The Conformal Einstein Field Equations for Trace-free Perfect Fluids

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Abstract A nonlinear stability analysis is carried out for the trace-free (radiation) perfect fluid Friedmann-Lemaître-Robertson-Walker models with a de Sitter-like cosmological constant. It is shown that the solutions close to the above FLRW spacetimes exist globally towards the future and are future geodesically complete. For this analysis we formulate the conformal Einstein field equations for a trace-free (radiation) perfect fluid in terms of the Levi-Civita connection of a conformally rescaled metric.

1 Introduction

During his time in Prague, 100 years ago, Einstein started his work on a general theory of relativity [1]. Since then many exact solutions have been found and analysed in detail—see e.g. [2, 3] for an overview. Typically these solutions lead to models which approximate certain features of our universe by making assumptions on the matter model or spacetime symmetries. As a result of this, a cosmological model is a representation of the universe at a particular averaging scale. This leads to the following important question: "How sensitive are the predictions derived from these models to perturbations". The answer is of interest for the following reasons. Firstly, the universe does not match an idealised model on all scales. Secondly, any observation of the universe (or a subsystem of it) gives rise to data that includes a certain margin of error. Thirdly, numerical calculations and simulations have made marked

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progress in recent years. However, any simulation is limited by the finite precision of the individual computations and hence has to repeatedly deal with numerical errors. In all three scenarios one hopes that the unavoidable deviation from reality has negligible consequences for the predictions, as long as this deviation is sufficiently small. However, the concern about stability is not at all limited to cosmological scales. It equally applies to stars and other compact objects. One of the main open problems concerns the behaviour of black holes under perturbations—see e.g. [4].

Due to the nonlinear nature of the Einstein field equations, the question of stability is not straightforward. Seminal work by Friedrich [5, 6], using conformal methods, and Christodoulos and Klainermann [7], using a detailed analysis of the structure of the underlying evolution equations, have provided an essential starting point for this problem. In recent years the topic has seen several papers addressing the stability problem for a range of matter models, dimensions and types of cosmological constants.

The problem of nonlinear stability of the Euler-Einstein system for de Sitter-like spacetimes has been analysed in [8, 9]. It was shown that the Friedmann-Lemaître-Robertson-Walker (FLRW) solutions with a barotropic equation of state of the form $\tilde{p} = (\gamma - 1)\tilde{\rho}$, where $1 < \gamma < \frac{4}{3}$, are future asymptotically stable under small perturbations. The case of a pure radiation perfect fluid ($\gamma = \frac{4}{3}$) was not covered by this analysis. This motivated the authors to investigate this case using conformal methods [10]. The details and the results are outlined in this article.

We address the question of nonlinear stability for small perturbations of the FLRW solutions, which describe a trace-free perfect fluid with de Sitter-like cosmological constant. The main result is given by the following theorem:

Theorem 1 *Suppose one is given Cauchy initial data for the Einstein-Euler system with a de Sitter-like cosmological constant and equation of state for pure radiation* $\tilde{p} = \frac{1}{3}\tilde{\rho}$. *If the initial data is sufficiently close to data for a FLRW cosmological model with the same equation of state, value of the cosmological constant and spatial curvature* $k = 1$, *then the development exists globally towards the future, is future geodesically complete and remains close to the FLRW solution.*

2 Methodology and Analysis

For a spacetime with a pure radiation perfect fluid the problem of nonlinear stability is addressed using the conformal methods of [5, 11, 12]. In this approach, which is summarised below, the physical spacetime $(\tilde{\mathscr{M}}, \tilde{g}_{\mu\nu})$ is conformally embedded into a manifold $(\mathscr{M}, g_{\mu\nu})$, referred to as the unphysical spacetime here, where the metrics are related by

$$g_{\mu\nu} = \theta^2 \tilde{g}_{\mu\nu}. \quad (1)$$

The physical Einstein field equations

$$\tilde{R}_{\mu\nu} - \tfrac{1}{2}\tilde{R}\tilde{g}_{\mu\nu} + \lambda\tilde{g}_{\mu\nu} = \tilde{T}_{\mu\nu} \tag{2}$$

