Notions of Metric Dimension of Corona Products: Combinatorial and Computational Results

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Abstract. The metric dimension is quite a well-studied graph parameter. Recently, the adjacency metric dimension and the local metric dimension have been introduced. We combine these variants and introduce the local adjacency metric dimension. We show that the (local) metric dimension of the corona product of a graph of order n and some non-trivial graph H equals n times the (local) adjacency metric dimension of H. This strong relation also enables us to infer computational hardness results for computing the (local) metric dimension, based on according hardness results for (local) adjacency metric dimension that we also give.

Keywords: (local) metric dimension, (local) adjacency dimension, NP-hardness.

1 Introduction and Preliminaries

Throughout this paper, we only consider undirected simple loop-free graphs and use standard graph-theoretic terminology. Less known notions are collected at the end of this section.

Let (X, d) be a metric space. The *diameter* of a point set $S \subseteq X$ is defined as $\mathrm{diam}(S) = \sup\{d(x, y) : x, y \in S\}$. A *generator* of (X, d) is a set $S \subseteq X$ such that every point of the space is uniquely determined by the distances from the elements of S. A point $v \in X$ is said to *distinguish* two points x and y of X if $d(v, x) \neq d(v, y)$. Hence, S is a generator if and only if any pair of points of X is distinguished by some element of S.

Four notions of dimension in graphs. Let $\mathbb{N}$ denote the set of non-negative integers. Given a connected graph $G = (V, E)$, we consider the function $d_G : V \times V \to \mathbb{N}$, where $d_G(x, y)$ is the length of a shortest path between u and v. Clearly, (V, d_G) is a metric space. The diameter of a graph is understood in this metric. A vertex set $S \subseteq V$ is said to be a *metric generator* for G if it is a generator of the metric space (V, d_G). A minimum metric generator is called a *metric basis*, and its cardinality the *metric dimension* of G, denoted by $\dim(G)$.

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Motivated by the problem of uniquely determining the location of an intruder in a network, the concept of metric dimension of a graph was introduced by Slater in [33], where the metric generators were called *locating sets*. Independently, Harary and Melter introduced this concept in [16], where metric generators were called *resolving sets*. Applications of this parameter to the navigation of robots in networks are discussed in [25] and applications to chemistry in [22,23]. This graph parameter was studied further in a number of other papers including recent papers like [1,10,13,20,35].

Keeping in mind the robot navigation scenario, where the robot can determine its position by knowing the distances to the vertices in the metric generator, it makes sense to consider local variants of this parameter, assuming that the robot has some idea about its current position. A set S of vertices in a connected graph G is a *local metric generator* for G (also called local metric set for G [29]) if every two adjacent vertices of G are distinguished by some vertex of S. A minimum local metric generator is called a *local metric basis* for G and its cardinality, the *local metric dimension* of G, is denoted by $\dim_l(G)$.

If the distances between vertices are hard to determine, then it might still be the case that the robot can sense whether or not it is within the range of some sender installed on some other vertex. This has motivated the next definition. A set S of vertices in a graph G is an *adjacency generator* for G (also adjacency resolving set for G [21]) if for every $x, y \in V(G) - S$ there exists $s \in S$ such that $|N_G(s) \cap \{x, y\}| = 1$. This concept is very much related to that of a 1-locating dominating set [5]. A minimum adjacency generator is called an *adjacency basis* for G and its cardinality, the *adjacency dimension* of G, is denoted by $\dim_A(G)$. Observe that an adjacency generator of a graph $G = (V, E)$ is also a generator in a suitably chosen metric space, namely by considering $(V, d_{G,2})$, with $d_{G,2}(x, y) = \min\{d_G(x, y), 2\}$, and vice versa.

Now, we combine the two variants of metric dimension defined so far and introduce the local adjacency dimension of a graph. We say that a set S of vertices in a graph G is a *local adjacency generator* for G if for every two adjacent vertices $x, y \in V(G) - S$ there exists $s \in S$ such that $|N_G(s) \cap \{x, y\}| = 1$. A minimum local adjacency generator is called a *local adjacency basis* for G and its cardinality, the *local adjacency dimension* of G, is denoted by $\dim_{A,l}(G)$.

