# On the Observability of Quantum-Gravitational Effects in the Cosmic Microwave Background

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**Abstract** In any approach to quantum gravity, it is crucial to look for observational effects in order to discriminate between different approaches. Here, we discuss how quantum-gravitational contributions to the anisotropy spectrum of the cosmic microwave background arise in the framework of canonical quantum gravity using the Wheeler–DeWitt equation. From the present non-observation of these contributions, we find a constraint on the Hubble parameter of inflation.

## 1 Introduction

One of today's most significant tasks in theoretical physics is to find the correct quantum theory of gravity. We have several approaches to such a theory at hand; however, there has not yet been a definite prediction which is testable with today's level of precision by experiment or observation. The reason for this is that quantum effects of gravity should only become sizable in situations where large curvature and very high energies approaching the Planck scale are involved. This effectively makes black hole physics and very early universe cosmology the two main applications for a theory of quantum gravity.

Here, we want to focus on cosmology, and in particular on the Cosmic Microwave Background (CMB), which has opened a new era of precision cosmology ever since its anisotropies have been detected. The power spectrum of these anisotropies has turned out to be a rich source of information about the very early universe and it is therefore a very suitable candidate to look for quantum-gravitational effects.

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We have chosen to use quantum geometrodynamics as our framework, a direct canonical quantization of gravity. It is unlikely that quantum geometrodynamics turns out to be the ultimate answer to the problem of quantum gravity; however, it should be able to be used at least as an effective theory, since it leads to Einstein's equations in the semiclassical limit, see e.g. [1].

Our aim is to calculate the dominant quantum-gravitational contribution for the primordial power spectrum of cosmological perturbations, which arises from a semi-classical approximation to the Wheeler–DeWitt equation of quantum cosmology.

This conference contribution is based on our papers [2] and [3].

## 2 The Quantum-Cosmological Model

In order to give a first estimate of how sizable quantum gravity effects for the CMB can be, we choose the simplest model, an inflationary universe with perturbations of only the scalar field $\phi$, which plays the role of the inflaton. The background universe is a flat Friedmann–Lemaître universe with a scale factor $a \equiv \exp(\alpha)$. Furthermore, we assume that the slow-roll approximation holds in the form of $\dot{\phi}^2 \ll |\mathscr{V}(\phi)|$, where $\mathscr{V}(\phi)$ is the inflaton potential, which we choose to be $\mathscr{V}(\phi) = \frac{1}{2}m^2\phi^2 \approx$ const. for definiteness.

After setting $\hbar = c = 1$, redefining the Planck mass as $m_{\rm P} = \sqrt{3\pi/2G} \approx 2.65 \times 10^{19}\,$GeV and rescaling the scalar field $\phi \to \phi/\sqrt{2\pi}$, one arrives at the following Wheeler–DeWitt equation in minisuperspace:

$$\mathscr{H}_0\Psi_0(\alpha,\phi) \equiv \frac{e^{-3\alpha}}{2}\left[\frac{1}{m_{\rm P}^2}\frac{\partial^2}{\partial\alpha^2} - \frac{\partial^2}{\partial\phi^2} + e^{6\alpha}m^2\phi^2\right]\Psi_0(\alpha,\phi) = 0\,. \tag{1}$$

Furthermore, we make the assumption that the kinetic term of the $\phi$-field can be neglected, as it is small compared to the potential term $\partial^2\Psi_0/\partial\phi^2 \ll e^{6\alpha}m^2\phi^2\Psi_0$. This allows us to substitute $m\phi$ by $m_{\rm P}H$, where $H$ denotes the quasi-static Hubble parameter during inflation, and our Wheeler–DeWitt equation for the background becomes

$$\mathscr{H}_0\Psi_0(\alpha) \equiv \frac{e^{-3\alpha}}{2}\left[\frac{1}{m_{\rm P}^2}\frac{\partial^2}{\partial\alpha^2} + e^{6\alpha}m_{\rm P}^2H^2\right]\Psi_0(\alpha) = 0\,. \tag{2}$$

