Chapter 5
Food-Limited Population Models

If a nonnegative quantity was so small that is smaller than any given one, then it certainly could not be anything but zero. To those who ask what the infinity small quantity in mathematics is, we answer that it is actually zero. Hence there are not so many mysteries hidden in this concept as they are usually believed to be.

Leonhard Euler (1707–1783)

The real end of science is the honor of the human mind.

Gustav J. Jacobi (1804–1851)

Smith [66] reasoned that a food-limited population in its growing stage requires food for both maintenance and growth, whereas, when the population has reached saturation level, food is needed for maintenance only. On the basis of these assumptions, Smith derived a model of the form

$$\frac{dN(t)}{dt} = rN(t)\frac{K - N(t)}{K + crN(t)} \tag{5.1}$$

which is called the "food limited" population. Here N, r, and K are the mass of the population, the rate of increase with unlimited food, and the value of N at saturation, respectively. The constant $1/c$ is the rate of replacement of mass in the population at saturation. Since a realistic model must include some of the past history of the population, Gopalsamy, Kulenovic and Ladas introduced the delay in (5.1) and considered the equation

$$\frac{dN(t)}{dt} = rN(t)\frac{K - N(t-\tau)}{K + crN(t-\tau)},$$

as the delay "food-limited" population model, where r, K, c, and τ are positive constants.

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In this chapter we discuss autonomous and nonautonomous "food-limited" population models with delay times.

5.1 Oscillation of Delay Models

Motivated by the model

$$N'(t) = r(t)N(t)\frac{K - N(h(t))}{1 + s(t)N(g(t))}, \quad t \geq 0, \tag{5.2}$$

in this section we consider

$$x'(t) = -r(t)x(h(t))\frac{1 + x(t)}{1 + s(t)[1 + x(g(t))]}, \quad t \geq 0, \tag{5.3}$$

with the following assumptions:

$(A1)$ $r(t)$ and $s(t)$ are Lebesgue measurable locally essentially bounded functions such that $r(t) \geq 0$ and $s(t) \geq 0$.

$(A2)$ $h, g : [0, \infty) \to \mathbf{R}$ are Lebesgue measurable functions such that $h(t) \leq t$, $g(t) \leq t$, $\lim_{t\to\infty} h(t) = \infty$, and $\lim_{t\to\infty} g(t) = \infty$.

Note the oscillation (or nonoscillation) of N about K is equivalent to oscillation (nonoscillation) of (5.3) about zero (let $x = N/K - 1$).

One could also consider for each $t_0 \geq 0$ the problem

$$x'(t) = -r(t)x(h(t))\frac{1 + x(t)}{1 + s(t)[1 + x(g(t))]}, \quad t \geq t_0, \tag{5.4}$$

with the initial condition

$$x(t) = \varphi(t), \quad t < t_0, \ x(t_0) = x_0. \tag{5.5}$$

We also assume that the following hypothesis holds:

$(A3)$ $\varphi : (-\infty, t_0) \to \mathbf{R}$ is a Borel measurable bounded function.

An absolutely continuous function $x(: \mathbf{R} \to \mathbf{R})$ on each interval $[t_0, b]$ is called a solution of problems (5.4) and (5.5), if it satisfies (5.4) for almost all $t \in [t_0, \infty)$ and the equality (5.5) for $t \leq t_0$. Equation (5.3) has a nonoscillatory solution if it has an eventually positive or an eventually negative solution. Otherwise, all solutions of (5.3) are oscillatory. The results in this section can be found in [10]. In the following, we assume that $(A1)$–$(A3)$ hold and we consider only such solutions of (5.3) for which the following condition holds:

$$1 + x(t) > 0. \tag{5.6}$$

The proof of the following lemma follows a standard argument (see the proof in Theorem 2.4.1 and see Lemma 2.6.1).

Lemma 5.1.1. *Let* $(A1)$ *and* $(A2)$ *hold for the equation*

$$x'(t) + r(t)x(h(t)) = 0, \quad t \geq 0. \tag{5.7}$$

Then the following hypotheses are equivalent:

(1) *The differential inequality*

$$x'(t) + r(t)x(h(t)) \leq 0, \quad t \geq 0 \tag{5.8}$$

has an eventually positive solution.

(2) *There exists* $t_0 \geq 0$ *such that the inequality*

$$u(t) \geq r(t) \exp\left\{\int_{h(t)}^{t} u(s)ds\right\}, \ t \geq t_0, \ u(t) = 0, t < t_0 \tag{5.9}$$

has a nonnegative locally integrable solution.

(3) *Equation (5.7) has a nonoscillatory solution.*
If

$$\lim_{t\to\infty} \sup \int_{h(t)}^{t} r(s)ds < \frac{1}{e}, \tag{5.10}$$

then (5.7) has a nonoscillatory solution. If

$$\lim_{t\to\infty} \inf \int_{h(t)}^{t} r(s)ds > \frac{1}{e}, \tag{5.11}$$

then all the solutions of (5.7) are oscillatory.

Lemma 5.1.2. *Let* $x(t)$ *be a nonoscillatory solution of* (5.3) *and suppose that*

$$\int_0^\infty \frac{r(t)}{1+s(t)} dt = \infty. \tag{5.12}$$

Then $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose first $x(t) > 0, t \geq t_1$. Then there exists $t_2 \geq t_1$ such that

$$h(t) \geq t_1, \quad g(t) \geq t_1, \ \text{for } t \geq t_2. \tag{5.13}$$

Let

$$u(t) = -\frac{x'(t)}{x(t)}, \quad t \geq t_2. \tag{5.14}$$

Then $u(t) \geq 0, t \geq t_2$ and

$$x(t) = x(t_2) \exp\left\{-\int_{t_2}^{t} u(s)ds\right\}, \ t \geq t_2. \tag{5.15}$$

Substituting this into (5.3) we obtain

$$u(t) = r(t)e^{\left(\int_{h(t)}^{t} u(s)ds\right)} \frac{\left[1 + c\exp\left\{-\int_{t_2}^{t} u(s)ds\right\}\right]}{\left[1 + s(t)\left(1 + c\exp\left\{-\int_{t_2}^{g(t)} u(s)ds\right\}\right)\right]}, \tag{5.16}$$

where $h(t) \leq t$, $g(t) \leq t$, for $t \geq t_2$, and $c = x(t_2) > 0$. Hence

$$u(t) \geq \frac{r(t)}{(1+c)(1+s(t))}. \tag{5.17}$$

From (5.12) we have $\int_{t_2}^{\infty} u(t)dt = \infty$.

Now suppose $-1 < x(t) < 0, t \geq t_1$. Then there exists $t_2 \geq t_1$ such that (5.13) holds for $t \geq t_2$. With $u(t)$ denoted in (5.14) and $c = x(t_2)$ we have $u(t) \geq 0$ and $-1 < c < 0$. Substituting (5.15) into (5.3) and using (5.16), we have

$$u(t) \geq \frac{(1+c)r(t)}{(1+s(t))}. \tag{5.18}$$

Thus $\int_{t_2}^{\infty} u(t)dt = \infty$. Equation (5.15) implies that $\lim_{t\to\infty} x(t) = 0$. The proof is complete. ∎

Theorem 5.1.1. *Suppose (5.12) holds and for some $\varepsilon > 0$, all solutions of the linear equation*

$$x'(t) + (1-\varepsilon)\frac{r(t)}{1+s(t)}x(h(t)) = 0 \tag{5.19}$$

are oscillatory. Then all solutions of (5.3) are oscillatory.

Proof. First suppose $x(t)$ is an eventually positive solution of (5.3). Lemma 5.1.2 implies that there exists $t_1 \geq 0$ such that $0 < x(t) < \varepsilon$ for $t \geq t_1$. We suppose (5.13) holds for $t \geq t_2 \geq t_1$. For $t \geq t_2$, we have

$$\frac{[1+s(t)](1+x(t))}{1+s(t)[1+x(g(t))]} \geq \frac{(1+s(t))}{1+s(t)(1+\varepsilon)}$$
$$\geq \frac{(1+s(t))}{(1+s(t))(1+\varepsilon)} = \frac{1}{(1+\varepsilon)} \geq 1-\varepsilon. \tag{5.20}$$

Equation (5.3) implies

$$x'(t) + (1-\varepsilon)\frac{r(t)}{1+s(t)}x(h(t)) \leq 0, \quad t \geq t_2. \tag{5.21}$$

Lemma 5.1.1 yields that (5.19) has a nonoscillatory solution. We have a contradiction.

Now suppose $-\varepsilon < x(t) < 0$ for $t \geq t_1$ and (5.13) holds for $t \geq t_2 \geq t_1$. Then for $t \geq t_2$

$$\frac{[1+s(t)](1+x(t))}{1+s(t))[1+x(g(t))]} \geq \frac{(1+s(t))(1-\varepsilon)}{1+s(t)} = 1-\varepsilon. \tag{5.22}$$

Hence, (5.19) has a nonoscillatory solution and we again obtain a contradiction which completes the proof. ■

Corollary 5.1.1. *If*

$$\liminf_{t\to\infty} \int_{h(t)}^{t} \frac{r(\tau)}{1+s(\tau)} d\tau > \frac{1}{e}, \tag{5.23}$$

then all solutions of (5.3) are oscillatory.

Theorem 5.1.2. *Suppose for some $\varepsilon > 0$ there exists a nonoscillatory solution of the linear delay differential equation*

$$x'(t) + (1+\varepsilon)\frac{r(t)}{1+s(t)} x(h(t)) = 0. \tag{5.24}$$

Then there exists a nonoscillatory solution of (5.3).

Proof. Lemma 5.1.1 implies that there exists $t_0 \geq 0$ such that

$$w_0(t) \geq 0, \text{ for } t \geq t_0, \text{ and } w_0(t) = 0, \text{ for } t \leq t_0,$$

and

$$w_0(t) \geq (1+\varepsilon)\frac{r(t)}{1+s(t)} \exp\left\{\int_{h(t)}^{t} w_0(s)ds\right\}. \tag{5.25}$$

Suppose $0 < c < \varepsilon$ and consider two sequences:

$$w_n(t) = r(t)\exp\left\{\int_{h(t)}^{t} w_{n-1}(s)ds\right\} \times \frac{1+c\exp\left\{-\int_{t_0}^{t} v_{n-1}(s)ds\right\}}{1+s(t)\left(1+c\exp\left\{-\int_{t_0}^{g(t)} w_{n-1}(s)ds\right\}\right)}$$

and

$$v_n(t) = r(t)\exp\left\{\int_{h(t)}^{t} v_{n-1}(s)ds\right\} \times \frac{1+c\exp\left\{-\int_{t_0}^{t} w_{m-1}(s)ds\right\}}{1+s(t)\left(1+c\exp\left\{-\int_{t_0}^{g(t)} v_{n-1}(s)ds\right\}\right)},$$

where w_0 is as defined above and $\upsilon_0(t) \equiv 0$. We have

$$\begin{aligned} w_1(t) &= \frac{r(t)}{1+s(t)} \exp\left\{\int_{h(t)}^{t} w_0(s)ds\right\} \\ &\quad \times \frac{(1+s(t))(1+c)}{1+s(t)\left(1+c\exp\left\{-\int_{t_0}^{g(t)} w_0(s)ds\right\}\right)} \\ &\le \frac{r(t)}{1+s(t)} \exp\left\{\int_{h(t)}^{t} w_0(s)ds\right\} \\ &\quad \times \frac{(1+s(t))(1+\varepsilon)}{1+s(t)\left(1+c\exp\left\{-\int_{t_0}^{g(t)} w_0(s)ds\right\}\right)} \\ &\le w_0(t) \end{aligned} \tag{5.26}$$

from (5.25). Clearly $\upsilon_1(t) \ge \upsilon_0(t)$ and $w_0(t) \ge \upsilon_0(t)$. Hence by induction

$$\begin{cases} 0 \le w_n(t) \le w_{n-1}(t) \le \ldots \le w_0(t), \\ \upsilon_n(t) \ge \upsilon_{n-1}(t) \ge \ldots \ge \upsilon_0(t) = 0, \\ \quad w_n(t) \ge \nu_n(t). \end{cases} \tag{5.27}$$

There exist pointwise limits of the nonincreasing nonnegative sequence $w_n(t)$ and of the nondecreasing sequence $\nu_n(t)$. Let

$$w(t) = \lim_{n\to\infty} w_n(t) \ \text{ and } \ \upsilon(t) = \lim_{n\to\infty} \upsilon(t).$$

Then by the Lebesgue Convergence Theorem, we conclude that

$$\begin{aligned} w(t) &= r(t)\exp\left\{\int_{h(t)}^{t} w(s)ds\right\} \\ &\quad \times \frac{1+c\exp\left\{-\int_{t_0}^{t} \nu(s)ds\right\}}{1+s(t)\left(1+c\exp\left\{-\int_{t_0}^{g(t)} w(s)ds\right\}\right)} \end{aligned}$$

and

$$\begin{aligned} \nu(t) &= r(t)\exp\left\{\int_{h(t)}^{t} \nu(s)ds\right\} \\ &\quad \times \frac{1+c\exp\left\{-\int_{t_0}^{t} w(s)ds\right\}}{1+s(t)\left(1+c\exp\left\{-\int_{t_0}^{g(t)} \nu(s)ds\right\}\right)}. \end{aligned}$$

We fix $b \geq t_0$ and define the operator $T : L_\infty[t_0, b] \to L_\infty[t_0, b]$ by

$$T(u(t)) = e^{\int_{h(t)}^{t} u(s)ds} \frac{r(t)\left(1 + c\exp\left\{-\int_{t_0}^{t} u(s)ds\right\}\right)}{1 + s(t)\left(1 + c\exp\left\{-\int_{t_0}^{g(t)} u(s)ds\right\}\right)}. \tag{5.28}$$

For every function u from the interval $\upsilon \leq u \leq w$, we have $\upsilon \leq Tu \leq w$. One can also check that T is a completely continuous operator on the space $L_\infty[t_0, b]$. Then by Schauder's Fixed Point Theorem there exists a nonnegative solution of equation $u = Tu$. Let

$$x(t) = \begin{cases} c\exp\left\{-\int_{t_0}^{t} u(s)ds\right\}, & \text{if } t \geq t_0, \\ 0, & \text{if } t < t_0, \end{cases} \tag{5.29}$$

and then $x(t)$ is a nonoscillatory solution of (5.3) which completes the proof. ∎

The results in this section apply to (5.2). For example by applying Theorem 5.1.1 we have the following result.

Theorem 5.1.3. *Suppose (5.12) holds and for some $\varepsilon > 0$, all solutions of the linear equation*

$$N'(t) + (1-\varepsilon)\frac{r(t)}{1+s(t)}N(h(t)) = 0 \tag{5.30}$$

are oscillatory. Then all solutions of (5.2) are oscillatory about K.

5.2 Oscillation of Impulsive Delay Models

In this section we consider the impulsive "food-limited" population model

$$\begin{cases} N'(t) = r(t)N(t)\dfrac{K - N(h(t))}{K + \sum_{i=1}^{m} p_i(t)N(g_i(t))}, \quad t \neq t_k, \\ N(t_k^+) - N(t_k) = b_k(N(t_k) - K), \text{ for } k = 1, 2, \ldots; \end{cases} \tag{5.31}$$

here $N(t_k) = N(t_k^-)$. In this section, we will assume that the following assumptions hold:

$(A1)$ $0 \leq t_0 < t_1 < t_2 < \ldots < t_k < \ldots$ are fixed points with $\lim_{k\to\infty} t_k = \infty$,
$(A2)$ $b_k > -1, k = 1, 2, \ldots$, K is a positive constant,
$(A3)$ $r(t)$ and $p_i, i = 1, 2, \ldots, m$, are Lebesgue measurable locally essentially bounded functions, in each finite interval $[0, b]$, $r(t) \geq 0$ and $p_i(t) \geq 0$, for $i = 1, 2, \ldots, m$,
$(A4)$ $h, g_i : [0, \infty) \to \mathbf{R}$ are Lebesgue measurable functions, $h(t) \leq t, g_i(t) \leq t$, $\lim_{t\to\infty} h(t) = \infty$, $\lim_{t\to\infty} g_i(t) = \infty, i = 1, 2, \ldots, m$.

In this section (motivated by (5.31) with $y(t) = \frac{N(t)}{K} - 1$) we consider the delay model with impulses

$$\begin{cases} y'(t) = -r(t) \dfrac{(1 + y(t))\, y(h(t))}{1 + \sum_{i=1}^{m} p_i(t)\,[1 + y(g_i(t))]}, \; t \neq t_k, \; t \geq T_0 \geq 0 \\ y(t_k^+) - y(t_k) = b_k y(t_k), \text{ for } k = 1, 2, \ldots, \end{cases} \tag{5.32}$$

where $b_k > -1$ and r, h, p_i for $m = 1, 2, \ldots$ are nonnegative real-valued functions. We consider (5.32) with the initial condition

$$y(t) = \varphi(t) \geq 0, \quad \varphi(T_0) > 0, \;\; t \in [T^-, T_0]. \tag{5.33}$$

Here for any $T_0 \geq 0$, $T^- = \min_{1 \leq i \leq m} \inf_{t \geq T_0} (g_i(t), h(t))$, and $\varphi : [T^-, T_0] \to \mathbf{R}_+$ is a Lebesgue measurable function.

For any $T_0 \geq 0$ and $\varphi(t)$, a function $y : [T^-, \infty] \to \mathbf{R}$ is said to be a solution of (5.32) on $[T, \infty]$ satisfying the initial value condition (5.33), if the following conditions are satisfied:

1. $y(t)$ satisfies (5.33);
2. $y(t)$ is absolutely continuous in each interval $(T_0, t_k), (t_k, t_{k+1}), t_k > T_0, k \geq k_0$, $y(t_k^+), y(t_k^-)$ exist and $y(t_k^-) = y(t_k), k > k_0$;
3. $y(t)$ satisfies the former equation of (5.32) in $[T, \infty) \backslash \{t_k\}$ and satisfies the latter equation for every $t = t_k, k = 1, 2, \ldots$.

For any $t \geq 0$, consider the nonlinear delay differential equation

$$x'(t) = -r(t) \frac{1 + \left(\prod_{T_0 \leq t_k < t} (1 + b_k) \right) x(t)}{1 + \Psi(x(g_i(t))} \times \prod_{h(t) \leq t_k < t} (1 + b_k)^{-1} x(h(t)), \tag{5.34}$$

where

$$\Psi(x(g_i(t)) = \sum_{i=1}^{m} p_i(t) \left[1 + (\prod_{T_0 \leq t_k < g_i(t)} (1 + b_k)) x(g_i(t)) \right].$$

The results in this section are adapted from [77] (in fact as we see below it is easy to extend the theory in the nonimpulsive case in Sect. 5.1 to the impulsive case).

Lemma 5.2.1. *Assume that $(A1)$–$(A4)$ hold. Then the solution $N(t)$ of (5.31) oscillates about K if and only if the solution $y(t)$ of (5.32) oscillates about zero.*

The proof (which is elementary and straightforward) of the next lemma can be found in [81].

Lemma 5.2.2. *Assume that* $(A1)$–$(A4)$ *hold. For any* $T_0 \geq 0, y(t)$ *is a solution of (5.32) on* $[T_0, \infty)$ *if and only if*

$$x(t) = \left(\prod_{T_0 \leq t_k < t} (1 + b_k) \right)^{-1} y(t) \tag{5.35}$$

is a solution of the nonimpulsive delay differential equation (5.34).

From Lemmas 5.2.1 and 5.2.2 we see that the solution $N(t)$ of (5.31) is oscillatory about K if and only if the solution $y(t)$ of (5.32) is oscillatory.

We consider only such solutions of (5.32) for which the following condition holds:

$$1 + y(t) > 0, \quad \text{for } t \geq T_0, \tag{5.36}$$

and hence, in view of (5.35),

$$1 + \left(\prod_{T_0 \leq t_k < t} (1 + b_k) \right) x(t) > 0, \quad \text{for } t \geq T_0. \tag{5.37}$$

With $y(t) = \frac{N(t)}{K} - 1$ then from (5.36) and (5.37), we see that

$$N(t) = K \left(1 + \prod_{T_0 \leq t_k < t} (1 + b_k) x(t) \right) > 0, \quad t \geq T_0.$$

Thus for the initial condition $N(t) = \varphi(t) : [T^-, T_0] \to \mathbf{R}_+, \varphi(T_0) > 0$, the solution of (5.31) is positive on $[T_0, \infty)$.

Lemma 5.2.3. *Assume that* $(A1)$–$(A4)$ *hold,*

$$\int_0^\infty r(t) \left(1 + \sum_{i=1}^m p_i(t) \right)^{-1} dt = \infty, \tag{5.38}$$

and

$$\prod_{T_0 \leq t_k < t} (1 + b_k) \textit{ is bounded and } \liminf_{t \to \infty} \prod_{T_0 \leq t_k < t} (1 + b_k) > 0. \tag{5.39}$$

If $y(t)$ *is a nonoscillatory solution of (5.32), then* $\lim_{t \to \infty} y(t) = 0$.

Proof. Suppose first $y(t) > 0$ for $t \geq T_1 \geq 0$. From (5.35) and $(A1)$, $x(t) > 0$ for $t \geq T_1$. Then there exists $T_2 \geq T_1$ such that

$$h(t) \geq T_2, \quad g_i(t) \geq T_2, \quad i = 1, 2, \ldots, m, \text{ for } t \geq T_2. \tag{5.40}$$

Let

$$u(t) = -\frac{x'(t)}{x(t)}, \quad \text{for } t \geq T_2. \tag{5.41}$$

Then $u(t) \geq 0$ for $t \geq T_2$ and

$$x(t) = x(T_2) \exp\left\{-\int_{T_2}^{t} u(s)ds\right\}, \quad \text{for } t \geq T_2. \tag{5.42}$$

Setting $c = x(T_2)$, we have

$$\begin{aligned}
u(t) &= \frac{r(t)}{x(t)} \left(\prod_{h(t) \leq t_k < t} (1 + b_k)^{-1} \right) x(h(t)) \\
&\quad \times \frac{1 + (\prod_{T_0 \leq t_k < t} (1 + b_k))x(t)}{1 + \sum_{i=1}^{m} p_i(t)[1 + (\prod_{T_0 \leq t_k < g_i(t)} (1 + b_k))x(g_i(t))]} \\
&\geq \frac{r(t)}{x(t)} \left(\prod_{h(t) \leq t_k < t} (1 + b_k)^{-1} \right) x(t) \\
&\quad \times \frac{1}{1 + \sum_{i=1}^{m} p_i(t)[1 + \prod_{T_0 \leq t_k < g_i(t)} (1 + b_k)c]} \\
&= \frac{r(t)}{1 + \sum_{i=1}^{m} p_i(t)} \left(\prod_{h(t) \leq t_k < t} (1 + b_k)^{-1} \right) \\
&\quad \times \frac{1 + \sum_{i=1}^{m} p_i(t)}{1 + \sum_{i=1}^{m} p_i(t)[1 + \prod_{T_0 \leq t_k < g_i(t)} (1 + b_k)c]} \\
&\geq \frac{r(t)}{1 + \sum_{i=1}^{m} p_i(t)} \frac{(\prod_{h(t) \leq t_k < t} (1 + b_k))^{-1}}{(1 + \sum_{i=1}^{m} p_i(t)(1 + (\prod_{T_0 \leq t_k < g_i(t)} (1 + b_k)c))}.
\end{aligned}$$

Then from (5.38) and (5.39), $\int_{T_2}^{\infty} u(t)dt = \infty$.

