The Lattice of Definability. Origins, Recent Developments, and Further Directions

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Abstract. The paper presents recent results and open problems on classes of definable relations (definability spaces, reducts, relational algebras) as well as sources for the research starting from the XIX century. Finiteness conditions are investigated, including quantifier alternation depth and number of arguments width. The infinite lattice of definability for integers with a successor function (a non ω-categorical structure) is described. Methods of investigation include study of automorphism groups of elementary extensions of structures under consideration, using Svenonius theorem and a generalization of it.

Keywords: Definability, definability space, reducts, Svenonius theorem, quantifier elimination, decidability, automorphisms.

"Mathematicians, in general, do not like to operate with the notion of definability; their attitude towards this notion is one of distrust and reserve."
Alfred Tarski [Tar4]

1 Introduction. The Basic Definition

One of the "most existential" problems of humanity is "How to define something through something being understood yet". It sounds even more important than "What is Truth?" [John 18:38].

Let us recollect our understanding of the problem in the context of modern mathematics.

Language (of definition):

- Logical symbols including connectives, (free) variables: x_0, x_1, ..., quantifiers and equality $=$.
- Names of relations (sometimes, names of objects and operations as well). The set of these names is called signature. Each name has its number of arguments (its arity).
- Formulas.

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Structure of signature Σ is a triple $\langle D, \Sigma, \mathsf{Int}\rangle$. Here D is a set (mostly countable infinite in this paper) called the *domain* of the structure, Int is an *interpretation* that maps every n-ary name of relation to n-ary relation on D, in other words a subset of D^n.

If the structure of signature Σ is given then every formula in the language with this signature defines a relation on D.

The most common language uses quantifiers over D. But sometimes we consider other options like quantifiers over subsets of D (monadic language), over finite subsets of D (weak monadic language), or no quantifiers at all (quantifier-free language).

Let us fix a domain D and any set of relations S on D. Then we can take any finite subset of S and give names to its elements. We have now a structure and interpretation for elements of its signature as was given beforehand. Any formula in the constructed language defines a relation on D. We call any relation on D obtained in this way *definable* in S (over D).

All relations definable in S constitute *closure* of S. Any closed set of relations is called *definability space* (over D).

Any definability space S has its *group of automorphisms* $\mathsf{Aut}(S)$ i. e. permutations on D that preserve all relations from the space S.

All definability spaces over D constitute the *lattice of definability* with natural lattice operations. Evidently lattices for different (countable infinite) D are isomorphic. In other words we have one lattice only.

Investigation of this lattice is the major topic of our paper. In particular we consider lattice of subspaces of a given definability space. The subspaces are called also *reducts* of the space.

We shall consider finitely generated spaces mostly.

If a set of generators for a space is given we can define the theory of the space. Choosing a system of generators is like choosing a coordinate system for a linear space. For finitely generated spaces such properties as decidability of their theory is invariant for the choosing different sets of generators.

Today we feel that the concept of definability space (independently of formal definition and terminology) is very basic and central for mathematical logic and even for mathematics in general. As we will see from the next chapter it was in use as long as the development of the very discipline of mathematical logic was happening. Nevertheless the major results concerning it, including precise definitions and fundamental theorems were obtained quite late and paid much less respect than those concerning notions of "Truth" and "Provability".

We try to keep our text self-contained and introduce needed definitions.

2 The History and Modern Reflections

In our short historical survey we use materials from [BuDa, Smi, Hod].

We try to trace back original sources and motivations. In some important cases understanding of problems and meaning of notions were changed considerably over time. It is important to consider original ideas along with their maturity 30 years and even much later. As we will see the scene of events was pretty much international.

2.1 Relations, Logic, Languages. Early Approaches of XIX Century

Our central notion of an assignment satisfying a formula is implicit in George Peacock [Pea] and explicit in George Boole [Boo], though without a precise notion of "formula" in either case.

In 1860 – 1890-s Frege developed understanding of relations and quantifiers [Fre, Fre1].

