Multidimensional Hilbert-Type Integral Inequalities and Their Operators Expressions

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Abstract In this chapter, by the use of the methods of weight functions and techniques of Real Analysis, we provide a general multidimensional Hilbert-type integral inequality with a non-homogeneous kernel and a best possible constant factor. The equivalent forms, the reverses and some Hardy-type inequalities are obtained. Furthermore, we consider the operator expressions with the norm, some particular inequalities with the homogeneous kernel and a large number of particular examples.

Keywords Multidimensional Hilbert-type integral inequality • Weight function • Equivalent form • Hilbert-type integral operator

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1 Introduction

Suppose that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(x), g(y) \geq 0, f \in L^p(\mathbf{R}_+), g \in L^q(\mathbf{R}_+)$,

$$||f||_p = \left\{ \int_0^\infty f^p(x)dx \right\}^{\frac{1}{p}} > 0,$$

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$||g||_q > 0$. We have the following Hardy–Hilbert's integral inequality (cf. [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} ||f||_p ||g||_q, \tag{1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. If $a_m, b_n \geq 0, a = \{a_m\}_{m=1}^\infty \in l^p, b = \{b_n\}_{n=1}^\infty \in l^q$,

$$||a||_p = \left\{ \sum_{m=1}^\infty a_m^p \right\}^{\frac{1}{p}} > 0,$$

$||b||_q > 0$, then we have the following discrete Hardy–Hilbert's inequality with the same best constant $\frac{\pi}{\sin(\pi/p)}$:

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} ||a||_p ||b||_q. \tag{2}$$

Inequalities (1) and (2) are important in Analysis and its applications (cf. [1–6]).

In 1998, by introducing an independent parameter $\lambda \in (0, 1]$, Yang [7] gave an extension of (1) for $p = q = 2$. In 2009 and 2011, Yang [3,4] gave some extensions of (1) and (2) as follows: If $\lambda_1, \lambda_2 \in \mathbf{R} = (-\infty, \infty), \lambda_1 + \lambda_2 = \lambda, k_\lambda(x, y)$ is a nonnegative homogeneous function of degree $-\lambda$ in $\mathbf{R}_+^2$, with

$$k(\lambda_1) = \int_0^\infty k_\lambda(t, 1) t^{\lambda_1 - 1} dt \in \mathbf{R}_+ = (0, \infty),$$

$$\phi(x) = x^{p(1-\lambda_1)-1}, \psi(y) = y^{q(1-\lambda_2)-1} (x, y \in \mathbf{R}_+),$$

$f(x), g(y) \geq 0$, satisfying

$$f \in L_{p,\phi}(\mathbf{R}_+) = \left\{ f; ||f||_{p,\phi} := \left\{ \int_0^\infty \phi(x) |f(x)|^p dx \right\}^{\frac{1}{p}} < \infty \right\},$$

$g \in L_{q,\psi}(\mathbf{R}_+), ||f||_{p,\phi}, ||g||_{q,\psi} > 0$, then we have

$$\int_0^\infty \int_0^\infty k_\lambda(x, y) f(x) g(y) dxdy < k(\lambda_1) ||f||_{p,\phi} ||g||_{q,\psi}, \tag{3}$$

where the constant factor $k(\lambda_1)$ is the best possible. Moreover, if $k_\lambda(x, y)$ is finite and $k_\lambda(x, y) x^{\lambda_1 - 1} (k_\lambda(x, y) y^{\lambda_2 - 1})$ is strict decreasing with respect to $x > 0 (y > 0)$, then for $a_{m,} b_n \geq 0$,

$$a = \{a_m\}_{m=1}^{\infty} \in l_{p,\phi} = \left\{a; ||a||_{p,\phi} := \left\{\sum_{n=1}^{\infty} \phi(n)|a_n|^p\right\}^{\frac{1}{p}} < \infty\right\},$$

$b = \{b_n\}_{n=1}^{\infty} \in l_{q,\psi}, ||a||_{p,\phi}, ||b||_{q,\psi} > 0$, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} k_{\lambda}(m,n)a_m b_n < k(\lambda_1)||a||_{p,\phi}||b||_{q,\psi}, \tag{4}$$

where the constant factor $k(\lambda_1)$ is still the best possible.

Clearly, for $\lambda = 1, k_1(x, y) = \frac{1}{x+y}, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$, (3) reduces to (1), while (4) reduces to (2). Some other results including multidimensional Hilbert-type integral inequalities are provided by Yang et al. [8], Krnić and Pečarić [9], Yang and Rassias [10, 11], Azar [12], Arpad and Choonghong [13], Kuang and Debnath [14], Zhong [15], Hong [16], Zhong and Yang [17], Yang and Krnić [18], and Li and He [19].

In this chapter, by the use of the methods of weight functions and techniques of real analysis, we give a general multidimensional Hilbert-type integral inequality with a nonhomogeneous kernel and a best possible constant factor. The equivalent forms, the reverses and some Hardy-type inequalities are obtained. Furthermore, we consider the operator expressions with the norm, some particular inequalities with the homogeneous kernel and a large number of particular examples.

2 Some Lemmas

If $i_0, j_0 \in \mathbf{N}$($\mathbf{N}$ is the set of positive integers), $\alpha, \beta > 0$, we put

$$||x||_{\alpha} := \left(\sum_{k=1}^{i_0} |x_k|^{\alpha}\right)^{\frac{1}{\alpha}} \quad (x = (x_1, \ldots, x_{i_0}) \in \mathbf{R}^{i_0}),$$

$$||y||_{\beta} := \left(\sum_{k=1}^{j_0} |y_k|^{\beta}\right)^{\frac{1}{\beta}} \quad (y = (y_1, \ldots, y_{j_0}) \in \mathbf{R}^{j_0}).$$

Lemma 1. *If $s \in \mathbf{N}, \gamma, M > 0$, $\Psi(u)$ is a nonnegative measurable function in* $(0, 1]$, *and*

$$D_M := \left\{x \in \mathbf{R}_+^s; 0 < u = \sum_{i=1}^{s}\left(\frac{x_i}{M}\right)^{\gamma} \leq 1\right\},$$

then we have the following expression (cf. [6]):

$$\int \cdots \int_{D_M} \Psi\left(\sum_{i=1}^{s}\left(\frac{x_i}{M}\right)^{\gamma}\right) dx_1 \cdots dx_s \\ = \frac{M^s \Gamma^s\left(\frac{1}{\gamma}\right)}{\gamma^s \Gamma\left(\frac{s}{\gamma}\right)} \int_0^1 \Psi(u) u^{\frac{s}{\gamma}-1} du. \tag{5}$$

In view of (5) and the conditions, it follows that

(i) for

$$\mathbf{R}_+^s = \left\{x \in \mathbf{R}_+^s; 0 < u = \sum_{i=1}^{s}\left(\frac{x_i}{M}\right)^{\gamma} \leq 1 (M \to \infty)\right\},$$

we have

$$\int \cdots \int_{\mathbf{R}_+^s} \Psi\left(\sum_{i=1}^{s}\left(\frac{x_i}{M}\right)^{\gamma}\right) dx_1 \cdots dx_s \\ = \lim_{M\to\infty} \frac{M^s \Gamma^s\left(\frac{1}{\gamma}\right)}{\gamma^s \Gamma\left(\frac{s}{\gamma}\right)} \int_0^1 \Psi(u) u^{\frac{s}{\gamma}-1} du; \tag{6}$$

(ii) for

$$\{x \in \mathbf{R}_+^s; ||x||_\gamma \geq 1\} \\ = \left\{x \in \mathbf{R}_+^s; \frac{1}{M^\gamma} < u = \sum_{i=1}^{s}\left(\frac{x_i}{M}\right)^{\gamma} \leq 1 (M \to \infty)\right\},$$

setting $\Psi(u) = 0\big(u \in \big(0, \frac{1}{M^\gamma}\big)\big)$, we have

$$\int \cdots \int_{\{x\in\mathbf{R}_+^s; ||x||_\gamma \geq 1\}} \Psi\left(\sum_{i=1}^{s}\left(\frac{x_i}{M}\right)^{\gamma}\right) dx_1 \cdots dx_s \\ = \lim_{M\to\infty} \frac{M^s \Gamma^s\left(\frac{1}{\gamma}\right)}{\gamma^s \Gamma\left(\frac{s}{\gamma}\right)} \int_{\frac{1}{M^\gamma}}^1 \Psi(u) u^{\frac{s}{\gamma}-1} du; \tag{7}$$

(iii) for

$$\{x \in \mathbf{R}_+^s; ||x||_\gamma \leq 1\} \\ = \left\{x \in \mathbf{R}_+^s; 0 < u = \sum_{i=1}^{s}\left(\frac{x_i}{M}\right)^{\gamma} \leq \frac{1}{M^\gamma}\right\},$$

setting $\Psi(u) = 0\big(u \in \big(\frac{1}{M^\gamma}, \infty\big)\big)$, we have

$$\begin{aligned}\int \cdots \int_{\{x\in\mathbf{R}_+^s;||x||_\gamma \le 1\}} \Psi\left(\sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma\right) dx_1 \cdots dx_s \\ = \frac{M^s \Gamma^s\left(\frac{1}{\gamma}\right)}{\gamma^s \Gamma\left(\frac{s}{\gamma}\right)} \int_0^{\frac{1}{M^\gamma}} \Psi(u) u^{\frac{s}{\gamma}-1} du.\end{aligned} \tag{8}$$

Lemma 2. *For $s \in \mathbf{N}, \gamma > 0$, $\varepsilon > 0$, we have*

$$\int \cdots \int_{\{x\in\mathbf{R}_+^s;||x||_\gamma \ge 1\}} ||x||_\gamma^{-s-\varepsilon} dx_1 \cdots dx_s = \frac{\Gamma^s\left(\frac{1}{\gamma}\right)}{\varepsilon \gamma^{s-1} \Gamma\left(\frac{s}{\gamma}\right)}, \tag{9}$$

$$\int \cdots \int_{\{x\in\mathbf{R}_+^s;||x||_\gamma \le 1\}} ||x||_\gamma^{-s+\varepsilon} dx_1 \cdots dx_s = \frac{\Gamma^s\left(\frac{1}{\gamma}\right)}{\varepsilon \gamma^{s-1} \Gamma\left(\frac{s}{\gamma}\right)}. \tag{10}$$

Proof. By (7), it follows

$$\begin{aligned}&\int \cdots \int_{\{x\in\mathbf{R}_+^s;||x||_\gamma \ge 1\}} ||x||_\gamma^{-s-\varepsilon} dx_1 \cdots dx_s \\ &= \int \cdots \int_{\{x\in\mathbf{R}_+^s;||x||_\gamma \ge 1\}} \left\{M\left[\sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma\right]^{\frac{1}{\gamma}}\right\}^{-s-\varepsilon} dx_1 \cdots dx_s \\ &= \lim_{M\to\infty} \frac{M^s \Gamma^s\left(\frac{1}{\gamma}\right)}{\gamma^s \Gamma\left(\frac{s}{\gamma}\right)} \int_{\frac{1}{M^\gamma}}^1 (Mu^{1/\gamma})^{-s-\varepsilon} u^{\frac{s}{\gamma}-1} du \\ &= \frac{\Gamma^s\left(\frac{1}{\gamma}\right)}{\varepsilon \gamma^{s-1} \Gamma\left(\frac{s}{\gamma}\right)}.\end{aligned}$$

By (8), we find

$$\begin{aligned}&\int \cdots \int_{\{x\in\mathbf{R}_+^s;||x||_\gamma \le 1\}} ||x||_\gamma^{-s+\varepsilon} dx_1 \cdots dx_s \\ &= \int \cdots \int_{\{x\in\mathbf{R}_+^s;||x||_\gamma \le 1\}} \left\{M\left[\sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma\right]^{\frac{1}{\gamma}}\right\}^{-s+\varepsilon} dx_1 \cdots dx_s \\ &= \frac{M^s \Gamma^s\left(\frac{1}{\gamma}\right)}{\gamma^s \Gamma\left(\frac{s}{\gamma}\right)} \int_0^{\frac{1}{M^\gamma}} (Mu^{1/\gamma})^{-s+\varepsilon} u^{\frac{s}{\gamma}-1} du = \frac{\Gamma^s\left(\frac{1}{\gamma}\right)}{\varepsilon \gamma^{s-1} \Gamma\left(\frac{s}{\gamma}\right)}.\end{aligned}$$

Hence, we have (9) and (10). The lemma is proved.

Note. By (9) and (10), for $\delta = \pm 1$, we have the following unified expression:

$$\int \cdots \int_{\{x \in \mathbf{R}_+^s; ||x||_\gamma^\delta \geq 1\}} ||x||_\gamma^{-s-\delta\varepsilon} dx_1 \cdots dx_s = \frac{\Gamma^s\left(\frac{1}{\gamma}\right)}{\varepsilon \gamma^{s-1} \Gamma\left(\frac{s}{\gamma}\right)}. \tag{11}$$

Definition 1. If $x = (x_1, \ldots, x_{i_0}) \in \mathbf{R}_+^{i_0}$, $y = (y_1, \ldots, y_{j_0}) \in \mathbf{R}_+^{j_0}$, $h(u)$ is a nonnegative measurable function in $\mathbf{R}_+$, $\sigma \in \mathbf{R}$, $\delta \in \{-1, 1\}$, then we define two weight functions $\omega_\delta(\sigma, y)$ and $\varpi_\delta(\sigma, x)$ as follows:

$$\omega_\delta(\sigma, y) := ||y||_\beta^\sigma \int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) \frac{dx}{||x||_\alpha^{i_0-\delta\sigma}}, \tag{12}$$

$$\varpi_\delta(\sigma, x) := ||x||_\alpha^{\delta\sigma} \int_{\mathbf{R}_+^{j_0}} h(||x||_\alpha^\delta ||y||_\beta) \frac{dy}{||y||_\beta^{j_0-\sigma}}. \tag{13}$$

By (6), we find

$$\begin{aligned}
\omega_\delta(\sigma, y) &= ||y||_\beta^\sigma \int_{\mathbf{R}_+^{i_0}} \frac{h\left(M^\delta \left[\sum_{i=1}^{i_0} \left(\frac{x_i}{M}\right)^\alpha\right]^{\frac{\delta}{\alpha}} ||y||_\beta\right)}{M^{i_0-\delta\sigma} \left[\sum_{i=1}^{i_0} \left(\frac{x_i}{M}\right)^\alpha\right]^{\frac{i_0-\delta\sigma}{\alpha}}} dx \\
&= ||y||_\beta^\sigma \lim_{M\to\infty} \frac{M^{i_0} \Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0} \Gamma\left(\frac{i_0}{\alpha}\right)} \int_0^1 \frac{h(M^\delta u^{\frac{\delta}{\alpha}} ||y||_\beta)}{M^{i_0-\delta\sigma} u^{\frac{i_0-\delta\sigma}{\alpha}}} u^{\frac{i_0}{\alpha}-1} du \\
&= ||y||_\beta^\sigma \lim_{M\to\infty} \frac{M^{\delta\sigma} \Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0} \Gamma\left(\frac{i_0}{\alpha}\right)} \int_0^1 h(M^\delta u^{\frac{\delta}{\alpha}} ||y||_\beta) u^{\frac{\delta\sigma}{\alpha}-1} du.
\end{aligned}$$

Setting $v = M^\delta u^{\frac{\delta}{\alpha}} ||y||_\beta$ in the above integral, in view of $\delta = \pm 1$, we obtain

$$\omega_\delta(\sigma, y) = K_2(\sigma) := \frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1} \Gamma\left(\frac{i_0}{\alpha}\right)} k(\sigma), \tag{14}$$

where $k(\sigma) = \int_0^\infty h(v) v^{\sigma-1} dv$.

