Approximating the Minimum Tour Cover with a Compact Linear Program

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Abstract. A tour cover of an edge-weighted graph is a set of edges which forms a closed walk and covers every other edge in the graph. The minimum tour cover problem is to find a minimum weight tour cover. This problem is introduced by Arkin, Halldórsson and Hassin (Information Processing Letters 47:275-282, 1993) where the author prove the NP-hardness of the problem and give a combinatorial 5.5-approximation algorithm. Later Könemann, Konjevod, Parekh, and Sinha [7] improve the approximation factor to 3 by using a linear program of exponential size. The solution of this program involves the ellipsoid method with a separation oracle. In this paper, we present a new approximation algorithm achieving a slightly weaker approximation factor of 3.5 but only dealing with a compact linear program.

1 Introduction

Let $G = (V, E)$ be an undirected graph with a (nonnegative) weight function $c : E \Rightarrow \mathbb{Q}_+$ defined on the edges. A *tour cover* of G is a subgraph $T = (U, F)$ such that

1. for every $e \in E$, either $e \in F$ or F contains an edge f adjacent to e, i.e. $F \cap N(e) \neq \emptyset$ where $N(e)$ is the set of the edges adjacent to e.
2. T is a closed walk.

A tour cover is hence actually a tour over a vertex cover of G. The *minimum tour cover problem* consists in finding a tour cover of minimum total weight :

$$\min \sum_{e \in F} c_e,$$

over subgraphs $H = (U, F)$ which form a tour cover of G.

The minimum tour cover problem were introduced by Arkin, Haldórsson and Hassin [1]. The motivation for their study comes from the close relation of the tour cover problem to vertex cover, watchman route and traveling purchaser problems. They prove that the problem is NP-hard and provide a fast combinatorial algorithm achieving an approximation factor of 5.5.

Improved approximations came from Könemann, Konjevod, Parekh, and Sinha [7] where they use an integer formulation and its linear programming relaxation

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to design a 3-approximation algorithm. However, as the linear programming relaxation is of exponential size, their algorithm needs an ellipsoid method and a separation oracle to solve it.

Several problems are closely related to the minimum tour cover problem. First, if instead of a tour, we need a tree then this is the *minimum tree cover problem*. Second, if we need just a edge subset over a vertex cover then this is the *minimum edge dominating set problem*. These two problems are all NP-hard. Approximation algorithms [1],[7],[5] has been designed for the minimum tree cover problem and the current best approximation factor is 2 [5]. Similarly, approximation algorithms for the minimum edge dominating set problem have been discussed in [2], [6] and the current best approximation factor is also 2 [6]. Another related problem is the well known *minimum edge cover problem* which consists in finding a minimum weight edge subset which covers every vertex of G. This problem can be solved in polynomial time and we know a complete linear programming for it [4].

In this paper, we present a new approximation algorithm achieving a factor 3.5 which is slightly weaker than the factor 3 obtained by Könemann et al. but our algorithm needs only to solve a compact linear program. Precisely, we use a compact linear relaxation of the formulation in [7]. From an optimal solution of this relaxation, we determine a vertex cover subset and use a reduction to the edge cover problem to find a forest F spanning it. Finally, to obtain a tour cover, we apply the Christofides heuristic [3] to find a tour connecting the connected components of F and eventually duplicate edges in each connected component of F. We prove that the weight of such a tour cover is at most 3.5 times the weight of the minimum tour cover.

The idea of reduction to the edge cover problem is first given by Carr et al. [2] in the context of the minimum edge dominating set problem. We borrow their idea here to apply to the minimum tour cover problem.

Let us introduce the notations which will be used in the paper. For a subset of vertices $S \subseteq V$, we write $\delta(S)$ for the set of edges with exactly one endpoint inside S et $E(S)$ for the set of edges with both endpoints inside S. If $x \in \mathbb{R}^{|E|}$ is a vector indexed by the edges of a graph $G = (V, E)$ and $F \subseteq E$ is a subset of edges, we use $x(F)$ to denote the sum of values of x on the edges in the set F, $x(F) = \sum_{e \in F} x_e$.

The paper is organized as follows. First, we present the first three steps of the algorithm and we explain the idea of the reduction to the edge cover problem. Second and lastly, we describe the last step and we give a proof for the approximation factor of 3.5.

