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Themistocles M. Rassias
László Tóth *Editors*

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Aims and Scope
Optimization has been expanding in all directions at an astonishing rate during the last few decades. New algorithmic and theoretical techniques have been developed, the diffusion into other disciplines has proceeded at a rapid pace, and our knowledge of all aspects of the field has grown even more profound. At the same time, one of the most striking trends in optimization is the constantly increasing emphasis on the interdisciplinary nature of the field. Optimization has been a basic tool in all areas of applied mathematics, engineering, medicine, economics, and other sciences.

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Topics in Mathematical Analysis and Applications

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Preface

This book entitled *Topics in Mathematical Analysis and Applications* consists of papers written by eminent scientists from the international mathematical community, who present significant advances in a number of theories and problems of mathematical analysis and its applications. These contributions focus on both old and recent developments of analytic inequalities, operator theory, functional analysis, approximation theory, functional equations, differential equations, wavelets, discrete mathematics and mechanics. Special emphasis is given to new results that have been obtained in the above-mentioned disciplines in which nonlinear analysis plays a central role. Furthermore some review papers are published in this volume which are particularly useful for a broader audience of readers in mathematical analysis as well as for graduate students who search for the latest information. It is a pleasure to express our deepest thanks to all of the mathematicians who, through their works, participated in this volume. We would also wish to acknowledge the superb assistance that the staff of Springer has provided in the preparation of the publication.

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Simple Proofs of Some Bernstein–Mordell Type Inequalities

Vandanjav Adiyasuren and Tserendorj Batbold

Abstract In this paper we give simple proofs of some Bernstein–Mordell type inequalities.

Keywords Orthogonal polynomial • Gamma function • Bernstein inequality • Mordell inequality • Cauchy–Schwarz inequality

Mathematics Subject Classification (2000): Primary 26D15, Secondary 33A65

1 Introduction

In 1926, Bernstein proved the following integral inequality (see [1]).

If $a_0, \ldots, a_n$ are real numbers, then

$$\int_{-1}^{1} (a_0 + a_1 x + \cdots + a_n x^n)^2 dx \geq \frac{2}{(n+1)^2}. \tag{1}$$

The following inequality was posed as a problem in the book [2] by Bowman and Gerard. A simple proof was given by Mordell in [6].

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If $a_0, \ldots, a_n$ are real numbers, then

$$\int_0^\infty e^{-x}(1 + b_1 x + b_2 x^2 + \cdots + b_n x^n)^2 dx \geq \frac{1}{n+1}. \tag{2}$$

In [7], Mordell has solved, in some cases, the problem of finding the minimum value of integrals of the form

$$\int_a^b p(x)(a_0 + a_1 x + \cdots + a_n x^n)^2 dx \tag{3}$$

where $p(x) \geq 0$ is such that the integrals $\int_a^b p(x)x^r dx$ $(r \geq 0)$ exist and the coefficient a_k of the term x^k in the bracket is given as 1. Mirsky [4] has found the minimum of the integral

$$\int_a^b p(t)(t^{k_0} + \lambda_1 t^{k_1} + \cdots + \lambda_n t^{k_n})^2 dt, \tag{4}$$

using the principle of linear algebra.

Inequalities involving (3) and (4) are known in the literature as the Bernstein–Mordell type inequalities (see [5]).

Vasić and Rakovich [8,9], and Janous [3] found the minimum values of (3) with the condition $a_0 + a_1 p + \cdots + a_n p^n = 1$ for any given real number p, and $a_k = 1$ for some integers $0 \leq k \leq n$, respectively. The method presented in their papers is based on the properties of orthogonal polynomials and the method of Lagrange multipliers.

In this paper, we give simple proofs of some Bernstein–Mordell type inequalities using the properties of orthogonal polynomials and Cauchy–Schwarz inequality.

2 Some Lemmas

In order to establish the proof of the propositions, we need the following lemmas:

Lemma 1 ([3]). *Let k and N be non-negative integers. Then*

$$\sum_{p=0}^{N}(2k + 2p + 1)\binom{2k+p}{p}^2 = (2k+1)\binom{2k+N+1}{N}^2. \tag{5}$$

Proof. For any $n \in \mathbb{N}$ we have

$$(2k+1)\binom{2k+n+1}{n}^2 = (2k+1)\cdot\frac{(2k+n+1)^2}{(2k+1)^2}\cdot\binom{2k+n}{n}^2$$

$$= \frac{n^2 + (2k+1)(2k+2n+1)}{2k+1} \binom{2k+n}{n}^2$$

$$= (2k+1)\binom{2k+n}{n-1}^2 + (2k+2n+1)\binom{2k+n}{n}^2,$$

i.e.,

$$(2k+1)\binom{2k+n+1}{n}^2 - (2k+1)\binom{2k+n}{n-1}^2 = (2k+2n+1)\binom{2k+n}{n}^2.$$

Summing up these equalities for $n = 1, \dots, N$ yields

$$\sum_{p=0}^{N} (2k+2p+1)\binom{2k+p}{p}^2 = (2k+1)\binom{2k+N+1}{N}^2.$$

Lemma 2 ([8]). *Let $\lambda > -1$ be real number. Then*

$$\sum_{j=0}^{n} \frac{\Gamma(j+2\lambda)(j+\lambda)}{j!} = \frac{(2\lambda+2n+1)\Gamma(2\lambda+n+1)}{2n!(2\lambda+1)}.$$

Proof. It is clear that

$$\sum_{j=0}^{n} \frac{\Gamma(j+2\lambda)(j+\lambda)}{j!}$$

$$= \Gamma(2\lambda) \sum_{j=0}^{n} \binom{j+2\lambda-1}{2\lambda-1}(j+\lambda)$$

$$= \Gamma(2\lambda)\left(2\lambda \sum_{j=0}^{n} \binom{j+2\lambda}{2\lambda} - \lambda \sum_{j=0}^{n} \binom{j+2\lambda-1}{2\lambda-1}\right)$$

$$= 2\lambda\Gamma(2\lambda)\left(\binom{2\lambda}{2\lambda} + \sum_{j=1}^{n}\left(\binom{j+2\lambda+1}{2\lambda+1} - \binom{j+2\lambda}{2\lambda+1}\right)\right)$$

$$-\lambda\Gamma(2\lambda)\left(\binom{2\lambda-1}{2\lambda-1} + \sum_{j=1}^{n}\left(\binom{j+2\lambda}{2\lambda} - \binom{j+2\lambda-1}{2\lambda}\right)\right)$$

$$= \Gamma(2\lambda)\cdot\lambda\left(2\binom{n+2\lambda+1}{2\lambda+1}-\binom{n+2\lambda}{2\lambda}\right)$$

$$= \frac{(2\lambda+2n+1)\Gamma(2\lambda+n+1)}{2n!(2\lambda+1)},$$

which completes the proof.

Lemma 3. *Let $\alpha > -1$ and $\beta > -1$ be real numbers. Then*

$$\sum_{j=0}^{n}\frac{\Gamma(\alpha+j+1)(\alpha+\beta+2j+1)\Gamma(\alpha+\beta+j+1)}{j!\Gamma(j+\beta+1)} = \frac{\Gamma(n+\alpha+\beta+2)\Gamma(n+\alpha+2)}{n!(\alpha+1)\Gamma(n+\beta+1)}. \tag{6}$$

Proof. We will use the method of mathematical induction. If $n = 1$, then identity (6) is true:

$$\frac{\Gamma(\alpha+1)\Gamma(\alpha+\beta+1)(\alpha+\beta+1)}{\Gamma(\beta+1)}+\frac{\Gamma(\alpha+2)\Gamma(\alpha+\beta+2)\Gamma(\alpha+\beta+3)}{\Gamma(\beta+2)}$$

$$= \frac{\Gamma(\alpha+2)\Gamma(\alpha+\beta+2)(\beta+1)}{(\alpha+1)\Gamma(\beta+2)}+\frac{\Gamma(\alpha+2)(\alpha+\beta+2)(\alpha+\beta+3)}{\Gamma(\beta+2)}$$

$$= \frac{\Gamma(\alpha+2)\Gamma(\alpha+\beta+2)}{(\alpha+1)\Gamma(\beta+2)}(\beta+1+(\alpha+\beta+3)(\alpha+1))$$

$$= \frac{\Gamma(\alpha+2)\Gamma(\alpha+\beta+2)(\alpha+2)(\alpha+\beta+2)}{(\alpha+1)\Gamma(\beta+2)}$$

$$= \frac{\Gamma(\alpha+3)\Gamma(\alpha+\beta+3)}{(\alpha+1)\Gamma(\beta+2)}.$$

Let us assume that

$$\sum_{j=0}^{n}\frac{\Gamma(\alpha+j+1)(\alpha+\beta+2j+1)\Gamma(\alpha+\beta+j+1)}{j!\Gamma(j+\beta+1)} = \frac{\Gamma(n+\alpha+\beta+2)\Gamma(n+\alpha+2)}{n!(\alpha+1)\Gamma(n+\beta+1)}$$

for some n. Then, we have

$$\sum_{j=0}^{n+1}\frac{\Gamma(\alpha+j+1)\Gamma(\alpha+\beta+j+1)(\alpha+\beta+2j+1)}{j!\Gamma(j+\beta+1)}$$

$$
\begin{aligned}
&= \frac{\Gamma(n+\alpha+\beta+2)\Gamma(n+\alpha+2)}{n!(\alpha+1)\Gamma(n+\beta+1)} \\
&\quad + \frac{\Gamma(\alpha+n+2)\Gamma(\alpha+\beta+n+2)(\alpha+\beta+2n+3)}{(n+1)!\Gamma(n+\beta+2)} \\
&= \frac{\Gamma(n+\alpha+2)\Gamma(n+\alpha+\beta+2)}{(n+1)!(\alpha+1)\Gamma(n+\beta+2)} \\
&\quad \times ((n+1)(n+\beta+1) + (\alpha+\beta+2n+3)(\alpha+1)) \\
&= \frac{\Gamma(n+\alpha+2)\Gamma(n+\alpha+\beta+2)}{(n+1)!(\alpha+1)\Gamma(n+\beta+2)} ((\alpha+\beta+n+2)(n+\alpha+2)) \\
&= \frac{\Gamma(n+\alpha+3)\Gamma(n+\alpha+\beta+3)}{(n+1)!(\alpha+1)\Gamma(n+\beta+2)}.
\end{aligned}
$$

This proves identity (6) for all n.

3 Main Results

Proposition 1. *Let A_n be a set of monic polynomials with real coefficients and degree at most n. Then*

$$\min_{A_n} \int_{-1}^{1} A_n^2(x)dx = \frac{2^{2n+1}\cdot(n!)^4}{(2n)!\cdot(2n+1)!}.$$

Proof. We start by noting that the normalized Legendre polynomials

$$P_n(x) = \frac{1}{n!\cdot 2^n}\cdot\sqrt{\frac{2n+1}{2}}\cdot\frac{d^n}{dx^n}\left(\left(x^2-1\right)^n\right), \quad n = 0, 1, \ldots$$

transformed to the interval $[-1, 1]$ form an orthonormal basis of all polynomials defined on $[-1, 1]$. We put

$$A_n(x) = \sum_{j=0}^{n} a_j P_j(x). \tag{7}$$

Then

$$\int_{-1}^{1} A_n^2(x)dx = \sum_{j=0}^{n} a_j^2. \tag{8}$$

From

$$P_j(x) = \frac{1}{j!\cdot 2^j}\sqrt{\frac{2j+1}{2}}\cdot\frac{d^j}{dx^j}\left(\sum_{p=0}^{j}(-1)^p C_j^p (x^2)^{j-p}\right)$$

$$= \frac{1}{j!\cdot 2^j}\sqrt{\frac{2j+1}{2}}\cdot\sum_{p=0}^{[\frac{j}{2}]}(-1)^p C_j^p (2j-2p)(2j-2p-1)\cdots(j-2p+1)x^{j-2p}$$

$$= \frac{1}{j!2^j}\sqrt{\frac{2j+1}{2}}\sum_{p=0}^{[\frac{j}{2}]}(-1)^p\frac{j!}{p!(j-p)!}\frac{(2j-2p)!}{(j-2p)!}x^{j-2p}$$

$$= \frac{1}{2^j}\sqrt{\frac{2j+1}{2}}\cdot\sum_{p=0}^{[\frac{j}{2}]}(-1)^p\frac{(2j-2p)!}{p!(j-p)!(j-2p)!}x^{j-2p}$$

we deduce that the coefficient of x^n in A_n equals

$$1 = a_n\cdot\frac{1}{2^n}\cdot\sqrt{\frac{2n+1}{2}}\cdot\frac{(2n)!}{(n!)^2}.$$

Hence, we find

$$a_n^2 = \frac{2^{2n+1}\cdot(n!)^4}{(2n)!\cdot(2n+1)!}. \tag{9}$$

From the equalities (8) and (9), we have

$$\int_{-1}^{1} A_n^2(x)dx = \sum_{j=0}^{n} a_j^2 \geq a_n^2 = \frac{2^{2n+1}\cdot(n!)^4}{(2n)!\cdot(2n+1)!}.$$

We conclude that

$$\min_{A_n}\int_{-1}^{1} A_n^2(x)dx = \frac{2^{2n+1}\cdot(n!)^4}{(2n)!\cdot(2n+1)!}.$$

Proposition 2 ([9]). *Let $\lambda > -1$ be a real number. Then*

$$\min_{b_k\in\mathbb{R},k=\overline{1,n}}\int_0^\infty x^\lambda e^{-x}(1+b_1x+\cdots+b_nx^n)^2dx = \frac{\Gamma(\lambda+1)}{\binom{n+\lambda+1}{n}}.$$

Proof. We start by noting that the Laguerre polynomials

$$L_n^\lambda(x) = x^{-\lambda} e^x \frac{d^n}{dx^n}(x^{\lambda+n} e^{-x})$$

transformed to the interval $[0, \infty)$ form an orthogonal basis of all polynomials defined on $[0, \infty)$. We put

$$1 + b_1 x + \cdots + b_n x^n = \sum_{j=0}^{n} a_j L_j^\lambda(x). \tag{10}$$

Then

$$\int_0^\infty x^\lambda e^{-x}(1 + b_1 x + \cdots + b_n x^n)^2 dx = \sum_{j=0}^{n} a_j^2 \cdot j!\Gamma(\lambda + j + 1).$$

By the Leibniz formula, we have

$$\begin{aligned} L_n^\lambda(x) &= \sum_{k=0}^{n} (-1)^k \cdot \binom{n}{k} (\lambda + k + 1) \cdots (\lambda + n) \cdot x^k \\ &= \sum_{k=0}^{n} \frac{(-1)^k \cdot \Gamma(n+1)\Gamma(\lambda+n+1)}{\Gamma(k+1)\Gamma(n-k+1)\Gamma(\lambda+k+1)} \cdot x^k. \end{aligned}$$

From the above, we deduce that the coefficient of x^0 in polynomial (10) equals

$$1 = \sum_{j=0}^{n} a_j \cdot \frac{\Gamma(\lambda + j + 1)}{\Gamma(\lambda + 1)}.$$

Using Cauchy–Schwarz inequality, we have

$$\begin{aligned} 1 &= \sum_{j=0}^{n} \left(a_j \cdot \sqrt{j!\Gamma(\lambda + j + 1)}\right) \cdot \left(\sqrt{\frac{\Gamma(\lambda + j + 1)}{j!}} \cdot \frac{1}{\Gamma(\lambda + 1)}\right) \\ &\le \sqrt{\sum_{j=0}^{n} a_j^2 j!\Gamma(\lambda + j + 1)} \cdot \sqrt{\sum_{j=0}^{n} \frac{\Gamma(\lambda + j + 1)}{j!} \cdot \frac{1}{(\Gamma(\lambda + 1))^2}}, \end{aligned}$$

i.e.

$$\sum_{j=0}^{n} a_j^2 \cdot j!\Gamma(\lambda+j+1) \ge \frac{(\Gamma(\lambda + 1))^2}{\sum_{j=0}^{n} \frac{\Gamma(\lambda+j+1)}{j!}} = \frac{\Gamma(\lambda + 1)}{\sum_{j=0}^{n} \frac{(\lambda+j)\cdot\ldots\cdot(\lambda+1)}{j!}} = \frac{\Gamma(\lambda + 1)}{\sum_{j=0}^{n} \binom{\lambda+j}{\lambda}}.$$

From the above inequality and following inequality

$$\sum_{j=0}^{n}\binom{\lambda+j}{\lambda}=\binom{\lambda}{\lambda}+\sum_{j=1}^{n}\left(\binom{\lambda+j+1}{\lambda+1}-\binom{\lambda+j}{\lambda+1}\right)$$
$$=\binom{\lambda+n+1}{\lambda+1}=\binom{\lambda+n+1}{n},$$

we find

$$\sum_{j=0}^{n}a_j^2\, j!\,\Gamma(\lambda+j+1)\geq\frac{\Gamma(\lambda+1)}{\binom{\lambda+n+1}{n}}.$$

Hence

$$\int_0^{\infty}x^{\lambda}e^{-x}\left(1+b_1x+\cdots+b_nx^n\right)dx\geq\frac{\Gamma(\lambda+1)}{\binom{\lambda+n+1}{n}}.$$

We conclude that

$$\min_{b_k\in\mathbb{R},k=\overline{1,n}}\int_0^{\infty}x^{\lambda}e^{-x}\left(1+b_1x+\cdots+b_nx^n\right)dx=\frac{\Gamma(\lambda+1)}{\binom{\lambda+n+1}{n}}.$$

Remark 1. If we substitute $\lambda=0$ in Proposition 2, we get the Mordell inequality (2).

