On Some Integral Operators

Khalida Inayat Noor

Abstract Let $P(n,\beta)$, $\quad 0 \leq \beta < 1$, be the class of functions $p : p(z) = 1 + c_n z^n + c_{n+1}z^{n+1} + \ldots$ analytic in the unit disc E such that $Re\{p(z)\} > \beta$. The class $P_k(n,\beta)$, $\quad k \geq 2$ is defined as follows: An analytic function $p \in P_k(n,\beta), k \geq 2$, $0 \leq \beta < 1$ if and only if there exist $p_1, p_2 \in P(n,\beta)$ such that

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z).$$

In this paper, we discuss some integral operators for certain classes of analytic functions defined in E and related with the class $P_k(n,\beta)$.

Keywords Analytic functions • Integral operators • Convolution • Libera operators

1 Introduction

Let $\mathscr{A}(n)$ denote the class of functions f of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \quad (n = N = \{1, 2, 3, \ldots,\}), \tag{1}$$

analytic in the unit disc $E = \{z : |z| < 1\}$. Let $P(n,\beta)$ be the class of functions $h(z)$ of the form

$$h(z) = 1 + c_n z^n + c_{n+1}z^{n+1} + \ldots, \tag{2}$$

K.I. Noor (✉)
COMSATS Institute of Information and Technology, Park Road, Islamabad, Pakistan
e-mail: khalidanoor@hotmail.com

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which are analytic in E and satisfy $Re\{h(z)\} > \beta, 0 \leq \beta < 1, z \in$ We note that $P(1,0) \equiv P$ is the class of functions with positive real part.

Let $P_k(n,\beta), k \geq 2, 0 \leq \beta < 1$, be the class of functions p, analytic in E, such that

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z)$$

if and only if $p_1, p_2 \in P(n,\beta)$ for $z \in E$. The class $P_k(1,0) \equiv P_k$ was introduced in [6]. We note that $p \in P_k(n,\beta)$ if and only if there exists $h \in P_k(n,0)$ such that

$$p(z) = (1-\beta)h(z) + \beta,$$

Let f and g be analytic in E with $f(z)$ given by (1) and $g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k$. Then the convolution (or Hadamard product) of f and g is defined by

$$(f \star g)(z) = z + \sum_{k=n+1}^{\infty} a_k b_k z^k.$$

A function $f \in \mathscr{A}(n)$ is said to belong to the class $R_k(n,\beta), k \geq 2, 0 \leq \beta < 1$, if and only if $\frac{zf'}{f} \in P_k(n,\beta)$ for $z \in E$.

We note that $R_k(1,0) \equiv R_k$ is the class of functions with bounded radius rotation, first discussed by Tammi, see [1] and $R_2(1,0)$ consists of starlike univalent functions.

Similarly $f \in \mathscr{A}(n)$ belongs to $V_k(n,\beta)$ for $z \in E$ if and only if $\frac{(f')'}{f'} \in P_k(n,\beta)$. It is obvious that

$$f \in V_k(n,\beta) \quad \text{if and only if} \quad zf' \in R_k(n,\beta). \tag{3}$$

It may be observed that $V_2(1,0) \equiv C$, the class of convex univalent functions and $V_k(1,0) \equiv V_k$ is the class of functions with bounded boundary rotation first discussed by Paatero, see [1].

2 Preliminary Results

We need the following results in our investigation.

Lemma 2.1 ([5]). *Let $u = u_1 + iu_2$, $v = v_1 + iv_2$ and $\Psi(u,v)$ be a complex-valued function satisfying the following conditions:*

(i). $\Psi(u,v)$ *is continuous in a domain* $D \subset \mathbf{C}^2$

(ii). $(1,0) \in D$ *and* $\Psi\{(1,0)\} > 0$.

(iii). $Re\Psi(iu_2, v_1) \leq 0$ *whenever* $(iu_2, v_1) \in D$ *and* $v_1 \leq \frac{-1}{2}(1+u_2^2)$.