Our main results. In this paper, we study the (local) metric dimension of corona product graphs via the (local) adjacency dimension of a graph. We show that the (local) metric dimension of the corona product of a graph of order n and some non-trivial graph H equals n times the (local) adjacency metric dimension of H. This relation is much stronger and under weaker conditions compared to the results of Jannesari and Omoomi [21] concerning the lexicographic product of graphs. This also enables us to infer NP-hardness results for computing the (local) metric dimension, based on corresponding NP-hardness results for (local) adjacency metric dimension that we also provide. To our knowledge, this is the first time combinatorial results on this particular form of graph product have been used to deduce computational hardness results. The obtained reductions are relatively simple and also allow us to conclude hardness results based on the Exponential Time Hypothesis. We also discuss NP-hardness results for planar graphs, which seem to be of some particular importance to the sketched applications. This also shows the limitations of using corona products to obtain hardness results. Finally, we indicate why computing the (local) adjacency metric dimension is in FPT (under the standard parameterization), contrasting what is known for computing the metric dimension.

Some notions from graph theory. Let $G = (V, E)$ be a graph. A vertex set $D \subseteq V$ is called a *dominating set* if $\bigcup_{v \in D} N_G[v] = V$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality among all dominating sets in G. A vertex set $C \subseteq V$ is called a *vertex cover* if for each edge $e \in E$, $C \cap e \neq \emptyset$, The *vertex cover number* of G, denoted by $\beta(G)$, is the minimum cardinality among all vertex covers of G.

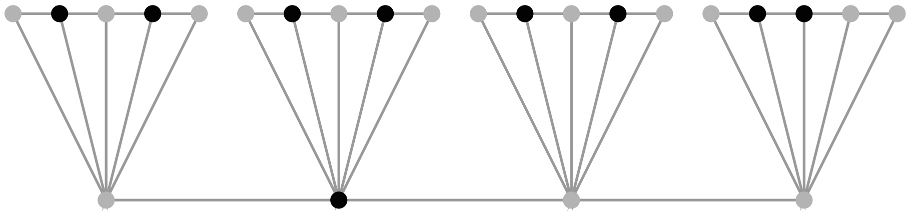

Fig. 1. The bold type forms an adjacency basis for $P_4 \odot P_5$ but not a dominating set

Let G and H be two graphs of order n and n', respectively. The *join (graph)* $G + H$ is defined as the graph obtained from vertex-disjoint graphs G and H by taking one copy of G and one copy of H and joining by an edge each vertex of G with each vertex of H. Graph products is one of the recurring themes in graph theory, see [15]. The *corona product (graph)* $G \odot H$ is defined as the graph obtained from G and H by taking one copy of G and n copies of H and joining by an edge each vertex from the i^{th} copy of H with the i^{th} vertex of G [11]. We will denote by $V = \{v_1, v_2, \ldots, v_n\}$ the set of vertices of G and by $H_i = (V_i, E_i)$ the i^{th} copy of H so that $N_{G \odot H}(v_i) = V_i \cup N_G(v_i)$ and $N_{G \odot H}(x) = \{v_i\} \cup N_{H_i}(x)$ for every $x \in V_i$. Notice that the corona graph $K_1 \odot H$ is isomorphic to the join graph $K_1 + H$. For our computational complexity results, it is important but easy to observe that these graph operations can be performed in polynomial time, given two input graphs. Some of the notions important in this paper are illustrated in Figure 1.

Simple facts. By definition, the following inequalities hold for any graph G:

- $\dim(G) \leq \dim_A(G)$;
- $\dim_l(G) \leq \dim_{A,l}(G)$;
- $\dim_l(G) \leq \dim(G)$;
- $\dim_{A,l}(G) \leq \dim_A(G)$.

Moreover, if S is an adjacency generator, then at most one vertex is not dominated by S, so that

$$\gamma(G) \leq \dim_A(G) + 1.$$

Namely, if x, y are not dominated by S, then no element in S distinguishes them.

We also observe that

$$\dim_{A,l}(G) \leq \beta(G),$$

because each vertex cover is a local adjacency generator.

However, all mentioned inequalities could be either equalities or quite weak bounds. Consider the following examples:

1. $\dim_l(P_n) = \dim(P_n) = 1 \leq \lfloor \frac{n}{4} \rfloor \leq \dim_{A,l}(P_n) \leq \lceil \frac{n}{4} \rceil \leq \lfloor \frac{2n+2}{5} \rfloor = \dim_A(P_n)$, $n \geq 7$;
2. $\dim_l(K_{1,n}) = \dim_{A,l}(K_{1,n}) = 1 \leq n-1 = \dim(K_{1,n}) = \dim_A(K_{1,n})$, $n \geq 2$;
3. $\gamma(P_n) = \lceil \frac{n}{3} \rceil \leq \lfloor \frac{2n+2}{5} \rfloor = \dim_A(P_n)$, $n \geq 7$;
4. $\lfloor \frac{n}{4} \rfloor \leq \dim_{A,l}(P_n) \leq \lceil \frac{n}{4} \rceil \leq \lfloor \frac{n}{2} \rfloor = \beta(P_n)$, $n \geq 2$.

