Loop Quantum Cosmology: Anisotropy and Singularity Resolution

Alejandro Corichi, Asieh Karami and Edison Montoya

Abstract In this contribution we consider the issue of singularity resolution within loop quantum cosmology (LQC) for different homogeneous models. We present results of numerical evolutions of effective equations for both isotropic as well as anisotropic cosmologies, with and without spatial curvature. To address the issue of singularity resolution we examine geometrical and curvature invariants that yield information about the spacetime geometry. We discuss generic behavior found for a variety of initial conditions.

1 Introduction

In general relativity (GR) the singularity theorems of Hawking, Penrose and Geroch tell us that, under reasonable assumptions, singularities are generic. A spacetime is said to be singular if it is not geodesically complete, which may happen when some geometrical curvature invariants diverge. The expectation is that, by quantizing the gravitational degrees of freedom, namely, with a complete theory unifying gravity and the quantum, the singularities shall be resolved. Loop quantization (as in Loop Quantum Gravity) of the homogeneous, isotropic and flat Friedman-Robertson-

A. Corichi (✉) · A. Karami · E. Montoya
Centro de Ciencias Matemáticas, Universidad Nacional Autónoma de México,
UNAM-Campus Morelia, Morelia, Mexico
e-mail: corichi@matmor.unam.mx

A. Karami · E. Montoya
Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo,
Morelia, Michoacán, Mexico
e-mail: karami@ifm.umich.mx

E. Montoya
e-mail: edison@ifm.umich.mx

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Walker (FRW) cosmology coupled to a massless scalar field ϕ, can be exactly solved [1]. For that model it was shown that:

- The matter density operator $\hat{\rho}$ has an absolute upper bound and the expansion θ is also bounded. One can conclude that curvature scalars do not diverge. This is a signal that a singularity is not present.
- All states undergo a bounce and with this, the big bang is replaced by a *big bounce*.
- The GR dynamics is recovered as we go away from the Planck scale, this means that we are recovering the original theory that we want.
- Dynamics of semiclassical states are well captured by an effective theory that retains information about the loop quantum geometry.
- With all these, one can conclude that the singularities are resolved: the geodesics are inextendible, and are well defined on the other side of the would be big bang.

The fact that the *effective theory* provides an accurate description of the dynamics at the Planck scale is strongly used to explore the anisotropic models. The effective theory is obtained from the quantum Hamiltonian operator by taking expectation values on appropriately defined states. The thus obtained effective Hamiltonian then generates the dynamics on a classical phase space. The solutions to the effective theory were shown in [2] to accurately describe the evolution of the expectation value of the observables in the quantum theory when they are considered on semiclassical states. Those results were extended to open, closed and flat FRW models with and without cosmological constant (see [3] for a review). Loop quantum cosmology (LQC) has been extended to the simplest anisotropic cosmological models, namely Bianchi I, II and IX [4–6]. But in none of these cases, the quantum theory has been solved, even numerically. Then, in order to study these models at semiclassical level, one generally assumes that the effective theory reproduces the solutions to the quantum theory when semiclassical states are considered. This is our working hypothesis, which is well justified by the results in the isotropic cases. It would be interesting to know whether the evolution of the semiclassical states reproduces the solutions which we get from the effective theory. From this point of view, the study of the effective theory can be seen as the first step in this direction.

The new issues to consider in the anisotropic models are: is the bounce generic? We now have anisotropy/Weyl curvature, how does it behave near the singularity/bounce? Can we have different kind of bounce, say, dominated by shear σ? Are the geometric scalars absolutely bounded? The goal of this contribution is to answer these questions using the effective theory for Bianchi I which has anisotropies, Bianchi II that has anisotropies and spatial curvature and Bianchi IX which has all the features of Bianchi I, II and is furthermore, spatially compact. Even more, the Bianchi IX model has a non trivial classical limit, in the sense that, Bianchi IX is chaotic in the classical theory and behaves like Bianchi I with Bianchi II transitions as one approaches the singularity.