Proposition 3 ([3]). *Let n and k be integers with $0\leq k\leq n$ and let $P_{n,k}$ be a set of all polynomials with real coefficients and degree at most n such that the coefficient of x^k is 1. Then*

$$\min_{P_{n,k}}\int_0^1\left(P_{n,k}(x)\right)^2dx=\left((2k+1)\binom{n+k+1}{n-k}^2\binom{2k}{k}^2\right)^{-1}.$$

Proof. Using similar way of proof of the Proposition 1 and following formula

$$P_j(x)=\frac{\sqrt{2j+1}}{j!}\cdot\frac{d^j}{dx^j}(x^2-x)^j,\ j=0,1,2,\ldots,$$

we have

$$P_{n,k}(x)=\sum_{j=0}^{n}a_jP_j(x).$$

Hence

$$\int_0^1 (P_{n,k}(x))^2 dx = \sum_{j=0}^{n} a_j^2 \geq \sum_{j=k}^{n} a_j^2. \tag{11}$$

From

$$\begin{aligned} P_j(x) &= \frac{\sqrt{2j+1}}{j!} \cdot \frac{d^j}{dx^j} \left(\sum_{p=0}^{j} (-1)^p \cdot \binom{j}{p} \cdot x^{2j-p} \right) \\ &= \sqrt{2j+1} \cdot \sum_{p=0}^{j} (-1)^p \cdot \frac{(2j-p)!}{p!(j-p)!)^2} \cdot x^{j-p}, \end{aligned}$$

we deduce that the coefficient of x^k in $P_{n,k}(x)$ equals

$$1 = \sum_{j=k}^{n} a_j (-1)^{j-k} \sqrt{2j+1} \cdot \frac{(j+k)!}{(j-k)!(k!)^2}.$$

Using Cauchy–Schwarz inequality, we have

$$\begin{aligned} 1 &= \sum_{j=k}^{n} a_j (-1)^{j-k} \cdot \sqrt{2j+1} \frac{(j+k)!}{(j-k)!(k!)^2}. \\ &\leq \sqrt{\sum_{j=k}^{n} a_j^2} \cdot \sqrt{\sum_{j=k}^{n} (2j+1) \cdot \frac{((j+k)!)^2}{((j-k)!)^2 (k!)^4}}. \end{aligned}$$

Therefore, using Lemma 1, we have

$$\begin{aligned} 1 &\leq \left(\sum_{j=k}^{n} a_j^2 \right) \cdot \left(\sum_{j=k}^{n} (2j+1) \binom{j+k}{j-k}^2 \binom{2k}{k}^2 \right) \\ &= \left(\sum_{j=k}^{n} a_j^2 \right) \cdot \binom{2k}{k}^2 \cdot \sum_{j=k}^{n} (2j+1) \binom{j+k}{j-k}^2 \\ &= \left(\sum_{j=k}^{n} a_j^2 \right) \cdot \binom{2k}{k}^2 \cdot \sum_{j=0}^{n-k} (2k+2j+1) \binom{2k+j}{j}^2 \\ &= \left(\sum_{j=k}^{n} a_j^2 \right) \cdot \binom{2k}{k}^2 \cdot (2k+1) \binom{n+k+1}{n-k}^2, \end{aligned}$$

i.e.,

$$\sum_{j=k}^{n} a_j^2 \geq \left((2k+1) \binom{2k}{k}^2 \binom{n+k+1}{n-k}^2 \right)^{-1}.$$

We conclude that

$$\min_{P_{n,k}} \int_0^1 (P_{n,k}(x))^2 dx = \left((2k+1) \binom{2k}{k}^2 \binom{n+k+1}{n-k}^2 \right)^{-1}.$$

Proposition 4 ([8]). *Let $\alpha > -1$ and $b_i, i = 0, \dots, n$ be real numbers such that $\sum_{i=0}^{n} b_i = 1$. Then*

$$\min_{b_i \in \mathbb{R},\ i=\overline{0,n}} \int_{-1}^{1} (1-x^2)^{\alpha} \left(\sum_{i=0}^{n} b_i x^i \right)^2 dx$$
$$= \frac{\pi n! \Gamma(2\alpha+2)\Gamma(2\alpha+3)}{\Gamma(2\alpha+2+n) \left(\Gamma\left(\alpha+\frac{3}{2}\right) \right)^2 (\alpha+n+1) 2^{2+2\alpha}}.$$

Proof. The Gegenbauer polynomials

$$C_n^{\lambda}(x) = \frac{(-2)^n}{n!} \frac{\Gamma(n+\lambda)\Gamma(n+2\lambda)}{\Gamma(\lambda)\Gamma(2n+2\lambda)} (1-x^2)^{-\lambda+\frac{1}{2}} \cdot \frac{d^n}{dx^n} \left((1-x^2)^{\lambda+n-\frac{1}{2}} \right)$$

transformed to the interval $[-1, 1]$ form an orthonormal basis of all polynomials defined on $[-1, 1]$. We put

$$\sum_{i=0}^{n} b_i x^i = \sum_{j=0}^{n} a_j C_j^{\lambda}(x). \tag{12}$$

Hence

$$\int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} \left(\sum_{i=0}^{n} b_i x^i \right)^2 dx = \sum_{j=0}^{n} a_j^2 \cdot \|C_j^{\lambda}\|^2$$
$$= \sum_{j=0}^{n} a_j^2 \cdot \frac{2^{1-2\lambda} \cdot \pi \cdot \Gamma(j+2\lambda)}{j!(j+\lambda)(\Gamma(\lambda))^2}. \tag{13}$$

By the Leibniz formula and formula $2^{2x-1}\Gamma(x)\Gamma(x+1/2)=\sqrt{\pi}\,\Gamma(2x)$, we have

$$C_n^\lambda(1)=\frac{\Gamma(n+2\lambda)}{n!\Gamma(2\lambda)}.$$

From the equality (12), we find

$$1=\sum_{i=0}^n b_i=\sum_{j=0}^n a_j C_j^\lambda(1)=\sum_{j=0}^n a_j\cdot\frac{\Gamma(j+2\lambda)}{j!\Gamma(2\lambda)}.$$

Therefore, using Cauchy–Schwarz inequality

$$\begin{aligned}1&=\sum_{j=0}^n\left(a_j\sqrt{\frac{2^{1-2\lambda}\pi\Gamma(j+2\lambda)}{j!(j+\lambda)(\Gamma(\lambda))^2}}\right)\cdot\left(\frac{\Gamma(j+2\lambda)}{j!\Gamma(2\lambda)}\cdot\sqrt{\frac{j!(j+\lambda)(\Gamma(\lambda))^2}{2^{1-2\lambda}\pi\Gamma(j+2\lambda)}}\right)\\&\le\sqrt{\sum_{j=0}^n a_j^2\cdot\frac{2^{1-2\lambda}\cdot\pi\cdot\Gamma(j+2\lambda)}{j!(j+\lambda)(\Gamma(\lambda))^2}}\cdot\sqrt{\sum_{j=0}^n\frac{\Gamma(j+2\lambda)(j+\lambda)(\Gamma(\lambda))^2}{j!(\Gamma(2\lambda))^2\cdot\pi\cdot2^{1-2\lambda}}},\end{aligned}$$

i.e.,

$$\sum_{j=0}^n a_j^2\cdot\frac{2^{1-2\lambda}\cdot\pi\cdot\Gamma(j+2\lambda)}{j!(j+\lambda)(\Gamma(\lambda))^2}\ge\frac{1}{\sum_{j=0}^n\frac{\Gamma(j+2\lambda)(j+\lambda)(\Gamma(\lambda))^2}{j!(\Gamma(2\lambda))^2\cdot\pi\cdot2^{1-2\lambda}}}.\qquad(14)$$

By Lemma 2, we find

$$\sum_{j=0}^n\frac{\Gamma(j+2\lambda)(j+\lambda)(\Gamma(\lambda))^2}{j!(\Gamma(2\lambda))^2\cdot\pi\cdot2^{1-2\lambda}}=\frac{\lambda\cdot\Gamma(2\lambda+n+1)\cdot(2\lambda+2n+1)(\Gamma(\lambda))^2}{2^{1-2\lambda}\pi n!\Gamma(2\lambda)\Gamma(2\lambda+2)}.\qquad(15)$$

From the relations (13)–(15), we have

$$\int_{-1}^1(1-x^2)^{\lambda-\frac12}\left(\sum_{i=0}^n b_i x^i\right)^2dx\ge\frac{2^{1-2\lambda}\pi n!\Gamma(2\lambda)\Gamma(2\lambda+2)}{\lambda\cdot\Gamma(2\lambda+n+1)(2\lambda+2n+1)(\Gamma(\lambda))^2}.$$

For $\lambda-\frac12=\alpha$, we have

$$\int_{-1}^1(1-x^2)^\alpha\left(\sum_{i=0}^n b_i x^i\right)^2dx\ge\frac{\pi n!\Gamma(2\alpha+2)\Gamma(2\alpha+3)}{\Gamma(2\alpha+2+n)\left(\Gamma(\alpha+\frac32)\right)^2\cdot(\alpha+n+1)\cdot2^{2+2\alpha}}.$$

We conclude that

$$\min_{b_i\in\mathbb{R},\ i=\overline{0,n}} \int_{-1}^{1} (1-x^2)^{\alpha} \left(\sum_{i=0}^{n} b_i x^i\right)^2 dx$$
$$= \frac{\pi n!\Gamma(2\alpha+2)\Gamma(2\alpha+3)}{\Gamma(2\alpha+2+n)\left(\Gamma\left(\alpha+\frac{3}{2}\right)\right)^2(\alpha+n+1)2^{2+2\alpha}}.$$

The proof is completed.

Proposition 5 ([9]). *Let $\alpha > -1, \beta > -1$, and $b_i, i = 0, \dots, n$ be real numbers such that $\sum_{i=0}^{n} b_i = 1$. Then*

$$\min_{b_i\in\mathbb{R},\ i=\overline{0,n}} \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} \left(\sum_{i=0}^{n} b_i x^i\right)^2 dx$$
$$= \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\alpha+2)\Gamma(n+\beta+1)\cdot n!}{\Gamma(n+\alpha+2)\Gamma(n+\alpha+\beta+2)}.$$

Proof. The Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n\cdot n!}(1-x)^{-\alpha}(1+x)^{-\beta}\cdot\frac{d^n}{dx^n}\left((1-x)^{\alpha+n}(1+x)^{\beta+n}\right)\ (\alpha,\beta>-1),$$

transformed to the interval $[-1, 1]$ form an orthogonal basis of all polynomials defined on $[-1, 1]$. We put

$$b_0 + b_1 x + \cdots + b_n x^n = \sum_{j=0}^{n} a_j \cdot P_j^{(\alpha,\beta)}(x). \tag{16}$$

Hence

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} \left(\sum_{i=0}^{n} b_i x^i\right)^2 dx$$
$$= \sum_{j=0}^{n} a_j^2 \| P_j^{(\alpha,\beta)} \|^2$$
$$= \sum_{j=0}^{n} a_j^2 \cdot \frac{2^{\alpha+\beta+1}\cdot\Gamma(j+\alpha+1)\Gamma(j+\beta+1)}{j!(\alpha+\beta+2j+1)\Gamma(\alpha+\beta+j+1)}. \tag{17}$$

By the Leibniz formula, we have

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n n!}\sum_{k=0}^{n} C_n^k \cdot \frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{\Gamma(\alpha+k+1)\Gamma(\beta+n-k+1)} \cdot (x-1)^k (x+1)^{n-k}$$

Moreover, for $x = 1$ we have

$$P_n^{(\alpha,\beta)}(1) = \frac{1}{n!}\cdot\frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{\Gamma(\alpha+1)\Gamma(\beta+n+1)} = \frac{1}{n!}\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)}.$$

From the equality (16), we find

$$1 = \sum_{i=0}^{n} b_i = \sum_{j=0}^{n} a_j P_j^{(\alpha,\beta)}(1) = \sum_{j=0}^{n} a_j \cdot \frac{1}{j!}\cdot\frac{\Gamma(\alpha+j+1)}{\Gamma(\alpha+1)}.$$

Therefore, using Cauchy–Schwarz inequality

$$\begin{aligned} 1 &= \sum_{j=0}^{n}\left(a_j\sqrt{\frac{2^{\alpha+\beta+1}\cdot\Gamma(j+\alpha+1)\Gamma(j+\beta+1)}{j!(\alpha+\beta+2j+1)\Gamma(\alpha+\beta+j+1)}}\right)\\ &\quad\times\left(\frac{1}{j!}\cdot\frac{\Gamma(\alpha+j+1)}{\Gamma(\alpha+1)}\cdot\sqrt{\frac{j!(\alpha+\beta+2j+1)\Gamma(\alpha+\beta+j+1)}{2^{\alpha+\beta+1}\Gamma(j+\alpha+1)\Gamma(j+\beta+1)}}\right)\\ &\le \sqrt{\sum_{j=0}^{n} a_j^2\cdot\frac{2^{\alpha+\beta+1}\Gamma(j+\alpha+1)\Gamma(j+\beta+1)}{j!(\alpha+\beta+2j+1)\Gamma(\alpha+\beta+j+1)}}\\ &\quad\times\sqrt{\sum_{j=0}^{n}\frac{\Gamma(\alpha+j+1)(\alpha+\beta+2j+1)\Gamma(\alpha+\beta+j+1)}{2^{\alpha+\beta+1}j!\Gamma(j+\beta+1)(\Gamma(\alpha+1))^2}}, \end{aligned}$$

i.e.,

$$\begin{aligned} &\sum_{j=0}^{n} a_j^2\cdot\frac{2^{\alpha+\beta+1}\Gamma(j+\alpha+1)\Gamma(j+\beta+1)}{j!(\alpha+\beta+2j+1)\Gamma(\alpha+\beta+j+1)}\\ &\quad\ge\left(\sum_{j=0}^{n}\frac{\Gamma(\alpha+j+1)(\alpha+\beta+2j+1)\Gamma(\alpha+\beta+j+1)}{2^{\alpha+\beta+1}j!\Gamma(j+\beta+1)(\Gamma(\alpha+1))^2}\right)^{-1}\\ &\quad= 2^{\alpha+\beta+1}(\Gamma(\alpha+1))^2\left(\sum_{j=0}^{n}\frac{\Gamma(\alpha+j+1)(\alpha+\beta+2j+1)\Gamma(\alpha+\beta+j+1)}{j!\Gamma(j+\beta+1)}\right)^{-1}. \end{aligned}$$

By Lemma 3, we find

$$\sum_{j=0}^{n} a_j^2 \cdot \frac{2^{\alpha+\beta+1}\Gamma(j+\alpha+1)\Gamma(j+\beta+1)}{j!(\alpha+\beta+2j+1)\Gamma(\alpha+\beta+j+1)}$$
$$\geq 2^{\alpha+\beta+1}(\Gamma(\alpha+1))^2 \cdot \frac{(\alpha+1)\Gamma(n+\beta+1)\cdot n!}{\Gamma(n+\alpha+\beta+2)\Gamma(n+\alpha+2)}$$
$$= \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\alpha+2)\Gamma(n+\beta+1)n!}{\Gamma(n+\alpha+\beta+2)\Gamma(n+\alpha+2)}. \tag{18}$$

From the relations (17) and (18), we have

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\left(\sum_{i=0}^{n} b_i x^i\right)^2 dx \geq \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\alpha+2)\Gamma(n+\beta+1)\cdot n!}{\Gamma(n+\alpha+2)\Gamma(n+\alpha+\beta+2)}.$$

We conclude that

$$\min_{b_i\in\mathbb{R},\ i=\overline{0,n}} \int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\left(\sum_{i=0}^{n} b_i x^i\right)^2 dx$$
$$= \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\alpha+2)\Gamma(n+\beta+1)\cdot n!}{\Gamma(n+\alpha+2)\Gamma(n+\alpha+\beta+2)}.$$

The proof is completed.

Remark 2. If we substitute $\alpha = \beta = 0$ in Proposition 5, we get the Bernstein inequality (1).

Corollary 1. *Let $b_i, i = 0, \ldots, n$ be real numbers such that $\sum_{i=0}^{n} b_i = 1$. Then*

$$\min_{b_i\in\mathbb{R},\ i=\overline{0,n}} \int_{0}^{1}(1-x^2)^{-\frac{1}{2}}\left(\sum_{i=0}^{n} b_i x^i\right)^2 dx = \frac{\pi}{2n+1}.$$

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Hilbert-Type Inequalities Including Some Operators, the Best Possible Constants and Applications: A Survey

Vandanjav Adiyasuren, Tserendorj Batbold, and Mario Krnić

Abstract The present work is a review article about some recent results dealing with Hilbert-type inequalities including certain operators in both integral and discrete case. A particular emphasis is given to inequalities including classical means operators. The constants appearing in all discussed inequalities are the best possible. For an illustration, some proofs are given, as well as some applications.

Keywords Hilbert inequality • Hilbert-type inequality • Hardy inequality • Knopp inequality • Carleman inequality • The best possible constant • Mean operator

Mathematics Subject Classification (2000): Primary 26D10, 26D15, Secondary 40A05, 33B15

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1 Introduction

The Hilbert inequality is one of the most important inequalities in mathematical analysis. Applications of this inequality in diverse fields of mathematics have certainly contributed to its importance. After its discovery, the Hilbert inequality was studied by numerous authors, who either reproved it using various techniques, or applied and generalized it in many different ways. For a comprehensive inspection of the initial development of the Hilbert inequality, the reader is referred to a classical monograph [14].

Nowadays, more than a century after its discovery, this problem area is still of interest to numerous authors. In 2005, Krnić and Pečarić [17] established a unified treatment of Hilbert-type inequalities with a general measurable kernel and weight functions. Here we just refer to Hilbert-type inequalities from [17] regarding a homogeneous kernel and power weight functions. Namely, if p and q are non-negative mutually conjugate exponents, that is, if $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, $q>1$, then

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} K_\lambda(x,y)f(x)g(y)dxdy$$
$$< L\left[\int_{\mathbb{R}_+} x^{1-\lambda+p(A_1-A_2)}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_{\mathbb{R}_+} y^{1-\lambda+q(A_2-A_1)}g^q(y)dy\right]^{\frac{1}{q}} \quad (1)$$

and

$$\left[\int_{\mathbb{R}_+} y^{(p-1)(\lambda-1)+p(A_1-A_2)}\left(\int_{\mathbb{R}_+} K_\lambda(x,y)f(x)dx\right)^p dy\right]^{\frac{1}{p}}$$
$$< L\left[\int_{\mathbb{R}_+} x^{1-\lambda+p(A_1-A_2)}f^p(x)dx\right]^{\frac{1}{p}}, \quad (2)$$

where $K_\lambda:\mathbb{R}_+^2\to\mathbb{R}$ is a non-negative homogeneous function of degree $-\lambda$, $\lambda>0$, $f,g:\mathbb{R}_+\to\mathbb{R}$ are non-negative functions such that $f,g\neq 0$ a.e. on $\mathbb{R}_+$, and $L=k_\lambda^{\frac{1}{p}}(pA_2)k_\lambda^{\frac{1}{q}}(2-\lambda-qA_1)$, $k_\lambda(\alpha)=\int_0^\infty K_\lambda(1,t)t^{-\alpha}dt$. Of course, A_1 and A_2 are real parameters such that all integrals in above inequalities converge.

Inequalities (1) and (2) are equivalent. Considering (1) with the kernel $K_\lambda(x,y)=(x+y)^{-1}$ and parameters $A_1=A_2=\frac{1}{pq}$, it follows that $\lambda=1$ and $L=\frac{\pi}{\sin\frac{\pi}{p}}$, so (1) reduces to one of the earliest versions of the Hilbert inequality (for more details, see [14]). Hence, inequalities related to (1) are usually referred to as Hilbert-type inequalities. On the other hand, inequality (2) and its consequences are referred to as Hardy–Hilbert-type inequalities, since (2) is a generalization of the classical Hardy inequality (for more details, see [17]). In this paper, for the reason of simplicity, a whole class of inequalities related to (1) and (2) will sometimes be referred to as Hilbert-type inequalities.

In paper [17], authors also derived discrete versions of inequalities (1) and (2), i.e., the relations

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K_\lambda(m,n)a_m b_n < L\left[\sum_{m=1}^{\infty} m^{1-\lambda+p(A_1-A_2)}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty} n^{1-\lambda+q(A_2-A_1)}b_n^q\right]^{\frac{1}{q}} \tag{3}$$

and

$$\left[\sum_{n=1}^{\infty} n^{(p-1)(\lambda-1)+p(A_1-A_2)}\left(\sum_{m=1}^{\infty} K_\lambda(m,n)a_m\right)^p\right]^{\frac{1}{p}} < L\left[\sum_{m=1}^{\infty} m^{1-\lambda+p(A_1-A_2)}a_m^p\right]^{\frac{1}{p}}, \tag{4}$$

which hold under some stronger conditions. Namely, when dealing with discrete Hilbert-type inequalities, some integral bounds are used for certain sums. Usually, such sums may be recognized as the lower Darboux sums for the corresponding integrals. Therefore, inequalities (3) and (4) hold if in addition K_λ is strictly decreasing in each argument, and parameters A_1 and A_2 are chosen so that $pA_2 \geq 0$ and $2-\lambda-qA_1 \geq 0$. Moreover, $(a_m)_{m\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are non-negative sequences, not identically equal to zero, and we assume convergence of all series appearing in (3) and (4).

Considering inequalities (1)–(4) with parameters A_1 and A_2 fulfilling condition $pA_2 + qA_1 = 2-\lambda$, the constant L reduces to $L = k_\lambda(pA_2)$. It was shown that such constant is the best possible in the corresponding inequalities (for more details, see [18, 24]). Hilbert-type inequalities may also be considered in the setting of non-conjugate exponents (see [10, 11]), but in that case there is no evidence that the constants appearing in the corresponding inequalities are the best possible. For comprehensive accounts on Hilbert inequality including history, different proofs, refinements and diverse applications, we refer to recent monograph [19] and references therein.

In the last few years, considerable attention is given to a class of Hilbert-type inequalities where the functions and sequences are replaced by certain integral or discrete operators. As an example, the classical Hardy operator $f \mapsto \frac{1}{x}\int_0^x f(t)dt$ represents the arithmetic mean in integral case. Such inequalities may be derived by virtue of Hilbert-type inequalities from this Introduction and several well-known classical inequalities, such as the Hardy, the Knopp inequality etc. But the most interesting fact in connection with this topic is that the constants appearing in these inequalities remain the best possible.

