Electric and Magnetic Weyl Tensors in Higher Dimensions

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Abstract Recent results on purely electric (PE) or magnetic (PM) spacetimes in n dimensions are summarized. These include: Weyl types; diagonalizability; conditions under which direct (or warped) products are PE/PM.

1 Definition and General Properties

The standard decomposition of the Maxwell tensor F_{ab} into its electric and magnetic parts **E** and **B** with respect to (wrt) an observer (i.e., a unit time-like vector u) can be extended to any tensor in an n-dimensional spacetime [1–3]. Here we summarize the results of [3] about the Weyl tensor, and the connection with the null alignment classification [4, 5].

Consider the u-orthogonal projector $h_{ab} = g_{ab} + u_a u_b$. The "electric" and "magnetic" parts of C_{abcd} can be defined, respectively, as [3]

$$(C_+)^{ab}{}_{cd} = h^{ae} h^{bf} h_c{}^g h_d{}^h C_{efgh} + 4u^{[a} u_{[c} C^{b]e}{}_{d]f} u_e u^f, \tag{1}$$

$$(C_-)^{ab}{}_{cd} = 2h^{ae} h^{bf} C_{efk[c} u_{d]} u^k + 2u_k u^{[a} C^{b]kef} h_{ce} h_{df}. \tag{2}$$

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These extend the well-known 4D definitions [6, 7]. In any orthonormal frame adapted to u the electric [magnetic] part accounts for the Weyl components with an even [odd] number of indices u. *At a spacetime point (or region) the Weyl tensor is called purely electric [magnetic] (from now on, PE [PM]) wrt u if* $C_- = 0\,[C_+ = 0]$. The corresponding spacetime is also called PE [PM]. Several conditions on PE/PM Weyl tensors follow.

Proposition 1 (Bel-Debever-like criteria [3]). *A Weyl tensor C_{abcd} is: (i) PE wrt u iff $u_a g^{ab} C_{bc[de}u_{f]} = 0$; (ii) PM wrt u iff $u_{[a}C_{bc][de}u_{f]} = 0$.*

Proposition 2 (Eigenvalues [3]). *A PE [PM] Weyl operator*[1] *is diagonalizable, and possesses only real [purely imaginary] eigenvalues. Moreover, a PM Weyl operator has at least $\frac{(n-1)(n-4)}{2}$ zero eigenvalues.*

Proposition 3 (Algebraic type [3]). *A Weyl tensor which is PE/PM wrt a certain u can only be of type G, I_i, D or O. In the type I_i and D cases, the second null direction of the timelike plane spanned by u and any WAND is also a WAND (with the same multiplicity). Furthermore, a type D Weyl tensor is PE iff it is type D(d), and PM iff it is type D(abc).*

Proposition 4 (Uniqueness of u [3]). *A PE [PM] Weyl tensor is PE [PM] wrt: (i) a unique u (up to sign) in the type I_i and G cases; (ii) any u belonging to the space spanned by all double WANDs (and only wrt such us) in the type D case (noting also that if there are more than two double WANDs the Weyl tensor is necessarily PE (type D(d)) [10]).*

2 PE Spacetimes

Proposition 5 ([3]). *All spacetimes admitting a shearfree, twistfree, unit timelike vector field u are PE wrt u. In coordinates such that $u = V^{-1}\partial_t$, the line-element reads*

$$ds^2 = -V(t,x)^2 dt^2 + P(t,x)^2 \xi_{\alpha\beta}(x) dx^\alpha dx^\beta. \tag{3}$$

The above metrics include, in particular, direct, warped and doubly warped products with a one-dimensional timelike factor, and thus all *static* spacetimes (see also [11]). For a warped spacetime (M,g) with $M = M^{(n_1)} \times M^{(n_2)}$, one has $g = e^{2(f_1+f_2)}\left(g^{(n_1)} \oplus g^{(n_2)}\right)$, where $g^{(n_i)}$ is a metric on the factor space $M^{(n_i)}$ ($i = 1, 2$) and f_i are functions on $M^{(n_i)}$ ($M^{(n_i)}$ has dimension n_i, $n = n_1 + n_2$, and $M^{(n_1)}$ is Lorentzian).

