Effective Vacuum Bianchi IX in Loop Quantum Cosmology

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Abstract In classical general relativity, the chaotic behavior of the Bianchi IX model can be useful to describe the generic local evolution near a singularity. However, one expects that quantum effects can modify it. In this contribution we show that the modifications which come from Loop Quantum Cosmology imply a non-chaotic effective behavior.

1 Introduction

Bianchi models are spatially homogeneous models such that the symmetry group $\mathscr{S}$ acts simply and transitively on a space manifold $\Sigma \cong \mathscr{S}$. The symmetry group for Bianchi IX model is generated by three spatial rotations on a 3-sphere. We identify this group with SU(2) to define fiducial frames and co-frames. The fiducial cell is a 3-sphere with radius a_o (=2) and its volume is $V_o = 2\pi^2 a_o^3$. We define $\ell_o = V_o^{1/3}$ and $\vartheta = \ell_o/a_o = (2\pi^2)^{1/3}$. In terms of the phase space variables used in loop quantum gravity (LQG) [1–3], a connection A_a^i and a densitized triad E_i^a, the classical constraint is

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$$\mathscr{C}_H = \int_{\mathscr{V}} N \left[-\frac{E_i^a E_j^b}{16\pi G \gamma^2 \sqrt{|q|}} \epsilon^{ij}_{\ k} \left(F_{ab}^k - (1+\gamma^2)\Omega_{ab}^k \right) + \mathscr{H}_{\text{matter}} \right] \mathrm{d}^3x \,, \quad (1)$$

where N is the lapse function, $\mathscr{H}_{\text{matter}} = \rho V$ and Ω_{ab} is the curvature of the spin connection Γ_a^i which is compatible with the triads. In what follows, we take $N = 1$ and since we work in vacuum, ρ is equal to zero. If we restrict ourselves to diagonal Bianchi IX model, we can parametrize A_a^i and E_i^a as $A_a^i = c^i {}^o\omega_a^i/\ell_o$ and $E_i^a = p_i \sqrt{{}^o q}\, {}^o e_i^a/\ell_o^2$, the p_i's in terms of scale factors a_i are $|p_i| = \ell_o^2 a_j a_k$, and the volume is $V = \sqrt{|p_1 p_2 p_3|}$. The nonzero Poisson brackets are given by $\{c_i, p_j\} = 8\pi G\gamma\delta_{ij}$, where γ is the Barbero-Immirzi parameter.

To quantize the Hamiltonian constraint in (1), we find an operator corresponding to F_{ab} and we express the connection A_a^i in terms of holonomies [4]. The operators associated to the connection are then given by $\hat{c}_i = \widehat{\sin \bar{\mu}_i c_i / \bar{\mu}_i}$, where $\bar{\mu}_i = \lambda\sqrt{p_i/p_j p_k}$, $i \neq j \neq k$ and $\lambda^2 = 4\sqrt{3}\pi\gamma l_p^2$ is the smallest eigenvalue of area in LQG.

For the term which contains the inverse of the metric determinant, and for those which contain the negative powers of p_i's, we use the Thiemann strategy [3]. The idea is to find some classical equivalent expression for them in terms of holonomies and positive powers of p's and then quantize this expression. For instance, to quantize a negative power of p_i we know that, classically, there is the following identity

$$|p_i|^{(\ell-1)/2} = -\frac{\sqrt{|p_i|}}{4\pi G\gamma\mu_i \ell j(j+1)} \tau_i h_i^{(\mu_i)} \{h_i^{(\mu_i)-1}, |p_i|^{\ell/2}\} \,, \quad (2)$$

where μ is the length of a curve which is used for calculating the holonomy, ℓ is a number between 0 and 1 and $j \in \frac{1}{2}\mathbb{N}$ labels the representation. For simplicity we take $j = 1/2$ and choose $\mu_i = \bar{\mu}_i \ell_o$ because they appear in the terms corresponding to curvature. Since the largest negative power of p's which appears in the constraint is $-1/4$ we will take $\ell = 1/2$ to obtain it directly from the above equation. After that, we express the other negative powers by it. The eigenvalues for the operator $\widehat{|p_i|^{-1/4}}$ are

$$J_i = \frac{h(V)}{V_c} \prod_{j\neq i} p_j^{1/4} \,, \text{ where } h(V) = \sqrt{V + V_c} - \sqrt{|V - V_c|},\ V_c = 2\pi\gamma\lambda\ell_p^2.$$

The correction term which comes from the $\epsilon_k^{ij} E_i^a E_j^b/\sqrt{|q|}$ is $A(V) = (V + V_c - |V - V_c|)/2V_c$. Hence, with these definitions one obtains the corresponding constraint operator.

In this work, we are interested in studying the classical effective Hamiltonian which has some modifications from the quantum theory to gain qualitative insights into the leading order quantum effects. Since the Hamiltonian is invariant under parity, we restrict ourselves to positive p_i's. The effective Hamiltonian which is derived from the quantum theory is

$$\mathscr{H}_{\text{eff}} = -\frac{V^4 A(V) h^6(V)}{8\pi G V_c^6 \gamma^2 \lambda^2}\Big(\sin\bar{\mu}_1 c_1 \sin\bar{\mu}_2 c_2 + \sin\bar{\mu}_1 c_1 \sin\bar{\mu}_3 c_3 + \sin\bar{\mu}_2 c_2 \sin\bar{\mu}_3 c_3\Big) + \frac{\vartheta A(V) h^4(V)}{4\pi G V_c^4 \gamma^2 \lambda}\Big(p_1^2 p_2^2 \sin\bar{\mu}_3 c_3 + p_2^2 p_3^2 \sin\bar{\mu}_1 c_1 + p_1^2 p_3^2 \sin\bar{\mu}_2 c_2\Big) - \frac{\vartheta^2 (1+\gamma^2) A(V) h^4(V)}{8\pi G V_c^4 \gamma^2}\Big(2V\Big[p_1^2 + p_2^2 + p_3^2\Big] - \Big[(p_1 p_2)^4 + (p_1 p_3)^4 + (p_2 p_3)^4\Big]\frac{h^6(V)}{V_c^6}\Big).$$

