The Gravitational Equation in Higher Dimensions

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Abstract Like the Lovelock Lagrangian which is a specific homogeneous polynomial in Riemann curvature, for an alternative derivation of the gravitational equation of motion, it is possible to define a specific homogeneous polynomial analogue of the Riemann curvature, and then the trace of its Bianchi derivative yields the corresponding polynomial analogue of the divergence free Einstein tensor defining the differential operator for the equation of motion. We propose that the general equation of motion is $G_{ab}^{(n)} = -\Lambda g_{ab} + \kappa_n T_{ab}$ for $d = 2n + 1$, $2n + 2$ dimensions with the single coupling constant κ_n, and $n = 1$ is the usual Einstein equation. It turns out that gravitational behavior is essentially similar in the critical dimensions for all n. All static vacuum solutions asymptotically go over to the Einstein limit, Schwarzschild-dS/AdS. The thermodynamical parameters bear the same relation to horizon radius, for example entropy always goes as r_h^{d-2n} and so for the critical dimensions it always goes as r_h, r_h^2. In terms of the area, it would go as $A^{1/n}$. The generalized analogues of the Nariai and Bertotti–Robinson solutions arising from the product of two constant curvature spaces, also bear the same relations between the curvatures $k_1 = k_2$ and $k_1 = -k_2$ respectively.

1 Introduction

What stands gravity apart from rest of the physics is its universal character that it links to everything including massless particles and hence it can only be described by the spacetime curvature, and its dynamics has therefore to follow from the geometric properties of the Riemann curvature tensor [1]. The Einstein gravitational equation

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could be deduced from the geometric property of Riemann curvature, known as the Bianchi identity, implying vanishing of its Bianchi derivative identically. Its trace yields the divergence-free second rank symmetric Einstein tensor. It defines the differential operator on the left hand side of the equation while the gravitational source – energy momentum distribution described by a second rank symmetric tensor with the condition of vanishing divergence—appears on the right hand side. This is the case for Einstein gravity which is linear in Riemann curvature, and its vacuum is trivially flat in 3 dimensions and it becomes dynamically non-trivial in 4 dimensions.

The question is, could this be generalized to a polynomial analogue of the Riemann tensor? Consider a tensor with the same symmetry properties as the Riemann which is a homogeneous polynomial of degree n in Riemann, and then demand that the trace of its Bianchi derivative vanishes. This will fix the coefficients in the polynomial and will give the divergence free second rank symmetric tensor $G_{ab}^{(n)}$, the nth order analogue of the Einstein tensor, which is the same as what one would get from the variation of the nth order Lovelock Lagrangian [2]. Thus we have the generalized polynomial Riemann curvature, $R_{abcd}^{(n)}$, which would describe gravitational dynamics in $d = 2n + 1, 2n + 2$ in the same manner as Riemann does for $d = 3, 4$. We can define corresponding vacuum as $R_{ab}^{(n)} = 0$, would it also be trivial in $d = 2n + 1$ dimension? The answer is indeed, yes [3]. It would be $R_{abcd}^{(n)}$ flat but not Riemann flat, and for that it would describe a global monopole [4].

What should be the gravitational equation in dimension > 4? Should it continue to be the Einstein equation which is linear in Riemann or should it include the one following from the higher order Riemann, $R_{abcd}^{(n)}$ yet giving the second order quasi-linear equation? A general abiding principle is that the equation be second order quasi-linear so that the initial value problem is well formulated giving unique evolution. This uniquely identifies the Lovelock polynomial Lagrangian or equivalently the above discussed polynomial Riemann curvature [2]. Should all orders that are non-trivial in the equation be included like the linear Einstein, quadratic Gauss-Bonnet, and so on, or the only highest one? Should it be $\sum G_{ab}^{(n)}$ or $G_{ab}^{(n)}$? In the former, each order will have its own coupling and so there would be n of them, and there is no obvious way to fix them. Since there is only one force which allows determination of only one coupling parameter by experimentally measuring its strength, gravity should therefore have only one dimensional coupling parameter and its dimension would however depend upon the spacetime dimension. Thus we propose the gravitational equation should in general be written as

$$G_{ab}^{(n)} = -\Lambda g_{ab} + \kappa_n T_{ab} \tag{1}$$

for $d = 2n + 1, 2n + 2$ dimensions. Note that Λ, which characterizes dynamics free spacetime, is part of the structure of spacetime on the same footing as the velocity of light [5]. In what follows we wish to demonstrate that this equation imbibes beautifully the general vacuum character [3] while the static vacuum solutions asymptotically go over to the right Einstein limit, even though the linear Einstein term is not included. This means higher order terms in curvature are only pertinent to the

high energy end near the black hole horizon while their effect weans out asymptotically at the low energy end approximating to the linear order Einstein solution, Schwarzschild-dS/AdS in d dimension [6, 7]. It is remarkable that the thermodynamical parameters, temperature and entropy bear universal relation to the horizon radius for static black holes in $d = 2n + 1, 2n + 2$, and interestingly this property also marks the characterization of this class of black holes [7, 8].