The proofs of results marked with an asterisk symbol ($*$) can be found in the long version of this paper that can be retrieved as a Technical Report [30].

2 The Metric Dimension of Corona Product Graphs versus the Adjacency Dimension of a Graph

The following is the first main combinatorial result of this paper and provides a strong link between the metric dimension of the corona product of two graphs and the adjacency dimension of the second graph involved in the product operation. A seemingly similar formula was derived in [20,35], but there, only the notion of metric dimension was involved (which makes it impossible to use the formula to obtain computational hardness results as we will do), and also, special conditions were placed on the second argument graph of the corona product.

Theorem 1. *For any connected graph G of order $n \geq 2$ and any non-trivial graph H,* $\dim(G \odot H) = n \cdot \dim_A(H)$.

Proof. We first need to prove that $\dim(G \odot H) \leq n \cdot \dim_A(H)$. For any $i \in \{1, \ldots, n\}$, let S_i be an adjacency basis of H_i, the i^{th}-copy of H. In order to show that $X := \bigcup_{i=1}^{n} S_i$ is a metric generator for $G \odot H$, we differentiate the following four cases for two vertices $x, y \in V(G \odot H) - X$.

1. $x, y \in V_i$. Since S_i is an adjacency basis of H_i, there exists a vertex $u \in S_i$ so that $|N_{H_i}(u) \cap \{x, y\}| = 1$. Hence,

$$d_{G \odot H}(x, u) = d_{\langle v_i \rangle + H_i}(x, u) \neq d_{\langle v_i \rangle + H_i}(y, u) = d_{G \odot H}(y, u).$$

2. $x \in V_i$ and $y \in V$. If $y = v_i$, then for $u \in S_j$, $j \neq i$, we have

$$d_{G \odot H}(x, u) = d_{G \odot H}(x, y) + d_{G \odot H}(y, u) > d_{G \odot H}(y, u).$$

   Now, if $y = v_j$, $j \neq i$, then we also take $u \in S_j$ and we proceed as above.

3. $x = v_i$ and $y = v_j$. For $u \in S_j$, we find that

$$d_{G\odot H}(x,u) = d_{G\odot H}(x,y) + d_{G\odot H}(y,u) > d_{G\odot H}(y,u).$$

4. $x \in V_i$ and $y \in V_j$, $j \neq i$. In this case, for $u \in S_i$ we have

$$d_{G\odot H}(x,u) \leq 2 < 3 \leq d_{G\odot H}(u,y).$$

Hence, X is a metric generator for $G \odot H$ and, as a consequence,

$$\dim(G \odot H) \leq \sum_{i=1}^{n} |S_i| = n \cdot \dim_A(H).$$

It remains to prove that $\dim(G \odot H) \geq n \cdot \dim_A(H)$. To do this, let W be a metric basis for $G \odot H$ and, for any $i \in \{1,\ldots,n\}$, let $W_i := V_i \cap W$. Let us show that W_i is an adjacency metric generator for H_i. To do this, consider two different vertices $x, y \in V_i - W_i$. Since no vertex $a \in V(G\odot H) - V_i$ distinguishes the pair x, y, there exists some $u \in W_i$ such that $d_{G\odot H}(x,u) \neq d_{G\odot H}(y,u)$. Now, since $d_{G\odot H}(x,u) \in \{1,2\}$ and $d_{G\odot H}(y,u) \in \{1,2\}$, we conclude that $|N_{H_i}(u) \cap \{x,y\}| = 1$ and consequently, W_i must be an adjacency generator for H_i. Hence, for any $i \in \{1,\ldots,n\}$, $|W_i| \geq \dim_A(H_i)$. Therefore,

$$\dim(G \odot H) = |W| \geq \sum_{i=1}^{n} |W_i| \geq \sum_{i=1}^{n} \dim_A(H_i) = n \cdot \dim_A(H).$$

Consequences of Theorem 1 We can now investigate $\dim(G \odot H)$ through the study of $\dim_A(H)$, and vice versa. In particular, results from [3,32,35] allow us to deduce the exact adjacency dimension for several special graphs. For instance, we find that $\dim_A(C_r) = \dim_A(P_r) = \left\lfloor \frac{2r+2}{5} \right\rfloor$ for any $r \geq 7$. Other combinatorial results of this type are collected in the long version of this paper [30].

A detailed analysis of the adjacency dimension of the corona product via the adjacency dimension of the second operand. We now analyze the adjacency dimension of the corona product $G \odot H$ in terms of the adjacency dimension of H.