The present work is a review article of research of several authors in this area. More precisely, this paper is based on some 15 significant papers dealing with Hilbert-type inequalities including some integral and discrete operators (such as above mentioned classical Hardy operator), published in the course of the last few years. All results that will be discussed refer to homogeneous kernels and involve the best constants on their right-hand sides.

The paper is divided into six sections as follows: After this Introduction, in Sect. 2 we introduce notation and list some important classical inequalities necessary for studying Hilbert-type inequalities including classical means operators. In Sect. 3, we present the recent result about a unified treatment of two-dimensional Hilbert-type inequalities including classical mean operators in both integral and discrete case. To illustrate the technique, some proofs are also given, as well as some applications. Further, Sect. 4 deals with the so-called half-discrete case, while in Sect. 5 we discuss an extension to a multidimensional case. Finally, in Sect. 6, we discuss several new Hilbert-type inequalities involving some differential operators.

Since the present work is based on numerous papers written by different authors, the terminology in the paper is not quite unified. However, to avoid misunderstandings, some extra notation and definitions are presented when it is necessary.

2 Notation and Preliminaries

Throughout this paper $L^p(\mathbb{R}_+)$, $p \geq 1$ denotes the space of all Lebesgue measurable functions $f : \mathbb{R}_+ \to \mathbb{R}$ such that $\|f\|_{L^p(\mathbb{R}_+)} = \left(\int_{\mathbb{R}_+} |f(t)|^p dt\right)^{\frac{1}{p}} < \infty$. Similarly, l^p, $p \geq 1$, denotes the space of all real sequences $a = (a_n)_{n\in\mathbb{N}}$ such that $\|a\|_{l^p} = \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{\frac{1}{p}} < \infty$. In addition, $L^p(\mathbb{R}_+, \varphi)$, where $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is a non-negative measurable function, stands for the weighted Lebesgue space with the norm $\|f\|_{L^p(\mathbb{R}_+,\varphi)} = \left(\int_{\mathbb{R}_+} \varphi(t)|f(t)|^p dt\right)^{\frac{1}{p}} < \infty$.

Hilbert-type inequalities we deal with in this article will often contain constants expressed in terms of some special functions. Throughout this article $B(\cdot,\cdot)$ stands for the usual Beta function $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$, $a, b > 0$, while $\Gamma(\cdot)$ denotes the usual Gamma function defined by $\Gamma(a) = \int_{\mathbb{R}_+} t^{a-1}e^{-t}\,dt$, $a > 0$.

Besides Hilbert-type inequalities presented in the Introduction, we will need several other important inequalities. The first of them is the well-known Hardy inequality

$$\int_{\mathbb{R}_+} \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_{\mathbb{R}_+} f^p(x)dx, \tag{5}$$

which holds for $p > 1$ and for all non-negative functions $f : \mathbb{R}_+ \to \mathbb{R}$, provided that $0 < \|f\|_{L^p(\mathbb{R}_+)} < \infty$. Its discrete version asserts that

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} a_k \right)^p < \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p, \tag{6}$$

where $p > 1$ and $a = (a_n)_{n \in \mathbb{N}}$ is a non-negative sequence such that $0 < \|a\|_{l^p} < \infty$. It should be noticed here that the constant $\left(\frac{p}{p-1}\right)^p$ is the best possible in both inequalities. For comprehensive accounts on Hardy inequality including history, different proofs, refinements, and diverse applications, we refer to recent monograph [21] and references therein.

Observe that the Hardy inequality includes arithmetic mean in integral and discrete case. We shall also be occupied with the corresponding inequalities including a geometric mean. The integral version of such inequality is known as the Knopp inequality, i.e.,

$$\int_{\mathbb{R}_+} \exp\left(\frac{1}{x} \int_0^x \log f(t) dt \right) dx < e \int_{\mathbb{R}_+} f(x) dx, \tag{7}$$

while its discrete version is known as the Carleman inequality:

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} a_k \right)^{\frac{1}{n}} < e \sum_{n=1}^{\infty} a_n. \tag{8}$$

The constant e appearing in both inequalities is the best possible (see [23]).

In 2005, Yang [31] derived the corresponding inequalities equipped with a generalized harmonic mean. Namely, integral version asserts that

$$\int_{\mathbb{R}_+} \left(\frac{x}{\int_0^x f^{-r}(t) dt} \right)^{\frac{1}{r}} dx < (1+r)^{\frac{1}{r}} \int_{\mathbb{R}_+} f(x) dx \tag{9}$$

holds for $r > 0$, while its discrete analogue holds for $0 < r \leq 1$:

$$\sum_{n=1}^{\infty} \left(\frac{n}{\sum_{k=1}^{n} a_k^{-r}} \right)^{\frac{1}{r}} < (1+r)^{\frac{1}{r}} \sum_{n=1}^{\infty} a_n. \tag{10}$$

Moreover, Yang also proved that inequalities (9) and (10) include the best constant $(1+r)^{\frac{1}{r}}$. In accordance to [31], inequalities (9) and (10) will be referred to as the integral and discrete Hardy–Carleman inequality.

For the reader's convenience, we define integral arithmetic, geometric, and harmonic mean operators $\mathscr{A}, \mathscr{G}, \mathscr{H} : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$ by

$$(\mathscr{A}f)(x) = \frac{1}{x}\int_0^x f(t)dt,$$

$$(\mathscr{G}f)(x) = \exp\left(\frac{1}{x}\int_0^x \log f(t)dt\right),$$

$$(\mathscr{H}f)(x) = \frac{x}{\int_0^x f^{-1}(t)dt}.$$

Obviously, the above operators are well-defined since inequalities (5), (7), and (9) may, respectively, be rewritten as

$$\|\mathscr{A}f\|_{L^p(\mathbb{R}_+)} < \frac{p}{p-1}\|f\|_{L^p(\mathbb{R}_+)}, \tag{11}$$

$$\|\mathscr{G}f\|_{L^p(\mathbb{R}_+)} < e^{\frac{1}{p}}\|f\|_{L^p(\mathbb{R}_+)}, \tag{12}$$

$$\|\mathscr{H}f\|_{L^p(\mathbb{R}_+)} < \left(1+\frac{1}{p}\right)\|f\|_{L^p(\mathbb{R}_+)}. \tag{13}$$

Moreover, since these inequalities include the best constants on their right-hand sides, we are able to compute norms of the corresponding integral operators. Namely, since $\|\mathscr{A}\| = \sup_{f\neq 0}\frac{\|\mathscr{A}f\|_{L^p(\mathbb{R}_+)}}{\|f\|_{L^p(\mathbb{R}_+)}}$, it follows that $\|\mathscr{A}\| = \frac{p}{p-1}$, and similarly $\|\mathscr{G}\| = e^{\frac{1}{p}}$, $\|\mathscr{H}\| = 1+\frac{1}{p}$.

Discrete versions of means operators $\mathscr{A}, \mathscr{G}, \mathscr{H} : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$, i.e., the operators $\overline{\mathscr{A}}, \overline{\mathscr{G}}, \overline{\mathscr{H}} : l^p \to l^p$ are defined by

$$(\overline{\mathscr{A}}a)_n = \frac{\sum_{k=1}^n a_k}{n},$$

$$(\overline{\mathscr{G}}a)_n = \left(\prod_{k=1}^n a_k\right)^{\frac{1}{n}},$$

$$(\overline{\mathscr{H}}a)_n = \frac{n}{\sum_{k=1}^n a_k^{-1}}.$$

With this notation, discrete inequalities (6), (8), and (10), respectively, read

$$\|\overline{\mathscr{A}}a\|_{l^p} < \frac{p}{p-1}\|a\|_{l^p}, \tag{14}$$

$$\|\overline{\mathscr{G}}a\|_{l^p} < e^{\frac{1}{p}}\|a\|_{l^p}, \tag{15}$$

$$\|\overline{\mathscr{H}}a\|_{l^p} < \left(1+\frac{1}{p}\right)\|a\|_{l^p}. \tag{16}$$

Clearly, due to the best constants, above inequalities provide norms of the corresponding operators, that is, $\|\overline{\mathscr{A}}\| = \frac{p}{p-1}$, $\|\overline{\mathscr{G}}\| = e^{\frac{1}{p}}$, and $\|\overline{\mathscr{H}}\| = 1 + \frac{1}{p}$.

3 Hilbert-Type Inequalities Involving Means Operators

In this section we deal with two-dimensional Hilbert-type inequalities, in both integral and discrete case, involving arithmetic, geometric, harmonic, as well as some related operators. It should be noticed here that the inequalities appearing in this section refer to non-negative conjugate parameters p and q, i.e., to parameters such that $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, and $q > 1$. A pair of non-negative conjugate parameters will be denoted in this way throughout the whole paper.

We start this overview with some particular results involving arithmetic mean operators $\mathscr{A}$ and $\overline{\mathscr{A}}$.

3.1 Some Particular Results

In 2010, based on the Hardy integral inequality, Das and Sahoo [12] obtained the following pair of Hilbert-type inequalities involving the arithmetic mean operator $\mathscr{A}$.

Theorem 1 ([12]). *If r, s, λ are positive real parameters such that $\lambda = r + s$, then the inequalities*

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{x^{r-\frac{1}{q}} y^{s-\frac{1}{p}}}{(x+y)^{\lambda}} (\mathscr{A} f)(x)(\mathscr{A} g)(y)\,dx\,dy < pqB(r,s)\|f\|_{L^p(\mathbb{R}_+)}\|g\|_{L^q(\mathbb{R}_+)} \tag{17}$$

and

$$\left[\int_{\mathbb{R}_+} y^{ps-1}\left(\int_{\mathbb{R}_+} \frac{x^{r-\frac{1}{q}}}{(x+y)^{\lambda}}(\mathscr{A} f)(x)\,dx\right)^p dy\right]^{\frac{1}{p}} < qB(r,s)\|f\|_{L^p(\mathbb{R}_+)} \tag{18}$$

hold for all non-negative functions $f, g : \mathbb{R}_+ \to \mathbb{R}$ such that $0 < \|f\|_{L^p(\mathbb{R}_+)} < \infty$ and $0 < \|g\|_{L^q(\mathbb{R}_+)} < \infty$. In addition, the constants $pqB(r,s)$ and $qB(r,s)$ are the best possible in the corresponding inequalities.

It should be noticed here that some particular cases of inequality (17) were studied in [28], few years earlier. Furthermore, with the assumption $\lambda > 2$, Das and Sahoo also proved a discrete version of Theorem 1.

Theorem 2 ([12]). *Let $r, s > 0$ and $\lambda > 2$ be real parameters such that $\lambda = r + s$. Then the inequalities*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^{r-\frac{1}{q}}n^{s-\frac{1}{p}}}{(m+n)^{\lambda}}(\overline{\mathscr{A}}a)_m(\overline{\mathscr{A}}b)_n < pqB(r,s)\|a\|_{l^p}\|b\|_{l^q} \tag{19}$$

and

$$\left[\sum_{n=1}^{\infty}n^{ps-1}\left(\sum_{m=1}^{\infty}\frac{m^{r-\frac{1}{q}}}{(m+n)^{\lambda}}(\overline{\mathscr{A}}a)_m\right)^p\right]^{\frac{1}{p}} < qB(r,s)\|a\|_{l^p} \tag{20}$$

hold for all non-negative sequences $a = (a_m)_{m\in\mathbb{N}}$ and $b = (b_n)_{n\in\mathbb{N}}$ satisfying $0 < \|a\|_{l^p} < \infty$ and $0 < \|b\|_{l^q} < \infty$. In addition, the constants $pqB(r,s)$ and $qB(r,s)$ are the best possible in the corresponding inequalities.

Observe also that reference [13] provides the corresponding result for the kernel $1/\max\{x^{\lambda}, y^{\lambda}\}$, with the best possible constant. Moreover, Adiyasuren and Batbold [3] also obtained some related inequalities:

Theorem 3 ([3]). *Let α and β be such that $p > \frac{1}{\alpha}, q > \frac{1}{\beta}, 0 < \alpha, \beta \leq 1$, and let $\lambda, s, r > 0$ with $s + r = \lambda$. Then the inequalities*

$$\begin{aligned}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}&\frac{x^{r-\frac{1}{q}}y^{s-\frac{1}{p}}}{\max\{x^{\lambda}, y^{\lambda}\}}(\mathscr{A}f)^{\alpha}(x)(\mathscr{A}g)^{\beta}(y)dxdy\\ &< \frac{\lambda}{rs}\left(\frac{\alpha p}{\alpha p-1}\right)^{\alpha}\left(\frac{\beta q}{\beta q-1}\right)^{\beta}\|f^{\alpha}\|_{L^p(\mathbb{R}_+)}\|g^{\beta}\|_{L^q(\mathbb{R}_+)}\end{aligned} \tag{21}$$

and

$$\begin{aligned}\left[\int_{\mathbb{R}_+}y^{ps-1}\left(\int_{\mathbb{R}_+}\frac{x^{r-\frac{1}{q}}}{\max\{x^{\lambda}, y^{\lambda}\}}(\mathscr{A}f)^{\alpha}(x)dx\right)^p dy\right]^{\frac{1}{p}}\\ < \left(\frac{\lambda}{rs}\right)\left(\frac{\alpha p}{\alpha p-1}\right)^{\alpha}\|f^{\alpha}\|_{L^p(\mathbb{R}_+)}\end{aligned} \tag{22}$$

hold for all non-negative functions $f, g : \mathbb{R}_+ \to \mathbb{R}$ such that $0 < \|f^{\alpha}\|_{L^p(\mathbb{R}_+)} < \infty$ and $0 < \|g^{\beta}\|_{L^q(\mathbb{R}_+)} < \infty$. In addition, the constants appearing on the right-hand sides of (21) and (22) are the best possible.

Theorem 4 ([3]). *With the assumptions of Theorem 3, inequalities*

$$\begin{aligned}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\frac{x^{r-\frac{1}{q}}y^{s-\frac{1}{p}}}{|x-y|^{\lambda}}(\mathscr{A}f)^{\alpha}(x)(\mathscr{A}g)^{\beta}(y)dxdy &< (B(s,1-\lambda)+B(r,1-\lambda))\\ \times\left(\frac{\alpha p}{\alpha p-1}\right)^{\alpha}\left(\frac{\beta q}{\beta q-1}\right)^{\beta}\|f^{\alpha}\|_{L^p(\mathbb{R}_+)}\|g^{\beta}\|_{L^q(\mathbb{R}_+)}\end{aligned} \tag{23}$$

and

$$\left[\int_{\mathbb{R}_+} y^{ps-1}\left(\int_{\mathbb{R}_+}\frac{x^{r-\frac{1}{q}}}{|x-y|^{\lambda}}(\mathscr{A}f)^{\alpha}(x)dx\right)^{p}dy\right]^{\frac{1}{p}}$$
$$< (B(s,1-\lambda)+B(r,1-\lambda))\left(\frac{\alpha p}{\alpha p-1}\right)^{\alpha}\|f^{\alpha}\|_{L^p(\mathbb{R}_+)} \tag{24}$$

hold for all non-negative functions $f, g : \mathbb{R}_+ \to \mathbb{R}$ *such that* $0 < \|f^{\alpha}\|_{L^p(\mathbb{R}_+)} < \infty$ *and* $0 < \|g^{\beta}\|_{L^q(\mathbb{R}_+)} < \infty$. *Moreover, the constants appearing on the right-hand sides of* (23) *and* (24) *are the best possible.*

All inequalities in this subsection are simple consequences of Hilbert-type inequalities (1)–(4) and the Hardy inequality. For the proofs of the best possible constants, the reader is referred to the corresponding references.

3.2 A General Homogeneous Kernel

Observe that all results in the previous subsection have a homogeneity in common. Now, we give an extension of Theorems 1–4 to a general homogeneous case. Throughout the whole paper, we deal with the constant

$$c_{\lambda}(s) = \int_{\mathbb{R}_+} K_{\lambda}(1,t)t^{s-1}dt,$$

where $K_{\lambda} : \mathbb{R}_+^2 \to \mathbb{R}$ is a non-negative homogeneous function of degree $-\lambda$ and s is a non-negative real parameter. Observe that $c_{\lambda}(s) = k_{\lambda}(1-s)$, where $k_{\lambda}(\cdot)$ is the constant appearing in relations (1)–(4).

It should be noticed here that Sulaiman (see [29, 30]) investigated some related results with a homogeneous kernel, without considering the problem of the best constants.

In already mentioned reference [3], Adiyasuren and Batbold derived a pair of Hilbert-type inequalities with the arithmetic mean operator $\mathscr{A}$, referring to a general homogeneous kernel.

Theorem 5 ([3]). *Let* α *and* β *be such that* $p > \frac{1}{\alpha}, q > \frac{1}{\beta}, 0 < \alpha, \beta \le 1$, *and let* $s + r = \lambda$, *where* $\lambda, s, r > 0$. *Further, let* $K_{\lambda} : \mathbb{R}_+^2 \to \mathbb{R}$ *be a non-negative homogeneous function of degree* $-\lambda$, *provided that*

$$0<c_{\lambda}(s) < \infty, 0<\int_{\mathbb{R}_+} K_{\lambda}(1,u)u^{s-\frac{1}{p}-\beta}du<\infty, 0<\int_{\mathbb{R}_+} K_{\lambda}(1,u)u^{r-\frac{1}{q}-\alpha}du<\infty.$$

Then the inequalities

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} K_\lambda(x,y)x^{r-\frac{1}{q}}y^{s-\frac{1}{p}}(\mathscr{A}f)^\alpha(x)(\mathscr{A}g)^\beta(y)dxdy$$
$$< \left(\frac{\alpha p}{\alpha p-1}\right)^\alpha \left(\frac{\beta q}{\beta q-1}\right)^\beta c_\lambda(s)\|f^\alpha\|_{L^p(\mathbb{R}_+)}\|g^\beta\|_{L^q(\mathbb{R}_+)} \tag{25}$$

and

$$\left[\int_{\mathbb{R}_+} y^{ps-1}\left(\int_{\mathbb{R}_+} K_\lambda(x,y)x^{r-\frac{1}{q}}(\mathscr{A}f)^\alpha(x)dx\right)^p dy\right]^{\frac{1}{p}}$$
$$< \left(\frac{\alpha p}{\alpha p-1}\right)^\alpha c_\lambda(s)\|f^\alpha\|_{L^p(\mathbb{R}_+)} \tag{26}$$

hold for all non-negative functions $f, g : \mathbb{R}_+ \to \mathbb{R}$ *such that* $0 < \|f^\alpha\|_{L^p(\mathbb{R}_+)} < \infty$ *and* $0 < \|g^\beta\|_{L^q(\mathbb{R}_+)} < \infty$. *In addition, the constants* $\left(\frac{\alpha p}{\alpha p-1}\right)^\alpha \left(\frac{\beta q}{\beta q-1}\right)^\beta c_\lambda(s)$ *and* $\left(\frac{\alpha p}{\alpha p-1}\right)^\alpha c_\lambda(s)$ *appearing in* (25) *and* (26) *are the best possible.*

Remark 1. Clearly, if $K_\lambda(x,y) = (x+y)^{-\lambda}$ and $\alpha = \beta = 1$, Theorem 5 reduces to Theorem 1. In a similar manner, Theorem 5 also represents an extension of Theorems 3 and 4.

Recently, Adiyasuren et al. [5] derived discrete analogues of relations (25) and (26), as well as the corresponding analogues with geometric and harmonic mean operators in both integral and discrete case. We first give the corresponding integral results including operators $\mathscr{G}$ and $\mathscr{H}$.