By (6), setting $v = M||x||_\alpha^\delta u^{\frac{1}{\beta}}$, we find

$$\varpi_\delta(\sigma, x) = ||x||_\alpha^{\delta\sigma} \int_{\mathbf{R}_+^{j_0}} \frac{h\left(M||x||_\alpha^\delta \left[\sum_{j=1}^{j_0} \left(\frac{y_j}{M}\right)^\beta\right]^{\frac{1}{\beta}}\right)}{M^{j_0-\sigma} \left[\sum_{j=1}^{j_0} \left(\frac{y_j}{M}\right)^\beta\right]^{\frac{j_0-\sigma}{\beta}}} dy$$

$$= ||x||_\alpha^{\delta\sigma} \lim_{M\to\infty} \frac{M^{j_0}\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0}\Gamma\left(\frac{j_0}{\beta}\right)} \int_0^1 \frac{h\left(M||x||_\alpha^\delta u^{\frac{1}{\beta}}\right)}{M^{j_0-\sigma}u^{\frac{j_0-\sigma}{\beta}}} u^{\frac{j_0}{\beta}-1} du$$

$$= ||x||_\alpha^{\delta\sigma} \lim_{M\to\infty} \frac{M^{\sigma}\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0}\Gamma\left(\frac{j_0}{\beta}\right)} \int_0^1 h\left(M||x||_\alpha^\delta u^{\frac{1}{\beta}}\right) u^{\frac{\sigma}{\beta}-1} du$$

$$= K_1(\sigma) := \frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)} k(\sigma). \tag{15}$$

Lemma 3. *As the assumptions of Definition 1, for $k(\sigma) \in \mathbf{R}_+$, $p \in \mathbf{R}\backslash\{0,1\}$, $\frac{1}{p} + \frac{1}{q} = 1$, setting*

$$\tilde{I} := \int_{\left\{x\in\mathbf{R}_+^{i_0};||x||_\alpha^\delta\geq 1\right\}} ||x||_\alpha^{\delta\sigma-\frac{\delta\varepsilon}{p}-i_0} \left[\int_{\left\{y\in\mathbf{R}_+^{j_0};||y||_\beta\leq 1\right\}} h\left(||x||_\alpha^\delta ||y||_\beta\right) ||y||_\beta^{\sigma+\frac{\varepsilon}{q}-j_0} dy\right] dx, \tag{16}$$

then we have

$$\varepsilon\tilde{I} \geq \tilde{K}(\sigma) + o(1)(\varepsilon \to 0^+), \tag{17}$$

where $\tilde{K}(\sigma) := L(\alpha,\beta)k(\sigma)$,

$$L(\alpha,\beta) := \frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)} \frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}. \tag{18}$$

Moreover, if there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, $k(\tilde{\sigma}) \in \mathbf{R}$, then we have

$$\varepsilon\tilde{I} = \tilde{K}(\sigma) + o(1)(\varepsilon \to 0^+). \tag{19}$$

Proof. For $\varepsilon > 0$, setting $\tilde{\sigma} = \sigma + \frac{\varepsilon}{q}$ and

$$H(||x||_\alpha^\delta) := ||x||_\alpha^{\delta\tilde{\sigma}} \int_{\left\{y\in\mathbf{R}_+^{j_0};||y||_\beta\leq 1\right\}} h\left(||x||_\alpha^\delta ||y||_\beta\right) ||y||_\beta^{\tilde{\sigma}-j_0} dy,$$

in view of (16), it follows

$$\tilde{I} = \int_{\left\{x\in\mathbf{R}_+^{i_0};||x||_\alpha^\delta\geq 1\right\}} ||x||_\alpha^{-\delta\varepsilon-i_0} H\left(||x||_\alpha^\delta\right) dx.$$

Putting

$$\Psi(u) = h(||x||_\alpha^\delta M u^{\frac{1}{\beta}}) M^{\tilde{\sigma}-j_0} u^{\frac{1}{\beta}(\tilde{\sigma}-j_0)},$$

by (8), we find

$$\begin{aligned} H(||x||_\alpha^\delta) &= ||x||_\beta^{\delta\tilde{\sigma}} \frac{M^{j_0}\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0}\Gamma\left(\frac{j_0}{\beta}\right)} \\ &\quad \times \int_0^{\frac{1}{M^\beta}} h(||x||_\alpha^\delta M u^{\frac{1}{\beta}}) M^{\tilde{\sigma}-j_0} u^{\frac{1}{\beta}(\tilde{\sigma}-j_0)} u^{\frac{j_0}{\beta}-1} du \\ &= ||x||_\alpha^{\delta\tilde{\sigma}} \frac{M^{\tilde{\sigma}}\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0}\Gamma\left(\frac{j_0}{\beta}\right)} \int_0^{\frac{1}{M^\beta}} h(||x||_\alpha^\delta M u^{\frac{1}{\beta}}) u^{\frac{\tilde{\sigma}}{\beta}-1} du. \end{aligned}$$

Setting $v = ||x||_\alpha^\delta M u^{\frac{1}{\beta}}$ in the above, it follows

$$L(||x||_\alpha^\delta) = \frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)} \int_0^{||x||_\alpha^\delta} h(v) v^{\tilde{\sigma}-1} dv.$$

Putting $\Psi(u) = M^{-\delta\varepsilon-i_0} u^{\frac{1}{\alpha}(-\delta\varepsilon-i_0)} H(M^\delta u^{\frac{\delta}{\alpha}})$, for $\delta = 1$, by (7), we obtain

$$\begin{aligned} \tilde{I} &= \lim_{M\to\infty} \frac{M^{i_0}\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0}\Gamma\left(\frac{i_0}{\alpha}\right)} \int_{\frac{1}{M^\alpha}}^1 M^{-\varepsilon-i_0} u^{\frac{1}{\alpha}(-\varepsilon-i_0)} H(M u^{\frac{1}{\alpha}}) u^{\frac{i_0}{\alpha}-1} du \\ &= \lim_{M\to\infty} \frac{M^{-\varepsilon}\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0}\Gamma\left(\frac{i_0}{\alpha}\right)} \int_{\frac{1}{M^\alpha}}^1 H(M u^{\frac{1}{\alpha}}) u^{\frac{-\varepsilon}{\alpha}-1} du \\ &= \frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)} \int_1^\infty H(t) t^{-\varepsilon-1} dt (t = M u^{\frac{1}{\alpha}}); \end{aligned}$$

for $\delta = -1$, by (8), we still find that

$$\begin{aligned} \tilde{I} &= \frac{M^{i_0}\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0}\Gamma\left(\frac{i_0}{\alpha}\right)} \int_0^{\frac{1}{M^\alpha}} M^{\varepsilon-i_0} u^{\frac{1}{\alpha}(\varepsilon-i_0)} H(M^{-1} u^{\frac{-1}{\alpha}}) u^{\frac{i_0}{\alpha}-1} du \\ &= \frac{M^{\varepsilon}\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0}\Gamma\left(\frac{i_0}{\alpha}\right)} \int_0^{\frac{1}{M^\alpha}} H(M^{-1} u^{\frac{-1}{\alpha}}) u^{\frac{\varepsilon}{\alpha}-1} du \\ &= \frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)} \int_1^\infty H(t) t^{-\varepsilon-1} dt (t = M^{-1} u^{\frac{-1}{\alpha}}). \end{aligned}$$

Hence, we find

$$\begin{aligned}\varepsilon\tilde{I} &= \varepsilon L(\alpha,\beta)\int_1^\infty t^{-\varepsilon-1}\int_0^t h(v)v^{\tilde{\sigma}-1}dvdt \\ &= \varepsilon L(\alpha,\beta)\left[\int_1^\infty t^{-\varepsilon-1}\int_0^1 h(v)v^{\tilde{\sigma}-1}dvdt\right. \\ &\quad \left. + \int_1^\infty t^{-\varepsilon-1}\int_1^t h(v)v^{\tilde{\sigma}-1}dvdt\right]\end{aligned}$$

$$= \varepsilon L(\alpha,\beta)\left[\frac{1}{\varepsilon}\int_0^1 h(v)v^{\tilde{\sigma}-1}dvdt + \int_1^\infty\left(\int_v^\infty t^{-\varepsilon-1}dt\right)h(v)v^{\tilde{\sigma}-1}dv\right]$$

$$= L(\alpha,\beta)\left[\int_0^1 h(v)v^{\tilde{\sigma}-1}dvdt + \int_1^\infty h(v)v^{\left(\sigma-\frac{\varepsilon}{p}\right)-1}dv\right]. \tag{20}$$

By Fatou lemma (cf. [20]), it follows

$$\begin{aligned}\underline{\lim}_{\varepsilon\to 0^+}\varepsilon\tilde{I} &= L(\alpha,\beta)\,\underline{\lim}_{\varepsilon\to 0^+}\left[\int_0^1 h(v)v^{\tilde{\sigma}-1}dvdt + \int_1^\infty h(v)v^{\left(\sigma-\frac{\varepsilon}{p}\right)-1}dv\right] \\ &\geq L(\alpha,\beta)\left[\int_0^1 \underline{\lim}_{\varepsilon\to 0^+} h(v)v^{\tilde{\sigma}-1}dvdt\right. \\ &\quad \left. + \int_1^\infty \underline{\lim}_{\varepsilon\to 0^+} h(v)v^{\left(\sigma-\frac{\varepsilon}{p}\right)-1}dv\right] = L(\alpha,\beta)k(\sigma),\end{aligned}$$

and then (17) follows.

Moreover, for $0 < \varepsilon < \delta_0\min\{|p|,|q|\}, \tilde{\sigma} \in \left(\sigma - \frac{1}{2}\delta_0, \sigma + \frac{1}{2}\delta_0\right)$, since

$$\begin{aligned}h(v)v^{\tilde{\sigma}-1} &\leq h(v)v^{(\sigma-\frac{1}{2}\delta_0)-1}(v\in(0,1]), \\ 0 &\leq \int_0^1 h(v)v^{(\sigma-\frac{1}{2}\delta_0)-1} \leq k\left(\sigma-\frac{1}{2}\delta_0\right) < \infty, \\ h(v)v^{\tilde{\sigma}-1} &\leq h(v)v^{(\sigma+\frac{1}{2}\delta_0)-1}(v\in[1,\infty)), \\ 0 &\leq \int_1^\infty h(v)v^{(\sigma+\frac{1}{2}\delta_0)-1} \leq k\left(\sigma+\frac{1}{2}\delta_0\right) < \infty,\end{aligned}$$

by Lebesgue control convergence theorem (cf. [20]), it follows that

$$\begin{aligned}\int_0^1 h(v)v^{\tilde{\sigma}-1}dv &= \int_0^1 h(v)v^{\sigma-1}dv + o_1(1)(\varepsilon\to 0^+), \\ \int_1^\infty h(v)v^{(\sigma-\frac{\varepsilon}{p})-1}dv &= \int_1^\infty h(v)v^{\sigma-1}dv + o_2(1)(\varepsilon\to 0^+).\end{aligned}$$

Then by (20), (19) follows. The lemma is proved.

Lemma 4. *As the assumptions of Definition 1, if* $p \in \mathbf{R}\backslash\{0,1\}$, $\frac{1}{p}+\frac{1}{q}=1$, $f(x)=f(x_1,\ldots,x_{i_0})\geq 0$, $g(y)=g(y_1,\ldots,y_{j_0})\geq 0$, *then*

(i) for $p>1$, *we have the following inequality:*

$$J_1 := \left\{ \int_{\mathbf{R}_+^{j_0}} \frac{||y||_\beta^{p\sigma-j_0}}{[\omega_\delta(\sigma,y)]^{p-1}} \left(\int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) f(x)dx \right)^p dy \right\}^{\frac{1}{p}}$$
$$\leq \left\{ \int_{\mathbf{R}_+^{i_0}} \varpi_\delta(\sigma,x) ||x||_\alpha^{p(i_0-\delta\sigma)-i_0} f^p(x)dx \right\}^{\frac{1}{p}}; \tag{21}$$

(ii) for $0<p<1$, *or* $p<0$, *we have the reverse of (21).*

Proof. (i) For $p>1$, by Hölder's inequality with weight (cf. [21]), it follows

$$\int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) f(x)dx$$
$$= \int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) \left[\frac{||x||_\alpha^{(i_0-\delta\sigma)/q} f(x)}{||y||_\beta^{(j_0-\sigma)/p}} \right] \left[\frac{||y||_\beta^{(j_0-\sigma)/p}}{||x||_\alpha^{(i_0-\delta\sigma)/q}} \right] dx$$
$$\leq \left\{ \int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) \frac{||x||_\alpha^{(i_0-\delta\sigma)(p-1)}}{||y||_\beta^{j_0-\sigma}} f^p(x)dx \right\}^{\frac{1}{p}}$$
$$\times \left\{ \int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) \frac{||y||_\beta^{(j_0-\sigma)(q-1)}}{||x||_\alpha^{i_0-\delta\sigma}} dx \right\}^{\frac{1}{q}}$$
$$= [\omega_\delta(\sigma,y)]^{\frac{1}{q}} ||y||_\beta^{\frac{j_0}{p}-\sigma}$$
$$\times \left\{ \int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) \frac{||x||_\alpha^{(i_0-\delta\sigma)(p-1)}}{||y||_\beta^{j_0-\sigma}} f^p(x)dx \right\}^{\frac{1}{p}}. \tag{22}$$

Then by Fubini theorem (cf. [20]), we have

$$J_1 \leq \left\{ \int_{\mathbf{R}_+^{j_0}} \left[\int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) \frac{||x||_\alpha^{(i_0-\delta\sigma)(p-1)}}{||y||_\beta^{j_0-\sigma}} f^p(x)dx \right] dy \right\}^{\frac{1}{p}}$$
$$= \left\{ \int_{\mathbf{R}_+^{i_0}} \left[\int_{\mathbf{R}_+^{j_0}} h(||x||_\alpha ||y||_\beta) \frac{||x||_\alpha^{(i_0-\delta\sigma)(p-1)}}{||y||_\beta^{j_0-\sigma}} dy \right] f^p(x)dx \right\}^{\frac{1}{p}}$$

$$= \left\{ \int_{\mathbf{R}_+^{i_0}} \varpi_\delta(\sigma, x) ||x||_\alpha^{p(i_0-\delta\sigma)-i_0} f^p(x) dx \right\}^{\frac{1}{p}}. \tag{23}$$

Hence, (21) follows.

(ii) For $0 < p < 1$, or $p < 0$, by the reverse Hölder's inequality with weight (cf. [21]), we obtain the reverse of (22). Then by Fubini theorem, we still can obtain the reverse of (21). The lemma is proved.

