A Reference for the Covariant Hamiltonian Boundary Term

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Abstract The Hamiltonian for dynamic geometry generates the evolution of a spatial region along a vector field. It includes a boundary term which determines both the value of the Hamiltonian and the boundary conditions. The value gives the quasi-local quantities: energy-momentum, angular-momentum/center-of-mass. The boundary term depends not only on the dynamical variables but also on their reference values; the latter determine the ground state (having vanishing quasi-local quantities). For our preferred boundary term for Einstein's GR we propose 4D isometric matching and extremizing the energy to determine the reference metric and connection values.

1 Introduction

Energy-momentum is the source of gravity. Gravitating bodies can exchange energy-momentum with gravity—*locally*—yet there is no well defined energy-momentum

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density for gravity itself. This inescapable conclusion can be understood as a consequence of the equivalence principle (for a discussion see Misner et al. [1], Sect. 20.4).

2 Quasi-Local Energy-Momentum

The standard approaches aimed at identifying an energy-momentum density for gravitating systems always led to various non-covariant, reference frame dependent, energy-momentum complexes (such expressions are generally referred to as pseudotensors). There are two types of ambiguity. First, there was no unique expression, but rather many that were found by various investigators—including Einstein [2], Papapetrou [3], Landau-Lifshitz [4], Bergmann-Thompson [5],break Møller [6], Goldberg [7], and Weinberg [8]—so which expression should be used? And second—in view of the fact all of these expressions are inherently reference frame dependent—for a chosen expression which reference frame should be used to give the proper physical energy-momentum localization?

The more modern idea is *quasi-local*, i.e., energy-momentum should be associated not with a local density but rather with a closed 2-surface; for a comprehensive review, see Szabados [9].

One approach to energy-momentum is via the *Hamiltonian* (the generator of time evolution). It turns out that this actually includes all the classical pseudotensors as special cases, while taming their ambiguities—providing clear physical/geometric meaning [10, 11].

3 The Covariant Hamiltonian Formulation Results

We have developed a covariant Hamiltonian formalism that is applicable to a large class of geometric gravity theories [10–16]. For such theories the Hamiltonian 3-form $\mathcal{H}(N)$ is both a conserved Noether current,

$$d\mathcal{H}(N) \propto \text{field eqns} \simeq 0, \tag{1}$$

as well as the generator of the evolution of a spatial region along a space-time displacement vector field. It has the general form

$$\mathcal{H}(N) = N^{\mu}\mathcal{H}_{\mu} + d\mathcal{B}(N), \tag{2}$$

where $N^{\mu}\mathcal{H}_{\mu}$—which generates the evolution equations—is itself proportional to certain field equations (initial value constraints) and thus vanishes "on shell". Consequently the *value* of the Hamiltonian is determined by the total differential (boundary) term:

$$E(N, \Sigma) := \int_{\Sigma} \mathcal{H}(N) = \oint_{\partial\Sigma} \mathcal{B}(N). \tag{3}$$

Thus, the value is *quasi-local*. From this boundary term, with suitable choices of the vector field on the boundary, one can determine the quasi-local energy-momentum and angular momentum/center-of-mass.

It should be noted that the boundary 2-form $\mathcal{B}(N)$ can be modified—*by hand*—in any way without destroying the conservation property. (This is a particular case of the usual Noether conserved current ambiguity.) With this freedom one can arrange for almost any conserved quasi-local values. Fortunately the Hamiltonian's role in generating evolution equations tames that freedom.

4 Boundary Variation Principle, Reference Values

One must look to the boundary term in the *variation of the Hamiltonian* (see Lanczos [17–19]). Requiring it to vanish yields the *boundary conditions*. The Hamiltonian is *functionally differentiable* on the phase space of fields *satisfying these boundary conditions*. Modifying the boundary term changes the boundary conditions. (The different classical pseudotensors are each associated with a specific "superpotential" which can serve as the Hamiltonian boundary term, thus they correspond to Hamiltonians with different boundary conditions [10].)