We include inhomogeneities by adding perturbations to the homogeneous background inflaton field $\phi \to \phi(t) + \delta\phi(\mathbf{x},t)$ and decompose them into Fourier modes, where we assume for simplicity that the space is compact and the spectrum for the wave vector $\mathbf{k}$, $k \equiv |\mathbf{k}|$, discrete: $\delta\phi(\mathbf{x},t) = \sum_k f_k(t)\, e^{i\mathbf{k}\cdot\mathbf{x}}$. Note that we use units in which $k$ is a dimensionless quantity. For each of the modes we have a Hamiltonian

$$\mathscr{H}_k = \frac{1}{2}e^{-3\alpha}\left[-\frac{\partial^2}{\partial f_k^2} + \left(k^2e^{4\alpha} + m^2e^{6\alpha}\right)f_k^2\right],$$

such that the Wheeler–DeWitt equation that includes the scalar field inhomogeneities reads [4]

$$\left[\mathscr{H}_0 + \sum_{k=1}^{\infty} \mathscr{H}_k\right] \Psi\left(\alpha, \{f_k\}_{k=1}^{\infty}\right) = 0 .$$

Due to the smallness of the fluctuations, one can neglect self-interactions of the respective modes and therefore make a product ansatz for the full wave function including the fluctuation modes: $\Psi\left(\alpha, \{f_k\}_{k=1}^{\infty}\right) = \Psi_0(\alpha) \prod_{k=1}^{\infty} \widetilde{\Psi}_k(\alpha, f_k)$. This ansatz allows us to write out a Wheeler–DeWitt equation for each fluctuation mode $\Psi_k(\alpha, f_k) := \Psi_0(\alpha)\widetilde{\Psi}_k(\alpha, f_k)$, which takes the form:

$$\frac{1}{2}\, \mathrm{e}^{-3\alpha} \left[\frac{1}{m_{\mathrm{P}}^2} \frac{\partial^2}{\partial \alpha^2} + \mathrm{e}^{6\alpha} m_{\mathrm{P}}^2 H^2 - \frac{\partial^2}{\partial f_k^2} + W_k(\alpha) f_k^2\right] \Psi_k(\alpha, f_k) = 0 ,$$

where we have defined $W_k(\alpha) := k^2 \mathrm{e}^{4\alpha} + m^2 \mathrm{e}^{6\alpha}$.

## 3 The Semiclassical Approximation

We are interested in finding quantum-gravitational correction terms to the standard expressions used to calculate the power spectrum of quantum fluctuations in an inflationary universe. Hence, it suffices to solve the Wheeler–DeWitt equation (3) by performing a Born–Oppenheimer type of approximation, as it was presented for the full Wheeler–DeWitt equation in [5]. The Born–Oppenheimer approximation is widely used in molecular physics, where one can separate the degrees of freedom of the molecules into slow ones (the nuclei) and fast ones (the electrons). In our quantum-cosmological setting, the slow variable is the scale factor, while the fast ones are the fluctuations $f_k$.

We implement the Born–Oppenheimer approximation by making the ansatz $\Psi_k(\alpha, f_k) = \mathrm{e}^{\mathrm{i}\, S(\alpha, f_k)}$ and expanding $S$ in terms of powers of $m_{\mathrm{P}}^2$: $S(\alpha, f_k) = m_{\mathrm{P}}^2 S_0 + m_{\mathrm{P}}^0 S_1 + m_{\mathrm{P}}^{-2} S_2 + \ldots$. Inserting this ansatz into equation (3) and comparing terms of equal power of $m_{\mathrm{P}}$, one obtains that at order $\mathscr{O}(m_{\mathrm{P}}^2)$ $S_0$ obeys the classical Hamilton–Jacobi equation

$$\left[\frac{\partial S_0}{\partial \alpha}\right]^2 - \mathrm{e}^{6\alpha} H^2 = 0 , \tag{3}$$

which describes the classical minisuperspace background on which the quantum fluctuations propagate. At the next order $\mathscr{O}(m_{\mathrm{P}}^0)$, we define $\psi_k^{(0)}(\alpha, f_k) \equiv \gamma(\alpha)\, \mathrm{e}^{\mathrm{i}\, S_1(\alpha, f_k)}$ and impose a condition in order to make $\gamma(\alpha)$ equal to the standard WKB prefactor. At this point, we can introduce a time parameter $t$ that arises from the approximate background defined by the Hamilton–Jacobi equation (3), using the definition

$$\frac{\partial}{\partial t} := -\,\mathrm{e}^{-3\alpha}\,\frac{\partial S_0}{\partial \alpha}\,\frac{\partial}{\partial \alpha}\,. \tag{4}$$

This limit can be compared with the limit of geometric optics that arises from wave optics. In that case, light rays result as an approximate concept from the eikonal equation. In our case, an approximate spacetime emerges, and one has an approximate time $t$ at one's disposal.

Consequently, we find that each $\psi_k^{(0)}(\alpha, f_k)$ obeys a Schrödinger equation with respect to $t$: $\mathrm{i}\,\frac{\partial}{\partial t}\,\psi_k^{(0)} = \mathcal{H}_k \psi_k^{(0)}$.