Lemma 5. *As the assumptions of Lemma 4, then*

(i) for $p > 1$, *we have the following inequality equivalent to (21):*

$$\begin{aligned} I &:= \int_{\mathbf{R}_+^{j_0}} \int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) f(x) g(y) dx dy \\ &\le \left\{ \int_{\mathbf{R}_+^{i_0}} \varpi_\delta(\sigma, x) ||x||_\alpha^{p(i_0-\delta\sigma)-i_0} f^p(x) dx \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \int_{\mathbf{R}_+^{j_0}} \omega_\delta(\sigma, y) ||y||_\beta^{q(j_0-\sigma)-j_0} g^q(y) dy \right\}^{\frac{1}{q}}; \end{aligned} \tag{24}$$

(ii) for $0 < p < 1$, *or* $p < 0$, *we have the reverse of (24) equivalent to the reverse of (21).*

Proof. (i) For $p > 1$, by Hölder's inequality (cf. [21]), it follows

$$\begin{aligned} I &= \int_{\mathbf{R}_+^{j_0}} \frac{||y||_\beta^{\frac{j_0}{q}-(j_0-\sigma)}}{[\omega_\delta(\sigma, y)]^{\frac{1}{q}}} \left[\int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) f(x) dx \right] \\ &\quad \times \left[[\omega_\delta(\sigma, y)]^{\frac{1}{q}} ||y||_\beta^{(j_0-\sigma)-\frac{j_0}{q}} g(y) \right] dy \\ &\le J_1 \left\{ \int_{\mathbf{R}_+^{j_0}} \omega_\delta(\sigma, y) ||y||_\beta^{q(j_0-\sigma)-j_0} g^q(y) dy \right\}^{\frac{1}{q}}. \end{aligned} \tag{25}$$

Then by (21), we have (24).

On the other hand, assuming that (24) is valid, we set

$$g(y) := \frac{||y||_\beta^{p\sigma-j_0}}{[\omega_\delta(\sigma, y)]^{p-1}} \left(\int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) f(x) dx \right)^{p-1}, y \in \mathbf{R}_+^{j_0}.$$

Then it follows

$$J_1^p = \int_{\mathbf{R}_+^{j_0}} \omega_\delta(\sigma, y)||y||_\beta^{q(j_0-\sigma)-j_0} g^q(y)dy.$$

If $J_1 = 0$, then (21) is trivially valid; if $J_1 = \infty$, then by (23), (21) keeps the form of equality ($= \infty$). Suppose that $0 < J_1 < \infty$. By (24), we have

$$0 < \int_{\mathbf{R}_+^{j_0}} \omega_\delta(\sigma, y)||y||_\beta^{q(j_0-\sigma)-j_0} g^q(y)dy = J_1^p = I$$
$$\leq \left\{ \int_{\mathbf{R}_+^{i_0}} \varpi_\delta(\sigma, x)||x||_\alpha^{p(i_0-\delta\sigma)-i_0} f^p(x)dx \right\}^{\frac{1}{p}}$$
$$\times \left\{ \int_{\mathbf{R}_+^{j_0}} \omega_\delta(\sigma, y)||y||_\beta^{q(j_0-\sigma)-j_0} g^q(y)dy \right\}^{\frac{1}{q}} < \infty.$$

It follows

$$J_1 = \left\{ \int_{\mathbf{R}_+^{j_0}} \omega_\delta(\sigma, y)||y||_\beta^{q(j_0-\sigma)-j_0} g^q(y)dy \right\}^{\frac{1}{p}}$$
$$\leq \left\{ \int_{\mathbf{R}_+^{i_0}} \varpi_\delta(\sigma, x)||x||_\alpha^{p(i_0-\delta\sigma)-i_0} f^p(x)dx \right\}^{\frac{1}{p}},$$

and then (21) follows. Hence, (21) and (24) are equivalent.

(ii) For $0 < p < 1$, or $p < 0$, by the same way, we can obtain the reverse of (24) equivalent to the reverse of (21). The lemma is proved.

3 Main Results and Operator Expressions

Setting

$$\Phi_\delta(x) := ||x||_\alpha^{p(i_0-\delta\sigma)-i_0},$$
$$\Psi(y) := ||y||_\beta^{q(j_0-\sigma)-j_0} (x \in \mathbf{R}_+^{i_0}, y \in \mathbf{R}_+^{j_0}),$$

by Lemmas 3–5, it follows

Theorem 1. *Suppose that* $\alpha, \beta > 0,\ \sigma \in \mathbf{R}, h(v) \geq 0,$

$$k(\sigma) = \int_0^\infty h(v)v^{\sigma-1}dv \in \mathbf{R}_+,$$

$$K(\sigma) := \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{q}} k(\sigma),$$

$\delta \in \{-1, 1\}$, $p \in \mathbf{R}\backslash\{0, 1\}$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x) = f(x_1, \ldots, x_{i_0}) \geq 0$, $g(y) = g(y_1, \ldots, y_{j_0}) \geq 0$,

$$0 < ||f||_{p,\Phi_\delta} = \left\{\int_{\mathbf{R}_+^{i_0}} \Phi_\delta(x) f^p(x)dx\right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||g||_{q,\Psi} = \left\{\int_{\mathbf{R}_+^{j_0}} \Psi(y)g^q(y)dy\right\}^{\frac{1}{q}} < \infty.$$

(i) If $p > 1$, then we have the following equivalent inequalities with the best possible constant factor $K(\sigma)$:

$$I = \int_{\mathbf{R}_+^{j_0}} \int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) f(x)g(y)dxdy < K(\sigma)||f||_{p,\Phi_\delta}||g||_{q,\Psi}, \quad (26)$$

$$J := \left\{\int_{\mathbf{R}_+^{j_0}} ||y||_\beta^{p\sigma-j_0} \left(\int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) f(x)dx\right)^p dy\right\}^{\frac{1}{p}}$$
$$< K(\sigma)||f||_{p,\Phi_\delta}; \quad (27)$$

(ii) if $0 < p < 1$, or $p < 0$, there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, $k(\tilde{\sigma}) \in \mathbf{R}$, then we still have the equivalent reverses of (26) and (27) with the same best constant factor $K(\sigma)$.

Proof. (i) For $p > 1$, by the conditions, we can prove that (22) takes the form of strict inequality for a.e. $y \in \mathbf{R}_+^{j_0}$. Otherwise, if (22) takes the form of equality for a $y \in \mathbf{R}_+^{j_0}$, then there exist constants A and B, which are not all zero, such that

$$A\frac{||x||_\alpha^{(i_0-\delta\sigma)(p-1)}}{||y||_\beta^{j_0-\sigma}} f^p(x) = B\frac{||y||_\beta^{(j_0-\sigma)(q-1)}}{||x||_\alpha^{i_0-\delta\sigma}} \quad \text{a.e. in } x \in \mathbf{R}_+^{i_0}. \quad (28)$$

If $A = 0$, then $B = 0$, which is impossible; if $A \neq 0$, then (28) reduces to

$$||x||_{\alpha}^{p(i_0-\delta\sigma)-i_0} f^p(x) = \frac{B||y||_{\beta}^{q(j_0-\sigma)}}{A||x||_{\alpha}^{i_0}} \text{ a.e. in } x \in \mathbf{R}_+^{i_0},$$

which contradicts the fact that $0 < ||f||_{p,\Phi_\delta} < \infty$. In fact, by (9) (for $\varepsilon \to 0^+$), it follows

$$\int_{\mathbf{R}_+^{i_0}} ||x||_{\alpha}^{-i_0} dx \geq \int_{\left\{x \in \mathbf{R}_+^{i_0}; ||x||_{\alpha} \geq 1\right\}} ||x||_{\alpha}^{-i_0} dx = \infty.$$

Hence (22) still takes the form of strict inequality. By (14) and (15), we obtain (27).

Similarly to (25), we still have

$$I \leq J \left\{ \int_{\mathbf{R}_+^{j_0}} ||y||_{\beta}^{q(j_0-\sigma)-j_0} g^q(y) dy \right\}^{\frac{1}{q}}. \tag{29}$$

Then by (29) and (27), we have (26). It is evident that by Lemma 5 and the assumptions, inequalities (27) and (26) are also equivalent.

For $\varepsilon > 0$, we set $\tilde{f}(x), \tilde{g}(y)$ as follows:

$$\tilde{f}(x) := \begin{cases} 0, & 0 < ||x||_{\alpha}^{\delta} < 1, \\ ||x||_{\alpha}^{\delta\left(\sigma - \frac{\varepsilon}{p}\right) - i_0}, & ||x||_{\alpha}^{\delta} \geq 1, \end{cases}$$

$$\tilde{g}(y) := \begin{cases} ||y||_{\beta}^{\sigma + \frac{\varepsilon}{q} - j_0}, & 0 < ||y||_{\beta} \leq 1, \\ 0, & ||y||_{\beta} \geq 1. \end{cases}$$

In view of (11) and (10), it follows

$$\begin{aligned} &||\tilde{f}||_{p,\Phi_\delta} ||\tilde{g}||_{q,\Psi} \\ &= \left\{ \int_{\left\{x \in \mathbf{R}_+^{i_0}; ||x||_{\alpha}^{\delta} \geq 1\right\}} ||x||_{\alpha}^{-i_0-\delta\varepsilon} dx \right\}^{\frac{1}{p}} \left\{ \int_{\left\{y \in \mathbf{R}_+^{j_0}; ||y||_{\beta} \leq 1\right\}} ||y||_{\beta}^{-j_0+\varepsilon} dy \right\}^{\frac{1}{q}} \\ &= \frac{1}{\varepsilon} \left\{ \frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)} \right\}^{\frac{1}{p}} \left\{ \frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)} \right\}^{\frac{1}{q}}. \end{aligned}$$

If there exists a constant $K \leq K(\sigma)$, such that (26) is valid when replacing $K(\sigma)$ by K, then in particular, by (16) and (17), we have

$$\begin{aligned}
\tilde{K}(\sigma) + o(1) &\le \varepsilon \tilde{I} \\
&= \varepsilon \int_{\mathbf{R}_+^{j_0}} \int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) \tilde{f}(x) \tilde{g}(y) dx dy \\
&< \varepsilon K ||\tilde{f}||_{p,\Phi_\delta} ||\tilde{g}||_{q,\Psi} \\
&= K \left\{ \frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)} \right\}^{\frac{1}{p}} \left\{ \frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)} \right\}^{\frac{1}{q}},
\end{aligned}$$

and then we find $K(\sigma) \le K(\varepsilon \to 0^+)$. Hence $K = K(\sigma)$ is the best possible constant factor of (26).

By the equivalency, we can prove that the constant factor $K(\sigma)$ in (27) is the best possible. Otherwise, we would reach a contradiction by (29) that the constant factor $K(\sigma)$ in (26) is not the best possible.

(ii) For $0 < p < 1$, or $p < 0$, by the same way, we still can obtain the equivalent reverses of (26) and (27). For $\varepsilon > 0$, we set $\tilde{f}(x), \tilde{g}(y)$ as the case of $p > 1$. If there exists a constant $K \ge K(\sigma)$, such that the reverse of (26) is valid when replacing $K(\sigma)$ by K, then in particular, by (16) and (19), we have

$$\begin{aligned}
\tilde{K}(\sigma) + o(1) &= \varepsilon \tilde{I} \\
&= \varepsilon \int_{\mathbf{R}_+^{j_0}} \int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) \tilde{f}(x) \tilde{g}(y) dx dy \\
&> \varepsilon K ||\tilde{f}||_{p,\Phi_\delta} ||\tilde{g}||_{q,\Psi} \\
&= K \left\{ \frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)} \right\}^{\frac{1}{p}} \left\{ \frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)} \right\}^{\frac{1}{q}},
\end{aligned}$$

and then we find $K(\sigma) \ge K(\varepsilon \to 0^+)$. Hence $K = K(\sigma)$ is the best possible constant factor of the reverse of (26). By the equivalency, we can prove that the constant factor $K(\sigma)$ in the reverse of (27) is the best possible. Otherwise, we would reach a contradiction by the reverse of (29) that the constant factor $K(\sigma)$ in the reverse of (26) is not the best possible. The theorem is proved.

In particular, for $\delta = 1$ in Theorem 1, we have

Corollary 1. *Suppose that* $\alpha, \beta > 0,\ \sigma \in \mathbf{R}, h(v) \ge 0$,

$$k(\sigma) = \int_0^\infty h(v) v^{\sigma-1} dv \in \mathbf{R}_+,$$

$$K(\sigma) = \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)} \right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)} \right]^{\frac{1}{q}} k(\sigma),$$

$p \in \mathbf{R}\backslash\{0,1\}$, $\frac{1}{p}+\frac{1}{q}=1$, $f(x)=f(x_1,\ldots,x_{i_0})\geq 0$, $g(y)=g(y_1,\ldots,y_{j_0})\geq 0$,

$$0<||f||_{p,\Phi_1}=\left\{\int_{\mathbf{R}_+^{i_0}}\Phi_1(x)f^p(x)dx\right\}^{\frac{1}{p}}<\infty,$$

$$0<||g||_{q,\Psi}=\left\{\int_{\mathbf{R}_+^{j_0}}\Psi(y)g^q(y)dy\right\}^{\frac{1}{q}}<\infty.$$

(i) *If $p>1$, then we have the following equivalent inequalities with the best possible constant factor $K(\sigma)$:*

$$I=\int_{\mathbf{R}_+^{j_0}}\int_{\mathbf{R}_+^{i_0}}h(||x||_\alpha||y||_\beta)f(x)g(y)dxdy<K(\sigma)||f||_{p,\Phi_1}||g||_{q,\Psi}, \quad (30)$$

$$J:=\left\{\int_{\mathbf{R}_+^{j_0}}||y||_\beta^{p\sigma-j_0}\left(\int_{\mathbf{R}_+^{i_0}}h(||x||_\alpha||y||_\beta)f(x)dx\right)^p dy\right\}^{\frac{1}{p}}$$
$$<K(\sigma)||f||_{p,\Phi_1}; \quad (31)$$

(ii) *if $0<p<1$, or $p<0$, there exists a constant $\delta_0>0$, such that for any $\tilde{\sigma}\in(\sigma-\delta_0,\sigma+\delta_0)$, $k(\tilde{\sigma})\in\mathbf{R}$, then we still have the equivalent reverses of (30) and (31) with the same best constant factor $K(\sigma)$.*

For $i_0=j_0=\alpha=\beta=1$ in Corollary 1, we have

Corollary 2. *Assuming that $\sigma\in\mathbf{R}, k(\sigma)\in\mathbf{R}_+, p\in\mathbf{R}\backslash\{0,1\}, \frac{1}{p}+\frac{1}{q}=1$, we set*

$$\varphi(x):=x^{p(1-\sigma)-1}, \psi(y):=y^{q(1-\sigma)-1}(x,y>0).$$

If $f(x)\geq 0$, $g(y)\geq 0$,

$$0<||f||_{p,\varphi}=\left\{\int_0^\infty\varphi(x)f^p(x)dx\right\}^{\frac{1}{p}}<\infty,$$

$$0<||g||_{q,\psi}=\left\{\int_0^\infty\psi(y)g^q(y)dy\right\}^{\frac{1}{q}}<\infty,$$

then (i) for $p>1$, we have the following equivalent inequalities with the best possible constant factor $k(\sigma)$:

$$\int_0^\infty\int_0^\infty h(xy)f(x)g(y)dxdy<k(\sigma)||f||_{p,\varphi}||g||_{q,\psi}, \quad (32)$$

$$\left\{\int_0^\infty y^{p\sigma-1}\left[\int_0^\infty h(xy)f(x)dx\right]^p dy\right\}^{\frac{1}{p}} < k(\sigma)||f||_{p,\varphi}; \quad (33)$$