Peirce established the fundamental laws of the calculus of classes and created the theory of relatives. Essentially it was the definability issue. Starting with his 1870 paper [Pei], Peirce presented the final form of his first-order logic in his 1885 paper [Pei1]. Pierce's theory was the frame that made possible the proof by Leopold Löwenheim of the first metamathematical theorem in his [Löw]. Löwenheim proved that every first-order expression [Zählausdrücke] is either contradictory or already satisfiable in a countable infinite domain (see [Sko]). So, the major concepts of semantics were used by Löwenheim as well as Thoralf Skolem, but were not explicitly presented in their papers.

Schröder proposed the first complete axiomatization of the calculus of classes and expanded considerably the calculus and the theory of relatives [Sch, Sch1].

2.2 Automorphisms. Isomorphisms

With the appearance of Klein's Erlangenprogramm in 1872 [Kle] it became clear that automorphism groups are useful means of studying mathematical theories.

The word "isomorphism" appeared in the definition of categoricity bt Huntington [Hun]. There he says that "special attention may be called to the discussion of the notion of isomorphism between two systems, and the notion of a sufficient, or categorical, set of postulates".

Alfred Tarski in his paper "What are Logical Notions?", presented first in 1963 [Tar4] explains the subject of logic as study of "everything" up to permutations: "I shell try to extend his [Klein's] method beyond geometry and apply it also to logic ... I use the term "notion" in a rather loose and general sense ... Thus notions include individuals, classes of individuals, relations on individuals".

2.3 How to Define Major Mathematical Structures? Geometry and Numbers. 1900-s. The Width

At the end of XIX century Italian (Giuseppe Peano, Alessandro Padoa, Mario Pieri, ...) and German (Gotlob Frege, Moritz Pasch, David Hilbert, ...) mathematicians tried to find "the best" set of primitive notions for Geometry and Arithmetic considered as deductive systems. This was about "how to define something through something".

In August 1900 The First International Congress of Philosophy [ICP] followed by the Second International Congress of Mathematicians [ICM] met in Paris.

At the mathematical congress Hilbert presented his list of problems [Hil], some of which became central to mathematical logic, Padoa gave two talks on the axiomatizations of the integers and of geometry.

At the philosophical congress Russell read a paper on the application of the theory of relations to the problem of order and absolute position in space and time. The Italian school of Peano and his disciples contributed papers on the logical analysis of mathematics. Peano and Burali-Forti spoke on definitions, Pieri spoke on geometry considered as a purely logical system. Padoa read his famous essay containing the "logical introduction to any theory whatever", where he states:

> "To prove that the system of undefined symbols is irreducible with respect to the system of unproved propositions [axioms] it is necessary and sufficient to find, for any undefined symbol, an interpretation of the system of undefined symbols that verifies the system of unproved propositions and that continues to do so if we suitably change the meaning of only the symbol considered."

Pieri formulated about 1900 and completed in his 1908 "Point and Sphere" memoir, a full axiomatization of Euclidean geometry based solely on the undefined notions point and equidistance of two points from a third point [Pie].

Tarski's undefined notions were point and two relations: congruence of two point pairs and betweenness of a triple. Tarski and Adolf Lindenbaum [LiTa] showed that in the first-order context, Pieri's selection of equidistance as the sole undefined relation for Euclidean geometry was optimal. No family of binary relations, however large, can serve as the sole undefined relations.

We considered the problem of minimization of maximal number of arguments in generators of a given definability space.

Definition 1. *Let a definability space S is given. Its* width *is the minimal n such as S can be generated by relations with n or less arguments.*

Theorem 1. [Sem] *There are definability spaces of any finite or countable width.*

Huntington and Oswald Veblen were part of a group of mathematicians known as the American Postulate Theorists. Huntington was concerned with providing "complete" axiomatizations of various mathematical systems, such as the theory of the algebra of logic and the theory of real numbers. In 1935 Hungtington published [Hun1] "Inter-relations among the four principal types of order", where he says:

> "The four types of order whose inter-relations are considered in this paper may be called, for brevity, (1) serial order; (2) betweenness; (3) cyclic order; and (4) separation."

These "four types of order" will play special role in the further developments discussed in the present paper.

2.4 The Exact Formulation of Definability

Indirectly the notion of truth and even more indirectly definability were present from beginning of 1900-s and even earlier. For example he word “satisfy” in this context may be due to Huntington (for example in [Hun2]). We mentioned works of Löwenheim and Skolem.