are reformulated in terms of the geometry of $(\mathscr{M}, g_{\mu\nu})$ and some rescaled matter variables, leading to a new set of equations referred to as the conformal Einstein field equations (CEFE). A key advantage of this approach is that one can study global problems in $(\tilde{\mathscr{M}}, \tilde{g}_{\mu\nu})$ in terms of a local analysis near conformal infinity in $(\mathscr{M}, g_{\mu\nu})$. For this analysis one has to show that the CEFE associated to the chosen matter model form a regular system of PDEs on $(\mathscr{M}, g_{\mu\nu})$. One then proceeds by formualting an evolution problem in the form of a first order symmetric hyperbolic (FOSH) system to deduce existence and uniqueness results. In order to derive the desired stability results, one shows that a chosen reference spacetime is a regular solution to the CEFE, respectively the associated FOSH system, and invokes Kato's stability theorem [13].

The CEFE have inherited degrees of freedom. One needs to fix a set of coordinates x^{μ}, a frame $\{e_k\}$ and the conformal factor θ. In the following, we use frame components with respect to the chosen frame $\{e_k\}$.

There are two main methods to fix these gauge freedoms. One method uses gauge source functions [5, 6, 11, 14] to evolve the coordinates, the frame and the conformal factor. In the other method, a congruence of conformal geodesics [15–17] or conformal curves [10] is used to construct a conformal Gaussian coordinate system. The congruence also induces a general Weyl connection, which is used to propagated the frame $\{e_k\}$, and a canonical choice of the conformal factor. Here we employ the first method. In particular, we fix the conformal factor θ locally by setting the unphysical Ricci scalar to $R = -\frac{1}{6}$ and work in the Levi-Civita connection induced by $g_{\mu\nu}$ in (1). Moreover, the frame $\{e_k\}$ will be g-orthonormal.

The general approach for trace-free matter models has been discussed in [6]. In [10] it was shown how to address the case of a pure radiation perfect fluid ($\gamma = \frac{4}{3}$). The energy momentum tensor is given by

$$\tilde{T}_{ij} = \frac{4}{3}\tilde{\rho}\tilde{u}_i\tilde{u}_j - \frac{1}{3}\tilde{\rho}\tilde{g}_{ij}, \tag{3}$$

where $\tilde{u}^{\mu}$ is the fluid flow velocity. As shown in [6], if one defines $T_{ij} = \theta^{-2}\tilde{T}_{ij}$, then

$$\nabla^i \tilde{T}_{ij} = 0 \quad \Leftrightarrow \quad \nabla^i T_{ij} = 0. \tag{4}$$

We thus define new variables $\rho = \theta^{-4}\tilde{\rho}$ and $u_i = \theta\tilde{u}_i$ so that

$$\theta^{-2}\tilde{T}_{ij} = T_{ij} = \frac{4}{3}\rho u_i u_j - \frac{1}{3}\rho g_{ij}. \tag{5}$$

It was shown in [18] how to obtain a FOSH system for (3) if $\tilde{\rho} \neq 0$ and $\tilde{u}_0 \neq 0$, where e_0 is timelike vector. The same method works for (5) using (4). Moreover the approach can be adapted to derive a FOSH system for $\rho_i = \nabla_i \rho$ and $u_{ij} = \nabla_i u_j$. It is necessary to introduce these new variables in order to close the overall FOSH system derived from the CEFE.

3 Existence and Uniqueness

Overall it can be shown that for $\rho \neq 0$ and $u_0 \neq 0$ the CEFE for a radiation fluid form a regular FOSH system. In particular this system is regular at conformal inifinity $\mathscr{I}$, where $\theta = 0$.

Given sufficiently smooth initial data, the CEFE have a unique solution $(\mathscr{M}, g_{\mu\nu})$, which implies a solution $(\tilde{\mathscr{M}}, \tilde{g}_{\mu\nu})$ to the Einstein field equations for a radiation fluid by setting $\tilde{\mathscr{M}} = \mathscr{M}|_{\{\theta>0\}}$, $\tilde{g}_{\mu\nu} = \theta^{-2} g_{\mu\nu}$. Thus, if the initial data is given at or near conformal infinity, then one can construct a spacetime $(\tilde{\mathscr{M}}, \tilde{g}_{\mu\nu})$ with a radiation fluid for which a part reaches all the way to conformal infinity $\mathscr{I}$.