(ii) for $0 < p < 1$*, or* $p < 0$*, there exists a constant* $\delta_0 > 0$*, such that for any* $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$*,* $k(\tilde{\sigma}) \in \mathbf{R}$*, we have the equivalent reverses of (32) and (33) with the same best constant factor.*

As the assumptions of Theorem 1, for $p > 1$, in view of $J < K(\sigma)||f||_{\Phi_\delta}$, we can give the following definition:

Definition 2. Define a multidimensional Hilbert-type integral operator

$$T : \mathbf{L}_{p,\Phi_\delta}(\mathbf{R}_+^{i_0}) \to \mathbf{L}_{p,\Psi^{1-p}}(\mathbf{R}_+^{j_0}) \quad (34)$$

as follows: For $f \in \mathbf{L}_{p,\Phi_\delta}(\mathbf{R}_+^{i_0})$, there exists a unique representation

$$Tf \in \mathbf{L}_{p,\Psi^{1-p}}(\mathbf{R}_+^{j_0}),$$

satisfying

$$(Tf)(y) := \int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) f(x) dx (y \in \mathbf{R}_+^{j_0}). \quad (35)$$

For $g \in \mathbf{L}_{q,\Psi}(\mathbf{R}_+^{j_0})$, we define the following formal inner product of Tf and g as follows:

$$(Tf, g) := \int_{\mathbf{R}_+^{j_0}} \int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) f(x) g(y) dx dy. \quad (36)$$

Then by Theorem 1, for $p > 1, 0 < ||f||_{p,\Phi_\delta}, ||g||_{q,\Psi} < \infty$, we have the following equivalent inequalities:

$$(Tf, g) < K(\sigma)||f||_{p,\Phi_\delta}||g||_{q,\Psi}, \quad (37)$$

$$||Tf||_{p,\Psi^{1-p}} < K(\sigma)||f||_{p,\Phi_\delta}. \quad (38)$$

It follows that T is bounded with

$$||T|| := \sup_{f(\neq\theta)\in\mathbf{L}_{p,\Phi_\delta}(\mathbf{R}_+^{i_0})} \frac{||Tf||_{p,\Psi^{1-p}}}{||f||_{p,\Phi_\delta}} \leq K(\sigma).$$

Since the constant factor $K(\sigma)$ in (38) is the best possible, we have

$$||T|| = K(\sigma) = \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)} \right]^{\frac{1}{p}} \times \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)} \right]^{\frac{1}{q}} k(\sigma). \tag{39}$$

4 A Corollary for $\delta = -1$

Corollary 3. *Suppose that $\alpha, \beta > 0$, $\mu, \sigma \in \mathbf{R}, \mu + \sigma = \lambda, k_\lambda(x, y) \geq 0$ is a homogeneous function of degree $-\lambda$,*

$$k_\lambda(\sigma) := \int_0^\infty k_\lambda(1, v) v^{\sigma-1} dv \in \mathbf{R}_+,$$

$$K_\lambda(\sigma) := \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)} \right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)} \right]^{\frac{1}{q}} k_\lambda(\sigma),$$

$p \in \mathbf{R}\backslash\{0, 1\}$, $\frac{1}{p} + \frac{1}{q} = 1, \Phi(x) := x^{p(i_0-\mu)-i_0}, F(x) = F(x_1, \ldots, x_{i_0}) \geq 0$, $g(y) = g(y_1, \ldots, y_{j_0}) \geq 0$,

$$0 < ||F||_{p,\Phi} = \left\{ \int_{\mathbf{R}_+^{i_0}} \Phi(x) F^p(x) dx \right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||g||_{q,\Psi} = \left\{ \int_{\mathbf{R}_+^{j_0}} \Psi(y) g^q(y) dy \right\}^{\frac{1}{q}} < \infty.$$

(i) If $p > 1$, then we have the following equivalent inequalities with the best possible constant factor $K(\sigma)$:

$$\int_{\mathbf{R}_+^{j_0}} \int_{\mathbf{R}_+^{i_0}} k_\lambda(||x||_\alpha, ||y||_\beta) F(x) g(y) dx dy < K_\lambda(\sigma) ||F||_{p,\Phi} ||g||_{q,\Psi}, \tag{40}$$

$$\left\{ \int_{\mathbf{R}_+^{j_0}} ||y||_\beta^{p\sigma-j_0} \left(\int_{\mathbf{R}_+^{i_0}} k_\lambda(||x||_\alpha, ||y||_\beta) F(x) dx \right)^p dy \right\}^{\frac{1}{p}} < K_\lambda(\sigma) ||F||_{p,\Phi}; \tag{41}$$

(ii) *if* $0 < p < 1$, *or* $p < 0$, *there exists a constant* $\delta_0 > 0$, *such that for any* $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, $k_\lambda(\tilde{\sigma}) \in \mathbf{R}$, *then we still have the equivalent reverses of (40) and (41) with the same best constant factor* $K_\lambda(\sigma)$.

In particular, for $i_0 = j_0 = \alpha = \beta = 1, \varphi_1(x) := x^{p(1-\mu)-1}$, if $F(x) \geq 0$, $g(y) \geq 0$,

$$0 < ||F||_{p,\varphi_1} = \left\{ \int_0^\infty \varphi_1(x) F^p(x) dx \right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||g||_{q,\psi} = \left\{ \int_0^\infty \psi(y) g^q(y) dy \right\}^{\frac{1}{q}} < \infty,$$

then (i) for $p > 1$, we have the following equivalent inequalities with the best possible constant factor $k_\lambda(\sigma)$:

$$\int_0^\infty \int_0^\infty k_\lambda(x, y) F(x) g(y) dx dy < k_\lambda(\sigma) ||F||_{p,\varphi_1} ||g||_{q,\psi}, \tag{42}$$

$$\left\{ \int_0^\infty y^{p\sigma - 1} \left[\int_0^\infty k_\lambda(x, y) F(x) dx \right]^p dy \right\}^{\frac{1}{p}} < k_\lambda(\sigma) ||F||_{p,\varphi_1}; \tag{43}$$

(ii) for $0 < p < 1$, or $p < 0$, there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, $k_\lambda(\tilde{\sigma}) \in \mathbf{R}$, we have the equivalent reverses of (42) and (43) with the same best constant factor $k_\lambda(\sigma)$.

Proof. For $\delta = -1$ in Theorem 1, setting $h(u) = k_\lambda(1, u)$ and $||x||_\alpha^\lambda f(x) = F(x)$, since $\mu = \lambda - \sigma$, by simplifications, we can obtain (40) and (41) (for $p > 1$). It is evident that (40) and (41) are equivalent with the same best constant factor $K_\lambda(\sigma)$. By the same way, we can show the cases in $0 < p < 1$ or $p < 0$. The corollary is proved.

Remark 1. Inequality (42), (43) is equivalent to (32), (33). In fact, Setting $x = \frac{1}{X}, h(u) = k_\lambda(1, u)$ in (32), (33), replacing $X^\lambda f(\frac{1}{X})$ by $F(X)$, by simplification, we obtain (42), (43). On the other hand, by (42), (43), we can deduce (32), (33).

5 Two Classes of Hardy-Type Inequalities

If $h(v) = 0 (v > 1)$, then

$$h(||x||_\alpha^\delta ||y||_\beta) = 0(||x||_\alpha^\delta > ||y||_\beta^{-1}),$$

by Theorem 1, we have the following first class of Hardy-type inequalities:

Corollary 4. *Suppose that* $\alpha, \beta > 0,\ \sigma \in \mathbf{R}, h(v) \geq 0,$

$$k_1(\sigma) := \int_0^1 h(v)v^{\sigma-1}dv \in \mathbf{R}_+,$$

$$H_1(\sigma) := \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{q}} k_1(\sigma),$$

$\delta \in \{-1, 1\}, p \in \mathbf{R}\backslash\{0, 1\},\ \frac{1}{p} + \frac{1}{q} = 1,\ f(x) = f(x_1, \ldots, x_{i_0}) \geq 0,\ g(y) = g(y_1, \ldots, y_{j_0}) \geq 0,$

$$0 < ||f||_{p,\Phi_\delta} = \left\{\int_{\mathbf{R}_+^{i_0}} \Phi_\delta(x) f^p(x)dx\right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||g||_{q,\Psi} = \left\{\int_{\mathbf{R}_+^{j_0}} \Psi(y) g^q(y)dy\right\}^{\frac{1}{q}} < \infty.$$

(i) If $p > 1$*, then we have the following equivalent inequalities with the best possible constant factor* $H_1(\sigma)$*:*

$$\int_{\mathbf{R}_+^{j_0}} \left[\int_{\left\{x \in \mathbf{R}_+^{i_0}; ||x||_\alpha^\delta \leq ||y||_\beta^{-1}\right\}} h(||x||_\alpha^\delta ||y||_\beta) f(x)dx\right] g(y)dy < H_1(\sigma)||f||_{p,\Phi_\delta}||g||_{q,\Psi}, \tag{44}$$

$$\left\{\int_{\mathbf{R}_+^{j_0}} ||y||_\beta^{p\sigma-j_0} \left(\int_{\left\{x \in \mathbf{R}_+^{i_0}; ||x||_\alpha^\delta \leq ||y||_\beta^{-1}\right\}} h\left(||x||_\alpha^\delta ||y||_\beta\right) f(x)dx\right)^p dy\right\}^{\frac{1}{p}} < H_1(\sigma)||f||_{p,\Phi_\delta}; \tag{45}$$

(ii) If $0 < p < 1$*, or* $p < 0$*, there exists a constant* $\delta_0 > 0$*, such that for any* $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$*,* $k_1(\tilde{\sigma}) \in \mathbf{R}$*, then we still have the equivalent reverses of (44) and (45) with the same best constant factor* $H_1(\sigma)$*.*

For $i_0 = j_0 = \alpha = \beta = 1, \delta = 1$ in Corollary 4, we have

Corollary 5. *Assuming that* $\sigma \in \mathbf{R}, k_1(\sigma) \in \mathbf{R}_+, p \in \mathbf{R}\backslash\{0, 1\}, \frac{1}{p} + \frac{1}{q} = 1,$ *we set*

$$\varphi(x) := x^{p(1-\sigma)-1}, \psi(y) := y^{q(1-\sigma)-1}(x, y > 0).$$

If $f(x) \geq 0,\ g(y) \geq 0,$

$$0 < ||f||_{p,\varphi} = \left\{ \int_0^\infty \varphi(x) f^p(x) dx \right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||g||_{q,\psi} = \left\{ \int_0^\infty \psi(y) g^q(y) dy \right\}^{\frac{1}{q}} < \infty,$$

then (i) for $p > 1$, we have the following equivalent inequalities with the best possible constant factor $k_1(\sigma)$:

$$\int_0^\infty \left(\int_0^{\frac{1}{y}} h(xy) f(x) dx \right) g(y) dy < k_1(\sigma) ||f||_{p,\varphi} ||g||_{q,\psi}, \quad (46)$$

$$\left\{ \int_0^\infty y^{p\sigma-1} \left[\int_0^{\frac{1}{y}} h(xy) f(x) dx \right]^p dy \right\}^{\frac{1}{p}} < k_1(\sigma) ||f||_{p,\varphi}; \quad (47)$$

(ii) for $0 < p < 1$, or $p < 0$, there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, $k_1(\tilde{\sigma}) \in \mathbf{R}$, we have the equivalent reverses of (46) and (47) with the same best constant factor $k_1(\sigma)$.

If $k_\lambda(x, y) = 0 (x < y)$, by (42) and (43), we have

Corollary 6. *Assuming that $\mu, \sigma \in \mathbf{R}, \mu + \sigma = \lambda$,*

$$k_\lambda^{(1)}(\sigma) := \int_0^1 k_\lambda(1, v) v^{\sigma-1} dv \in \mathbf{R}_+,$$

$p \in \mathbf{R} \backslash \{0, 1\}$, $\frac{1}{p} + \frac{1}{q} = 1$, $\varphi_1(x) := x^{p(1-\mu)-1}$, *if* $F(x) \geq 0$, $g(y) \geq 0$,

$$0 < ||F||_{p,\varphi_1} = \left\{ \int_0^\infty \varphi_1(x) F^p(x) dx \right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||g||_{q,\psi} = \left\{ \int_0^\infty \psi(y) g^q(y) dy \right\}^{\frac{1}{q}} < \infty,$$

then (i) for $p > 1$, we have the following equivalent inequalities with the best possible constant factor $k_\lambda^{(1)}(\sigma)$:

$$\int_0^\infty \left[\int_y^\infty k_\lambda(x, y) F(x) dx \right] g(y) dy < k_\lambda^{(1)}(\sigma) ||F||_{p,\varphi_1} ||g||_{q,\psi}, \quad (48)$$

$$\left\{ \int_0^\infty y^{p\sigma-1} \left[\int_y^\infty k_\lambda(x, y) F(x) dx \right]^p dy \right\}^{\frac{1}{p}} < k_\lambda^{(1)}(\sigma) ||F||_{p,\varphi_1}; \quad (49)$$

(ii) for $0 < p < 1$*, or* $p < 0$*, there exists a constant* $\delta_0 > 0$*, such that for any* $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$*,* $k_\lambda^{(1)}(\tilde{\sigma}) \in \mathbf{R}$*, we have the equivalent reverses of (48) and (49) with the same best constant factor* $k_\lambda^{(1)}(\sigma)$*.*

If $h(v) = 0 (0 < v < 1)$, then

$$h(||x||_\alpha^\delta ||y||_\beta) = 0(||x||_\alpha^\delta < ||y||_\beta^{-1}),$$

by Theorem 1, we have the following second class of Hardy-type inequalities:

Corollary 7. *Suppose that* $\alpha, \beta > 0,\ \sigma \in \mathbf{R}, h(v) \geq 0,$

$$k_2(\sigma) := \int_1^\infty h(v) v^{\sigma - 1} dv \in \mathbf{R}_+,$$

$$H_2(\sigma) := \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0 - 1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0 - 1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{q}} k_2(\sigma),$$

$\delta \in \{-1, 1\}, p \in \mathbf{R}\backslash\{0, 1\}, \frac{1}{p} + \frac{1}{q} = 1,\ f(x) = f(x_1, \ldots, x_{i_0}) \geq 0,\ g(y) = g(y_1, \ldots, y_{j_0}) \geq 0,$

$$0 < ||f||_{p,\Phi_\delta} = \left\{\int_{\mathbf{R}_+^{i_0}} \Phi_\delta(x) f^p(x) dx\right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||g||_{q,\Psi} = \left\{\int_{\mathbf{R}_+^{j_0}} \Psi(y) g^q(y) dy\right\}^{\frac{1}{q}} < \infty.$$

(i) If $p > 1$*, then we have the following equivalent inequalities with the best possible constant factor* $H_2(\sigma)$ *:*

$$\int_{\mathbf{R}_+^{j_0}} \left[\int_{\left\{x \in \mathbf{R}_+^{i_0}; ||x||_\alpha^\delta \geq ||y||_\beta^{-1}\right\}} h\left(||x||_\alpha^\delta ||y||_\beta\right) f(x) dx\right] g(y) dy < H_2(\sigma) ||f||_{p,\Phi_\delta} ||g||_{q,\Psi}, \tag{50}$$

$$\left\{\int_{\mathbf{R}_+^{j_0}} ||y||_\beta^{p\sigma - j_0} \left(\int_{\left\{x \in \mathbf{R}_+^{i_0}; ||x||_\alpha^\delta \geq ||y||_\beta^{-1}\right\}} h(||x||_\alpha^\delta ||y||_\beta) f(x) dx\right)^p dy\right\}^{\frac{1}{p}} < H_2(\sigma) ||f||_{p,\Phi_\delta}; \tag{51}$$

(ii) if $0 < p < 1$, *or* $p < 0$, *there exists a constant* $\delta_0 > 0$, *such that for any* $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, $k_2(\tilde{\sigma}) \in \mathbf{R}$, *then we still have the equivalent reverses of (50) and (51) with the same best constant factor* $H_2(\sigma)$.