Theorem 2. (∗) *Let G be a connected graph of order $n \geq 2$ and let H be a non-trivial graph. If there exists an adjacency basis S for H which is also a dominating set, and if for every $v \in V(H) - S$, it is satisfied that $S \not\subseteq N_H(v)$, then* $\dim_A(G \odot H) = n \cdot \dim_A(H)$.

Corollary 1. (∗) *Let $r \geq 7$ with $r \not\equiv 1 \bmod 5$ and $r \not\equiv 3 \bmod 5$. For any connected graph G of order $n \geq 2$,* $\dim_A(G \odot C_r) = \dim_A(G \odot P_r) = n \cdot \left\lfloor \frac{2r+2}{5} \right\rfloor$.

Theorem 3. *Let G be a connected graph of order $n \geq 2$ and let H be a non-trivial graph. If there exists an adjacency basis for H which is also a dominating set and if, for any adjacency basis S for H, there exists some $v \in V(H) - S$ such that $S \subseteq N_H(v)$, then* $\dim_A(G \odot H) = n \cdot \dim_A(H) + \gamma(G)$.

Proof. Let W be an adjacency basis for $G \odot H$ and let $W_i = W \cap V_i$ and $U = W \cap V$. Since two vertices belonging to V_i are not distinguished by any $u \in W - V_i$, the set W_i must be an adjacency generator for H_i. Now consider the partition $\{V', V''\}$ of V defined as follows:

$$V' = \{v_i \in V : \quad |W_i| = \dim_A(H)\} \text{ and } V'' = \{v_j \in V : \quad |W_j| \geq \dim_A(H)+1\}.$$

Note that, if $v_i \in V'$, then W_i is an adjacency basis for H_i, thus in this case there exists $u_i \in V_i$ such that $W_i \subseteq N_{H_i}(u_i)$. Then the pair u_i, v_i is not distinguished by the elements of W_i and, as a consequence, either $v_i \in U$ or there exists some $v_j \in U$ adjacent to v_i. Hence, $U \cup V''$ must be a dominating set and, as a result, $|U \cup V''| \geq \gamma(G)$. So we obtain the following:

$$\begin{aligned}\dim_A(G \odot H) = |W| &= \bigcup_{v_i \in V'} |W_i| + \bigcup_{v_j \in V''} |W_j| + |U| \\ &\geq \sum_{v_i \in V'} \dim_A(H) + \sum_{v_j \in V''} (\dim_A(H) + 1) + |U| \\ &= n \cdot \dim_A(H) + |V''| + |U| \geq n \cdot \dim_A(H) + |V'' \cup U| \\ &\geq n \cdot \dim_A(H) + \gamma(G).\end{aligned}$$

To conclude the proof, we consider an adjacency basis S for H which is also a dominating set, and we denote by S_i the copy of S corresponding to H_i. We claim that for any dominating set D of G of minimum cardinality $|D| = \gamma(G)$, the set $D \cup (\bigcup_{i=1}^n S_i)$ is an adjacency generator for $G \odot H$ and, as a result,

$$\dim_A(G \odot H) \leq \left| D \cup \left(\bigcup_{i=1}^{n} S_i \right) \right| = n \cdot \dim_A(H) + \gamma(G).$$

This can be seen by some case analysis. Let $S' = D \cup \bigcup_{i=1}^n S_i$ and let us prove that S' is an adjacency generator for $G \odot H$. We differentiate the following cases for any pair x, y of vertices of $G \odot H$ not belonging to S'.

1. $x, y \in V_i$. Since S_i is an adjacency basis of H_i, there exists $u_i \in S_i$ such that u_i is adjacent to x or to y but not to both.
2. $x \in V_i$, $y \in V_j$, $j \neq i$. As S_i is a dominating set of H_i, there exists $u \in S_i$ such that $u \in N_{H_i}(x)$ and, obviously, $u \notin N_{G \odot H}(y)$.
3. $x \in V_i$, $y = v_i \in V$. As $y = v_i \notin D$, $v_j \in N_G(v_i)$ distinguishes the pair x, y.
4. $x \in V_i \cup \{v_i\}$, $y = v_j \in V$, $i \neq j$. In this case, every $u \in S_j$ is a neighbor of y but not of x.

Corollary 2. *Let $r \geq 2$. Let G be a connected graph of order $n \geq 2$. Then,* $\dim_A(G \odot K_r) = n(r-1) + \gamma(G)$.

Theorem 4. (∗) *Let G be a connected graph of order $n \geq 2$ and let H be a non-trivial graph. If no adjacency basis for H is a dominating set, then we have:* $\dim_A(G \odot H) = n \cdot \dim_A(H) + n - 1$.

