Geometric Inequalities for Black Holes

Sergio Dain

Abstract It is well known that the parameters that characterize the Kerr black hole satisfy several important geometric inequalities. Remarkably enough, some of these inequalities also hold for dynamical black holes. This kind of inequalities play an important role in the characterization of the gravitational collapse, they are closely related with the cosmic censorship conjecture. We briefly review recent results in this subject.

The Kerr black hole is characterized by two parameters, the mass m and the angular momentum J. These parameters satisfy the following inequality

$$m \geq \sqrt{|J|}. \tag{1}$$

It is important to emphasize that the Kerr metric is a solution of Einstein vacuum equation for any choice of m and J, but it only describes a black hole if (1) holds. From Newtonian considerations, we can interpret this inequality as follows (see [1]): in a collapse the gravitational attraction ($\approx m^2/r^2$) at the horizon ($r \approx m$) dominates over the centrifugal repulsive forces ($\approx J^2/mr^3$).

Black holes are very simple macroscopic objects. The black hole uniqueness theorem ensures that Kerr is the only stationary black hole in vacuum, and hence stationary black holes are characterized by the two parameters, m and J. However, black holes are not stationary in general. Astrophysical phenomena like the formation of a black hole by gravitational collapse or a binary black hole collision are highly dynamical. For such systems, the black hole cannot be characterized by few parameters as in the stationary case.

S. Dain (✉)
Facultad de Matemática, Astronomía y Física, FaMAF, Universidad Nacional de Córdoba,
Instituto de Física Enrique Gaviola, IFEG, CONICET, Ciudad Universitaria,
(5000) Córdoba, Argentina
e-mail: dain@famaf.unc.edu.ar

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Remarkably, inequality (1) extends (under appropriate assumptions) to fully dynamical axially symmetric black holes. Moreover, this inequality is deeply connected with properties of the global evolution of Einstein equations, in particular with the cosmic censorship conjecture.

Inequality (1) is a global inequality for two reasons. First, it involves the total mass m of the spacetime. Second it assumes global restrictions on the initial data: axial symmetry and vacuum.

The area A and the angular momentum J in axial symmetry are quasi-local quantities. Namely they carry information on a bounded region of the spacetime. In contrast with a local quantity like a tensor field which depends on a point of the spacetime or a global quantities (like the total mass) which depends on the whole initial conditions. The area of the horizon of the Kerr black hole satisfies the following inequality

$$A \geq 8\pi |J|. \tag{2}$$

A natural question is whether dynamical black holes satisfy purely quasi-local inequalities. The relevance of this kind of inequalities is that they provide a much finer control on the dynamics of black holes than the global versions. It turns out that inequality (2) also holds for dynamical, axially symmetric black holes.

The equality in (1) and (2) is achieved only for the extreme Kerr black hole. In the dynamical regime, this rigidity also holds. Extreme black holes lie in the boundary between black holes and naked singularities. They play an important role as minimizer in the corresponding energy in the proof of these inequalities.

Both inequalities (1) and (2) can be extended to include an electric charge. The quasi-local inequality (2) is also valid when generic matter sources are present at the horizon of black holes. For more details see the recent review article [2] on the whole subject.

References

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