Let $h(z)$, *given by (2), be analytic in* E *such that* $(h(z), zh'(z)) \in D$ *and* $Re\Psi(h(z), zh'(z)) > 0$ *for all* $z \in E$, *then* $Re\{h(z)\} > 0$ *in* E.

We shall need the following result which is a modified version of Theorem 3.3e in [4, p113].

Lemma 2.2. *Let* $\beta > 0, \beta + \delta > 0$ *and* $\alpha \in [\alpha_0, 1)$, *where*

$$\alpha_0 = \max\left\{\frac{\beta - \delta - 1}{2\beta}, \frac{-\delta}{\beta}\right\}.$$

If $\left\{h(z) + \frac{zh'(z)}{\beta h(z)+\delta}\right\} \in P(1, \alpha)$ *for* $z \in E$, *then* $h \in P(1, \sigma)$ *in* E, *where*

$$\sigma(\alpha, \beta, \delta) = \left[\frac{(\beta + \delta)}{\beta\{{}_2F_1(2\beta(1-\alpha), 1, \beta + \delta + 1; \frac{r}{1+r}\}} - \frac{\delta}{\beta}\right], \tag{4}$$

where ${}_2F_1$ *denotes hypergeometric function. This result is sharp and external function is given as*

$$p_0(z) = \frac{1}{\beta g(z)} - \frac{\delta}{\beta}, \tag{5}$$

with

$$\begin{aligned} g(z) &= \int_0^1 \left(\frac{1-z}{1-tz}\right)^{2\beta(1-\alpha)} t^{(\beta+\delta-1)} dt \\ &= {}_2F_1\left(2\beta(1-\alpha), 1, \beta + \delta + 1; \frac{z}{z-1}\right). (\beta + \delta)^{-1}. \end{aligned}$$

3 Main Results

Theorem 3.1. *Let* $f \in R_k(n, \beta)$, $g \in R_k(n, \beta), \alpha, c, \delta$ *and* ν *be positively real and* $\delta = \nu = \alpha$. *Then the function* F *defined by*

$$[F(z)]^{\alpha} = cz^{\alpha - c} \int_0^z t^{(c-\delta-\nu)-1} (f(t))^{\delta} (g(t))^{\nu} dt \tag{6}$$

belongs to $R_k(n, \sigma)$, *where*

$$\sigma = \frac{2(2\beta c_1 + n\alpha_1)}{(n\alpha_1 - 2\beta + 2c_1) + \sqrt{(n\alpha_1 - 2\beta + 2c_1)^2 + 8(2\beta c_1 + n\alpha_1)}}, \tag{7}$$

with

$$c_1 = \frac{c-\alpha}{\alpha}, \quad \alpha_1 = \frac{1}{\alpha}.$$

Proof. First we show that there exists a function $F \in \mathscr{A}(n)$ satisfying (6). Let

$$G(z) = z^{-(\nu+\delta)} (f(z))^{\delta} (g(z))^{\nu} = 1 + \alpha_n z^n + \alpha_{n+1} z^{n+1} + \dots,$$

and choose the branches which equal 1 when $z = 0$. For

$$K(z) = z^{(c-\nu-\delta)-1} (f(z))^{\delta} (g(z))^{\nu} = z^{c-1} G(z),$$

we have

$$L(z) = \frac{c}{z^c} \int_0^z K(t)dt = 1 + \frac{c}{n+1} \alpha_n z^n + \dots,$$

where L is well defined and analytic in E. Now let

$$F(z) = [z^{\alpha} L(z)]^{\frac{1}{\alpha}} = z [L(z)]^{\frac{1}{\alpha}},$$

where we choose the branch of $[L(z)]^{\frac{1}{\alpha}}$ which equals 1 when $z = 0$. Thus $F \in \mathscr{A}(n)$ and satisfies (6).

Now, from (6), we have

$$z^{(c-\alpha-1)} [F(z)]^{\alpha} \left[(c-\alpha) + \alpha \frac{zF'(z)}{F(z)} \right] = c \left[z^{(c-\delta-\nu)-1} (f(z))^{\delta} (g(z))^{\nu} \right]. \quad (8)$$

We write

$$\frac{zF'(z)}{F(z)} = p(z) = \left(\frac{k}{4} + \frac{1}{2} \right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2} \right) p_2(z). \quad (9)$$

Then $p(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \dots,$ is analytic in E.

