being formed outside a context of formal instruction. For sure, their experiences as learners of mathematics are situated within a setting of formal instruction at school, but here the focus of that instruction was on mathematical contents and not on the nature of mathematical knowledge or the question of how mathematics should be taught or learned. Hence, mathematical beliefs are tacitly constructed and, therefore, the condition of "lived experience" is met. In comparison to the original theory of conceptual change in learning and instruction proposed by Posner et al. (1982), we do not aim to judge beliefs as inadequate or inappropriate, as "misbeliefs". Rather, we would like to emphasize that – referring to the above-mentioned classification of mathematical beliefs according to Dionne (1984) and Ernest (1989, 1991) – all three (or four) aspects in some sense do play a valuable role in answering the question "What is mathematics?" In some papers found in the literature, the toolbox aspect is presented as being rather unacceptable. However, which mathematician would, indeed, claim that mathematics has nothing to do with numbers, rules or calculations? It is important to note here that a sole view on mathematics as toolbox is certainly insufficient and it is strongly desirable to enrich this view with ideas from the system and process aspect. In the case of pre-service mathematics teachers, it can be noted that they often come to teacher education courses with a dominance of the toolbox and system aspect (Rolka et al. 2006).

The criteria (b) and (c) are equally given. The teachers must feel some discomfort with their existing beliefs and they must experience that they can benefit from alternative beliefs that are useful, plausible and fruitful for them. However, in line with our remarks above, belief rejection would not be an appropriate term as the notion of incorrect beliefs as such is not appropriate.

Another approach also focusing on conceptual change is given by Appleton (1997), who elaborated a model for describing and analysing students' learning especially during science lessons. This model offers, according to Piaget's terms of assimilation and accommodation, different possibilities of what happens when students are confronted with new information and experiences. When this new information is processed, one of three possibilities is likely to occur:

- **Identical fit**: The new information may form an apparent identical fit with an existing idea. This means that the students are able to make sense of the new information on the basis of their existing knowledge. This does not imply the correctness of the students' explanations.
- **Approximate fit**: The new information forms an approximate fit with an existing idea in which aspects are seen to be related, but details may be unclear. These students encounter new ideas but do not give up old ones. However, even if contradictory, they do not reach a situation where a cognitive conflict could take place. Hence, new information is assimilated but not accommodated.
- **Incomplete fit**: The new information is acknowledged as not being explained by the ideas tried so far. This incomplete fit of information results in a cognitive conflict. When students experience an incomplete fit they try to reduce the conflict by seeking information which might provide a solution.

The main mechanism for change in Appleton's model is *cognitive conflict*. Although it was originally conceived in the context of knowledge change, we explore in the following how the categories can be applied to capture belief changes. The theory of Appleton (1997) enables categorizing the different reactions of students when confronted with new ideas. The main difference between identical, approximate and incomplete fit is the presence of *cognitive conflict*, which proves to be also the decisive tool for change in beliefs.

Illustrating the Networking of the Two Theories with Empirical Data

The data that are used to illustrate the networking of the beliefs classification and the conceptual change approaches come from a research study that looked more broadly at initiating changes in pre-service teachers' beliefs (Liljedahl et al. 2007a, b). Participants in this study were 39 pre-service elementary school teachers enrolled in a Designs for Learning Elementary Mathematics course which was taught with the implicit goal of teaching for conceptual change in beliefs. The students were immersed into a problem-solving environment for initiating metacognitive discourse about their mathematics-related beliefs. In addition, the students encountered different instructional strategies so that they were encouraged to change their conceptions about the meaning of mathematics teaching and learning. Throughout the course the participants kept a reflective journal in which they documented their beliefs. In the first and final week of the course, they were asked to respond to the following questions:

- What is mathematics?
- What does it mean to learn mathematics?
- What does it mean to teach mathematics?