Now suppose $-1 < y(t) < 0$. Hence in view of (5.36),

$$-1 < \prod_{T_0 \le t_k < g_i(t)} (1 + b_k)x(t) < 0, \quad t \ge T_1.$$

Then there exists $T_2 > T_1$ such that (5.40) holds for $t > T_2$. With $u(t)$ denoted in (5.41) and $c = x(T_2)$, then from (5.37) $u(t) \ge 0$, $-1 < c < 0$, and we obtain

$$u(t) = \frac{r(t)}{x(t)} \left(\prod_{h(t) \le t_k < t} (1 + b_k)^{-1} \right) x(h(t))$$
$$\times \frac{1 + (\prod_{T_0 \le t_k < t} (1 + b_k))x(t)}{1 + \sum_{i=1}^{m} p_i(t)[1 + (\prod_{T_0 \le t_k < g_i(t)} (1 + b_k))x(g_i(t))]}$$

$$\ge \frac{r(t)}{x(h(t))} \left(\prod_{h(t) \le t_k < t} (1 + b_k)^{-1} \right) x(h(t)) \frac{1 + (\prod_{T_0 \le t_k < t} (1 + b_k))c}{1 + \sum_{i=1}^{m} p_i(t)}$$
$$= \left(\prod_{h(t) \le t_k < t} (1 + b_k)^{-1} \right) \left(1 + (\prod_{T_0 \le t_k < t} (1 + b_k))c \right)$$
$$\times \frac{r(t)}{1 + \sum_{i=1}^{m} p_i(t)}.$$

Then by (5.37)–(5.39), we have $\int_{T_2}^{\infty} u(t)dt = \infty$. Equation (5.42) implies $\lim_{t \to \infty} x(t) = 0$. Use (5.35), and then we have $\lim_{t \to \infty} y(t) = 0$. The proof is complete. ∎

Theorem 5.2.1. *Assume that* $(A1)$ *and* $(A2)$*, (5.38) hold and for some* $\epsilon > 0$*, all solutions of the linear equation*

$$x'(t) + (1-\epsilon) \prod_{h(t)\le t_k<t} (1+b_k)^{-1} \frac{r(t)x(h(t))}{1+\sum_{i=1}^{m} p_i(t)} = 0 \tag{5.43}$$

are oscillatory. Then all solutions of (5.32) are oscillatory.

Proof. Suppose $y(t)$ is an eventually positive solution of (5.32). Then $x(t)$ is an eventually positive solution of (5.34). Lemma 5.2.3 implies that there exists $T_1 \ge 0$, such that

$$0 < (\prod_{T_0\le t_k<t} (1+b_k))x(t) < \epsilon, \quad \text{for } t \ge T_1.$$

We suppose (5.40) holds for $t \ge T_2$, and we have

$$\begin{aligned} &\frac{(1+\sum_{i=1}^{m} p_i(t))(1+(\prod_{T_0\le t_k<t}(1+b_k))x(t))}{1+\sum_{i=1}^{m} p_i(t)[1+(\prod_{T_0\le t_k<g_i(t)}(1+b_k))x(g_i(t))]} \\ &\ge \frac{1+\sum_{i=1}^{m} p_i(t)}{1+\sum_{i=1}^{m} p_i(t)(1+\epsilon)} \ge \frac{1+\sum_{i=1}^{m} p_i(t)}{(1+\epsilon)(1+\sum_{i=1}^{m} p_i(t))} \\ &= \frac{1}{1+\epsilon} \ge 1-\epsilon. \end{aligned} \tag{5.44}$$

Equation (5.34) implies

$$x'(t) + (1-\epsilon) \prod_{h(t)\le t_k<t} (1+b_k)^{-1} \frac{r(t)x(h(t))}{1+\sum_{i=1}^{m} p_i(t)} \le 0, \quad t \ge T_2. \tag{5.45}$$

This implies that the (5.43) has a positive solution, which is a contradiction.

Now, we suppose

$$-\epsilon < (\prod_{T_0\le t_k<t} (1+b_k))x(t) < 0, \quad \text{for } t \ge T_1,$$

and (5.38) holds for $t \ge T_2 \ge T_1$. Then for $t \ge T_2$, we also get

$$\frac{(1+\sum_{i=1}^{m} p_i(t))(1+(\prod_{T_0\le t_k<t}(1+b_k))x(t))}{1+\sum_{i=1}^{m} p_i(t)[1+(\prod_{T_0\le t_k<g_i(t)}(1+b_k))x(g_i(t))]}$$

$$\ge \frac{(1+\sum_{i=1}^{m} p_i(t))(1-\epsilon)}{1+\sum_{i=1}^{m} p_i(t)} = 1-\epsilon. \tag{5.46}$$

Thus (5.43) has a nonoscillatory solution and we again obtain a contradiction. The proof is complete. ∎

Theorem 5.2.2. *Assume that* $(A1)$ *and* $(A2)$ *hold and*

$$\prod_{h(t)\le t_k<t}(1+b_k) \quad \textit{is convergent.} \tag{5.47}$$

Moreover, for some $\epsilon > 0$ *if there exists a nonoscillatory solution of the linear delay differential equation*

$$x'(t) + (1+\epsilon)\prod_{h(t)\le t_k<t}(1+b_k)^{-1}\frac{r(t)x(h(t))}{1+\sum_{i=1}^{m} p_i(t)} = 0, \tag{5.48}$$

then there exists a nonoscillatory solution of (5.32).

Proof. Suppose that $x(t) > 0$ for $t > T_0$ is a solution of (5.48). Then by (5.34) there exist $T_0 \ge 0$ and $\omega_0(t) \ge 0, t \ge T_0, \omega_0(t) = 0, T_0^- \le t \le T_0$ such that

$$\omega_0(t) \ge \frac{(1+\epsilon)r(t)}{1+\sum_{i=1}^{m} p_i(t)}(\prod_{h(t)\le t_k<t}(1+b_k)^{-1})\exp\left\{\int_{h(t)}^{t}\omega_0(s)ds\right\}. \tag{5.49}$$

Since $\prod_{T_0\le t_k<g_i(t)}(1+b_k)$ is convergent, there exists a positive constant c such that

$$0 < c(\prod_{T_0\le t_k<g_i(t)}(1+b_k)) < \epsilon.$$

Consider the two sequences:

$$\omega_n(t) = r(t)\left(\prod_{h(t)\le t_k<t}(1+b_k)^{-1}\right)\exp\left\{\int_{h(t)}^{t}\omega_{n-1}(s)ds\right\}$$
$$\cdot\frac{1+c(\prod_{T_0\le t_k<g_i(t)}(1+b_k))\exp\left\{-\int_{T_0}^{t}\upsilon_{n-1}(s)ds\right\}}{1+\sum_{i=1}^{m}p_i(t)(1+c(\prod_{T_0\le t_k<g_i(t)}(1+b_k))\exp-\{\int_{T_0}^{g_i(t)}\omega_{n-1}(s)ds\})},$$
$$n=1,2,\ldots,$$

$$\upsilon_n(t) = r(t)\left(\prod_{h(t)\le t_k<t}(1+b_k)^{-1}\right)\exp\left\{\int_{h(t)}^{t}\upsilon_{n-1}(s)ds\right\} \tag{5.50}$$
$$\cdot\frac{1+c(\prod_{T_0\le t_k<g_i(t)}(1+b_k))\exp\left\{-\int_{T_0}^{t}\omega_{n-1}(s)ds\right\}}{1+\sum_{i=1}^{m}p_i(t)(1+c\left(\prod_{T_0\le t_k<g_i(t)}(1+b_k)\right)\exp-\{\int_{T_0}^{g_i(t)}\upsilon_{n-1}(s)ds\})},$$
$$n=1,2,\ldots,$$

where ω_0 is defined above and $\upsilon_0 \equiv 0$. Thus we have

$$\omega_1(t) = \frac{r(t)}{1+\sum_{i=1}^{m}p_i(t)}\left(\prod_{h(t)\le t_k<t}(1+b_k)^{-1}\right)\exp\left\{\int_{h(t)}^{t}\omega_0(s)ds\right\}$$
$$\times\frac{(1+\sum_{i=1}^{m}p_i(t))(1+c(\prod_{T_0\le t_k<g_i(t)}(1+b_k))}{1+\sum_{i=1}^{m}p_i(t)(1+c(\prod_{T_0\le t_k<g_i(t)}(1+b_k))\exp-\{\int_{T_0}^{g_i(t)}\omega_0(s)ds\})}$$

$$\leq \frac{r(t)(\prod_{h(t)\leq t_k<t}(1+b_k)^{-1})}{1+\sum_{i=1}^{m}p_i(t)} e^{\int_{h(t)}^{t}\omega_0(s)ds} \frac{(1+\sum_{i=1}^{m}p_i(t))(1+\epsilon)}{1+\sum_{i=1}^{m}p_i(t)}$$

$$\leq \omega_0(t). \tag{5.51}$$

Clearly $\upsilon_1(t) \geq \upsilon_0(t), \omega_0(t) \geq \upsilon_0(t)$. Hence by induction

$$\begin{cases} 0 \leq \omega_n(t) \leq \omega_{n-1}(t) \leq \ldots \leq \omega_0(t), \\ \upsilon_n(t) \geq \upsilon_{n-1}(t) \geq \ldots \geq \upsilon_0(t) = 0, \quad n = 1, 2, \ldots, \\ \omega_n(t) \geq \upsilon_n(t). \end{cases}$$

There exist pointwise limits of the nonincreasing nonnegative sequence $\omega_n(t)$ and of the nondecreasing sequence $\upsilon_n(t)$. Let $\omega(t) = \lim_{n\to\infty}\omega_n(t)$ and $\upsilon(t) = \lim_{n\to\infty}\upsilon_n(t)$. Then by the Lebesgue Convergence Theorem, we deduce that

$$\omega(t) = r(t)\left(\prod_{h(t)\leq t_k<t}(1+b_k)\right)\exp\left\{\int_{h(t)}^{t}\omega(s)ds\right\}$$
$$\cdot\frac{1+c(\prod_{T_0\leq t_k<g_i(t)}(1+b_k))\exp\left\{-\int_{T_0}^{t}\upsilon(s)ds\right\}}{1+\sum_{i=1}^{m}p_i(t)(1+c(\prod_{T_0\leq t_k<g_i(t)}(1+b_k))\exp-\{\int_{T_0}^{g_i(t)}\omega(s)ds\})},$$

$$\upsilon(t) = r(t)\left(\prod_{h(t)\leq t_k<t}(1+b_k)\right)\exp\left\{\int_{h(t)}^{t}\upsilon(s)ds\right\}$$
$$\times\frac{1+c(\prod_{T_0\leq t_k<g_i(t)}(1+b_k))\exp\left\{-\int_{T_0}^{t}\omega(s)ds\right\}}{1+\sum_{i=1}^{m}p_i(t)(1+c(\prod_{T_0\leq t_k<g_i(t)}(1+b_k))e^{-\int_{T_0}^{g_i(t)}\upsilon(s)ds})}. \tag{5.52}$$

We fix $b \geq T_0$ and define the operator $T : L_\infty[T_0, b] \to L_\infty[T_0, b]$ by the following

$$(Tu)(t) = r(t)\left(\prod_{h(t)\leq t_k<t}(1+b_k)\right)\exp\left\{\int_{h(t)}^{t} u(s)ds\right\}$$
$$\times \frac{1 + c\prod_{T_0\leq t_k<g_i(t)}(1+b_k)\exp\left\{-\int_{T_0}^{t} u(s)ds\right\}}{1+\sum_{i=1}^{m} p_i(t)(1+c\prod_{T_0\leq t_k<g_i(t)}(1+b_k)e^{-\int_{T_0}^{g_i(t)} u(s)ds})}. \quad (5.53)$$

For every function u from the interval $\upsilon \leq u \leq \omega$, we have $\upsilon \leq Tu \leq \omega$. Also T is a completely continuous operator on the space $L_\infty[T_0, b]$, and then by the Schauder Fixed Point Theorem there exists a nonnegative solution of the equation $u = Tu$. Let

$$x(t) = \begin{cases} c\exp\{-\int_{T_0}^{t} u(s)ds\}, & t \geq T_0, \\ c, & T^- \leq t \leq T_0. \end{cases} \quad (5.54)$$

Then $x(t)$ is a nonoscillatory solution of (5.34). Thus by Lemma 5.2.1

$$y(t) = \left(\prod_{h(t)\leq t_k<t}(1+b_k)^{-1}\right)x(t)$$

is a nonoscillatory solution of (5.32) which completes the proof of Theorem 5.2.2. ■

The results in this section apply to (5.31).

5.3 $\frac{3}{2}$-Global Stability

In this section we examine the global attractivity of the “food-limited” population model

$$N'(t) = r(t)N(t)\frac{1 - N(t-\tau)}{1 + c(t)N(t-\tau)},\ t \geq 0, \quad (5.55)$$

where

$$r(t) \in C([0,\infty),(0,\infty)),\ c(t) \in C([0,\infty),(0,\infty)),\ \tau > 0.$$

We consider solutions of (5.55) which correspond to the initial condition

$$\begin{cases} N(t) = \phi(t), \ t \in [\tau, 0], \\ \phi \in C([\tau, 0], [0, \infty)), \ \phi(0) > 0. \end{cases} \tag{5.56}$$

Motivated by (5.55) in this section, we will study the global stability of the general equation

$$x'(t) + [1 + x(t)][1 - cx(t)]F(t, x(g(t))] = 0, \tag{5.57}$$

where $F(t, \varphi)$ is a continuous functional on $[0, \infty) \times C_t$, such that $F(t, 0) = 0$ for $t \geq 0$ and satisfies a York-type condition

$$-\frac{r(t)}{1+c} M_t(-\varphi) \leq F(t, \varphi) \leq \frac{r(t)}{1+c} M_t(-\varphi), \tag{5.58}$$

where $g : [0, \infty) \to (-\infty, \infty)$ is a nondecreasing continuous function with $g(t) < t$ for $t \geq 0$ and $\lim_{t \to \infty} g(t) = \infty$, $M_t(\varphi) = \max\{0, \sup_{s \in [g(t), t]} \varphi(s)\}$, $c \in (0, \infty)$ and $r \in C([0, \infty), (0, \infty))$. The class C_t is the set of all continuous functions $\varphi : [g(t), t] \to [-1, \infty)$ with the sup-norm $\|\varphi\|_t = \sup_{s \in [g(t), t]} |\varphi(s)|$.

Let $\tau = -g(0)$. We consider solutions of (5.57) which correspond to the initial condition

$$\begin{cases} x(t) = \phi(t), \ t \in [-\tau, 0], \\ \phi \in C([-\tau, 0], [-1, \frac{1}{c})), \ \phi(0) > -1. \end{cases} \tag{5.59}$$

In the following, we will establish a $3/2$-global attractivity condition for (5.57), and then apply this condition on equation (5.55) to establish a $3/2$-global attractivity condition. The results in this section are adapted from [73]. To prove the results, we need the following results (whose proofs are standard; for Lemma 5.3.7 see Lemma 5.7.3 with $c = 1$).

Lemma 5.3.1. *Assume that* $c \in (0, 1]$. *Then for any* $v \in [0, 1)$

$$(1 - v) \ln \frac{(1 + c)e^{-cv(1 - cv/2)} - 1}{c} \geq -(1 + c)v \left(1 - \frac{1 + c}{2} v - \frac{1 - c}{6} v^2\right).$$

Lemma 5.3.2. *Assume that* $c \in (0, 1]$. *Then for any* $u \in [0, \infty)$

$$(1 + u) \ln \frac{(1 + c)e^{cu(1 + cu/2)} - 1}{c} \geq (1 + c)u \left(1 + \frac{1 + c}{2} u - \frac{1 - c}{6} u^2\right).$$

Lemma 5.3.3. *Assume that* $c \in (0, 1]$ *and* $v \in (0, 1)$. *Then for any* $x \in [0, \infty)$

$$\ln \frac{1 + [(1 + c)e^{-cv(1 - cv/2)} - 1]e^{-vx}}{1 + ce^{-vx}} \leq -cv(1 - \frac{cv}{2}) + \frac{cv^2}{1 + c} x.$$

Lemma 5.3.4. *Assume that* $c \in (0,1]$. *Then for* $0 < v < \left[1-\frac{c}{2}+\sqrt{\frac{2(1-c)}{3}+\frac{c^2}{4}}\right]^{-1}$

$$-\frac{1}{v}\ln\frac{(1+c)e^{-cv(1-cv/2)}-1}{c} \le \frac{3}{2}(1+c).$$

Lemma 5.3.5. *Assume that* $c \in (0,1]$. *Then for any* $x \in [0,\infty)$

$$\ln\frac{c+e^x}{1+c} \le \frac{x}{1+c}+\frac{cx^2}{2(1+c)^2}-\frac{c(1-c)x^3}{6(1+c)^3}+\frac{c(1-4c+c^2)x^4}{24(1+c)^4} - \frac{c(1-11c+11c^2-c^3)}{120(1+c)^5}x^5+\frac{c(1+14c^2+c^4)}{720(1+c)^6}x^6.$$

Lemma 5.3.6. *Assume that* $c \in (0,1]$ *and*

$$1 \ge v \ge \left[1-\frac{c}{2}+\sqrt{\frac{2(1-c)}{3}+\frac{c^2}{4}}\right]^{-1}.$$

Then

$$\frac{81(1-11c+11c^2-c^3)}{160}v^3 \ge 1-\frac{19(1-c)v}{6}+\frac{27(1-4c+c^2)v^2}{16} + \frac{81(1+14c^2+c^4)}{640}v^4.$$

Lemma 5.3.7. *The system of inequalities*

$$\begin{cases} \ln\frac{1+y}{1-cy} \le (1+c)\left(x-\frac{1-c}{6}x^2\right), \\ -\ln\frac{1-x}{1+cx} \le (1+c)\left(y+\frac{1-c}{6}y^2\right) \end{cases}$$

has only a unique solution $x=y=0$ *in the region* $\{(x,y): 0 \le x \le 1, 0 \le y < 1/c\}$.

Theorem 5.3.1. *Assume that (5.58) holds. Then the solution* $x(t,0,\varphi)$ *of (5.57), (5.59) exists on* $[0,\infty)$ *and satisfies* $-1 < x(t,0,\varphi) < 1/c$.

Theorem 5.3.2. *Assume that (5.58) holds and there exists a function* $r^* \in C([0,\infty),(0,\infty))$ *such that for each* $\varepsilon > 0$ *there is a* $\eta = \eta(\varepsilon) > 0$ *satisfying*

$$\inf_{s\in[g(t),t]}\varphi(s) \ge \varepsilon \Rightarrow F(t,\varphi) \ge \eta r^*(t),\ F(t,-\varphi) \le -\eta r^*(t) \text{ for } t \ge 0 \quad (5.60)$$

and

$$\int_0^\infty r^*(s)ds = \infty. \quad (5.61)$$

Then every nonoscillatory solution of IVP (5.57) and (5.59) tends to zero.

Theorem 5.3.3. *Assume that (5.58), (5.60), and (5.61) hold. If there exists a constant M such that*

$$\int_{g(t)}^{t} r(s)ds \le M, \tag{5.62}$$

then the solutions of (5.57), (5.59) satisfy

$$\frac{-1+\exp\left(\frac{M(1-e^M)}{1+ce^M}\right)}{1+c\exp\left(\frac{M(1-e^M)}{1+ce^M}\right)} \le x(t) \le \frac{e^M-1}{1+ce^M}. \tag{5.63}$$

We now prove our main result in this section.

Theorem 5.3.4. *Assume that (5.58)–(5.61) hold, and*

$$\int_{g(t)}^{t} r(s)ds \le \frac{3}{2}(1+c) \text{ for large } t. \tag{5.64}$$

Then every solution of (5.57), (5.59) tends to zero.

Proof. Let $x(t)$ be a solution of (5.57) and (5.59) (note also Theorem 5.3.1 so $-1 < x(t) << 1/c$, $t \ge 0$). By Theorem 5.3.2, we only consider the case when $x(t)$ is oscillatory. First assume that $0 < c \le 1$. Set

$$u = \limsup_{t\to\infty} x(t) \text{ and } v = \liminf_{t\to\infty} x(t). \tag{5.65}$$

By Theorem 5.3.3, $0 \le u < \infty$ and $0 \le v < 1$. It suffices to prove that $u = v = 0$. For any $0 < \varepsilon < 1 - v$, by (5.64) and (5.65) there exists a $t_0 = t_0(\varepsilon) > g^{-2}(0)$ such that

$$\int_{g(t)}^{t} r(s)ds \le \delta_0 \equiv \frac{3}{2}(1+c), \quad t \ge g(t_0), \tag{5.66}$$

$$-v_1 \equiv -(v+\varepsilon) < x(t) < u+\varepsilon \equiv u_1, \quad t \ge g(t_0). \tag{5.67}$$

From (5.57), (5.58), and (5.67), we have

$$\frac{x'(t)}{(1+x(t))(1-cx(t))} \le \frac{r(t)v_1}{1+c}, \quad t \ge t_0, \tag{5.68}$$

and

$$\frac{x'(t)}{(1+x(t))(1-cx(t))} \ge \frac{-r(t)u_1}{1+c}, \quad t \ge t_0. \tag{5.69}$$

Let $\{l_n\}$ be an increasing infinite sequence of real numbers such that $g(l_n) > t_0$, $x(l_n) > 0, x'(l_n) = 0$, and $\lim_{n\to\infty} x(l_n) = u$. We may assume that l_n is a left local maximum point of $x(t)$. It is easy to show that there exists $\zeta_n \in [g(l_n), l_n)$ such that $x(\zeta_n) = 0$ and $x(t) > 0$ for $t \in (\zeta_n, l_n]$. By (5.68), we have

$$x(t) \geq \frac{-1 + \exp\left(-v_1 \int_t^{\zeta_n} r(s)ds\right)}{1 + c\exp\left(-v_1 \int_t^{\zeta_n} r(s)ds\right)}, \ t_0 \leq t \leq \zeta_n,$$

and [see also (5.57) and (5.58)] for $\zeta_n \leq t \leq l_n$ we have

$$\frac{x'(t)}{(1+x(t))(1-cx(t))} \leq \frac{r(t)}{1+c} \frac{1 - \exp\left(-v_1 \int_{g(t)}^{\zeta_n} r(s)ds\right)}{1 + c\exp\left(-v_1 \int_{g(t)}^{\zeta_n} r(s)ds\right)},$$

which together with (5.68) yields for $\zeta_n \leq t \leq l_n$

$$\frac{x'(t)}{(1+x(t))(1-cx(t))} \leq \min\left\{\frac{r(t)v_1}{1+c}, \frac{r(t)}{1+c} \frac{1 - \exp\left(-v_1 \int_{g(t)}^{\zeta_n} r(s)ds\right)}{1 + c\exp\left(-v_1 \int_{g(t)}^{\zeta_n} r(s)ds\right)}\right\}. \tag{5.70}$$

There are two cases to consider.