Logarithmic differentiation of (8) and use of (9) yields

$$(c-\alpha-1) + \alpha p(z) + \frac{\alpha z p'(z)}{(c-\alpha) + \alpha p(z)} = (c-\delta-\nu-1) + \frac{\delta z f'(z)}{f(z)} + \frac{\nu z g'(z)}{g(z)}.$$

Since $\nu + \delta = \alpha : f, g \in P_k(n, \beta)$ and it is known [2] that $P_k(n, \beta)$ is a convex set, it follows that

$$\left\{ p + \frac{\frac{1}{\alpha} z p'}{p + \left(\frac{c-\alpha}{\alpha}\right)} \right\} \in P_k(n, \beta), \quad z \in E.$$

Define

$$\Phi_{\alpha,c}(z) = \frac{1}{1+c_1}\frac{z}{(1-z)^{\alpha_1+1}} + \frac{c_1}{1+c_1}\frac{z}{(1-z)^{\alpha_1+2}},$$

with $\alpha_1 = \frac{1}{\alpha}$, $c_1 = \frac{c-\alpha}{\alpha}$.

Then, using (9), we have

$$\begin{aligned}\left(p \star \frac{\Phi_{\alpha,c}}{z}\right) &= p(z) + \frac{\alpha_1 z p'(z)}{p(z)+c_1} \\ &= \left(\frac{k}{4}+\frac{1}{2}\right)\left[p_1(z) + \frac{\alpha_1 z p_1'(z)}{p_1(z)+c_1}\right] \\ &\quad -\left(\frac{k}{4}-\frac{1}{2}\right)\left[p_2(z) + \frac{\alpha_1 z p_2'(z)}{p_2(z)+c_1}\right].\end{aligned}$$

Since $\left\{p + \frac{\alpha_1 z p'}{p+c_1}\right\} \in P_k(n,\beta)$, it follows that

$$\left\{p_i + \frac{\alpha_1 z p_i'}{p_i + c_1}\right\} \in P_k(n,\beta), \quad \text{for} \quad i = 1,2, \quad z \in E.$$

Writing $p_i(z) = (1-\sigma)H_i(z) + \sigma$, $i = 1,2$, we have, for $z \in E$,

$$\left[(1-\sigma)H_i + \sigma + \frac{\alpha_1(1-\sigma)H_i'}{(1-\sigma)H_i + \sigma + c_1} - \beta\right] \in P(n,0).$$

We now form the functional $\Psi(u,v)$ by taking $u = H_i$ and $v = zH_i'$ and so

$$\Psi(u,v) = (\sigma-\beta) + (1-\sigma)u + \frac{\alpha_1(1-\sigma)v}{(1-\sigma)u + \sigma + c_1}.$$

It can easily be seen that:

(i) $\Psi(u,v)$ is continuous in $\mathscr{D} = \left(\mathscr{C} - \left\{\frac{\sigma+c_1}{1-\sigma}\right\}\right) \times \mathscr{C}$.

(ii) $(i,0) \in \mathscr{D}$ and $Re\{\Psi(i,0) = 1-\beta > 0$.

To verify the condition (iii) of Lemma 2.1, we proceed as follows:

For all $(iu_2, v_1) \in \mathscr{D}$ such that $v_1 \leq \frac{-n(1+u_2^2)}{2}$, and

$$\begin{aligned}\Re\{\Psi(iu_2,v_1)\} &= (\sigma-\beta) + \frac{\alpha_1(1-\sigma)(\sigma+c_1)v_1}{(\sigma+c_1)^2 + (1-\sigma)^2u_2^2} \\ &\leq \frac{2(\sigma-\beta)\left\{(\sigma+c_1)^2+(1-\sigma)^2u_2^2\right\} - n\alpha_1(1-\sigma)(\sigma+c_1)(1+u_2^2)}{2(\sigma+c_1)^2+(1-\sigma)^2u_2^2} \\ &= \frac{A+Bu_2^2}{2C} \qquad (10) \\ &\leq 0, \quad \text{if} \quad A \leq 0 \quad \text{and} \quad B \leq 0,\end{aligned}$$

where

$$A = 2(\sigma - \beta)(\sigma + c_1)^2 - n\alpha_1(1-\sigma)(\sigma + c_1),$$
$$B = 2(\sigma - \beta)(1-\sigma)^2 - n\alpha_1(1-\sigma)(\sigma + c_1)$$
$$C = (\sigma + c_1)^2 + (1-\sigma)^2 u_2^2 > 0.$$