For $i_0 = j_0 = \alpha = \beta = 1, \delta = 1$ in Corollary 7, we have

Corollary 8. *Assuming that* $\sigma \in \mathbf{R}, k_2(\sigma) \in \mathbf{R}_+, p \in \mathbf{R}\backslash\{0,1\}, \frac{1}{p} + \frac{1}{q} = 1$, *we set*

$$\varphi(x) = x^{p(1-\sigma)-1}, \psi(y) = y^{q(1-\sigma)-1}(x, y > 0).$$

If $f(x) \geq 0$, $g(y) \geq 0$,

$$0 < ||f||_{p,\varphi} = \left\{\int_0^\infty \varphi(x) f^p(x) dx\right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||g||_{q,\psi} = \left\{\int_0^\infty \psi(y) g^q(y) dy\right\}^{\frac{1}{q}} < \infty.$$

then (i) for $p > 1$, *we have the following equivalent inequalities with the best possible constant factor* $k_2(\sigma)$:

$$\int_0^\infty \left(\int_{\frac{1}{y}}^\infty h(xy) f(x) dx\right) g(y) dy < k_2(\sigma) ||f||_{p,\varphi} ||g||_{q,\psi}, \tag{52}$$

$$\left\{\int_0^\infty y^{p\sigma-1} \left[\int_{\frac{1}{y}}^\infty h(xy) f(x) dx\right]^p dy\right\}^{\frac{1}{p}} < k_2(\sigma) ||f||_{p,\varphi}; \tag{53}$$

(ii) for $0 < p < 1$, *or* $p < 0$, *there exists a constant* $\delta_0 > 0$, *such that for any* $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, $k_2(\tilde{\sigma}) \in \mathbf{R}$, *we have the equivalent reverses of (52) and (53) with the same best constant factor* $k_2(\sigma)$.

If $k_\lambda(x, y) = 0 (x > y)$, by (42) and (43), we have

Corollary 9. *Assuming that* $\mu, \sigma \in \mathbf{R}, \mu + \sigma = \lambda$,

$$k_\lambda^{(2)}(\sigma) := \int_1^\infty k_\lambda(1, v) v^{\sigma-1} dv \in \mathbf{R}_+,$$

$p \in \mathbf{R}\backslash\{0,1\}, \frac{1}{p} + \frac{1}{q} = 1$, $\varphi_1(x) := x^{p(1-\mu)-1}$, *if* $F(x) \geq 0$, $g(y) \geq 0$,

$$0 < ||F||_{p,\varphi_1} = \left\{\int_0^\infty \varphi_1(x) F^p(x) dx\right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||g||_{q,\psi} = \left\{\int_0^\infty \psi(y) g^q(y) dy\right\}^{\frac{1}{q}} < \infty,$$

then (i) for $p > 1$, we have the following equivalent inequalities with the best possible constant factor $k_\lambda^{(2)}(\sigma)$:

$$\int_0^\infty \left[\int_0^y k_\lambda(x,y)F(x)dx\right] g(y)dy < k_\lambda^{(2)}(\sigma)||F||_{p,\varphi_1}||g||_{q,\psi}, \tag{54}$$

$$\left\{\int_0^\infty y^{p\sigma-1}\left[\int_0^y k_\lambda(x,y)F(x)dx\right]^p dy\right\}^{\frac{1}{p}} < k_\lambda^{(2)}(\sigma)||F||_{p,\varphi_1}; \tag{55}$$

(ii) for $0 < p < 1$, or $p < 0$, there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, $k_\lambda^{(2)}(\tilde{\sigma}) \in \mathbf{R}$, we have the equivalent reverses of (54) and (55) with the same best constant factor $k_\lambda^{(2)}(\sigma)$.

6 Multidimensional Hilbert-Type Inequalities with Two Variables

Suppose that $u_i(s_i), u_i'(s_i) > 0, u_i(a_i^+) = 0, u_i(b_i^-) = \infty(-\infty \le a_i < b_i \le \infty, i = 1, \ldots, i_0)$, $u(s) = (u_1(s_1), \ldots, u_{i_0}(s_{i_0})), v_j(t_j), v_j'(t_j) > 0, v_j(c_j^+) = 0, v_j(d_j^-) = \infty \quad (-\infty \le c_j < d_j \le \infty, j = 1, \ldots, j_0)$,
$v(t) = (v_1(t_1), \ldots, v_{j_0}(t_{j_0}))$,

$$\tilde{\Phi}_\delta(s) := \frac{||u(s)||_\alpha^{p(i_0-\delta\sigma)-i_0}}{\left[\Pi_{i=1}^{i_0} u_i'(s_i)\right]^{p-1}}, \tilde{\Psi}(t) := \frac{||v(t)||_\alpha^{q(j_0-\sigma)-j_0}}{\left[\Pi_{j=1}^{j_0} v_j'(t_j)\right]^{q-1}}.$$

Setting $x = u(s)$, $y = v(t)$ in Theorem 1, for

$$F(s) := \Pi_{i=1}^{i_0} u_i'(s_i) f(u(s)), G(t) := \Pi_{j=1}^{j_0} v_j'(t_j) g(v(t)),$$

we have

Theorem 2. *Suppose that $\alpha, \beta > 0$, $\sigma \in \mathbf{R}, h(v) \ge 0$,*

$$k(\sigma) = \int_0^\infty h(v)v^{\sigma-1}dv \in \mathbf{R}_+,$$

$$K(\sigma) = \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{q}} k(\sigma),$$

$\delta \in \{-1, 1\}, p \in \mathbf{R}\backslash\{0, 1\}$, $\frac{1}{p} + \frac{1}{q} = 1$, $F(s) = F(s_1, \ldots, s_{i_0}) \ge 0$, $G(t) = G(t_1, \ldots, t_{j_0}) \ge 0$,

$$0 < ||F||_{p,\tilde{\Phi}_\delta} = \left\{ \int_{\{s \in \mathbf{R}^{i_0}; a_i < s_i < b_i\}} \tilde{\Phi}_\delta(s) F^p(s) ds \right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||G||_{q,\tilde{\Psi}} = \left\{ \int_{\{t \in \mathbf{R}^{j_0}; c_j < t_j < d_j\}} \tilde{\Psi}(t) G^q(t) dt \right\}^{\frac{1}{q}} < \infty.$$

(i) *If $p > 1$, then we have the following equivalent inequalities with the best possible constant factor $K(\sigma)$:*

$$\int_{\{t \in \mathbf{R}^{j_0}; c_j < t_j < d_j\}} \int_{\{s \in \mathbf{R}^{i_0}; a_i < s_i < b_i\}} h(||u(s)||_\alpha^\delta ||v(t)||_\beta) F(s) G(t) ds dt < K(\sigma) ||F||_{p,\tilde{\Phi}_\delta} ||g||_{q,\tilde{\Psi}}, \tag{56}$$

$$\left\{ \int_{\{t \in \mathbf{R}^{j_0}; c_j < t_j < d_j\}} ||v(t)||_\beta^{p\sigma - j_0} \Pi_{j=1}^{j_0} v_j'(t_j) \left(\int_{\{s \in \mathbf{R}^{i_0}; a_i < s_i < b_i\}} h(||u(s)||_\alpha^\delta ||v(t)||_\beta) F(s) ds \right)^p dt \right\}^{\frac{1}{p}} < K(\sigma) ||F||_{p,\tilde{\Phi}_\delta}; \tag{57}$$

(ii) *if $0 < p < 1$, or $p < 0$, there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, $k(\tilde{\sigma}) \in \mathbf{R}$, then we still have the equivalent reverses of (56) and (57) with the same best constant factor $K(\sigma)$.*

In particular, for $i_0 = j_0 = \alpha = \beta = 1$,

$$\tilde{\phi}_\delta(s) := \frac{(u(s))^{p(1-\delta\sigma)-1}}{[u'(s)]^{p-1}}, \tilde{\psi}(t) := \frac{(v(t))^{q(1-\sigma)-1}}{[v'(t)]^{q-1}},$$

$$0 < ||F||_{p,\tilde{\phi}_\delta} = \left\{ \int_a^b \tilde{\phi}_\delta(s) F^p(s) ds \right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||G||_{q,\tilde{\psi}} = \left\{ \int_c^d \tilde{\Psi}(t) G^q(t) dt \right\}^{\frac{1}{q}} < \infty,$$

(i) if $p > 1$, then we have the following equivalent inequalities with the best possible constant factor $k(\sigma)$:

$$\int_c^d \int_a^b h(u^\delta(s) v(t)) F(s) G(t) ds dt < k(\sigma) ||F||_{p,\tilde{\phi}_\delta} ||G||_{q,\tilde{\psi}}, \tag{58}$$

$$\left\{ \int_c^d (v(t))^{p\sigma-1} v'(t) \left(\int_a^b h(u^\delta(s)v(t)) F(s) ds \right)^p dt \right\}^{\frac{1}{p}} < k(\sigma) ||F||_{p,\tilde{\Phi}_\delta}; \tag{59}$$

(ii) if $0 < p < 1$, or $p < 0$, there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, $k(\tilde{\sigma}) \in \mathbf{R}$, then we still have the equivalent reverses of (58) and (59) with the same best constant factor $k(\sigma)$.

In particular, for $\gamma, \eta > 0, u_i(s_i) = s_i^\gamma, u_i'(s_i) = \gamma s_i^{\gamma-1}, u_i(0^+) = 0, u_i(\infty) = \infty (a_i = 0, b_i = \infty, i = 1, \ldots, i_0)$, $\hat{u}(s) = (s_1^\gamma, \ldots, s_{i_0}^\gamma), v_j(t_j) = t_j^\eta, v_j'(t_j) = \eta t_j^{\eta-1}, v_j(0^+) = 0, v_j(\infty) = \infty (c_j = 0, d_j = \infty, j = 1, \ldots, j_0)$, $\hat{v}(t) = (t_1^\eta, \ldots, t_{j_0}^\eta)$, and

$$\tilde{\Phi}_\delta(s) = \frac{1}{\gamma^{i_0(p-1)}} \hat{\Phi}_\delta(s), \hat{\Phi}_\delta(s) := \frac{||\hat{u}(s)||_\alpha^{p(i_0-\delta\sigma)-i_0}}{\left(\Pi_{i=1}^{i_0} s_i^{\gamma-1}\right)^{p-1}},$$

$$\tilde{\Psi}(t) = \frac{1}{\eta^{j_0(q-1)}} \hat{\Psi}(t), \hat{\Psi}(t) := \frac{||\hat{v}(t)||_\alpha^{q(j_0-\sigma)-j_0}}{\left(\Pi_{j=1}^{j_0} t_j^{\eta-1}\right)^{q-1}}$$

in Theorem 2, we have

Corollary 10. *Suppose that* $\alpha, \beta, \gamma, \eta > 0,\ \sigma \in \mathbf{R}, h(v) \geq 0$,

$$k(\sigma) = \int_0^\infty h(v) v^{\sigma-1} dv \in \mathbf{R}_+,$$

$$K(\sigma) = \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)} \right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)} \right]^{\frac{1}{q}} k(\sigma),$$

$\delta \in \{-1, 1\}, p \in \mathbf{R} \backslash \{0, 1\},\ \frac{1}{p} + \frac{1}{q} = 1,\ F(s) = F(s_1, \ldots, s_{i_0}) \geq 0,\ G(t) = G(t_1, \ldots, t_{j_0}) \geq 0$,

$$0 < ||F||_{p,\hat{\Phi}_\delta} = \left\{ \int_{\mathbf{R}_+^{i_0}} \hat{\Phi}_\delta(s) F^p(s) ds \right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||G||_{q,\hat{\Psi}} = \left\{ \int_{\mathbf{R}_+^{j_0}} \hat{\Psi}(t) G^q(t) dt \right\}^{\frac{1}{q}} < \infty.$$

(i) If $p > 1$, then we have the following equivalent inequalities with the best possible constant factor $\frac{1}{\gamma^{i_0/q}\eta^{j_0/p}}K(\sigma)$:

$$\begin{aligned}&\int_{\mathbf{R}_+^{j_0}}\int_{\mathbf{R}_+^{i_0}} h(||\hat{u}(s)||_\alpha^\delta ||\hat{v}(t)||_\beta)F(s)G(t)dsdt\\&< \frac{1}{\gamma^{i_0/q}\eta^{j_0/p}}K(\sigma)||F||_{p,\hat{\Phi}_\delta}||G||_{q,\hat{\Psi}},\end{aligned} \tag{60}$$

$$\begin{aligned}&\left\{\int_{\mathbf{R}_+^{j_0}} ||\hat{v}(t)||_\beta^{p\sigma-j_0}\Pi_{j=1}^{j_0}t_j^{\eta-1}\left(\int_{\mathbf{R}_+^{i_0}} h(||\hat{u}(s)||_\alpha^\delta ||\hat{v}(t)||_\beta)\right.\right.\\&\left.\left.F(s)ds\right)^p dt\right\}^{\frac{1}{p}} < \frac{1}{\gamma^{i_0/q}\eta^{j_0/p}}K(\sigma)||F||_{p,\hat{\Phi}_\delta};\end{aligned} \tag{61}$$

(ii) if $0 < p < 1$, or $p < 0$, there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, $k(\tilde{\sigma}) \in \mathbf{R}$, then we still have the equivalent reverses of (60) and (61) with the same best constant factor $\frac{1}{\gamma^{i_0/q}\eta^{j_0/p}}K(\sigma)$.

In particular, for $i_0 = j_0 = \alpha = \beta = 1$,

$$\hat{\phi}_\delta(s) := s^{p(1-\delta\gamma\sigma)-1}, \hat{\psi}(t) := t^{q(1-\eta\sigma)-1},$$

$$0 < ||F||_{p,\hat{\phi}_\delta} = \left\{\int_0^\infty \hat{\phi}_\delta(s)F^p(s)ds\right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||G||_{q,\hat{\psi}} = \left\{\int_0^\infty \hat{\psi}(t)G^q(t)dt\right\}^{\frac{1}{q}} < \infty,$$

(i) if $p > 1$, then we have the following equivalent inequalities with the best possible constant factor $\frac{1}{\gamma^{1/q}\eta^{1/p}}k(\sigma)$:

$$\int_0^\infty\int_0^\infty h(s^{\gamma\delta}t^\eta)F(s)G(t)dsdt < \frac{1}{\gamma^{1/q}\eta^{1/p}}k(\sigma)||F||_{p,\hat{\phi}_\delta}||G||_{q,\hat{\psi}}, \tag{62}$$

$$\left\{\int_0^\infty t^{p\eta\sigma-1}\left(\int_0^\infty h(s^{\gamma\delta}t^\eta)F(s)ds\right)^p dt\right\}^{\frac{1}{p}} < \frac{1}{\gamma^{1/q}\eta^{1/p}}k(\sigma)||F||_{p,\hat{\phi}_\delta}; \tag{63}$$

(ii) if $0 < p < 1$, or $p < 0$, there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, $k(\tilde{\sigma}) \in \mathbf{R}$, then we still have the equivalent reverses of (62) and (63) with the same best constant factor $\frac{1}{\gamma^{1/q}\eta^{1/p}}k(\sigma)$.

