Plane Gravitational Waves and Flat Space in Loop Quantum Gravity

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Abstract Classically a system of arbitrary plane gravitational waves propagating in the same or opposite directions can be restricted by first-class constraints to unidirectional waves, which travel without dispersion on a flat background. The unidirectionality constraints are formulated as well-defined Loop Quantum Gravity operators, together with criteria for an anomaly-free implantation, which is crucial for the occurrence or non-occurrence of dispersion, and more generally, of local Lorentz invariance violations due to (loop) quantum effects. By a set of further first-class constraints of the same kind we construct a quantum model of a no-wave state, i.e. of empty space.

1 Introduction

The motivation behind this contribution is the search for quantum effects of gravity in the form of dispersion of pure, unidirectional gravitational waves. The existence or non-existence of gravitational wave dispersion, derived for a solvable system from first loop quantum gravity (LQG) principles is an important criterion in the issue of Lorentz invariance at the Planck scale in quantum gravity.

Our approach consists in a symmetry reduction to 1+1 dimensions on the classical level, taken over from [1, 2], vacuum solutions in this model represent plane gravitational waves moving back and forth in one direction. Further reduction to unidirectional waves is achieved by a set of first-class constraints, derived from the Killing equations that describe the special symmetry of space-time with unidi-

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rectional waves [3]. These constraints are to be imposed on the quantum states of one-dimensional, but possibly colliding and interacting waves, and the question is whether or not these constraints can single out waves propagating uniformly at the speed of light.

As a by-product, by imposing one more set of Killing constraints, one can model flat space. A successful quantum solution will show how LQG can predict gravitational fluctuations of Minkowski geometry.

2 Classical Polarized Two-Way Waves in Ashtekar Variables

We are considering plane gravitational waves propagating in the positive and negative z direction, the system is homogeneous in the x and the y directions, all metric components depend only on z and t.

2.1 Variables

The metric is formulated in terms of adapted densitized triad variables with the nonzero components

$$\mathcal{E} = E^z{}_3 \tag{1}$$

along the inhomogeneous direction, orthogonal to the components in the (x, y) plane

$$\begin{aligned} E^x{}_1 &= E^x \cos\eta, \quad E^x{}_2 = E^x \sin\eta, \\ E^y{}_1 &= -E^y \sin\eta, \; E^y{}_2 = E^y \cos\eta. \end{aligned} \tag{2}$$

The mutual orthogonality of these two triad vectors means that we are dealing with polarized waves. In terms of these variables the spatial metric reads

$$\mathrm{d}\sigma^2 = \mathcal{E}\frac{E^y}{E^x}\mathrm{d}x^2 + \mathcal{E}\frac{E^x}{E^y}\mathrm{d}y^2 + \frac{E^x E^y}{\mathcal{E}}\mathrm{d}z^2. \tag{3}$$

The canonically conjugate variables are the connection components $\mathcal{A}$, K_x, K_y, and P with the equal-time Poisson brackets

$$\begin{aligned} &\{K_a(z), E^b(z')\} = \kappa\, \delta^b_a\, \delta(z - z'), \quad a, b = x, y, \\ &\{\mathcal{A}(z), \mathcal{E}(z')\} = \{P(z), \eta(z')\} = \kappa\, \delta(z - z'). \end{aligned} \tag{4}$$

κ is the gravitational constant. The symmetry-reduced model has four phase-space degrees of freedom.

2.2 The Constraints

The standard constraints of canonical general relativity, adapted to the above model, are the Gauß constraint

$$G = \frac{1}{\kappa\gamma}\,(\mathscr{E}' + P), \tag{5}$$

which generates rotations in the (x, y) plane, the diffeomorphism constraint

$$C = \frac{1}{\kappa}\left[K_x' E^x + K_y' E^y - \mathscr{E}'\mathscr{A} + \frac{\eta'}{\gamma}\,P\right], \tag{6}$$

and the Hamiltonian constraint

$$\begin{aligned} H = &-\frac{1}{\kappa\sqrt{\mathscr{E}E^xE^y}}\left[E^xK_xE^yK_y + (E^xK_x + E^yK_y)\mathscr{E}\left(\mathscr{A} + \frac{\eta'}{\gamma}\right) - \frac{1}{4}\mathscr{E}'^2\right. \\ &\left. -\mathscr{E}\mathscr{E}'' - \frac{1}{4}\mathscr{E}^2\left[\left(\ln\frac{E^y}{E^x}\right)'\right]^2\right] - \frac{\kappa}{4\sqrt{\mathscr{E}E^xE^y}}\,G^2 - \gamma\left(\sqrt{\frac{\mathscr{E}}{E^xE^y}}\,G\right)'. \end{aligned} \tag{7}$$

A prime denotes the derivative with respect to z, γ is the Barbero-Immirzi parameter. H is partially expressed by the Gauß constraint G. These first-class constraints reduce the number of degrees of freedom to one, the correct number for polarized plane waves.