From $A = 0$, we obtain σ as given by (7) and $B \leq 0$ ensures that $0 \leq \sigma < 1$. Thus using Lemma 2.1, it follows that $H_i \in P(n, 0)$ and therefore $p_i \in P(n, \sigma)$, $\quad i = 1, 2$. Consequently $p \in P_k(n, \sigma)$ and this completes the proof. □

Corollary 3.1. *For* $0 = c = n = 1, \beta = 0$ *and* $f = g, F \in V_k$ *implies that* $F \in R_k(\frac{1}{2})$ *and this, with* $k = 2$, *gives us a well-known result that every convex function is starlike of order* $\frac{1}{2}$ *in* E.

Corollary 3.2. *For* $n = 1$, *let* $f \in R_k(1, \sigma)$ *in Theorem 3.1. Then* $F \in R_k(1, \sigma_0)$, *where* σ_0 *is given by (2.1) with* $\beta = \alpha, \delta = (1 - \alpha)$. *This result is sharp.*

Corollary 3.3. *In (2), we take* $\nu + \delta = 1, c = 2, f = g$ *and obtain Libera's integral operator [3, 6] as:*

$$F(z) = \frac{2}{z}\int_0^z f(t)dt, \tag{11}$$

where $f \in R_k(n, \beta)$. *Then, by Theorem 3.1, it follows that* $F \in R_k(n, \sigma_1)$, *where*

$$\sigma_1 = \frac{2(2\beta + n)}{\left[(n - 2\beta + 2) + \sqrt{(n - 2\beta + 2)^2 + 8(2\beta + n)}\right]}. \tag{12}$$

For $\beta = 0$ *and* $n = 1$, *we have Libera's operator for the class* R_k *of bounded radius rotation. That is, if* $f \in R_k$ *and* F *is given by (3.6), then*

$$F \in R_k(1, \sigma_2), \quad \textit{with} \quad \sigma_2 = \frac{2}{3 + \sqrt{17}}.$$

Using Theorem 3.1 and relation (3), we can prove the following.

Theorem 3.2. *Let* f *and* g *belong to* $V_k(n, \beta)$, *and let* F *be defined by (6) with* α, c, δ, ν *positively real,* $\delta + \nu = \alpha$. *Then* $F \in V_k(n, \sigma)$, *where* σ *is given by (7).*

By taking $\alpha = 1, c + \frac{1}{\lambda}, \nu + \delta = \alpha = 1$ and $f = g$ in (6), we obtain the integral operator $I_\lambda(f) = F$, defined as:

$$F(z) = \frac{1}{\lambda}\int_0^z t^{\frac{1}{\lambda}-2} f(t)dt, \quad (\lambda > 0). \tag{13}$$

With the similar techniques, we can easily prove the following result which is stronger version than the one proved in Theorem 3.1.

Theorem 3.3. *Let* $f \in R_k(n, \gamma)$ *and let, for* $0 < \lambda \leq 1$, F *be defined by (13). Then* $F \in R_k(n, \delta^*)$, *where* δ^* *satisfies the conditions given below:*

(i) *If* $0 < \lambda \leq \frac{1}{2}$ *and* $\frac{n\lambda}{2(\lambda-1)} \leq \gamma < 1$, *then*

$$\delta^* = \delta_1 = \frac{1}{4\lambda}\left[A_1 + \sqrt{A_1^2 + 8B_1}\right] \geq 0,$$

where

$$A_1 = 2\gamma\lambda + 2\lambda - n\lambda$$
$$B_1 = \lambda\{2\gamma(1-\lambda) + n\lambda\}.$$