4 The Reference Spacetimes

The family of FLRW metrics will be considered as the reference spacetime (sometimes also referred to as the background solution) against which the stability analysis is carried out. The FLRW metric may be written in the form (see e.g. [3], page 471):

$$ds^2_{FLRW} = dt^2 - \frac{a(t)^2}{(1 + \frac{1}{4}kr^2)^2}(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)). \tag{6}$$

As is well known, the FLRW spacetimes are conformally flat and they can be suitably rescaled to conformally embed them into the Einstein cosmos, whose metric is given by:

$$ds^2_{EC} = d\tau^2 - d\sigma^2_{\mathbb{S}^3} = d\tau^2 - (d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2)).$$

In this article we shall focus on FLRW cosmologies whose spatial sections have positive curvature ($k = 1$). Changing coordinates we have

$$ds^2_{FLRW} = a(t)^2 \left[d\tau^2 - d\sigma^2_{\mathbb{S}^3} \right] = a(t)^2 ds^2_{EC}. \tag{7}$$

Thus, the conformal factor is given by $\theta = 1/a(t)$. Our analysis is restricted to trace-free perfect fluids ($\gamma = \frac{4}{3}$) with $\lambda < 0$ in (2). There exists a static solution amongst the remaining FLRW solutions for which

$$a(t) = a_0 = \text{constant}, \qquad \lambda = \lambda_0 \equiv \tfrac{3}{2} a_0^{-2}.$$

For the dynamical FLRW solutions in our family we can state the following:

Proposition 1 *For a FLRW cosmology with* $k = 1$, $\gamma = \frac{4}{3}$, $\dot{a}(t_0) > 0$ *and* $\lambda < 0$, $\lambda \neq \lambda_0$, *the scale factor,* $a(t)$, *is a smooth, non-vanishing and monotonically increasing function for* $t \in [t_0, \infty)$, *with* $t = t_0 > 0$ *and* $a_0 = a(t_0) > 0$. *Furthermore,*

$$\tau_\infty := \int_{t_0}^{\infty} \frac{ds}{a(s)} < \infty,$$

and one has the limits

$$a \to \infty, \quad \dot{a}/a \to \sqrt{-\tfrac{1}{3}\lambda}, \quad \ddot{a}/a \to -\tfrac{1}{3}\lambda,$$

as $t \to \infty$. *The density for these models is given by*

$$\tilde{\rho} = \tilde{\rho}_0 a_0^4 / a^4,$$

where $\tilde{\rho}_0 = \tilde{\rho}(t_0)$. *In particular, one has that* $\tilde{\rho} \to 0$ *as* $t \to \infty$.

The proof of this proposition follows from direct inspection of the explicit solutions—see e.g. [3, p. 78]. Moreover, the spacetime is de Sitter-like for late times, that is conformal infinity is given by a spacelike hypersurface.

It is straightforward to check directly that the Einstein cosmos is a regular solution of the CEFE for a radiation fluid and satisfies our gauge choices (in particular $R[g_{EC}] = -\frac{1}{6}$).

5 Stability

Note that the spatial slices of the Einstein cosmos and our FLRW solutions are 3-spheres. Hence for the stability analysis we consider an initial surface $\mathscr{S}$ with the topology of $\mathbb{S}^3$ and $\tilde{g}$-unit normal $\tilde{n}$.