But only the formal (“usual” inductive) definition of truth by Tarski gives the modern (model-theoretic) understanding of semantics of a formula as a relation over a domain [Tar].

Complications in understanding today of Tarski and Lindenbaum meaning of Padoa’s method (relevant for our considerations) are discussed in [Hod1].

2.5 Elimination of Quantifiers

In the attempts to describe meaning of logical formulas and to obtain “decidability” (in the sense “to prove or disprove”) versions of quantifier elimination were developed in 1910 – 1930-s. Remarkable results were published in the end of 1920-s.

C. H. Langford used this method in 1927 [Lan, Lan1] to prove decidability of the theory of dense order without a first or last element.

Mojżesz Presburger [Pre] proved elimination of quantifiers for the additive theory of the integers.

Not using the formal (Tarski-style) definition Skolem illustrated in 1929 an elimination [Sko1] for the theory of order and multiplication on the natural numbers. The complete proof was obtained by of Mostowski in 1952 [Mos].

Tarski himself announced in 1931 a decision procedure for elementary algebra and geometry (published however only in 1948, see [Tar1]).

Elimination of quantifiers was considered as a part of introducing semantics. A natural modern definition appealing to finite signature was not used. In fact, both Presburger and Tarski structures DO NOT permit elimination of quantifiers in this sense. But in these cases you can either choose using operations and terms in atomic formulas, or take a finite set of generators,

then every formula can be effectively transform to an equivalent of a limited quantifier depth (the number of quantifier changes).

Let S be a definability space generated by a finite set of relations F. Consider a quantifier hierarchy of subsets of S: $F_0, F_1, \ldots$. Here F_0 is a quantifier-free (Boolean) closure of F, for every $i = 0, 1, \ldots F_{i+1}$ is obtained from F_i by taking all projections of its relations (adding of existential quantifiers) and then getting Boolean closure. (An alternative definition can be given by counting quantifier alternations.) The hierarchy can be of infinite length if F_{i+1} differs of F_i for all i, or finite length n – minimal for which $F_{n+1} = F_n$.

Here are several well-known examples. We indicate informally the structure and the – length of the hierarchy for it:

- $\langle \mathbb{Q}; < \rangle$ – 0.
- Dense order $[0, 1]$ – 0 (if we include these elements into signature).

- $\langle \mathbb{Z}; +1 \rangle$ – 1.
- Presburger arithmetic – 1. Linear forms, congruences module m can be introduced via existential quantifiers. Extensions of + with rapidly growing functions [Sem1].
- Tarski algebra – 1. Again, polynomials can be explained with existential quantifiers only.
- Skolem arithmetic – 1.
- Multiple successor arithmetic (automata proofs) – 1.
- Arithmetic of + and $\times$ – infinity.

A priory the length of the hierarchy for the space S can depend of the choice of (finite) F.

Problem 1. Can the hierarchy length be really different for different choices of F?

Definition 2. *The* depth *of a definability space is the minimal (over all finite sets of generators) length of the quantifier hierarchy for it.*

In [Sem] a problem on existing of other options was formulated. The answer was obtained in 2010:

Theorem 2. [SeSo] *There are spaces of arbitrary finite or infinite depth.*

Problem 2. Are there “natural” examples of “big” (2, 3, 4, . . .) finite depth?

Problem 3. What is the depth of Rabin Space [Rab], [Muc]?

2.6 Decidability

Decidability in the sense of existing an algorithm to decide is a statement (closed formula) true or false was a key question of study. For example, Tarski result on the field of reals implies the decidability of Geometry. The decidability results for multiple successor arithmetic led Elgot and Rabin to the following problem

Problem 4. [ElRa] Does there exist a structure with maximally decidable theory?

We say that a finitely generated definability space has a *maximally decidable theory* iff its theory is decidable and any greater finitely generated definability space does not have a decidable theory.