Theorem 6 ([5]). *Let* r, s, λ *be non-negative real parameters such that* $\lambda = r + s$. *Further, suppose* $K_\lambda : \mathbb{R}_+^2 \to \mathbb{R}$ *is a non-negative homogeneous function of degree* $-\lambda$ *such that* $0 < c_\lambda(s) < \infty$. *Then the inequalities*

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} K_\lambda(x,y)x^{r-\frac{1}{q}}y^{s-\frac{1}{p}}(\mathscr{G}f)(x)(\mathscr{G}g)(y)dxdy$$
$$< e\cdot c_\lambda(s)\|f\|_{L^p(\mathbb{R}_+)}\|g\|_{L^q(\mathbb{R}_+)} \tag{27}$$

and

$$\left[\int_{\mathbb{R}_+} y^{ps-1}\left(\int_{\mathbb{R}_+} K_\lambda(x,y)x^{r-\frac{1}{q}}(\mathscr{G}f)(x)dx\right)^p dy\right]^{\frac{1}{p}} < e^{\frac{1}{p}}c_\lambda(s)\|f\|_{L^p(\mathbb{R}_+)} \tag{28}$$

hold for all non-negative functions $f, g : \mathbb{R}_+ \to \mathbb{R}$ *such that* $0 < \|f\|_{L^p(\mathbb{R}_+)} < \infty$ *and* $0 < \|g\|_{L^q(\mathbb{R}_+)} < \infty$*. In addition, the constants* $e \cdot c_\lambda(s)$ *and* $e^{\frac{1}{p}} c_\lambda(s)$ *are the best possible in* (27) *and* (28)*.*

Theorem 7 ([5]). *With the assumptions of Theorem 6, the inequalities*

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} K_\lambda(x, y) x^{r-\frac{1}{q}} y^{s-\frac{1}{p}} (\mathscr{H} f)(x)(\mathscr{H} g)(y) dx dy$$
$$< \left(2 + \frac{1}{pq}\right) c_\lambda(s) \|f\|_{L^p(\mathbb{R}_+)} \|g\|_{L^q(\mathbb{R}_+)} \tag{29}$$

and

$$\left[\int_{\mathbb{R}_+} y^{ps-1} \left(\int_{\mathbb{R}_+} K_\lambda(x, y) x^{r-\frac{1}{q}} (\mathscr{H} f)(x) dx\right)^p dy\right]^{\frac{1}{p}}$$
$$< \left(1 + \frac{1}{p}\right) c_\lambda(s) \|f\|_{L^p(\mathbb{R}_+)} \tag{30}$$

hold for all non-negative functions $f, g : \mathbb{R}_+ \to \mathbb{R}$ *such that* $0 < \|f\|_{L^p(\mathbb{R}_+)} < \infty$ *and* $0 < \|g\|_{L^q(\mathbb{R}_+)} < \infty$*. In addition, the constants* $\left(2+\frac{1}{pq}\right)c_\lambda(s)$ *and* $\left(1+\frac{1}{p}\right)c_\lambda(s)$ *are the best possible in the corresponding inequalities.*

The methods of proving Theorems 5–7 are quite similar. For an illustration, we give the proof of Theorem 6.

Proof (Proof of Theorem 6). The starting point in the proof is inequality (1) with parameters $A_1 = \frac{1-r}{q}$, $A_2 = \frac{1-s}{p}$, and with functions f and g, respectively, replaced by $x^{r-\frac{1}{q}}(\mathscr{G} f)(x)$ and $y^{s-\frac{1}{p}}(\mathscr{G} g)(y)$, that is, the inequality

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} K_\lambda(x, y) x^{r-\frac{1}{q}} y^{s-\frac{1}{p}} (\mathscr{G} f)(x)(\mathscr{G} g)(y) dx dy$$
$$< k_\lambda(1-s) \|\mathscr{G} f\|_{L^p(\mathbb{R}_+)} \|\mathscr{G} g\|_{L^q(\mathbb{R}_+)} = c_\lambda(s) \|\mathscr{G} f\|_{L^p(\mathbb{R}_+)} \|\mathscr{G} g\|_{L^q(\mathbb{R}_+)}.$$

Now, due to the Knopp inequality (12), it follows that $\|\mathscr{G} f\|_{L^p(\mathbb{R}_+)} < e^{\frac{1}{p}} \|f\|_{L^p(\mathbb{R}_+)}$ and $\|\mathscr{G} g\|_{L^q(\mathbb{R}_+)} < e^{\frac{1}{q}} \|g\|_{L^q(\mathbb{R}_+)}$, which yields inequality (27). Similarly, inequality (28) follows from Hardy–Hilbert-type inequality (2) and the Knopp inequality.

In order to prove that inequalities (27) and (28) involve the best constants on their right-hand sides, we first suppose that there exists a positive constant C, smaller than $e \cdot c_\lambda(s)$, such that the inequality

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} K_\lambda(x,y)x^{r-\frac{1}{q}}y^{s-\frac{1}{p}}(\mathscr{G}f)(x)(\mathscr{G}g)(y)dxdy < C\|f\|_{L^p(\mathbb{R}_+)}\|g\|_{L^q(\mathbb{R}_+)} \tag{31}$$

holds for all non-negative functions $f, g : \mathbb{R}_+ \to \mathbb{R}$ with $0 < \|f\|_{L^p(\mathbb{R}_+)} < \infty$ and $0 < \|g\|_{L^q(\mathbb{R}_+)} < \infty$.

Considering the above inequality with functions $\tilde{f}, \tilde{g} : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$\tilde{f}(x) = \begin{cases} 1, & 0 < x < 1 \\ e^{-\frac{1}{p}} x^{\frac{-\varepsilon-1}{p}}, & x \ge 1 \end{cases}, \quad \tilde{g}(y) = \begin{cases} 1, & 0 < y < 1 \\ e^{-\frac{1}{q}} y^{\frac{-\varepsilon-1}{q}}, & y \ge 1 \end{cases},$$

where $\varepsilon > 0$ is sufficiently small number, its right-hand side reduces to

$$C\|\tilde{f}\|_{L^p(\mathbb{R}_+)}\|\tilde{g}\|_{L^q(\mathbb{R}_+)} = \frac{C}{\varepsilon}\left(\varepsilon + \frac{1}{e}\right). \tag{32}$$

On the other hand, since

$$(\mathscr{G}\tilde{f})(x) = \begin{cases} 1, & 0 < x < 1 \\ e^{\frac{\varepsilon}{p} - \frac{\varepsilon}{xp}} x^{\frac{-\varepsilon-1}{p}}, & x \ge 1 \end{cases}$$

and

$$(\mathscr{G}\tilde{g})(y) = \begin{cases} 1, & 0 < y < 1 \\ e^{\frac{\varepsilon}{q} - \frac{\varepsilon}{yq}} y^{\frac{-\varepsilon-1}{q}}, & y \ge 1 \end{cases},$$

the well-known Fubini theorem and the change of variables $t = \frac{y}{x}$ imply the following series of relations:

$$\begin{aligned}
&\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} K_\lambda(x,y)x^{r-\frac{1}{q}}y^{s-\frac{1}{p}}(\mathscr{G}\tilde{f})(x)(\mathscr{G}\tilde{g})(y)dxdy \\
&\quad > \int_1^\infty\int_1^\infty K_\lambda(x,y)x^{r-\frac{1}{q}}y^{s-\frac{1}{p}}(\mathscr{G}\tilde{f})(x)(\mathscr{G}\tilde{g})(y)dxdy \\
&\quad = \int_1^\infty\int_1^\infty K_\lambda(x,y)x^{r-\frac{\varepsilon}{p}-1}y^{s-\frac{\varepsilon}{q}-1}e^{\varepsilon-\frac{\varepsilon}{xp}-\frac{\varepsilon}{yq}}dxdy \\
&\quad \ge \int_1^\infty\int_1^\infty K_\lambda(x,y)x^{r-\frac{\varepsilon}{p}-1}y^{s-\frac{\varepsilon}{q}-1}dxdy
\end{aligned}$$

$$= \int_1^\infty x^{-\varepsilon-1} \int_{\frac{1}{x}}^\infty K_\lambda(1,t) t^{s-\frac{\varepsilon}{q}-1} dt dx$$
$$= \frac{1}{\varepsilon} \int_1^\infty K_\lambda(1,t) t^{s-\frac{\varepsilon}{q}-1} dt + \int_1^\infty x^{-\varepsilon-1} \int_{\frac{1}{x}}^1 K_\lambda(1,t) t^{s-\frac{\varepsilon}{q}-1} dt dx$$
$$= \frac{1}{\varepsilon} \left(\int_1^\infty K_\lambda(1,t) t^{s-\frac{\varepsilon}{q}-1} dt + \int_0^1 K_\lambda(1,t) t^{s+\frac{\varepsilon}{p}-1} dt \right). \tag{33}$$

Now, multiplying both sides of inequality (31) by ε, relations (32) and (33) yield inequality

$$\int_1^\infty K_\lambda(1,t) t^{s-\frac{\varepsilon}{q}-1} dt + \int_0^1 K_\lambda(1,t) t^{s+\frac{\varepsilon}{p}-1} dt < C\left(\varepsilon + \frac{1}{e}\right).$$

Finally, when ε goes to 0, it follows that $e \cdot c_\lambda(s) \leq C$, which is in contrast to our hypothesis. Therefore, the constant $e \cdot c_\lambda(s)$ is the best possible in (27).

It remains to show that $e^{\frac{1}{p}} c_\lambda(s)$ is the best possible constant in (28). Similarly to above discussion, suppose that there exists a constant C' smaller than $e^{\frac{1}{p}} c_\lambda(s)$ such that inequality

$$\left[\int_{\mathbb{R}_+} y^{ps-1} \left(\int_{\mathbb{R}_+} K_\lambda(x,y) x^{r-\frac{1}{q}} (\mathscr{G} f)(x) dx \right)^p dy \right]^{\frac{1}{p}} < C' \|f\|_{L^p(\mathbb{R}_+)}$$

holds for all non-negative functions $f : \mathbb{R}_+ \to \mathbb{R}$ such that $0 < \|f\|_{L^p(\mathbb{R}_+)} < \infty$. Then, utilizing the well-known Hölder and the Knopp inequality, we have

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} K_\lambda(x,y) x^{r-\frac{1}{q}} y^{s-\frac{1}{p}} (\mathscr{G} f)(x) (\mathscr{G} g)(y) dx dy$$
$$= \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} K_\lambda(x,y) x^{r-\frac{1}{q}} y^{s-\frac{1}{p}} (\mathscr{G} f)(x) dx \right) (\mathscr{G} g)(y) dy$$
$$\leq \left[\int_{\mathbb{R}_+} y^{ps-1} \left(\int_{\mathbb{R}_+} K_\lambda(x,y) x^{r-\frac{1}{q}} (\mathscr{G} f)(x) dx \right)^p dy \right]^{\frac{1}{p}} \|\mathscr{G} g\|_{L^q(\mathbb{R}_+)}$$
$$< C' e^{\frac{1}{q}} \|f\|_{L^p(\mathbb{R}_+)} \|g\|_{L^q(\mathbb{R}_+)},$$

which results that the constant $e \cdot c_\lambda(s)$ is not the best possible in (27), since $C' e^{\frac{1}{q}} < c_\lambda(s) e^{\frac{1}{p}} e^{\frac{1}{q}} = e \cdot c_\lambda(s)$. This contradiction completes the proof.

In order to prove Theorem 5, it is necessary to use Hardy integral inequality (5), while the proof of Theorem 7 is accompanied with Hardy–Carleman integral inequality (9). Of course, to establish the best constants in these theorems, it is necessary to find suitable functions to obtain contradiction, as in the proof of Theorem 6 (for more details, see [5]).

Paper [5] also provides discrete versions of Theorems 5–7, including discrete means operators $\overline{\mathscr{A}}$, $\overline{\mathscr{G}}$, and $\overline{\mathscr{H}}$. Discrete Hilbert-type inequalities are more complicated than the integral ones. Namely, in order to derive discrete forms of the corresponding integral inequalities, it is necessary to estimate certain sums by integrals, which requires some extra conditions regarding a kernel and the weight functions.

Theorem 8 ([5]). *Let r, s, λ be real parameters such that $0 < r, s \leq 1$ and $\lambda = r + s$, and let $K_\lambda : \mathbb{R}_+^2 \to \mathbb{R}$ be a non-negative homogeneous function of degree $-\lambda$, strictly decreasing in each argument, such that $0 < c_\lambda(s) < \infty$. Then the inequalities*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K_\lambda(m,n) m^{r-\frac{1}{q}} n^{s-\frac{1}{p}} (\overline{\mathscr{A}}a)_m (\overline{\mathscr{A}}b)_n < pq c_\lambda(s) \|a\|_{l^p} \|b\|_{l^q} \tag{34}$$

and

$$\left[\sum_{n=1}^{\infty} n^{ps-1} \left(\sum_{m=1}^{\infty} K_\lambda(m,n) m^{r-\frac{1}{q}} (\overline{\mathscr{A}}a)_m\right)^p\right]^{\frac{1}{p}} < q c_\lambda(s) \|a\|_{l^p} \tag{35}$$

hold for all non-negative sequences $a = (a_m)_{m\in\mathbb{N}}$ and $b = (b_n)_{n\in\mathbb{N}}$ satisfying $0 < \|a\|_{l^p} < \infty$ and $0 < \|b\|_{l^q} < \infty$. In addition, the constants $pqc_\lambda(s)$ and $qc_\lambda(s)$ are the best possible in the corresponding inequalities.

Theorem 9 ([5]). *With the assumptions as in Theorem 8, the inequalities*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K_\lambda(m,n) m^{r-\frac{1}{q}} n^{s-\frac{1}{p}} (\overline{\mathscr{G}}a)_m (\overline{\mathscr{G}}b)_n < e \cdot c_\lambda(s) \|a\|_{l^p} \|b\|_{l^q} \tag{36}$$

and

$$\left[\sum_{n=1}^{\infty} n^{ps-1} \left(\sum_{m=1}^{\infty} K_\lambda(m,n) m^{r-\frac{1}{q}} (\overline{\mathscr{G}}a)_m\right)^p\right]^{\frac{1}{p}} < e^{\frac{1}{p}} c_\lambda(s) \|a\|_{l^p} \tag{37}$$

hold for all non-negative sequences $a = (a_m)_{m\in\mathbb{N}}$ and $b = (b_n)_{n\in\mathbb{N}}$, $0 < \|a\|_{l^p} < \infty$, $0 < \|b\|_{l^q} < \infty$. In addition, the constants $e \cdot c_\lambda(s)$ and $e^{\frac{1}{p}} c_\lambda(s)$ are the best possible in the corresponding inequalities.

Theorem 10 ([5]). *With the assumptions of Theorem 8, the inequalities*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K_\lambda(m,n) m^{r-\frac{1}{q}} n^{s-\frac{1}{p}} (\overline{\mathscr{H}}a)_m (\overline{\mathscr{H}}b)_n < \left(2 + \frac{1}{pq}\right) c_\lambda(s) \|a\|_{l^p} \|b\|_{l^q} \tag{38}$$

and

$$\left[\sum_{n=1}^{\infty} n^{ps-1}\left(\sum_{m=1}^{\infty} K_\lambda(m,n)m^{r-\frac{1}{q}}(\overline{\mathscr{H}}a)_m\right)^p\right]^{\frac{1}{p}} < \left(1+\frac{1}{p}\right)c_\lambda(s)\|a\|_{l^p} \tag{39}$$

hold for all non-negative sequences $a = (a_m)_{m\in\mathbb{N}}$ *and* $b = (b_n)_{n\in\mathbb{N}}$*, provided that* $0 < \|a\|_{l^p} < \infty$ *and* $0 < \|b\|_{l^q} < \infty$*. In addition, the constants* $\left(2+\frac{1}{pq}\right)c_\lambda(s)$ *and* $\left(1+\frac{1}{p}\right)c_\lambda(s)$ *are the best possible in the corresponding inequalities.*

To illustrate the discrete case, we provide the proof of Theorem 8. For the proofs of the remaining theorems, the reader is referred to [5].

Proof (Proof of Theorem 8). Utilizing discrete Hilbert-type inequality (3) with sequences $m^{r-\frac{1}{q}}(\overline{\mathscr{A}}a)_m$, $n^{s-\frac{1}{p}}(\overline{\mathscr{A}}b)_n$, and with parameters $A_1 = \frac{1-r}{q}$, $A_2 = \frac{1-s}{p}$, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K_\lambda(m,n)m^{r-\frac{1}{q}}n^{s-\frac{1}{p}}(\overline{\mathscr{A}}a)_m(\overline{\mathscr{A}}b)_n < c_\lambda(s)\|\overline{\mathscr{A}}a\|_{l^p}\|\overline{\mathscr{A}}b\|_{l^q}.$$

Now, double use of discrete Hardy inequality (14) yields (34). Similarly, inequality (35) follows by virtue of discrete Hardy–Hilbert-type inequality (4).

Now, we prove that the constants appearing in (34) and (35) are the best possible. First, suppose that there exists a positive constant $0 < K < pqc_\lambda(s)$ so that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K_\lambda(m,n)m^{r-\frac{1}{q}}n^{s-\frac{1}{p}}(\overline{\mathscr{A}}a)_m(\overline{\mathscr{A}}b)_n < K\|a\|_{l^p}\|b\|_{l^q} \tag{40}$$

holds for $0 < \|a\|_{l^p} < \infty$ and $0 < \|b\|_{l^q} < \infty$. Let $\tilde{L}$ and $\tilde{R}$, respectively, denote the left-hand side and the right-hand side of (40) equipped with the sequences

$$\tilde{a}_m = \begin{cases} m^{-\frac{1}{p}}, & m \le N \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \tilde{b}_n = \begin{cases} n^{-\frac{1}{q}}, & n \le N \\ 0, & \text{otherwise} \end{cases}, \tag{41}$$

where $N \in \mathbb{N}$ is fixed. Then, the right-hand side of (40) may be bounded from above by a natural logarithm function:

$$\tilde{R} = K\|\tilde{a}\|_{l^p}\|\tilde{b}\|_{l^q} = K\left(\sum_{m=1}^{N}\frac{1}{m}\right) = K\left(1+\sum_{m=2}^{N}\frac{1}{m}\right)$$
$$< K\left(1+\int_1^N\frac{dx}{x}\right) = K(1+\log N). \tag{42}$$

Our next intention is to estimate the left-hand side of inequality (40) from below. More precisely, considering $\sum_{k=1}^{m} k^{-\frac{1}{p}}$ as the upper Darboux sum for the function $h(x) = x^{-\frac{1}{p}}$ on the segment $[1, m+1]$, we have

$$\sum_{k=1}^{m} k^{-\frac{1}{p}} > \int_1^{m+1} x^{\frac{1}{p}} dx > \int_1^{m} x^{-\frac{1}{p}} dx = q(m^{\frac{1}{q}} - 1),$$

and consequently,

$$(\overline{\mathscr{A}}\tilde{a})_m > \frac{q(m^{\frac{1}{q}} - 1)}{m} = qm^{-\frac{1}{p}}(1 - m^{-\frac{1}{q}}), \quad m \le N,$$

$$(\overline{\mathscr{A}}\tilde{b})_n > \frac{p(n^{\frac{1}{p}} - 1)}{n} = pn^{-\frac{1}{q}}(1 - n^{-\frac{1}{p}}), \quad n \le N.$$

Therefore, $\tilde{L}$ may be estimated as follows:

$$\tilde{L} > pq \sum_{m=1}^{N} \sum_{n=1}^{N} K_\lambda(m, n) m^{r-1} n^{s-1} (1 - m^{-\frac{1}{q}})(1 - n^{-\frac{1}{p}}).$$

Moreover, since $(1 - m^{-\frac{1}{q}})(1 - n^{-\frac{1}{p}}) > 1 - m^{-\frac{1}{q}} - n^{-\frac{1}{p}}$, the above relation implies inequality

$$\begin{aligned} \frac{\tilde{L}}{pq} > & \sum_{m=1}^{N} \sum_{n=1}^{N} K_\lambda(m, n) m^{r-1} n^{s-1} \\ & - \sum_{m=1}^{N} \sum_{n=1}^{N} K_\lambda(m, n) m^{r-1-\frac{1}{q}} n^{s-1} \\ & - \sum_{m=1}^{N} \sum_{n=1}^{N} K_\lambda(m, n) m^{r-1} n^{s-1-\frac{1}{p}}. \end{aligned} \tag{43}$$