It is easy to check that any adjacency basis of a star graph $K_{1,r}$ is composed of $r-1$ leaves, with the last leaf non-dominated. Thus, Theorem 4 implies:

Corollary 3. *For a connected graph G of order $n \geq 2$, $\dim_A(G \odot K_{1,r}) = n \cdot r - 1$.*

Given a vertex $v \in V$ we denote by $G - v$ the subgraph obtained from G by removing v and the edges incident with it. We define the following auxiliary domination parameter: $\gamma'(G) := \min_{v \in V(G)}\{\gamma(G - v)\}$.

Theorem 5. $(*)$ *Let H be a non-trivial graph such that some of its adjacency bases are also dominating sets, and some are not. If there exists an adjacency basis S' for H such that for every $v \in V(H) - S'$ it is satisfied that $S' \not\subseteq N_H(v)$, and for any adjacency basis S for H which is also a dominating set, there exists some $v \in V(H) - S$ such that $S \subseteq N_H(v)$, then for any connected graph G of order $n \geq 2$, $\dim_A(G \odot H) = n \cdot \dim_A(H) + \gamma'(G)$.*

As indicated in Figure 1, $H = P_5$ satisfies the premises of Theorem 5, as in particular there are adjacency bases that are also dominating set (see the leftmost copy of a P_5 in Figure 1) as well as adjacency bases that are not dominating sets (see the rightmost copy of a P_5 in that drawing). Hence, we can conclude:

Corollary 4. *For any connected graph G of order $n \geq 2$, $\dim_A(G \odot P_5) = 2n + \gamma'(G)$.*

Since the assumptions of Theorems 2, 3, 4 and 5 are complementary and for any graph G of order $n \geq 3$ it holds that $0 < \gamma'(G) \leq \gamma(G) \leq \frac{n}{2} < n - 1$, we can conclude that in fact, Theorems 2 and 5 are equivalences for $n \geq 3$ (or even $n \geq 2$ in the first case). Therefore, we obtain:

Theorem 6. *Let G be a connected graph of order $n \geq 2$ and let H be a non-trivial graph. The following statements are equivalent:*

(i) *There exists an adjacency basis S for H, which is also a dominating set, such that for every $v \in V(H) - S$ it is satisfied that $S \not\subseteq N_H(v)$.*
(ii) $\dim_A(G \odot H) = n \cdot \dim_A(H)$.
(iii) $\dim_A(G \odot H) = \dim(G \odot H)$.

This should be conferred to the combinatorial results in [20], as it exactly tells when they could possibly apply.

As an example of applying Theorem 6 we can take H as the cycle graphs C_r or the path graphs P_r, where $r \geq 7$, $r \not\equiv 1 \bmod 5$, $r \not\equiv 3 \bmod 5$, see Cor. 1.

Theorem 7. *Let G be a connected graph of order $n \geq 3$ and let H be a non-trivial graph. The following statements are equivalent:*

(i) *No adjacency basis for H is a dominating set.*
(ii) $\dim_A(G \odot H) = n \cdot \dim_A(H) + n - 1$.
(iii) $\dim_A(G \odot H) = \dim(G \odot H) + n - 1$.

3 Locality in Dimensions

First, we consider some straightforward cases. If H is an empty graph, then $K_1 \odot H$ is a star graph and $\dim_l(K_1 \odot H) = 1$. Moreover, if H is a complete graph of order n, then $K_1 \odot H$ is a complete graph of order $n+1$ and $\dim_l(K_1 \odot H) = n$. It was shown in [31] that for any connected nontrivial graph G and any empty graph H, $\dim_l(G \odot H) = \dim_l(G)$. We are going to state results similar to the non-local situation as discussed in the previous section. We omit all proofs as they are along similar lines.

Theorem 8. (∗) *For any connected graph G of order $n \geq 2$ and any non-trivial graph H,* $\dim_l(G \odot H) = n \cdot \dim_{A,l}(H)$.

Based on [31], this allows to deduce quite a number of combinatorial results for the new notion of a local adjacency dimension, as contained in [30].

Fortunately, the comparison of the local adjacency dimension of the corona product with the one of the second argument is much simpler in the local version as in the previously studied non-local version.

Theorem 9. (∗) *Let G be a connected graph of order $n \geq 2$ and let H be a non-trivial graph. If there exists a local adjacency basis S for H such that for every $v \in V(H) - S$ it is satisfied that $S \not\subseteq N_H(v)$, then* $\dim_{A,l}(G \odot H) = n \cdot \dim_{A,l}(H)$.

Theorem 10. (∗) *Let G be a connected graph of order $n \geq 2$ and let H be a non-trivial graph. If for any local adjacency basis for H, there exists some $v \in V(H) - S$ which satisfies that $S \subseteq N_H(v)$, then* $\dim_{A,l}(G \odot H) = n \cdot \dim_{A,l}(H) + \gamma(G)$.