Soprunov proved in [Sop] (using forcing arguments) that every space in which a regular ordering is definable is not maximal. A partial ordering $\langle B; < \rangle$ is said to be *regular* if for every $a \in B$ there exist distinct elements $b_1, b_2 \in B$ such that $b_1 < a, \quad b_2 < a$, and no element $c \in B$ satisfies both $c < b_1$ and $c < b_2$. As a corollary he also proved that there is no maximal decidable space if we use weak monadic language for definability instead of our standard language.

In [BeCe], Bès and Cégielski consider a weakening of the Elgot – Rabin question, namely the question of whether all structures M whose theory is decidable can be expanded by some constant in such a way that the resulting structure still has a decidable theory. They answer this question negatively by proving that there exists a structure M with a decidable theory (even monadic theory) and such that any expansion of M by a constant has an undecidable theory.

In [BeCe1] they indicate a sufficient condition for a space with decidable theory no to be maximal.

In our context it is natural to consider also decidability of elements of a definability space. Of course we need a "constructivisation" of the domain D. For example, we can take natural numbers as it.

Definition 3. *We call a space* decidable *if all its elements are decidable. We call a finitely generated space* uniformly decidable *if there is an algorithm providing a decision procedure for any formula (using the generators) and any vector of its arguments.*

Problem 5. [Sem], 2003. Are there spaces of arbitrary finite or infinite depth with decidable theory?

Problem 6. Are there decidable and uniformly decidable spaces of arbitrary finite or infinite depth?

Problem 7. Does there exist a maximal decidable structure?

We say that a finitely generated definability space is *maximal decidable* iff it's decidable but any greater finitely generated definability space is not decidable.

As it was shown in [Sem1] there is an unary predicate R for which the space generated by $+$, R on the domain of natural numbers is decidable, but not uniformly, and has an undecidable theory.

3 General Fundamental Theorems on Definability vs. Provability and Automorphisms. 1950 -s

Buchi and Danhof [BuDa] outlined the transition between end of 1930-s and end of 1950-s:

> "At this time it might have seemed that most of the basic problems of elementary axiom systems were solved. A more careful observer however, upon reading the papers of Tarski [Tar2, Tar3], might have wondered about the existence of general theorems which would explain elementary definability as the above theorems explain the basic properties of elementary logical consequence.
>
> One such theorem, the completeness, in the sense of definability, of elementary logic was proved by Beth in 1953 [Bet]. In 1959 Svenonius [Sve] published a further result on elementary definability. Just as with the earlier results of Beth and Craig, logicians seem slow in recognizing Svenonius' theorem as a basic tool in the theory of definability, perhaps because it is not generally known to be available."

These results are generally considered as realization of Padoa's idea (or "method").

Let Σ is a signature, we say that $M' = \langle D', \Sigma, \mathsf{Int}' \rangle$ is an *extension* of $M = \langle D, \Sigma, \mathsf{Int} \rangle$ if D is a subset of D' and $\mathsf{Int}(R)$ is the restriction of $\mathsf{Int}'(R)$ on D for any $R \in \Sigma$.

We say that M' is an *elementary extension* of M if the previous condition holds for any definable relation, i. e. if R is definable in M relation, then R is the restriction on D of the relation, definable in M' by the same formula.

In our context Svenonius' theorem is the most useful tool. Here is its suitable formulation.

Theorem 3. (Svenonius Theorem) *Let M — countable structure with signature Σ^+ and let $\Sigma \subset \Sigma^+, R \in \Sigma^+$. The following statements are equivalent:*

(i) Relation R belongs to closure of Σ in M,

(ii) For any M' countable elementary extension of M and any permutation of the domain of M' which preserves Σ, preserves R.

The idea here is to use an additional structure to the original one and consider its elementary extensions. The additional structure narrows the class of extensions and makes the extensions more comprehensible, so we can find the needed automorphism.

In fact, we can use one universe only in a modification of the theorem as was shown in [SeSo1].

By $\mathcal{F}$ we denote the set of everywhere defined functions $f\colon \mathbb{N} \to D$. If R is n-ary relation on D and φ is a mapping $\mathcal{F} \to \mathcal{F}$ then we say that φ *almost preserves* R if $\{i \mid R(f_1(i), \dots, f_n(i)) \not\equiv R(\varphi(f_1)(i), \dots, \varphi(f_n)(i))\}$ is finite for any $f_1, \dots, f_n$ in $\mathsf{Dom}(\varphi)$.