2 Preliminaries

In this section we briefly review the quantization of some cosmological models which include $k = 0$ and $k = 1$ FRW and Bianchi I, II and IX models by using loop quantum gravity methods. Let us consider the spacetime as $M = \Sigma \times \mathbb{R}$ where Σ is a spatial 3-manifold which can be identified by the symmetry group of the chosen model and is endowed with a fiducial metric ${}^{o}q_{ab}$ and associated fixed fiducial basis of 1-forms ${}^{o}\omega_a^i$ and vectors ${}^{o}e_i^a$. If Σ is non-compact then we fix a fiducial cell, $\mathscr{V}$, adapted to the fiducial triads with finite volume V_o. In GR, the gravitational phase space consists of pairs (A_a^i, E_i^a) on Σ where A_a^i is a SU(2) connection and E_i^a is a densitized triad of weight 1. Since all of the models in which we are interested are homogeneous and, if we restrict ourselves to diagonal metrics, one can fix the gauge in such a way that $A_a^i = \frac{c^i}{V_o^{1/3}}{}^{o}\omega_a^i$ and $E_i^a = \frac{p_i}{V_o^{2/3}}\sqrt{{}^{o}q}\ {}^{o}e_i^a$, where p_i in terms of the scale factors a_i are $|p_i| = V_o^{2/3} a_j a_k$. Note that in isotropic cases, each of phase space variables has only one independent component. The Poisson brackets can be expressed as $\{c^i, p_j\} = 8\pi G\gamma\delta^i_j$ and for isotropic models, the Poisson bracket is $\{c, p\} = 8\pi G\gamma/3$ where γ is Barbero-Immirzi parameter. With this choice of variables and gauge fixing, the Gauss and diffeomorphism constraints are satisfied and the only constraint is the Hamiltonian constraint

$$\mathscr{C}_H = \int_{\mathscr{V}} N\left[-\frac{\varepsilon^{ij}_{\ \ k}E_i^a E_j^b}{16\pi G\gamma^2\sqrt{|q|}}\left(F_{ab}^k - (1+\gamma^2)\Omega_{ab}^k\right) + \mathscr{H}_{\text{matter}}\right]\mathrm{d}^3x, \qquad (1)$$

where N is the lapse function, $\mathscr{H}_{matter} = \rho V$ and Ω_{ab} is the curvature of spin connection Γ_a^i compatible with the triads.

To construct the quantum kinematics, we have to select a set of elementary observables such that their associated operators are unambiguous. In loop quantum gravity they are the holonomies h_e defined by the connection A_a^i along edges e and the fluxes of the densitized triad E_i^a across surfaces. For our homogeneous models we choose holonomies and p_i. To have the corresponding constraint operator, one needs to express it in terms of the chosen phase space functions h_e and p_i. The first term, $\varepsilon^{ij}_{\ \ k}E_i^a E_j^b/\sqrt{|q|}$, as in loop quantum gravity, can be treated by using Thiemann's strategy [7].

$$\varepsilon_{ijk}\frac{E^{ai}E^{bj}}{\sqrt{|q|}} = \sum_i \frac{1}{2\pi\gamma G\mu}\varepsilon^{abc}\omega_c^i \mathrm{Tr}(h_i^{(\mu)}\{h^{(\mu)-1}, V\}\tau_k), \qquad (2)$$

where $h_i^{(\mu)}$ is the holonomy along the edge parallel to ith vector basis with length μ and V is the volume, which is equal to $\sqrt{|p_1 p_2 p_3|}$. Note that μ is arbitrary. Now to define an operator related to the first term, we can use the right hand side of (2) and replace Poisson brackets with commutation relations. To find an operator related to the curvature F_{ab}^k, for isotropic models and Bianchi I, one can consider a square $\Box_{ij}$

in the i–j plane which is spanned by two of the fiducial triads (for closed isotropic model since triads do not commute, to define this plane we use a triad and a right invariant vector ${}^{o}\xi^{a}_{i}$), each of its sides has length $\bar{\mu}_i$. Therefore, F^k_{ab} is given by

$$F^k_{ab} = 2 \lim_{Ar\square\to 0} \varepsilon_{ij}{}^{k}\mathrm{Tr}\left(\frac{h^{\bar{\mu}}_{\square_{ij}} - \mathbb{I}}{\bar{\mu}_i\bar{\mu}_j}\tau^k\right){}^{o}\omega^{i}_{a}\,{}^{o}\omega^{j}_{b}. \tag{3}$$

Since in loop quantum gravity, the area operator does not have a zero eigenvalue, one can take the limit of (3) when the area is equal to the smallest eigenvalue of area operator, $\lambda^2 = 4\sqrt{3}\pi\gamma l_p^2$, instead of zero. Then, $V_o^{1/3}\bar{\mu}_i a_i = \lambda$ where a_i is the scale factor. For Bianchi II and IX, we cannot use this method because the resulting operator is not almost periodic, therefore we express connection A^i_a in terms of holonomies and then use the standard definition of curvature F^k_{ab}. The operators corresponding to the connection are given by

$$\hat{c}_i = \frac{\widehat{\sin\bar{\mu}_i c_i}}{\bar{\mu}_i}, \text{ where } \bar{\mu}_i = \lambda\sqrt{\left|\frac{p_i}{p_j p_k}\right|}\ ,\quad i \neq j \neq k. \tag{4}$$