3 Reduction to Unidirectional Waves

Unidirectional waves are characterized by the existence of a null Killing vector field in the direction of propagation. This corresponds to a dependence of the metric functions either on $t - z$ or on $t + z$. To formulate such fields, we add an orthogonal timelike direction and construct a space-time metric with lapse function $N(t, z)$.

On a manifold with this metric we assume a null Killing vector field k^μ with $\nabla_{(\mu}k_{\nu)} = 0$. Two of the Killing equations give rise to nontrivial conditions on the phase space variables, for propagation in the positive z direction they are

$$U_x := E^xK_x - \frac{1}{2}\mathscr{E}' - \frac{1}{2}\mathscr{E}\left(\frac{E^{y\prime}}{E^y} - \frac{E^{x\prime}}{E^x}\right) = 0, \tag{8}$$

$$U_y := E^yK_y - \frac{1}{2}\mathscr{E}' + \frac{1}{2}\mathscr{E}\left(\frac{E^{y\prime}}{E^y} - \frac{E^{x\prime}}{E^x}\right) = 0. \tag{9}$$

In addition to the standard constraints there can be at most one more first-class constraint which, of course, cannot be a gauge generator, because an associated gauge condition would reduce the number of degrees of freedom to zero.

To extract from U_x and U_y a relation that can be added as a set of first-class constraints to the standard constraints, we take the linear combinations

$$U_+ := U_x + U_y \qquad \text{and} \qquad U_- := U_x - U_y, \tag{10}$$

explicitly

$$U_+ = E^x K_x + E^y K_y - \mathscr{E}', \tag{11}$$

$$U_- = E^x K_x - E^y K_y - \mathscr{E}\left(\ln \frac{E^y}{E^x}\right)'. \tag{12}$$

The Poisson brackets of these expressions, smeared out by test functions,

$$U_a[f] := \int \mathrm{d}z\, f(z) U_a(z), \tag{13}$$

are

$$\{U_+[f], U_+[g]\} = \{U_+[f], U_-[g]\} = 0 \tag{14}$$

and

$$\{U_-[f], U_-[g]\} = 2 \int \mathrm{d}z (f'g - fg')\, \mathscr{E}. \tag{15}$$

The function U_+ weakly Poisson-commutes also with G, C, and H:

$$\{U_+[f], G[g]\} = 0, \qquad \{U_+[f], C[g]\} = -\frac{1}{\kappa}\, U_+[f'g] \approx 0, \tag{16}$$

$$\{U_+[f], H[g]\} = \frac{1}{\kappa}\, U_+\left[\sqrt{\frac{\mathscr{E}}{E^x E^y}}\, f'g\right] - H[fg] \approx 0. \tag{17}$$

This qualifies $U_+(z)$ as a set of first-class constraints that have to be added to G, C, and H, when we want to restrict counter-current waves to unidirectional ones at the classical level.

Not being a gauge generator, but a restriction of the number of the physical degrees of freedom, the new constraint reduces their number to one half, i.e. to one phase space function. This corresponds to the original formulation [4], which contains two functions, the so-called "wave factor" and the "background factor", connected by one non-trivial Einstein equation.

4 Preparation for Quantization

After the formulation of unidirectional waves as a classical system with first-class constraints we start the Dirac quantization programme, which distinguishes physical states as those that are annihilated by the constraint operators.

This gives rise to two kinds of problems: The formulation of the constraints as well-defined operators on a suitable Hilbert space of unconstrained states, and the problem of non-trivial structure functions in the constraint algebra. For the standard constraints these problems are solved in general LQG [5], for U_+ they will be dealt with in the following.

Both parts of U_+, $E^x K_x + E^y K_y$ as well as $\mathscr{E}'$ are scalar densities, which can be naturally integrated along z in order to construct an operator. The integral over some interval $\mathscr{I}$ is

$$U_+[\mathscr{I}] = \int_{\mathscr{I}} \mathrm{d}z\, (E^x K_x + E^y K_y) - \mathscr{E}_+ + \mathscr{E}_-, \tag{18}$$

where $\mathscr{E}_\pm$ are the values at the endpoints of $\mathscr{I}$. $\mathscr{E}$ has a meaningful operator equivalent in the adapted LQG framework [2]. In analogy to full LQG the integral can be obtained as the Poisson bracket

$$\int_{\mathscr{I}} \mathrm{d}z (E^x K_x + E^y K_y) = 2 \left\{ \int_{\mathscr{I}} \mathrm{d}z\, \frac{E^x K_x E^y K_y}{\sqrt{\mathscr{E} E^x E^y}}, \int_{\mathscr{I}} \mathrm{d}z'\, \sqrt{\mathscr{E} E^x E^y} \right\}. \tag{19}$$

The first expression is part of the kinetic Hamiltonian constraint, denoted by H_1 in the following, the second part is the volume of a slice of space, constructed from a fiducial area in the (x, y) plane and the interval $\mathscr{I}$ in the z direction. Both have an operator interpretation on one-dimensional spin network functions [2].