Theorem 4. (CH) *Let S be a definability space. The following conditions are equivalent:*

(1) Relation $R \in S$,

(2) any permutation φ on $\mathcal{F}$ which almost preserves all relations from S almost preserves R.

The remarkable feature of this form of Svenonius Theorem is that the condition (2) is purely combinatorial, not appealing to any logical language.

4 The Definability Lattice

Numerous results were devoted to the study of specific definability spaces. For example, Inan Korec in [Kor] surveyed different natural generation sets for the definability space generated by addition and multiplication of integers.

Cobham — Semenov's theorem [Sem2] states that nontrivial intersection of spaces generated by automata working in different bases should be exactly the space generated by +. (This will be considered later in the context of self-definability of Muchnik.)

4.1 Authomorphisms and Galois Correspondence. ω-categoricity

As we see in Svenonius theorem the authomorphism group is an important object in the study of definability spaces.

The *symmetric group* $\mathsf{Sym}(D)$ on a set D is the group consisting of all permutations of D.

There is a natural topology on the symmetric group, we mean the topology of pointwise convergence: a basis of neighborhoods of an element consists of all permutations that coincide with the element on a finite set.

It's easy to see that for spaces S and T we have $S \subseteq T \Rightarrow \mathsf{Aut}(S) \supseteq \mathsf{Aut}(T)$ and that automorphism groups for spaces are closed. So, we can call groups corresponding to reducts of a space S supergroups of $\mathsf{Aut}(S)$.

Groups for different spaces can coincide.

An ω*-categorical* structure is one for which all countable structures that are elementary equivalent to it are isomorphic to it.

For ω-categorical structures definability subspaces are in one-to-one correspondence with closed automorphism groups, so $S \subseteq T$ iff $\mathsf{Aut}(S) \supseteq \mathsf{Aut}(T)$, i.e the correspondence between definability spaces and their automorphism groups is an antitone Galois connection.

It immediately follows from Svenonius theorem, but in the special case of ω-categoricity it may be concluded from so called Engeler – Ryll-Nardzewski – Svenonius Theorem (see e. g. [Hod2]).

4.2 The Rational Order. Homogeneous Structures

We start with a case of the most famous definability space where all subspaces were discovered first. This result describing the lattice of subspaces of $\langle \mathbb{Q}; < \rangle$ was obtained by Claude Frasnay in 1965 [Fra]. All subspaces of rational order are given by the following descriptions:

- One may view the ordering up to reversal, and so obtain a (ternary) linear Betweenness relation B on $\mathbb{Q}$, where $B(x; y, z)$ holds if and only if $y < x < z$ or $z < x < y$.
- Alternatively, by bending the rational line into a Circle one obtains a natural (ternary) circular ordering K on $\mathbb{Q}$; here, $K(x, y, z) \Longleftrightarrow (x < y < z) \vee (y < z < x) \vee (z < x < y)$.
- The latter too may be viewed up to reversal, to obtain the (quaternary) Separation relation S: $S(x, y; z, w)$ if the points x, y in the circular ordering separate the points z, w.

The remarkable fact is that these are exactly the structures that in axiomatic form were described by Huntington in 1935 [Hun1] (as was mentioned above).

The structure $\langle \mathbb{Q}; < \rangle$ is ω-categorical. The method of proof for this is "back-and-forth" argument discovered by Huntington (not Cantor) [Hun3]. In fact the proof shows that $\langle \mathbb{Q}; < \rangle$ is homogeneous in the following sense.

Definition 4. *A structure M is* homogeneous *if every isomorphism between its finite substructures extends to an automorphism of M.*

This definition is a generalization of its "group counterpart".

Definition 5. *A permutation group is* homogeneous *iff any finite subset of its domain can be translated to any other subset of the same cardinality with an element of the group.*

It's obvious, that if $\mathsf{Aut}(S)$ is homogeneous, then the structure is homogeneous as well. Actually not only the structure $\langle\mathbb{Q};<\rangle$ is homogeneous, but also the group $\mathsf{Aut}(\langle\mathbb{Q};<\rangle)$ is homogeneous.