Case 1. $\int_{\zeta_n}^{l_n} r(s)ds \leq -\frac{1}{v_1} \ln \frac{(1+c)e^{-cv_1(1-cv_1/2)}-1}{c} \equiv A$

Then by (5.66) and (5.70), we have

$$\ln \frac{1+x(l_n)}{1-cx(l_n)} \leq \int_{\zeta_n}^{l_n} r(s)ds - \frac{1+c}{cv_1} \ln \frac{1 + c\exp\left[-v_1\left(\delta_0 - \int_{\zeta_n}^{l_n} r(s)ds\right)\right]}{1 + ce^{-\delta_0 v_1}}. \tag{5.71}$$

If $\int_{\zeta_n}^{l_n} r(s)ds \leq A \leq \delta_0 = \frac{3}{2}(1+c)$, then by Lemmas 5.3.1 and 5.3.3

$$\ln \frac{1+x(l_n)}{1-cx(l_n)} \leq A - \frac{1+c}{cv_1} \ln \frac{1+ce^{-v_1(\delta_0 - A)}}{1+ce^{-\delta_0 v_1}} \leq (1+c)\left(v_1 - \frac{1-c}{6}v_1^2\right).$$

If $\int_{\zeta_n}^{l_n} r(s)ds \leq \delta_0 = \frac{3}{2}(1+c) < A$, then

$$-\frac{1}{v_1} \ln \frac{(1+c)e^{-cv_1(1-cv_1/2)}}{c} - 1 > \frac{3}{2}(1+c).$$

From Lemma 5.3.4 we have that

$$v_1 > \left[1 - \frac{c}{2} + \sqrt{\frac{2(1-c)}{3} + \frac{c^2}{4}}\right]^{-1}.$$

Hence from (5.71), Lemmas 5.3.5 and 5.3.6, we have

$$\ln\frac{1+x(l_n)}{1-cx(l_n)} \le \delta_0 - \frac{1+c}{cv_1}\ln\frac{1+c}{1+ce^{-\delta_0 v_1}} \le (1+c)\left(v_1 - \frac{1-c}{6}v_1^2\right).$$

Case 2. $A < \int_{\zeta_n}^{l_n} r(s)ds \le \delta_0$

Choose $\eta_n \in (\zeta_n, l_n)$ such that $\int_{\eta_n}^{l_n} r(s)ds = A$. Then by (5.66), (5.70), and Lemma 5.3.1 we have

$$\begin{aligned}
&\ln\frac{1+x(l_n)}{1-cx(l_n)}\\
&\le v_1\int_{\zeta_n}^{\eta_n} r(s)ds + \int_{\eta_n}^{l_n}\frac{r(t)\left[1-\exp\left(-v_1\int_{g(t)}^{\zeta_n} r(s)ds\right)\right]}{1+c\exp\left(-v_1\int_{g(t)}^{\zeta_n} r(s)ds\right)}dt\\
&\le -(1+c)\left(1-\frac{3+c}{2}\right) - \frac{1-v_1}{v_1}\ln\frac{(1+c)\,e^{-cv_1(1-cv_1/2)}-1}{c}\\
&\le (1+c)\left(v_1-\frac{1-c}{6}v_1^2\right).
\end{aligned}$$

Combining the above cases we see that

$$\ln\frac{1+x(l_n)}{1-cx(l_n)} \le (1+c)\left(v_1 - \frac{1-c}{6}v_1^2\right).$$

Letting $n\to\infty$ and $\varepsilon \to 0$, we have

$$\ln\frac{1+u}{1-cu} \le (1+c)\left(v - \frac{1-c}{6}v^2\right). \tag{5.72}$$

Now, we show that

$$-\ln\frac{1-v}{1+cv} \le (1+c)\left(u + \frac{1-c}{6}u^2\right). \tag{5.73}$$

Let $\{s_n\}$ be an increasing infinite sequence of real numbers such that $g(s) > t_0$, $x(s_n) < 0, x'(s_n) = 0$ and $\lim_{n\to\infty} x(s_n) = -v$. We may assume that s_n is a left local minimum point of $x(t)$. It is easy to show that there exists $\eta_n \in [g(s_n), s_n)$ such that $x(\eta_n) = 0$ and $x(t) < 0$ for $t \in (\eta_n, s_n]$. By (5.69), we get

$$x(t) \le \frac{\exp\left(u_1\int_t^{\eta_n} r(s)ds\right)-1}{1+c\exp\left(u_1\int_t^{\eta_n} r(s)ds\right)},\ t_0 \le t \le \eta_n,$$

which together with (5.58) yields

$$\frac{-x'(t)}{(1+x(t))\,(1-cx(t))} \leq \frac{r(t)}{1+c}\,\frac{\exp\left(u_1\int_{g(t)}^{\eta_n} r(s)ds\right)-1}{1+c\exp\left(u_1\int_{g(t)}^{\eta_n} r(s)ds\right)},\ \ \eta_n < t < s_n. \tag{5.74}$$

Note that u_1 is bounded and note

$$\frac{1}{u_1}\ln\frac{(1+c)e^{cu_1(1+cu_1/2)}-1}{c} \leq \frac{3(1+c)}{2}.$$

We consider two cases.

Case I. $\int_{\eta_n}^{s_n} r(s)ds < \frac{3(1+c)}{2} - \frac{1}{u_1}\ln\frac{(1+c)e^{cu_1(1+cu_1/2)}-1}{c} \equiv B.$

From (5.69) and Lemma 5.3.2, we have

$$\begin{aligned}-\ln\frac{1+x(s_n)}{(1-cx(s_n))} &\leq u_1\int_{\eta_n}^{s_n} r(s)ds\\ &\leq u_1\frac{3(1+c)}{2} - \ln\frac{(1+c)e^{cu_1(1+cu_1/2)}-1}{c}\\ &\leq (1+c)\left(u_1+\frac{1-c}{6}u_1^2\right).\end{aligned}$$

Case II. $B < \int_{\eta_n}^{s_n} r(s)ds < \frac{3(1+c)}{2}$

Choose $h_n \in (\eta_n, s_n)$ such that $\int_{\eta_n}^{h_n} r(s)ds = B$. Then by (5.69) and (5.74) we have

$$\begin{aligned}-\ln\frac{1+x(s_n)}{(1-cx(s_n))} &\leq u_1\int_{\eta_n}^{h_n} r(s)ds + \int_{h_n}^{s_n}\frac{r(t)\left[\exp\left(u_1\int_{g(t)}^{\eta_n} r(s)ds\right)-1\right]}{1+c\exp\left(u_1\int_{g(t)}^{\eta_n} r(s)ds\right)}\\ &\leq (1+c) + \frac{(1+c)(3+c)}{2}u_1\\ &\quad -\frac{1+u_1}{u_1}\ln\frac{(1+c)e^{cu_1(1+cu_1/2)}-1}{c}\\ &\leq (1+c)\left(u_1+\frac{1-c}{6}u_1^2\right).\end{aligned}$$

Combining these two cases we have

$$-\ln\frac{1+x(s_n)}{(1-cx(s_n))} \leq (1+c)\left(u_1+\frac{1-c}{6}u_1^2\right).$$

Letting $n \to \infty$ and $\varepsilon \to 0$ we see that (5.73) holds. In view of Lemma 5.3.7, we see from (5.72) to (5.73) that $u = v = 0$.

Next assume that $c > 1$. Set $y(t) = -cx(t)$. Then (5.57) reduces to

$$y'(t) + [1 + y(t)][1 - c^* y(t)]F^*(t, y(g(t))) = 0, \ t \geq 0, \tag{5.75}$$

where $c^* = 1/c \in (0, 1)$ and $F^*(t, \varphi) = -cF(t, -\frac{1}{c}\varphi)$ satisfies the York-type condition

$$-\frac{r^*(t)}{1 + c^*} M_t(-\varphi) \leq F^*(t, \varphi) \leq \frac{r^*(t)}{1 + c^*} M_t(-\varphi). \tag{5.76}$$

Note for large t that

$$\int_{g(t)}^{t} r^*(s)ds \leq \frac{3}{2}(1 + c^*), \tag{5.77}$$

so we have $\lim_{t\to\infty} y(t) = 0$, and this implies that $\lim_{t\to\infty} x(t) = 0$. The proof is complete. ■

Applying Theorem 5.3.4 on (5.55) we have the following result.

Theorem 5.3.5. *Assume that*

$$\int_0^\infty \frac{r(t)}{1 + c(t)} dt = \infty$$

and

$$\int_{t-\tau}^{t} r(s)ds \leq \frac{3}{2}(1 + c_0) \textit{ for large } t, \tag{5.78}$$

where $c_0 = \inf\{c(t) : t \geq 0\}$. *Then every solution of (5.55), (5.56) tends to 1.*

5.4 $\frac{3}{2}$-Uniform Stability

In this section we discuss the uniform stability of the "food-limited" population model

$$N'(t) = r(t)N(t)\frac{k - N^l(t - \tau)}{k + s(t)N^l(t - \tau)}, \quad t \geq 0, \tag{5.79}$$

where $r(t)$ and $s(t)$ are positive functions, $l, \tau > 0$ are positive constants, and $k^{1/l}$ is the unique positive equilibrium point of (5.79). The results in this section are adapted from [67].

Motivated by (5.79) (let $x(t) = (N(t)/k^{1/l}) - 1$) in this section we examine

$$x'(t) = r(t)[1 + x(t)]\frac{1 - (1 + x(t-\tau))^l}{1 + s(t)(1 + x(t-\tau))^l}, \quad t \geq 0. \tag{5.80}$$

We consider solutions of (5.80), which correspond to the initial condition for any $t_0 \geq 0$

$$\begin{cases} x(t) = \varphi(t), \text{ for } t_0 - \tau \leq t \leq t_0, \ \varphi \in C[t_0 - \tau, t_0] \\ 1 + \varphi(t) \geq 0 \text{ for } t_0 - \tau \leq t \leq t_0 \text{ and } 1 + \varphi(t_0) > 0. \end{cases} \tag{5.81}$$

The zero solution of (5.80) is said to be uniformly stable if, for $\varepsilon > 0$, there exists a $\delta(\varepsilon)$ such that $t_0 > 0$ and $\|\phi\| = \sup_{s\in[t_0-\tau,t_0]} |\varphi(s)| < \delta$ imply $|y(t; t_0, \varphi)| < \varepsilon$ for all $t \geq t_0$ where $y(t; t_0, \varphi)$ is a solution of (5.80) with the initial value φ at t_0.

Theorem 5.4.1. *If*

$$l\int_{t-\tau}^{t} \frac{r(u)}{1 + s(u)}du \leq \alpha < \frac{3}{2}, \quad t \geq \tau, \tag{5.82}$$

then the zero solution of (5.80) is uniformly stable.

Proof. Since $\alpha < \frac{3}{2}$, there exist $\alpha_1 > 1$ and $0 < p < 1$, such that

$$\alpha_1 \frac{(1+p)\,\alpha}{(1-p)^l} < \frac{3}{2} \tag{5.83}$$

and

$$\left|(1+x)^l - 1\right| \leq l\alpha_1 |x|, \quad \text{for } |x| \leq p.$$

For $0 < \varepsilon < p$, we choose a $\delta = \delta(\varepsilon) > 0$ sufficiently small so that $\delta < p$,

$$p_1 \equiv (1+\delta)e^{h_1\alpha} - 1 < \varepsilon, \text{ and } p_2 \equiv (1+p_1)e^{h_2\alpha} - 1 < \varepsilon,$$

where

$$h_1 \equiv \alpha_1\delta/(1-\delta)^l > 0, \text{ and } h_2 \equiv \alpha_1 p_1/(1-p_1)^l > 0.$$

Clearly, $\delta < p_1 < p_2 < \varepsilon$. Consider a solution $x(t) = x(t; t_0, \varphi)$ of (5.80) with initial condition φ at t_0, where $t_0 \geq 0$ and $\|\varphi\| = \sup_{s\in[t_0-\tau,t_0]} |\varphi(s)| < \delta$. We need to prove that

$$|x(t)| < \varepsilon, \quad \text{for all } t \geq t_0. \tag{5.84}$$

For $t \in [t_0, t_0 + \tau]$, we have

$$\left|[\ln(1 + x(t))]'\right| \leq h_1 \frac{lr(t)}{1 + s(t)},$$

since

$$\left|1-(1+\varphi)^l\right| \le l\alpha_1\delta$$

and

$$1+s(t)(1+\varphi)^l \ge 1+s(t)(1-\delta)^l \ge (1+s(t))(1-\delta)^l.$$

Hence

$$\left|\ln\frac{1+x(t)}{1+x(t_0)}\right| \le h_1 l\int_{t_0}^{t}\frac{r(u)}{1+s(u)}du \le h_1\alpha,\ \text{for } t\in[t_0,t_0+\tau].$$

It follows that

$$\begin{aligned}1-(1+\delta)e^{h_1\alpha} &< (1-\delta)e^{h_1\alpha}-1\\ &< x(t) < (1+\delta)e^{h_1\alpha}-1,\ \text{for } t\in[t_0,t_0+\tau]\end{aligned}$$

and so

$$|x(t)| < p_1 < \varepsilon, \quad \text{for } t\in[t_0,t_0+\tau].$$

Repeating the previous argument, we have $|x(t)| < p_2 < \varepsilon$ for all $t\in[t_0+\tau,t_0+2\tau]$ and thus

$$|x(t)| < p_2 < \varepsilon, \quad \text{for } t\in[t_0,t_0+2\tau].$$

There are two cases to consider.

Case 1. $x(t)$ has no zeros on $[t_0+\tau,t_0+2\tau]$.

Without loss of generality, we assume that $x(t)>0$ for $t\in[t_0+\tau,t_0+2\tau]$ (the case when $x(t)<0$ is similar). Then by (5.80)

$$x'(t)<0 \quad \text{for } t\in[t_0+2\tau,t_0+3\tau].$$

If $x(t)>0$ for all $t\ge t_0+\tau$, then $x'(t)<0$ for all $t\ge t_0+2\tau$ and

$$0<x(t)\le x(t_0+2\tau)<p_2<\varepsilon, \text{for } \ t\ge t_0+2\tau.$$

Now let t_1 be the smallest zero of $x(t)$ on $(t_0+2\tau,\infty)$. Clearly, $0<x(t)<p_2$ for $t\in[t_0+2\tau,t_1)$ since $x(t)$ is decreasing on $[t_0+2\tau,t_1)$. Thus $|x(t)|<p_2$ for $t\in[t_0,t_1]$. Assume that (5.84) does not hold. Then there must exist $t_2>t_1$ such that $|x(t_2)|=p_2$ and $x(t_2)x'(t_2)\ge 0$ and $|x(t)|<p_2$, for $t_0\le t<t_2$. By (5.80), we have that $x(t)$ has a zero in $[t_2-\tau,t_2]$, which we call ξ. Since

$$\left|x'(t_2)\right| \le (1+p_2)r(t)\frac{l\alpha_1 p_2}{1+s(t)(1-p_2)^l}$$
$$\le \frac{(1+p_2)\alpha_1 l p_2}{(1-p_2)^l}\frac{r(t)}{1+s(t)}, \text{ for } t_0 \le t < t_2,$$

we have for $t \in [\xi, t_2]$ that

$$|-x(t-\tau)| \le \frac{(1+p_2)\alpha_1 l p_2}{(1-p_2)^l}\int_{t-\tau}^{\xi}\frac{r(u)}{1+s(u)}du,$$

and so

$$\left|x'(t)\right| \le (1+p_2)\frac{\alpha_1 l}{(1-p_2)^l}\frac{r(t)}{1+s(t)}|x(t-\tau)|$$
$$\le \left[\frac{\alpha_1 l(1+p_2)}{(1-p_2)^l}\right]^2 p_2\frac{r(t)}{1+s(t)}\int_{t-\tau}^{\xi}\frac{r(u)}{1+s(u)}du.$$

Thus, we get for $t \in [\xi, t_2]$ that

$$\left|x'(t)\right| \le \min\left\{\frac{(1+p_2)\alpha_1 l p_2}{(1-p_2)^l}\frac{r(t)}{1+s(t)}, \mu(t,s)\int_{t-\tau}^{\xi}\frac{r(u)}{1+s(u)}du\right\},$$

and therefore

$$|x(t_2)| \le \int_{\xi}^{t_2}\min\left\{\frac{(1+p_2)\alpha_1 l p_2}{(1-p_2)^l}\frac{r(t)}{1+s(t)}, \mu(t,s)\int_{t-\tau}^{\xi}\frac{r(u)}{1+s(u)}du\right\}dt,$$

where

$$\mu(t,s) := \left[\frac{\alpha_1 l(1+p_2)}{(1-p_2)^l}\right]^2 p_2\frac{r(t)}{1+s(t)}.$$

There are two possibilities.

Case I.

$$\int_{\xi}^{t_2}\frac{r(t)}{1+s(t)}dt\frac{(1+p_2)\alpha_1 l}{(1-p_2)^l} \le 1.$$

Then

$$|x(t_2)| \le \left[\frac{\alpha_1 l(1+p_2)}{(1-p_2)^l}\right]^2$$
$$\times p_2\int_{\xi}^{t_2}\frac{r(t)}{1+s(t)}\int_{t-\tau}^{\xi}\frac{r(u)}{1+s(u)}dudt$$

$$= \left[\frac{\alpha_1 l(1+p_2)}{(1-p_2)^l}\right]^2$$
$$\times p_2 \left[\int_\xi^{t_2} \frac{r(t)}{1+s(t)} \left(\int_{t-\tau}^t \frac{r(u)}{1+s(u)} du - \int_\xi^t \frac{r(u)}{1+s(u)} du\right) dt\right]$$

$$< \left[\frac{\alpha_1 l(1+p_2)}{(1-p_2)^l}\right]^2$$
$$\times p_2 \left[\frac{3}{2}\frac{(1-p_2)^l}{\alpha_1 l(1+p_2)} \int_\xi^{t_2} \frac{r(t)}{1+s(t)} dt - \frac{1}{2}\left(\int_\xi^{t_2} \frac{r(t)}{1+s(t)} dt\right)^2\right],$$

since

$$\int_{t-\tau}^t \frac{r(u)}{1+s(u)} du < \frac{3}{2}\frac{(1-p_2)^l}{\alpha_1 l(1+p_2)}$$

and

$$\int_\xi^{t_2} \frac{r(t)}{1+s(t)} \int_\xi^t \frac{r(u)}{1+s(u)} dudt$$
$$= \int_\xi^{t_2} d\left(\frac{1}{2}\left(\int_\xi^{t_2} \frac{r(u)}{1+s(u)} du\right)^2\right) = \frac{1}{2}\left(\int_\xi^{t_2} \frac{r(t)}{1+s(t)} dt\right)^2.$$

Using the fact that $\frac{3}{2}az - \frac{1}{2}z^2$ (here $a > 0$) is an increasing function for $0 < z < \frac{3}{2}a$, we have

$$|x(t_2)|$$
$$< \left[\frac{\alpha_1 l(1+p_2)}{(1-p_2)^l}\right]^2 p_2 \left[\frac{3}{2}\left(\frac{(1-p_2)^l}{(1+p_2)\alpha_1 l}\right)^2 - \frac{1}{2}\left(\frac{(1-p_2)^l}{(1+p_2)\alpha_1 l}\right)^2\right] = p_2,$$

which is a contradiction.

Case II.

$$\int_\xi^{t_2} \frac{r(t)}{1+s(t)} dt \frac{(1+p_2)\alpha_1 l}{(1-p_2)^l} > 1.$$

Choose $\eta \in (\xi, t_2)$ such that

$$\int_\eta^{t_2} \frac{r(t)}{1+s(t)} dt \frac{(1+p_2)\alpha_1 l}{(1-p_2)^l} = 1.$$

Then

$$
\begin{aligned}
&|x(t_2)| \\
&\le \int_\xi^\eta \frac{(1+p_2)\alpha_1 l p_2}{(1-p_2)^l} \frac{r(t)}{1+s(t)} \\
&\quad + \int_\eta^{t_2} \left[\frac{\alpha_1 l(1+p_2)}{(1-p_2)^l}\right]^2 p_2 \frac{r(t)}{1+s(t)} \int_{t-\tau}^{\xi} \frac{r(u)}{1+s(u)} du dt \\
&= \left[\frac{\alpha_1 l(1+p_2)}{(1-p_2)^l}\right]^2 p_2 \int_\eta^{t_2} \frac{r(t)}{1+s(t)} dt \int_\xi^{\eta} \frac{r(u)}{1+s(u)} du dt \\
&\quad + \left[\frac{\alpha_1 l(1+p_2)}{(1-p_2)^l}\right]^2 p_2 \int_\eta^{t_2} \frac{r(t)}{1+s(t)} \int_{t-\tau}^{\xi} \frac{r(u)}{1+s(u)} du dt
\end{aligned}
$$

$$
\begin{aligned}
&= \left[\frac{(1+p_2)\alpha_1 l}{(1-p_2)^l}\right]^2 p_2 \int_\eta^{t_2} \frac{r(t)}{1+s(t)} dt \int_{t-\tau}^{\eta} \frac{r(u)}{1+s(u)} du dt \\
&< \left[\frac{(1+p_2)\alpha_1 l}{(1-p_2)^l}\right]^2 p_2 \left[\int_\eta^{t_2} \frac{r(t)}{1+s(t)} \left(\frac{3}{2} \frac{(1-p_2)^l}{\alpha_1 l(1+p_2)} - \int_\eta^t \frac{r(u)}{1+s(u)} du\right) dt\right] \\
&= \left[\frac{(1+p_2)\alpha_1 l}{(1-p_2)^l}\right]^2 p_2 \left[\frac{3}{2}\left(\frac{(1-p_2)^l}{\alpha_1 l(1+p_2)}\right)^2 - \frac{1}{2}\left(\frac{(1-p_2)^l}{\alpha_1 l(1+p_2)}\right)^2\right] = p_2,
\end{aligned}
$$

which is a contradiction.