According to its factor-ordering, the operator formulation of the structure function in (17) raises potentially an anomaly problem. When the factor ordering is chosen analogous to that of the Hamilton constraint operator—connection components to the left of triad components (see [5][1]), then the operator constructed from

$$U_+ \sqrt{\frac{\mathscr{E}}{E^x E^y}} = \frac{(K_x E^x + K_y E^y - \mathscr{E}')\, \mathscr{E}}{\sqrt{\mathscr{E} E^x E^y}} \tag{20}$$

does not obviously annihilate solutions of the gauge constraints and U_+ and its action on them must be examined.

As in the case of U_+, the first step is a consistent operator formulation: The first part of (20) can be written as a Poisson bracket of the second part H_2 of the Hamiltonian constraint (7) with test function 1 and $\mathscr{E}(z)$,

$$-\frac{1}{\kappa} \left(\frac{K_x E^x + K_y E^y}{\sqrt{\mathscr{E} E^x E^y}}\, \mathscr{E} \right) = \{H_2[1], \mathscr{E}(z)\}, \tag{21}$$

[1] A different factor ordering is presented in [6].

so we can write

$$U_+\sqrt{\frac{\mathscr{E}}{E^x E^y}} = \kappa\{\mathscr{E}(z), H_2[1]\} - \frac{\mathscr{E}\mathscr{E}'}{\sqrt{\mathscr{E}E^x E^y}}. \tag{22}$$

Both expressions have operator equivalents, the second term is part of H.

5 Quantum States and the Action of Operators

In LQG a suitable basis of kinematical quantum states is provided by spin network functions, based on three-dimensional graphs. In the present case we have one-dimensional graphs with $U(1)$-holonomies $h_e^{(k)}(\mathscr{A}) = \exp\left(i\frac{k}{2}\int_e \mathscr{A}\right)$, $k \in Z$ associated to its edges. Holonomies along curves in the (x, y) plane are shrunk to "point holonomies" at the vertices v: $h_v^{(\mu)}(X) = \exp\left(i\frac{\mu}{2}X(v)\right)$, $h_v^{(\rho)}(Y) = \exp\left(i\frac{\rho}{2}Y(v)\right)$, and $h_v^{(\lambda)}(\eta) = \exp(i\lambda\eta(v))$. $X = \gamma K_x$, $Y = \gamma K_y$. η is an angular variable, its holonomy has values in $U(1)$, $\mu, \rho \in R$, their holonomies lie in the Bohr compactification of the reals, see [5].

Connection components act in the form of holonomy operators, which add one of the above holonomies to a given state. States, denoted by $|s\rangle$, depend on the graph G and the labels k, μ, ρ, and λ. Triad components and the conjugate variable to η act as flux operators in the following way

$$\hat{\mathscr{E}}(z)\,|s\rangle = \frac{\gamma\ell_{\mathrm{P}}^2}{2}\frac{k_+(z)+k_-(z)}{2}\,|s\rangle, \qquad \int_{\mathscr{I}}\hat{P}\,|s\rangle = \gamma\ell_{\mathrm{P}}^2\sum_v \lambda_v\,|s\rangle$$
$$\int_{\mathscr{I}}\hat{E^x}\,|s\rangle = \frac{\gamma\ell_{\mathrm{P}}^2}{2}\sum_v \mu_v\,|s\rangle, \qquad \int_{\mathscr{I}}\hat{E^y}\,|s\rangle = \frac{\gamma\ell_{\mathrm{P}}^2}{2}\sum_v \nu_v\,|s\rangle. \tag{23}$$

$\mathscr{E}$ is a scalar quantity, the other ones are scalar densities and have to be integrated over an interval $\mathscr{I}$ to give raise to an operator, $k_\pm(z)$ are the representation labels of the edge holonomies left and right to z, ℓ_{P} is the Planck length, the sum is taken over all vertices of G in the interval $\mathscr{I}$.