The next goal is to establish suitable estimates for double sums on the right-hand side of inequality (43). The first double sum may be regarded as the upper Darboux sum for the function $K_\lambda(x, y) x^{r-1} y^{s-1}$ defined on square $[1, N+1] \times [1, N+1]$, since this two-variable function is strictly decreasing in each argument. Hence, utilizing suitable variable changes and the well-known Fubini theorem, we have

$$\begin{aligned} & \sum_{m=1}^{N} \sum_{n=1}^{N} K_\lambda(m, n) m^{r-1} n^{s-1} \\ & > \int_1^N \int_1^N K_\lambda(x, y) x^{r-1} y^{s-1} dx dy \end{aligned}$$

$$= \int_1^N \frac{dx}{x} \int_{\frac{1}{x}}^{\frac{N}{x}} K_\lambda(1,t)t^{s-1}dt$$

$$= \int_{\frac{1}{N}}^1 \left(\int_{\frac{1}{t}}^N \frac{dx}{x} \right) K_\lambda(1,t)t^{s-1}dt + \int_1^N \left(\int_1^{\frac{N}{t}} \frac{dx}{x} \right) K_\lambda(1,t)t^{s-1}dt$$

$$= \log N \int_{\frac{1}{N}}^1 K_\lambda(1,t)t^{s-1} \left(1 + \frac{\log t}{\log N} \right) dt$$

$$+ \log N \int_1^N K_\lambda(1,t)t^{s-1} \left(1 - \frac{\log t}{\log N} \right) dt. \tag{44}$$

The second sum on the right-hand side of (43) may be rewritten as

$$\sum_{m=1}^N \sum_{n=1}^N K_\lambda(m,n)m^{r-1-\frac{1}{q}}n^{s-1} = \sum_{n=1}^N K_\lambda(1,n)n^{s-1} + \sum_{m=2}^N \sum_{n=1}^N K_\lambda(m,n)m^{r-1-\frac{1}{q}}n^{s-1},$$

and both sums on the right-hand side of this relation may be regarded as the lower Darboux sums for the corresponding functions. More precisely, we have

$$\sum_{n=1}^N K_\lambda(1,n)n^{s-1} < \int_0^N K_\lambda(1,t)t^{s-1}dt < \int_0^\infty K_\lambda(1,t)t^{s-1}dt = c_\lambda(s)$$

and

$$\sum_{m=2}^N \sum_{n=1}^N K_\lambda(m,n)m^{r-1-\frac{1}{q}}n^{s-1} < \int_1^N \int_0^N K_\lambda(x,y)x^{r-1-\frac{1}{q}}y^{s-1}dxdy$$

$$= \int_1^N \frac{dx}{x^{1+\frac{1}{q}}} \int_0^{\frac{N}{x}} K_\lambda(1,t)t^{s-1}dt$$

$$< \int_1^N \frac{dx}{x^{1+\frac{1}{q}}} \int_0^\infty K_\lambda(1,t)t^{s-1}dt$$

$$= \left(q - \frac{q}{N^{\frac{1}{q}}} \right) c_\lambda(s),$$

so that

$$\sum_{m=1}^N \sum_{n=1}^N K_\lambda(m,n)m^{r-1-\frac{1}{q}}n^{s-1} < \left(1 + q - \frac{q}{N^{\frac{1}{q}}} \right) c_\lambda(s). \tag{45}$$

In a similar manner, it follows that

$$\sum_{m=1}^{N}\sum_{n=1}^{N} K_\lambda(m,n)m^{r-1}n^{s-1-\frac{1}{p}} < \left(1+p-\frac{p}{N^{\frac{1}{p}}}\right)c_\lambda(s). \tag{46}$$

Now, relations (40), (42)–(46) yield inequality

$$\begin{aligned}\frac{K(1+\log N)}{pq} > \log N \int_{\frac{1}{N}}^{1} K_\lambda(1,t)t^{s-1}\left(1+\frac{\log t}{\log N}\right)dt \\ + \log N \int_{1}^{N} K_\lambda(1,t)t^{s-1}\left(1-\frac{\log t}{\log N}\right)dt \\ -\left(2+pq-\frac{p}{N^{\frac{1}{p}}}-\frac{q}{N^{\frac{1}{q}}}\right)c_\lambda(s).\end{aligned} \tag{47}$$

Dividing inequality (47) by $\log N$ and letting N to infinity, it follows that $\frac{K}{pq} \geq c_\lambda(s)$, which contradicts with the assumption that K is smaller than $pqc_\lambda(s)$. Therefore, the constant $pqc_\lambda(s)$ is the best possible in (34).

It remains to prove that $qc_\lambda(s)$ is the best constant in (35). For this reason, suppose that there exists a positive constant $0 < K' < qc_\lambda(s)$ such that inequality

$$\left[\sum_{n=1}^{\infty} n^{ps-1}\left(\sum_{m=1}^{\infty} K_\lambda(m,n)m^{r-\frac{1}{q}}(\overline{\mathscr{A}}a)_m\right)^p\right]^{\frac{1}{p}} < K'\|a\|_{l^p}$$

holds for all non-negative sequences $a = (a_m)_{m\in\mathbb{N}}$, provided that $0 < \|a\|_{l^p} < \infty$. Then, utilizing the Hölder and the Hardy inequality, we have

$$\begin{aligned}&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K_\lambda(m,n)m^{r-\frac{1}{q}}n^{s-\frac{1}{p}}(\overline{\mathscr{A}}a)_m(\overline{\mathscr{A}}b)_n \\ &= \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} K_\lambda(m,n)m^{r-\frac{1}{q}}n^{s-\frac{1}{p}}(\overline{\mathscr{A}}a)_m\right)(\overline{\mathscr{A}}b)_n \\ &\leq \left[\sum_{n=1}^{\infty} n^{ps-1}\left(\sum_{m=1}^{\infty} K_\lambda(m,n)m^{r-\frac{1}{q}}(\overline{\mathscr{A}}a)_m\right)^p\right]^{\frac{1}{p}} \|\overline{\mathscr{A}}b\|_{l^q} \\ &< K'p\|a\|_{l^p}\|b\|_{l^q},\end{aligned}$$

which is impossible since $K'p < pqc_\lambda(s)$ and $pqc_\lambda(s)$ is the best constant in (34).

As the application of Theorems 8–10, we consider the function $K_\lambda : \mathbb{R}_+^2 \to \mathbb{R}$, defined by $K_\lambda(x,y) = \frac{\ln y - \ln x}{y-x}$. Evidently, it is homogeneous of degree -1 and strictly decreasing in both arguments, $c_\lambda(s)$ converges for all $s \in (0,1)$, and we have $c_\lambda(s) = \frac{\pi^2}{\sin^2 \pi s}$, (see [1, 5]). Now, Theorems 8–10 equipped with this kernel and parameters $r = \frac{1}{q}$, $s = \frac{1}{p}$ read as follows:

Corollary 1 ([5]). *The series of inequalities*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\log \frac{m}{n}}{m-n} (\overline{\mathscr{A}}a)_m (\overline{\mathscr{A}}b)_n < \frac{pq\pi^2}{\sin^2 \frac{\pi}{p}} \|a\|_{l^p} \|b\|_{l^q},$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\log \frac{m}{n}}{m-n} (\overline{\mathscr{G}}a)_m (\overline{\mathscr{G}}b)_n < \frac{e\pi^2}{\sin^2 \frac{\pi}{p}} \|a\|_{l^p} \|b\|_{l^q},$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\log \frac{m}{n}}{m-n} (\overline{\mathscr{H}}a)_m (\overline{\mathscr{H}}b)_n < \left(2 + \frac{1}{pq}\right) \frac{\pi^2}{\sin^2 \frac{\pi}{p}} \|a\|_{l^p} \|b\|_{l^q},$$

and

$$\left[\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} \frac{\log \frac{m}{n}}{m-n} (\overline{\mathscr{A}}a)_m\right)^p\right]^{\frac{1}{p}} < \frac{q\pi^2}{\sin^2 \frac{\pi}{p}} \|a\|_{l^p},$$

$$\left[\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} \frac{\log \frac{m}{n}}{m-n} (\overline{\mathscr{G}}a)_m\right)^p\right]^{\frac{1}{p}} < \frac{e^{\frac{1}{p}}\pi^2}{\sin^2 \frac{\pi}{p}} \|a\|_{l^p},$$

$$\left[\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} \frac{\log \frac{m}{n}}{m-n} (\overline{\mathscr{H}}a)_m\right)^p\right]^{\frac{1}{p}} < \left(1 + \frac{1}{p}\right) \frac{\pi^2}{\sin^2 \frac{\pi}{p}} \|a\|_{l^p}$$

hold for all non-negative sequences $a = (a_m)_{m\in\mathbb{N}}$ and $b = (b_n)_{n\in\mathbb{N}}$, provided that $0 < \|a\|_{l^p} < \infty$ and $0 < \|b\|_{l^q} < \infty$. Moreover, above inequalities include the best constants on their right-hand sides.

3.3 Inequalities with Some Related Integral Operators

We continue our discussion with a few related Hilbert-type inequalities involving some other integral operators. In 2012, Adiyasuren and Batbold [2] gave the following analogue of Theorem 5, where the arithmetic mean operator $\mathscr{A}$ is replaced by integral operator $(\mathscr{A}_1 f)(x) = \frac{1}{x}\int_0^x (x-t) f(t)dt$.

Theorem 11 ([2]). *Let $s+r = \lambda$, $\lambda, s, r > 0$, and $K_\lambda : \mathbb{R}_+^2 \to \mathbb{R}$ be a non-negative homogeneous function of degree $-\lambda$, provided that*

$$0<c_\lambda(s)<\infty, 0<\int_{\mathbb{R}_+} K_\lambda(1,u)u^{s-\frac{1}{p}-1}du<\infty, 0<\int_{\mathbb{R}_+} K_\lambda(1,u)u^{r-\frac{1}{q}-1}du<\infty.$$

Then the inequalities

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} K_\lambda(x,y)x^{r-\frac{1}{q}-1}y^{s-\frac{1}{p}-1}(\mathscr{A}_1 f)(x)(\mathscr{A}_1 g)(y)dxdy$$
$$< \frac{(pq)^2}{1+2pq}c_\lambda(s)\|f\|_{L^p(\mathbb{R}_+)}\|g\|_{L^q(\mathbb{R}_+)} \tag{48}$$

and

$$\left[\int_{\mathbb{R}_+} y^{ps-1}\left(\int_{\mathbb{R}_+} K_\lambda(x,y)x^{r-\frac{1}{q}-1}(\mathscr{A}_1 f)(x)dx\right)^p dy\right]^{\frac{1}{p}}$$
$$< \frac{p^2}{1+p}c_\lambda(s)\|f\|_{L^p(\mathbb{R}_+)}, \tag{49}$$

hold for all non-negative functions $f, g : \mathbb{R}_+ \to \mathbb{R}$ *such that* $0 < \|f\|_{L^p(\mathbb{R}_+)} < \infty$ *and* $0 < \|g\|_{L^q(\mathbb{R}_+)} < \infty$. *The constants* $\frac{(pq)^2}{1+2pq}c_\lambda(s)$ *and* $\frac{p^2}{1+p}c_\lambda(s)$ *are the best possible in* (48) *and* (49).

It should be noticed here that Theorem 11 follows from inequalities (1), (2), and general version of the Hardy integral inequality (5). On the other hand, Liu and Yang [22] obtained a pair of inequalities, based on the so-called dual Hardy inequality (see relation (83), for more details see also [21]). The following result deals with an integral operator $\mathscr{A}_\lambda^*$ defined by $(\mathscr{A}_\lambda^* f)(x) = \frac{1}{x}\int_x^\infty \frac{f(t)}{t^\lambda}dt$.

Theorem 12 ([22]). *Let* $\lambda_1 + \lambda_2 = \lambda < 2$, *and let* $K_\lambda : \mathbb{R}_+^2 \to \mathbb{R}$ *be a non-negative homogeneous function of degree* $-\lambda$, *provided that* $0 < c_\lambda(\lambda_1) < \infty$ *for any* $\lambda_1 \in (\lambda - 1, 1)$. *Then, for* $\varphi(x) = x^{p(2-\lambda-\lambda_1)-1}$, $\psi(y) = y^{q(1-\lambda_2)-1}$, $0 < \|f\|_{L^p(\mathbb{R}_+,\varphi)} < \infty$, *and* $0 < \|g\|_{L^q(\mathbb{R}_+,\psi)} < \infty$, *the inequalities*

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} K_\lambda(x,y)xy(\mathscr{A}_\lambda^* f)(x)(\mathscr{A}_\lambda^* g)(y)dxdy$$
$$< \frac{c_\lambda(\lambda_1)}{(1-\lambda_1)(1-\lambda_2)}\|f\|_{L^p(\mathbb{R}_+,\varphi)}\|g\|_{L^q(\mathbb{R}_+,\psi)} \tag{50}$$

and

$$\left[\int_{\mathbb{R}_+} \psi^{1-p}(y)\left(\int_{\mathbb{R}_+} K_\lambda(x,y)x(\mathscr{A}_\lambda^* f)(x)dx\right)^p dy\right]^{\frac{1}{p}} < \frac{c_\lambda(\lambda_1)}{1-\lambda_1}\|f\|_{L^p(\mathbb{R}_+,\varphi)} \tag{51}$$

hold and the constants $\frac{c_\lambda(\lambda_1)}{(1-\lambda_1)(1-\lambda_2)}$ *and* $\frac{c_\lambda(\lambda_1)}{1-\lambda_1}$ *are the best possible.*

Observe also that Yang and Xie proved discrete versions of inequalities (50) and (51) in [34].

3.4 Applications

In Sect. 2 we have defined a class of operators representing arithmetic, geometric, and harmonic mean in both integral and discrete case. Their norms were determined as a simple consequences of the corresponding inequalities. With the same reasoning, Hardy–Hilbert-type inequalities established in this section enable us to define another class of integral and discrete operators and to determine their norms.

Regarding notations from this section and Sect. 2, we define integral operators $\mathbf{A}, \mathbf{G}, \mathbf{H} : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$ by

$$(\mathbf{A}f)(y) = y^{s-\frac{1}{p}} \int_0^\infty K_\lambda(x,y) x^{r-\frac{1}{q}} (\mathscr{A}f)(x) dx,$$

$$(\mathbf{G}f)(y) = y^{s-\frac{1}{p}} \int_0^\infty K_\lambda(x,y) x^{r-\frac{1}{q}} (\mathscr{G}f)(x) dx,$$

$$(\mathbf{H}f)(y) = y^{s-\frac{1}{p}} \int_0^\infty K_\lambda(x,y) x^{r-\frac{1}{q}} (\mathscr{H}f)(x) dx.$$

Due to inequalities (26), (28), and (30), the above operators are well-defined. Moreover, since the corresponding inequalities include the best constants, it follows that $\|\mathbf{A}\| = qc_\lambda(s)$, $\|\mathbf{G}\| = e^{\frac{1}{p}} c_\lambda(s)$, and $\|\mathbf{H}\| = \big(1 + \frac{1}{p}\big) c_\lambda(s)$.

Similarly to integral case, we also define discrete operators $\overline{\mathbf{A}}, \overline{\mathbf{G}}, \overline{\mathbf{H}} : l^p \to l^p$ by

$$(\overline{\mathbf{A}}a)_n = n^{s-\frac{1}{p}} \sum_{m=1}^\infty K_\lambda(m,n) m^{r-\frac{1}{q}} (\overline{\mathscr{A}}a)_m,$$

$$(\overline{\mathbf{G}}a)_n = n^{s-\frac{1}{p}} \sum_{m=1}^\infty K_\lambda(m,n) m^{r-\frac{1}{q}} (\overline{\mathscr{G}}a)_m,$$

$$(\overline{\mathbf{H}}a)_n = n^{s-\frac{1}{p}} \sum_{m=1}^\infty K_\lambda(m,n) m^{r-\frac{1}{q}} (\overline{\mathscr{H}}a)_m.$$

Due to inequalities (35), (37), and (39), these operators are well-defined. Moreover, by virtue of the best constants, it follows that $\|\overline{\mathbf{A}}\| = qc_\lambda(s)$, $\|\overline{\mathbf{G}}\| = e^{\frac{1}{p}} c_\lambda(s)$, and $\|\overline{\mathbf{H}}\| = \big(1 + \frac{1}{p}\big) c_\lambda(s)$.

4 Half-Discrete Versions

Nowadays, considerable attention is given to the so-called half-discrete Hilbert-type inequalities, that is, to inequalities which include both integral and sum. Recently, Krnić et al. [20] provided a unified treatment of half-discrete Hilbert-type inequalities with a homogeneous kernel and in the setting with non-conjugate exponents. In this article we only refer to a conjugate version, since in this case one may obtain the best constants.

More precisely, if $K_\lambda : \mathbb{R}_+^2 \to \mathbb{R}$ is a non-negative homogeneous function of degree $-\lambda$, $\lambda > 0$, Krnić et al. [20] have showed that the following triple of half-discrete Hilbert-type inequalities

$$\sum_{n=1}^{\infty} a_n \int_{\mathbb{R}_+} K_\lambda(x,n) f(x)dx = \int_{\mathbb{R}_+} f(x) \left(\sum_{n=1}^{\infty} K_\lambda(x,n) a_n \right) dx$$
$$< L \left[\int_{\mathbb{R}_+} x^{1-\lambda+p(\alpha_1-\alpha_2)} f^p(x)dx \right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} n^{1-\lambda+q(\alpha_2-\alpha_1)} a_n^q \right]^{\frac{1}{q}}, \quad (52)$$

$$\left[\sum_{n=1}^{\infty} n^{(p-1)(\lambda-1)+p(\alpha_1-\alpha_2)} \left(\int_{\mathbb{R}_+} K_\lambda(x,n) f(x)dx \right)^p \right]^{\frac{1}{p}}$$
$$< L \left[\int_{\mathbb{R}_+} x^{1-\lambda+p(\alpha_1-\alpha_2)} f^p(x)dx \right]^{\frac{1}{p}}, \quad (53)$$

and

$$\left[\int_{\mathbb{R}_+} x^{(q-1)(\lambda-1)+q(\alpha_2-\alpha_1)} \left(\sum_{n=1}^{\infty} K_\lambda(x,n) a_n \right)^q dx \right]^{\frac{1}{q}}$$
$$< L \left[\sum_{n=1}^{\infty} n^{1-\lambda+q(\alpha_2-\alpha_1)} a_n^q \right]^{\frac{1}{q}}, \quad (54)$$

where $L = k_\lambda^{\frac{1}{p}}(p\alpha_2) k_\lambda^{\frac{1}{q}}(2-s-q\alpha_1)$, $k_\lambda(\alpha) = \int_{\mathbb{R}_+} K_\lambda(1,t) t^{-\alpha} dt$, and α_1, α_2 are real parameters such that the function $K(x,y)y^{-q'\alpha_2}$ is decreasing on $\mathbb{R}_+$ for any $x \in \mathbb{R}_+$, holds for any non-negative measurable function $f : \mathbb{R}_+ \to \mathbb{R}$ and a non-negative sequence $a = (a_n)_{n \in \mathbb{N}}$. Clearly, in the above inequalities all integrals and sums are assumed to be convergent, and the function and the sequence are not equal to zero. For some related half-discrete Hilbert-type inequalities, regarding some particular classes of kernels and weight functions, the reader is referred to the following references: [15, 26, 32, 33].

Based on the above half-discrete inequalities, Adiyasuren et al. [8] derived half-discrete versions of inequalities from Sect. 3. Clearly, the following set of inequalities include both integral and discrete mean operators.