For $\delta = -1, h(u) = k_\lambda(1,u), ||u(s)||_\alpha^\lambda F(s) = f(s), \mu = \lambda - \sigma$ and

$$\tilde{\Phi}(s) := \frac{||u(s)||_\alpha^{p(i_0-\mu)-i_0}}{\left[\Pi_{i=1}^{i_0} u_i'(s_i)\right]^{p-1}}$$

in Theorem 2, by simplifications, we have

Corollary 11. *Suppose that* $\alpha, \beta > 0,\ \lambda, \mu, \sigma \in \mathbf{R}, \mu + \sigma = \lambda, k_\lambda(x,y)(\geq 0)$ *is a homogeneous function of degree* $-\lambda$ *in* $\mathbf{R}_+^2$*, with*

$$k_\lambda(\sigma) = \int_0^\infty k_\lambda(1,v)v^{\sigma-1}dv \in \mathbf{R}_+,$$

$$K_\lambda(\sigma) = \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{q}} k_\lambda(\sigma),$$

$p \in \mathbf{R}\backslash\{0,1\}$*,* $\frac{1}{p} + \frac{1}{q} = 1$*,* $f(s) = f(s_1, \ldots, s_{i_0}) \geq 0$*,* $G(t) = G(t_1, \ldots, t_{j_0}) \geq 0$*,*

$$0 < ||f||_{p,\tilde{\Phi}} = \left\{\int_{\{s\in\mathbf{R}^{i_0};a_i<s_i<b_i\}} \tilde{\Phi}(s) f^p(s)ds\right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||G||_{q,\tilde{\Psi}} = \left\{\int_{\{t\in\mathbf{R}^{j_0};c_j<t_j<d_j\}} \tilde{\Psi}(t) G^q(t)dt\right\}^{\frac{1}{q}} < \infty.$$

(i) If $p > 1$*, then we have the following equivalent inequalities with the best possible constant factor* $K(\sigma)$ *:*

$$\int_{\{t\in\mathbf{R}^{j_0};c_j<t_j<d_j\}} \int_{\{s\in\mathbf{R}^{i_0};a_i<s_i<b_i\}} k_\lambda(||u(s)||_\alpha, ||v(t)||_\beta) f(s)G(t)dsdt < K_\lambda(\sigma)||f||_{p,\tilde{\Phi}}||G||_{q,\tilde{\Psi}}, \tag{64}$$

$$\left\{\int_{\{t\in\mathbf{R}^{j_0};c_j<t_j<d_j\}} ||v(t)||_\beta^{p\sigma-j_0} \Pi_{j=1}^{j_0} v_j'(t_j) \left(\int_{\{s\in\mathbf{R}^{i_0};a_i<s_i<b_i\}} k_\lambda(||u(s)||_\alpha, ||v(t)||_\beta) \times f(s)ds\right)^p dt\right\}^{\frac{1}{p}} < K_\lambda(\sigma)||f||_{p,\tilde{\Phi}}; \tag{65}$$

(ii) if $0 < p < 1$*, or* $p < 0$*, there exists a constant* $\delta_0 > 0$*, such that for any* $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$*,* $k_\lambda(\tilde{\sigma}) \in \mathbf{R}$*, then we still have the equivalent reverses of (64) and (65) with the same best constant factor* $K_\lambda(\sigma)$*.*

In particular, for $i_0 = j_0 = \alpha = \beta = 1$,

$$\tilde{\phi}(s) := \frac{(u(s))^{p(1-\mu)-1}}{[u'(s)]^{p-1}}, \tilde{\psi}(t) = \frac{(v(t))^{q(1-\sigma)-1}}{[v'(t)]^{q-1}},$$

$$0 < ||f||_{p,\tilde{\phi}} = \left\{ \int_a^b \tilde{\phi}(s) f^p(s) ds \right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||G||_{q,\tilde{\psi}} = \left\{ \int_c^d \tilde{\Psi}(t) G^q(t) dt \right\}^{\frac{1}{q}} < \infty,$$

(i) if $p > 1$, then we have the following equivalent inequalities with the best possible constant factor $k_\lambda(\sigma)$:

$$\int_c^d \int_a^b k_\lambda(u(s), v(t)) f(s) G(t) ds dt < k_\lambda(\sigma) ||f||_{p,\tilde{\phi}} ||G||_{q,\tilde{\psi}}, \tag{66}$$

$$\left\{ \int_c^d (v(t))^{p\sigma-1} v'(t) \left(\int_a^b k_\lambda(u(s), v(t)) f(s) ds \right)^p dt \right\}^{\frac{1}{p}} < k_\lambda(\sigma) ||f||_{p,\tilde{\phi}}; \tag{67}$$

(ii) if $0 < p < 1$, or $p < 0$, there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, $k_\lambda(\tilde{\sigma}) \in \mathbf{R}$, then we still have the equivalent reverses of (66) and (67) with the same best constant factor $k_\lambda(\sigma)$.

In particular, for $u_i(s_i) = \ln s_i, u_i'(s_i) = s_i^{-1}, u_i(1^+) = 0, u_i(\infty) = \infty (a_i = 1, b_i = \infty, i = 1, \ldots, i_0)$, $U(s) = (\ln s_1, \ldots, \ln s_{i_0})$, $v_j(t_j) = \ln t_j, v_j'(t_j) = t_j^{-1}, v_j(1^+) = 0, v_j(\infty) = \infty (c_j = 1, d_j = \infty, j = 1, \ldots, j_0)$, $V(t) = (\ln t_1, \ldots, \ln t_{j_0})$, and

$$\tilde{\Phi}(s) = \hat{\Phi}(s) := \frac{||U(s)||_\alpha^{p(i_0-\mu)-i_0}}{\left(\Pi_{i=1}^{i_0} s_i\right)^{1-p}},$$

$$\tilde{\Psi}(t) = \hat{\Psi}(t) := \frac{||V(t)||_\alpha^{q(j_0-\sigma)-j_0}}{\left(\Pi_{j=1}^{j_0} t_j\right)^{1-q}}$$

in Corollary 10, we have

Corollary 12. *Suppose that* $\alpha, \beta > 0,\ \lambda, \mu, \sigma \in \mathbf{R}, \mu + \sigma = \lambda, k_\lambda(x, y) (\geq 0)$ *is a homogeneous function of degree* $-\lambda$ *in* $\mathbf{R}_+^2$*, with*

$$k_\lambda(\sigma) = \int_0^\infty k_\lambda(1, v) v^{\sigma-1} dv \in \mathbf{R}_+,$$

$$K_\lambda(\sigma) = \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{q}} k_\lambda(\sigma),$$

$p \in \mathbf{R}\backslash\{0, 1\}$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(s) = f(s_1, \ldots, s_{i_0}) \geq 0$, $G(t) = G(t_1, \ldots, t_{j_0}) \geq 0$,

$$0 < ||f||_{p,\hat{\Phi}} = \left\{\int_{\{s\in\mathbf{R}^{i_0};1<s_i<\infty\}} \hat{\Phi}(s) f^p(s) ds\right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||G||_{q,\hat{\Psi}} = \left\{\int_{\{t\in\mathbf{R}^{j_0};1<t_j<\infty\}} \hat{\Psi}(t) G^q(t) dt\right\}^{\frac{1}{q}} < \infty.$$

(i) *If* $p > 1$, *then we have the following equivalent inequalities with the best possible constant factor* $K(\sigma)$:

$$\int_{\{t\in\mathbf{R}^{j_0};1<t_j<\infty\}} \int_{\{s\in\mathbf{R}^{i_0};1<s_i<\infty\}} k_\lambda(||U(s)||_\alpha, ||V(t)||_\beta) f(s)G(t) ds dt < K_\lambda(\sigma)||f||_{p,\hat{\Phi}}||G||_{q,\hat{\Psi}}, \tag{68}$$

$$\left\{\int_{\{t\in\mathbf{R}^{j_0};1<t_j<\infty\}} ||V(t)||_\beta^{p\sigma-j_0} \Pi_{j=1}^{j_0} t_j^{-1} \left(\int_{\{s\in\mathbf{R}^{i_0};1<s_i<\infty\}} k_\lambda(||U(s)||_\alpha, ||V(t)||_\beta) \times f(s) ds\right)^p dt\right\}^{\frac{1}{p}} < K_\lambda(\sigma)||f||_{p,\hat{\Phi}}; \tag{69}$$

(ii) *if* $0 < p < 1$, *or* $p < 0$, *there exists a constant* $\delta_0 > 0$, *such that for any* $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, $k_\lambda(\tilde{\sigma}) \in \mathbf{R}$, *then we still have the equivalent reverses of (68) and (69) with the same best constant factor* $K_\lambda(\sigma)$.

In particular, for $i_0 = j_0 = \alpha = \beta = 1$,

$$\tilde{\phi}(s) = \hat{\phi}(s) := \frac{(\ln s)^{p(1-\mu)-1}}{s^{1-p}}, \tilde{\psi}(t) = \hat{\psi}(t) := \frac{(\ln t)^{q(1-\sigma)-1}}{t^{1-q}},$$

$$0 < ||f||_{p,\hat{\phi}} = \left\{\int_1^\infty \hat{\phi}(s) f^p(s) ds\right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||G||_{q,\hat{\psi}} = \left\{\int_1^\infty \hat{\psi}(t) G^q(t) dt\right\}^{\frac{1}{q}} < \infty,$$

(i) if $p > 1$, then we have the following equivalent inequalities with the best possible constant factor $k_\lambda(\sigma)$:

$$\int_1^\infty \int_1^\infty k_\lambda(\ln s, \ln t) f(s) G(t) ds dt < k_\lambda(\sigma) ||f||_{p,\hat{\phi}} ||G||_{q,\hat{\psi}}, \tag{70}$$

$$\left\{ \int_1^\infty (\ln t)^{p\sigma-1} \frac{1}{t} \left(\int_1^\infty k_\lambda(\ln s, \ln t) f(s) ds \right)^p dt \right\}^{\frac{1}{p}} < k_\lambda(\sigma) ||f||_{p,\hat{\phi}}; \tag{71}$$

(ii) if $0 < p < 1$, or $p < 0$, there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, $k_\lambda(\tilde{\sigma}) \in \mathbf{R}$, then we still have the equivalent reverses of (70) and (71) with the same best constant factor $k_\lambda(\sigma)$.

7 Some Particular Examples on the Norm

Example 1. For $h(v) = \frac{|\ln v|^\gamma}{(1+v)^\lambda} (\gamma \geq 0, \mu, \sigma > 0, \mu + \sigma = \lambda)$, we have

$$k(\sigma) = k_\gamma(\sigma) := \int_0^\infty \frac{|\ln v|^\gamma}{(1+v)^\lambda} v^{\sigma-1} dv.$$

Since $\frac{|\ln v|^\gamma}{(1+v)^{\lambda/2}} v^{\frac{\sigma}{2}} \to 0 (v \to 0^+ \text{ or } v \to \infty)$, there exists a constant number $L > 0$, such that

$$0 < \frac{|\ln v|^\gamma}{(1+v)^{\lambda/2}} v^{\frac{\sigma}{2}} \leq L (v \in \mathbf{R}_+).$$

Then it follows that

$$0 < k_\gamma(\sigma) \leq L \int_0^\infty \frac{v^{(\sigma/2)-1} dv}{(1+v)^{\lambda/2}} = LB\left(\frac{\sigma}{2}, \frac{\mu}{2}\right) < \infty,$$

and $k_\gamma(\sigma) \in \mathbf{R}_+$. We find

$$k_0(\sigma) = \int_0^\infty \frac{1}{(1+v)^\lambda} v^{\sigma-1} dv = B(\sigma, \mu). \tag{72}$$

For $\gamma \geq 0$, we obtain

$$\begin{aligned} k_\gamma(\sigma) &= \int_0^1 \frac{(-\ln v)^\gamma v^{\sigma-1}}{(1+v)^\lambda} dv + \int_1^\infty \frac{(\ln v)^\gamma v^{\sigma-1}}{(1+v)^\lambda} dv \\ &= \int_0^1 \frac{(-\ln v)^\gamma}{(1+v)^\lambda} \left(v^{\sigma-1} + v^{\mu-1}\right) dv \end{aligned}$$

$$= \int_0^1 (-\ln v)^\gamma \sum_{k=0}^{\infty} \binom{-\lambda}{k} \left(v^{k+\sigma-1} + v^{k+\mu-1}\right) dv$$
$$= \sum_{k=0}^{\infty} \binom{-\lambda}{k} \int_0^1 (-\ln v)^\gamma \left(v^{k+\sigma-1} + v^{k+\mu-1}\right) dv.$$

Setting $t = -\ln v$, we find

$$\begin{aligned} k_\gamma(\sigma) &= \sum_{k=0}^{\infty} \binom{-\lambda}{k} \int_0^\infty t^{(\gamma+1)-1} \left[e^{-t(k+\sigma)} + e^{-t(k+\mu)}\right] dt \\ &= \Gamma(\gamma+1) \sum_{k=0}^{\infty} \binom{-\lambda}{k} \left[\frac{1}{(k+\sigma)^{\gamma+1}} + \frac{1}{(k+\mu)^{\gamma+1}}\right]. \end{aligned} \tag{73}$$

In view of Theorem 1 and (39), we have

$$\begin{aligned} ||T|| = K_\gamma(\sigma) := \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \\ \times \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{,q}} k_\gamma(\sigma). \end{aligned} \tag{74}$$

Example 2. For $h(v) = \frac{|\ln v|^\gamma}{1+v^\lambda} (\gamma \geq 0, \mu, \sigma > 0, \mu + \sigma = \lambda)$, we have

$$k(\sigma) = l_\gamma(\sigma) := \int_0^\infty \frac{|\ln v|^\gamma}{1+v^\lambda} v^{\sigma-1} dv.$$

Since $\frac{|\ln v|^\gamma}{(1+v^\lambda)^{1/2}} v^{\frac{\sigma}{2}} \to 0 (v \to 0^+$ or $v \to \infty)$, there exists a constant number $L > 0$, such that

$$0 < \frac{|\ln v|^\gamma}{(1+v^\lambda)^{1/2}} v^{\frac{\sigma}{2}} \leq L (v \in \mathbf{R}_+).$$

Then it follows that

$$\begin{aligned} 0 < l_\gamma(\sigma) &\leq L \int_0^\infty \frac{v^{(\sigma/2)-1} dv}{(1+v^\lambda)^{1/2}} \\ &= \frac{L}{\lambda} \int_0^\infty \frac{u^{(\sigma/2\lambda)-1} dv}{(1+u)^{1/2}} = \frac{L}{\lambda} B\left(\frac{\sigma}{2\lambda}, \frac{\mu}{2\lambda}\right) < \infty, \end{aligned}$$

and $l_\gamma(\sigma) \in \mathbf{R}_+$. We find

$$l_0(\sigma) = \int_0^\infty \frac{1}{1+v^\lambda} v^{\sigma-1} dv = \frac{\pi}{\lambda \sin\left(\frac{\pi\sigma}{\lambda}\right)}. \tag{75}$$