2 The Effective Potential

It is helpful to use the potential term of the constraint to study the solutions. The classical potential which comes from the spin connection's curvature in the classical constraint, in terms of Misner variables is [5]

$$W = \frac{1}{2} e^{-4\Omega}\Big(e^{-4\beta_+} - 4e^{-\beta_+}\cosh\sqrt{3}\beta_- + 2e^{-2\beta_+}[\cosh 2\sqrt{3}\beta_- - 1]\Big), \quad (3)$$

where $\Omega = -\frac{1}{3}\log V$ and the anisotropies $\beta_\pm$ are defined via $a_1 = e^{-\Omega+(\beta_+ + \sqrt{3}\beta_-)/2}$, $a_2 = e^{-\Omega+(\beta_+ - \sqrt{3}\beta_-)/2}$ and $a_3 = e^{-\Omega-\beta_+}$. Since the Ω dependence factorizes, one can obtain an anisotropy potential $\mathscr{V}(\beta_+, \beta_-)$ which exhibits exponential walls for large anisotropies. The universe can be seen as a particle moving in such a potential (W) that presents reflections at the walls. An infinite number of these reflections implies that the system behaves chaotically. When the volume becomes small, the quantum effects become important and one should work with the full quantum theory, but one can use the effective equations to have a qualitative view of what happens near the classical singularity. From the effective Hamiltonian, the modified potential can be derived as a function of p_i:

$$W = -\frac{V^2 A(V) h^4(V)}{V_c^4}\Big(p_1^2 + p_2^2 + p_3^2 - \Big[(p_1 p_2)^4 + (p_1 p_3)^4 + (p_2 p_3)^4\Big]\frac{h^6(V)}{2V V_c^6}\Big).$$

For a simple case, when $\beta_- = 0$ and $\beta_+ \to -\infty$, the classical potential is $W(\beta_+, \Omega) \sim \frac{1}{2}e^{-4\Omega-4\beta_+}$. If we rewrite the modified potential in terms of Misner variables we can see that in this limit, the modified potential behaves as $\frac{1}{2V_c^9}e^{-52\Omega-4\beta_+}$ where the β_+-dependency of both classical and modified potential are the same, so we have an infinite wall for the modified potential too, (see Fig. 1, left). Note however that, for small volumes, the modified potential can be negative at some points.

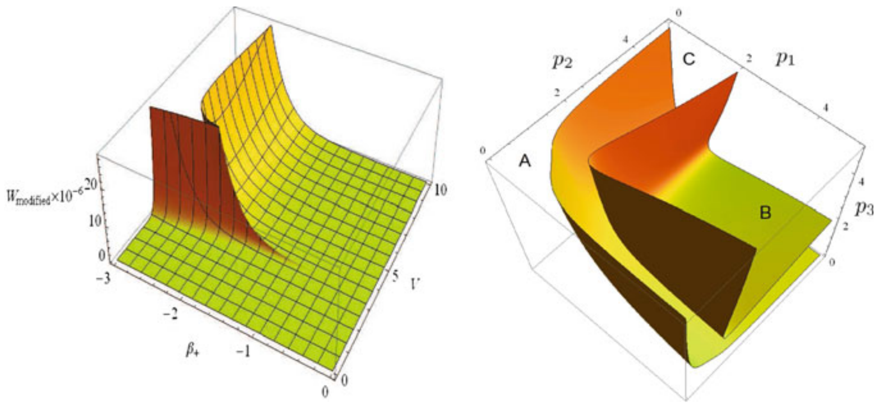

Fig. 1 *Left* modified potential when $\beta_- = 0$. *Right* zero surfaces of the maximum allowed density; density in regions A and B can be non-negative but in region C it is always negative; both in Planck units

3 Density

In the general case, as can be seen in Fig. 1 (right), the maximum allowed density (which arises from the modified Hamiltonian by choosing the sine functions equal to -1), has two distinct disconnected regions with positive values. Therefore, if we impose the weak energy condition and start the evolution within one region, the universe cannot reach the other region. To study the vacuum Bianchi IX, we start from large volumes which lie in region B of Fig. 1 (right) and, as we go to smaller volumes, we cannot reach zero volume because 'crossing' to region A is not allowed. Therefore, there is a smallest reachable volume in region B and, since very large anisotropies are not allowed near this smallest volume, and the modified potential is not too large there, then we have, at most, finite oscillations before reaching the bounce. On the other hand, in the internal region A, the anisotropies are very large when some of the p_i are very small, and then the volume of the universe cannot be large enough to start the evolution from there.

4 Conclusions

We have studied the behavior of a modified potential for the Bianchi IX model when quantum effects become important. We showed that the potential wall does not disappear and we have potential chaotic behavior near the classical singularity. However, if the weak energy condition holds and if we start from large volumes and evolve the equations into small volumes, there will be a lower bound for volumewithin region B, and one does not reach region A (connected to zero volume). Since there are no large anisotropies near the smallest allowed volume, the solutions will *not* exhibit chaotic behavior.

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