Remark 1. As a concrete example for the previous theorem, consider $H = K_{n'}$. Clearly, $\dim_{A,l}(H) = n' - 1$, and the neighborhood of the only vertex that is not in the local adjacency basis coincides with the local adjacency basis. For any connected graph G of order $n \geq 2$, we can deduce that

$$\dim_{A,l}(G \odot K_{n'}) = n \cdot \dim_{A,l}(K_{n'}) + \gamma(G) = n(n' - 1) + \gamma(G).$$

Since the assumptions of Theorems 9 and 10 are complementary, we obtain the following property for $\dim_{A,l}(G \odot H)$.

Theorem 11. *Let G be a connected graph of order $n \geq 2$ and let H be a non-trivial graph. Then the following assertions are equivalent.*

- (i) *There exists a local adjacency basis S for H such that for every $v \in V(H) - S$ it is satisfied that $S \not\subseteq N_H(v)$.*
- (ii) $\dim_{A,l}(G \odot H) = n \cdot \dim_{A,l}(H)$.
- (iii) $\dim_l(G \odot H) = \dim_{A,l}(G \odot H)$.

Theorem 12. *Let G be a connected graph of order $n \geq 2$ and let H be a non-trivial graph. Then the following assertions are equivalent.*

- (i) *For any local adjacency basis S for H, there exists some $v \in V(H) - S$ which satisfies that $S \subseteq N_H(v)$.*

(ii) $\dim_{A,l}(G \odot H) = n \cdot \dim_{A,l}(H) + \gamma(G)$.
(iii) $\dim_l(G \odot H) = \dim_{A,l}(G \odot H) - \gamma(G)$.

As a concrete example of graph H where we can apply the above result is the star $K_{1,r}$, $r \geq 2$. In this case, for any connected graph G of order $n \geq 2$, we find that $\dim_{A,l}(G \odot K_{1,r}) = n \cdot \dim_{A,l}(K_{1,r}) + \gamma(G) = n + \gamma(G)$.

4 Computational Complexity of the Dimension Variants

In this section, we not only prove NP-hardness of all dimension variants, but also show that the problems (viewed as minimization problems) cannot be solved in time $O(\text{poly}(n+m)2^{o(n)})$ on any graph of order n (and size m). Yet, it is straightforward to see that each of our computational problems can be solved in time $O(\text{poly}(n+m)2^n)$, simply by cycling through all vertex subsets by increasing cardinality and then checking if the considered vertex set forms an appropriate basis. More specifically, based on our reductions we can conclude that these trivial brute-force algorithms are in a sense optimal, assuming the validity of the Exponential Time Hypothesis (ETH). A direct consequence of ETH (using the sparsification lemma) is the hypothesis that 3-SAT instances cannot be solved in time $O(\text{poly}(n+m)2^{o(n+m)})$ on instances with n variables and m clauses; see [19,4].

From a mathematical point of view, the most interesting fact is that most of our computational results are based on the combinatorial results on the dimensional graph parameters on corona products of graphs that are derived above.

Due to the practical motivation of the parameters, we also study their computational complexity on planar graph instances.

We are going to investigate the following problems:
DIM: Given a graph G and an integer k, decide if $\dim(G) \leq k$ or not.
LOCDIM: Given a graph G and an integer k, decide if $\dim_l(G) \leq k$ or not.
ADJDIM: Given a graph G and an integer k, decide if $\dim_A(G) \leq k$ or not.
LOCADJDIM: Given a graph G and an integer k, decide if $\dim_{A,l}(G) \leq k$ or not.

As auxiliary problems, we will also consider:
VC: Given a graph G and an integer k, decide if $\beta(G) \leq k$ or not.
DOM: Given a graph G and an integer k, decide if $\gamma(G) \leq k$ or not.
1-LOCDOM: Given a graph G and an integer k, decide if there exists a 1-locating dominating set of G with at most k vertices or not. (A dominating set $D \subseteq V$ in a graph $G = (V, E)$ is called a *1-locating dominating set* if for every two vertices $u, v \in V \setminus D$, the symmetric difference of $N(u) \cap D$ and $N(v) \cap D$ is non-empty.)

Theorem 13. DIM *is NP-complete, even when restricted to planar graphs.*

Different proofs of this type of hardness result appeared in the literature. While this result is only mentioned in the textbook of Garey and Johnson [12], a proof was first published in [25]. For planar instances, we refer to [9] where this result is stated.