Theorem 13 ([8]). *Let $K_\lambda : \mathbb{R}_+^2 \to \mathbb{R}$ be a non-negative homogeneous function of degree $-\lambda, \lambda > 0$, and let α_1 and α_2 be real parameters fulfilling condition $p\alpha_2 + q\alpha_1 = 2 - \lambda$. If the function $K_\lambda(x, y)y^{-p\alpha_2}$ is decreasing on $\mathbb{R}_+$ for any fixed $x \in \mathbb{R}_+$, then the inequalities*

$$\begin{aligned}\sum_{n=1}^{\infty} n^{\frac{1-pq\alpha_2}{q}} (\overline{\mathscr{A}} a)_n \int_{\mathbb{R}_+} K_\lambda(x, n) x^{\frac{1-pq\alpha_1}{p}} (\mathscr{A} f)(x) dx \\ = \int_{\mathbb{R}_+} x^{\frac{1-pq\alpha_1}{p}} (\mathscr{A} f)(x) \left(\sum_{n=1}^{\infty} K_\lambda(x, n) n^{\frac{1-pq\alpha_2}{q}} (\overline{\mathscr{A}} a)_n \right) dx \\ < c_\lambda(1 - p\alpha_2) pq \| f \|_{L^p(\mathbb{R}_+)} \|a\|_{l^q},\end{aligned} \tag{55}$$

$$\left[\sum_{n=1}^{\infty} \left(n^{\frac{1-pq\alpha_2}{q}} \int_{\mathbb{R}_+} K_\lambda(x, n) x^{\frac{1-pq\alpha_1}{p}} (\mathscr{A} f)(x) dx \right)^p \right]^{\frac{1}{p}} < c_\lambda(1-p\alpha_2) q \| f \|_{L^p(\mathbb{R}_+)}, \tag{56}$$

and

$$\left[\int_{\mathbb{R}_+} \left(x^{\frac{1-pq\alpha_1}{p}} \sum_{n=1}^{\infty} K_\lambda(x, n) n^{\frac{1-pq\alpha_2}{q}} (\overline{\mathscr{A}} a)_n \right)^q dx \right]^{\frac{1}{q}} < c_\lambda(1 - p\alpha_2) p \|a\|_{l^q} \tag{57}$$

hold for any non-negative measurable function $f : \mathbb{R}_+ \to \mathbb{R}$ and a non-negative sequence $a = (a_n)_{n\in\mathbb{N}}$, provided $0 < \| f \|_{L^p(\mathbb{R}_+)} < \infty$ and $0 < \|a\|_{l^q} < \infty$. In addition, the constants $c_\lambda(1 - p\alpha_2) pq$, $c_\lambda(1 - p\alpha_2)q$, and $c_\lambda(1 - p\alpha_2) p$ are the best possible in the corresponding inequalities.

Theorem 14 ([8]). *Under the same assumptions as in Theorem 13, inequalities*

$$\begin{aligned}\sum_{n=1}^{\infty} n^{\frac{1-pq\alpha_2}{q}} (\overline{\mathscr{G}} a)_n \int_{\mathbb{R}_+} K_\lambda(x, n) x^{\frac{1-pq\alpha_1}{p}} (\mathscr{G} f)(x) dx \\ = \int_{\mathbb{R}_+} x^{\frac{1-pq\alpha_1}{p}} (\mathscr{G} f)(x) \left(\sum_{n=1}^{\infty} K_\lambda(x, n) n^{\frac{1-pq\alpha_2}{q}} (\overline{\mathscr{G}} a)_n \right) dx \\ < c_\lambda(1 - p\alpha_2) e \| f \|_{L^p(\mathbb{R}_+)} \|a\|_{l^q},\end{aligned} \tag{58}$$

$$\left[\sum_{n=1}^{\infty} \left(n^{\frac{1-pq\alpha_2}{q}} \int_{\mathbb{R}_+} K_\lambda(x, n) x^{\frac{1-pq\alpha_1}{p}} (\mathscr{G} f)(x) dx \right)^p \right]^{\frac{1}{p}} < c_\lambda(1-p\alpha_2) e^{\frac{1}{p}} \| f \|_{L^p(\mathbb{R}_+)}, \tag{59}$$

and

$$\left[\int_{\mathbb{R}_+}\left(x^{\frac{1-pq\alpha_1}{p}}\sum_{n=1}^{\infty}K_\lambda(x,n)n^{\frac{1-pq\alpha_2}{q}}(\overline{\mathscr{G}}a)_n\right)^q dx\right]^{\frac{1}{q}} < c_\lambda(1-p\alpha_2)e^{\frac{1}{q}}\|a\|_{l^q} \quad (60)$$

hold and the constants appearing on their right-hand sides are the best possible.

Theorem 15 ([8]). *With the assumptions of Theorem 13, inequalities*

$$\begin{aligned}
&\sum_{n=1}^{\infty}n^{\frac{1-pq\alpha_2}{q}}(\overline{\mathscr{H}}a)_n\int_{\mathbb{R}_+}K_\lambda(x,n)x^{\frac{1-pq\alpha_1}{p}}(\mathscr{H}f)(x)dx\\
&\quad=\int_{\mathbb{R}_+}x^{\frac{1-pq\alpha_1}{p}}(\mathscr{H}f)(x)\left(\sum_{n=1}^{\infty}K_\lambda(x,n)n^{\frac{1-pq\alpha_2}{q}}(\overline{\mathscr{H}}a)_n\right)dx\\
&\quad< c_\lambda(1-p\alpha_2)\left(2+\frac{1}{pq}\right)\|f\|_{L^p(\mathbb{R}_+)}\|a\|_{l^q}, \qquad (61)
\end{aligned}$$

$$\begin{aligned}
&\left[\sum_{n=1}^{\infty}\left(n^{\frac{1-pq\alpha_2}{q}}\int_{\mathbb{R}_+}K_\lambda(x,n)x^{\frac{1-pq\alpha_1}{p}}(\mathscr{H}f)(x)dx\right)^p\right]^{\frac{1}{p}}\\
&\quad< c_\lambda(1-p\alpha_2)\big(1+\frac{1}{p}\big)\|f\|_{L^p(\mathbb{R}_+)}, \qquad (62)
\end{aligned}$$

and

$$\left[\int_{\mathbb{R}_+}\left(x^{\frac{1-pq\alpha_1}{p}}\sum_{n=1}^{\infty}K_\lambda(x,n)n^{\frac{1-pq\alpha_2}{q}}(\overline{\mathscr{H}}a)_n\right)^q dx\right]^{\frac{1}{q}} < c_\lambda(1-p\alpha_2)\big(1+\frac{1}{q}\big)\|a\|_{l^q} \quad (63)$$

hold and the constants appearing on their right-hand sides are the best possible.

The idea of proving Theorems 13–15 is quite similar to theorems from Sect. 3, except that we utilize half-discrete inequalities (52)–(54) instead of integral and discrete Hilbert-type inequalities. Moreover, to obtain the best constants, we simultaneously plug the appropriate function and the sequence in the corresponding inequality. For detailed proofs of these theorems, the reader is referred to [8].

Remark 2. Similarly to Sect. 3.4, by virtue of Hardy–Hilbert-type inequalities from Theorems 13–15, one can define certain half-discrete operators and determine their norms.

Namely, with the assumptions of Theorem 13, it follows from (56) and (57), that a pair of half-discrete arithmetic operators $\mathbf{A_1} : L^p(\mathbb{R}_+) \to l^p$ and $\mathbf{A_2} : l^q \to L^q(\mathbb{R}_+)$,

$$(\mathbf{A_1} f)_n = n^{\frac{1-pq\alpha_2}{q}} \int_{\mathbb{R}_+} K(x,n) x^{\frac{1-pq\alpha_1}{p}} (\mathscr{A} f)(x) dx,$$

$$(\mathbf{A_2} a)(x) = x^{\frac{1-pq\alpha_1}{p}} \sum_{n=1}^{\infty} K(x,n) n^{\frac{1-pq\alpha_2}{q}} (\overline{\mathscr{A}} a)_n,$$

is well-defined. Moreover, inequalities (56) and (57) may be rewritten as $\|\mathbf{A_1} f\|_{l^p} < c_\lambda(1-p\alpha_2)q\|f\|_{L^p(\mathbb{R}_+)}$ and $\|\mathbf{A_2} a\|_{L^q(\mathbb{R}_+)} < c_\lambda(1-p\alpha_2)p\|a\|_{l^q}$. Due to the best constants, it follows that $\|\mathbf{A_1}\| = c_\lambda(1-p\alpha_2)q$ and $\|\mathbf{A_2}\| = c_\lambda(1-p\alpha_2)p$.

In the same way, Theorems 14 and 15 are utilized to define the corresponding half-discrete geometric and harmonic operators. For more details, the reader is referred to [8].

5 Extension to a Multidimensional Case

The main goal of this section is to present extensions of Theorems 5–7 to a multidimensional case. Such results are consequences of multidimensional Hilbert-type inequalities.

In 2005, Brnetić et al. [10] (see also [11]) provided a unified treatment of multidimensional Hilbert-type inequalities with non-conjugate exponents, with a basic result including a general non-negative measurable kernel and weight functions. Moreover, Perić and Vuković [25] studied the latter inequalities for the case of a homogeneous kernel. For some related multidimensional Hilbert-type inequalities, regarding some particular classes of kernels and weight functions, the reader is referred to the following references: [19, 27].

Before we state the basic result, we need some conventions. Recall that the function $K_\lambda : \mathbb{R}_+^n \to \mathbb{R}$ is said to be homogeneous of degree $-\lambda$, $\lambda > 0$, if $K(t\mathbf{x}) = t^{-\lambda} K(\mathbf{x})$ for all $t > 0$ and $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n$. If $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$, we define

$$k_i(\mathbf{a}) = \int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{u}}^i) \prod_{j=1, j\neq i}^{n} u_j^{a_j} \hat{d}^i \mathbf{u}, \quad i = 1, 2, \dots, n, \tag{64}$$

where $\hat{\mathbf{u}}^i = (u_1, \dots, u_{i-1}, 1, u_{i+1}, \dots, u_n)$, $\hat{d}^i \mathbf{u} = du_1 \dots du_{i-1} du_{i+1} \dots du_n$, and provided that the above integral converges. Further, in the sequel $d\mathbf{u}$ is the abbreviation for $du_1 du_2 \dots du_n$.

Although the general Hilbert-type inequalities are derived in the setting with non-conjugate exponents, we consider here only the conjugate case. More precisely, in this section $\{p_1, p_2, \ldots, p_n\}$ represents the set of non-negative conjugate parameters, that is, $\sum_{i=1}^n \frac{1}{p_i} = 1$, $p_i > 1$, $i = 1, 2, \ldots, n$. The parameters p_i' are defined as associated conjugates, that is, $\frac{1}{p_i} + \frac{1}{p_i'} = 1$.

Here we refer to a pair of inequalities derived in [25], regarding a homogeneous kernel K_λ and some particular parameters. More precisely, the authors obtained inequalities

$$\int_{\mathbb{R}_+^n} K_\lambda(\mathbf{x}) \prod_{i=1}^n f_i(x_i) d\mathbf{x} \leq k_1(\tilde{\mathbf{A}}) \prod_{i=1}^n \| f_i \|_{L^{p_i}(\mathbb{R}_+, x_i^{-1-p_i \tilde{A}_i})} \tag{65}$$

and

$$\left[\int_{\mathbb{R}_+} x_n{}^{(p_n'-1)(1+p_n \tilde{A}_n)} \left(\int_{\mathbb{R}_+^{n-1}} K_\lambda(\mathbf{x}) \prod_{i=1}^{n-1} f_i(x_i) \hat{d}^n \mathbf{x} \right)^{p_n'} dx_n \right]^{1/p_n'}$$
$$\leq k_1(\tilde{\mathbf{A}}) \prod_{i=1}^{n-1} \| f_i \|_{L^{p_i}(\mathbb{R}_+, x_i^{-1-p_i \tilde{A}_i})}, \tag{66}$$

where the parameters $\tilde{A}_i$, $i = 1, \ldots, n$, fulfill conditions

$$k_1(\tilde{\mathbf{A}}) < \infty \text{ for } \tilde{A}_2, \ldots, \tilde{A}_n > -1, \sum_{i=2}^n \tilde{A}_i < \lambda - n + 1, \text{and} \sum_{i=1}^n \tilde{A}_i = \lambda - n. \tag{67}$$

In addition, the constant $k_1(\tilde{\mathbf{A}})$, appearing in (65) and (66) is the best possible in both inequalities.

Utilizing the above two inequalities, Krnić [16] obtained the following multidimensional version of Theorem 5.

Theorem 16 ([16]). *Let $K_\lambda : \mathbb{R}_+^n \to \mathbb{R}$ be a non-negative measurable homogeneous function of degree $-\lambda$, $\lambda > 0$, such that for every $i = 2, 3, \ldots, n$,*

$$K_\lambda(1, t_2, \ldots, t_i, \ldots, t_n) \leq C_K K_\lambda(1, t_2, \ldots, 0, \ldots, t_n), \; 0 \leq t_i \leq 1,$$

where C_K is a positive constant. Further, let $1/p_i < \mu_i \leq 1$, $i = 1, 2, \ldots, n$, and let the parameters $\tilde{A}_i$, $i = 1, 2, \ldots, n$, fulfill conditions as in (67). If $f_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1, 2, \ldots, n$, are non-negative measurable functions, then

$$\int_{\mathbb{R}_+^n} K_\lambda(\mathbf{x}) \prod_{i=1}^n x_i^{\frac{1}{p_i} + \tilde{A}_i} (\mathscr{A} f_i)^{\mu_i} (x_i) d\mathbf{x} \leq \overline{m}_n^\lambda(\mathbf{p}, \tilde{\mathbf{A}}, \boldsymbol{\mu}) \prod_{i=1}^n \| f_i^{\mu_i} \|_{L^{p_i}(\mathbb{R}_+)}, \tag{68}$$

and

$$\left[\int_{\mathbb{R}_+} x_n^{(p_n'-1)(1+p_n\tilde{A}_n)} \left(\int_{\mathbb{R}_+^{n-1}} K_\lambda(\mathbf{x}) \prod_{i=1}^{n-1} x_i^{\frac{1}{p_i}+\tilde{A}_i} (\mathscr{A} f_i)^{\mu_i} (x_i) \hat{d}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{\frac{1}{p_n'}}$$

$$\leq \overline{m}_{n-1}^{\lambda}(\mathbf{p}, \tilde{\mathbf{A}}, \boldsymbol{\mu}) \prod_{i=1}^{n-1} \| f_i^{\mu_i} \|_{L^{p_i}(\mathbb{R}_+)}, \tag{69}$$

where the constants

$$\overline{m}_n^{\lambda}(\mathbf{p}, \tilde{\mathbf{A}}, \boldsymbol{\mu}) = k_1(\tilde{\mathbf{A}}) \prod_{i=1}^{n} \left(\frac{p_i \mu_i}{p_i \mu_i - 1}\right)^{\mu_i},$$

and

$$\overline{m}_{n-1}^{\lambda}(\mathbf{p}, \tilde{\mathbf{A}}, \boldsymbol{\mu}) = k_1(\tilde{\mathbf{A}}) \prod_{i=1}^{n-1} \left(\frac{p_i \mu_i}{p_i \mu_i - 1}\right)^{\mu_i}$$

are the best possible in the corresponding inequalities.

Remark 3. Considering inequality (68) with the kernel $K_\lambda(\mathbf{x}) = (x_1 + x_2 + \cdots + x_n)^{-\lambda}$, $\lambda > 0$, and the parameters $\tilde{A}_i = s_i - 1$, $i = 1, 2, \ldots, n$, the constant on its right-hand side reduces to $\frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma(s_i) \prod_{i=1}^n \left(\frac{p_i \mu_i}{p_i \mu_i - 1}\right)^{\mu_i}$, where Γ stands for the usual Gamma function. This particular result was obtained by Adiyasuren and Batbold [4], in 2012.

Recently, Adiyasuren et al. [6] gave analogues of inequalities (68) and (69), with the weighted geometric and harmonic mean operators, instead of the arithmetic operator. The weighted geometric mean operator $\mathscr{G}_\alpha$, $\alpha > 0$ is defined by

$$(\mathscr{G}_\alpha f)(x) = \exp\left[\frac{\alpha}{x^\alpha} \int_0^x t^{\alpha-1} \log f(t) dt\right], \tag{70}$$

while the weighted harmonic operator $\mathscr{H}_\alpha$, $\alpha > 0$ is given by

$$(\mathscr{H}_\alpha f)(x) = \frac{x^\alpha}{\int_0^x t^{\alpha-1} f^{-1}(t) dt}. \tag{71}$$

Theorem 17 ([6]). *Let $K_\lambda : \mathbb{R}_+^n \to \mathbb{R}$ be a non-negative measurable homogeneous function of degree $-\lambda$, $\lambda > 0$. Further, let ν_i, μ_i, and $\alpha > 0$ be real parameters such that $\tilde{A}_i \leq \nu_i \leq \frac{\alpha}{p_i} + \tilde{A}_i$, $i = 1, 2, \ldots, n$, where the parameters $\tilde{A}_i$, $i = 1, 2, \ldots, n$, fulfill conditions as in (67). If $f_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1, 2, \ldots, n$ are non-negative measurable functions, then the following two inequalities hold*

$$\int_{\mathbb{R}_+^n} K_\lambda(\mathbf{x}) \prod_{i=1}^{n} x_i^{\nu_i} \, (\mathscr{G}_\alpha f_i)^{\mu_i} \, (x_i) d\mathbf{x} \le m_n^\lambda(\mathbf{p}, \tilde{\mathbf{A}}, \boldsymbol{\nu}) \prod_{i=1}^{n} \| f_i^{\mu_i} \|_{L^{p_i}(\mathbb{R}_+, \varphi_i(x_i))} \tag{72}$$

and

$$\left[\int_{\mathbb{R}_+} x_n^{(p_n'-1)(1+p_n \tilde{A}_n)} \left(\int_{\mathbb{R}_+^{n-1}} K_\lambda(\mathbf{x}) \prod_{i=1}^{n-1} x_i^{\nu_i} \, (\mathscr{G}_\alpha f_i)^{\mu_i} \, (x_i) \hat{d}^n \mathbf{x} \right)^{p_n'} dx_n \right]^{1/p_n'}$$
$$\le m_{n-1}^\lambda(\mathbf{p}, \tilde{\mathbf{A}}, \boldsymbol{\nu}) \prod_{i=1}^{n-1} \| f_i^{\mu_i} \|_{L^{p_i}(\mathbb{R}_+, \varphi_i(x_i))}, \tag{73}$$

where $\varphi_i(x_i) = x_i^{p_i \nu_i - p_i \tilde{A}_i - 1}$*, and the constants*

$$m_n^\lambda(\mathbf{p}, \tilde{\mathbf{A}}, \boldsymbol{\nu}) = k_1(\tilde{\mathbf{A}}) e^{\frac{1}{\alpha}[-\lambda + n + \sum_{i=1}^{n} \nu_i]}$$

and

$$m_{n-1}^\lambda(\mathbf{p}, \tilde{\mathbf{A}}, \boldsymbol{\nu}) = k_1(\tilde{\mathbf{A}}) e^{\frac{1}{\alpha}[-\lambda + n + \tilde{A}_n + \sum_{i=1}^{n-1} \nu_i]}$$

are the best possible in the corresponding inequalities.