This shows that if $x(t)$ has no zero in $[t_0+\tau, t_0+2\tau]$, then $|x(t)| < p_2 < \varepsilon$ for all $t \ge t_0$.

Case 2. $x(t)$ has a zero $\bar{t} \in [t_0+\tau, t_0+2\tau]$.

We prove that

$$
|x(t)| < p_2, \text{ for all } t \ge \bar{t}. \tag{5.85}
$$

In fact, if (5.85) does not hold, then there must be a point $t^* > \bar{t}$ such that $|x(t^*)| = p_2$, $x(t^*)\,x'(t^*) \ge 0$ and $|x(t)| < p_2$ for $t \in [t_0, t^*)$. Following the reasoning in Case 1 we derive a similar contradiction. The proof of Theorem 5.4.1 is now complete. ■

Theorem 5.4.2. *Assume that*

$$
\int_0^\infty \frac{r(t)}{1+s(t)} dt = \infty. \tag{5.86}
$$

If (5.82) holds, then the zero solution of (5.80) is uniformly and asymptotically stable.

Proof. In view of Theorem 5.4.1, it suffices to prove that there exists a $\delta_0 > 0$ such that the solution of (5.80) with the initial condition $\|\varphi\| = \sup_{t\in[t_0-\tau,t_0]} |\varphi(t)| < \delta_0$ satisfies

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x(t;t_0,\varphi) = 0, \quad t_0 \geq 0.$$

Let $\alpha_1 > 1$ and $0 < p < 1$ be such that

$$\alpha^* \equiv \max\left\{1, \frac{\alpha\alpha_1}{(1-p)^l}\right\} < \frac{3}{2}$$

and

$$\left|(1+x)^l - 1\right| \leq l\alpha_1 |x|, \text{ for } |x| \leq p.$$

Since the zero solution of (5.80) is uniformly stable, it follows that for $0 < \varepsilon < p$, there exists $\delta_0 > 0$ such that

$$|x(t)| = |x(t;t_0,\varphi)| < \frac{\varepsilon}{2}, \text{ for } t \geq t_0$$

provided $\|\varphi\| = \sup_{t\in[t_0-\tau,t_0]} |\varphi(t)| < \delta_0$. Set

$$\Delta := \limsup_{t\to\infty} |x(t)|. \tag{5.87}$$

Clearly $0 \leq \Delta < \varepsilon$. We prove that $\Delta = 0$.

If $x(t)$ is eventually nonnegative, then by (5.80), $x(t)$ is eventually decreasing and hence $\lim_{t\to\infty} x(t) = \Delta_1$ exists. Suppose $\Delta_1 > 0$. Then there exists $t_1 > t_0$ such that

$$\frac{1}{2}\Delta_1 < x(t) < 2\Delta_1, \quad \text{for } t \geq t_1.$$

By (5.80), we have for $t \geq t_1 + \tau$ that

$$\begin{aligned}(\ln[1+x(t)])' &= r(t)\frac{1-(1+x(t-\tau))^l}{1+s(t)(1+x(t-\tau))^l} \\ &\leq \frac{-[(1+\frac{1}{2}\Delta_1)^l - 1]}{(1+2\Delta_1)^l}\frac{r(t)}{1+s(t)}.\end{aligned}$$

Using (5.86), we have

$$\ln[1+x(t)] \to -\infty, \text{ as } t \to \infty,$$

which contradicts $\Delta_1 > 0$. Hence $\lim_{t\to\infty} x(t) = \Delta_1 = 0$. Similarly, one can show that if $x(t)$ is eventually nonpositive then $\lim_{t\to\infty} x(t) = 0$.

Now assume that $x(t)$ is oscillatory. For any $0 < \eta < \varepsilon - \Delta$, by (5.87) there exists $t_2 > t_0$ such that $|x(t)| < \Delta + \eta$ for $t \geq t_2$. Let $\{t_n^*\}$ be an increasing sequence such that $t_n^* \geq t_2 + 2\tau$, $x'(t_n^*) = 0$, $\lim_{n\to\infty} |x(t_n^*)| = \Delta$ and $t_n^* \to \infty$ as $n \to \infty$. By (5.80), $x(t_n^* - \tau) = 0$. Thus, we have

$$\begin{aligned} &\left|(\ln[1 + x(t)])'\right| \\ &\leq \frac{l\alpha_1}{(1 - \Delta - \eta)^l} \frac{r(t)}{1 + s(t)} |x(t - \tau)|, \quad \text{for} \ \ t \geq t_2 + \tau. \end{aligned} \tag{5.88}$$

This yields

$$\begin{aligned} &|-\ln(1 + x(t - \tau))| \\ &\leq \frac{l(\Delta + \eta)\alpha_1}{(1 - \Delta - \eta)^l} \int_{t-\tau}^{t_n^*-\tau} \frac{r(u)}{1 + s(u)} du, \ \text{for} \ \ t \in [t_n^* - \tau, t_n^*]. \end{aligned}$$

Consequently,

$$|x(t - \tau)| \leq \exp\left(\frac{l(\Delta + \eta)\alpha_1}{(1 - \Delta - \eta)^l} \int_{t-\tau}^{t_n^*-\tau} \frac{r(u)}{1 + s(u)} du\right) - 1,$$

since $|\ln(1 + z)| \leq a$ implies $|z| \leq e^a - 1$. Thus for $t \in [t_n^* - \tau, t_n^*]$

$$\begin{aligned} &\left|(\ln[1 + x(t)])'\right| \\ &\leq \frac{l\alpha_1}{(1 - \Delta - \eta)^l} \frac{r(t)}{1 + s(t)} \\ &\quad \times \left[\exp\left(\frac{l(\Delta + \eta)\alpha_1}{(1 - \Delta - \eta)^l} \int_{t-\tau}^{t_n^*-\tau} \frac{r(u)}{1 + s(u)} du\right) - 1\right], \end{aligned}$$

which implies for $t \in [t_n^* - \tau, t_n^*]$ that

$$\left|(\ln[1 + x(t)])'\right| \leq \min\{C_1, C_2\}, \tag{5.89}$$

where

$$\begin{aligned} C_1 &:= \frac{l(\Delta + \eta)\alpha_1}{(1 - \Delta - \eta)^l} \frac{r(t)}{1 + s(t)}, \\ C_2 &:= \frac{l\alpha_1}{(1 - \Delta - \eta)^l} \frac{r(t)}{1 + s(t)} \left[\exp\left(\frac{l(\Delta + \eta)\alpha_1}{(1 - \Delta - \eta)^l} \int_{t-\tau}^{t_n^*-\tau} \frac{r(u)}{1 + s(u)} du\right) - 1\right]. \end{aligned}$$

There are three cases to consider:

Case I.

$$\frac{l\alpha_1}{(1 - \Delta - \eta)^l} \int_{t_n^*-\tau}^{t_n^*} \frac{r(t)}{1 + s(t)} dt \leq 1.$$

Then

$$
\begin{aligned}
&|\ln(1+x(t_n^*))| \\
&\le \frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t_n^*}\frac{r(t)}{1+s(t)} \\
&\quad\times\left[\exp\left(\frac{l(\Delta+\eta)\alpha_1}{(1-\Delta-\eta)^l}\int_{t-\tau}^{t_n^*-\tau}\frac{r(u)}{1+s(u)}du\right)-1\right]dt \\
&\le \frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t_n^*}\frac{r(t)}{1+s(t)} \\
&\quad\times\left[\exp\left((\Delta+\eta)\left(\alpha^*-\frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t}\frac{r(u)}{1+s(u)}du\right)\right)-1\right]dt \\
&= \frac{-1}{\Delta+\eta}\int_{t_n^*-\tau}^{t_n^*} d\left[\exp\left(-\frac{l\alpha_1(\Delta+\eta)}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t}\frac{r(u)}{1+s(u)}du\right)-1\right]e^{(\Delta+\eta)\alpha^*} \\
&\qquad -\frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t_n^*}\frac{r(t)}{1+s(t)}dt \\
&\quad = \frac{1}{\Delta+\eta}e^{(\Delta+\eta)\alpha^*}\left[1-\exp\left(-\frac{l\alpha_1(\Delta+\eta)}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t}\frac{r(u)}{1+s(u)}du\right)\right] \\
&\qquad -\frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t_n^*}\frac{r(t)}{1+s(t)}dt \\
&\quad \le \frac{1}{\Delta+\eta}e^{(\Delta+\eta)\alpha^*}(1-e^{(\Delta+\eta)})-1,
\end{aligned}
$$

since the function

$$z \to \frac{1}{\Delta+\eta}e^{(\Delta+\eta)\alpha^*}[1-e^{(\Delta+\eta)z}]-z$$

is increasing for $0 \le z \le \alpha^*$ and

$$\frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t_n^*}\frac{r(u)}{1+s(u)}du \le 1 \le \alpha^*.$$

Thus,

$$|x(t_n^*)| \le \exp\left(\frac{1}{\Delta+\eta}e^{(\Delta+\eta)\alpha^*}(1-e^{(\Delta+\eta)})-1\right)-1.$$

Case II.

$$1 < \frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t_n^*}\frac{r(t)}{1+s(t)}dt \le \alpha^* - \frac{\ln(1+\Delta+\eta)}{\Delta+\eta}.$$

Then

$$\begin{aligned}|\ln(1+x(t_n^*))| &\le \frac{l(\Delta+\eta)\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t_n^*}\frac{r(t)}{1+s(t)}dt \\ &\le \alpha^*(\Delta+\eta) - \ln(1+\Delta+\eta)\end{aligned}$$

or

$$|x(t_n^*)| \le \frac{1}{1+\Delta+\eta}e^{(\Delta+\eta)\alpha^*} - 1.$$

Case III.

$$\alpha^* - \frac{\ln(1+\Delta+\eta)}{\Delta+\eta} < \frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t_n^*}\frac{r(t)}{1+s(t)}dt \le \alpha^*.$$

Choose $h \in (0,\tau)$ such that

$$\frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t_n^*-h}\frac{r(t)}{1+s(t)}dt = \alpha^* - \frac{\ln(1+\Delta+\eta)}{\Delta+\eta}.$$

Then by (5.89)

$$\begin{aligned}&|\ln(1+x(t_n^*))| \\ &\le \int_{t_n^*-\tau}^{t_n^*-h}\frac{r(t)}{1+s(t)}dt\,\frac{l(\Delta+\eta)\alpha_1}{(1-\Delta-\eta)^l} \\ &\quad + \frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-h}^{t_n^*}\frac{r(t)}{1+s(t)} \\ &\quad \times\left[\exp\left(\frac{l(\Delta+\eta)\alpha_1}{(1-\Delta-\eta)^l}\int_{t-\tau}^{t_n^*-\tau}\frac{r(u)}{1+s(u)}du\right)-1\right]dt \\ &\quad \le (\Delta+\eta)\left(\alpha^* - \frac{\ln(1+\Delta+\eta)}{\Delta+\eta}\right) \\ &\qquad + e^{(\Delta+\eta)\alpha^*}\int_{t_n^*-h}^{t_n^*}\frac{r(t)}{1+s(t)}\end{aligned}$$

$$\times \exp\left(-\frac{l(\Delta+\eta)\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t}\frac{r(u)}{1+s(u)}du\right)dt$$
$$-\frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-h}^{t_n^*}\frac{r(t)}{1+s(t)}dt$$

$$= (\Delta+\eta)\left(\alpha^* - \frac{\ln(1+\Delta+\eta)}{\Delta+\eta}\right)$$
$$+\frac{e^{(\Delta+\eta)\alpha^*}}{(\Delta+\eta)}\exp\left(-\frac{l\alpha_1(\Delta+\eta)}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t_n^*-h}\frac{r(u)}{1+s(u)}du\right)$$
$$-\frac{e^{(\Delta+\eta)\alpha^*}}{(\Delta+\eta)}\exp\left(-\frac{l\alpha_1(\Delta+\eta)}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t_n^*}\frac{r(u)}{1+s(u)}du\right)$$
$$-\frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-h}^{t_n^*}\frac{r(t)}{1+s(t)}dt$$

$$= (\Delta+\eta)\left(\alpha^* - \frac{\ln(1+\Delta+\eta)}{\Delta+\eta}\right)$$
$$+\frac{1}{(\Delta+\eta)}\exp\left((\Delta+\eta)\left(\alpha^*-\frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t_n^*-h}\frac{r(u)}{1+s(u)}du\right)\right)$$
$$-\frac{1}{(\Delta+\eta)}\exp\left((\Delta+\eta)\left(\alpha^*-\frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t_n^*}\frac{r(u)}{1+s(u)}du\right)\right)$$
$$-\frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-h}^{t_n^*}\frac{r(t)}{1+s(t)}dt, \quad \text{since } e^x \geq 1+x \text{ for all } x,$$

$$\leq (\Delta+\eta)\left(\alpha^* - \frac{\ln(1+\Delta+\eta)}{\Delta+\eta}\right) + \frac{1+\Delta+\eta-1}{(\Delta+\eta)}$$
$$-(\Delta+\eta)\left(\alpha^*-\frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t_n^*}\frac{r(u)}{1+s(u)}du\right)$$
$$-\frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-h}^{t_n^*}\frac{r(t)}{1+s(t)}dt$$

$$= (\Delta+\eta)\left(\alpha^* - \frac{\ln(1+\Delta+\eta)}{\Delta+\eta}\right)$$

$$+1-\alpha^* + \frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t_n^*-h}\frac{r(t)}{1+s(t)}dt$$

$$= (\Delta+\eta)\left(\alpha^* - \frac{\ln(1+\Delta+\eta)}{\Delta+\eta}\right)$$

$$+1-\alpha^*+\alpha^*-\frac{\ln(1+\Delta+\eta)}{\Delta+\eta}$$

$$= 1+\alpha^*(\Delta+\eta)-\frac{(1+\Delta+\eta)\ln(1+\Delta+\eta)}{\Delta+\eta}$$

and so

$$|x(t_n^*))| \le \exp\left(1+\alpha^*(\Delta+\eta)-\frac{(1+\Delta+\eta)\ln(1+\Delta+\eta)}{\Delta+\eta}\right)-1.$$

Combining all the three cases, we have

$$|x(t_n^*))| \le \max\{A,\ B, C\}, \tag{5.90}$$

where

$$A = \exp\left(\frac{1}{\Delta+\eta}e^{(\Delta+\eta)\alpha^*}(1-e^{(\Delta+\eta)})-1\right)-1,$$

$$B = \frac{1}{1+\Delta+\eta}e^{(\Delta+\eta)\alpha^*}-1,$$

$$C = \exp\left(1+\alpha^*(\Delta+\eta)-\frac{(1+\Delta+\eta)\ln(1+\Delta+\eta)}{\Delta+\eta}\right)-1.$$

Since

$$\lim_{z\to 0}\frac{1}{z}\left\{\exp\left(\frac{1}{z}e^{\alpha^* z}(1-e^z)-1\right)-1\right\} = \alpha^*-\frac{1}{2}<1,$$

$$\lim_{z\to 0}\frac{1}{z}\left\{\frac{1}{z+1}e^{\alpha^* z}-1\right\} = \alpha^*-1<1,$$

and

$$\lim_{z\to 0}\frac{1}{z}\left\{\exp\left(1+\alpha^* z-\frac{(1+z)\ln(1+z)}{z}\right)-1\right\} = \alpha^*-\frac{1}{2}<1$$

it follows that there exists $\alpha_0 < 1$ such that, for sufficiently small $\varepsilon > 0$, we have

$$\exp\left(\frac{1}{z}e^{\alpha^* z}(1-e^{-z})-1\right)-1 < \alpha_0 z, \ \ \frac{1}{z+1}e^{\alpha^* z}-1 < \alpha_0 z,$$

and

$$\exp\left(1+\alpha^* z-\frac{(1+z)\ln(1+z)}{z}\right)-1 < \alpha_0 z, \ \text{for all } 0 < z < \varepsilon.$$

Thus by (5.90), we get

$$|x(t_n^*))| < \alpha_0(\Delta+\eta).$$

Letting $n \to \infty$ and $\eta \to 0$, we have

$$\Delta \le \alpha_0 \Delta,$$

which, together with $\alpha_0 < 1$, implies $\Delta = 0$. The proof is now complete. ■

5.5 Models with Periodic Coefficients

The variation of the environment plays an important role in many biological and ecological dynamical systems. The assumption of periodicity of the parameters in the system (in a way) incorporates the periodicity of the environment. It is realistic to assume that the parameters in the models are periodic functions of period ω. We consider the nonautonomous "food-limited" population model

$$\frac{dN(t)}{dt} = r(t)N(t)\frac{K(t)-N(t-m\omega)}{K(t)+c(t)r(t)N(t-m\omega)}. \tag{5.91}$$

In this section we discuss (5.91) when K is a periodic function. The results in this section are adapted from [28]. We first consider the nondelay case.

Theorem 5.5.1. *Suppose r, c, and K are continuous and positive periodic function of period ω. Then there exists a unique ω-periodic solution $N^*(t)$ of the periodic differential equation*

$$\frac{dN(t)}{dt} = r(t)N(t)\frac{K(t)-N(t)}{K(t)+c(t)r(t)N(t)}, \tag{5.92}$$

such that all other positive solutions of (5.92) satisfy

$$\lim_{n\to\infty}[N(t)-N^*(t)] = 0. \tag{5.93}$$

Proof. Let $N(t, 0, N_0)$ denote the unique solution of (5.92) through the initial point $(0, N_0)$. Let

$$K_* = \min_{0 \le t \le \omega} K(t) \quad \text{and} \quad K^* = \max_{0 \le t \le \omega} K(t).$$

Then it follows from (5.92) that

$$N_0 \in [K_*, K^*] \Rightarrow N(t, 0, N_0) \in [K_*, K^*], \text{ for } t \ge 0$$

and in particular

$$N_\omega \equiv N(\omega, 0, N_0) \in [K_*, K^*].$$

Define the function

$$f : [K_*, K^*] \to [K_*, K^*]$$

by

$$f(N_0) = N_\omega.$$

As $N(t; 0, N_0)$ depends continuously on N_0, it follows that f is a continuous function mapping $[K_*, K^*]$ into itself. Therefore f has a fixed point N_0^*. In view of the ω-periodic of r, c, and K, it follows that the unique solution $N^*(t) \equiv N(t, 0, N_0^*)$ of (5.92) through the initial point $(0, N_0^*)$ is positive and ω-periodic. This completes the proof of the existence of a positive and ω-periodic solution $N^*(t)$ of (5.92).

Let $N(t)$ be an arbitrary positive solution of (5.92). We let

$$N(t) = N^*(t)e^{x(t)} \tag{5.94}$$

and note

$$\frac{dx(t)}{dt} = F(N^*(t)e^{x(t)}) - F(N^*(t)), \tag{5.95}$$

where

$$F(u) = r(t)\frac{K(t) - u}{K(t) + c(t)r(t)u}.$$

By the mean-value theorem of differential calculus, we can rewrite (5.95) in the form

$$\frac{dx(t)}{dt} = -A(t)[e^{x(t)} - 1], \tag{5.96}$$

where

$$A(t) = \frac{1 + r(t)c(t)}{[K(t) + r(t)c(t)\xi(t)]^2} r(t)N^*(t)K(t), \tag{5.97}$$

and $\xi(t)$ lies between $N^*(t)$ and $N^*(t)e^{x(t)}$. Define a Lyapunov function V for (5.96) in the form

$$V(t) = V(x(t)) = [e^{x(t)} - 1]^2.$$

Calculating the rate of change of V along the solutions of (5.96) we obtain for $x(t) \neq 0$ that

$$\frac{dV(t)}{dt} = -2A(t)[e^{x(t)} - 1]^2 e^{x(t)} < 0. \tag{5.98}$$

One can easily see that every positive solution of this equation is bounded. Therefore $x(t)$ is also bounded. As r, K, and N^* are positive functions and $\xi(t)$ lies between $N^*(t)$ and $N^*(t)e^{x(t)}$, it follows from (5.97) that there exists a positive number μ such that

$$A(t) \geq \mu, \quad \text{for } t \geq 0.$$

Thus from (5.98) we have

$$\frac{dV(t)}{dt} \leq -2\mu e^{x(t)}[e^{x(t)} - 1]^2,$$

so

$$V(t) + 2\mu \int_0^t e^{x(s)}[e^{x(s)} - 1]^2 ds \leq V(0) < \infty.$$

Hence

$$e^{x(t)}[e^{x(t)} - 1]^2 \in L_1(0, \infty).$$

Since $x(t)$ and $\dot{x}(t)$ are bounded in $[0, \infty)$, it follows from Barbalats' Theorem (see Sect. 1.4) that

$$e^{x(t)}[e^{x(t)} - 1]^2 \to 0 \quad as \ t \to \infty.$$

Thus $x(t) \to 0$ as $t \to \infty$ and the result follows from (5.94). This completes the proof. ∎

Now we consider the periodic delay differential equation (5.91), namely

$$N'(t) = r(t)N(t)\frac{K(t) - N(t - m\omega)}{K(t) + c(t)r(t)N(t - m\omega)}, \tag{5.99}$$

together with the initial condition

$$\begin{cases} N(t) = \varphi(t), & \text{for } -m\omega \leq t \leq 0, \\ \varphi \in C[[-m\omega, 0], \mathbf{R}^+], & \text{and } \varphi(0) > 0. \end{cases} \tag{5.100}$$

Note the unique positive periodic solution $N^*(t)$ of (5.92) is also a periodic solution of (5.99).

For convenience, we introduce the notations

$$r^* = \max\{r(t) : t \in [0,\omega]\}, \quad r_* = \min\{r(t) : t \in [0,\omega]\},$$

$$K^* = \max\{K(t) : t \in [0,\omega]\}, \quad K_* = \min\{K(t) : t \in [0,\omega]\},$$

$$N^u = K^* \exp[K^* (\frac{r}{K})_{av} m\omega], \text{ where } (\frac{r}{K})_{av} = \frac{1}{m\omega}\int_0^{m\omega} \frac{r(s)}{K(s)} ds, \tag{5.101}$$

$$N_l = K_* \exp[\frac{K_* - N^u}{K_*} r_{av} m\omega], \text{ where } r_{av} = \frac{1}{m\omega}\int_0^{m\omega} r(s) ds. \tag{5.102}$$

Theorem 5.5.2. *If $N(t)$ is a solution of the initial value problems (5.99) and (5.100) then there exists a number $T = T(\varphi)$ such that*

$$N_l \le N(t) \le N^u, \quad \text{for } t \ge T. \tag{5.103}$$

Proof. We note that any solution of (5.99) satisfies the differential inequality

$$N'(t) \le \frac{r(t)N(t)[K^* - N(t-m\omega)]}{K(t) + c(t)r(t)N(t-m\omega)}. \tag{5.104}$$

Solutions of (5.104) can be either oscillatory or nonoscillatory about K^*.