Remark 2. In fact, we can offer a further proof for the NP-hardness of DIM (on general graphs), based upon Theorem 1 and the following reasoning. If there were a polynomial-time algorithm for computing $\dim(G)$, then we could compute $\dim_A(H)$ for any (non-trivial) graph H by computing $\dim(K_2 \odot H)$ with the assumed polynomial-time algorithm, knowing that this is just twice as much as $\dim_A(H)$. As every NP-hardness proof adds a bit to the understanding of the nature of the problem, this one does so, as well. It shows that DIM is NP-complete even on the class of graphs that can be written as $G \odot H$, where G is some connected graph of order $n \geq 2$ and H is non-trivial.

Theorem 14. (∗) 1-LOCDOM *is NP-hard, even when restricted to planar graphs. Moreover, assuming ETH, there is no $O(poly(n + m)2^{o(n)})$ algorithm solving* 1-LOCDOM *on general graphs of order n and size m.*

Proof. (Sketch) Recall the textbook proof for the NP-hardness of VC (see [12]) that produces from a given 3-SAT instance I with n variables and m clauses a graph G with two adjacent vertices per variable gadget and three vertices per clause gadget forming a C_3 (and $3m$ more edges that interconnect these gadgets to indicate which literals occur in which clauses). So, G has $3m + 2n$ vertices and $3m + n + 3m = 6m + n$ edges. We modify G to obtain G' as follows: Each edge that occurs inside of a variable gadget or of a clause gadget is replaced by a triangle, so that we add $3m+n$ new vertices of degree two. All in all, this means that G' has $(3m+2n)+(3m+n) = 6m+3n$ vertices and $9m+3n+3m = 12m+3n$ edges. Now, assuming (w.l.o.g.) that I contains, for each variable x, at least one clause with x as a literal and another clause with $\bar{x}$ as a literal, we can show that I is satisfiable iff G has a vertex cover of size at most $2m + n$ iff G' has a 1-locating dominating set of size at most $2m + n$. □

The general case was treated in [7], but that proof (starting out again from 3-SAT) does not preserve planarity, as the variable gadget alone already contains a $K_{2,3}$ subgraph that inhibits non-crossing interconnections with the clause gadgets. However, although not explicitly mentioned, that reduction also yields the non-existence of $O(\text{poly}(n + m)2^{o(n)})$ algorithms based on ETH. In [30], we also provide a reduction that works for planar graphs, working on a variant of Lichtenstein's reduction [27] that shows NP-hardness of VC on planar graph instances.

Theorem 15. ADJDIM *is NP-complete, even when restricted to planar graphs. Assuming ETH, there is no $O(poly(n + m)2^{o(n)})$ algorithm solving* ADJDIM *on graphs of order n and size m.*

Proof. (Sketch) From an instance $G = (V, E)$ and k of 1-LOCDOM, produce an instance (G', k) of ADJDIM by obtaining G' from G by adding a new isolated vertex $x \notin V$ to G. We claim that G has a 1-locating dominating set of size at most k if and only if $\dim_A(G') \leq k$. □

Alternatively, NP-hardness of ADJDIM (and even the ETH-result) can be deduced from the strong relation between the domination number and the adjacency dimension as stated in Cor. 2, based on the NP-hardness of DOM.

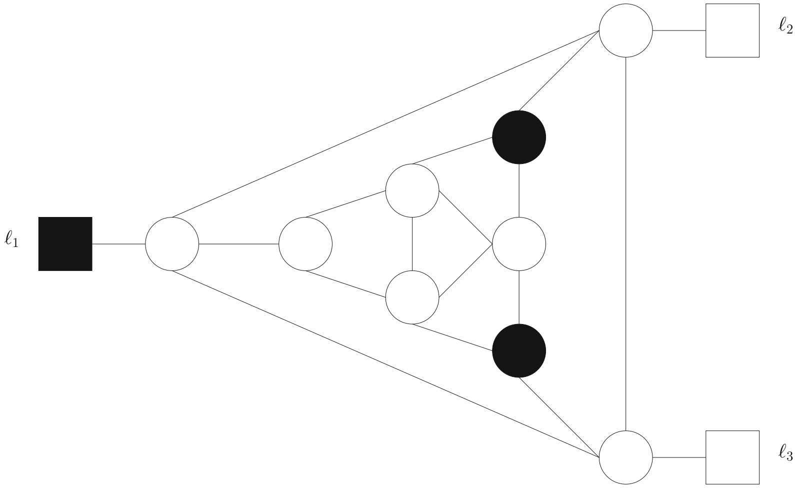

Fig. 2. The clause gadget illustration. The square-shaped vertices do not belong to the gadget, but they are the three literal vertices in variable gadgets that correspond to the three literals in the clause.

As explained in Remark 2, Theorem 1 can be used to deduce furthermore:

Corollary 5. *Assuming ETH, there is no* $O(poly(n+m)2^{o(n)})$ *algorithm solving* DIM *on graphs of order* n *and size* m.

