A Remark on Some Simultaneous Functional Inequalities in Riesz Spaces

Bogdan Batko and Janusz Brzdęk

Abstract We study continuous at a point functions that take values in a Riesz space and satisfy some systems of two simultaneous functional inequalities. In this way we obtain in particular generalizations and extensions of some earlier results of Krassowska, Matkowski, Montel, and Popoviciu.

Keywords Functional inequality • Riesz space • σ-Ideal

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1 Introduction

In what follows $\mathbb{N}_0$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{R}_+$ denote, as usual the sets of nonnegative integers, positive integers, integers, rationals, reals and nonnegative reals, respectively. Moreover, let $a, b \in \mathbb{R} \setminus \{0\}$ with $ab^{-1} \notin \mathbb{Q}$ and $ab < 0$ be fixed. Montel [13] (see also [14] and [11, p. 228]) proved that a function $f : \mathbb{R} \to \mathbb{R}$, that is continuous at a point and satisfies the system of functional inequalities

$$f(x + a) \leq f(x), \qquad f(x + b) \leq f(x), \qquad x \in \mathbb{R}, \tag{1}$$

has to be constant. A similar (but more abstract) result for measurable functions has been proved in [2].

In [7–9] (see also [10]) the result of Montel has been generalized and extended in several ways. In particular, motivated by some problem arising in a characterization of L^p norm, Krassowska and Matkowski [8] (cf. also [7]) have proved that if $\alpha, \beta \in$

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$\mathbb{R}$ and $\alpha b \leq \beta a$, then a continuous at a point function $f : \mathbb{R} \to \mathbb{R}$ satisfies the following two functional inequalities

$$f(x+a) \leq f(x) + \alpha, \qquad f(x+b) \leq f(x) + \beta, \qquad x \in \mathbb{R}, \tag{2}$$

if and only if $\alpha b = \beta a$; moreover f has to be of the form $f(x) = cx + d$ for $x \in \mathbb{R}$, with some $c, d \in \mathbb{R}$.

In this paper we investigate the possibility to obtain results analogous to those in [2, 8, 13] for functions taking values in Riesz spaces. Moreover, we consider the system (2) in a conditional form and almost everywhere. We obtain outcomes that correspond somewhat to the results in [3, 4] and to the problem of stability of functional equations and inequalities (for some further information concerning that problem we refer to, e.g., [1, 5, 6]).

2 Preliminaries

For the readers convenience we present the definition and some basic properties of Riesz spaces (see [12]).

Definition 1 (cf. [12, Definitions 11.1 and 22.1]). We say that a real linear space L, endowed with a partial order $\leq \subset L^2$, is a *Riesz space* if $\sup\{x, y\}$ exists for all $x, y \in L$ and

$$ax + y \leq az + y, \qquad x, y, z \in X, x \leq z, a \in \mathbb{R}_+;$$

we define the absolute value of $x \in L$ by the formula $|x| := \sup\{x, -x\} \geq 0$. Next, we write $x < z$ provided $x \leq z$ and $x \neq z$.

A Riesz space L is called *Archimedean* if, for each $x \in L$, the inequality $x \leq 0$ holds whenever the set $\{nx : n \in \mathbb{N}\}$ is bounded from above.

In the following it will be assumed that L is an Archimedean Riesz space. It is easily seen that $\alpha u \leq \beta u$ for every $u \in L_+ := \{x \in L : x > 0\}$ and $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$. Moreover, given $u \in L_+$ we can define an extended (i.e., admitting the infinite value) norm $\|\cdot\|_u$ on L by

$$\|v\|_u := \inf\{\lambda \in \mathbb{R}_+ : |v| \leq \lambda u\}, \qquad v \in L,$$

where it is understood that $\inf \emptyset = +\infty$ and $0 \cdot (+\infty) = 0$.

Let us yet recall some further necessary definitions.

Definition 2. Let $E \subset \mathbb{R}$ be nonempty and let $\mathscr{I} \subset 2^{\mathbb{R}}$. We say that a property $p(x)$ $(x \in E)$ holds $\mathscr{I}$-almost everywhere in E (abbreviated in the sequel to $\mathscr{I}$-a.e. in E) provided there exists a set $A \in \mathscr{I}$ such that $p(x)$ holds for all $x \in E \setminus A$.

Definition 3. $\mathscr{I} \subset 2^{\mathbb{R}}$ is a σ-ideal provided $2^A \subset \mathscr{I}$ for $A \in \mathscr{I}$ and

$$\bigcup_{n\in\mathbb{N}} A_n \in \mathscr{I}\,, \qquad \{A_n\}_{n\in\mathbb{N}} \subset \mathscr{I}\,.$$

Moreover, if $\mathscr{I} \neq 2^{\mathbb{R}}$, then we say that $\mathscr{I}$ is proper; if $\mathscr{I} \neq \{\emptyset\}$, then we say that $\mathscr{I}$ is nontrivial. Finally, $\mathscr{I}$ is translation invariant (abbreviated to t.i. in the sequel) if $x + A \in \mathscr{I}$ for $A \in \mathscr{I}$ and $x \in \mathbb{R}$.