For $\gamma \geq 0$, we obtain

$$\begin{aligned} l_\gamma(\sigma) &= \int_0^1 \frac{(-\ln v)^\gamma v^{\sigma-1}}{1+v^\lambda} dv + \int_1^\infty \frac{(\ln v)^\gamma v^{\sigma-1}}{1+v^\lambda} dv \\ &= \int_0^1 \frac{(-\ln v)^\gamma}{1+v^\lambda} \left(v^{\sigma-1} + v^{\mu-1}\right) dv \\ &= \int_0^1 (-\ln v)^\gamma \sum_{k=0}^\infty (-1)^k \left(v^{k\lambda+\sigma-1} + v^{k\lambda+\mu-1}\right) dv \\ &= \sum_{k=0}^\infty (-1)^k \int_0^1 (-\ln v)^\gamma \left(v^{k\lambda+\sigma-1} + v^{k\lambda+\mu-1}\right) dv. \end{aligned}$$

Setting $t = -\ln v$, we find

$$\begin{aligned} l_\gamma(\sigma) &= \sum_{k=0}^\infty (-1)^k \int_0^\infty t^{(\gamma+1)-1} \left[e^{-t(k\lambda+\sigma)} + e^{-t(k\lambda+\mu)}\right] dt \\ &= \Gamma(\gamma+1) \sum_{k=0}^\infty (-1)^k \left[\frac{1}{(k\lambda+\sigma)^{\gamma+1}} + \frac{1}{(k\lambda+\mu)^{\gamma+1}}\right]. \end{aligned} \tag{76}$$

In view of Theorem 1 and (39), we have

$$\begin{aligned} \|T\| = L_\gamma(\sigma) &:= \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \\ &\times \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]_\gamma^{\frac{1}{,q}} l_\gamma(\sigma). \end{aligned} \tag{77}$$

Example 3. For $h(v) = \frac{|\ln v|^\gamma}{(\max\{1,v\})^\lambda} (\gamma \geq 0, \mu, \sigma > 0, \mu + \sigma = \lambda)$, we have

$$\begin{aligned} k(\sigma) &= \int_0^\infty \frac{|\ln v|^\gamma}{(\max\{1,v\})^\lambda} v^{\sigma-1} dv \\ &= \int_0^1 (-\ln v)^\gamma v^{\sigma-1} dv + \int_1^\infty \frac{(\ln v)^\gamma}{v^\lambda} v^{\sigma-1} dv \\ &= \int_0^1 (-\ln v)^\gamma \left(v^{\sigma-1} + v^{\mu-1}\right) dv. \end{aligned}$$

Setting $t = -\ln v$, we find

$$k(\sigma) = \int_0^\infty t^\gamma \left[e^{-(\sigma-1)t} + e^{-(\mu-1)t}\right] e^{-t} dt$$
$$= \int_0^\infty t^{(\gamma+1)-1} \left(e^{-\sigma t} + e^{-\mu t}\right) dt$$
$$= \Gamma(\gamma+1)\left(\frac{1}{\sigma^{\gamma+1}} + \frac{1}{\mu^{\gamma+1}}\right) \in \mathbf{R}_+. \tag{78}$$

In view of Theorem 1 and (39), we have

$$||T|| = K(\sigma) = \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \times \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]_\gamma^{\frac{1}{,q}} \Gamma(\gamma+1)\left(\frac{1}{\sigma^{\gamma+1}} + \frac{1}{\mu^{\gamma+1}}\right). \tag{79}$$

Example 4. For $h(v) = \frac{|\ln v|^\gamma}{|1-v|^\lambda} (\gamma \geq 0, \mu, \sigma > 0, \mu + \sigma = \lambda < 1)$, we have

$$k(\sigma) = \tilde{k}_\gamma(\sigma) := \int_0^\infty \frac{|\ln v|^\gamma}{|1-v|^\lambda} v^{\sigma-1} dv.$$

We find

$$\tilde{k}_0(\sigma) = \int_0^\infty \frac{v^{\sigma-1}}{|1-v|^\lambda} dv$$
$$= \int_0^1 (1-v)^{-\lambda} v^{\sigma-1} dv + \int_1^\infty \frac{v^{\sigma-1}}{(v-1)^\lambda} dv$$
$$= \int_0^1 (1-v)^{(1-\lambda)-1} v^{\sigma-1} dv + \int_0^1 (1-u)^{(1-\lambda)-1} u^{\mu-1} du$$
$$= B(1-\lambda, \sigma) + B(1-\lambda, \mu). \tag{80}$$

For $\gamma \geq 0$, we obtain

$$\tilde{k}_\gamma(\sigma) = \int_0^1 \frac{(-\ln v)^\gamma v^{\sigma-1}}{(1-v)^\lambda} dv + \int_1^\infty \frac{(\ln v)^\gamma v^{\sigma-1}}{(v-1)^\lambda} dv$$
$$= \int_0^1 \frac{(-\ln v)^\gamma}{(1-v)^\lambda} \left(v^{\sigma-1} + v^{\mu-1}\right) dv.$$

Setting $0 < \delta < \min\{\mu, \sigma\}$, since $(-\ln v)^{\gamma} v^{\delta} \to 0 (v \to 0^{+})$, there exists a constant $L > 0$, such that $0 < (-\ln v)^{\gamma} v^{\delta} \leq L (v \in (0, 1])$, and then it follows

$$\begin{aligned} 0 < \tilde{k}_{\gamma}(\sigma) &\leq L \int_{0}^{1} \frac{v^{\sigma-\delta-1} + v^{\mu-\delta-1}}{(1-v)^{\lambda}} dv \\ &= L(B(1-\lambda, \sigma-\delta) + B(1-\lambda, \mu-\delta). \end{aligned}$$

Hence $\tilde{k}_{\gamma}(\sigma) \in \mathbf{R}_{+}$, and

$$\begin{aligned} \tilde{k}_{\gamma}(\sigma) &= \int_{0}^{1} (-\ln v)^{\gamma} \sum_{k=0}^{\infty} (-1)^{k} \binom{-\lambda}{k} \left(v^{k+\sigma-1} + v^{k+\mu-1}\right) dv \\ &= \sum_{k=0}^{\infty} (-1)^{k} \binom{-\lambda}{k} \int_{0}^{1} (-\ln v)^{\gamma} \left(v^{k+\sigma-1} + v^{k+\mu-1}\right) dv. \end{aligned}$$

Setting $t = -\ln v$, we find

$$\begin{aligned} \tilde{k}_{\gamma}(\sigma) &= \sum_{k=0}^{\infty} (-1)^{k} \binom{-\lambda}{k} \int_{0}^{\infty} t^{(\gamma+1)-1} \left[e^{-t(k+\sigma)} + e^{-t(k+\mu)}\right] dt \\ &= \Gamma(\gamma+1) \sum_{k=0}^{\infty} (-1)^{k} \binom{-\lambda}{k} \left[\frac{1}{(k+\sigma)^{\gamma+1}} + \frac{1}{(k+\mu)^{\gamma+1}}\right]. \end{aligned} \tag{81}$$

In view of Theorem 1 and (39), we have

$$\begin{aligned} ||T|| = \tilde{K}_{\gamma}(\sigma) &:= \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \\ &\times \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{q}} \tilde{k}_{\gamma}(\sigma) (\gamma \geq 0). \end{aligned} \tag{82}$$

Example 5. For $h(v) = \frac{|\ln v|^{\gamma}}{|v^{\lambda}-1|} (\gamma > 0, \mu, \sigma > 0, \mu + \sigma = \lambda)$, we have

$$k(\sigma) = \hat{k}_{\gamma}(\sigma) := \int_{0}^{\infty} \frac{|\ln v|^{\gamma}}{|v^{\lambda}-1|} v^{\sigma-1} dv.$$

We find

$$\hat{k}_1(\sigma) = \int_0^\infty \frac{(\ln v)v^{\sigma-1}}{v^\lambda - 1} dv \\ = \frac{1}{\lambda^2}\int_0^\infty \frac{(\ln u)u^{(\sigma/\lambda)-1}du}{u-1} = [\frac{\pi}{\lambda \sin\left(\frac{\pi\sigma}{\lambda}\right)}]^2. \tag{83}$$

For $\gamma > 0$, we obtain

$$\hat{k}_\gamma(\sigma) = \int_0^1 \frac{(-\ln v)^\gamma v^{\sigma-1}}{1-v^\lambda} dv + \int_1^\infty \frac{(\ln v)^\gamma v^{\sigma-1}}{v^\lambda - 1} dv \\ = \int_0^1 \frac{(-\ln v)^\gamma}{1-v^\lambda}\left(v^{\sigma-1} + v^{\mu-1}\right) dv$$

$$= \int_0^1 (-\ln v)^\gamma \sum_{k=0}^\infty \left(v^{k\lambda+\sigma-1} + v^{k\lambda+\mu-1}\right) dv \\ = \sum_{k=0}^\infty \int_0^1 (-\ln v)^\gamma \left(v^{k\lambda+\sigma-1} + v^{k\lambda+\mu-1}\right) dv.$$

Setting $t = -\ln v$, we find

$$\hat{k}_\gamma(\sigma) = \sum_{k=0}^\infty \int_0^\infty t^{(\gamma+1)-1}\left[e^{-t(k\lambda+\sigma)} + e^{-t(k\lambda+\mu)}\right] dt \\ = \Gamma(\gamma+1)\sum_{k=0}^\infty \left[\frac{1}{(k\lambda+\sigma)^{\gamma+1}} + \frac{1}{(k\lambda+\mu)^{\gamma+1}}\right] \in \mathbf{R}_+. \tag{84}$$

In view of Theorem 1 and (39), we have

$$||T|| = \hat{K}_\gamma(\sigma) := \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \\ \times \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{,q}} \hat{k}_\gamma(\sigma). \tag{85}$$

Lemma 6. *If* $\mathbf{C}$ *is the set of complex numbers and* $\mathbf{C}_\infty = \mathbf{C} \cup \{\infty\}$, $z_k \in \mathbf{C}\backslash\{z|Rez \geq 0, Imz = 0\}(k = 1, 2, \ldots, n)$ *are different points, the function* $f(z)$ *is analytic in* $\mathbf{C}_\infty$ *except for* $z_i(i = 1, 2, \ldots, n)$*, and* $z = \infty$ *is a zero point of* $f(z)$ *whose order is not less than 1, then for* $\alpha \in \mathbf{R}$*, we have*

$$\int_0^\infty f(x)x^{\alpha-1}dx = \frac{2\pi i}{1-e^{2\pi\alpha i}}\sum_{k=1}^{n} Res[f(z)z^{\alpha-1}, z_k], \tag{86}$$

where $0 < Im \ln z = \arg z < 2\pi$. *In particular, if* $z_k(k = 1, \ldots, n)$ *are all poles of order 1, setting* $\varphi_k(z) = (z - z_k)f(z)(\varphi_k(z_k) \neq 0)$, *then*

$$\int_0^\infty f(x)x^{\alpha-1}dx = \frac{\pi}{\sin\pi\alpha}\sum_{k=1}^{n}(-z_k)^{\alpha-1}\varphi_k(z_k). \tag{87}$$

Proof. By Pan et al. [22, p. 118], we have (86). We find

$$\begin{aligned} 1-e^{2\pi\alpha i} &= 1-\cos 2\pi\alpha - i\sin 2\pi\alpha \\ &= -2i\sin\pi\alpha(\cos\pi\alpha + i\sin\pi\alpha) \\ &= -2ie^{i\pi\alpha}\sin\pi\alpha. \end{aligned}$$

In particular, since $f(z)z^{\alpha-1} = \frac{1}{z-z_k}(\varphi_k(z)z^{\alpha-1})$, it is obvious that

$$Res[f(z)z^{\alpha-1}, -a_k] = {z_k}^{\alpha-1}\varphi_k(z_k) = -e^{i\pi\alpha}(-z_k)^{\alpha-1}\varphi_k(z_k).$$

Then by (86), we obtain (87). The lemma is proved.

Example 6. For $s \in \mathbf{N}$, $0 < a_1 < \cdots < a_s$, we set

$$h(v) = \frac{1}{\prod_{k=1}^{s}(v^{\lambda/s} + a_k)}(0 < \sigma < \lambda)$$

By (87), setting $u = v^{\lambda/s}$, we find

$$\begin{aligned} k(\sigma) = k_s(\sigma) &:= \int_0^\infty \frac{1}{\prod_{k=1}^{s}\left(v^{\lambda/s} + a_k\right)}v^{\sigma-1}dv \\ &= \frac{s}{\lambda}\int_0^\infty \frac{1}{\prod_{k=1}^{s}(u+a_k)}u^{\frac{s\sigma}{\lambda}-1}du \end{aligned}$$

$$= \frac{\pi s}{\lambda\sin\left(\frac{\pi s\sigma}{\lambda}\right)}\sum_{k=1}^{s} a_k^{\frac{s\sigma}{\lambda}-1}\prod_{j=1(j\neq k)}^{s}\frac{1}{a_j - a_k} \in \mathbf{R}_+. \tag{88}$$

In view of Theorem 1 and (39), we have

$$||T|| = K_s(\sigma) := \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \times \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{,q}} k_s(\sigma). \tag{89}$$

Example 7. For $c > 0, 0 < \gamma < \pi$, We set

$$h(v) = \frac{1}{v^\lambda + \sqrt{c}v^{\lambda/2}\cos\gamma + \frac{c}{4}}(0 < \sigma < \lambda).$$

Putting $z_1 = -\frac{\sqrt{c}}{2}e^{i\gamma}, z_2 = -\frac{\sqrt{c}}{2}e^{-i\gamma}$, by (87), it follows

$$\begin{aligned} k(\sigma) = c_\gamma(\sigma) &:= \int_0^\infty \frac{v^{\sigma-1}}{v^\lambda + \sqrt{c}v^{\lambda/2}\cos\gamma + \frac{c}{4}}dv \\ &= \frac{2}{\lambda}\int_0^\infty \frac{u^{\frac{2\sigma}{\lambda}-1}}{u^2 + \sqrt{c}u\cos\gamma + \frac{c}{4}}du \\ &= \frac{2}{\lambda}\int_0^\infty \frac{u^{\frac{2\sigma}{\lambda}-1}}{(u-z_1)(u-z_2)}du \\ &= \frac{2\pi}{\lambda\sin\left(\frac{2\pi\sigma}{\lambda}\right)}\left[\left(\frac{\sqrt{c}}{2}e^{i\gamma}\right)^{\frac{2\sigma}{\lambda}-1}\frac{\sqrt{c}}{2(e^{-i\gamma}-e^{i\gamma})}\right. \\ &\quad \left. + \left(\frac{\sqrt{c}}{2}e^{-i\gamma}\right)^{\frac{2\sigma}{\lambda}-1}\frac{\sqrt{c}}{2(e^{i\gamma}-e^{-i\gamma})}\right] \\ &= \left(\frac{\sqrt{c}}{2}\right)^{\frac{2\sigma}{\lambda}}\frac{2\pi\sin\gamma\left(1-\frac{2\sigma}{\lambda}\right)}{\lambda\sin\gamma\sin\left(\frac{2\pi\sigma}{\lambda}\right)} \in \mathbf{R}_+. \end{aligned} \tag{90}$$

In view of Theorem 1 and (39), we have

$$||T|| = C_\gamma(\sigma) := \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \times \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{,q}} c_\gamma(\sigma). \tag{91}$$

Example 8. We set

$$h(v) = \frac{(\min\{v,1\})^{\eta}}{(\max\{v,1\})^{\lambda+\eta}}(\eta > -\min\{\sigma,\mu\}, \sigma+\mu=\lambda).$$

