Source Integrals of Asymptotic Multipole Moments

Norman Gürlebeck

Abstract We derive source integrals for multipole moments that describe the behaviour of static and axially symmetric spacetimes close to spatial infinity. We assume isolated non-singular sources but will not restrict the matter content otherwise. Some future applications of these source integrals of the asymptotic multipole moments are outlined as well.

1 Introduction

In experiments that measure general relativistic effects, some parameters characterizing the spacetime have to be determined. The multipole moments are one set of such parameters. They are measured in the exterior region of astrophysical objects like neutron stars or galaxies but also planets and describe the gravitational field near spatial infinity. They will be called here *asymptotic multipole moments* (AMM). The bending of light and the gravitational lensing proved particularly useful for their measurement, see, e.g., [1–3] and references therein. But what information can be gathered about the matter distribution and the metric in its interior by their measurement? What does it mean to measure a certain value of the quadrupole moment? In Newtonian theory, the answers are provided by the source integrals of the AMM. These integrals determine the asymptotic multipole moments by an integration over the mass density. In general relativity, similar expressions for the AMM are only known in approximations to general relativity, see e.g. [4]. Here we will present

N. Gürlebeck (✉)
ZARM, University of Bremen, Am Fallturm, 28359 Bremen, Germany
e-mail: norman.guerlebeck@zarm.uni-bremen.de

N. Gürlebeck
EU and Institute of Theoretical Physics, Charles University, V Holešovičkách 2,
18000 Praha, Czech Republic

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source integrals of the AMM for static and axially symmetric spacetimes in full general relativity. At the same time these provide quasi-local definitions of *all* asymptotic multipole moments.

2 Preliminaries

In this section, we shortly review several concepts necessary in the derivation of the source integrals. We use throughout this article geometric units $G = c = 1$ and the signature of the metric is $(-, +, +, +)$.

2.1 *The Line Element and the Field Equations*

We concentrate on axially symmetric and static spacetimes of the Weyl form, i.e.,

$$ds^2 = \mathrm{e}^{2k-2U}\left(d\rho^2 + d\zeta^2\right) + W^2\mathrm{e}^{-2U}d\varphi^2 - \mathrm{e}^{2U}dt^2. \tag{1}$$

We do not restrict the type of matter except in that the line element (1) can be introduced, see [5]. The metric functions e^{2U} and W can be expressed by the timelike Killing vector $\xi^\alpha = (\partial_t)^\alpha$ and the Killing vector of the axial symmetry $\eta^\alpha = (\partial_\varphi)^\alpha$

$$\mathrm{e}^{2U} = -\xi_\alpha\xi^\alpha, \quad W^2 = -\eta_\alpha\eta^\alpha\xi_\beta\xi^\beta. \tag{2}$$

Let us choose a sphere $\mathscr{S}_0$ of finite radius $r = R_0$ $(\rho = r\sin\theta,\ \zeta = r\cos\theta)$ that covers the entire matter distribution, cf. Fig. 1. Outside of $\mathscr{S}_0$, canonical Weyl coordinates $(W = \rho)$ are introduced by virtue of one of the vacuum field equations. This allows still a shift in the $\zeta-$coordinate, which enables us later to move the origin with respect to which the AMM are measured. The vacuum field equations in canonical Weyl coordinates read

$$\Delta U = 0, \quad k_{,\zeta} = 2\rho U_{,\rho}U_{,\zeta}, \quad k_{,\rho} = \rho\left((U_{,\rho})^2 - (U_{,\zeta})^2\right), \tag{3}$$

where $\Delta = \left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial\zeta^2}\right)$. The function k is determined via a line integration, cf. the last two equations of (3), once U is known. Hence, only a Laplace equation for U remains to be solved in practice.