First, suppose that $N(t)$ is oscillatory about K^*. Then there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ of zeros of $N(t) - K^*$ such that $N(t) - K^*$ takes both positive and negative values on (t_n, t_{n+1}) for $n = 1, 2, \ldots$. Let $N(t_n^*)$ denote a local maximum of $N(t)$ on (t_n, t_{n+1}). Then from (5.104), we obtain

$$0 = N'(t_n^*) \le \frac{r(t_n^*)N(t_n^*)[K^* - N(t_n^* - m\omega)]}{K(t_n^*) + c(t_n^*)r(t_n^*)N(t_n^* - m\omega)},$$

which implies that

$$N(t_n^* - m\omega) \le K^*.$$

This shows that there exists a point $\xi \in [t_n^* - m\omega, t_n^*]$ such that $N(\xi) = K^*$. Integrating (5.104) over $[\xi, t_n^*]$ we obtain

$$\ln \frac{N(t_n^*)}{N(\xi)} \le \int_\xi^{t_n^*} K^* \frac{r(s)}{K(s)} ds \le K^* \int_{t_n^* - m\omega}^{t_n^*} \frac{r(s)}{K(s)} ds$$

and

$$N(t_n^*) \le K^* \exp[K^* (r/K)_{av} m\omega]. \tag{5.105}$$

Since the right side of (5.105) is independent of t_n, we conclude that

$$N(t) \le K^* \exp[K^*(r/K)_{av} m\omega] = N^u, \quad \text{for } t > t_1 + 2m\omega. \tag{5.106}$$

Next assume that $N(t)$ is non oscillatory about K^*. Then it is easily seen that for every $\varepsilon > 0$ there exists a $T_1 = T_1(\varepsilon)$ such that

$$N(t) < K^* + \varepsilon, \quad \text{for } t > T_1.$$

This and (5.106) imply that there exists a $T = T(\varphi)$ such that

$$N(t) \le N^u \quad for \ \ t > T.$$

In a similar way we can derive a lower bound for positive solutions of (5.99). In fact from (5.99) we find

$$N'(t) \ge r(t)N(t)\frac{K_* - N(t - m\omega)}{K(t) + c(t)r(t)N(t - m\omega)}. \tag{5.107}$$

Let $N(t)$ be an oscillatory solution about K_* and let $\{s_n\} \to \infty$ as $n \to \infty$ be such that

$$N(s_n) - K_* = 0, \text{ for } n = 1, 2, \dots,$$

and $N(t) - K_*$ takes both positive and negative values on (t_n, t_{n+1}). Let s_n^* be such that $N(s_n^*)$ is a local minimum of $N(t)$. Then from (5.107), we obtain

$$0 = N'(s_n^*) \ge r(s_n^*)N(s_n^*)\frac{K_* - N(s_n^* - m\omega)}{K(s_n^*) + c(s_n^*)r(s_n^*)N(s_n^* - m\omega)},$$

which implies that

$$N(s_n^* - m\omega) \ge K_*.$$

This show that there exists a point $\eta \in [s_n^* - m\omega, s_n^*]$ such that $N(\eta) = K_*$. Integrating (5.107) over $[\eta, s_n^*]$ we find

$$\ln\frac{N(s_n^*)}{K_*} \ge \int_{\eta}^{s_n^*} \frac{r(s)(K_* - N^u)}{K_*} ds$$
$$= \frac{K_* - N^u}{K_*}\int_{\eta}^{s_n^*} r(s) \ge \frac{K_* - N^u}{K_*}\int_{s_n^* - m\omega}^{s_n^*} r(s)ds$$

and

$$N(s_n^*) \ge K_* \exp\left(\frac{K_* - N^u}{K_*}\int_{s_n^* - m\omega}^{s_n^*} r(s)ds\right) = N_l.$$

Hence

$$N(s) \geq N_l, \quad \text{for} \;\; t \geq t_1 + 2m\omega. \tag{5.108}$$

Next, assume that $N(t)$ is nonoscillatory about K_*. One can easily show in this case that for every positive ε there exists a $T_2 = T_2(\varepsilon)$ such that

$$N(t) > K_* - \varepsilon, \qquad \text{for} \;\; t > T_2.$$

This and (5.108) imply that there exists a $T_2 = T_2(\varphi)$ such that

$$N(t) \geq N_l - \varepsilon, \qquad \text{for} \;\; t \geq T_2.$$

The proof is complete. ■

We will derive sufficient conditions for the global attractivity of $N^*(t)$ with respect to all other positive solutions of (5.99) and (5.100). As before we set

$$N(t) \equiv N^*(t)e^{x(t)}, \tag{5.109}$$

in (5.99) and note that

$$x'(t) = G(x(t - m\omega)) - G(0), \tag{5.110}$$

where

$$G(u) = r(t)\frac{K(t) - N^*(t)e^u}{K(t) + c(t)r(t)N^*(t)e^u}. \tag{5.111}$$

We can rewrite (5.110) in the form

$$x'(t) = -B(t)\, x(t - m\omega), \tag{5.112}$$

where

$$B(t) = \frac{K(t)r(t)[1 + r(t)c(t)]\zeta(t)}{[K(t) + c(t)r(t)\zeta(t)]^2} \tag{5.113}$$

and $\zeta(t)$ lies between $N^*(t)$ and $N(t - m\omega)$. Clearly

$$B_l = \frac{K_* r_* (1 + r_* c_*) N_l}{(K^* + c^* r^* N^u)^2} \leq B(t) \leq \frac{K^* r^* (1 + r^* c^*) N^u}{(K_* + c_* r_* N_l)^2} = B^u. \tag{5.114}$$

Theorem 5.5.3. *Assume that the positive periodic functions $r(t)$, $K(t)$, and $c(t)$ satisfy the condition*

$$\mu \equiv K^* \exp\left[K^* \left(\frac{r}{K}\right)_{av} m\omega\right] \int_0^{m\omega} [1 + r(s)c(s)] \frac{r(s)}{K(s)} ds < 1. \tag{5.115}$$

Then every solution of (5.99) and (5.100) satisfies

$$\lim_{t\to\infty}[N(t) - N^*(t)] = 0. \tag{5.116}$$

Proof. It suffices to prove that every solution x of (5.112) and (5.113) satisfies

$$\lim_{t\to\infty} x(t) = 0. \tag{5.117}$$

Consider $V(t) = V(x(t))$ given by

$$V(t) = \left[x(t) - \int_{t-m\omega}^{t} B(s+m\omega)x(s)ds\right]^2 + \int_{t-m\omega}^{t} B(s+2m\omega)\left(\int_{s}^{t} B(u+m\omega)x^2(t)du\right)ds, \tag{5.118}$$

which in view of (5.112) yields

$$\begin{aligned}\frac{dV(t)}{dt} &= 2\left[x(t) - \int_{t-m\omega}^{t} B(s+m\omega)x(s)\,ds\right][-B(t+m\omega)x(t)]\\ &\quad +B(t+m\omega)x^2(t)\int_{t-m\omega}^{t} B(s+2m\omega)ds\\ &\quad -B(t+m\omega)\int_{t-m\omega}^{t} B(u+m\omega)x^2(u)du.\end{aligned} \tag{5.119}$$

Using the inequality

$$2x(t)x(s) \le x^2(t) + x^2(s),$$

and simplifying (5.119) we obtain

$$\begin{aligned}\frac{dV(t)}{dt} &\le -B(t+m\omega)x^2(t)\\ &\quad\times\left[2 - \int_{t-m\omega}^{t} B(s+m\omega)ds - \int_{t-m\omega}^{t} B(s+m\omega)ds\right]\\ &\le -B(t+m\omega)x^2(t)(1-\mu).\end{aligned} \tag{5.120}$$

It follows from (5.115) that V is eventually nonincreasing say for $t \ge T$. Clearly all solutions of (5.99) are bounded and so by (5.109) and (5.110), x is uniformly continuous on $[0,\infty)$. Integrating (5.120) over $[T,t]$ and taking into account the inequality (5.115), we get

$$V(t) + 2B_l(1-\mu)\int_{T}^{t} x^2(s)ds \le V(T) < \infty.$$

Hence $x^2 \in L_1(T, \infty)$ and by Barbalat's Theorem (see Sect. 1.4)

$$\lim_{t\to\infty} x^2(t) = 0.$$

The proof is complete. ∎

5.6 Global Stability of Models with Impulses

In this section, we are concerned with the global stability of "food-limited" population models with impulsive effects. We consider the model

$$\begin{cases} N'(t) = p(t)N(t)\dfrac{1 - N(t-\tau)}{1 + \lambda N(t-\tau)}, & t \geq 0, \quad t \neq t_k, \\ N(t_k^+) = N(t_k)^{1+b_k}, \quad k \in \mathbf{N}, \end{cases} \tag{5.121}$$

where $p \in C[0, \infty)$ with $p > 0$, $\lambda \in (0, \infty)$, $\tau > 0$, $b_k > -1$ for all $k \in \mathbf{N}$. The aim in this section is to establish some sufficient conditions which ensure that every solution of (5.121) tends to 1 as $t \to \infty$. The results in this section are adapted from [41]. Let the sequence $t_k (k \in \mathbf{N})$ be fixed and satisfy the condition,

$$0 < t_1 < t_2 < \ldots < t_{k+1} \to \infty, \quad \text{as } k \to \infty.$$

We only consider solutions of (5.121) with initial conditions of the form

$$\begin{cases} N(t) = \phi(t), & \text{for} \quad -\tau \leq t \leq 0, \\ \phi \in C([-\tau, 0], [0, \infty)), & \text{and } \phi(0) > 0. \end{cases} \tag{5.122}$$

Lemma 5.6.1. *Suppose that any $\epsilon > 0$ there exists an integer N such that*

$$\prod_{k=n}^{n+m} (1 + b_k) < 1 + \epsilon, \quad \textit{for } n > N \textit{ and } m \geq 0. \tag{5.123}$$

If in addition

$$\int_0^{+\infty} p(s) \prod_{0 \leq t_k < s} (1 + b_k)^{-1} ds = \infty, \tag{5.124}$$

then every non-oscillatory solution of

$$\begin{cases} x'(t) = p(t)\dfrac{1 - e^{x(t-\tau)}}{1 + \lambda e^{x(t-\tau)}}, & t \neq t_k, \\ x(t_k^+) = (1 + b_k)x(t_k), & k \in \mathbf{N} \end{cases} \tag{5.125}$$

tends to zero as t tends to infinity.

Proof. Without loss of generality, suppose that $x(t)$ is an eventually positive solution of (5.125). Then there is a $T_1 \geq 0$ such that $x(t-\tau) > 0$ for $t \geq T_1,\ t \neq t_k$. Thus (5.125) implies that $x(t)$ is decreasing in $(t_k, t_{k+1}]$ with $t_k \geq T_1$. Let

$$\lim_{t\to+\infty}\inf x(t) = \alpha.$$

Then $\alpha \geq 0$. First we prove $\alpha = 0$. Since $x(t_k)$ is a left locally minimum value of $x(t)$, there is a subsequence $\{x(t_{k_j})\}$ such that

$$\lim_{j\to+\infty} x(t_{k_j}) = \alpha.$$

If $\alpha \neq 0$, then $\alpha > 0$. Choose $\epsilon > 0$ such that $\alpha - \epsilon > 0$. Again there is a $T > T_1,\ T \neq t_k$ such that $x(t-\tau) > \alpha - \epsilon$, for $t \geq T$. Hence (5.125) implies

$$x'(t) \leq p(t)\frac{1-e^{\alpha-\epsilon}}{1+\lambda e^{\alpha-\epsilon}},\quad t \geq T,\quad t \neq t_k.$$

Integrating the above inequality from T to t_{k_j}, we get

$$\begin{aligned}&\prod_{T\leq t_k<t_{k_j}}(1+b_k)^{-1}x(t_{k_j}) - x(T)\\ &\leq \frac{1-e^{\alpha-\epsilon}}{1+\lambda e^{\alpha-\epsilon}}\int_T^{t_{k_j}} p(s)\prod_{T\leq t_k<s}(1+b_k)^{-1}ds.\end{aligned}$$

Let either

$$\lim_{j\to+\infty}\sup\prod_{T\leq t_k<t_{k_j}}(1+b_k) = 0 \ \ or\ \ \lim_{j\to+\infty}\sup\prod_{T\leq t_k<t_{k_j}}(1+b_k) \neq 0,$$

and it follows that $\infty \leq -\infty$ or $-x(T) \leq -\infty$, a contradiction. Then $\alpha = 0$.

Now for any $t \geq T$, there is a t_{k_j} such that $t_{k_j} \leq t < t_{k_{j+1}}$. Suppose that $t_{k_j} < t_{k_j+1} < \ldots < t_{k_j+l} \leq t$. Then

$$\begin{aligned}0 < x(t) &< x(t^+_{k_j+l}) = (1+b_{k_j+l})x(t_{k_j+l})\\ &\leq (1+b_{k_j+l})x(t^+_{k_j+l-1})\\ &= (1+b_{k_j+l})(1+b_{k_j+l-1})x(t_{k_j+l-1})\\ &\leq \ldots \leq \prod_{s=0}^{l}(1+b_{k_j+s})x(t_{k_j}).\end{aligned}$$

From (5.123), there is a constant $A > 0$ such that $\prod_{s=0}^{l}(1 + b_{k_j+s}) \leq A$ for any l and any k_j. Thus $0 < x(t) \leq Ax(t_{k_j})$. Then $\lim_{t\to+\infty} x(t) = 0$. The proof is complete. ■

Lemma 5.6.2. *Suppose that (5.123), (5.124) hold and there is a constant $M > 0$ such that*

$$\int_{t-\tau}^{t} p(s) \prod_{s\leq t_k<t} (1 + b_k)ds \leq M, \quad t \geq 0. \tag{5.126}$$

Then every oscillatory solution of (5.125) is bounded.

Proof. Let $x(t)$ be oscillatory solution of (5.125). Equation (5.125) implies

$$x'(t) \leq p(t),\ t \geq 0, \quad t \neq t_k. \tag{5.127}$$

Choose a sequence $\{c_n\}$ such that

$$x(c_n) = 0,\ \text{where } 0 < c_1 < c_2 < \ldots,\ \text{with } \lim_{n\to+\infty} c_n = +\infty,$$

$$x(t) \geq 0,\ \text{for } t \in [c_{2i-1}, c_{2i}],\ \text{and } x(t) \leq 0,\ \text{for } t \in [c_{2i}, c_{2i+1}].$$

Let

$$\hat{x}_i = \sup_{t\in[c_{2i-1},c_{2i}]} x(t) \ \text{and}\ \tilde{x}_i = \inf_{t\in[c_{2i},c_{2i+1}]} x(t).$$

It suffices to prove that $\{\hat{x}_i\}$ and $\{\tilde{x}_i\}$ are bounded. First, we prove that $\{\hat{x}_i\}$ is bounded above. In this step, there are two cases to consider.

Case 1. $\hat{x}_i$ is the maximum value of $x(t)$ in $[c_{2i-1}, c_{2i}]$.

In this case, there is a $c \in (c_{2i-1}, c_{2i})$ such that $\hat{x}_i = x(c) > 0$, $x'(c) \geq 0$. Equation (5.125) implies $x(t - \tau) \leq 0$. Then there is a $\xi \in (c - \tau, c)$ such that $x(\xi) = 0$. Integrating (5.127) from ξ to c, we get

$$\hat{x}_i = x(c) \leq \int_{\xi}^{c} p(t) \prod_{t\leq t_k<c} (1 + b_k)dt \leq M.$$

Case 2. $\hat{x}_i$ is not the maximum value of $x(t)$ in $[c_{2i-1}, c_{2i}]$.

In this case, there is a $t_{k+l} \in (c_{2i-1}, c_{2i})$ such that $\hat{x}_i = x(t_{k+l}^+)$. We suppose that

$$c_{2i-1} < t_{k+1} < \ldots < t_{k+l}.$$

There are two cases to consider.

Subcase 2.1: $x(t_{k+j-1}^{+}) \geq x(t_{k+j}),\ j = 2, \ldots, l$

Then $x(t)$ has maximum $x(c)$ in $[c_{2i-1}, t_{k+1}]$. By Case 1 we have $x(c) \leq M$. Hence

$$\hat{x}_i = x(t_{k+l}^{+}) = (1 + b_{k+l})x(t_{k+l}) \ldots \leq \prod_{s=1}^{l}(1 + b_{k+s})x(t_{k+1})$$

$$\leq M \prod_{s=1}^{l}(1 + b_{k+s}).$$

Subcase 2.2: There is an integer $j^{\star} \in \{2, \ldots, l\}$ with $x(t_{k+j^{\star}-1}^{+}) < x(t_{t+j^{\star}})$ and $x(t_{k+j-1}^{+}) \geq x(t_{k+j}),\ j = j^{\star} + 1, \ldots, l.$

Then $x(t)$ has maximum $x(c)$ in $[t_{k+j^{\star}-1}, t_{k+j^{\star}}]$. By Case 1 we have $x(c) \leq M$. Hence

$$\hat{x}_i = x(t_{k+l}^{+}) = (1 + b_{k+l})x(t_{k+l}) \leq \ldots \leq \prod_{s=j^{\star}}^{l}(1 + b_{k+s})x(t_{k+j^{\star}})$$

$$\leq M \prod_{s=j^{\star}}^{l}(1 + b_{k+s}).$$

From condition (5.123), from Cases 1 and 2, one gets that there is a constant $A > 0$ such that

$$\hat{x}_i = x(t_{k+l}) \leq M \quad \text{or} \quad \hat{x}_i = x(t_{k+l}) \leq AM. \tag{5.128}$$

Next, we prove that $\{\tilde{x}_i\}$ is bounded below. From (5.128), there is a constant $B > 0$ such that $x(t) \leq B$, for all $t \geq 0$. Equation (5.125) implies

$$x'(t) \geq \frac{1 - e^{B}}{1 + \lambda e^{B}} p(t), \quad t \geq 0, \quad t \neq t_k. \tag{5.129}$$

Using a method similar to that in Cases 1 and 2, we get

$$\tilde{x}_i \geq \frac{1 - e^{B}}{1 + \lambda e^{B}} M$$

or

$$\tilde{x}_i \geq \frac{1 - e^{B}}{1 + \lambda e^{B}} AM.$$

This shows that $\{\tilde{x}_i\}$ is bounded below. The proof is complete. ■

The following result is well known.

Lemma 5.6.3. *The system of inequalities*

$$v \le (1+\lambda)\frac{1-e^{u}}{1+\lambda e^{u}} \text{ and } u \ge (1+\lambda)\frac{1-e^{v}}{1+\lambda e^{v}}$$

has only a unique solution $u = v = 0$ in the region $-\infty < u \le 0 \le v < +\infty$.

Lemma 5.6.4. *Suppose that $\lambda \in (0, 1]$ and (5.123), (5.124) hold. If*

$$\limsup_{t\to+\infty} \int_{t-\tau}^{t} p(s) \prod_{s\le t_k<t} (1+b_k)ds \le 1+\lambda, \tag{5.130}$$

then every oscillatory solution of (5.125) tends to zero as t tends to infinity.

Proof. Let $x(t)$ be an oscillatory solution of (5.125). By Lemma 5.6.2, $x(t)$ is bounded. Let

$$\lim \inf_{t\to+\infty} x(t) = u \quad \text{and} \quad \lim \sup_{t\to+\infty} x(t) = v.$$

Then

$$-\infty < u \le 0 \le v < +\infty.$$

For any $\epsilon > 0$, (5.123) implies that there is a $N > 0$ such that

$$\prod_{k=n}^{n+m} (1+b_k) < 1+\epsilon, \ \text{ for } n \ge N \text{ and } m \ge 0.$$

In addition, for this ϵ there is a $T > t_N$ such that

$$\int_{t-\tau}^{t} p(s) \prod_{s\le t_k<t} (1+b_k)ds < (1+\lambda)(1+\epsilon), \ \text{ for all } t \ge T,$$

and

$$u_1 \equiv u - \epsilon < u(t-\tau) < v + \epsilon \equiv v_1.$$

Then (5.125) implies

$$x'(t) \le p(t)\frac{1-e^{u_1}}{1+\lambda e^{u_1}}, \quad t \ge T, \quad t \ne t_k, \tag{5.131}$$

and

$$x'(t) \ge p(t)\frac{1-e^{v_1}}{1+\lambda e^{v_1}}, \quad t \ge T, \quad t \ne t_k.$$

Choose a sequence $\{c_n\}$ such that $x(c_n) = 0$, $T < c_1 < c_2 < \ldots$, $c_n \to +\infty$, $x(t) \geq 0$, for $t \in (c_{2i-1}, c_{2i})$ and $x(t) \leq 0$ for $t \in (c_{2i}, c_{2i+1})$. Let

$$\hat{x}_i = \sup_{t\in(c_{2i-1},c_{2i})} x(t), \quad \tilde{x}_i = \inf_{t\in(c_{2i},c_{2i+1})} x(t).$$

Then

$$\lim_{i\to\infty} \sup \hat{x}_i = v, \quad \lim_{i\to\infty} \inf \tilde{x}_i = u.$$

We divide the proof into two steps.

Case 1. $\hat{x}_i$ is the maximum value of $x(t)$ in (c_{2i-1}, c_{2i}).

In this case, there is a $c \in (c_{2i-1}, c_{2i})$ such that $\hat{x}_i = x(c) > 0$, $x'(c) \geq 0$, and $x(t-\tau) \leq 0$. Then there is a $\xi \in (c-\tau, c)$ such that $x(\xi) = 0$. Integrating (5.131) from ξ to c, we get

$$\begin{aligned}\hat{x}_i &= x(c) \leq \frac{1-e^{u_1}}{1+\lambda e^{u_1}} \int_{\xi}^{c} p(s) \prod_{s\leq t_k<c} (1+b_k)ds \\ &\leq (1+\lambda)(1+\epsilon)\frac{1-e^{u_1}}{1+\lambda e^{u_1}}.\end{aligned}$$

Case 2. $\hat{x}_i$ is not the maximum value of $x(t)$ in (c_{2i-1}, c_{2i}).