Peter Cameron [Cam1] showed that there are just four homogeneous nontrivial groups of permutations on a countable set. As the corollary we get, that in the case of $\langle\mathbb{Q};<\rangle$ apart from $\mathsf{Aut}(\langle\mathbb{Q};<\rangle)$ and $\mathsf{Sym}(\mathbb{Q})$, there are just three homogeneous groups. The first is the group of all permutations of $\mathbb{Q}$ which either preserve the order or reverse it. The second is the group of all permutations which preserve the cyclic relation "$x < y < z$ or $y < z < x$ or $z < x < y$"; this corresponds to taking an initial segment of $\mathbb{Q}$ and moving it to the end. The third is the group generated by these other two: it consists of those permutations which preserve the relation "exactly one of x, y lies between z and w".

All countable homogeneous structures are ω-categorical, if they have a finite signature or signature finite for any fixed number of variables. For ω-categorical structures homogeneity is equivalent to quantifier elimination. All reducts of $\langle\mathbb{Q};<\rangle$ are homogeneous and have quantifier elimination.

A good source for information related to homogeneous structures is [Mac].

4.3 The Random Graph. Thomas Conjecture

Our next example is one more remarkable homogeneous structure.

Definition 6. *We call a countable graph* random *iff given two finite disjoint sets U, V of vertices, there exists a vertex z joined to every vertex in U and to no vertex in V.*

This Is called "Alice's Restaurant Property". The term was coined by Peter Winkler [Win], in reference to a popular song by Arlo Guthrie. The refrain of the song "You can get anything you want at Alice's restaurant" catches the spirit of this property.

Any two random graphs are isomorphic. The proof is similar to the isomorphism proof for every two countable dense unlimited orders (the $\mathbb{Q}$ case). The term "random" can be explained by the following property:

> If a graph X on a fixed countable vertex set is chosen by selecting edges independently at random with probability $1/2$ from the unordered pairs of vertices, then $\mathsf{Prob}(X{=}R) = 1$.

An explicit construction of R in [Rad]:

> The set of vertices is $\mathbb{N}$, and x is connected to y if and only if the x-th digit in the base 2 expansion of y is equal to 1 or vice versa.

Here are the subspaces of the random graph R

Let $R^{(k)}$ be the k-ary relation that contains all k-tuples of pairwise distinct elements $x_1, \ldots, x_k$ in V such that the number of (undirected) edges between those elements is odd.

$R(a,b)$ – "(ab) is an edge in R"; $R^{(3)}$;$R^{(4)}$;$R^{(5)}$;Sym — equality

This description is given in [Tho].

It easy to see that structure of $R^{(3)}$ is not homogeneous and does not have quantifier elimination.

Simon Thomas proved obtained this description in [Tho1]. and suggested the following conjecture:

> If M is a finitely generated homogeneous structure then M has finitely many reducts.

Problem 8. Verify Thomas conjecture.

4.4 Further Examples

In order to verify Thomas conjecture the superposition of two homogeneous structures: $\langle \mathbb{Q}; < \rangle$ and random graph $\langle G; E \rangle$ was considered in [BoPiPo]. They presented a complete classification of the reducts of this random ordered graph up to equivalence. It was shown that without counting obvious reducts $\langle D; <, E \rangle$ and $\langle D; = \rangle$ there are precisely 42 such reducts.

In [JuZi] was described a complete lattice of the reducts of expansion of the structure $\langle \mathbb{Q}; < \rangle$ by a constant. This expansion can be considered as expansion by three unary predicates: "$x < a$"; "$x = a$"; and "$x > a$". Actually in this paper different expansions of $\langle \mathbb{Q}; < \rangle$ by unary predicates that have quantifier elimination were studied. They classified the reducts of such expansions and showed that there are only finitely many such. In particular it shows that in the simplest case: expansion of rational numbers by two convex subsets (a cut of the rational numbers) there are exactly 53 reducts, generated by the 5 standard reducts on the elements of the cut as well as permutations preserving, swapping and mixing elements of the cut.

Let us mention the example of an ω-categorical structure, which shows that the condition of quantifier elimination in the Thomas' Conjecture is necessary: [AhZi] describes infinitely many reducts of a doubled infinite-dimensional projective space over binary field (F_2).