We have the following (see [3, Propositions 2.1 and 2.2]).

Proposition 1. *Let $\mathscr{I} \subset 2^{\mathbb{R}}$ be a proper t.i. σ-ideal and let $U \subset \mathbb{R}$ be open and nonempty. Then*

$$\operatorname{int}\,[(U \setminus T) - V] \neq \emptyset\,, \qquad V \in 2^{\mathbb{R}} \setminus \mathscr{I},\ T \in \mathscr{I}\,, \tag{3}$$

where $(U \setminus T) - V = \{u - v : u \in U \setminus T, v \in V\}$.

3 The Main Result

Let us start with an auxiliary result.

Theorem 1. *Let P be a dense subset of $\mathbb{R}$, $\mathscr{J} \subset 2^{\mathbb{R}}$ be a proper t.i. σ-ideal and let E be a subset of a nontrivial interval $I \subset \mathbb{R}$ with $H := I \setminus E \in \mathscr{J}$. We assume that $v : I \to L$ satisfies*

$$v(p + x) \leq v(x)\,, \qquad x \in E \cap (E - p), p \in P\,. \tag{4}$$

If there exists $u \in L_+$ such that v is continuous at a point $x_0 \in I$, with respect to the extended norm $\|\cdot\|_u$, then $v(x) = v(x_0)$ $\mathscr{J}$-a.e. in I.

Proof. Note that (4) yields

$$v(y) \leq v(y + q)\,, \qquad y \in E \cap (E - q), q \in -P\,, \tag{5}$$

where $-P := \{-p : p \in P\}$. Since $\mathscr{J}$ is proper and t.i., we deduce that $I \notin \mathscr{J}$, whence $E \notin \mathscr{J}$.

For each $n \in \mathbb{N}$ we write

$$D_n := \left\{z \in I : \|v(z) - v(x_0)\|_u < \frac{1}{n}\right\}\,,$$

$$E_n' := \left\{z \in E : v(z) - v(x_0) < \frac{1}{n}u\right\}\,,$$

$$F_n' := \left\{z \in E : v(x_0) - v(z) < \frac{1}{n}u\right\}\,,$$

$C_n := D_n \setminus H$, $E_n := E \setminus E'_n$ and $F_n := E \setminus F'_n$. Clearly, int $D_n \neq \emptyset$ for $n \in \mathbb{N}$, because v is continuous at x_0.

Suppose that there exists $k \in \mathbb{N}$ with $E_k \notin \mathscr{J}$. Then, on account of Proposition 1, there is $p \in P$ such that $-p \in \operatorname{int}(C_k - E_k)$, whence $p + c = e \in E_k \subset E$ with some $c \in C_k$ and $e \in E_k$. Hence, by (4),

$$v(e) - v(x_0) = v(p + c) - v(x_0) \leq v(c) - v(x_0) < \frac{1}{k}u\,.$$

This is a contradiction.

Next, suppose that $F_k \notin \mathscr{J}$ for some $k \in \mathbb{N}$. Then, on account of Proposition 1, there is $q \in -P$ with $-q \in \operatorname{int}(C_k - F_k)$, whence $q + c = e \in F_k \subset E$ with some $c \in C_k$ and $e \in F_k$. Hence, by (5),

$$v(x_0) - v(e) = v(x_0) - v(c + q) \leq v(x_0) - v(c) < \frac{1}{k}u\,.$$

This is a contradiction, too.

In this way we have shown that $G_k := E_k \cup F_k \in \mathscr{J}$ for $k \in \mathbb{N}$. Clearly

$$V := v^{-1}(L \setminus \{v(x_0)\}) = I \setminus \bigcap_{n\in\mathbb{N}} D_n$$
$$= \bigcup_{n\in\mathbb{N}} I \setminus D_n \subset H \cup \bigcup_{k\in\mathbb{N}} G_k \in \mathscr{J}$$

and $v(x) = v(x_0)$ for $x \in I \setminus V$. □

The next theorem is the main result of this paper.

Theorem 2. *Let I be a real infinite interval, $\mathscr{J} \subset 2^{\mathbb{R}}$ be a proper t.i. σ-ideal, L be an Archimedean Riesz space, $v : I \to L$, $a_1, a_2, \alpha_1, \alpha_2 \in \mathbb{R}$, $a_1 < 0 < a_2$, $a_1 a_2^{-1} \notin \mathbb{Q}$ and*

$$c_i := \frac{1}{a_i}\alpha_i, \qquad i = 1, 2\,. \tag{6}$$

If $c_1 \geq c_2$ and there exist $\omega, u \in L_+$ such that $\|\omega\|_u < \infty$, v is continuous at some point $x_0 \in I$, with respect to the extended norm $\|\cdot\|_u$, and the following two conditional inequalities

$$\textit{if}\ \ a_1 + x \in I, \ \ \textit{then}\ \ v(a_1 + x) - v(x) \leq \alpha_1\omega\,, \tag{7}$$

$$\textit{if}\ \ a_2 + x \in I, \ \ \textit{then}\ \ v(a_2 + x) - v(x) \leq \alpha_2\omega \tag{8}$$

are valid $\mathscr{J}$*-a.e. in* I*, then* $c_2 = c_1$ *and*

$$v(x) = c_1(x - x_0)\,\omega + v(x_0)\,, \qquad \mathscr{J}\text{-a.e. in } I\ . \tag{9}$$