In order to accommodate suitable boundary conditions one must, in general, also introduce certain *reference values* which represent the ground state of the field—the "vacuum" (or background field) values. To this end for any quantity α we let $\Delta\alpha := \alpha - \bar{\alpha}$ where $\bar{\alpha}$ is the reference value.

5 Preferred Boundary Term for GR

Some time ago we identified for GR two *covariant-symplectic* boundary terms [12]; one, which was also found[1] at about the same time by Katz, Bičák and Lynden-Bell [20, 21], has become our preferred choice[2]:

$$\mathcal{B}(N) = \frac{1}{2\kappa}(\Delta\Gamma^{\alpha}{}_{\beta} \wedge i_N\eta_{\alpha}{}^{\beta} + \bar{D}_{\beta}N^{\alpha}\Delta\eta_{\alpha}{}^{\beta}), \tag{4}$$

This choice corresponds to fixing the orthonormal coframe ϑ^{μ} (equivalently the metric) on the boundary:

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[1] Via a different route, using a Noether type argument with a global reference.

[2] Here $\Gamma^{\alpha}{}_{\beta}$ is the connection one-form, $\eta^{\alpha\beta\cdots} := *(\vartheta^{\alpha} \wedge \vartheta^{\beta} \wedge \cdots)$ and i_N denotes the interior product (aka contraction) with the vector field N.

$$\delta \mathscr{H}(N) \sim d i_N (\Delta \Gamma^\alpha{}_\beta \wedge \delta \eta_\alpha{}^\beta). \tag{5}$$

Like other choices, at spatial infinity it gives the ADM [22], MTW [1], Regge-Teitelboim [18], Beig-Ó Murchadha [23], Szabados [24, 25] energy, momentum, angular-momentum, center-of-mass.

Its special virtues include (i) at null infinity it directly gives the Bondi-Trautman energy and the Bondi energy flux [15], (ii) it is "covariant", (iii) it has a positive energy property, (iv) for small spheres it gives a positive multiple of the Bel-Robinson tensor, (v) it yields the first law of thermodynamics for black holes [13], (vi) for spherically symmetric solutions it has the hoop property [26].

6 The Reference and the Quasi-Local Quantities

For all other fields it is appropriate to choose vanishing reference values as the reference ground state—the vacuum. But for geometric gravity the standard ground state is the non-vanishing Minkowski metric. Thus a non-trivial reference is essential.

Using standard Minkowski coordinates y^i, a Killing field of the reference has the form $N^k = N_0^k + \lambda_{0l}^k y^l$, where the translation parameters N_0^k and the boost-rotation parameters $\lambda_0^{kl} = \lambda_0^{[kl]}$ are constants. The 2-surface integral of the Hamiltonian boundary term then gives a value of the form

$$\oint_S \mathscr{B}(N) = -N_0^k p_k(S) + \frac{1}{2}\lambda_0^{kl} J_{kl}(S), \tag{6}$$

which yields not only a quasi-local energy-momentum but also a quasi-local angular momentum/center-of-mass. The integrals $p_k(S)$, $J_{kl}(S)$ in the spatial asymptotic limit agree with accepted expressions for these quantities [1, 18, 23–25].

7 The Reference

For energy-momentum one takes N to be a translational Killing field of the Minkowski reference. Then the second term in our quasi-local boundary expression (4) vanishes. Let us note in passing that holonomically (with vanishing reference connection coefficients) the first term in (4) reduces to Freud's 1939 superpotential [27]. Thus we are in effect here making a proposal for good coordinates for the Einstein pseudotensor.

To construct a reference, choose, in a neighborhood of the desired spacelike boundary 2-surface S, four smooth functions y^i, $i = 0, 1, 2, 3$ with $dy^0 \wedge dy^1 \wedge dy^2 \wedge dy^3 \neq 0$; they define a Minkowski reference by $\bar{g} = -(dy^0)^2 + (dy^1)^2 + (dy^2)^2 + (dy^3)^2$. This is equivalent to finding a diffeomorphism for a neighborhood of the 2-surface

into Minkowski space. The reference connection can now be obtained from the pullback of the flat Minkowski connection.