In the sequel, it will be assumed that one has a solution $(\mathscr{S}, \tilde{h}_{\alpha\beta}, \tilde{K}_{\alpha\beta}, \tilde{\rho}, \tilde{u}^\alpha)$ to the (physical) $\lambda < 0$ Einstein-Euler perfect fluid constraint equations

$$\begin{aligned} \tilde{r} + \tilde{K}^2 - \tilde{K}_{\alpha\beta}\tilde{K}^{\alpha\beta} &= 2(\lambda - \tfrac{1}{3}\tilde{\rho}(4\tilde{u}_\parallel - 1)), \\ \tilde{D}^\alpha \tilde{K}_{\alpha\beta} - \tilde{D}_\beta \tilde{K} &= \tfrac{4}{3}\tilde{\rho}\tilde{u}_\parallel \tilde{u}_\beta, \end{aligned}$$

where $\tilde{D}_\beta$ and $\tilde{r}$ denote the Levi-Civita covariant derivative and the Ricci scalar of the intrinsic 3-metric $\tilde{h}_{\alpha\beta}$ of $\mathscr{S}$. $\tilde{K}_{\alpha\beta}$ is a symmetric 3-dimensional tensor corresponding to the extrinsic curvature of $\mathscr{S}$ with respect to the $\tilde{g}-$unit normal $\tilde{n}_\mu$. In addition, $\tilde{u}_\parallel \equiv \tilde{u}^\mu \tilde{n}_\mu$. Thus, one has Cauchy initial data for the Einstein-Euler system with a de Sitter-like cosmological constant and equation of state for pure radiation, $\tilde{p} = \frac{1}{3}\tilde{\rho}$.

From this data one can obtain a solution to the corresponding constraints induced by the CEFE.

For the stability analysis we make use of the Hilbert space $H^m(\mathbb{S}^3, \mathbb{R}^N)$ with $m \geq 4$—for more details see e.g. [5]. Using an extension of the general existence and stability Theorem by Kato [13] provided in [12] we are then able to prove the following more detailed version of Theorem 1.

Theorem 2 *Let* $\mathbf{w}_0$ *be initial data on* $\mathbb{S}^3$ *for CEFE for radiation fluids with* $\lambda < 0$ *such that* $\mathbf{w}_0$ *is sufficiently close to* $\mathring{\mathbf{w}}_0$ *(FLRW data with* $\lambda < 0$ *and* $k = 1$*). Then*

(i) *a solution* $\mathbf{w}$ *to the CEFE exists on* $[0, T] \times \mathbb{S}^3$ *with* $T > \tau_\infty$*,*
(ii) $\mathbf{w}$ *implies a* C^{m-2} *solution of the Einstein equations for a radiation fluid on* $\tilde{M} = \{p \in [0, T] \times \mathbb{S}^3 : \theta(p) > 0\}$*,*
(iii) *the development exists globally towards the future,*
(iv) $\tilde{\mathscr{M}}$ *is future geodesically complete and* $\mathscr{I}^+$ *is a space-like hypersurface.*
(v) $\mathbf{w}$ *remains close to the FLRW solution, and hence, it is nonlinearly stable.*

6 Discussion

As originally envisaged, the above theorem provides a proof for the case $\gamma = \frac{4}{3}$ missing in [9] using the conformal method. It should be noted that in the mean time the results in [9] have been extended to cover $\gamma = \frac{4}{3}$ as well [19]. In a next step one might consider spacetimes with null dust, such as the Robinson-Trautman and the Vaidya spacetimes, which describe radiating black holes. It remains to be seen whether the conformal method can in turn be generalised to cover more general perfect fluids with $\gamma \neq \frac{4}{3}$. This requires a formulation of the CEFE for matter models that have non-vanishing trace, which is an open problem in itself.

It is of interest, whether the CEFE for radiation fluid can be formulated using the alternative method with a congruence of conformal curves [10]. This method has the advantage that some of the PDEs reduce to transport equations along the congruence. Moreover, it allows one to prescribe/pretedict the location of conformal infinity, since the conformal factor is known a priori in terms of the specified initial data. This can simplify the explicit analysis of conformal infinity for the spacetime one is dealing with.

The use of the CEFE for the stability analysis presented here is restricted to regions near conformal infinity. The region near the initial singularity cannot be analysed as the CEFE are not regular any longer in the form given here. For an analysis of polytropic perfect fluid spacetimes near an initial singularity we refer the reader to [20]. The approach in [20] is based on conformal methods as well. However instead of the conformal factor vanishing, as it does at $\mathscr{I}$, it diverges to ∞ at the initial singularity. Nevertheless, the conformal geometry is perfectly regular in this region. It would be interesting to know whether a variant of the CEFE exists which is regular at the singularity and corresponds to the analysis in [20].