Proposition 6 (Warps with $n_1 = 2$ [3, 11]). *A (doubly) warped spacetime with $n_1 = 2$ is either type O, or type D(d) and PE wrt any u living in $M^{(n_1)}$; the uplifts of the null directions of the tangent space to ($M^{(n_1)}$,$g^{(n_1)}$) are double WANDs of*

[1] In the sense of the Weyl operator approach of [8] (see also [9]).

(M,g). If $(M^{(n_2)},g^{(n_2)})$ is Einstein the type specializes to D(bd), and if it is of constant curvature to D(bcd).

In particular, all spherically, hyperbolically or plane symmetric spacetimes belong to the latter special case.

Proposition 7 (Warps with $n_1 = 3$ [3, 11]). *A (doubly) warped spacetime with $(M^{(n_1)},g^{(n_1)})$ Einstein and $n_1 = 3$ is of type D(d) or O. The uplift of* any *null direction of the tangent space to $(M^{(n_1)},g^{(n_1)})$ is a double WAND of (M,g), which is PE wrt any u living in $M^{(n_1)}$.*

Proposition 8 (Warps with $n_1 > 3$ [3, 11]). *In a (doubly) warped spacetime*

(i) *if $(M^{(n_1)},g^{(n_1)})$ is an Einstein spacetime of type D, (M,g) can be only of type D (or O) and the uplift of a double WAND of $(M^{(n_1)},g^{(n_1)})$ is a double WAND of (M,g)*
(ii) *if $(M^{(n_1)},g^{(n_1)})$ is of constant curvature, (M,g) is of type D(d) (or O) and the uplifts of* any *null direction of the tangent space to $(M^{(n_1)},g^{(n_1)})$ is a double WAND of (M,g); (M,g) is PE wrt any u living in $M^{(n_1)}$.*

Proposition 9 (PE direct products [3]). *A direct product spacetime $M^{(n)} = M^{(n_1)} \times M^{(n_2)}$ is PE wrt a u that lives in $M^{(n_1)}$ iff u is an eigenvector of $R_{ab}^{(n_1)}$, and $M^{(n_1)}$ is PE wrt u. (u is then also an eigenvector of the Ricci tensor R_{ab} of $M^{(n)}$, i.e., $R_{ui} = 0$.)*

A conformal transformation (e.g., to a (doubly) warped space) will not, of course, affect the above conclusions about the Weyl tensor. There exist also direct products which are PE wrt a vector *u not* living in $M^{(n_1)}$ [3].

Also the presence of certain (Weyl) isotropies (e.g., $SO(n-2)$ for $n > 4$) implies that the spacetime is PE, see [3, 8] for details and examples.

3 PM Spacetimes

Proposition 10 (PM direct products [3]). *A direct product spacetime $M^{(n)} = M^{(n_1)} \times M^{(n_2)}$ is PM wrt a u that lives in $M^{(n_1)}$ iff all the following conditions hold (where $R_{(n_i)}$ is the Ricci scalar of $M^{(n_i)}$):*

(i) *$M^{(n_1)}$ is PM wrt u and has a Ricci tensor of the form $R_{ab}^{(n_1)} = \frac{R_{(n_1)}}{n_1} g_{ab}^{(n_1)} + u_{(a}q_{b)}$ (with $u^a q_a = 0$).*
(ii) *$M^{(n_2)}$ is of constant curvature and $n_2(n_2-1)R_{(n_1)} + n_1(n_1-1)R_{(n_2)} = 0$.*

Further, $M^{(n)}$ is PM Einstein iff $M^{(n_1)}$ is PM Ricci-flat and $M^{(n_2)}$ is flat.

See [3] for explicit (non-Einstein) examples. However, in general PM spacetimes are most elusive. For example,

Proposition 11 ([3]). *PM Einstein spacetimes of type D do not exist.*

In [3] also several results for PE/PM Ricci and Riemann tensors have been worked out, along with corresponding examples. In general, we observe that PE/PM tensors provide examples of *minimal tensors* [12]. Thanks to the *alignment theorem* [13], the latter are of special interest since they are precisely the *tensors characterized by their invariants* [13] (cf. also [3]). This in turn sheds new light on the classification of the Weyl tensor [5], providing a further invariant characterization that distinguishes the (minimal) types G/I/D from the (non-minimal) types II/III/N.

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