In this case, there is a $t_{k+l} \in (c_{2i-1}, c_{2i})$ such that $\hat{x}_i = x(t_{k+l}^+)$. Suppose $c_{2i-1} < t_{k+1} < \ldots < t_{k+l}$. As in Case 2 in Lemma 5.6.2, there is a $c \in (c_{2i-1}, t_{k+l})$ such that $x(c)$ is a left locally maximum value of $x(t)$, and we have that there is a $j \in \{1, 2, \ldots, l\}$ such that

$$\hat{x}_i \leq \prod_{s=j}^{l}(1+b_{k+s})x(c) \leq \prod_{s=j}^{l}(1+b_{k+s})(1+\epsilon)(1+\lambda)\frac{1-e^{u_1}}{1+\lambda e^{u_1}}.$$

Then by (5.123), we get

$$\hat{x}_i \leq (1+\epsilon)^2(1+\lambda)\frac{1-e^{u_1}}{1+\lambda e^{u_1}}.$$

Let $i \to +\infty$, $\epsilon \to 0$, and we get

$$v \leq (1+\lambda)\frac{1-e^{u}}{1+\lambda e^{u}}. \tag{5.132}$$

Similarly, we have

$$u \geq (1+\lambda)\frac{1-e^{v}}{1+\lambda e^{v}}. \tag{5.133}$$

From Lemma 5.6.3, we get from (5.132) and (5.133) that $u = v = 0$. Then $\lim_{t\to+\infty} x(t) = 0$. This completes the proof. ∎

Lemma 5.6.5. *Suppose that* $\lambda > 1$ *and (5.123), (5.124), and (5.130) hold. Then every oscillatory solution of (5.125) tends to zero as* t *tends to infinity.*

Proof. Since $\lambda \in (1, +\infty)$, let $M(t) = \dfrac{1}{N(t)}$, and (5.121) becomes

$$M'(t) = \frac{1}{\lambda} p(t) M(t) \frac{1 - M(t-\tau)}{1 + \frac{1}{\lambda} M(t-\tau)}. \tag{5.134}$$

We note $\dfrac{1}{\lambda} \in (0, 1)$. Then by Lemma 5.6.4, we get Lemma 5.6.5. The proof is complete. ∎

Lemma 5.6.6. *Suppose that* $\lambda \in (0, 1]$, *and (5.123), (5.124) holds. If*

$$\limsup_{t\to+\infty} \int_{t-\tau}^{t} p(s) \prod_{t-\tau \le t_k < t} (1 + b_k)^{-1} ds \le \frac{3}{2}(1 + \lambda), \tag{5.135}$$

then every oscillatory solution of (5.125) tends to zero as $t \to +\infty$.

Proof. Let $x(t)$ be an oscillatory solution of (5.125). By Lemma 5.6.2, $x(t)$ is bounded. Let

$$\limsup_{t\to+\infty} x(t) = v \quad \text{and} \quad \liminf_{t\to+\infty} x(t) = u.$$

Then

$$-\infty < u \le 0 \le v < +\infty.$$

From (5.123), for any $\epsilon > 0$, there is a N such that

$$\prod_{k=n}^{n+m} (1 + b_k) < 1 + \epsilon, \quad n \ge N, \quad m \ge 0.$$

Again for this $\epsilon > 0$, there is a $T \ge t_N$ such that

$$\begin{cases} \displaystyle\int_{t-\tau}^{t} \frac{p(s)}{\prod_{t-\tau \le t_k < s} (1+b_k)} ds \le \frac{3}{2}(1 + \lambda)(1 + \epsilon) := \delta(1 + \epsilon), \ t \ge T, \\ u_1 \equiv u - \epsilon < x(t-\tau) < v + \epsilon \equiv v_1, \qquad t \ge T. \end{cases} \tag{5.136}$$

Then (5.125) implies

$$x'(t) \le \frac{1 - e^{u_1}}{1 + \lambda e^{u_1}} p(t), \quad t \ge T, \quad t \ne t_k. \tag{5.137}$$

Choose a sequence $\{c_n\}$ such that $x(c_n) = 0,\ T < c_1 < c_2 < \dots,\ c_n \to +\infty, n \to +\infty$, $x(t) \geq 0$ for $t \in (c_{2i-1}, c_{2i})$ and $x(t) \leq 0$ for $t \in (c_{2i}, c_{2i+1})$. Let

$$\hat{x}_i = \sup_{t \in (c_{2i-1}, c_{2i})} x(t), \quad \tilde{x}_i = \inf_{t \in (c_{2i}, c_{2i+1})} x(t).$$

Then

$$\lim_{i \to \infty} \sup \hat{x}_i = v, \ \lim_{i \to \infty} \inf \tilde{x}_i = u.$$

We first prove

$$\hat{x}_i \leq (1+\lambda)\left(A - \frac{1-\lambda}{6}A^2\right)(1+\epsilon) \tag{5.138}$$

or

$$\hat{x}_i \leq (1+\lambda)(1+\epsilon)^2\left(A - \frac{1-\lambda}{6}A^2\right), \text{ where } A = \frac{1-e^{u_1}}{1+\lambda e^{u_1}}. \tag{5.139}$$

There are two cases to be considered.

Case 1. $\hat{x}_i$ is the maximum value of $x(t)$ in (c_{2i-1}, c_{2i}).

In this case, there is a $c \in (c_{2i-1}, c_{2i})$ such that $\hat{x}_i = x(c) > 0,\ x'(c) \geq 0$. By (5.125) we have $x(t-\tau) \leq 0$. Then there is a $\xi \in (c-\tau, c)$ such that $x(\xi) = 0$. If $t \in [\xi, c]$, then $t - \tau \leq \xi$. Integrating (5.137) from $t-\tau$ to ξ, one gets

$$-\prod_{t-\tau \leq t_k < \xi}(1+b_k)x(t-\tau) \leq A\int_{t-\tau}^{\xi} p(s)\prod_{s \leq t_k < \xi}(1+b_k)ds. \tag{5.140}$$

Equation (5.125) implies for $t \geq 0$ that

$$x'(t) \leq p(t)\frac{1-\exp(-A\displaystyle\int_{t-\tau}^{\xi} p(s)\prod_{t-\tau \leq t_k < s}(1+b_k)^{-1}ds)}{1+\lambda\exp(-A\displaystyle\int_{t-\tau}^{\xi} p(s)\prod_{t-\tau \leq t_k < s}(1+b_k)^{-1}ds)}. \tag{5.141}$$

Integrating (5.141) from ξ to c and noting that $\dfrac{1-e^x}{1+\lambda e^x}$ is decreasing, we get

$$x(c) \leq \int_{\xi}^{c} p(t) \frac{1 - \exp(-A \int_{t-\tau}^{\xi} p(s) \prod_{t-\tau \leq t_k < s} (1+b_k)^{-1} ds)}{1 + \lambda \exp(-A \int_{t-\tau}^{\xi} p(s) \prod_{t-\tau \leq t_k < s} (1+b_k)^{-1} ds)} \prod_{t \leq t_k < c} (1+b_k) dt$$

$$\leq \int_{\xi}^{c} p(t) \prod_{t \leq t_k < c} (1+b_k)$$

$$\times \frac{1 - e^{-A\delta} \exp(A \int_{\xi}^{t} p(s) \prod_{t-\tau \leq t_k < s} (1+b_k)^{-1} ds)}{1 + \lambda e^{-A\delta} \exp(A \int_{\xi}^{t} p(s) \prod_{t-\tau \leq t_k < s} (1+b_k)^{-1} ds)} dt$$

$$\leq \int_{\xi}^{c} p(t) \frac{1 - e^{-A\delta} \exp(A \int_{\xi}^{t} p(s) \prod_{s \leq t_k < c} (1+b_k) ds \prod_{t-\tau \leq t_k < c} (1+b_k)^{-1})}{1 + \lambda e^{-A\delta} \exp(A \int_{\xi}^{t} p(s) \prod_{s \leq t_k < c} (1+b_k) ds \prod_{t-\tau \leq t_k < c} (1+b_k)^{-1})}$$

$$\times \prod_{t \leq t_k < c} (1+b_k) dt$$

$$\leq \int_{\xi}^{c} p(t) \prod_{t \leq t_k < c} (1+b_k) \frac{1 - e^{-A\delta} \exp(A(1+\epsilon)^{-1} \int_{\xi}^{t} p(s) \prod_{s \leq t_k < c} (1+b_k) ds)}{1 + \lambda e^{-A\delta} \exp(A(1+\epsilon)^{-1} \int_{\xi}^{t} p(s) \prod_{s \leq t_k < c} (1+b_k) ds)} dt$$

$$= \int_{\xi}^{c} p(t) \prod_{t \leq t_k < c} (1+b_k) dt - \frac{1+\lambda}{\lambda A (1+\epsilon)^{-1}}$$

$$\times \ln \frac{1 + \lambda e^{-A\delta} \exp(A(1+\epsilon)^{-1} \int_{\xi}^{c} p(s) \prod_{s \leq t_k < c} (1+b_k) ds)}{1 + \lambda e^{-A\delta}} dt.$$

Subcase 1.1:

$$\int_{\xi}^{c} p(t) \prod_{t \le t_k < c} (1+b_k)dt \le -\frac{1}{A} \ln \frac{(1+\lambda)e^{-\lambda A(1-\frac{\lambda}{A})}}{\lambda}(1+\epsilon)$$
$$\equiv \alpha(1+\epsilon) \le \delta(1+\epsilon).$$

By the monotone property of the function

$$x - \frac{(1+\lambda)}{\lambda A(1+\epsilon)^{-1}} \ln\left(1+\lambda e^{-A\delta + Ax(1+\epsilon)^{-1}}\right),$$

and using $\lambda e^{-A\alpha} = (1+\lambda)e^{-\lambda A(1-\frac{\lambda A}{2})} - 1$, we get that

$$x(c) \le (1+\epsilon)\left(\alpha - \frac{1+\lambda}{\lambda A} \ln \frac{1+\lambda e^{-A\delta + A\alpha}}{1+\lambda e^{-A\delta}}\right)$$
$$= (1+\epsilon)(\alpha + \frac{1+\lambda}{\lambda A} \ln \frac{1+((1+\lambda)e^{-\lambda A(1-\frac{\lambda A}{2})}-1)e^{-A\delta+A\alpha}}{1+\lambda e^{-A\delta+A\alpha}}).$$

Then Lemma 5.3.3 gives us that

$$\hat{x}_i = x(c) \le (1+\epsilon)\left[\alpha + \frac{1+\lambda}{\lambda A}(-\lambda A(1-\frac{\lambda A}{2}) + \frac{\lambda A^2}{1+\lambda}(\delta - \alpha))\right]$$
$$= (1+\epsilon)\left[\alpha - (1+\lambda)(1-\frac{\lambda A}{2}) + A\delta - A\alpha\right]$$
$$= -(1+\epsilon)(1+\lambda)\left[1 - \frac{\lambda A}{2} - \frac{3}{2}A\right]$$
$$-(1+\epsilon)\frac{1-A}{A} \ln \frac{(1+\lambda)e^{-\lambda A(1-\frac{\lambda A}{2})}-1}{\lambda}$$
$$= (1+\epsilon)\left[-(1+\lambda)\left(1-\frac{3+\lambda}{2}A\right) - \frac{1-A}{A} \ln \frac{(1+\lambda)e^{-\lambda A(1-\frac{\lambda A}{2})}-1}{\lambda}\right].$$

Then from Lemma 5.3.1

$$x(c) \le -(1+\lambda)(1+\epsilon)\left(1-\frac{3+\lambda}{2}A\right)$$
$$+(1+\epsilon)\frac{1+\lambda}{A}A\left(1-\frac{1+\lambda}{2}A-\frac{1-\lambda}{6}A^2\right)$$
$$= (1+\epsilon)(1+\lambda)\left(A-\frac{1-\lambda}{6}A^2\right),$$

i.e.,

$$x(c) \leq (1+\epsilon)(1+\lambda)\left(A - \frac{1-\lambda}{6}A^2\right). \tag{5.142}$$

Subcase 1.2:

$$\int_{\xi}^{c} p(t) \prod_{t \leq t_k < c} (1+b_k)dt \leq \delta(1+\epsilon) < \alpha(1+\epsilon).$$

In this case $\alpha > \frac{3}{2}(1+\lambda)$, i.e.,

$$-\frac{1}{A}\ln\frac{(1+\lambda)e^{-\lambda A(1-\frac{\lambda}{A})} - 1}{\lambda} > \frac{3}{2}(1+\lambda).$$

From Lemma 5.3.4 we have that

$$A > \left(1 - \frac{\lambda}{2} + \sqrt{\frac{2(1-\lambda)}{3} + \frac{\lambda^2}{4}}\right)^{-1}.$$

Integrating (5.141) from ξ to c, we get

$$\begin{aligned} \hat{x}_i &= x(c) \leq \delta(1+\epsilon) - \frac{1+\lambda}{\lambda A(1+\epsilon)^{-1}} \\ &\quad \times \ln\frac{1+\lambda e^{-A\delta}\exp(A(1+\epsilon)^{-1}\delta(1+\epsilon))}{1+\lambda e^{-A\delta}} \\ &= (1+\epsilon)\left(\delta - \frac{1+\lambda}{\lambda A}\ln\frac{1+\lambda}{1+\lambda e^{-A\delta}}\right) \\ &= (1+\epsilon)\left(\delta + \frac{1+\lambda}{\lambda A}\left(\ln\frac{\lambda+e^{A\delta}}{1+\lambda} - A\delta\right)\right). \end{aligned}$$

By a method similar to that in Lemmas 5.3.5 and 5.3.6, we get

$$\begin{aligned} \hat{x}_i &= x(c) \leq (1+\epsilon)(1+\lambda) \\ &\quad \times A\,[\,1 - \frac{1-\lambda}{6}A + \frac{1}{8}(1 - \frac{19(1-\lambda)}{6}A + \frac{27(1-4\lambda+\lambda^2)}{16}A^2 \\ &\quad - \frac{81(1-11\lambda+11\lambda^2-\lambda^3)}{160}A^3 + \frac{81(1+14\lambda^2+\lambda^4)}{640}A^4\,)\,], \end{aligned}$$

i.e.,

$$x(c) \leq (1+\epsilon)(1+\lambda)\left(A - \frac{1-\lambda}{6}A^2\right). \tag{5.143}$$

Subcase 1.3:

$$\delta(1+\epsilon) \geq \int_{\xi}^{c} p(t) \prod_{t \leq t_k < c} (1+b_k)dt > \alpha(1+\epsilon).$$

Choose $\eta \in (\xi, c)$ such that

$$\int_{\eta}^{c} p(t) \prod_{t \leq t_k < c} (1+b_k)dt = \alpha(1+\epsilon).$$

Integrating (5.137) from ξ to η, one gets

$$x \leq A \int_{\xi}^{\eta} p(t) \prod_{t \leq t_k < \eta} (1+b_k)dt.$$

Integrating (5.137) from η to c, we get

$$\begin{aligned} &x(c) - x(\eta) \prod_{\eta \leq t_k < c} (1+b_k) \\ &\leq \int_{\eta}^{c} p(t) \prod_{t \leq t_k < c} (1+b_k) \frac{1-\exp\left(-A \int_{t-\tau}^{\xi} p(s) \prod_{t-\tau \leq t_k < s} (1+b_k)^{-1} ds\right)}{1+\lambda \exp\left(-A \int_{t-\tau}^{\xi} p(s) \prod_{t-\tau \leq t_k < s} (1+b_k)^{-1} ds\right)} dt. \end{aligned}$$

By deleting $x(\eta)$ and noting

$$e^{-A\alpha} = \frac{(1+\lambda)e^{-\lambda A(1-\frac{\lambda A}{2})} - 1}{\lambda},$$

we have

$$\begin{aligned} \hat{x}_i &= x(c) \\ &\leq A \int_{\xi}^{\eta} p(t) \prod_{t \leq t_k < \eta} (1+b_k)dt + \int_{\eta}^{c} p(t) \prod_{t \leq t_k < c} (1+b_k) \\ &\quad \times \frac{1+\lambda e^{-A\delta} \exp(A(1+\epsilon)^{-1} \int_{\xi}^{c} p(s) \prod_{s \leq t_k < c} (1+b_k)ds)}{1+\lambda e^{-A\delta} \exp(A(1+\epsilon)^{-1} \int_{\xi}^{\eta} p(s) \prod_{s \leq t_k < c} (1+b_k)ds)} dt \end{aligned}$$

$$= A\int_{\xi}^{\eta} p(t)\prod_{t\le t_k<c}(1+b_k)dt + \int_{\eta}^{c} p(t)\prod_{t\le t_k<c}(1+b_k)dt$$

$$-\frac{1+\lambda}{\lambda A(1+\epsilon)^{-1}}\ln\frac{1+\lambda e^{-A\delta}\exp(A(1+\epsilon)^{-1}\int_{\xi}^{c} p(s)\prod_{s\le t_k<c}(1+b_k)ds)}{1+\lambda e^{-A\delta}\exp(A(1+\epsilon)^{-1}\int_{\xi}^{\eta} p(s)\prod_{s\le t_k<c}(1+b_k)ds)}.$$

Using the monotone property of the function

$$Ax - \frac{(1+\lambda)}{\lambda A(1+\epsilon)^{-1}}\ln\frac{1+\lambda e^{-A\delta+Ax(1+\epsilon)^{-1}}}{1+\lambda e^{-A\delta-A\alpha+Ax(1+\epsilon)^{-1}}},\ \text{on } [0,\delta(1+\epsilon)]$$

and by Lemma 5.3.1, it follows that

$$\begin{aligned}
\hat{x}_i &= x(c)\\
&\le (1+\epsilon)\left(A\delta+(1-A)\alpha-\frac{1+\lambda}{\lambda A}\ln\frac{1+\lambda}{1+\lambda e^{-A\alpha}}\right)\\
&= (1+\epsilon)\left(A\delta+(1-A)\alpha-(1+\lambda)(1-\frac{\lambda A}{2})\right)\\
&= (1+\epsilon)\left(-(1+\lambda)(1-\frac{3+\lambda}{2}A)-\frac{1-A}{A}\varpi\right)\\
&\le (1+\epsilon)(1+\lambda)(A-\frac{1-\lambda}{6}A^2),
\end{aligned}$$

where

$$\varpi = \ln\frac{(1+\lambda)e^{-\lambda A(1-\frac{\lambda A}{2})}-1}{\lambda},$$

i.e.,

$$x(c) \le (1+\epsilon)(1+\lambda)(A-\frac{1-\lambda}{6}A^2). \tag{5.144}$$

Case 2. $\hat{x}_i$ is not the maximum value of $x(t)$ in (c_{2i-1}, c_{2i}).

In this case, there is a $t_{k+l} \in (c_{2i-1}, c_{2i})$ such that $\hat{x}_i = x(t_{k+l}^+)$. Suppose $c_{2i-1} < t_{k+1} < \ldots < t_{k+l}$. As in Case 2 in Lemma 5.6.2, there is a $c \in (c_{2i-1}, t_{k+l})$ such that $x(c)$ is a locally maximum value of $x(t)$, and there is a $j \in \{1, 2, \ldots, l\}$ such that

$$\hat{x}_i \leq \prod_{s=j}^{l}(1+b_{k+s})x(c),$$

where $x(c)$ satisfies (5.138). Then by (5.123), we get

$$\hat{x}_i \leq (1+\epsilon)x(c) \leq (1+\epsilon)^2(1+\lambda)(A - \frac{1-\lambda}{6}A^2).$$

Let $i \to +\infty,\ \epsilon \to 0$ in (5.138) and (5.139) to obtain

$$v \leq (1+\lambda)\left(\frac{1-e^u}{1+\lambda e^u} - \frac{1-\lambda}{6}\left(\frac{1-e^u}{1+\lambda e^u}\right)^2\right). \tag{5.145}$$

Next we prove

$$u \geq (1+\lambda)\left(\frac{1-e^u}{1+\lambda e^u} - \frac{1-\lambda}{6}\left(\frac{1-e^u}{1+\lambda e^u}\right)^2\right). \tag{5.146}$$

Let $B = \dfrac{1-e^v}{1+\lambda e^v}$. Then by (5.125), we have

$$x'(t) \geq Bp(t),\quad t \geq T,\quad t \neq t_k. \tag{5.147}$$

There are two cases to consider.

Case 1. $\tilde{x}_i$ is the minimum value of $x(t)$ in (c_{2i}, c_{2i+1}).

In this case, there is a $c \in (c_{2i}, c_{2i+1})$ such that $x(c) = \tilde{x}_i < 0,\ x'(c) \leq 0$, and then there is a $\xi \in (c-\tau, c)$ such that $x(\xi) = 0$. If $t \in [\xi, c]$, then $t - \tau \leq \xi$. Integrating (5.137) from $t-\tau$ to c, we get

$$-\prod_{t-\tau \leq t_k < \xi}(1+b_k)x(t-\tau) \geq B\int_{t-\tau}^{\xi} p(s)\prod_{s \leq t_k < \xi}(1+b_k)ds.$$

Then, we get for $t \in [\xi, c],\ t \neq t_k$, that

$$x'(t) \geq p(t)\frac{1-\exp(-B\displaystyle\int_{t-\tau}^{\xi} p(s)\prod_{t-\tau \leq t_k < s}(1+b_k)^{-1}ds)}{1+\lambda\exp(-B\displaystyle\int_{t-\tau}^{\xi} p(s)\prod_{t-\tau \leq t_k < s}(1+b_k)^{-1}ds)}. \tag{5.148}$$

We consider two subcases.