2 The Lovelock Curvature Polynomial and the Equation of Motion

Following Ref. [2], we define the Lovelock curvature polynomial

$$R_{abcd}^{(n)} = F_{abcd}^{(n)} - \frac{n-1}{n(d-1)(d-2)} F^{(n)} (g_{ac} g_{bd} - g_{ad} g_{bc}),$$
$$F_{abcd}^{(n)} = Q_{ab}{}^{mn} R_{cdmn}, \tag{2}$$

where

$$Q^{ab}{}_{cd} = \delta_{cdc_1 d_1 \ldots c_n d_n}^{aba_1 b_1 \ldots a_n b_n} R_{a_1 b_1}{}^{c_1 d_1}, \ldots, R_{a_{n-1} b_{n-1}}{}^{c_{n-1} d_{n-1}},$$
$$Q^{abcd}{}_{;d} = 0. \tag{3}$$

It follows that the trace of the Bianchi derivative yields the divergence-free $G_{ab}^{(n)}$; i.e.

$$g^{ac} g^{bd} R_{abcd;e}^{(n)} = G^{(n)b}{}_{e;b} = 0, \tag{4}$$

where the analogue of nth order Einstein tensor is given by

$$G_{ab}^{(n)} = n(R_{ab}^{(n)} - \frac{1}{2} R^{(n)} g_{ab}). \tag{5}$$

Note that

$$R^{(n)} = \frac{d - 2n}{n(d-2)} F^{(n)}, \tag{6}$$

which vanishes for $D = 2n$ while $F^{(n)}$, the Lovelock action polynomial, is non-zero but its variation, $G_{ab}^{(n)}$ vanishes identically. Since $R^{(n)} = g^{ab} R_{ab}^{(n)} = 0$ for $d = 2n$ for arbitrary g_{ab}, it implies $R_{ab}^{(n)} = 0$ identically as it involves apart from the metric its first and second derivatives which are arbitrary.

Since $G_{ab}^{(n)}$ is divergence free, we could write

$$G_{ab}^{(n)} = \kappa_n T_{ab} - \Lambda g_{ab}, \quad T_{;b}^{ab} = 0. \tag{7}$$

This is the gravitational equation for $d = 2n + 1, 2n + 2$ dimensions with κ_n as the gravitational constant, and $n = 1$ is the Einstein equation for 3 and 4 dimensions. What degree of polynomial in Riemann should the equation have is thus determined by the spacetime dimension. It is linear for 3, 4, quadratic for 5, 6, and so on.

3 Universal Features

The first universal feature studied was that of gravitational field inside a uniform density sphere and it was shown that it was always given by the Schwarzschild interior solution in Einstein as well as in Einstein–Gauss–Bonnet/Lovelock theories [9]. Here we shall consider the cases of static black holes, and product spaces describing the Nariai and Bertotti–Robinson spacetimes.

3.1 Static Black Holes

The static spherically symmetric solution of the vacuum (1) is given by

$$g_{tt} = -1/g_{rr} = V = 1 - r^2(\Lambda + M/r^{d-1})^{1/n}, \tag{8}$$

which asymptotically takes the form of the Schwarzschild-dS/AdS solution in d dimensions showing the correct Einstein limit. The solution for the general case of the Einstein–Lovelock equation can also be written in terms of the nth order algebraic polynomial equation which cannot be solved in general for $n > 4$. It is therefore clear that we cannot carry on with arbitrarily large number of coupling parameters. For the case of dimensionally continued black holes [10], it was proposed that all the couplings are determined in terms of the unique ground state Λ, and the solution is then given by $V = 1 - r^2\Lambda - M/r^{d-1/2}$ which clearly does not go over to the Einstein solution for large r. This corresponded to the algebraic polynomial being degenerate. It turns out that the proper Einstein limit could be brought in simply by considering the polynomial to be derivative degenerate [7]. Then the solution agrees near the horizon with the dimensionally continued black hole and asymptotically with the proper Einstein limit, and it is the solution of equation (1).

Further, the thermodynamical parameters, temperature and entropy bear the universal relation to the horizon radius for the critical $d = 2n + 1, 2n + 2$ dimensions [8]. For instance, the entropy always goes as r_h^{d-2n} which for the critical dimensions would always go as r_h, r_h^2. In terms of the area, it would however go as $A^{1/n}$, and hence the entropy is proportional to area only for the $n = 1$ Einstein theory. Interestingly, this universality is also the characterizing property of this class of pure Lovelock black holes [7, 8].

We would like to conjecture that the above universality property would also be true for the rotating black hole solution as and when it is found.