Lemma 1. *[28] Assuming ETH, there is no* $O(poly(n+m)2^{o(n)})$ *algorithm solving* DOM *on graphs of order* n *and size* m.

From Remark 1 and Lemma 1, we can conclude:

Theorem 16. LOCADJDIM *is NP-complete. Moreover, assuming ETH, there is no* $O(poly(n+m)2^{o(n)})$ *algorithm solving* LOCADJDIM *on graphs of order* n *and size* m.

We provide an alternative proof of the previous theorem in [30]. That proof is a direct reduction from 3-SAT and is, in fact, very similar to the textbook proof for the NP-hardness of VC, also see the proof of Theorem 14. This also proves that LOCADJDIM is NP-complete when restricted to planar instances. More precisely, the variable gadgets are paths on four vertices, where the middle two ones interconnect to the clause gadgets in which they occur. The clause gadgets are a bit more involved, as shown in Fig. 2.

As explained in Remark 2, we can (now) use Theorem 8 together with Theorem 16 to conclude the following hitherto unknown complexity result.

Theorem 17. LOCDIM *is NP-complete. Moreover, assuming ETH, there is no* $O(poly(n+m)2^{o(n)})$ *algorithm solving* LOCDIM *on graphs of order* n *and size* m.

Notice that the reduction explained in Remark 2 does not help find any hardness results on planar graphs. Hence, we leave it as an open question whether or not LOCDIM is NP-hard also on planar graph instances.

5 Conclusions

We have studied four dimension parameters in graphs. In particular, establishing concise formulae for corona product graphs, linking (local) metric dimension with (local) adjacency dimension of the involved graphs, allowed to deduce NP-hardness results (and similar hardness claims) for all these graph parameters, based on known results, in particular on Vertex Cover and on Dominating Set problems. We hope that the idea of using such types of non-trivial (combinatorial) formulae for computational hardness proofs can be also applied in other situations.

For instance, observe that reductions based on formulae as derived in Theorem 1 clearly preserve the natural parameter of these problems, which makes this approach suitable for Parameterized Complexity. However, let us notice here that Dim is unlikely to be fixed-parameter tractable under the natural parameterization (i.e., an upper bound on the metric dimension) even for subcubic graph instances; see [17]. Conversely, it is not hard to see that the natural parameterization of AdjDim can be shown to be in FPT by reducing it to Test Cover. Namely, let $G = (V, E)$ be a graph and k be an integer, defining an instance of AdjDim. We construct a Test Cover instance as follows: Let $S = \binom{V}{2}$ be the substances and define the potential test set $T = \{t_v \mid v \in V\}$ by letting

$$t_v(\{x, y\}) = \begin{cases} 1, \text{ if } v \in N[x] \triangle N[y] \\ 0, \text{ otherwise} \end{cases}$$

Now, if D is some adjacency generator, then $T_D = \{t_v \mid v \in D\}$ is some test cover solution, i.e., for any pair of substances, we find a test that differentiates the two. The converse is also true. Test Cover has received certain interest recently in Parameterized Complexity [8,14]. Does AdjDim admit a polynomial-size kernel, or does it rather behave like Test Cover?

From a computational point of view, let us mention (in-)approximability results as obtained in [26,34]. In particular, inapproximability of 1-LocDom readily transfers to inapproximability of AdjDim and this in turn leads to inapproximability results for Dim as in Remark 2; also see [17].

Also, 1-locating dominating sets have been studied (actually, independently introduced) in connection with coding theory [24]. Recall that these sets are basically adjacency bases. Therefore, it might be interesting to try to apply some of the information-theoretic arguments on variants of metric dimension, as well. Conversely, the notion of locality used in this paper connects to the idea of correcting only 1-bit errors in codes. These interconnections deserve further studies.

All these computational hardness results, as well as the various different applications that led to the introduction of these graph dimension parameters, also open up the quest for moderately exponential-time algorithms, i.e., algorithms that should find an optimum solution for any of our dimension problems in time $O(\text{poly}(n + m)c^n)$ on graphs of size m and order n for some $c < 2$, or also to finding polynomial-time algorithms for special graph classes. In this context, we mention results on trees, series-parallel and distance-regular graphs [7,13,18].

In view of the original motivation for introducing these graph parameters, it would be interesting to study their complexity on geometric graphs. Notice that the definition of a metric generator is not exclusively referring to (finite) graphs, which might lead us even back to the common roots of graph theory and topology.

In view of the many different motivations, also the study of computational aspects of other variants of dimension parameters could be of interest. We only mention here the notions of resolving dominating sets [2] and independent resolving sets [6].

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