Hence, the order $\mathcal{O}(m_{\mathrm{P}}^0)$ corresponds to the limit of quantum theory in an external background. It is at this order where we will obtain the standard results for quantum fluctuations in an inflationary universe.

But before that, we will take the semiclassical approximation one step further, to the order $\mathcal{O}(m_{\mathrm{P}}^{-2})$, where we use a decomposition of $S_2(\alpha, f_k)$ as follows: $S_2(\alpha, f_k) \equiv \varsigma(\alpha) + \eta(\alpha, f_k)$. After demanding that $\varsigma(\alpha)$ be the standard second-order WKB correction, we find that the wave functions $\psi_k^{(1)}(\alpha, f_k) := \psi_k^{(0)}(\alpha, f_k)\,\mathrm{e}^{\mathrm{i}\, m_{\mathrm{P}}^{-2}\,\eta(\alpha, f_k)}$ obey a quantum-gravitationally corrected Schrödinger equation of the form

$$\mathrm{i}\,\frac{\partial}{\partial t}\,\psi_k^{(1)} = \mathcal{H}_k \psi_k^{(1)} - \frac{\mathrm{e}^{3\alpha}}{2 m_{\mathrm{P}}^2 \psi_k^{(0)}}\left[\frac{\left(\mathcal{H}_k\right)^2}{V}\psi_k^{(0)} + \mathrm{i}\,\frac{\partial}{\partial t}\left(\frac{\mathcal{H}_k}{V}\right)\psi_k^{(0)}\right]\psi_k^{(1)}\,, \tag{5}$$

where $V := \mathrm{e}^{6\alpha} H^2$. The first term in this equation gives the dominant contribution, while the second one corresponds to a small violation of unitarity with respect to the standard inner $\mathcal{L}^2$-product for the modes $f_k$. Since it is usually negligible with respect to the first term, we will neglect the unitarity-violation term in the following.

## 4 Calculation of the Power Spectrum

In order to calculate the power spectrum of the scalar field fluctuations, we have to solve the uncorrected Schrödinger equation. We express $\alpha$ in terms of $t$ and use the Gaussian ansatz $\psi_k^{(0)}(t, f_k) = \mathcal{N}_k^{(0)}(t)\,\mathrm{e}^{-\frac{1}{2}\,\Omega_k^{(0)}(t)\, f_k^2}$.

This leads to the following system of equations:

$$\dot{\mathcal{N}}_k^{(0)}(t) = -\frac{\mathrm{i}}{2}\,\mathrm{e}^{-3\alpha}\,\mathcal{N}_k^{(0)}(t)\,\Omega_k^{(0)}(t)\,, \tag{6}$$

$$\dot{\Omega}_k^{(0)}(t) = \mathrm{i}\,\mathrm{e}^{-3\alpha}\left[-\left(\Omega_k^{(0)}(t)\right)^2 + W_k(t)\right]. \tag{7}$$

Equation (7) has the following solution

$$\Omega_k^{(0)}(t) = \frac{k^2 a^2}{k^2 + H^2 a^2}\,(k + \mathrm{i}\,Ha) + \mathscr{O}\left(\frac{m^2}{H^2}\right), \tag{8}$$

while (6) together with the normalization of the states yields the solution $|\mathscr{N}_k^{(0)}(t)|^2 = (\Re\mathfrak{e}\,\Omega_k^{(0)}(t)/\pi)^{1/2}$. In order to use equation (7) to calculate the power spectrum, we use the definition of the density contrast in the slow-roll regime given by

$$\delta_k(t) \approx \frac{\delta\rho_k(t)}{\mathscr{V}_0} = \frac{\dot{\phi}(t)\,\dot{\sigma}_k(t)}{\mathscr{V}_0}\,.$$

Here, $\mathscr{V}_0$ represents the scalar-field potential evaluated at the background solution $\phi(t)$, and $\sigma_k(t)$ denotes the classical quantity related to the quantum-mechanical variable $f_k(t)$. We implement this relation by taking the expectation value of $f_k$ with respect to a Gaussian state:

$$\sigma_k^2(t) := \left\langle \psi_k | f_k^2 | \psi_k \right\rangle = \sqrt{\frac{\Re\mathfrak{e}\,\Omega_k}{\pi}} \int_{-\infty}^{\infty} f_k^2\, \mathrm{e}^{-\frac{1}{2}\left[\Omega_k^*(t)+\Omega_k(t)\right] f_k^2}\,\mathrm{d}f_k = \frac{1}{2\,\Re\mathfrak{e}\,\Omega_k(t)}\,.$$