Theorem 18 ([6]). *Suppose* $K_\lambda : \mathbb{R}_+^n \to \mathbb{R}$ *is a non-negative measurable homogeneous function of degree* $-\lambda$, $\lambda > 0$, *and let* α, ν_i, *and* $\mu_i > 0$ *be real parameters such that* $\alpha + \frac{1}{\mu_i}(\nu_i - \tilde{A}_i) > 0$, $i = 1, 2, \ldots, n$, *where the parameters* $\tilde{A}_i$, $i = 1, 2, \ldots, n$, *fulfill conditions as in (67). If* $f_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1, 2, \ldots, n$ *are non-negative measurable functions, then the following inequalities hold*

$$\int_{\mathbb{R}_+^n} K_\lambda(\mathbf{x}) \prod_{i=1}^{n} x_i^{\nu_i} \, (\mathscr{H}_\alpha f_i)^{\mu_i} \, (x_i) d\mathbf{x} \le \tilde{m}_n^\lambda(\mathbf{p}, \tilde{\mathbf{A}}, \boldsymbol{\nu}, \boldsymbol{\mu}) \prod_{i=1}^{n} \| f_i^{\mu_i} \|_{L^{p_i}(\mathbb{R}_+, \varphi_i(x_i))} \tag{74}$$

and

$$\left[\int_{\mathbb{R}_+} x_n^{(p_n'-1)(1+p_n \tilde{A}_n)} \left(\int_{\mathbb{R}_+^{n-1}} K_\lambda(\mathbf{x}) \prod_{i=1}^{n-1} x_i^{\nu_i} \, (\mathscr{H}_\alpha f_i)^{\mu_i} \, (x_i) \hat{d}^n \mathbf{x} \right)^{p_n'} dx_n \right]^{1/p_n'}$$
$$\le \tilde{m}_{n-1}^\lambda(\mathbf{p}, \tilde{\mathbf{A}}, \boldsymbol{\nu}, \boldsymbol{\mu}) \prod_{i=1}^{n-1} \| f_i^{\mu_i} \|_{L^{p_i}(\mathbb{R}_+, \varphi_i(x_i))}, \tag{75}$$

where $\varphi_i(x_i) = x_i^{p_i\nu_i - p_i\tilde{A}_i - 1}$, *and the constants*

$$\tilde{m}_n^{\lambda}(\mathbf{p}, \tilde{\mathbf{A}}, \boldsymbol{\nu}, \boldsymbol{\mu}) = k_1(\tilde{\mathbf{A}}) \prod_{i=1}^{n} \left[\alpha + \frac{1}{\mu_i}\left(\nu_i - \tilde{A}_i\right)\right]^{\mu_i}$$

and

$$\tilde{m}_{n-1}^{\lambda}(\mathbf{p}, \tilde{\mathbf{A}}, \boldsymbol{\nu}, \boldsymbol{\mu}) = k_1(\tilde{\mathbf{A}}) \prod_{i=1}^{n-1} \left[\alpha + \frac{1}{\mu_i}\left(\nu_i - \tilde{A}_i\right)\right]^{\mu_i}$$

are the best possible in the corresponding inequalities.

The methodology of proving Theorems 16–18 follows the lines of proofs of Theorems 5–7, except that multidimensional Hilbert-type inequalities (65) and (66) are utilized instead of two-dimensional ones. In addition, in Theorems 16–18 one deals with the weighted versions of the Hardy, Knopp, and Hardy–Carleman inequality (for more details, see [6]). It should also be noticed here that the condition regarding a homogeneous kernel in Theorem 16 may be omitted (see [16]). However, the proofs of these multidimensional theorems are technically more complicated. As an illustration, we give the part of the proof of Theorem 18 referring to the best constant.

Proof (Proof of the Best Constant in (74)*).* Suppose that the inequality

$$\int_{\mathbb{R}_+^n} K_\lambda(\mathbf{x}) \prod_{i=1}^{n} x_i^{\nu_i} \left(\mathscr{H}_\alpha f_i\right)^{\mu_i}(x_i)d\mathbf{x} \le C_n \prod_{i=1}^{n} \|f_i^{\mu_i}\|_{L^{p_i}(\mathbb{R}_+,\varphi_i(x_i))}, \tag{76}$$

holds with the constant $0 < C_n < \tilde{m}_n^{\lambda}(\mathbf{p}, \tilde{\mathbf{A}}, \boldsymbol{\nu}, \boldsymbol{\mu})$. Considering this inequality with the functions

$$f_i^{\varepsilon}(x_i) = \begin{cases} x_i^{\frac{\tilde{A}_i - \nu_i}{\mu_i} + \frac{\varepsilon}{p_i\mu_i}}, & 0 < x_i \le 1, \\ 0, & x_i > 1, \end{cases}$$

where ε is sufficiently small number, its right-hand side reduces to

$$C_n \prod_{i=1}^{n} \|(f_i^{\varepsilon})^{\mu_i}\|_{L^{p_i}(\mathbb{R}_+,\varphi_i(x_i))} = \frac{C_n}{\varepsilon}. \tag{77}$$

Moreover, since

$$(\mathscr{H}_\alpha f_i^{\varepsilon})(x_i) = \begin{cases} \left[\alpha + \frac{\nu_i - \tilde{A}_i}{\mu_i} - \frac{\varepsilon}{\mu_i p_i}\right] x_i^{\frac{\tilde{A}_i - \nu_i}{\mu_i} + \frac{\varepsilon}{\mu_i p_i}}, & 0 < x_i \le 1, \\ 0, & x_i > 1, \end{cases}$$

the left-hand side of (76), denoted here by L, reads

$$L = \int_{\mathbb{R}_+^n} K_\lambda(\mathbf{x}) \prod_{i=1}^n x_i^{\nu_i} \left(\mathscr{H}_\alpha f_i^\varepsilon\right)^{\mu_i} (x_i) d\mathbf{x}$$
$$= \varphi(\varepsilon) \cdot I,$$

where

$$\varphi(\varepsilon) = \prod_{i=1}^n \left[\alpha + \frac{\nu_i - \tilde{A}_i}{\mu_i} - \frac{\varepsilon}{\mu_i p_i}\right]^{\mu_i}$$

and

$$I = \int_{(0,1]^n} K_\lambda(\mathbf{x}) \prod_{i=1}^n x_i^{\tilde{A}_i + \frac{\varepsilon}{p_i}} d\mathbf{x}.$$

Obviously, the integral I can be rewritten as

$$I = \int_0^1 x_1^{\varepsilon-1} \left[\int_{(0,1/x_1]^{n-1}} K_\lambda(\hat{\mathbf{u}}^1) \prod_{i=2}^n u_i^{\tilde{A}_i+\varepsilon/p_i} \hat{d}^1\mathbf{u}\right] dx_1,$$

providing the estimate

$$\begin{aligned} I &\geq \int_0^1 x_1^{\varepsilon-1} \left[\int_{\mathbb{R}_+^{n-1}} K_\lambda(\hat{\mathbf{u}}^1) \prod_{i=2}^n u_i^{\tilde{A}_i+\varepsilon/p_i} \hat{d}^1\mathbf{u}\right] dx_1 \\ &\quad - \int_0^1 x_1^{\varepsilon-1} \left[\sum_{i=2}^n \int_{\mathbb{E}_i} K_\lambda(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{\tilde{A}_j+\varepsilon/p_j} \hat{d}^1\mathbf{u}\right] dx_1 \\ &\geq \frac{1}{\varepsilon} \int_{\mathbb{R}_+^{n-1}} K_\lambda(\hat{\mathbf{u}}^1) \prod_{i=2}^n u_i^{\tilde{A}_i+\varepsilon/p_i} \hat{d}^1\mathbf{u} \\ &\quad - \int_0^1 x_1^{-1} \left[\sum_{i=2}^n \int_{\mathbb{E}_i} K_\lambda(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{\tilde{A}_j+\varepsilon/p_j} \hat{d}^1\mathbf{u}\right] dx_1, \end{aligned} \tag{78}$$

where $\mathbb{E}_i = \{(u_2, u_3, \ldots, u_n); 1/x_1 \leq u_i < \infty, u_j > 0, j \neq i\}$, $\mathbf{1/p} = (1/p_1, \ldots, 1/p_n)$.

Clearly, it suffices to estimate the integral $\int_{\mathbb{E}_2} K_\lambda(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{\tilde{A}_j+\varepsilon/p_j} \hat{d}^1\mathbf{u}$. Namely, choosing $\alpha > 0$ so that $\tilde{A}_2 + 1 > -\varepsilon/p_2 - \alpha$, since $-u_2^{-\alpha} \log \frac{1}{u_2} \to 0$ $(u_2 \to \infty)$, there exists $M \geq 0$ such that $-u_2^{-\alpha} \log \frac{1}{u_2} \leq M$ $(u_2 \in [1, \infty))$. Further, considering the parameters $a_2 = \tilde{A}_2 + (\varepsilon/p_2 + \alpha)$ and $a_i = \tilde{A}_i + \varepsilon/p_i$, $i = 3, \ldots, n$, we have

$$\int_0^1 x_1^{-1} \int_{\mathbb{E}_2} K_\lambda(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{\tilde{A}_j + \varepsilon/p_j} \hat{d}^1 \mathbf{u} dx_1$$
$$\leq M \cdot k_1(\tilde{A}_2 + (\varepsilon/p_2 + \alpha), \tilde{A}_3 + \varepsilon/p_3, \ldots, \tilde{A}_n + \varepsilon/p_n) < \infty,$$

and utilizing (78), it follows that

$$L \geq \varphi(\varepsilon) \cdot \left(\frac{1}{\varepsilon} k_1 \left(\tilde{\mathbf{A}} + \varepsilon \mathbf{1}/\mathbf{p} \right) - O(1) \right). \tag{79}$$

Finally, taking into account (77) and (79), we have that $\tilde{m}_n^\lambda(\mathbf{p}, \tilde{\mathbf{A}}, \boldsymbol{\nu}, \boldsymbol{\mu}) \leq C_n$ when $\varepsilon \to 0^+$, which is in contrast to our hypothesis.

6 Hilbert-Type Inequalities with Differential Operators

So far, we have discussed Hilbert-type inequalities with certain operators on their left-hand sides. To conclude the paper, in this section we deal with some related inequalities accompanied with operators on their right-hand sides.

Recently, Azar [9] derived several new forms of Hilbert-type inequalities accompanied with some operators on their right-hand sides. The constants appearing in these inequalities are also the best possible.

His first result refers to the homogeneous kernel $K_\lambda(x, y) = (x + y)^{-\lambda}$, $\lambda > 0$ and the Hardy (or integration) operator $(\mathscr{H} f)(x) = \int_0^x f(t) dt$.

Theorem 19 ([9]). *If* $\lambda > 0$, $\|\mathscr{H} f\|_{L^p(\mathbb{R}_+, x^{-\lambda-1})} < \infty$, *and* $\|\mathscr{H} g\|_{L^q(\mathbb{R}_+, y^{-\lambda-1})} < \infty$, *then*

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{f(x) g(y)}{(x + y)^\lambda} dx dy$$
$$\leq \frac{\lambda^2}{pq} B\left(\frac{\lambda}{q}, \frac{\lambda}{p}\right) \|\mathscr{H} f\|_{L^p(\mathbb{R}_+, x^{-\lambda-1})} \|\mathscr{H} g\|_{L^q(\mathbb{R}_+, y^{-\lambda-1})}, \tag{80}$$

where the constant $\frac{\lambda^2}{pq} B\left(\frac{\lambda}{q}, \frac{\lambda}{p}\right)$ *is the best possible.*

In the same paper, Azar also obtained an analogue of Theorem 19, with a differential operator instead of the Hardy integration operator. Moreover, Adiyasuren et al. [7], extended that result to hold for an arbitrary homogeneous kernel. Before we state the corresponding pair of Hilbert-type inequalities, we first introduce some notation.

We denote by $\mathscr{D}_+^n$, $n \geq 0$, a differential operator defined by $\mathscr{D}_+^n f(x) = f^{(n)}(x)$, where $f^{(n)}$ stands for the n-th derivative of a function $f : \mathbb{R}_+ \to \mathbb{R}$. In addition, throughout this section, Λ_+^n denotes the set of non-negative measurable functions $f : \mathbb{R}_+ \to \mathbb{R}$ such that $f^{(n)}$ exists a.e. on $\mathbb{R}_+$, $f^{(k)}(x) > 0$, $k = 0, 1, 2, \ldots, n$, a.e. on $\mathbb{R}_+$, and $f^{(k)}(0) = 0$, $k = 0, 1, 2, \ldots, n-1$.

The following theorem deals with a homogeneous kernel $K_\lambda : \mathbb{R}_+^2 \to \mathbb{R}$, of degree $-\lambda$, $\lambda > 0$, such that the integral

$$k(\alpha) = \int_{\mathbb{R}_+} K_\lambda(1,t)t^\alpha dt,$$

converges for $-1 < \alpha < \lambda - 1$.

Theorem 20 ([7]). *Let α_1, α_2 be real parameters such that $\alpha_1, \alpha_2 \in (n-1, \lambda-1)$ and $\alpha_1 + \alpha_2 = \lambda - 2$, where n is a fixed non-negative integer and $\lambda > n$, and let $\varphi(x) = x^{p(n-\alpha_1)-1}, \psi(y) = y^{q(n-\alpha_2)-1}$. If $K_\lambda : \mathbb{R}_+^2 \to \mathbb{R}$ is a non-negative measurable homogeneous function of degree $-\lambda$, then the inequalities*

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} K_\lambda(x,y)f(x)g(y)dxdy < M\|\mathscr{D}_+^n f\|_{L^p(\mathbb{R}_+,\varphi(x))}\|\mathscr{D}_+^n g\|_{L^q(\mathbb{R}_+,\psi(y))} \tag{81}$$

and

$$\left[\int_{\mathbb{R}_+} y^{(p-1)(1+q\alpha_2)}\left(\int_{\mathbb{R}_+} K_\lambda(x,y)f(x)dx\right)^p dy\right]^{\frac{1}{p}} < m\|\mathscr{D}_+^n f\|_{L^p(\mathbb{R}_+,\varphi(x))} \tag{82}$$

hold for all non-negative functions $f, g \in \Lambda_+^n$. In addition, the constants $M = k(\alpha_2)\frac{\Gamma(\alpha_1-n+1)\Gamma(\alpha_2-n+1)}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}$ and $m = k(\alpha_2)\frac{\Gamma(\alpha_1-n+1)}{\Gamma(\alpha_1+1)}$ are the best possible in the corresponding inequalities.

Inequalities (81) and (82) are consequences of Hilbert-type inequalities (1) and (2), equipped with the weighted Hardy inequality. The idea of proving the best constants in (81) and (82) is similar to the proofs presented in this article. Namely, starting from the opposite assumption, it is necessary to plug suitable functions in inequality to obtain a contradiction (for more details, see [7]).

Remark 4. Considering (81) with a homogeneous kernel $K_\lambda(x,y) = (x+y)^{-\lambda}$, $\lambda > 0$, and the parameters $\alpha_1 = \frac{\lambda}{p} - 1$, $\alpha_2 = \frac{\lambda}{q} - 1$, where $\lambda > n\max\{p,q\}$, the above constant M reduce to $\frac{\Gamma\left(\frac{\lambda}{p}-n\right)\Gamma\left(\frac{\lambda}{q}-n\right)}{\Gamma(\lambda)}$. This particular case was studied by Azar [9], and it was derived by a different technique.

Observe that Theorem 20 covers the case when the degree of homogeneity of the kernel, i.e. $-\lambda$ is less than $-n$, for a fixed non-negative integer n. The next result from [7], that is in some way complementary to Theorem 20, covers the case $0 < \lambda \leq 1$, and it follows by virtue of the weighted dual Hardy inequality.

The dual Hardy inequality, accompanied with the dual integration operator or the dual Hardy operator $\mathscr{H}^* f(x) = \int_x^\infty f(t)dt$, asserts that

$$\int_{\mathbb{R}_+} x^{-r} \left(\mathscr{H}^* f(x)\right)^p dx < \left(\frac{p}{1-r}\right)^p \int_{\mathbb{R}_+} x^{p-r} f^p(x)dx, \tag{83}$$

holds for $p > 1$ and $r < 1$, provided that $0 < \int_{\mathbb{R}_+} x^{p-r} f^p(x)dx < \infty$. We define a differential operator $\mathscr{D}_\pm^n$ by $\mathscr{D}_\pm^n f(x) = (-1)^n f^{(n)}(x)$, where n is a non-negative integer. Moreover, the following theorem holds for all non-negative functions $f : \mathbb{R}_+ \to \mathbb{R}$ such that the n-th derivative $f^{(n)}$ exists a.e. on $\mathbb{R}_+$, $\mathscr{D}_\pm^k f(x) > 0$, $k = 0, 1, 2, \ldots, n$, a.e. on $\mathbb{R}_+$, and $\lim_{x\to\infty} f^{(k)}(x) = 0$ for $k = 0, 1, 2, \ldots, n-1$. This set of functions will be denoted by $\Lambda_\pm^n$.

Theorem 21 ([7]). *Suppose that α_1, α_2 are real parameters such that $\alpha_1, \alpha_2 \in (-1, \lambda - 1)$ and $\alpha_1 + \alpha_2 = \lambda - 2$, where $0 < \lambda \leq 1$, and let $\varphi(x) = x^{p(n-\alpha_1)-1}$, $\psi(y) = y^{q(n-\alpha_2)-1}$. If $K_\lambda : \mathbb{R}_+^2 \to \mathbb{R}$ is a non-negative homogeneous function of degree $-\lambda$, then the inequalities*

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} K_\lambda(x, y) f(x)g(y)dxdy$$
$$< M^* \|\mathscr{D}_\pm^n f\|_{L^p(\mathbb{R}_+,\varphi(x))} \|\mathscr{D}_\pm^n g\|_{L^q(\mathbb{R}_+,\psi(y))} \tag{84}$$

and

$$\left[\int_{\mathbb{R}_+} y^{(p-1)(1+q\alpha_2)} \left(\int_{\mathbb{R}_+} K_\lambda(x, y) f(x)dx\right)^p dy\right]^{\frac{1}{p}}$$
$$< m^* \|\mathscr{D}_\pm^n f\|_{L^p(\mathbb{R}_+,\varphi(x))} \tag{85}$$

hold for all non-negative functions $f, g \in \Lambda_\pm^n$, where n is a fixed non-negative integer. In addition, the constants $M^* = k(\alpha_2)\frac{\Gamma(-\alpha_1)\Gamma(-\alpha_2)}{\Gamma(n-\alpha_1)\Gamma(n-\alpha_2)}$ and $m^* = k(\alpha_2)\frac{\Gamma(-\alpha_1)}{\Gamma(n-\alpha_1)}$, appearing in (84) and (85) are the best possible.

For an illustration, we only give the proof of inequality (84).

Proof (Proof of Inequality (84)). The starting point is inequality (1) accompanied with the dual Hardy inequality (83). Namely, utilizing (1) with parameters $A_1 = -\frac{\alpha_1}{q}$ and $A_2 = -\frac{\alpha_2}{p}$, its right-hand side may be rewritten as

$$k(\alpha_2)\left[\int_{\mathbb{R}_+} x^{-p\alpha_1-1} f^p(x)dx\right]^{\frac{1}{p}}\left[\int_{\mathbb{R}_+} y^{-q\alpha_2-1} g^q(y)dy\right]^{\frac{1}{q}}$$
$$= k(\alpha_2)\left[\int_{\mathbb{R}_+} x^{-(p\alpha_1+1)}(\mathscr{H}^*(\mathscr{D}_\pm f)(x))^p dx\right]^{\frac{1}{p}}$$
$$\times\left[\int_{\mathbb{R}_+} y^{-(q\alpha_2+1)}(\mathscr{H}^*(\mathscr{D}_\pm g)(y))^q dy\right]^{\frac{1}{q}}, \tag{86}$$

since $\mathscr{H}^*(\mathscr{D}_\pm f)(x) = -\int_x^\infty f'(t)dt = f(x)$. Moreover, applying the dual Hardy inequality to the expressions on right-hand side of (86) n times, it follows that

$$\left[\int_{\mathbb{R}_+} x^{-(p\alpha_1+1)}(\mathscr{H}^*(\mathscr{D}_\pm f)(x))^p dx\right]^{\frac{1}{p}}$$
$$< \frac{1}{(-\alpha_1)^{\overline{n}}}\left[\int_{\mathbb{R}_+} x^{p(n-\alpha_1)-1}(\mathscr{D}_\pm^n f(x))^p dx\right]^{\frac{1}{p}} \tag{87}$$

and

$$\left[\int_{\mathbb{R}_+} y^{-(q\alpha_2+1)}(\mathscr{H}^*(\mathscr{D}_\pm g)(y))^q dy\right]^{\frac{1}{q}}$$
$$< \frac{1}{(-\alpha_2)^{\overline{n}}}\left[\int_{\mathbb{R}_+} y^{q(n-\alpha_2)-1}(\mathscr{D}_\pm^n g(y))^q dy\right]^{\frac{1}{q}}, \tag{88}$$

where $x^{\overline{n}}$ stands for a rising factorial power or a Pochhammer symbol, that is, $x^{\overline{n}} = x(x+1)(x+2)\cdots(x+n-1)$. Now, since $(-\alpha_1)^{\overline{n}} = \frac{\Gamma(n-\alpha_1)}{\Gamma(-\alpha_1)}$ and $(-\alpha_2)^{\overline{n}} = \frac{\Gamma(n-\alpha_2)}{\Gamma(-\alpha_2)}$, the inequality (84) holds due to (1), (86)–(88).