Subcase 1.1:

$$\int_{\xi}^{c} p(t) \prod_{t \le t_k < c} (1 + b_k)dt \le (1 + \epsilon) \left(\delta + \frac{1}{B} \ln \frac{(1 + \lambda)e^{-\lambda B(1-\frac{\lambda B}{2})} - 1}{\lambda} \right).$$

In this case, it is easy to see that

$$\begin{aligned} & -\frac{(1 + \lambda)B}{1 - B} \left(1 - \frac{1 + \lambda}{2} B - \frac{1 - \lambda}{6} B^2 \right) \\ & > -\frac{(1 + \lambda)B}{2} \left(1 + \frac{1 - \lambda}{3} B \right). \end{aligned}$$

Then by Lemma 5.3.2, we get

$$\ln \frac{(1 + \lambda)e^{-\lambda B(1-\frac{\lambda B}{2})} - 1}{\lambda} > \frac{1 + \lambda}{2} (B - \frac{1 - \lambda}{3} B^2).$$

Integrating (5.147) from ξ to c, one gets

$$\begin{aligned} \tilde{x}_i &= x(c) \ge B \int_{\xi}^{c} p(t) \prod_{t \le t_k < c} (1 + b_k)dt \\ &\ge \left[\delta B + \ln \frac{(1 + \lambda)e^{-\lambda B(1-\frac{\lambda B}{2})} - 1}{\lambda} \right] (1 + \epsilon) \\ &\ge (1 + \lambda)(1 + \epsilon)(B - \frac{1 - \lambda}{6} B^2). \end{aligned}$$

Then

$$x(c) = \tilde{x}_i \ge (1 + \lambda)(1 + \epsilon)(B - \frac{1 - \lambda}{6} B^2). \tag{5.149}$$

Subcase 1.2:

$$\begin{aligned} \delta(1 + \epsilon) &\ge \int_{\xi}^{c} p(t) \prod_{t \le t_k < c} (1 + b_k)dt \\ &> (\delta + \frac{1}{B} \ln \frac{(1 + \lambda)e^{-\lambda B(1-\frac{\lambda B}{2})} - 1}{\lambda})(1 + \epsilon). \end{aligned}$$

Choose $\eta \in (\xi, c)$ such that

$$\int_{\eta}^{c} p(t) \prod_{t \le t_k < c} (1 + b_k) dt = \left[\delta + \frac{1}{B} \ln \frac{(1+\lambda)e^{-\lambda B(1-\frac{\lambda B}{2})} - 1}{\lambda} \right] (1+\epsilon).$$

Integrating (5.147) from ξ to η, integrating (5.148) from $\eta\ to\ c$, and deleting $x(\eta)$, we get

$$\begin{aligned}
\tilde{x}_i &= x(c) \\
&\ge B \int_{\xi}^{\eta} p(t) \prod_{t \le t_k < \eta} (1 + b_k) dt + \int_{\eta}^{c} p(t) \prod_{t \le t_k < c} (1 + b_k) \\
&\quad \times \frac{1 - \exp(-B \int_{t-\tau}^{\eta} p(s) \prod_{t-\tau \le t_k < s} (1 + b_k)^{-1} ds)}{1 + \lambda \exp(-B \int_{t-\tau}^{\eta} p(s) \prod_{t-\tau \le t_k < s} (1 + b_k)^{-1} ds)} dt \\
&\ge B \int_{\xi}^{\eta} p(t) \prod_{t \le t_k < \eta} (1 + b_k) dt \\
&\quad + \int_{\eta}^{c} p(t) \prod_{t \le t_k < c} (1 + b_k) dt \\
&\quad - \frac{1+\lambda}{\lambda B(1+\epsilon)^{-1}} \ln \frac{1+\lambda e^{-B\delta} \exp\left(B(1+\epsilon)^{-1} \int_{\xi}^{c} p(s) \prod_{s \le t_k < c} (1+b_k) ds\right)}{1+\lambda e^{-B\delta} \exp\left(B(1+\epsilon)^{-1} \int_{\xi}^{\eta} p(s) \prod_{s \le t_k < c} (1+b_k) ds\right)} \\
&= B \int_{\xi}^{\eta} p(t) \prod_{t \le t_k < \eta} (1 + b_k) dt + \int_{\eta}^{c} p(t) \prod_{t \le t_k < c} (1 + b_k) dt \\
&\quad - \frac{1+\lambda}{\lambda B(1+\epsilon)^{-1}} \ln \frac{1+\lambda e^{-B(1+\epsilon)^{-1}\delta} \exp\left(B \int_{\xi}^{c} p(s) \prod_{s \le t_k < c} (1+b_k) ds\right)}{(1+\lambda) e^{-B\lambda(1-\frac{\lambda B}{2})}}
\end{aligned}$$

$$= -(1-B)\int_{\xi}^{\eta} p(t)\prod_{t\le t_k<\eta}(1+b_k)dt$$

$$+\int_{\eta}^{c} p(t)\prod_{t\le t_k<c}(1+b_k)dt-(1+\lambda)(1+\epsilon)(1-\frac{\lambda B}{2})$$

$$-\frac{(1+\lambda)(1+\epsilon)}{\lambda B}$$

$$\times\ln\frac{1+\lambda e^{-B\delta}\exp(B(1+\epsilon)^{-1}\int_{\xi}^{c} p(s)\prod_{s\le t_k<c}(1+b_k)ds)}{1+\lambda}.$$

Using the monotone property of the function

$$x-\frac{(1+\lambda)(1+\epsilon)}{\lambda B}\ln\frac{1+\lambda e^{-B\delta}e^{B(1+\epsilon)^{-1}x}}{1+\lambda},\ x\in[0,\delta(1+\epsilon)],$$

we get

$$x(c)$$

$$\ge -(1-B)\int_{\xi}^{\eta} p(t)\prod_{t\le t_k<c}(1+b_k)dt$$

$$+\delta(1+\epsilon)-(1+\lambda)(1+\epsilon)(1-\frac{\lambda B}{2})$$

$$=(1+\epsilon)\left[-(1+\lambda)+\tfrac{(1+\lambda)(3+\lambda)}{2}B-\frac{1-B}{B}\ln\tfrac{(1+\lambda)e^{-\lambda B(1-\frac{\lambda B}{2})}-1}{\lambda}\right].$$

By Lemma 5.3.2, we get

$$\tilde{x}_i=x(c)\ge(1+\epsilon)(1+\lambda)(B-\frac{1-\lambda}{6}B^2). \tag{5.150}$$

Case 2. $\tilde{x}_i$ is not the minimum value of $x(t)$ in (c_{2i}, c_{2i+1}).

In this case, there is a $t_{k+l}\in(c_{2i-1}, c_{2i})$ such that $\tilde{x}_i = x(t_{k+l}^{+})$. Suppose $c_{2i} < t_{k+1} < \ldots < t_{k+l}$. As in Case 2 in Lemma 5.6.2, there is a $c\in(c_{2i-1}, t_{k+l})$ such that $x(c)$ is a locally minimum value of $x(t)$, and $x(c)$ satisfies (5.149) [(5.150)]. Then there is a $j\in\{1,2,\ldots,l\}$ such that

$$\tilde{x}_i\ge\prod_{s=j}^{l}(1+b_{k+s})(1+\epsilon)x(c).$$

By (5.123), we have

$$\tilde{x}_i \geq (1+\epsilon)x(c) \geq (1+\epsilon)^2(1+\lambda)(B - \frac{1-\lambda}{6}B^2). \tag{5.151}$$

Let $i \to +\infty$, $\epsilon \to 0$ in (5.149) and (5.151) and we get (5.146). Let

$$\frac{1-e^u}{1+\lambda e^u} = x, \quad \frac{1-e^v}{1+\lambda e^v} = -y.$$

Then (5.145) and (5.146) become

$$\begin{cases} \ln \dfrac{1+y}{1-\lambda y} \leq (1+\lambda)(x - \dfrac{1-\lambda}{6}x^2), \\ \ln \dfrac{1-x}{1+\lambda x} \geq (1+\lambda)(-y - \dfrac{1-\lambda}{6}y^2). \end{cases} \tag{5.152}$$

By Lemma 5.3.7, then $x = y = 0$. Thus $u = v = 0$. Then $x(t)$ tends to zero as t tends to infinity. The proof is complete. ■

Lemma 5.6.7. *Suppose that $\lambda \in (1, \infty)$ and (5.123), (5.130) holds. Then every oscillatory solution of (5.125) tends to zero as t tends to infinity.*

Theorem 5.6.1. *Assume $-1 < b_k \leq 0$ for every $k \in \mathbf{N}$ and $\sum_{k=1}^{\infty} b_k = -\infty$. In addition if*

$$\int_{t-\tau}^{t} p(s) \prod_{s \leq t_k < t} (1+b_k) ds$$

is bounded, then every positive solution of (5.121) tends to 1 as t tends to infinity.

Proof. It follows from $-1 < b_k \leq 0$ and $\int_{t-\tau}^{t} p(s) \prod_{s \leq t_k < t} (1+b_k) ds$ is bounded that (5.123) holds. Let

$$y(t) = x(t) \prod_{0 \leq t_k < t} (1+b_k)^{-1}.$$

An argument similar to that in the proof of Lemma 5.6.2 yields that $y(t)$ is bounded. If $-1 < b_k \leq 0$, then $\prod_{k=1}^{\infty}(1+b_k) = 0$, if and only if $\sum_{k=1}^{\infty} b_k = -\infty$. Hence

$$x(t) = y(t) \prod_{0 \leq t_k < t} (1+b_k),$$

and the conditions of this theorem imply that $x(t)$ tends to zero as t tends to infinity. This completes the proof. ■

Theorem 5.6.2. *Suppose (5.123), (5.124), and (5.135) hold. Then every positive solution of (5.121) tends to 1 as t tends to infinity.*

5.7 Global Stability of Generalized Models

In this section we establish some global attractivity conditions of the generalized "food-limited" population model

$$N'(t) = r(t)N(t)\left(\frac{1 - N(t-\tau)}{1 + \lambda(t)N(t-\tau)}\right)^{\alpha}, \quad t \geq 0, \tag{5.153}$$

where

$$r \in C([0,\infty),(0,\infty)),\ \lambda(t) \in C([0,\infty),[0,\infty)),\ \tau > 0,$$

and α is a ratio of two odd positive integers so that $\alpha \geq 1$. The results in this section are adapted from [39]. We consider solutions of (5.153) under the initial condition

$$\begin{cases} N(t) = \phi(t),\ t \in [-\tau, 0], \\ \phi \in C([-\tau,0],[0,\infty)),\ \phi(0) > 0. \end{cases} \tag{5.154}$$

Lemma 5.7.1. *For any $v \in [0,1)$,*

$$\ln(2e^{-v(1-v/2)} - 1) \geq -2v,$$

and for any $u \in [0,\infty)$,

$$\ln(2e^{u(1+u/2)} - 1) \geq 2u.$$

Proof. Let

$$f(v) = 2e^{-v(1-v/2)} - e^{-2v} \text{ and } g(v) = (1-v)e^{v(1+v/2)}.$$

It is easy to see that

$$g(0) = 1, g'(v) = -v^2 e^{v(1+v/2)} \leq 0$$

and

$$f'(v) = 2e^{-2v}[1 - g(v)] = -2e^{-2v} g'(\xi)v \geq 0,\ \text{for some } \xi \in (0,v).$$

It follows that $f(v) \geq f(0) = 1$ for $v \in [0,1)$. The other assertion can be similarly proved. The proof is complete. ■

Lemma 5.7.2. *Assume that* $v \in (0, 1)$. *Then for any* $x \in [0, \infty)$,

$$\ln \frac{1 + [2e^{-v(1-v/2)} - 1]e^{-vx}}{1 + e^{-vx}} \le -v\left(1 - \frac{v}{2}\right) + \frac{v^2}{2}x \tag{5.155}$$

Proof. Set

$$a := 2e^{-v(1-v/2)} - 1$$

and

$$f(x) := \ln((1 + ae^{-vx})/(1 + e^{-vx})).$$

Note

$$f(0) = -v(1 - v/2), \quad f'(0) = \frac{v}{2}[e^{-v(1-v/2)} - 1],$$

and

$$f''(x) = \left[\frac{a}{(a + e^{vx})^2} - \frac{1}{(1 + e^{vx})^2}\right] v^2 e^{vx}.$$

Since $\alpha \le 1$, it follows that $f''(x) \le 0$ for $x \ge 0$. By the mean-value theorem and the fact that

$$e^{x(1-x/2)} \le 1 + x, \quad \text{for } x \ge 0,$$

we have

$$f(x) \le f(0) + f'(0)x = -v(1 - \frac{v}{2}) + \frac{vx}{2}[e^{v(1-v/2)} - 1]$$
$$\le -v(1 - \frac{v}{2}) + \frac{v^2 x}{2}.$$

The proof is complete. ■

The following result follows the usual argument in the literature (for completeness we include it here; see also Lemma 5.3.7).

Lemma 5.7.3. *The system of inequalities*

$$\begin{cases} \ln \frac{1+u}{1-u} \le 2v, \\ -\ln \frac{1-v}{1+v} \le 2u \end{cases}. \tag{5.156}$$

has a unique solution $(u, v) = (0, 0)$ *in the region* $\{(u, v) : -1 < v \le 0 \le u < 1\}$.

Proof. Set

$$g(x) = \exp(2(1-x)/(1+x)),\ \ f(x) = x - g(g(x))$$

and

$$h(x) = (1+x)^2[1+g(x)]^2 - 16g(x)g(g(x)).$$

Observe that $h(1) = 0$,

$$f'(x) = 1 - g'(x)g'(g(x)) = 1 - \frac{16g(x)g(g(x))}{(1+x)^2[1+g(x)]^2},$$

and for $x > 1$

$$\begin{aligned} h'(x) &= 2[1+g(x)][(1+x)(1+g(x)) - 4g(x)] \\ &\quad + \frac{64}{(1+x)^2} g(x)g(g(x)) \frac{[1-g(x)]^2}{[1+g(x)]^2} \\ &> 0. \end{aligned}$$

It follows that $h(x) > h(1) = 0$ for $x > 1$, and so $f'(x) > 0$ for $x > 1$. This shows that $f(x) > f(1) = 0$ for $x > 1$. From (5.156), we have

$$g(\mu) \le \lambda \le 1 \le \mu \le g(\lambda),$$

where

$$\lambda = (1-v)/(1+v) \text{ and } \mu = (1+u)/(1-u).$$

If $u > 0$, then $\mu > 1$, and so

$$\mu \le g(\lambda) \le g(g(\mu)) < \mu.$$

This contradiction implies that $u = v = 0$. The proof is complete. ■

The following result follows the usual argument.

Lemma 5.7.4. *Suppose that*

$$\int_0^{+\infty} \frac{r(t)}{[1+\lambda(t)]^\alpha} dt = \infty. \tag{5.157}$$

Then every solution of (5.153) and (5.154) that does not oscillate about 1 tends to 1 as $t \to \infty$.

Lemma 5.7.5. *Suppose* $0 < \lambda(t) \leq 1$ *for* $t \geq 0$ *and*

$$\limsup_{t\to+\infty} \int_{t-\tau}^{t} \frac{r(s)}{(\lambda(s))^{\alpha}} ds \leq 3. \tag{5.158}$$

Let $N(t) = N(t; 0, \phi)$ *be a solution of (5.153) and (5.154) which is oscillatory about 1. Then* $N(t)$ *is bounded above and is strictly bounded below by 0.*

Proof. Let t_0 be large enough so that

$$\int_{t-\tau}^{t} \frac{r(s)}{(\lambda(t))^{\alpha}} ds \leq 4, \text{ for all } t \geq t_0.$$

Let t^* be a local maximum point of $N(t)$ for $t \geq t_0 + \tau$. Then

$$N'(t^*) = 0 \text{ and } N(t^* - \tau) = 1.$$

Integrating (5.153) from $t^* - \tau$ to t^* yields

$$\begin{aligned} N(t^*) &= \exp\left(\int_{t^*-\tau}^{t^*} r(s)N(s)\left[\frac{1 - N(s-\tau)}{\lambda(s)N(s-\tau)}\right]^{\alpha} ds\right) \\ &\leq \exp\left(\int_{t^*-\tau}^{t^*} r(s)ds\right) \leq e^4. \end{aligned}$$

Consequently,

$$\limsup_{t\to\infty} N(t) \leq e^4.$$

Next, let t_* be a local minimum point of $N(t)$ for $t \geq t_0 + 3\tau$. Then $N'(t_*) = 0$ and $N(t_* - \tau) = 1$. Proceeding as before and using the fact that

$$\frac{1 - N(t-\tau)}{1 + \lambda(t)N(t-\tau)} \geq \frac{1 - e^4}{1 + \lambda(t)e^4} \geq \frac{1 - e^4}{\lambda(t)(1 + e^4)},$$

for $t \geq t_0 + \tau$, we have

$$\begin{aligned} N(t_*) &\geq \exp\left(\int_{t_*-\tau}^{t^*} \frac{r(s)}{\lambda^{\alpha}(s)}\left[\frac{1 - e^4}{\lambda(s)(1 + e^4)}\right]^{\alpha} ds\right) \\ &\geq \exp\left(4\left[\frac{1 - e^4}{1 + e^4}\right]^{\alpha}\right). \end{aligned}$$

Hence

$$\liminf_{t\to\infty} N(t) \geq \exp\left(4\left[\frac{1-e^4}{1+e^4}\right]^{\alpha}\right) > 0.$$

The proof is complete. ■

The proof of next result is similar to the proof of Lemma 5.7.5 and is thus omitted.

Lemma 5.7.6. *Assume that* $\lambda(t) \geq 1$ *for* $t \geq 1$ *and*

$$\limsup_{t\to+\infty} \int_{t-\tau}^{t} r(s)ds \leq 3. \tag{5.159}$$

Let $N(t) = N(t, 0, \phi)$ *be a solution of (5.153) and (5.154) which is oscillatory about 1. Then* $N(t)$ *is bounded above and strictly bounded below by 0.*

Theorem 5.7.1. *Suppose* $0 < \lambda(t) \leq 1$, *for* $t \geq 0$, *and (5.157) holds. If (5.158) holds, then every solution of (5.153) and (5.154) tends to* 1 *as* t *tends to* $+\infty$.

Proof. Let

$$u = \lim\sup_{t\to\infty} N(t) \text{ and } v = \lim\inf_{t\to\infty} N(t).$$

Then by Lemma 5.7.5, $0 < v \leq 1$ and $u \geq 1$. It suffices to show that $u = v = 1$. For any $\varepsilon \in (0, v)$, choose $t_0 = t_0(\varepsilon)$ such that

$$v_1 \equiv v - \varepsilon < N(t-\tau) < u + \varepsilon \equiv u_1,\ t \geq t_0 \tag{5.160}$$

and

$$\int_{t-\tau}^{t} \frac{r(s)}{\lambda^{\alpha}(t)} ds \leq 3 + \varepsilon, \quad t \geq t_0 - \tau. \tag{5.161}$$

Note that

$$\frac{(1-x)}{(1+\lambda x)} \leq \frac{(1-x)}{(\lambda(1+x))} \quad \text{for } x \leq 1$$

and

$$\frac{(1-x)}{(1+\lambda x)} \geq \frac{(1-x)}{\lambda(1+x)} \quad \text{for } x \geq 1.$$

Thus

$$N'(t) \le r(t)N(t)\left(\frac{1-v_1}{1+\lambda(t)v_1}\right)^{\alpha} \le r(t)N(t)\left(\frac{1-v_1}{\lambda(t)(1+v_1)}\right)^{\alpha},\ t \ge t_0, \tag{5.162}$$

and

$$N'(t) \ge r(t)N(t)\left(\frac{1-u_1}{1+\lambda(t)u_1}\right)^{\alpha} \ge r(t)N(t)\left(\frac{1-u_1}{\lambda(t)(1+u_1)}\right)^{\alpha},\ t \ge t_0. \tag{5.163}$$

Consequently,

$$N'(t) \le \frac{r(t)}{\lambda^{\alpha}(t)}N(t)\frac{1-v_1}{1+v_1}, \quad t \ge t_0, \tag{5.164}$$

and

$$N'(t) \ge \frac{r(t)}{\lambda^{\alpha}(t)}N(t)\frac{1-u_1}{1+u_1}, \quad t \ge t_0. \tag{5.165}$$

Let $R(t) = r(t)/\lambda^{\alpha}(t)$. Let $\{p_n\}$ be an increasing sequence such that $p_n \ge t_0 + \tau$

$$\lim_{n\to\infty} p_n = +\infty,\ N'(p_n) = 0 \text{ and } \lim_{n\to\infty} N(p_n) = u.$$

By (5.153), $N(p_n - \tau) = 1$. For $p_n - \tau \le t \le p_n$, by integrating (5.164) from $t-\tau$ to $p_n - \tau$, we get

$$N(t-\tau) \ge \exp\left(-\frac{1-v_1}{1+v_1}\int_{t-\tau}^{p_n-\tau} R(s)ds\right),\ (p_n-\tau) \le t \le p_n.$$

Substituting this into (5.153), if $N(t-\tau) \le 1$, we have

$$N'(t) \le R(t)N(t)\left[\frac{1-N(t-\tau)}{1+N(t-\tau)}\right]^{\alpha} \le R(t)N(t)\frac{1-N(t-\tau)}{1+N(t-\tau)}$$

$$\le R(t)N(t)\frac{1-\exp\left(-\frac{1-v_1}{1+v_1}\displaystyle\int_{t-\tau}^{p_n-\tau} R(s)ds\right)}{1+\exp\left(-\frac{1-v_1}{1+v_1}\displaystyle\int_{t-\tau}^{p_n-\tau} R(s)ds\right)}.$$

If $N(t-\tau) > 1$, by (5.153), $N'(t) < 0$, and thus

$$N'(t) \le R(t)N(t)\frac{1-\exp\left(-\frac{1-v_1}{1+v_1}\int_{t-\tau}^{p_n-\tau} R(s)ds\right)}{1+\exp\left(-\frac{1-v_1}{1+v_1}\int_{t-\tau}^{p_n-\tau} R(s)ds\right)}.$$

If $t \in (p_n - \tau, p_n)$, we have

$$N'(t) \le \min\left\{R(t)N(t)\frac{1-v_1}{1+v_1}, R(t)N(t)A(t)\right\}, \tag{5.166}$$

where

$$A(t) = \frac{1 - \exp\left(-\frac{1-v_1}{1+v_1}\int\limits_{t-\tau}^{p_n-\tau} R(s)ds\right)}{1 + \exp\left(-\frac{1-v_1}{1+v_1}\int\limits_{t-\tau}^{p_n-\tau} R(s)ds\right)}.$$

Since

$$0 < x = (1-v_1)/(1+v_1) < 1,$$

it follows from Lemma 5.7.1 that

$$\ln 2e^{-x(1-x/2)-1} \ge -2x,$$

and so

$$0 < -\frac{1}{x}\ln(2e^{-x(1-x/2)} - 1) \le 2.$$

There are two possibilities.

Case 1.

$$\int\limits_{p_n-\tau}^{p_n} R(s)ds \le -\frac{1}{v_0}\ln(2e^{-v_0(1-v_0/2)} - 1) \equiv A \le 3 + \varepsilon,$$

where $v_0 = (1-v_1)/(1+v_1)$.