The density contrast is then evaluated at the time $t_{\text{enter}}$, when the corresponding mode re-enters the Hubble radius during the radiation-dominated phase. By using a standard relation,

$$\delta_k(t_{\text{enter}}) = \frac{4}{3}\,\frac{\mathscr{V}_0}{\dot{\phi}^2}\,\delta_k(t_{\text{exit}}) = \frac{4}{3}\,\frac{\dot{\sigma}_k(t)}{\dot{\phi}(t)}\bigg|_{t=t_{\text{exit}}},$$

we can relate $t_{\text{enter}}$ to the time $t_{\text{exit}}$, when the mode exits the Hubble radius during the inflationary phase.

We therefore evaluate $\dot{\sigma}_k^{(0)}(t)$ at the Hubble-scale crossing $t_{\text{exit}}$. Using $\xi(t_{\text{exit}}) = 2\pi$, we arrive at $\left|\dot{\sigma}_k^{(0)}(t)\right|_{t=t_{\text{exit}}} \propto H^2\,k^{-3/2}$. Since the power spectrum is defined as $\Delta^2_{(0)}(k) := 4\pi k^3\,|\delta_k(t_{\text{enter}})|^2 \propto H^4\left|\dot{\phi}(t)\right|_{t_{\text{exit}}}^{-2}$, we immediately see that we obtain a scale-invariant power spectrum, which is the standard result for the simplest models of inflation.

## 5 The Quantum-Gravitationally Corrected Power Spectrum

In order to calculate the quantum-gravitational correction to the power spectrum we determined above, we have to look for an approximate solution to equation (5), where we ignore the unitarity-violating term as mentioned. We assume that we can accommodate the correction by the following modified Gaussian ansatz

$$\psi_k^{(1)}(t, f_k) = \left(\mathcal{N}_k^{(0)}(t) + \frac{1}{m_{\mathrm{P}}^2}\,\mathcal{N}_k^{(1)}(t)\right)\exp\left[-\frac{1}{2}\left(\Omega_k^{(0)}(t) + \frac{1}{m_{\mathrm{P}}^2}\,\Omega_k^{(1)}(t)\right)f_k^2\right].$$

Inserting this ansatz into equation (5) leads to the following equation for the correction term $\Omega_k^{(1)}(t)$:

$$\dot{\Omega}_k^{(1)}(t) \approx -2\,\mathrm{i}\,\mathrm{e}^{-3\alpha}\,\Omega_k^{(0)}(t)\left(\Omega_k^{(1)}(t) - \frac{3}{4V(t)}\left[\left(\Omega_k^{(0)}(t)\right)^2 - W_k(t)\right]\right). \tag{9}$$

We assume that this correction vanishes for late times, $\Omega_k^{(1)}(t) \to 0$ as $t \to \infty$, and can then solve equation (9) by the method of variation of constants, which reduces the problem to a numerical integration.

The relevant quantum-gravitationally corrected quantity for determining the power spectrum is given by

$$\left|\dot{\sigma}_k^{(1)}(t)\right| = \left|\frac{1}{\sqrt{2}}\,\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\Re\mathfrak{e}\,\Omega_k^{(0)}(\xi) + \frac{1}{m_{\mathrm{P}}^2}\,\Re\mathfrak{e}\,\Omega_k^{(1)}(\xi)\right)^{-\frac{1}{2}}\right]\right| \tag{10}$$

and we can incorporate the quantum-gravitational correction into a correction term $C_k$ relating the uncorrected quantity $\dot{\sigma}_k^{(0)}$ to the corrected one $\dot{\sigma}_k^{(1)}$ in the following way $\left|\dot{\sigma}_k^{(1)}\right|_{t_{\mathrm{exit}}} \simeq |C_k|\left|\dot{\sigma}_k^{(0)}\right|_{t_{\mathrm{exit}}}$.
The correction term can then be numerically calculated

$$C_k := \left(1 - 43.56\,\frac{1}{k^3}\,\frac{H^2}{m_{\mathrm{P}}^2}\right)^{-\frac{3}{2}}\left(1 - 189.18\,\frac{1}{k^3}\,\frac{H^2}{m_{\mathrm{P}}^2}\right), \tag{11}$$

which allows us to immediately determine the quantum-gravitationally corrected power spectrum $\Delta^2_{(1)}(k) = \Delta^2_{(0)}(k)\,C_k^2$. Performing a Taylor expansion of $C_k$ with respect to $(H/m_{\mathrm{P}})^2$ leads to

$$\Delta^2_{(1)}(k) \simeq \Delta^2_{(0)}(k)\left[1 - 123.83\,\frac{1}{k^3}\,\frac{H^2}{m_{\mathrm{P}}^2} + \frac{1}{k^6}\,\mathcal{O}\!\left(\frac{H^4}{m_{\mathrm{P}}^4}\right)\right]^2. \tag{12}$$

We therefore see that the quantum-gravitational correction explicitly breaks the scale independence of the uncorrected power spectrum and leads to a *suppression of power at large scales* (small $k$). However, our approximation breaks down when the zero point is approached and one would have to take into account higher orders of $(H/m_{\mathrm{P}})^2$ to suitably describe this limit.