In this section we start by presenting an example for beliefs change in terms of the classification by Grigutsch, Raatz and Törner, the four criteria (a) to (d) of the theory of conceptual change, and the model of learning by Appleton. In the following, we present excerpts from David's journal entries where he answers the above-mentioned three questions.

At the beginning of the course, David writes the following:

> When first pondering the question, "What is mathematics?" I initially thought that mathematics is about numbers and rules. It is something that you just do and will do well as long as you follow the rules or principles that were created by some magical man thousands of years ago. […] To be honest, I don't like math. I found it so boring and so robotic. Lessons were set up in a robotic way. The teachers would show us the principles and then we would do the exercises.

David nicely articulates that his view on mathematics is strongly informed by his experiences with mathematics learning. Using the above-mentioned classification of mathematical beliefs, his answer is coded as "toolbox aspect". Using the conceptual change approach, this is an adequate example of illustrating the first condition for a possible change, namely the role of experiences made by an individual. His lived experience as a student of mathematics is informing his current understanding of what mathematics is. It is also informing his understanding of what it means to teach mathematics – robotic.

He continues his remarks by stating:

> I wish my initial definition could be different but this is the kind of math that I was exposed to.

Here it becomes obvious that David is not satisfied with his view on mathematics or – as he calls it – definition of mathematics, which is part of the second condition of the conceptual change approach. Interestingly, David entered the course already expressing a certain discomfort with his beliefs about mathematics and the teaching and learning of mathematics. However, as there exists no alternative for him, he has not yet fully let go these beliefs, as becomes apparent from further analysis of his journal entries. Although not initiated through instruction, it could be said that David has already experienced cognitive conflict with respect to his beliefs.

In his last journal entry, David expresses and explicitly reflects his change:

> However, after experiencing a couple of challenging problems and exciting classes, I have to say that my definition [of mathematics] can be summed up very simply. To me, mathematics is not about answers, it's about process. Mathematics is about exploring, investigating, representing, and explaining problems and solutions. Learning math is about inquiry and the development of strategies. It is about using your intuition, experimenting with strategies and discussing the outcome. It is about risk taking and experimenting. To teach mathematics is to welcome all ideas that are generated and facilitate discussion. It is about letting the students make sense of the math in their own way, not 'my way'. The teacher's role is about guiding the process, but handing the problem over to the students.

Not only can a change be noted, but using the classification introduced above one can say that this answer corresponds to the process aspect. In terms of the conceptual change approach, it becomes clear that he not only distances himself from his original beliefs ("math is not about answers") but also expresses another belief instead ("it's about process") which illustrates the third condition.

Finally, in order to make sense of David's change in terms of the model established by Appleton (1997), his entries are exemplary of an incomplete change.

In the following, we concentrate on providing more evidence that the theory of conceptual change can be adapted to describe changes in beliefs by examples for the two other possibilities of change introduced by Appleton (1997).

Jacqueline is an example where we observed an identical fit. In her first journal entry, Jacqueline writes the following:

> To teach mathematics, is to guide the learner through the process. It is not the job of the teacher to supply the answer, but to scaffold the process in order for the learners to be successful problem solvers. Guiding the students through the process also allows the learners to discover at their own pace and be at the centre of their learning.

Jacqueline focuses on the role of the teacher as a guide. In her last entry she still remains in this position:

> Finally to teach mathematics is to teach through facilitation. The teacher is there to guide students through the process and supply them with the most efficient tools to solve a problem. It is ultimately up to the student to discover for themselves. […] It is also the role of the teacher not to provide the answer but put this on the students to solve in the way that best suits them.

This example shows that the ideas offered by the course seem to fit perfectly with what Jacqueline has experienced so far. There is no apparent need for her to change her beliefs.

Aleksandra is an example where an approximate fit took place. She writes in her first journal entry the following:

> I think mathematics is something more than just the use of numbers. It is a way of thinking, a way of knowing things and figuring things out. I believe that it is one of the many ways that some people understand life, connected to multiple intelligences. What I mean is that it is beyond just looking at the world "numerically" and calculating things – it is logical reasoning. Mathematics is a belief that everything has a rational explanation. It is an abstract and conceptual way of thinking about the world around us and solving logical problems.