With constant N^k our quasi-local expression now takes the form

$$\mathscr{B}(N) = N^k x^\mu{}_k (\Gamma^\alpha{}_\beta - x^\alpha{}_j \, dy^j{}_\beta) \wedge \eta_{\mu\alpha}{}^\beta , \tag{7}$$

where $dy^k = y^k{}_\alpha dx^\alpha$ has the inverse $dx^\alpha = x^\alpha{}_k dy^k$.

8 Isometric Matching of the 2-Surface

The reference metric on the dynamical space has the components

$$\bar{g}_{\mu\nu} = \bar{g}_{ij} y^i{}_\mu y^j{}_\nu. \tag{8}$$

Consider the usual embedding restriction: isometric matching of the 2-surface S. This can be expressed quite simply in terms of quasi-spherical foliation adapted coordinates t, r, θ, φ as

$$g_{AB} \doteq \bar{g}_{AB} = \bar{g}_{ij} y^i_A y^j_B = -y^0_A y^0_B + \delta_{ij} y^i_A y^j_B , \tag{9}$$

where S is given by constant values of t, r, and A, B range over θ, φ. We use $\doteq$ to indicate a relation which holds only on the 2-surface S.

From a classic closed 2-surface into $\mathbb{R}^3$ embedding theorem—as long as one restricts S and $y^0(x^\mu)$ so that on S

$$g'_{AB} := g_{AB} + y^0_A y^0_B \tag{10}$$

is convex—one has a unique embedding. Wang and Yau have discussed in detail this type of embedding of a 2-surface into Minkowski controlled by one function in their recent quasi-local work [28, 29].

9 Complete 4D Isometric Matching

Our "new" proposal is: complete 4D isometric matching on S. (We remark that this was already suggested by Szabados back in 2000,[3] and he has since extensively explored this idea [30] in unpublished work.)

Complete 4D isometric matching imposes 10 constraints,

$$g_{\mu\nu}|_S \doteq \bar{g}_{\mu\nu}|_S \doteq \bar{g}_{ij} y^i{}_\mu y^j{}_\nu|_S ,$$

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[3] At a workshop in Hsinchu, Taiwan.

on the 16 $y^i_\alpha(t_0, r_0, \theta, \varphi)$. On the 2-surface S these 16 quantities are actually determined by 12 independent embedding functions: y^i, $y^i{}_t$, $y^i{}_r$ (since from y^i on S one can get $y^i{}_\theta$, $y^i{}_\varphi$). There remain $2 = 12$–10 degrees of freedom in choosing the reference.

One could as an alternative use orthonormal frames. Then the 4D isometric matching can be represented by $\vartheta^\alpha \doteq \bar\vartheta^\alpha$. But the reference coframe has the form $\bar\vartheta^\alpha = dy^\alpha$. Thus one should Lorentz transform the coframe ϑ^α to match dy^α on the 2-surface S. This leads to an integrability condition: the 2-forms $d\vartheta^\alpha$ should vanish when restricted to the 2-surface:

$$d\vartheta^\alpha|_S \doteq 0. \tag{11}$$

This is 4 conditions restricting the 6 parameter local Lorentz gauge freedom. Which again reveals that after 4D isometric matching there remains $2 = 6$–4° of freedom in choosing our reference. By the way, this orthonormal frame formulation shows that our procedure can alternatively be viewed as finding a good frame for the "teleparallel gauge current" [31].

10 The Best Matched Reference Geometry

There are 12 embedding variables subject to 10 4D isometric matching conditions, or equivalently, 6 local Lorentz gauge parameters subject to 4 frame embedding conditions. To fix the remaining 2, one can regard the quasi-local value as a measure of the difference between the dynamical and the reference boundary values. So we propose taking the optimal "best matched" embedding as the one which gives the extreme value to the associated invariant mass $m^2 = -p_i p_j \bar g^{ij}$. This is reasonable, as one expects the quasi-local energy to be non-negative and to vanish only for Minkowski space.