Then

$$\ln N(p_n) \le \int\limits_{p_n-\tau}^{p_n} \frac{R(t)\left[1 - \exp\left(-v_0 \int\limits_{t-\tau}^{p_n-\tau} R(s)ds\right)\right]}{1 + \exp\left(-v_0 \int\limits_{t-\tau}^{p_n-\tau} R(s)ds\right)} dt$$

$$= \int\limits_{p_n-\tau}^{p_n} \frac{R(t)\left[1 - \exp\left(-v_0\left(\int\limits_{t-\tau}^{t} r(s)ds - \int\limits_{p_n-\tau}^{t} R(s)ds\right)\right)\right]}{1 + \exp\left(-v_0\left(\int\limits_{t-\tau}^{t} R(s)ds - \int\limits_{p_n-\tau}^{t} R(s)ds\right)\right)} dt$$

$$\leq \int_{p_n-\tau}^{p_n} \frac{R(t)\left[1-\exp\left(-v_0\left(3+\varepsilon-\int_{p_n-\tau}^{t} R(s)ds\right)\right)\right]}{1+\exp\left(-v_0\left(3+\varepsilon-\int_{p_n-\tau}^{t} R(s)ds\right)\right)}dt$$

$$= \int_{p_n-\tau}^{p_n} R(s)ds - \frac{2}{v_0}\ln\frac{1+\exp\left(-v_0\left(3+\varepsilon-\int_{p_n-\tau}^{t} R(s)ds\right)\right)}{1+e^{-(3+\varepsilon)v_0}}.$$

Note that the function

$$f(x) = x - \frac{(2\ln[1+e^{-v_1(3+\varepsilon-x)}])}{v_1}$$

is increasing in $[0, 3+\varepsilon]$ and we have by Lemmas 5.7.1 and 5.7.2, that

$$\begin{aligned}
\ln N(p_n) &\leq A - \frac{2}{v_0}\ln\frac{1+e^{-v_0(3+\varepsilon-A)}}{1+e^{-(3+\varepsilon)v_0}} \\
&= A + \frac{2}{v_0}\ln\frac{1+[2e^{-v_0(1-v_0/2)}-1]e^{-v_0(3+\varepsilon-A)}}{1+e^{-v_0(3+\varepsilon-A)}} \\
&\leq A + \frac{2}{v_0}\left[-v_0\left(1-\frac{v_0}{2}\right)+\frac{v_0^2}{2}(3+\varepsilon-A)\right] \\
&= -2+(4+\varepsilon)v_0 - \frac{1-v_0}{v_0}\ln(2e^{-v_0(1-v_0/2)}-1) \\
&\leq (2+\varepsilon)v_1.
\end{aligned}$$

Case 2.

$$A < \int_{p_n-\tau}^{p_n} R(s)ds \leq 3+\varepsilon.$$

Choose $\xi_n \in (p_n-\tau, p_n)$ such that

$$\int_{\xi_n}^{p_n} R(s)ds \equiv A.$$

Then by (5.166) and Lemma 5.7.1,

$$\ln N(p_n) \le \int_{p_n-\tau}^{\xi_n} R(s)ds + \int_{\xi_n}^{p_n} \frac{R(t)\left[1-\exp\left(-v_0 \int_{t-\tau}^{p_n-\tau} R(s)ds\right)\right]}{1+\exp\left(-v_0 \int_{t-\tau}^{p_n-\tau} R(s)ds\right)} dt$$

$$\le v_0 \int_{p_n-\tau}^{\xi_n} R(s)ds + \int_{\xi_n}^{p_n} \frac{R(t)\left[1-\exp\left(-v_0\left(3+\varepsilon-\int_{p_n-\tau}^{t} R(s)ds\right)\right)\right]}{1+\exp\left(-v_0\left(3+\varepsilon-\int_{p_n-\tau}^{t} R(s)ds\right)\right)} dt$$

$$= v_0 \int_{p_n-\tau}^{\xi_n} R(s)ds + \int_{\xi_n}^{p_n} R(s)ds - \frac{2}{v_0}\ln B_0$$

$$= v_0 \int_{p_n-\tau}^{p_n} R(s)ds + (1-v_0)A - \frac{2}{v_0} B_0$$

$$\le (3+\varepsilon)v_0 + (1-v_0)A - \frac{2}{v_0}\ln\frac{2}{1+e^{-Av_0}}$$

$$= -2 + (4+\varepsilon)v_0 - \frac{1-v_0}{v_0}\ln(2e^{-v_0(1-v_0/2)} - 1)$$

$$\le (2+\varepsilon)v_1,$$

where

$$B_0 = \frac{1 + \exp\left(-v_0\left(3 + \varepsilon - \int\limits_{p_n-\tau}^{p_n} R(s)ds\right)\right)}{1 + \exp\left(-v_0\left(3 + \varepsilon - \int\limits_{p_n-\tau}^{\xi_n} R(s)ds\right)\right)}$$

and we have used the fact that the function

$$g(x) = -\frac{2}{v_1}\ln\frac{1 + \exp[-v_1\,(3 + \varepsilon - x)]}{1 + \exp[-v_1\,(3 + \varepsilon + A - x)]} + v_1 x$$

is increasing on $[0, 3 + \varepsilon]$.

In either cases, we have proved that

$$\ln N(p_n) \le (2 + \varepsilon)v_1 \text{ for } n = 1, 2, \ldots .$$

Letting $n \to \infty$ and $\varepsilon \to 0$, we have

$$\ln u \le 2\frac{1 - v}{1 + v}. \tag{5.167}$$

Next, let $\{q_n\}$ be an increasing sequence such that $q_n \ge t_0 + \tau$, $\lim\limits_{n\to\infty} q_n = +\infty$, $N'(q_n) = 0$, and $\lim_{n\to\infty} N(q_n) = -v$. By (5.153), $N(q_n - \tau) = 1$. For $q_n - \tau \le t \le p_n$, integrating (5.165) from $t - \tau$ to $q_n - \tau$, we have

$$N(t-\tau) \le \exp\left(-\frac{1 - u_1}{1 + u_1}\int\limits_{t-\tau}^{p_n-\tau} R(s)ds\right),\ q_n - \tau \le t \le q_n.$$

Substituting this into (5.153), if $N(t - \tau) \ge 1$, we have

$$\begin{aligned} N'(t) &= r(t)N(t)\left[\frac{1 - N(t-\tau)}{1 + \lambda(t)N(t-\tau)}\right]^{\alpha} \\ &\ge R(t)N(t)\frac{1 - N(t-\tau)}{1 + \lambda(t)N(t-\tau)} \\ &\ge R(t)N(t)\frac{1 - \exp(-u_0 \int\limits_{t-\tau}^{q_n-\tau} R(s)ds)}{1 + \exp\left(-u_0 \int\limits_{t-\tau}^{q_n-\tau} R(s)ds\right)} \end{aligned}$$

for $q_n - \tau \le t \le q_n$. If $N(t - \tau) < 1$, then by (5.153), $N'(t) > 0$, and thus

$$N'(t) \geq R(t)N(t)\frac{1-\exp\left(-u_0\int\limits_{t-\tau}^{q_n-\tau}R(s)ds\right)}{1+\exp\left(-u_0\int\limits_{t-\tau}^{q_n-\tau}R(s)ds\right)},$$

where $u_0 = (1-u_1)/(1+u_1)$. Thus

$$-N'(t) \leq \min\left\{-R(t)N(t)u_0, -R(t)N(t)\frac{1-\exp\left(-u_0\int\limits_{t-\tau}^{q_n-\tau}R(s)ds\right)}{1+\exp\left(-u_0\int\limits_{t-\tau}^{q_n-\tau}R(s)ds\right)}\right\} \tag{5.168}$$

for $q_n - \tau \leq t \leq q_n$. Note that $0 < -u_0 < 1$, and one can easily see that

$$0 < -\frac{1}{u_0}\ln(2e^{-u_0(1-u_0/2)}-1) < 3.$$

There are two cases to consider.

Case 1.

$$\int\limits_{q_n-\tau}^{q_n} R(s)ds \leq (3+\varepsilon) + \frac{1}{u_0}\ln(2e^{-u_0(1-u_0/2)}-1) \equiv B.$$

By (5.168) and Lemma 5.7.1,

$$\begin{aligned}-\ln N(q_n) &\leq -u_0\int\limits_{q_n-\tau}^{q_n}R(s)ds \leq -(3+\varepsilon)u_0 - \ln(2e^{-u_0(1-u_0/2)}-1)\\ &\leq -(1+\varepsilon)u_0.\end{aligned}$$

Case 2.

$$B < \int\limits_{q_n-\tau}^{q_n} R(s)ds \leq 3+\varepsilon.$$

We choose $\eta_n \in (q_n-\tau, q_n)$ such that

$$B = \int\limits_{q_n-\tau}^{\eta_n} R(s)ds.$$

Then by (5.155) and Lemma 5.7.1, we have

$$-\ln N(q_n) \le -u_0 \int_{q_n-\tau}^{\eta_n} R(s)ds + \int_{\eta_n}^{q_n} \frac{R(t)[\exp(-u_0 \int_{t-\tau}^{q_n-\tau} R(s)ds) - 1]}{1 + \exp(-u_0 \int_{t-\tau}^{q_n-\tau} R(s)ds)} dt$$

$$\le -u_0 \int_{q_n-\tau}^{\eta_n} R(s)ds$$

$$+ \int_{\eta_n}^{q_n} \frac{R(t)[\exp -u_0(3+\varepsilon - \int_{q_n-\tau}^{t} R(s)ds)] - 1}{1 + \exp(-u_0(3+\varepsilon - \int_{q_n-\tau}^{t} R(s)ds))} dt$$

$$= -u_0 \int_{q_n-\tau}^{\eta_n} R(s)ds - \int_{\eta_n}^{q_n} R(s)ds$$

$$-\frac{2}{u_0} \ln \frac{1 + \exp(-u_0(3+\varepsilon - \int_{q_n-\tau}^{q_n} R(s)ds))}{1 + \exp(-u_0(3+\varepsilon - \int_{q_n-\tau}^{\eta_n} R(s)ds))} - \ln N(q_n)$$

$$= (1-u_0)B - \int_{\eta_n}^{q_n} R(s)ds + 2\left(1 - \frac{u_0}{2}\right)$$

$$+\frac{2}{u_0} \ln \frac{1 + \exp(-u_0(3+\varepsilon - \int_{q_n-\tau}^{q_n} R(s)ds))}{2}$$

$$\le 2 - (4+\varepsilon)u_0 + \left(\frac{1-u_0}{u_0}\right) \ln\left(2e^{-u_0(1-u_0/2)-1}\right)$$

$$\le (2+\varepsilon)u_0,$$

where we have used the fact that

$$h(x) = -x - \frac{2}{u_0} \ln \frac{1 + \exp\left(-u_0\left(3+\varepsilon - x\right)\right)}{2}$$

is increasing on $[0, 3+\varepsilon]$.

In either cases, we have proved that $-\ln N(p_n) \le -(2+\varepsilon)u_0$ for $n = 1, 2, \ldots$. Letting $n \to \infty$ and $\varepsilon \to 0$, we have

$$-\ln v \le -2\frac{1-u}{1+u}. \tag{5.169}$$

Let

$$y = -(1-u)/(1+u)$$

and

$$x = (1-v)/(1+v),$$

then in view of (5.167), (5.169), and Lemma 5.7.3, we get $x = y = 0$. This shows that $u = v = 1$. The proof is complete. ■

By methods similar to those in the proof of Theorem 5.7.1, and by noting that if $\lambda \ge 1$, then

$$(1-x)/(1+\lambda x) \le (1-x)/(1+x), \text{ for } x \le 1,$$

and

$$(1-x)/(1+\lambda x) \ge (1-x)/(1+x), \text{ for } x \ge 1,$$

one can prove the next result. The details are omitted.

Theorem 5.7.2. *Suppose $\lambda(t) \ge 1$ for $t \ge 0$, (5.157), and (5.159) hold. Then every solution of (5.153) and (5.154) tends to 1 as t tends to $+\infty$.*

5.8 Existence of Periodic Solutions

In this section, we consider the equation

$$\frac{dN(t)}{dt} = N(t)\frac{r(t) - a(t)N(t) - b(t)N(t-\tau(t))}{k(t) + c(t)N(t) + d(t)N(t-\tau(t))} \tag{5.170}$$

and establish some sufficient condition which ensures the existence of periodic solutions. Here a, b, c, d, k, r are continuous ω-periodic functions with $r > 0$, $k > 0$, $a > 0$, $b \ge 0$, $c \ge 0$, and $d \ge 0$. The results in this section are adapted from [22]. Considering the biological significance of system (5.170), we always assume that $N(0) > 0$. The main results will be proved by applying Theorem 1.4.11. To prove the main results we present some useful lemmas.

Let f be a ω-periodic function and define

$$f^l = \min_{t\in[0,\omega]} f(t), \quad f^u = \max_{t\in[0,\omega]} f(t).$$

Lemma 5.8.1. *There exists a unique* $u^* > 0$ *such that*

$$\int_0^\omega \frac{r(t) - [a(t) + b(t)]u^*}{k(t) + [c(t) + d(t)]u^*} dt = 0.$$

Proof. Let

$$f(u) = \int_0^\omega \frac{r(t) - [a(t) + b(t)]u}{k(t) + [c(t) + d(t)]u} dt.$$

It is clear that

$$f(0) = \int_0^\omega \frac{r(t)}{k(t)} dt > 0,$$

$$f\left(\frac{r^u + 1}{a^l + b^l}\right) = \int_0^\omega \frac{r(t) - [a(t) + b(t)]\frac{r^u + 1}{a^l + b^l}}{k(t) + [c(t) + d(t)]\frac{r^u + 1}{a^l + b^l}} dt$$

$$\leq \int_0^\omega \frac{r(t) - (r^u + 1)}{k(t) + [c(t) + d(t)]\frac{r^u + 1}{a^l + b^l}} dt < 0,$$

and then from the zero point theorem, it follows that there exists a $u^* \in \left(0, \frac{r^u + 1}{a^l + b^l}\right)$ such that $f(u^*) = 0$. Moreover,

$$\frac{df}{du} = -\int_0^\omega \frac{k(t)[a(t) + b(t)] + r(t)[c(t) + d(t)]}{\{k(t) + [c(t) + d(t)]u\}^2} dt < 0,$$

that is, $f(u)$ is monotonically decreasing with respect to u, and hence u^* is unique. The proof is complete. ∎

Theorem 5.8.1. *Equation (5.170) has at least one positive periodic solution of period ω*

Proof. Let $N(t) = \exp\{x(t)\}$. Then (5.170) may be reformulated as

$$\frac{dx(t)}{dt} = \frac{r(t) - a(t)\exp\{x(t)\} - b(t)\exp\{x(t-\tau(t))\}}{k(t) + c(t)\exp\{x(t)\} + d(t)\exp\{x(t-\tau(t))\}}. \tag{5.171}$$

In order to apply Theorem 1.4.11 to (5.171), we first let

$$\mathbb{X} = \mathbb{Y} = \{x(t) \in C(\mathbb{R}, \mathbb{R}),\ x(t+\omega) = x(t)\}$$

and

$$\|x\| = \max_{t\in[0,\omega]} |x(t)|, \quad x \in \mathbb{X}\ (or\ \mathbb{Y}).$$

Then $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces with the norm $\|.\|$. Let

$$N\,x = \frac{r(t) - a(t)\exp\{x(t)\} - b(t)\exp\{x(t-\tau(t))\}}{k(t) + c(t)\exp\{x(t)\} + d(t)\exp\{x(t-\tau(t))\}}, \quad x \in \mathbb{X},$$

$$L\,x = x' = \frac{dx(t)}{dt}, \qquad P\,x = \frac{1}{\omega}\int_0^\omega x(t)dt, \quad x \in \mathbb{X},$$

$$Q\,z = \frac{1}{\omega}\int_0^\omega z(t)dt, \quad z \in \mathbb{Y}.$$

Then it follows that

$$Ker\ L = \mathbb{R}, \quad \text{Im}\ L = \left\{ z \in \mathbb{Y} : \int_0^\omega z(t)dt = 0 \right\} \text{ is closed in } \mathbb{Y},$$

$$\dim\ Ker\ L = 1 = co\dim\ \text{Im}\ L,$$

and $P,\ Q$ are continuous projectors such that

$$\text{Im}\,P = Ker\ L, \quad Ker\ Q = \text{Im}\ L = \text{Im}\,(I-Q).$$

Therefore, L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (of L)

$$K_P : \text{Im}\ L \to KerP \cap Dom\ L$$

is

$$K_P(z) = \int_0^t z(s)ds - \frac{1}{\omega}\int_0^\omega\int_0^t z(s)\ ds\ dt.$$

Also

$$QN\,x = \frac{1}{\omega}\int_0^{\omega} \frac{r(s) - a(s)\exp\{x(s)\} - b(s)\exp\{x(s-\tau(s))\}}{k(s) + c(s)\exp\{x(s)\} + d(s)\exp\{x(s-\tau(s))\}}\,ds$$

and

$$K_P(I-Q)N\,x = \int_0^{t} \frac{r(s) - a(s)\exp\{x(s)\} - b(s)\exp\{x(s-\tau(s))\}}{k(s) + c(s)\exp\{x(s)\} + d(s)\exp\{x(s-\tau(s))\}}\,ds$$

$$-\frac{1}{\omega}\int_0^{\omega}\int_0^{t} \frac{r(s) - a(s)\exp\{x(s)\} - b(s)\exp\{x(s-\tau(s))\}}{k(s) + c(s)\exp\{x(s)\} + d(s)\exp\{x(s-\tau(s))\}}\,ds\,dt$$

$$-\left(\frac{t}{\omega} - \frac{1}{2}\right)\int_0^{\omega} \frac{r(s) - a(s)\exp\{x(s)\} - b(s)\exp\{x(s-\tau(s))\}}{k(s) + c(s)\exp\{x(s)\} + d(s)\exp\{x(s-\tau(s))\}}\,ds.$$

By the Arzela–Ascoli Theorem, it is easy to see that $\overline{K_P(I-Q)N(\overline{\Omega})}$ is compact for any open bounded subset Ω of $\mathbb{X}$ and $QN(\overline{\Omega})$ is bounded. Thus, N is L-compact on $\overline{\Omega}$ for any open bounded set $\Omega \in \mathbb{X}$.

Consider the operator equation $L\,x = \lambda N\,x,\ \ \lambda \in (0,1)$, that is,

$$\frac{dx(t)}{dt} = \lambda\frac{r(t) - a(t)\exp\{x(t)\} - b(t)\exp\{x(t-\tau(t))\}}{k(t) + c(t)\exp\{x(t)\} + d(t)\exp\{x(t-\tau(t))\}}. \tag{5.172}$$

Let $x = x(t) \in \mathbb{X}$ be a solution of (5.172) for a certain $\lambda \in (0,1)$. Integrating (5.172) with respect to t over the interval $[0,\omega]$ yields

$$\int_0^{\omega} \frac{r(t) - a(t)\exp\{x(t)\} - b(t)\exp\{x(t-\tau(t))\}}{k(t) + c(t)\exp\{x(t)\} + d(t)\exp\{x(t-\tau(t))\}}\,dt = 0, \tag{5.173}$$

and therefore

$$\int_0^{\omega} \frac{a(t)\exp\{x(t)\} + b(t)\exp\{x(t-\tau(t))\}}{k(t) + c(t)\exp\{x(t)\} + d(t)\exp\{x(t-\tau(t))\}}\,dt$$

$$= \int_0^{\omega} \frac{r(t)}{k(t) + c(t)\exp\{x(t)\} + d(t)\exp\{x(t-\tau(t))\}}\,dt$$

$$\le \int_0^{\omega} \frac{r(t)}{k(t)}\,dt \le \frac{\omega r^u}{k^l}, \tag{5.174}$$

which together with (5.172) implies

$$\int_0^{\omega} |x'(t)|\,dt = \lambda \int_0^{\omega} \left| \frac{r(t) - a(t)\exp\{x(t)\} - b(t)\exp\{x(t-\tau(t))\}}{k(t) + c(t)\exp\{x(t)\} + d(t)\exp\{x(t-\tau(t))\}} \right| dt < \frac{2\omega r^u}{k^l}.$$

From (5.173) and the mean-value theorem for integral, we see that there exists $\xi \in [0,\omega]$ such that

$$\frac{r(\xi) - a(\xi)\exp\{x(\xi)\} - b(\xi)\exp\{x(\xi-\tau(\xi))\}}{k(\xi) + c(\xi)\exp\{x(\xi)\} + d(\xi)\exp\{x(\xi-\tau(\xi))\}}\omega = 0,$$

and therefore

$$r(\xi) = a(\xi)\exp\{x(\xi)\} + b(\xi)\exp\{x(\xi-\tau(\xi))\}. \tag{5.175}$$

Since $x(t) \in \mathbb{X}$, there exist $t_1, t_2 \in [0,\omega]$ such that $x(t_1) = x^l,\ x(t_2) = x^u$, and then from (5.175) it follows that

$$x(t_1) \le \ln\left\{\frac{r(\xi)}{a(\xi) + b(\xi)}\right\} \le \ln\left\{\frac{r^u}{a^l + b^l}\right\},$$

$$x(t_2) \ge \ln\left\{\frac{r(\xi)}{a(\xi) + b(\xi)}\right\} \ge \ln\left\{\frac{r^l}{a^u + b^u}\right\},$$

from which we derive

$$x(t) \le x(t_1) + \int_0^{\omega} |x'(t)|\,dt \le \ln\left\{\frac{r^u}{a^l + b^l}\right\} + \frac{2\omega r^u}{k^l} := M_1,$$

$$x(t) \ge x(t_2) - \int_0^{\omega} |x'(t)|\,dt \ge \ln\left\{\frac{r^l}{a^u + b^u}\right\} - \frac{2\omega r^u}{k^l} := M_2,$$

and hence

$$\|x\| = \max_{t\in[0,\omega]} |x(t)| \le \max\{|M_1|, |M_2|\} := B_1.$$

Clearly, B_1 is independent of the choice of λ. Take $B = B_1 + B_2$, where $B_2 > 0$ is taken sufficiently large such that $|\ln(u^*)| < B_2$ and define

$$\Omega := \{x(t) \in \mathbb{X} :\ \|x\| < B\}.$$

When $x \in \partial\Omega \cap Ker\ L = \partial\Omega \cap \mathbb{R},\ x = B$ or $x = -B$, and then

$$QN\,x = \frac{1}{\omega}\int_0^{\omega} \frac{r(t) - a(t)\exp\{x(t)\} - b(t)\exp\{x\}}{k(t) + c(t)\exp\{x(t)\} + d(t)\exp\{x\}}dt \neq 0.$$

Furthermore, a direct calculation reveals that

$$\deg\{JQN, \Omega \cap Ker\ L, 0\}$$
$$= sign\left\{-\frac{1}{\omega}\int_0^{\omega} \frac{k(t)[a(t) + b(t)] + r(t)[c(t) + d(t)]}{\{k(t) + [c(t) + d(t)]u^*\}^2}dt\right\} \neq 0;$$

here J is the identity mapping since $\Im P = KerL$. Thus all the requirements in Theorem 1.4.11 are satisfied. Hence (5.171) has at least one solution $x^*(t) \in Dom\ L \cap \overline{\Omega}$. Set $N^*(t) = \exp\{x^*(t)\}$. Then $N^*(t)$ is a positive ω-periodic solution of (5.170). The proof is complete. ■