The Gauß constraint relates the labels k and λ,

$$\lambda_v = -(k_+(v) - k_-(v))/2, \tag{24}$$

so gauge-invariant states are of the form

$$|s\rangle = \prod_e \exp\left[\frac{ik_e}{2}\int_e(\mathscr{A}(z) - \eta'(z))\right]\prod_v\left(\exp\left[\frac{i\mu_v}{2}X(v)\right]\exp\left[\frac{i\rho_v}{2}Y(\rho)\right]\right).$$

In this formula $\int_e \eta' = \eta_+(e) - \eta_-(e)$ was used, where $\eta_\pm$ are the values of η at the endpoints of the edge e.

6 Flat Space

In the case of unidirectional plane waves a null Killing vector field prevents waves in the opposite direction. A second null Killing field in the opposite direction characterizes a no-wave state, namely Minkowski space. The corresponding constraint is

$$\bar{U_+} = K_x E^x + K_y E^y + \mathscr{E}' = 0. \tag{25}$$

Classically this one additional first-class constraint reduces the number of degrees of freedom to zero, i.e. to one state. From U_+ and $\bar{U}_+$ together follow the constraints

$$\mathscr{E}' = 0 \qquad \text{and} \qquad K_x E^x + K_y E^y = 0. \tag{26}$$

The operator version of the latter expression is given by (19), the former one is just the derivative of a flux operator.

In the following we consider these constraints separately, which is easier than in the combination U_+. To qualify as a model for flat space, a solution to them must also be a solution to the Hamiltonian constraint and to the Poisson brackets with the Hamiltonian constraint.

$$\{\mathscr{E}'[f], H[g]\} = \int \mathrm{d}z\, f'(z) g(z) \left(\frac{(K_x E^x + K_y E^y)\mathscr{E}}{\kappa \sqrt{\mathscr{E} E^x E^y}} \right)(z), \tag{27}$$

$$\{(K_x E^x + K_y E^y)[f], H[g]\} = H[fg]. \tag{28}$$

From (27) a quantum anomaly may arise. An operator version is already given in (21).

The first one of the constraints (26) is solved by states with the same label k for all edges.

The second one is formulated by replacing the right-hand side of (19) by the commutator of $\hat{H}_1$ with the volume operator. The action of $\hat{H}_1$ on a state $|s\rangle$ of the form (5) with $k_+ = k_- = k$ is given by

$$\hat{H}_1|s\rangle = \frac{\ell_{\mathrm{P}} \gamma^{-\frac{3}{2}}}{2\mu_0 \rho_0} \sum_v \sqrt{|\mu_v||\rho_v|} \left(\sqrt{|2k+1|} - \sqrt{|2k-1|} \right) \\ \times \sin(\mu_0 X) \sin(\rho_0 Y)|s\rangle, \tag{29}$$

where μ_0 and ρ_0 are arbitrarily chosen, fixed values. The action of the volume operator is

$$\hat{V}|s\rangle = \frac{\gamma^{\frac{3}{3}}\ell_{\rm Pl}^3}{2}\sum_v (|\mu_v||\rho_v||k|)^{\frac{1}{2}}\,. \tag{30}$$

Now assume the state function at a certain vertex to be given by the superposition

$$|s\rangle_v = \sum_{m,n} a_{mn}|k, m, n\rangle, \tag{31}$$

where $m = \frac{\mu_v}{\mu_0}$ and $n = \frac{\rho_v}{\rho_0}$. The action of the commutator $[\hat{V}, \hat{H}_1]$ on such a function set equal to zero yields the following difference equation for the coefficients a_{mn},

$$\begin{aligned}
&\sqrt{|m-2||n-2|}\left(\sqrt{|m||n|} - \sqrt{|m-2||n-2|}\right) a_{m-2,n-2} - \\
&\sqrt{|m+2||n-2|}\left(\sqrt{|m||n|} - \sqrt{|m+2||n-2|}\right) a_{m+2,n-2} - \\
&\sqrt{|m-2||n+2|}\left(\sqrt{|m||n|} - \sqrt{|m-2||n+2|}\right) a_{m-2,n+2} + \\
&\sqrt{|m+2||n+2|}\left(\sqrt{|m||n|} - \sqrt{|m+2||n+2|}\right) a_{m+2,n+2} = 0,
\end{aligned} \tag{32}$$

where the k-dependence has dropped out.

The equation resulting from the structure function (27) contains H_2, whose operator version is more ambiguous, compare [1] and [2]. Anyway, after deciding for one version, the action of the structure function operator on solutions of (26) may already lead to an ambiguity, before the Hamiltonian constraint has to be solved. This would indicate gravitational fluctuations of the Minkowski vacuum. Work is ongoing.

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