2 A 3.5-Approximation Algorithm

2.1 First Three Steps of the Algorithm

Let x be a vector in $\mathbb{R}^E$ which is an incidence vector of a tour cover $\mathcal{C}$ of G. Let $e = uv$ be an edge of G. We can see that e can belong or not to $\mathcal{C}$, but in the two cases, there must be at least two edges in $\delta(\{u, v\})$ belonging to $\mathcal{C}$, i.e. $|\delta(\{u, v\}) \cap \mathcal{C}| \geq 2$.

Hence, the vector x satisfies the following constraint: $x(\delta(\{u,v\})) \geq 2$. The following linear program which consists of all these constraints applying for every edge in G :

$$\min \sum_{e \in E} c_e x_e$$

$$\text{s.t.} \quad x(\delta(\{u,v\})) \geq 2 \quad \text{for all edge } uv \in G, \tag{1}$$
$$0 \leq x_e \leq 2 \text{ for all } e \in E.$$

This linear program (1) is thus a linear programming relaxation of the tour cover problem. We can see that this is a compact linear program, its size is even linear since the number of constraints is in $O(|E|)$. Note that the linear programming relaxation used in [7] has, in addition of the box constraints $0 \leq x_e \leq 2$ for all $e \in E$,

$$x(\delta(S)) \geq 2 \quad \text{for all } S \subset V \text{ s.t. } E(S) \neq \emptyset, \tag{2}$$

as constraints and then is clearly of exponential size and the set of the inequalities(1) is the subset of the inequalities in (2) having $|S| = 2$.
Now let us consider the following

$$\min \sum_{e \in E} c_e x_e$$

$$\text{s.t.} \quad x(\delta(\{u,v\})) \geq 1 \quad \text{for all edge } uv \in G, \tag{3}$$
$$0 \leq x_e \leq 1 \text{ for all } e \in E.$$

It is clear that an optimal solution of (1) is two times a optimal solution of (3) and a solution of (3) is a half of a optimal solution of (1).
Let x^* be an optimal solution of (3) found by usual linear programming techniques. Consequently $2x^*$ is an optimal solution of (1). Let $V_+ \subseteq V = \{u \in V | x^*(\delta(u)) \geq 1/2\}$ and let $V_- = V \setminus V_+$. It is not difficult to prove the following lemma.

Lemma 1. *The set V_+ is a vertex cover of G.*

Hence, a tour in G containing all vertices of V_+ is thus a tour cover. Our algorithm will build such a tour by building first an edge subset $D_+ \subseteq E$ covering V_+ (i.e. V_+ is a subset of the set of the end vertices of the edges in D_+) with weight not greater than $2\sum_{e\in E} c_e x_e^*$ and after finding a tour connecting the connected components of D_+ by Christofides algorithm. The idea of build the edge set D_+ is borrowed from [2] where the authors apply it for designing an $2\frac{1}{10}$-approximation algorithm for the minimum edge dominating problem. Let us examine it in details.

Let V'_- be a copy of V_- where $v \in V_-$ corresponds to $v' \in V'_-$ and E' be the set of zero-weight edges, one between each $v \in V_-$ and its copy $v' \in V'_-$. We construct then the graph $\bar{G} = (\bar{V} = V \cup V', \bar{E} = E \cup E')$. We have the following lemma.

Lemma 2. *There is a one-to-one weight preserving correspondence between the edge subsets which cover V_+ in G and the edge cover subsets (that cover $\bar{V}$) of $\bar{G}$.*

Proof. If D_- is an edge cover of $\bar{G}$, then $D_+ = D_- \cap E$ must be an edge set of equal weight covering all the vertices in V_+. Conversely, if D_+ is an edge set covering all the vertices in V_+, then $D_+ \cup E'$ is an edge cover of $\bar{G}$ of equal weight, since the edges in E' cost nothing.

We can now describe the first three steps of the algorithm which can be stated as follows:

Step 1. Compute an optimal solution x^* of (3).
Step 2. Compute V_+.
Step 3. Build the graph $\bar{G}$. As the minimum edge cover problem can be solved in polynomial time [4], compute a minimum-weight edge cover D_- in $\bar{G}$ and set $D_+ = D_- \cap E$.