From equation (12), one also sees that the quantum-gravitational effect only becomes significant if the inflationary Hubble parameter $H$ approaches the Planck scale. From the observational bound of the scalar-to-tensor ratio we can deduce an

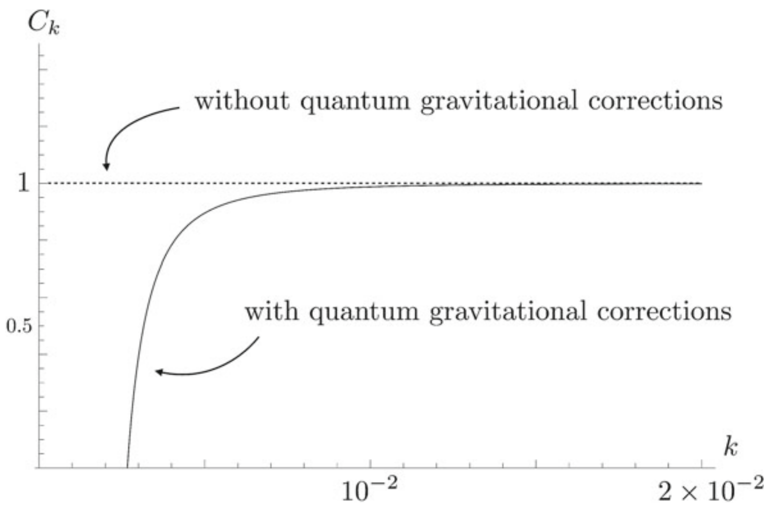


**Fig. 1** The function $C_k$ for $H = 10^{14}$ GeV, from [3]

upper bound on $H$, $H \lesssim 10^{-5} m_{\mathrm{P}} \sim 10^{14}$ GeV. Figure 1 shows the correction term $C_k$ for this value of $H$. The corrected power spectrum in this limiting case takes the following form:

$$\Delta^2_{(1)}(k) \simeq \Delta^2_{(0)}(k) \left[1 - 1.76 \times 10^{-9}\, \frac{1}{k^3} + \frac{1}{k^6}\, \mathcal{O}\left(10^{-15}\right)\right]^2 .$$

We thus see that even in this limiting case the quantum-gravitational effect is extremely small and if one adds that at large scales measurement accuracy is fundamentally limited by cosmic variance, we have to conclude that one will not be able to see this effect even with future, more precise measurements of the CMB anisotropies by satellite missions like PLANCK. A more elaborate discussion of the observable bounds from this calculation can be found in [6].

However, one can use our analysis to derive an upper limit on the Hubble parameter independently of the observational bound based on the tensor-to-scalar ratio. Given the fact that one has not yet unambiguously observed an effect as derived here in the CMB anisotropy spectrum, one can assume for a rough estimate that $C_k^2$ has to be not less than 0.95 for $k \sim 1$ since one has observed that the power spectrum deviates by less than 5 % from a scale-invariant spectrum [7]. In order to fulfill this condition, one finds that $H \lesssim 1.4 \times 10^{-2}\, m_{\mathrm{P}} \sim 4 \times 10^{17}$ GeV, which is, however, weaker than the bound from the tensor-to-scalar ratio.

Other approaches to quantum gravity also lead to effects in the CMB anisotropy spectrum. While non-commutative geometry and string theory give a similar suppression of power on the largest scales [8–10], loop quantum cosmology predicts an enhancement [11, 12].

## 6 Conclusion

We have seen that the Wheeler–DeWitt quantization of a model of an inflationary universe with scalar field perturbations modifies the power spectrum of these perturbations. While the suppression of power at large scales is not observable due to cosmic variance, we can derive an upper bound on the Hubble parameter during inflation, albeit weak. The comparison with other approaches to quantum gravity showed that loop quantum cosmology leads to a qualitatively opposite effect. This shows that looking for quantum-gravitational imprints in the cosmic microwave background could help us to discriminate between different approaches to quantum gravity.

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