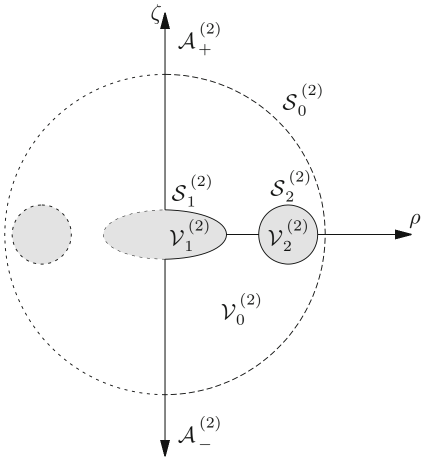

Fig. 1 An example of the physical situations discussed here: The surfaces of the individual matter components are denoted by $\mathscr{S}_i^{(2)} = \partial \mathscr{V}_i^{(2)}$ with $i \geq 1$ and their respective volumes by $\mathscr{V}_i^{(2)}$. The surface $\mathscr{S}_0^{(2)}$ describes a *circle* with sufficiently large but finite radius enclosing all matter components, cf. Sect. 2.1

2.2 The Physical Setting

We depict in Fig. 1 an example of a physical situation that will be covered by the subsequent considerations. The relevant surfaces and volumes are defined there as well. For simplicity, we allow only non-singular sources. However, such can be incorporated into the formalism as we showed in [6]. The 3-dimensional projection of the matter region into an hypersurface of constant Killing-time t is obtained by a rotation around the $\zeta-$axis in Fig. 1. In this way, the quantities $\mathscr{A}_\pm^{(3)}$, $\mathscr{S}_i^{(3)}$ and $\mathscr{V}_i^{(3)}$ are defined starting from $A_\pm^{(2)}$, $\mathscr{S}_i^{(2)}$ and $\mathscr{V}_i^{(2)}$, respectively.

2.3 The Multipole Moments

For asymptotically flat and static spacetimes a geometric definition of AMM was given by Geroch in [7]. This definition was generalized and applied by many authors, see the reviews [4, 8] and references therein. In the axially symmetric case with the line element (1), Geroch's multipole moments M_r can be obtained by an expansion of U along the symmetry axis in $|\zeta|^{-1}$:

$$U(\rho = 0, \zeta) = \sum_{r=0}^{\infty} U^{(r)} |\zeta|^{-r-1}. \tag{4}$$

The M_r follow uniquely from Weyl's multipole moments $U^{(r)}$ and vice versa as was shown in [9]. Therefore, we consider only the $U^{(r)}$ here.

2.4 The Inverse Scattering Technique

Lastly, we shortly review the inverse scattering technique (IST), see e.g. [10] for a recent account. Even though the Laplace equation is linear and the use of the IST seems artificial, the IST proves nonetheless beneficial, because it is easily generalizable to the non-linear case of the Ernst equation. This equation is of special interest in relativistic astrophysics, since it describes the exterior of rotating stars. The starting point of the IST in the present setting, i.e. the linear problem of the Laplace equation, is given by

$$\sigma_{,z} = (1+\lambda)U_{,z}\sigma, \quad \sigma_{,\bar{z}} = \left(1+\frac{1}{\lambda}\right)U_{,\bar{z}}\sigma, \tag{5}$$

where $z = \rho + \mathrm{i}\zeta$, the spectral parameter $\lambda = \sqrt{\frac{K-\mathrm{i}\bar{z}}{K+\mathrm{i}z}}$, $K \in \mathbb{C}$ and a bar denotes complex conjugation. The complex valued function σ depends on z, $\bar{z}$ and λ. The integrability condition of (5) is the Laplace equation for U. Therefore, having a solution σ of (5) yields also a solution U of the Laplace equation and vice versa. The main technical steps of the IST as described in [10] are to integrate (5) along $\mathscr{A}_{\pm}^{(2)}$, along a circle with sufficiently large radius and along a compact curve connecting $\mathscr{A}_{+}^{(2)}$ with $\mathscr{A}_{-}^{(2)}$. This scheme can be carried out partially and we quote only the results (simplified to the static case), which are relevant for us, from [10]:

$$\begin{aligned}
(0,\zeta) \in \mathscr{A}^{+}: \ & \sigma\,(\lambda = +1, \rho = 0, \zeta) = F(K)\mathrm{e}^{2U(\rho=0,\zeta)}, \\
& \sigma\,(\lambda = -1, \rho = 0, \zeta) = 1, \\
(0,\zeta) \in \mathscr{A}^{-}: \ & \sigma\,(\lambda = +1, \rho = 0, \zeta) = \mathrm{e}^{2U(\rho=0,\zeta)}, \\
& \sigma\,(\lambda = -1, \rho = 0, \zeta) = F(K).
\end{aligned} \tag{6a}$$

The function $F : \mathbb{C} \to \mathbb{C}$ is given for $K \in \mathbb{R}$ with $(\rho = 0, \zeta = K) \in \mathscr{A}^{\pm}$ by

$$F(K) = \begin{cases} \mathrm{e}^{-2U(\rho=0,\zeta=K)} & (0,K) \in \mathscr{A}^{+} \\ \mathrm{e}^{2U(\rho=0,\zeta=K)} & (0,K) \in \mathscr{A}^{-} \end{cases}. \tag{6b}$$

The integration along $\mathscr{S}_0^{(2)}$ does not enter the derivation of these formulas and it forms the crucial part of our considerations in the next section.

3 Source Integrals of Weyl's Multipole Moments

The derivation of the source integrals consists of several steps. First, the AMM are expressed as line integrals along $\mathscr{S}_0^{(2)}$. This is the most important step, because it makes it possible to determine the AMM quasi-locally. Then these integrals will be rewritten in a coordinate independent form as surface integrals over $\mathscr{S}_0^{(3)}$ by virtue of the axial symmetry. Subsequently, Stokes' theorem is used to rewrite these as volume integrals over $\mathscr{V}^{(0)}$. In the final step, it is shown that the contributions in the vacuum regions vanish. Thus, the steps from before can be retraced to obtain the contributions in source integral form of each individual matter component. We suppress the details of these derivations and show only the crucial steps.

The linear problem (5) is well-defined along $\mathscr{S}_0^{(2)}$ and reads:

$$\sigma_{,s} = \left[U_{,s} + \frac{1}{2}\left(\left(\frac{1}{\lambda} + \lambda\right)U_{,s} + \mathrm{i}\left(\frac{1}{\lambda} - \lambda\right)U_{,n}\right)\right]\sigma, \tag{7}$$

where $U_{,s}$ and $U_{,n}$ are the tangential and the (outward pointing) normal derivative of U along $\mathscr{S}_0^{(2)}$ with respect to a parametrisation $[s_N, s_S] \to \mathscr{S}_0^{(2)}$. The indices N and S refer to the values of a parameter or function at the "north" and "south" pole of $\mathscr{S}_0^{(2)}$, i.e., to the intersection points ($\rho = 0$, $\zeta = \zeta_{N/S}$) of $\mathscr{S}_0^{(2)}$ and the symmetry axis. Equation (7) is easily integrated using the boundary values from (6):

$$U(0, K) = \frac{U_N - U_S}{2} + \frac{1}{4}\int_{s_N}^{s_S}\left(\left(\lambda^{-1} + \lambda\right)U_{,s} + \mathrm{i}\left(\lambda^{-1} - \lambda\right)U_{,n}\right)\mathrm{d}s. \tag{8}$$

If we expand this equation in $|K|^{-1}$, we obtain expression for Weyl's multipole moments in terms of a line integration. Let us introduce the abbreviations $N_+^{(r)}$ and $N_-^{(r)}$ for the expansion coefficients $(\lambda^{-1} + \lambda)^{(r)}$ and $\mathrm{i}(\lambda^{-1} - \lambda)^{(r)}$ to order $r + 1$, respectively. After a lengthy but straightforward calculation they evaluate to

$$\begin{aligned} N_-^{(r)} &= \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \frac{2(-1)^{k+1} r! \rho^{2k+1} \zeta^{r-2k}}{4^k (k!)^2 (r-2k)!}, \\ N_+^{(r)} &= \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} \frac{2(-1)^{k+1} r! \rho^{2k+2} \zeta^{r-2k-1}}{4^k (k!)^2 (r-2k-1)!(2k+2)}. \end{aligned} \tag{9}$$