(ii) *If* $\frac{1}{2} < \lambda \leq 1, \quad \frac{n(\lambda-1)}{2\lambda} \leq \frac{n(3\lambda-\sqrt{8\lambda})}{2\lambda} \leq \gamma$, *then*

$$\delta^* = \delta_2 = \frac{1}{4\lambda}\left[A_2 + \sqrt{A_2^2 + 8B_2}\right] \geq 0,$$

where

$$A_2 = 2\lambda + 2\lambda\gamma - n\lambda$$
$$B_2 = \lambda(2\lambda\gamma + n - n\lambda).$$

(iii) *If* $\frac{1}{2} < \lambda \leq 1, \quad \frac{n(\lambda-1)}{2\lambda} < \frac{n(3\lambda-\sqrt{8\lambda})}{2\lambda} < \gamma < 1$, *then* $\delta_3 = \delta_1$.

Special Cases

(1). Let $\lambda = \frac{1}{2}$ in (13). Then we have Libera's operator and (i) gives us

$$\delta^* = \delta_1 = \frac{2(2\gamma + n)}{(n - 2\gamma + 2) + \sqrt{(n - 2\gamma + 2)^2 + 8(2\gamma + n)}}.$$

(2). When $\gamma = 0, \lambda = \frac{1}{2}, n = 1$, and $f \in R_k$, then $F \in R_k(1, \delta_1)$, where

$$\delta^* = \delta_1 = \frac{2}{3 + \sqrt{17}}.$$

(3). Let $\lambda = 1, \gamma = 0, n = 1$ and $f \in R_k$. Then, from (3.8), it follows that

$$F(z) = \int_0^z \frac{f(t)}{t} dt$$

and, by Theorem 3.3, $F \in R_k(\frac{1}{2})$. By using relation (3) and $k = 2$, we obtain a well-known result that every convex function is starlike of order $\frac{1}{2}$.

Theorem 3.4. *Let* $f \in R_k(n, 0), g \in R_k(n, \alpha), 0 \le \alpha \le 1$. *Let the function* F, *for* $b \ge 0$, *be defined as*

$$F(z) = \frac{1+b}{z^b} \int_0^z f^\alpha(t) t^{b-\alpha-1} g(t) dt. \tag{14}$$

Then $F \in R_k(n, \eta), \quad z \in E$, *where*

$$\eta = \frac{2n}{(2b+n) + \sqrt{(2b+n)^2 + 8n}}. \tag{15}$$

Proof. Set

$$\frac{zF'(z)}{F(z)} = p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z).$$

Then $p(z)$ is analytic in E and $p(0) = 1$. From (14), we have

$$\begin{aligned} p(z) + \frac{zp'(z)}{p(z)+b} &= \left[\alpha \frac{zf'(z)}{f(z)} + (1-\alpha)\right] + \frac{zg'(z)}{g(z)} - 1 \\ &= [\alpha h_1 + (1-\alpha)] + [(1-\alpha)h_2(z) + \alpha] - 1 \\ &= \alpha h_1(z) + (1-\alpha)h_2(z) = h(z), \quad h \in P_k(n, 0). \end{aligned}$$

Since $g \in P_k(n, \alpha), \quad f \in R_k(n, 0)$, it follows that $h_1, h_2 \in P_k(n, 0)$ and $P_k(n, 0)$ is a convex set. Now following the similar technique of Theorem 3.1 and using Lemma 2.1, we obtain the required result that $\frac{zF'(z)}{F(z)} = p(z) \in P_k(n, \eta)$, where η is given by (15). □

Remark 3.1. When $n = 1$, we obtain best possible value of $\eta = \sigma$ given by (2.1) with $\alpha = 0, \beta = 1, \delta = b$.

Conclusion. In this paper, we have introduced and considered a new class $P_k(n, \beta)$ of analytic function. We have discussed several special cases of this new class. We have discussed some integral operators for certain classes of analytic functions in the unit disc E and related with the new class $P_k(n, \beta)$. Results obtained in this paper can be viewed as an refinement and improvement of the previously known results in this field.

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