Note that using this quantization method for flat FRW and Bianchi I models, one has the same result as the direct quantization of curvature F^k_{ab}, but for closed FRW it leads to a different quantum theory which is more compatible with the isotropic limit of Bianchi IX. We call the first method of quantization *curvature based quantization* and the second one *connection based*. In Bianchi II and Bianchi IX models the terms related to the curvatures, F^k_{ab} and Ω^k_{ab}, contain some negative powers of p_i which are not well defined operators. To solve this problem we use the same idea as Thiemann's strategy. Write

$$|p_i|^{(\ell-1)/2} = -\frac{\sqrt{|p_i|}}{4\pi G\gamma j(j+1)\tilde{\mu}_i\ell}\tau_i h_i^{(\tilde{\mu}_i)}\{h_i^{(\tilde{\mu}_i)-1}, |p_i|^{\ell/2}\}, \tag{5}$$

where $\tilde{\mu}_i$ is the length of a curve, $\ell \in (0, 1)$ and $j \in \frac{1}{2}\mathbb{N}$ is for the representation. Therefore, for these three different operators we have three different curve lengths $(\mu, \bar{\mu}, \tilde{\mu})$ where μ and $\tilde{\mu}$ can be some arbitrary functions of p_i, so for simplicity we can choose all of them to be equal to $\bar{\mu}$. On the other hand we have another free parameter in the definition of negative powers of p_i, where for simplicity we take $j = 1/2$, and since the largest negative power of p_i which appears in the constraint is $-1/4$ we will take $\ell = 1/2$ to have them directly from (5) and after that to express the other negative powers by them. The eigenvalues for the operator $\widehat{|p_i|^{-1/4}}$ are given by

$$J_i = \frac{h(V)}{V_c}\prod_{j\neq i} p_j^{1/4}\ , \text{ where } h(V) = \sqrt{V + V_c} - \sqrt{|V - V_c|},\ V_c = 2\pi\gamma\lambda\ell_p^2.$$

By using these results and choosing some factor ordering, we can construct the total constraint operator. Note that with a different choice of factor ordering we will have different operators but the main results will remain almost the same. By solving the constraint equation $\hat{\mathscr{C}}_H \cdot \Psi = 0$, we have the physical states in physical Hilbert space $\mathscr{H}_{\text{phys}}$. Then we need to identify the physical observables. To test singularity resolution we will study some geometric observables: expansion θ, shear σ^2, curvature scalars and also volume of the universe V and matter density ρ, as relational observables in terms of ϕ, a massless scalar field.

Since working with full quantum theories of the models is difficult and, as shown in [2] for some models, the behavior of the effective or semiclassical equations, which are classical equations with some quantum corrections, are good approximations to the numerical quantum evolutions even near the Planck scale, we will work with the effective equations.

3 Effective Theories

Isotropic Flat and Closed Models

In the FRW model with $k = 0$, the *effective* Hamiltonian is given by

$$\mathscr{H}_{\text{k=0}} = \frac{3}{8\pi G\gamma^2\lambda^2} V^2 \sin(\lambda\beta)^2 - \frac{p_\phi^2}{2} \approx 0, \tag{6}$$

where p_ϕ is the momentum of the field, V is the volume and β its conjugate variable. They are related to the c and p variables by the equations $V = p^{3/2}$, $\beta = c/\sqrt{p}$ and satisfy the Poisson bracket $\{\beta, V\} = 4\pi G\gamma$ and $\{\phi, p_\phi\} = 1$. It was shown [2] that the dynamics of semiclassical states are well captured by the effective Friedman equation $H^2 = \frac{8\pi G}{3}\rho\left(1 - \frac{\rho}{\rho_{\text{crit}}}\right)$, where $H = \dot{V}/3V$ is the Hubble parameter and $\rho = p_\phi^2/2V^2$ is the matter density. The GR dynamics is recovered as we go away from the Planck scale $\rho < \rho_{\text{crit}}/10$, with $\rho_{\text{crit}} = \frac{3}{8\pi G\gamma^2\lambda^2}$.