4.5 Not ω-categorical Spaces. Integers with Successor – Depth 1

We don't know too much about the reducts of not ω-categorical structure. Answering the dual question to Thomas' one [BoMa] constructs an example of not ω-categorical structure with the finite reducts lattice — actually the lattice contains only two items. This example is based on tree of valency three structure.

Another (more simple) example was demonstrated in the [KaSi]. Answering a question from [BoMa] they show that the structure $\langle \mathbb{Q}; S(x,y,z) \rangle$, where

$S(x, y, z) \equiv (z{=}(x{+}y)/2)$ (or, the same, the structure $\langle \mathbb{Q}; f(x, y, z)\rangle$ where $f(x, y, z) = x{-}y{+}z$) admits no definable reduct. Though Svenonious theorem is not used explicitly in the proof, the approach is rather similar. They note that the structure $\langle \mathbb{Q}^{<\omega}; +\rangle$ is the saturated elementary extension of the $\langle \mathbb{Q}; +\rangle$, so it's enough to consider permutations of the structure $\langle \mathbb{Q}^{<\omega}; +\rangle$ only. Now the fact that $\mathsf{Aut}(\langle \mathbb{Q}^{<\omega}; f\rangle)$ is maximal closed nontrivial subgroup (proved in the same paper) is used.

The structure $\langle \mathbb{Z}; +1\rangle$ — integer numbers with the successor relation is not ω-categorical, and has depth 1. For any natural number n we define spaces by their generators
"$x_1{-}x_2 = n$" — A_n,
"$x_1{-}x_2 = x_3{-}x_4 = n \vee x_1{-}x_2 = x_3{-}x_4 = -n$" — B_n, and
"$|x_1{-}x_2| = n$" — C_n.

Theorem 5. [SeSo2] *Any subspaces of $\langle \mathbb{Z}; +1\rangle$ is A_n or B_n or C_n for a natural n.*
$A_n \succ B_n \succ C_n$ for any n and if $n \neq m$ then $A_n \succ A_m$, $B_n \succ B_m$, $C_n \succ C_m$ iff n is a divisor of m.

Problem 9. Describe the lattice of subspaces for $\langle \mathbb{N}; +1\rangle$.

Problem 10. Describe the lattice of subspaces for natural numbers with multiple successors.

We leave out the researches on the reducts of the field of real [MaPe, Pet] and complex [MaPi] numbers.

4.6 Decidability of the Lattice Problems. Muchnik's Self-definability

A natural algorithmic problem for an algebraic structure of definability lattice is does an element of a space (given by a formula in or case) belong to a subspace generated by a given set of elements? Positive and negative results on this for homogeneous structures were obtained in [BoPiTs].

Andrei Muchnik in his work [Muc1] introduced the following

Definition 7. *A definability space S is called* self-definable *iff there is a finite signature (set of generators) Σ for S and sequence of formulas $F_1, \ldots, F_n, \ldots$ such that for any $n = 1, 2, \ldots$*
1. F_n is a closed formula in signature $\Sigma \cup \{P\}$, where P is an n-ary symbol
2. F_n is true iff we take as interpretation of P an element from S.

He proved

Theorem 6. *The space $\langle \mathbb{N}; +\rangle$ is self-definable.*

He writes:

> "Unfortunately, we do not know any other examples of nice self-definable structures.

Structures with unsolvable elementary theory are usually mutually interpretable with the arithmetic of addition and multiplication of integers, the non-self-definability of which is proved in [Add] (using category arguments and [Tan] using measure arguments).

We believe that the structure formed by algebraic real numbers (with addition and multiplication) is not self-definable; however, a formal proof is missing (and seems to be rather complicated).

(Note that it is easy to prove that the structure formed by all real numbers with addition and multiplication is not self-definable. Indeed, let us assume that $\Phi(A)$ is true if and only if A is definable. Now we replace $A(x)$ by $x = y$. The new formula $\Phi'(y)$ is true if and only if y is algebraic. But we can eliminate quantifiers in $\Phi'(y)$ and get a finite union of segments. So we come to a contradiction.)"

Problem 11. Give more examples of structures with self-definability property.

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