Then we find

$$\begin{aligned} k(\sigma) &= \int_0^{\infty} \frac{(\min\{v,1\})^{\eta} v^{\sigma-1}}{(\max\{v,1\})^{\lambda+\eta}} dv = \int_0^1 v^{\eta+\sigma-1} dv + \int_1^{\infty} \frac{v^{\sigma-1} dv}{v^{\lambda+\eta}} \\ &= \frac{\lambda+2\eta}{(\sigma+\eta)(\mu+\eta)} \in \mathbf{R}_+. \end{aligned} \tag{92}$$

In view of Theorem 1 and (39), we have

$$\begin{aligned} ||T|| = K_{\eta}(\sigma) &:= \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \\ &\times \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{,q}} \frac{\lambda+2\eta}{(\sigma+\eta)(\mu+\eta)}. \end{aligned} \tag{93}$$

Example 9. We set

$$h(v) = \ln\left(\frac{b+v^{\gamma}}{a+v^{\gamma}}\right)(0 \le a < b, 0 < \sigma < \gamma).$$

We find

$$\begin{aligned} k(\sigma) &= \int_0^{\infty} \ln\left(\frac{b+v^{\gamma}}{a+v^{\gamma}}\right) v^{\sigma-1} dv \\ &= \frac{1}{\sigma}\int_0^{\infty} \ln\left(\frac{b+v^{\gamma}}{a+v^{\gamma}}\right) dv^{\sigma} \\ &= \frac{1}{\sigma}\Bigg[v^{\sigma}\ln\left(\frac{b+v^{\gamma}}{a+v^{\gamma}}\right)|_0^{\infty} \\ &\qquad + \gamma\int_0^{\infty}\left(\frac{1}{a+v^{\gamma}} - \frac{1}{b+v^{\gamma}}\right) v^{\sigma+\gamma-1} dv\Bigg] \\ &= \frac{b-a}{\sigma}\int_0^{\infty} \frac{u^{\left(1+\frac{\sigma}{\gamma}\right)-1}}{(u+a)(u+b)} du. \end{aligned}$$

For $a > 0$, by (87), we have

$$k(\sigma) = \frac{(b-a)\pi}{\sigma \sin \pi\left(1+\frac{\sigma}{\gamma}\right)}\left(\frac{a^{\frac{\sigma}{\gamma}}}{b-a} + \frac{b^{\frac{\sigma}{\gamma}}}{-b+a}\right)$$
$$= \frac{\left(b^{\frac{\sigma}{\gamma}} - a^{\frac{\sigma}{\gamma}}\right)\pi}{\sigma \sin\left(\frac{\pi\sigma}{\gamma}\right)} \in \mathbf{R}_+. \tag{94}$$

By using the simple way, we still can obtain (94) for $a = 0$.

In view of Theorem 1 and (39), we have

$$||T|| = K(\sigma) := \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \times \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{q}} \frac{\left(b^{\frac{\sigma}{\gamma}} - a^{\frac{\sigma}{\gamma}}\right)\pi}{\sigma \sin\left(\frac{\pi\sigma}{\gamma}\right)}. \tag{95}$$

Example 10. We set

$$h(v) = e^{-\rho v^{\gamma}} (\rho, \gamma, \sigma > 0).$$

Setting $u = \rho v^{\gamma}$, we find

$$k(\sigma) = \int_0^{\infty} e^{-\rho v^{\gamma}} v^{\sigma-1} dv = \frac{1}{\gamma e^{\sigma/\gamma}} \int_0^{\infty} e^{-u} u^{\frac{\sigma}{\gamma}-1} du$$
$$= \frac{1}{\gamma\rho^{\sigma/\gamma}} \Gamma\left(\frac{\sigma}{\gamma}\right) \in \mathbf{R}_+. \tag{96}$$

In view of Theorem 1 and (39), we have

$$||T|| = K(\sigma) := \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \times \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{q}} \frac{1}{\gamma\rho^{\sigma/\gamma}} \Gamma\left(\frac{\sigma}{\gamma}\right). \tag{97}$$

Example 11. We set

$$h(v) = \arctan \rho v^{-\gamma} (\rho > 0, 0 < \sigma < \gamma).$$

We find

$$\begin{aligned} k(\sigma) &= \int_0^\infty v^{\sigma-1}(\arctan \rho v^{-\gamma})dv \\ &= \frac{1}{\sigma}\int_0^\infty (\arctan \rho v^{-\gamma})dv^\sigma \\ &= \frac{1}{\sigma}\left[(\arctan \rho v^{-\gamma})v^\sigma|_0^\infty + \int_0^\infty \frac{\gamma\rho v^{\sigma-\gamma-1}}{1+(\rho v^{-\gamma})^2}dv\right] \\ &= \frac{\rho^{\frac{\sigma}{\gamma}}}{2\sigma}\int_0^\infty \frac{1}{1+u}u^{\left(\frac{1}{2}-\frac{\sigma}{2\gamma}\right)-1}du \\ &= \frac{\rho^{\frac{\sigma}{\gamma}}}{2\sigma}\frac{\pi}{\sin\pi\left(\frac{1}{2}-\frac{\sigma}{2\gamma}\right)} = \frac{\rho^{\frac{\sigma}{\gamma}}\pi}{2\sigma\cos\left(\frac{\pi\sigma}{2\gamma}\right)} \in \mathbf{R}_+, \end{aligned} \tag{98}$$

In view of Theorem 1 and (39), we have

$$\begin{aligned} ||T|| = K(\sigma) &:= \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \\ &\times \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{q}} \frac{\rho^{\frac{\sigma}{\gamma}}\pi}{2\sigma\cos\pi\left(\frac{\sigma}{2\gamma}\right)}. \end{aligned} \tag{99}$$

Example 12. We set

$$h(v) = \csc h(\rho v^\gamma) = \frac{2}{e^{\rho v^\gamma} - e^{-\rho v^\gamma}}(\rho > 0, \sigma > \gamma > 0),$$

where $\csc h(u) = \frac{2}{e^u - e^{-u}}$ is hyperbolic cosecant function [23]. We find

$$\begin{aligned} k(\sigma) = a_\gamma(\sigma) &:= \int_0^\infty v^{\sigma-1}\csc h(\rho v^\gamma)dv \\ &= \int_0^\infty \frac{2v^{\sigma-1}}{e^{\rho v^\gamma} - e^{-\rho v^\gamma}}dv \\ &= \int_0^\infty \frac{2v^{\sigma-1}e^{-\rho v^\gamma}dv}{1-e^{-2\rho v^\gamma}} = 2\int_0^\infty v^{\sigma-1}\sum_{k=0}^\infty e^{-(2k+1)\rho v^\gamma}dv \\ &= 2\sum_{k=0}^\infty \int_0^\infty v^{\sigma-1}e^{-(2k+1)\rho v^\gamma}dv. \end{aligned}$$

Setting $u = (2k+1)\rho v^{\gamma}$, we obtain

$$\begin{aligned} a_{\gamma}(\sigma) &= \frac{2}{\gamma\rho^{\sigma/\gamma}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{\sigma/\gamma}} \int_0^{\infty} u^{\frac{\sigma}{\gamma}-1} e^{-u} du \\ &= \frac{2}{\gamma\rho^{\sigma/\gamma}} \Gamma\left(\frac{\sigma}{\gamma}\right) \left[\sum_{k=1}^{\infty} \frac{1}{k^{\sigma/\gamma}} - \sum_{k=1}^{\infty} \frac{1}{(2k)^{\sigma/\gamma}}\right] \\ &= \frac{2}{\gamma\rho^{\sigma/\gamma}} \Gamma\left(\frac{\sigma}{\gamma}\right) \left(1 - \frac{1}{2^{\sigma/\gamma}}\right) \zeta\left(\frac{\sigma}{\gamma}\right) \in \mathbf{R}_{+}, \end{aligned} \tag{100}$$

where, $\zeta\left(\frac{\sigma}{\gamma}\right) = \sum_{k=1}^{\infty} \frac{1}{k^{\sigma/\gamma}} \left(\frac{\sigma}{\gamma} > 1\right)$ ($\zeta(\cdot)$ is the Riemann's zeta function [24]).

In view of Theorem 1 and (39), we have

$$\begin{aligned} ||T|| = A_{\gamma}(\sigma) &:= \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \\ &\times \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{q}} a_{\gamma}(\sigma). \end{aligned} \tag{101}$$

Example 13. We set

$$h(v) = \sec h(\rho v^{\gamma}) = \frac{2}{e^{\rho v^{\gamma}} + e^{-\rho v^{\gamma}}} (\rho, \sigma, \gamma > 0),$$

where $\sec h(u) = \frac{2}{e^u + e^{-u}}$ is hyperbolic secant function. We find

$$\begin{aligned} k(\sigma) = b_{\gamma}(\sigma) &:= \int_0^{\infty} v^{\sigma-1} \sec h(\rho v^{\gamma}) dv \\ &= \int_0^{\infty} \frac{2v^{\sigma-1} dv}{e^{\rho v^{\gamma}} + e^{-\rho v^{\gamma}}} = \int_0^{\infty} \frac{2v^{\sigma-1} e^{-\rho v^{\gamma}} dv}{1 + e^{-2\rho v^{\gamma}}} \\ &= 2\int_0^{\infty} v^{\sigma-1} \sum_{k=0}^{\infty} (-1)^k e^{-(2k+1)\rho v^{\gamma}} dv \\ &= 2\sum_{k=0}^{\infty} (-1)^k \int_0^{\infty} v^{\sigma-1} e^{-(2k+1)\rho v^{\gamma}} dv. \end{aligned}$$

Setting $u = (2k+1)\rho v^{\gamma}$, we obtain

$$\begin{aligned} b_{\gamma}(\sigma) &= \frac{2}{\gamma\rho^{\sigma/\gamma}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{\sigma/\gamma}} \int_0^{\infty} u^{\frac{\sigma}{\gamma}-1} e^{-u} du \\ &= \frac{1}{\gamma\rho^{\sigma/\gamma} 2^{(\sigma/\gamma)-1}} \Gamma\left(\frac{\sigma}{\gamma}\right) \varsigma\left(\frac{\sigma}{\gamma}\right) \in \mathbf{R}_+, \end{aligned} \tag{102}$$

where

$$\varsigma\left(\frac{\sigma}{\gamma}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{\sigma/\gamma}} \left(\frac{\sigma}{\gamma} > 0\right).$$

In view of Theorem 1 and (39), we have

$$\begin{aligned} ||T|| = B_{\gamma}(\sigma) &:= \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \\ &\times \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{q}} b_{\gamma}(\sigma). \end{aligned} \tag{103}$$

Example 14. We set

$$\begin{aligned} h(v) &= \coth h(\rho v^{\gamma}) - 1 = \frac{e^{\rho v^{\gamma}} + e^{-\rho v^{\gamma}}}{e^{\rho v^{\gamma}} - e^{-\rho v^{\gamma}}} - 1 \\ &= \frac{2e^{-\rho v^{\gamma}}}{e^{\rho v^{\gamma}} - e^{-\rho v^{\gamma}}} (\rho > 0, \sigma > \gamma > 0), \end{aligned}$$

where $\coth h(u) = \frac{e^u + e^{-u}}{e^u - e^{-u}}$ is hyperbolic cotangent function. We find

$$\begin{aligned} k(\sigma) = c_{\gamma}(\sigma) &:= \int_0^{\infty} v^{\sigma-1} (\coth h(\rho v^{\gamma}) - 1) dv \\ &= \int_0^{\infty} \frac{2e^{-\rho v^{\gamma}} v^{\sigma-1}}{e^{\rho v^{\gamma}} - e^{-\rho v^{\gamma}}} dv = \int_0^{\infty} \frac{2e^{-2\rho v^{\gamma}} v^{\sigma-1}}{1 - e^{-2\rho v^{\gamma}}} dv \\ &= 2\int_0^{\infty} v^{\sigma-1} \sum_{k=0}^{\infty} e^{-2(k+1)\rho v^{\gamma}} dv \\ &= 2\sum_{k=1}^{\infty} \int_0^{\infty} v^{\sigma-1} e^{-2k\rho v^{\gamma}} dv. \end{aligned}$$

Setting $u = 2k\rho v^\gamma$, we obtain

$$c_\gamma(\sigma) = \frac{2}{\gamma\rho^{\sigma/\gamma}} \sum_{k=1}^{\infty} \frac{1}{(2k)^{\sigma/\gamma}} \int_0^\infty u^{\frac{\sigma}{\gamma}-1} e^{-u} du$$

$$= \frac{1}{\gamma\rho^{\sigma/\gamma} 2^{(\sigma/\gamma)-1}} \Gamma\left(\frac{\sigma}{\gamma}\right) \zeta\left(\frac{\sigma}{\gamma}\right) \in \mathbf{R}_+. \tag{104}$$

In view of Theorem 1 and (39), we have

$$||T|| = C_\gamma(\sigma) := \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \times \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{q}} c_\gamma(\sigma). \tag{105}$$

Example 15. We set

$$h(v) = 1 - \tan h(\rho v^\gamma) = 1 - \frac{e^{\rho v^\gamma} - e^{-\rho v^\gamma}}{e^{\rho v^\gamma} + e^{-\rho v^\gamma}}$$

$$= \frac{2e^{-\rho v^\gamma}}{e^{\rho v^\gamma} + e^{-\rho v^\gamma}} (\rho, \sigma, \gamma > 0),$$

where $\tan h(u) = \frac{e^u + e^{-u}}{e^u - e^{-u}}$ is hyperbolic tangent function. We find

$$k(\sigma) = d_\gamma(\sigma) := \int_0^\infty v^{\sigma-1}(1 - \tan h(\rho v^\gamma)) dv$$

$$= \int_0^\infty \frac{2e^{-\rho v^\gamma} v^{\sigma-1}}{e^{\rho v^\gamma} + e^{-\rho v^\gamma}} dv = \int_0^\infty \frac{2e^{-2\rho v^\gamma} v^{\sigma-1}}{1 + e^{-2\rho v^\gamma}} dv$$

$$= 2\int_0^\infty v^{\sigma-1} \sum_{k=0}^{\infty} (-1)^k e^{-2(k+1)\rho v^\gamma} dv$$

$$= 2\sum_{k=1}^{\infty} (-1)^{k-1} \int_0^\infty v^{\sigma-1} e^{-2k\rho v^\gamma} dv.$$

Setting $u = 2k\rho v^\gamma$, we obtain

$$d_\gamma(\sigma) = \frac{2}{\gamma\rho^{\sigma/\gamma}} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k)^{\sigma/\gamma}} \int_0^\infty u^{\frac{\sigma}{\gamma}-1} e^{-u} du$$

$$= \frac{1}{\gamma\rho^{\sigma/\gamma} 2^{(\sigma/\gamma)-1}} \Gamma\left(\frac{\sigma}{\gamma}\right) \xi\left(\frac{\sigma}{\gamma}\right) \in \mathbf{R}_+, \tag{106}$$

where, $\xi(\frac{\sigma}{\gamma}) := \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^{\sigma/\gamma}}$.

In view of Theorem 1 and (39), we have

$$||T|| = D_{\gamma}(\sigma) := \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)} \right]^{\frac{1}{p}} \times \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)} \right]^{\frac{1}{q}} d_{\gamma}(\sigma). \quad (107)$$

Note. The following references [24–31] provide an extensive theory and applications of Analytic Number Theory relating to Riemann's zeta function that will provide a source study for further research on Hilbert-type inequalities.

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