Now, for the isotropic closed model, as we discussed in previous section, there are two different quantum theories depending on the two different methods of quantization of the curvature F_{ab}^k. The Hamiltonians are,

$$\mathscr{H}^{(1)}_{\text{k=1}} = \frac{3V^2}{8\pi G\gamma^2\lambda^2}[\sin^2(\lambda\beta - D) - \sin^2 D + (1+\gamma^2)D^2] - \frac{p_\phi^2}{2} \approx 0, \tag{7}$$

$$\mathscr{H}^{(2)}_{\text{k=1}} = \frac{3V^2}{8\pi G\gamma^2\lambda^2}[\sin^2 \lambda\beta - 2D\sin\lambda\beta + (1+\gamma^2)D^2] - \frac{p_\phi^2}{2} \approx 0, \tag{8}$$

where $D = \lambda\vartheta/V^{-1/3}$ and $\vartheta = (2\pi^2)^{1/3}$. Since for both effective theories there are some geometric observables which are not absolutely bounded, we go further and use more corrections which come from the inverse triad term in the full theory, to see if the unboundedness of those observables is generic, or whether it improves by adding more corrections. Therefore the Hamiltonian constraints change to

$$\mathscr{H}^{(1)}_{\mathrm{k=1}} = \frac{3A(V)V}{8\pi G\gamma^2\lambda^2}[\sin^2(\lambda\beta - D) - \sin^2 D + (1+\gamma^2)D^2] - \frac{p_\phi^2}{2} \approx 0, \tag{9}$$

$$\mathscr{H}^{(2)}_{\mathrm{k=1}} = \frac{3A(V)V}{8\pi G\gamma^2\lambda^2}[\sin^2 \lambda\beta - 2D\sin\lambda\beta + (1+\gamma^2)D^2] - \frac{p_\phi^2}{2} \approx 0, \tag{10}$$

where $A(V) = \frac{1}{2V_c}(V + V_c - |V - V_c|)$ is a correction term which comes from the operator $\varepsilon_k^{ij} E_i^a E_j^b/\sqrt{|q|}$.

Bianchi I and II

The Hamiltonian for Bianchi I and II can be written in a single expression,

$$\begin{aligned}\mathscr{H}_{\mathrm{BII}} &= \frac{p_1p_2p_3}{8\pi G\gamma^2\lambda^2}\left[\sin\bar\mu_1 c_1 \sin\bar\mu_2 c_2 + \sin\bar\mu_2 c_2 \sin\bar\mu_3 c_3 + \sin\bar\mu_3 c_3 \sin\bar\mu_1 c_1\right] \\ &\quad + \frac{1}{8\pi G\gamma^2}\left[\frac{\alpha(p_2p_3)^{3/2}}{\lambda\sqrt{p_1}}\sin\bar\mu_1 c_1 - (1+\gamma^2)\left(\frac{\alpha p_2 p_3}{2p_1}\right)^2\right] - \frac{p_\phi^2}{2} \approx 0,\end{aligned}$$

where the parameter α allows us to distinguish between Bianchi I ($\alpha = 0$) and Bianchi II ($\alpha = 1$). This Hamiltonian together with the Poisson Brackets $\{c^i, p_j\} = 8\pi G\gamma\delta^i_j$ and $\{\phi, p_\phi\} = 1$ gives the effective equations of motion.

Bianchi IX

In the previous Hamiltonians we choose the lapse $N = V$. But now in Bianchi IX, we choose $N = 1$ to include more inverse triad corrections, then the effective Hamiltonian is given by

$$\begin{aligned}\mathscr{H}_{\mathrm{BIX}} &= -\frac{V^4A(V)h^6(V)}{8\pi G V_c^6\gamma^2\lambda^2}\Big(\sin\bar\mu_1 c_1 \sin\bar\mu_2 c_2 + \sin\bar\mu_1 c_1 \sin\bar\mu_3 c_3 \\ &\quad + \sin\bar\mu_2 c_2 \sin\bar\mu_3 c_3\Big) + \frac{\vartheta A(V)h^4(V)}{4\pi G V_c^4\gamma^2\lambda}\Big(p_1^2p_2^2\sin\bar\mu_3 c_3 + p_2^2p_3^2\sin\bar\mu_1 c_1 \\ &\quad + p_1^2p_3^2\sin\bar\mu_2 c_2\Big) - \frac{\vartheta^2(1+\gamma^2)A(V)h^4(V)}{8\pi G V_c^4\gamma^2}\Big(2V\left[p_1^2+p_2^2+p_3^2\right] \\ &\quad - \left[(p_1p_2)^4 + (p_1p_3)^4 + (p_2p_3)^4\right]\frac{h^6(V)}{V_c^6}\Big) + \frac{h^6(V)V^2}{2V_c^6}p_\phi^2 \approx 0.\end{aligned}$$