The $r = -1$ order in $|K|^{-1}$ of (8) is satisfied trivially and will not be considered subsequently. The orders $r \geq 0$ of (8) yield the desired quasi-local definitions of Weyl's multipole moments:

$$U^{(r)} = \frac{1}{4}\int_{\mathscr{S}_0^{(2)}} \left(N_+^{(r)} U_{,\hat{s}} + N_-^{(r)} U_{,\hat{n}}\right)\mathrm{d}\mathscr{S}_0^{(2)}, \tag{10}$$

where $U_{,\hat{s}}$ and $U_{,\hat{n}}$ are the tangential and normal derivatives along $\mathscr{S}_0^{(2)}$ with respect to the unit tangent vector and the unit normal vector, which are defined with the induced metric on $\mathscr{S}_0^{(2)}$; $\mathrm{d}\mathscr{S}_0^{(2)}$ denotes the proper distance along $\mathscr{S}_0^{(2)}$. The functions $N_\pm^{(r)}$ and U are to be read as functions along $\mathscr{S}_0^{(2)}$, i.e. as functions of $(\rho(s), \zeta(s))$.

To make the coordinate independence apparent, we express ρ and ζ by scalars built from the Killing vectors. Firstly, observe that (2) holds everywhere and that in vacuum we have $W = \rho$. Additionally, the 1-form

$$Z_\alpha = \varepsilon_{\alpha\beta\gamma\delta} W^{,\beta} W^{-1} \eta^\gamma \xi^\delta \tag{11}$$

is well-defined and hypersurface orthogonal everywhere as well as exact in the vacuum region. Hence, there exist a potential Z and an integrating factor X such that $Z_{,\alpha} = X Z_\alpha$, where $X = 1$ in the exterior of $\mathscr{S}_0^{(3)}$. In the vacuum region and in canonical Weyl coordinates, we find $Z = \zeta + \text{const}$. Since we can shift the ζ-coordinate freely, we can drop the constant of integration, which specifies the origin with respect to which the AMM are measured. Thus, W and Z coincide with ρ and ζ in the vacuum region and can be used as their continuation into the interior of the matter. This choice is not unique and other continuations are possible, although they do not alter the values of the source integrals, which we present below.

Using W and Z along $\mathscr{S}_0^{(2)}$ instead of ρ and ζ, respectively, we can rewrite (10) as surface integrals:

$$U^{(r)} = \frac{1}{8\pi} \int\limits_{\mathscr{S}_0^{(3)}} \frac{\mathrm{e}^U}{W} \left(N_-^{(r)} U_{,\hat{n}} - N_{+,W}^{(r)} Z_{,\hat{n}} U + N_{+,Z}^{(r)} W_{,\hat{n}} U \right) \mathrm{d}\mathscr{S}_0^{(3)}. \tag{12}$$

An integration by parts, the axial symmetry and the vacuum field equations are necessary for this step.