References

1. Einstein, A., Die Grundlage der allgemeinen Relativitätstheorie. Ann. Phys. (Leipzig) **49**, 769 (1916). doi:10.1002/andp.19163540702
2. Stephani, H., Kramer, D., MacCallum, M., Hoenselaers, C., Herlt, E.: Exact solutions of Einstein's field equations (2nd edn). Cambridge Monographs on Mathematical Physics. Cambridge University Press, New York (2003)
3. Griffiths, J., Podolský, J.: Exact space-times in Einstein's general relativity. Cambridge Monographs on Mathematical Physics. Cambridge University Press, New York (2009)
4. Dafermos, M., Rodnianski, I.: Lectures on black holes and linear waves, ArXiv e-prints arXiv:0811.0354 [gr-qc] (2008)
5. Friedrich, H.: On the existence of n-geodesically complete or future complete solutions of Einstein's field equations with smooth asymptotic structure. Commun. Math. Phys. **107**, 587 (1986). doi:10.1007/BF01205488
6. Friedrich, H.: On the global existence and the asymptotic behaviour of solutions to the Einstein-Maxwell-Yang-Mills equations. J. Differ. Geom. **34**, 275 (1991)
7. Christodoulou, D., Klainerman, S.: The Global Nonlinear Stability of the Minkowski Space, Princeton Mathematical Series, vol. 41. Princeton University Press, Princeton (1993)
8. Rodnianski, I., Speck, J.: The Stability of the irrotational Euler-Einstein system with a positive cosmological constant, ArXiv e-prints arXiv:0911.5501 [math-ph] (2009)
9. Speck, J.: The nonlinear future stability of the FLRW family of solutions to the Euler-Einstein system with a positive cosmological constant. Selecta Mathematica **18**, 633 (2012). doi:10.1007/s00029-012-0090-6
10. Lübbe, C., Valiente Kroon, J.A.: The extended conformal Einstein field equations with matter: the Einstein-Maxwell field. J. Geom. Phys. **62**, 1548 (2012). doi:10.1016/j.geomphys.2012.01.009
11. Friedrich, H.: Cauchy problems for the conformal vacuum field equations in general relativity. Commun. Math. Phys. **91**, 445 (1983). doi:10.1007/BF01206015
12. Friedrich, H.: On purely radiative space-times. Commun. Math. Phys. **103**, 35 (1986). doi:10.1007/BF01464281
13. Kato, T.: The Cauchy problem for quasi-linear symmetric hyperbolic systems. Arch. Ration. Mech. Anal. **58**, 181 (1975). doi:10.1007/BF00280740
14. Friedrich, H.: On the hyperbolicity of Einstein's and other gauge field equations. Commun. Math. Phys. **100**, 525 (1985). doi:10.1007/BF01217728
15. Friedrich, H.: Einstein equations and conformal structure: existence of anti-de Sitter-type space-times. J. Geom. Phys. **17**, 125 (1995). doi:10.1016/0393-0440(94)00042-3
16. Friedrich, H.: Conformal Einstein Evolution. In: Frauendiener, J., Friedrich, H. (eds.) The Conformal Structure of Space-Time: Geometry, Analysis, Numerics. Lecture Notes in Physics, pp. 1–102. Springer, New York (2002)
17. Lübbe, C., Valiente Kroon J.: A stability result for purely radiative spacetimes. J. Hyperbol. Differ. Equations **7**, 545 (2010). doi:10.1142/S0219891610002220
18. Choquet-Bruhat, Y.: General Relativity and the Einstein Equations. Oxford Mathematical Monographs. Oxford University Press, New York (2009)
19. Speck J.: Private communication
20. Anguige, K., Tod, K.: Isotropic cosmological singularities: I. Polytropic perfect fluid space-times. Ann. Phys. (N.Y.) **276**, 257 (1999). doi:10.1006/aphy.1999.5946