Conversely, if $c_1 \le c_2$ *and* (9) *holds for some* $x_0 \in I$*, then* v *satisfies inequalities* (7) *and* (8) $\mathscr{J}$*-a.e. in* I*.*

Proof. Since $\mathscr{J}$ is a proper and t.i. σ-ideal, it is easily seen that we have the following property

$$\operatorname{int} T = \emptyset\,, \qquad T \in \mathscr{J}\,. \tag{10}$$

Next, there is a set $T \in \mathscr{J}$ such that conditions (7) and (8) hold for $x \in F := I \setminus T$.

Let

$$w_i(x) := v(x) - c_i x\,\omega\,, \qquad i = 1, 2, x \in I\ .$$

Clearly w_i is continuous at x_0 with respect to $\|\cdot\|_u$. Further, for every $i, j \in \{1, 2\}$, we have $\alpha_j \le c_i a_j$ and consequently

$$\begin{aligned} w_i(x + a_j) &= v(x + a_j) - c_i(x + a_j)\omega \\ &\le v(x) + \alpha_j\omega - c_i x\omega - c_i a_j\omega \\ &\le w_i(x)\,, \qquad x \in F \cap (F - a_j)\ . \end{aligned} \tag{11}$$

Let $E := I \setminus H$, where

$$H := \bigcup_{m,n\in\mathbb{Z}} (T + na_1 + ma_2) \in \mathscr{J}\ .$$

If we write $P := \{na_1 + ma_2 : n, m \in \mathbb{N}_0\}$, then

$$H + p = H\,, \qquad p \in P\ , \tag{12}$$

the set P is dense in $\mathbb{R}$ (see, e.g., [7–9]) and, in view of (11) and (12), it is easy to notice that

$$w_i(x + p) \le w_i(x)\,, \qquad x \in E \cap (E - p), p \in P, i = 1, 2\ . \tag{13}$$

Hence, on account of Theorem 1, there are $V_1, V_2 \in \mathscr{J}$ such that

$$w_i(x) = w_i(x_0)\,, \qquad x \in E \setminus V_i, i = 1, 2\ ,$$

which implies (9).

Further, observe that, by (10), we have $\mathrm{int}(H \cup V_1 \cup V_2) = \emptyset$ and

$$v(x_0) - c_i x_0 \omega = v(x) - c_i x \omega\,, \qquad x \in E_0 := I \setminus (H \cup V_1 \cup V_2), i = 1, 2\,.$$

Hence

$$(c_1 - c_2) x \omega = (c_1 - c_2) x_0 \omega\,, \qquad x \in E_0\,,$$

whence we get $c_1 = c_2$.

The converse is easy to check. □

Remark 1. Let $a_1, a_2 \in \mathbb{R}$ and $\alpha_1, \alpha_2 \in (0, \infty)$. Then every function $v : I \to \mathbb{R}$ with

$$\sup_{x \in \mathbb{R}} |v(x)| \le \frac{1}{2} \min\{\alpha_1, \alpha_2\}$$

fulfils (7) and (8) for each real interval I. This shows that some assumptions concerning a_1, a_2, c_1, c_2 are necessary in Theorem 2.

Taking $\mathscr{J} = \{\emptyset\}$ in Theorem 2 we obtain the following corollary.

Corollary 1. *Let $a_1, a_2, \alpha_1, \alpha_2 \in \mathbb{R}$ be such that $a_1 < 0 < a_2$, $a_1 a_2^{-1} \notin \mathbb{Q}$ and $c_1 \ge c_2$, where c_1, c_2 are given by (6). Let I be a real infinite interval, L be an Archimedean Riesz space, $u, \omega \in L_+$ and $\|\omega\|_u < \infty$. Then a function $v : I \to L$, that is continuous (with respect to the extended norm $\|\cdot\|_u$) at a point $x_0 \in I$, satisfies the inequalities*

$$\text{if } a_1 + x \in I, \text{ then } v(a_1 + x) - v(x) \le \alpha_1 \omega\,,$$

$$\text{if } a_2 + x \in I, \text{ then } v(a_2 + x) - v(x) \le \alpha_2 \omega$$

if and only if $c_2 = c_1$ and

$$v(x) = c_1 (x - x_0)\, \omega + v(x_0)\,, \qquad x \in I\,.$$

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