Using Stokes' theorem and the field equations we obtain

$$\begin{aligned} U^{(r)} = & \frac{1}{8\pi} \int\limits_{\mathscr{V}_0^{(3)}} \mathrm{e}^U \left[-\frac{N_-^{(r)}}{W} R_{\alpha\beta} \frac{\xi^\alpha \xi^\beta}{\xi^\gamma \xi_\gamma} + N_{+,Z}^{(r)} U \left(\frac{W^{,\alpha}}{W} \right)_{;\alpha} \right. \\ & \left. - N_{+,W}^{(r)} U \left(\frac{Z^{,\alpha}}{W} \right)_{;\alpha} + N_{+,WZ}^{(r)} \frac{U}{W} \left(W^{,\alpha} W_{,\alpha} - Z^{,\alpha} Z_{,\alpha} \right) \right] \mathrm{d}\mathscr{V}_0^{(3)} \\ = & \frac{1}{8\pi} \sum_i \int\limits_{\mathscr{V}_i^{(3)}} \mathrm{e}^U \left[8\pi \frac{N_-^{(r)}}{W} (T g_{\alpha\beta} - T_{\alpha\beta}) \frac{\xi^\alpha \xi^\beta}{\xi^\gamma \xi_\gamma} + N_{+,Z}^{(r)} U \left(\frac{W^{,\alpha}}{W} \right)_{;\alpha} \right. \\ & \left. - N_{+,W}^{(r)} U \left(\frac{Z^{,\alpha}}{W} \right)_{;\alpha} + N_{+,WZ}^{(r)} \frac{U}{W} \left(W^{,\alpha} W_{,\alpha} - Z^{,\alpha} Z_{,\alpha} \right) \right] \mathrm{d}\mathscr{V}_i^{(3)}. \end{aligned} \tag{13}$$

The $\mathrm{d}\mathscr{V}_i^{(3)}$ are the proper volume elements of $\mathscr{V}_i^{(3)}$ and a semicolon denotes the covariant derivative with respect to the line element (1). The last equality is due to Einstein's equations, which imply that the integrand vanishes in vacuum. The integrals (13) are the desired source integrals. They determine the AMM from the geometry inside the matter regions alone. Of course, Stokes' theorem can again be used to rewrite the source integrals as surface integrals over $\mathscr{S}_i^{(3)}$ of the respective matter component. In turn, these can be reformulated as line integrals, cf. Sect. 4. The fact that the contributions of the individual matter components, $\mathscr{V}_i^{(3)}$, to the asymptotic multipole moments superpose linearly is due to the choice of Weyl's multipole moments. If we employ the method from [9] to calculate Geroch's multipole moments M_r from Weyl's multipole moments $U^{(r)}$, we obtain a mixing of the contributions $U_i^{(k)}$ of the individual matter components with $k < r$ in the M_r. This is already apparent for the quadrupole moment M_2, which depends non-linearly on $U^{(0)}$:

$$M_2 = U^{(2)} - \frac{1}{3} {U^{(0)}}^3. \tag{14}$$

The Geroch mass M_0 equals $U^{(0)}$ and is given by the (negative) Komar integral. This follows also from (12) with $r = 0$.

4 Applications

We conclude the paper by discussing one possible application of the source integrals (12). Assume a matter distribution is given, where the metric is known in the interior or the Dirichlet and the Neumann data for U are known at the surface. Even then it is far from trivial (at least in the stationary case, see [11]) to obtain a global asymptotically flat solution, if it exists. The source integrals for the AMM provide a tool to solve this task. As a simple example serves here the case of static dust without any surface distributions. In Weyl coordinates (not necessarily canonical) the energy momentum tensor is given by

$$T_{\alpha\beta} = \mu e^{2U} \delta^t_\alpha \delta^t_\beta. \tag{15}$$

The contracted Bianchi identities imply U = const. in the interior and, thus, the gradient of U vanishes at $\mathscr{S}_i^{(3)}$ in all coordinates. Using the line integrals for Weyl's AMM, which follow from (13), we get:

$$U^{(r)} = \frac{1}{4} \sum_i \int\limits_{\mathscr{S}_i^{(2)}} \left(N_-^{(r)} U_{,\hat{n}} + N_+^{(r)} U_{,\hat{s}} \right) \mathrm{d}\mathscr{S}_i^{(2)} = 0. \tag{16}$$

Thus, all AMM vanish and the spacetime is flat in the exterior. This contradicts the presence of a dust distribution with positive mass density. Of course, this result is already known and more general non-existence results for dust including the rotating case can be found in [12, 13] and references therein. Although the non-existence is proved here, this example shows in a concise way how the source integrals can be applied in more difficult physical situations like rotating stars. This and other applications, e.g. to tidal distortions of black holes, will be investigated in future work.

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