In few words, Aleksandra views mathematics as a way of thinking. In her last entry, she states:

> I now realize that my understanding of what mathematics is has not really changed but expanded through the course of this class. I would add to this definition [that she used in her first entry] that it is also the way we examine information and analyse it. It is the use of mathematical concepts in real life situations and the flexible way of thinking about numbers, algorithms, patterns, etc. that apply to life. It is an abstract way of looking at the world, through the visualization of number and spatial concepts. It is also using logical and deductive reasoning and making inferences, evaluation problems and situations and making judgments and decisions in given situations. It is the ability to predict and plan and visualize things that are not necessarily presented to us visually.

Aleksandra articulates that her understanding of what mathematics is has not really changed but she emphasizes that she added some beliefs to her already existing ones. Hence, the course did not succeed in producing a fundamental change in her beliefs.

The last example was chosen to give evidence for our remark above that beliefs are not simply replaced but how reflection serves as an important catalyst to allow for changes. In her first entry, Nicola answers the question "What is mathematics?" as follows:

> The first thing that comes to my mind is numbers. I think of math as being calculations such as adding and subtracting or dividing and multiplying using numbers. […]

Concerning the question "Why do we teach mathematics?" she states:

> I believe that we encounter math everyday in our lives. For example, buying groceries, we need to know how to add and the value of coins and dollars. We need to know how to budget our money. We need to measure cups and table spoons when we are cooking and add up the calories we are eating. […] Therefore I believe we teach mathematics to function in our daily lives.

In her last entry she states:

> I realize that math is more of a process. There often is a right or a wrong answer but we can't focus on that. We need to value the fact that there is a thinking process of how we feel and of what we did to solve the problem. […] Mathematics is a set of tools. The more we use the tools, the better that we become with them. […] I still do believe that math involves an element of memorization. What I do think as well is that before memorization happens, comprehension and the "why" needs to happen. There is no sense in memorizing things we don't understand because we will be sure to get stuck later down the road. […] So if learning math needs a "why" then we must teach to the "why". I think the best way to figure out why is through self- discovery. I think as a teacher it is important to have interactive thought provoking activities that provide a time for students to ask themselves, their classmates and their teacher questions about why and how.

It becomes obvious that Nicola justifies her former beliefs in some sense but also makes clear that new beliefs have been added.

Conclusions

The field of beliefs research has developed into different directions which can roughly be sketched by the different research paradigms that were developed in psychology and mathematics education. In addition, much discussion on mathematics beliefs research has concentrated on the difference between knowledge and beliefs and has led to cognitive theories that mainly omitted beliefs research. Our approach has been to extend theories initially developed for knowledge categorization and development to the field of beliefs.

We showed two examples for networking prominent theories from psychology and mathematics education that helped to extend the understanding of the role of beliefs. First, we elaborated on Schoenfeld's theory of *Teaching-In-Context* that captures teacher behaviour from a local view as a function of teacher knowledge, goals and beliefs by extending the knowledge categorization provided by Shulman (1986) to the fields of beliefs and goals. In addition, we used Shulman's (2005a, b) overarching theory of signature pedagogies to understand the significant role of beliefs from a global view. By analysing a specific teaching episode we showed that those theoretical lenses helped to clarify the turning point that occurred during the lesson under observation.

Second, we developed further ideas of conceptual change originally conceived in the context of knowledge (Ernest 1989; Vosniadou and Verschaffel 2004; Appleton 1997) to explore belief changes which were analysed by using the classification of mathematical beliefs based on the work by Dionne (1984) and Ernest (1989). Changes in beliefs could then be illustrated in more detail. As suggested in the literature, we used different theoretical lenses to analyse the same data set. We found evidence in our data that broadening the theoretical approach is fruitful for gaining a deeper understanding of the construct.

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