4 Results

Now we will compare the results of the effective theories for the isotropic FRW $k = 0$ and $k = 1$, diagonal Bianchi I, II and IX. All of them with a matter content a massless scalar field satisfying the Klein-Gordon equation. A good starting point to compare the results is to answer the questions that we asked in the introduction, later we will mention other important results:

- Is the bounce generic? Yes. All solutions have a bounce. In other words, singularities are resolved. In the closed FRW and the Bianchi IX model, there are infinite number of bounces and recollapses due to the compactness of the spatial manifold.
- How does anisotropy/Weyl curvature behave near the bounce? These quantities far from the bounce approach their classical values, but when they reach the region near the bounce they behave differently. In Bianchi I, they present only one maximum. In Bianchi II, they exhibit a richer behavior because now they can be zero at the bounce or near to it, and have more than one maximum (for the shear there are up to 4 maxima and for the scalar curvature up to 2 maxima [8]). For Bianchi IX, if we restrict the analysis to one of the infinite number of bounces, it can be shown that anisotropy and curvature behave in the same way as Bianchi I or II. The subject of current research is whether there are new behaviors [9].
- Can we have different kind of bounce, say, dominated by shear σ? Yes, but only in Bianchi II and IX. In Bianchi I the dynamical contribution from matter is always bigger than the one from the shear, even in the solution which reaches the maximal shear at the bounce [8].
- Are geometric scalars θ, σ and ρ absolutely bounded? In the flat isotropic model all the solutions to the effective equations have a maximal density equal to the critical density, and a maximal expansion ($\theta^2_{\text{max}} = 6\pi G\rho_{\text{crit}} = 3/(2\gamma\lambda)$) when $\rho = \rho_{\text{crit}}/2$. For FRW $k = 1$ model, every solution has its maximum density but in general the density is not absolutely bounded. In the effective theory which comes from connection based quantization, expansion can tend to infinity. For the other case, expansion has the same bound as the flat FRW model. However, by adding some more corrections from inverse triad term, one can show that actually in both effective theories the density and the expansion have finite values. For Bianchi I, in all the solutions ρ and θ are upperly bounded by their values in the isotropic case and σ is bounded by $\sigma^2_{\text{max}} = 10.125/(3\gamma^2\lambda^2)$ [10]. For Bianchi II, θ, σ and ρ are also bounded, but for larger values than the ones in Bianchi I, i.e., there are solutions where the matter density is larger than the critical density. With point-like and cigar-like classical singularities [8], the density can achieve the maximal value ($\rho \approx 0.54\rho_{Pl}$) as a consequence of the shear being zero at the bounce and curvature different from zero. For Bianchi IX the behavior is the same as in closed FRW, if the inverse triad corrections are not used, then the geometric scalars are not absolutely bounded. But if the inverse triad corrections are used, then on each solution the geometric scalars are bounded but there is not an absolute bound for all the solutions [9, 10].

- Bianchi I, II and therefore the isotropic case $k = 0$ are limiting cases of Bianchi IX, but they are not contained within Bianchi IX. While the isotropic FRW $k = 1$ is contained within Bianchi IX only if the inverse triad corrections are not included, when they are included then the $k = 1$ universe is a limiting case, like the $k = 0$ universe.
- A set of quantities that are very useful are the Kasner exponents (in classical Bianchi I, the scale factors are $a_i = t^{k_i}$, where k_i are the Kasner exponents), because they can be used to determine which kind of solution is obtained. The Kasner exponents tell us about the Bianchi I transitions (if they exist) and particularly in Bianchi IX, they are used to study the BKL behavior in the vacuum case.

5 Conclusions

One of the main issues that a quantum theory of gravity is expected to address is that of singularity resolution. Loop quantum cosmology has provided a complete description in the case of isotropic cosmological models and singularity resolution has been shown to be generic. A pressing question is whether these results can be generalized to anisotropic models. In this case we lack a complete quantum theory, but one can rely on the existence of an effective description, capturing the main (loop) quantum geometric features. In this contribution we have described the main features of such effective solutions. Singularities seem to be generically resolved as the time evolution of geometrical scalars is well behaved past the would be classical singularity. With the study of these anisotropic models, a question that still arises is whether this behavior is generic for non-homogeneous configurations. That is, are we a step forward toward generic quantum singularity resolution?

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