More precisely, we note two different situations:
I: Given a 2-surface S find the critical points of m^2. This should determine the reference up to Poincaré transformations.
II: Given a 2-surface S and a vector field N, then one can look to the choices of the embedding variables that are a critical point of $E(N, S)$. (Afterward one could extremize over the choice of N.)

Based on some physical and practical computational arguments it seems reasonable to expect a unique solution in general.

For spherically symmetric systems (both static and dynamic), using this and some other related strategies we have found reasonable quasi-local energy results [32–35].

References

1. Misner, C., Thorne, K., Wheeler, J.: Gravitation. W.H. Freeman, San Francisco (1973)
2. Trautman, A.: Conservation laws in general relativity. In: Witten, L. (ed.) Gravitation: An Introduction to Current Research, pp. 169–198. Wiley, New York (1962)
3. Papapetrou, A.: Einstein's theory of gravitation and flat space. Proc. R. Irish Acad. A **52**, 11 (1958)
4. Landau, L., Lifshitz, E.: The Classical Theory of Fields, 2nd edn. Pergamon Press; Addison-Wesley, Oxford; Reading, MA (1962)
5. Bergmann, P., Thompson, R.: Spin and angular momentum in general relativity. Phys. Rev. **89**, 400 (1953). doi:10.1103/PhysRev.89.400
6. Møller, C.: On the localization of the energy of a physical system in general theory of relativity. Ann. Phys. (N.Y.) **4**, 347 (1958). doi:10.1016/0003-4916(58)90053-8
7. Goldberg, J.: Conservation laws in general relativity, Phys. Rev. **111**, 315 (1958). doi:10.1103/PhysRev.111.315
8. Weinberg, S.: Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity. Wiley, New York (1972)
9. Szabados, L.: Quasi-local energy-momentum and angular momentum in general relativity. Living Rev. Relativity **12**(4), lrr-2009-4 (2009). http://www.livingreviews.org/lrr-2009-4
10. Chang, C., Nester, J., Chen, C.M.: Pseudotensors and quasi-local energy-momentum, Phys. Rev. Lett. **83**, 1897 (1999). doi:10.1103/PhysRevLett.83.1897
11. Nester, J.: General pseudotensors and quasilocal quantities, Class Quantum Grav. **21**, S261 (2004). doi:10.1088/0264-9381/21/3/016
12. Chen, C.M., Nester, J., Tung, R.: Quasilocal energy-momentum for geometric gravity theories, Phys. Lett. A **203**, 5 (1995). doi:10.1016/0375-9601(95)92844-T
13. Chen, C.M., Nester, J.: Quasilocal quantities for general relativity and other gravity theories, Class. Quantum Grav. **16**, 1279 (1999). doi:10.1088/0264-9381/16/4/018
14. Chen, C.M., Nester, J.: A symplectic Hamiltonian derivation of quasilocal energy-momentum for GR. Grav. Cosmol. **6**, 257 (2000)
15. Chen, C.M., Nester, J., R.S. Tung, Hamiltonian boundary term and quasilocal energy flux, Phys. Rev. D **72**, 104020 (2005). doi:10.1103/PhysRevD.72.104020
16. Nester, J.: A manifestly covariant Hamiltonian formalism for dynamical geometry, Prog. Theor. Phys. Suppl. **172**, 30 (2008). doi:10.1143/PTPS.172.30
17. Lanczos, C.: The Variational Principles of Mechanics, Mathematical Expositions, vol. 4, 1st edn. University of Toronto Press, London (1949)
18. Regge, T., Teitelboim, C.: Role of surface integrals in the Hamiltonian formulation of general relativity. Ann. Phys. (N.Y.) **88**, 286 (1974). doi:10.1016/0003-4916(74)90404-7
19. Kijowski, J., Tulczyjew, W.: A Symplectic Framework for Field Theories, Lecture Notes in Physics, vol. 107. Springer, Berlin (1979). doi:10.1007/3-540-09538-1
20. Lynden-Bell, D., Katz, J., Bičák, J.: Mach's principle from the relativistic constraint equations, Mon. Not. R. Astron. Soc. **272**, 150 (1995). Erratum: ibid. 277, 1600 (1995)
21. Katz, J., Bičák, J., Lynden-Bell, D.: Relativistic conservation laws and integral constraints for large cosmological perturbations. Phys. Rev. D **55**, 5957 (1997). doi:10.1103/PhysRevD.55.5957
22. Arnowitt, R., Deser, S., Misner, C.: Energy and the criteria for radiation in general relativity. Phys. Rev. **118**, 1100 (1960). doi:10.1103/PhysRev.118.1100
23. Beig, R., Ó Murchadha, N.: The Poincaré group as the symmetry group of canonical general relativity. Ann. Phys. (N.Y.) **174**, 463 (1987). doi:10.1016/0003-4916(87)90037-6
24. Szabados, L.: On the roots of the Poincaré structure of asymptotically flat spacetimes, Class. Quantum Grav. **20**, 2627 (2003). doi:10.1088/0264-9381/20/13/312
25. Szabados, L.: The Poincaré structure and the centre-of-mass of asymptotically flat spacetimes. In: Frauendiener, J., Giulini, D., Perlick, V. (eds.) Analytical and Numerical Approaches to Mathematical Relativity vol. 692, pp. 157–184 (2006). doi:10.1007/3-540-33484-X8