3.2 Product Spaces: Nariai and Bertotti–Robinson Solutions

The Nariai and Bertotti–Robinson solutions arise as product of two constant curvature spaces. When the two curvatures are equal, $k_1 = k_2$, it is the Nariai solution of (1) with $T_{ab} = 0$ for $n = 1$, and when the curvatures are equal and opposite, $k_1 = -k_2$, it is the Bertotti–Robinson solution describing the uniform electric field. The former is the Λ vacuum spacetime but is not conformally flat while the latter is the Einstein–Maxwell solution for uniform electric field which is conformally flat. It turns out the generalized pure Lovelock solutions of (1) for any n bear out the same curvature relations for the Nariai vacuum ($k_1 = k_2$) and Bertotti–Robinson uniform electric field ($k_1 = -k_2$), and the condition for conformal flatness is also $k_1 k_2 = 0$ [11].

In $d = 2n + 2$ dimensions, we have the following general relation connecting the two curvatures, Λ and the electric field E,

$$(k_1 + k_2)E^2 = -4(k_1 - k_2)\Lambda. \tag{9}$$

This clearly indicates $k_1 = k_2$ for $E = 0$, the Nariai vacuum spacetime and $k_1 = -k_2$ for $\Lambda = 0$, the Bertotti–Robinson uniform electric field spacetime. The metric is given by

$$ds^2 = (1 - k_1 r^2)dt^2 - \frac{dr^2}{1 - k_1 r^2} - \frac{1}{k_2}d\Sigma_{(d-2)}^2. \tag{10}$$

4 Discussion

We have proposed that equation (1) is the proper equation for gravity in higher dimensions. The correct equation should have the following properties: (a) it should be second order quasi-linear, (b) for a given dimension, it should be of degree $n = [(d-1)/2]$ in the Riemann curvature, (c) it should have only one coupling constant which could be determined by experimentally measuring the strength of the force, and (d) since higher order curvature contributions are the high energy corrections to the linear order in Riemann Einstein gravity which should wean out asymptotically, hence solutions should tend to the corresponding Einstein solution for large r. The proposed equation satisfies all these properties. The latter feature of the asymptotic Einstein limit is verified for the static black hole solutions which, however, is also true for the Einstein–Gauss–Bonnet black hole. What is remarkable here is that the equation is free of the Einstein term, yet asymptotically solutions go over to the proper Einstein limit. This means high energy effects which come through the higher order curvature terms are fully and properly taken care by the highest order

$n = [(d − 1)/2]$ term, and they could be realized only in higher dimensions [12]. It is interesting that gravity asks for higher dimensions for realization of its high energy effects. This is because inclusion of higher orders in Riemann curvature and the demand that the equation continues to be second order quasi-linear naturally lead to higher dimensions. This does not happen for any other force that one has to consider higher dimension for realization of its high energy corrections. It happens for gravity because the spacetime curvature is the basic field variable, and hence high energy effects involve higher orders in it and their contribution in the equation, if it continues to retain its second order quasi-linear character, can be realized only in higher dimensions [12]. We would like to emphasize that higher dimensions and high energy effects seem to be intimately connected. Since high energy effects ask for higher dimensions, quantum gravity should also involve higher dimensions. This is because quantum gravity should approach the classical limit via the high energy intermediate limit.

One of the problems with the Einstein–Lovelock solutions is number of coupling constants and there is no way to fix them. For the dimensionally continued static black holes, all the couplings were prescribed in terms of the unique ground state Λ [10]. These solutions were, however, not asymptotically Einstein, Schwarzschild-dS/AdS. Instead, the corresponding solutions of (1) have the right limits at both ends, nearer to horizon agreeing with the dimensionally continued and asymptotically to Schwarzschild-dS/AdS. This is indicative of the inherent correctness of the equation. The universal character of gravity in the critical dimensions is another very attractive feature of the equation. That the vacuum, $G_{ab}^{(n)} = 0$, in the odd critical dimension is always trivial, $R_{abcd}^{(n)} = 0$ [3]. All this taken together points to the fact that (1) is right equation for gravitation in higher dimensions.

For a given order n in the Riemann curvature, the critical dimensions are $d = 2n + 1, \ 2n + 2$ and it is trivial/kinematic in the former and it becomes dynamic in the latter. This is a universal general feature. In the critical dimensions, gravity has the similar behavior as indicated by universality of the thermodynamic parameters in terms of the horizon radius and of the Nariai and Bertotti–Robinson solutions. It is interesting to note that in terms of black hole area, entropy is always proportional to $A^{1/n}$ and so it is proportional to area only for the $n = 1$ Einstein gravity. This is an interesting general result that entropy always goes as the nth root of area of the black hole. In an intuitive sense we can say that it is nth root of the Einstein gravity for the critical $d = 2n + 1, \ 2n + 2$ dimensions.

All this we have established for the simple case of static black hole but we believe that it is indeed a general feature and hence should be true for the stationary rotating black hole as well. So far there exists no rotating pure Lovelock black hole solution, and this conjecture would be verified as and when a solution is found.

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