2.2 Analysis on the Quality of D_+

It is known that the minimum edge cover problem in $\bar{G}$ can be formulated by the following linear program [4]:

$$\min \sum_{e \in E} c_e x_e,$$

$EC(\bar{G})$ s.t.

$$x(\delta(u)) \geq 1 \; u \in \bar{V}, \tag{4}$$

$$x(E(S)) + x(\delta(S)) \geq \frac{|S|+1}{2} \quad S \subseteq \bar{V},\ |S| \geq 3 \text{ odd}, \tag{5}$$

$$0 \leq x_e \leq 1 \; e \in \bar{E}.$$

Theorem 1. *Le point $2x^*$ which is an optimal solution of* (1) *is feasible pour $EC(\bar{G})$.*

Proof. Let $y^* = 2x^*$. Suppose u is a vertex in $\bar{V}$. If $u \in V_+$, we have $x^*(\delta(u)) \geq \frac{1}{2}$, otherwise $u \in V_- \cup V'_-$, and we have $x^*_e = 1$ for all $e \in E'$, so in either case

$$y^*(\delta(u)) \geq 1, \tag{6}$$

So y satisfies the constraints (4). As x^* is a solution of (3), hence y^* satisfies

$$y^*(\delta(u)) + y^*(\delta(v)) \geq 2 + 2y^*_{uv}. \tag{7}$$

Suppose that S is a subset of $\bar{V}$ of odd cardinality; let $s = |S|$. When $s = 1$, the constraints (4) are trivially satisfied by y^*, so suppose that $s \geq 3$. By combining (6) and (7) we see

$$y^*(\delta(u)) + y^*(\delta(v)) \geq \begin{cases} 2 + 2y^*_{uv} & \text{if } uv \in \bar{E}, \\ 2 & \text{otherwise.} \end{cases}$$

Summing the approriate inequality above for each pair $\{u, v\}$ in $S \times S$, where $u \neq v$, we get

$$\begin{aligned}(s-1)y^*(\delta(S)) + 2(s-1)y^*(\bar{E}(S)) &= (s-1)\sum_{u \in S} y^*(\delta(u)) \\ &= \sum_{\{uv \in S \times S | u \neq v\}} y^*(\delta(u)) + y^*(\delta(v)) \\ &\geq s(s-1) + 2y^*(\bar{E}(S)).\end{aligned}$$

Isolating the desired left hand side yields

$$y^*(\delta(S)) + y^*(\bar{E}(S)) \geq \tfrac{s(s-1)+(s-3)y^*(\delta(S))}{2s-4} \geq \tfrac{s(s-1)}{2s-4}, \text{ for } s \geq 3.$$

We can see that for $s \geq 3$, $s(s-1) > (s+1)(s-2)$, thus

$$\frac{s(s-1)}{2(s-2)} > \frac{(s+1)(s-2)}{2(s-2)} > \frac{s+1}{2}.$$

Hence y^* satisfies (5).

Corollary 1. *The weight of D_+ is a lower bound for the weight of an optimal solution of* (1).

2.3 Last Step

We can see that D_+ forms a forest.

Step 4. The last step of the algorithm consists of shrinking the connected componnents of D_+ along its edges into vertices to obtain a contracted graph G'. The contraction of a graph G along a set of edge D_+ produces a graph $G' = (V', E')$ and a set of vertices $S \subseteq V'$ defined as follows: For each edge in D_+, merge its two endpoints into a single vertex, whose adjacency list is the union of the two adjacency lists. If parallel edges occur, retain the edge with smaller weight. The resulting graph is G'. The set S consists of the vertices formed by edge contraction (i.e., the nodes in $V' \setminus V$). We now proceed to find an approximation of the optimal tour T' going through all the vertices of S in G' with distances modified (if necessary) to the shortest paths distances. This can be done by using Christofides heuristic [3]. Map this solution back to a partial tour T of G. For each component formed by the edges in D_+, form an Eulerian walk from the entry point to the exit point of Q in the component by duplicate some of its edges. Output the tour Q formed by T and the Eulerian walk.

Let APX be the weight of Q and OPT be the weight of an optimal tour cover in G.

Theorem 2. *$APX \leq 3.5\times$ OPT.*

Proof. We can see in the worst case, Q contains the tour T and 2 times the edge set D_+. Any tour cover in G should take visit to all the connected component of D_+, hence as the weight of T can not be worse than 1.5×OPT [3]. By Corollary 1, the weight of D_+ is less than OPT, then 2 times the edge set D_+ is of weight at most 2×OPT. Thus, overall $APX \leq 1.5\times\text{OPT} + 2\times\text{OPT} = 3.5\times$ OPT.

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