26. Ó Murchadha, N., Tung, R.S., Xie, N., Malec, E.: Brown-York mass and the Thorne hoop conjecture, Phys. Rev. Lett. **104**, 041101 (2010). doi:10.1103/PhysRevLett.104.041101
27. Freud, P.: Uber die Ausdrücke der Gesamtenergie und des Gesamtimpulses eines materiellen Systems in der allgemeinen Relativitätstheorie, Ann. Math. **40**, 417 (1939). doi:10.2307/1968929
28. Wang, M.T., Yau, S.T.: Isometric embeddings into the Minkowski space and new quasi-local mass, Commun. Math. Phys. **288**, 919 (2009). doi:10.1007/s00220-009-0745-0
29. Wang, M.T., Yau, S.T.: Quasilocal mass in general relativity. Phys. Rev. Lett. **102**, 021101 (2009). doi:10.1103/PhysRevLett.102.021101
30. Szabados, L.: Quasi-Local Energy-Momentum and Angular Momentum in GR: The Covariant Lagrangian Approach (2005). Unpublished draft.
31. de Andrade, V., Guillen, L., Pereira, J.: Gravitational energy-momentum density in teleparallel gravity, Phys. Rev. Lett. **84**, 4533 (2000). doi:10.1103/PhysRevLett.84.4533
32. Wu, M.F., Chen, C.M., Liu, J.L., Nester, J.: Optimal choices of reference for a quasi-local energy, Phys. Lett. A **374**, 3599 (2010). doi:10.1016/j.physleta.2010.07.004
33. Liu, J.L., Chen, C.M., Nester, J.: Quasi-local energy and the choice of reference. Class. Quantum Grav. **28**, 195019 (2012). doi:10.1088/0264-9381/28/19/195019
34. Wu, M.F., Chen, C.M., Liu, J.L., Nester, J.: Optimal choices of reference for a quasilocal energy: Spherically symmetric spacetimes, Phys. Rev. D **84**, 084047 (2011). doi:10.1103/PhysRevD.84.084047
35. Wu, M.F., Chen, C.M., Liu, J.L., Nester, J.: Quasi-local energy for spherically symmetric spacetimes, Gen. Relativ. Gravit. **44**, 2401 (2012). doi:10.1007/s10714-012-1399-3