Remark 5. Considering dual inequalities (84) and (85) accompanied with the kernel $K_\lambda(x,y) = (x+y)^{-\lambda}$, $\lambda > 0$, the constants M^* and m^* become, respectively,

$$M_1^* = \frac{\pi^2}{\sin(\alpha_1\pi)\sin(\alpha_2\pi)}\cdot\frac{1}{\Gamma(\lambda)\Gamma(n-\alpha_1)\Gamma(n-\alpha_2)}$$
$$m_1^* = -\frac{\pi}{\sin(\alpha_1\pi)}\cdot\frac{\Gamma(\alpha_2+1)}{\Gamma(\lambda)\Gamma(n-\alpha_1)}, \quad \alpha_1,\alpha_2 \in (-1,\lambda-1), 0<\lambda\le 1.$$

In addition, if $K_\lambda(x,y) = \max\{x,y\}^{-\lambda}$, $\lambda > 0$, these constants M^* and m^* reduce to

$$M_2^* = \frac{\lambda}{(\alpha_1+1)(\alpha_2+1)} \cdot \frac{\Gamma(-\alpha_1)\,\Gamma(-\alpha_2)}{\Gamma(n-\alpha_1)\,\Gamma(n-\alpha_2)}$$

$$m_2^* = \frac{\lambda}{(\alpha_1+1)(\alpha_2+1)} \cdot \frac{\Gamma(-\alpha_1)}{\Gamma(n-\alpha_1)}, \quad \alpha_1, \alpha_2 \in (-1, \lambda-1), 0 < \lambda \le 1.$$

For some other applications of Theorems 20 and 21, the reader is referred to [7].

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A Fixed Point Approach to Stability of the Quadratic Equation

M. Almahalebi, A. Charifi, S. Kabbaj, and E. Elqorachi

Abstract In this paper, by using the fixed point method in Banach spaces, we prove the Hyers–Ulam–Rassias stability for the quadratic functional equation

$$f\left(\sum_{i=1}^{m} x_i\right) = \sum_{i=1}^{m} f(x_i) + \frac{1}{2} \sum_{1 \le i < j \le m} \{f(x_i + x_j) - f(x_i - x_j)\}.$$

The concept of the Hyers–Ulam–Rassias stability originated from Rassias' stability theorem that appeared in his paper: On the stability of the linear mapping in Banach spaces. Proc. Am. Math. Soc. 72(2), 297–300 (1978).

Keywords Hyers-Ulam-Rassias stability • fixed point alternative theorem • quadratic functional equation • quadratic mapping

1 Introduction and Preliminaries

Under what conditions does there exist a group homomorphism near an approximate group homomorphism? This question concerning the stability of group homomorphisms was posed by Ulam [60]. In 1941, Ulam's problem for the case of approximately additive mappings was solved by Hyers [25] on Banach spaces. In 1950 Aoki [6] provided a generalization of the Hyers' theorem for additive mappings and in 1978 Rassias [53] generalized the Hyers' theorem for linear

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mappings by considering an unbounded Cauchy difference. The result of Rassias' theorem has been generalized by Găvruta [23] who permitted the Cauchy difference to be bounded by a general control function in the spirit of Rassias' approach.

Since then, the stability problems of various types of functional equations have been extensively investigated by different authors, we refer, for example, to [8, 9, 12, 13, 22, 26–33, 35–40, 42–47, 51–59, 61].

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1}$$

is called a *quadratic functional equation*, and every solution of the quadratic functional equation is said to be a *quadratic mapping*.

A Hyers–Ulam stability problem for the quadratic functional equation was proved by Skof [56] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space.

Cholewa [15] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group.

Czerwik [17] obtained the Hyers–Ulam–Rassias stability of the quadratic functional equation. The stability problem of quadratic functional equations have been investigated by a number of authors, we refer, for example, to [1, 4, 5, 14, 21, 24, 29, 31, 35, 39, 44].

In 2003 Cădariu and Radu [10] noticed that a fixed point alternative method is very essential for the solution of the Hyers–Ulam stability problem. Subsequently, this method was applied to investigate the Hyers–Ulam–Rassias stability for Jensen functional equation, as well as for the additive Cauchy functional equation [11] by considering a general control function $\varphi(x, y)$, with appropriate properties. By applying this idea, several mathematicians applied the method to investigate the stability of certain functional equations, see for example [2, 3, 16, 34, 41, 48].

The *fixed point method* was used for the first time by Baker [7] who applied a variant of Banach's fixed point theorem to obtain the Hyers Ulam stability of a functional equation in a single variable.

In the present paper, we shall study the following functional equations:

$$f\left(\sum_{i=1}^{m} x_i\right)=\sum_{i=1}^{m} f(x_i)+\frac{1}{2}\sum_{1\le i<j\le m}\{f(x_i+x_j)-f(x_i-x_j)\} \tag{2}$$

$$\begin{aligned} f\left(\sum_{i=1}^{k} x_i+\sum_{j=1}^{h} y_j\right)&=\sum_{i=1}^{k} f(x_i)+\sum_{j=1}^{h} f(y_j)\\ &\quad+\frac{1}{2}\sum_{1\le i<j\le k}\{f(x_i+x_j)-f(x_i-x_j)\}\end{aligned}$$

$$+\frac{1}{2}\sum_{1\le i<j\le h}\{f(y_i+y_j)-f(y_i-y_j)\}$$
$$+\frac{1}{2}\sum_{j=1}^{h}\sum_{i=1}^{k}\{f(x_i+y_j)-f(x_i-y_j)\}, \tag{3}$$

and we consider the difference operator $Df : X^m \to Y$ as:

$$Df(x_1, x_2, \ldots, x_m) := f\left(\sum_{i=1}^{m} x_i\right) - \sum_{i=1}^{m} f(x_i) - \frac{1}{2}\sum_{1\le i<j\le m}\{f(x_i+x_j)-f(x_i-x_j)\}.$$

The main purpose of this paper is to provide the general solution of (2), (3) and to apply the fixed point method as in [10] to prove the Hyers–Ulam–Rassias stability of the functional equations (2) and (3).

In the following we recall one of the fundamental results of fixed point theory.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies the following:

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1 ([19]). *Suppose we are given a complete generalized metric space (X, d) and a strictly contractive mapping $J : X \to X$, with the Lipshitz constant $L < 1$. If there exists a nonnegative integer k such that $d(J^k x, J^{k+1} x) < \infty$ for some $x \in X$, then the following are true:*

(1) the sequence $J^n x$ converges to a fixed point x^* of J;
(2) x^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^k x, y) < \infty\}$;
(3) $d(y, x^*) \le \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

2 Solutions of Eq. (2)

Throughout this section, X and Y will be real vector spaces. The functional equation (2) is connected with the functional equation (1) as it is shown below:

Theorem 2. *A function $f : X \to Y$ satisfies the functional equation:*

$$f\left(\sum_{i=1}^{m} x_i\right) = \sum_{i=1}^{m} f(x_i) + \frac{1}{2}\sum_{1\le i<j\le m}\{f(x_i+x_j)-f(x_i-x_j)\} \tag{4}$$

for all $x_1, x_2, \ldots, x_m \in X$ *if and only if* f *satisfies the quadratic functional equation:*

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{5}$$

for $x, y \in X$.

Proof. Suppose that the function $f : X \longrightarrow Y$ satisfies (4) for all $x_1, \ldots, x_m \in X$. Letting $x_i = x_j$ for $i, j \in \{1, \ldots, m\}$ in (4), we get that $f(0) = 0$. Setting $x_1 = x$, $x_2 = y$ and $x_i = 0, i = 3, 4, \ldots, m$ in (4), we conclude that f is a solution of (5). Conversely, let $f : X \longrightarrow Y$ be a quadratic function, so $f(0) = 0$ and f is even. Now, by induction, we will prove (4). f is a quadratic function, then we have

$$f(x+y) = f(x) + f(y) + \frac{1}{2}(f(x+y) - f(x-y)),$$

this proves (4) for $m = 1$. Assume that (4) is true for m and written

$$f\left(\sum_{i=1}^{m+1} x_i\right) = f\left(\sum_{i=1}^{m} y_i\right), \tag{6}$$

with $y_1 = x_1 + x_{m+1}$ and $y_i = x_i, 2 \le i \le m$. Then,

$$\begin{aligned} f\left(\sum_{i=1}^{m} y_i\right) &= \sum_{i=1}^{m} f(y_i) + \frac{1}{2} \sum_{1 \le i < j \le m} (f(y_i + y_j) - f(y_i - yj)) \\ &= f(x_1 + x_{m+1}) + \sum_{i=2}^{m} f(x_i) \\ &\quad + \frac{1}{2} \sum_{j=2}^{m} \big(f(x_1 + x_{m+1} + x_j) - f(x_1 + x_{m+1} - x_j)\big) \\ &\quad + \frac{1}{2} \sum_{2 \le i < j \le m} \{f(x_i + x_j) - f(x_i - x_j)\}. \end{aligned} \tag{7}$$

By using the induction assumption for $m = 3$, and the fact that f is an even function we obtain

$$\begin{aligned} f(x_1 \pm x_j + x_{m+1}) = f(x_1) + f(x_j) + f(x_{m+1}) \\ + \frac{1}{2} \big(f(x_1 \pm x_j) - f(x_1 \mp x_j)\big) \\ + \frac{1}{2} (f(x_1 + x_{m+1}) - f(x_1 - x_{m+1})) \\ + \frac{1}{2} \big(f(x_j \pm x_{m+1}) - f(x_j \mp x_{m+1})\big) \end{aligned} \tag{8}$$

so, we have

$$\begin{aligned}\frac{1}{2}\sum_{j=2}^{m}\big(f(x_1+x_{m+1}+x_j)-f(x_1+x_{m+1}-x_j)\big)\\ =\frac{1}{2}\sum_{j=2}^{m}\big(f(x_1+x_j)-f(x_1-x_j)\big)\\ +\frac{1}{2}\sum_{j=2}^{m}\big(f(x_j+x_{m+1})-f(x_j-x_{m+1})\big),\end{aligned}\tag{9}$$

$$f(x_1+x_{m+1})=f(x_1)+f(x_{m+1})+\frac{1}{2}\left(f(x_1+x_{m+1})-f(x_1-x_{m+1})\right).\tag{10}$$

It follows from (7), (9) and (10) that

$$\begin{aligned}f\left(\sum_{i=1}^{m}y_i\right)=f(x_1)+f(x_{m+1})+\sum_{i=2}^{m}f(x_i)\\ +\frac{1}{2}\left(f(x_1+x_{m+1})-f(x_1-x_{m+1})\right)\\ +\frac{1}{2}\sum_{j=2}^{m}\big(f(x_j+x_{m+1})-f(x_j-x_{m+1})\big)\\ +\frac{1}{2}\sum_{j=2}^{m}\big(f(x_1+x_j)-f(x_1-x_j)\big)\\ +\frac{1}{2}\sum_{2\le i<j\le m}\{f(x_i+x_j)-f(x_i-x_j)\}.\end{aligned}\tag{11}$$

Therefore,

$$\begin{aligned}f\left(\sum_{i=1}^{m}y_i\right)=\sum_{i=1}^{m+1}f(x_i)+\frac{1}{2}\sum_{j=1}^{m}\big(f(x_j+x_{m+1})-f(x_j-x_{m+1})\big)\\ +\frac{1}{2}\sum_{1\le i<j\le m}\{f(x_i+x_j)-f(x_i-x_j)\}.\end{aligned}\tag{12}$$

Consequently,

$$f\left(\sum_{i=1}^{m+1} x_i\right) = \sum_{i=1}^{m+1} f(x_i) + \frac{1}{2} \sum_{1\le i<j\le m+1} \{f(x_i + x_j) - f(x_i - x_j)\}. \tag{13}$$

which proves (4) for $m + 1$.

In the following, by using [50] we find the general solution of (4).

Corollary 1. *A function $f\colon X \longrightarrow Y$ satisfies (4) if and only if there exists a symmetric b-additive function $B\colon X \times X \longrightarrow Y$ such that $f(x) = B(x, x)$ for all $x \in X$.*

3 Hyers–Ulam–Rassias Stability of the Quadratic Functional Equation (4)

Throughout this section, we assume that X is a normed space and Y is a Banach space. In the following theorem, we will apply the fixed point method as in [19] to prove the Hyers–Ulam–Rassias stability of the quadratic functional equation (4). For convenience, we use the following abbreviation. For a given function $f\colon X \longrightarrow Y$

$$Df(x_1,\ldots,x_m) = f\left(\sum_{i=1}^{m} x_i\right) - \sum_{i=1}^{m} f(x_i) - \frac{1}{2} \sum_{1\le i<j\le m} \{f(x_i + x_j) - f(x_i - x_j)\}.$$

Theorem 3. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exists a function $\phi : X^m \to [0,\infty)$, where $m \ge 2$ be an integer, such that there exists an $L < 1$ and*

$$\phi(x, -x, \ldots, \pm x) \le 4L\phi\left(\frac{x}{2}, \frac{-x}{2}, \ldots, \frac{\pm x}{2}\right) \tag{14}$$

for all $x \in X$;

$$\lim_{n\to+\infty} 4^{-n}\phi(2^n x_1, 2^n x_2, \ldots, 2^n x_m) = 0 \tag{15}$$

$$\|Df(x_1, x_2, \ldots, x_m)\| \le \phi(x_1, x_2, \ldots, x_m) \tag{16}$$

for all $x_1, x_2, \ldots, x_m \in X$. Then, there exists a unique quadratic mapping Q which satisfies

(1)

$$\|f(x) - Q(x)\| \le \frac{1}{m - mL}\phi(x, -x, x, \dots, x, -x) \tag{17}$$

for all $x \in X$, where m is an even integer;

(2)

$$\|f(x) - Q(x)\| \le \frac{1}{(m-1)(1-L)}\phi(x, -x, x, \dots, -x, x) \tag{18}$$

for all $x \in X$, where m is an odd integer.

Proof. If $f : X \longrightarrow Y$ be an even mapping, then we study two cases as follows:

Case 1: *m* is even

Let us consider the set $S := \{g : X \to Y\}$ and introduce the generalized metric on S as follows:

$$d(g, h) = \inf\{K \in [0, \infty) : \|g(x) - h(x)\| \le K\phi(x, -x, \dots, x, -x), \forall x \in X\}. \tag{19}$$

It is easy to show that (S, d) is complete (see for example [11]). Now, we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4}g(2x) \tag{20}$$

for all $x \in X$. First we assert that J is strictly contractive on S. Given $g, h \in S$, let $K \in [0, \infty)$ be an arbitrary constant with $d(g, h) \le K$, that is $\| g(x) - h(x) \| \le K\phi(x, -x, \dots, x, -x)$. So, we have

$$\begin{aligned} \| Jg(x) - Jh(x) \| &= \frac{1}{4} \| g(2x) - h(2x) \| \le \frac{1}{4}K\phi(2x, -2x, \dots, 2x, -2x) \\ &\le KL\phi(x, -x, \dots, x, -x) \end{aligned}$$

for all $x \in X$, that is, $d(Jg, Jh) \le Ld(g, h)$, for any $g, h \in S$.

For i odd and j even, we let $x_i = x$ and $x_j = -x$ in (16), so by using the evenness of f we get

$$\left\|\frac{m}{4}f(2x) - mf(x)\right\| \le \phi(x, -x, \dots, x, -x) \tag{21}$$

for all $x \in X$ and we obtain,

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \le \frac{1}{m}\phi(x, -x, \dots, x, -x) \tag{22}$$

for all $x \in X$, that is

$$d(f, Jf) \leq \frac{1}{m} < \infty \tag{23}$$

By Theorem 1, there exists a mapping $Q : X \to Y$ satisfying the following:

1. Q is a fixed point of J, that is,

$$Q(2x) = 4Q(x) \tag{24}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}. \tag{25}$$

This implies that Q is a unique mapping satisfying (24) such that there exists $K \in (0, \infty)$ which satisfies

$$\| f(x) - Q(x)\| \leq K\phi(x, -x, \ldots, x, -x) \tag{26}$$

for all $x \in X$.

2.

$$d(J^n f, Q) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

This implies that

$$\lim_{n\to+\infty} J^n f(x) = \lim_{n\to+\infty} \frac{f(2^n x)}{4^n} = Q(x) \tag{27}$$

for all $x \in X$.

3.

$$d(f, Q) \leq \frac{1}{1-L} d(f, Jf), \tag{28}$$

which implies the inequality

$$d(f, Q) \leq \frac{1}{m - mL} \tag{29}$$

This implies that the inequality (17) holds.

It follows from (15), (16) and (27) that

$$\| DQ(x_1, x_2, \ldots, x_m) \| = \lim_{n\to+\infty} \frac{1}{4^n} \|Df(2^n x_1, 2^n x_2, \ldots, 2^n x_m)\|$$

$$\leq \lim_{n\to+\infty} \frac{1}{4^n}\phi(2^n x_1, 2^n x_2, \ldots, 2^n x_m) = 0 \tag{30}$$

for all $x_1, x_2, \ldots, x_m \in X$. So, $DQ(x_1, x_2, \ldots, x_m) = 0$ for all $x_1, x_2, \ldots, x_m \in X$. By Theorem 2, we get that the mapping $Q : X \to Y$ is a quadratic function. Therefore, there exists a unique quadratic mapping $Q : X \to Y$ satisfying (17), as desired.

Case 2: *m* is odd

Let us consider the set $S := \{g : X \to Y\}$ and introduce the generalized metric on S as follows:

$$d(g, h) = \inf\{K \in [0, \infty) : \|g(x) - h(x)\| \leq K\phi(x, -x, \ldots, -x, x), \forall x \in X\}. \tag{31}$$

It is easy to show that (S, d) is complete (see for example [11]). Now, we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4}g(2x) \tag{32}$$

for all $x \in X$. Given $g, h \in S$ and $K \in [0, \infty)$ such that $d(g, h) \leq K$, so we get

$$\| Jg(x) - Jh(x) \| = \frac{1}{4} \| g(2x) - h(2x) \| \leq \frac{1}{4}K\phi(2x, -2x, \ldots, -2x, 2x)$$
$$\leq KL\phi(x, -x, \ldots, -x, x)$$

for all $x \in X$. Hence we see that $(Jg, Jh) \leq Ld(g, h)$ for all $g, h \in S$. So J is a strictly contractive operator.

Putting $x_i = x$ and $x_j = -x$ in (16), for i odd and j even, we have

$$\left\| \frac{(m-1)}{4} f(2x) - (m-1) f(x) \right\| \leq \phi(x, -x, \ldots, -x, x) \tag{33}$$

for all $x \in X$. So,

$$\left\| f(x) - \frac{1}{4} f(2x) \right\| \leq \frac{1}{m-1}\phi(x, -x, \ldots, -x, x) \tag{34}$$

for all $x \in X$. That is,

$$d(f, Jf) \leq \frac{1}{m-1} < \infty \tag{35}$$

for all $f \in S$. The rest of the proof is similar to the proof of case 1.

Corollary 2. *Let* $0 < p < 2$ *and* $\theta \geq 0$ *be real numbers, and let* $f : X \to Y$ *be a mapping such that*

$$\| Df(x_1, x_2, \ldots, x_m) \| \leq \theta \sum_{i=1}^{m} \| x_i \|^p \tag{36}$$

for all $x_1, x_2, \ldots, x_m \in X - \{0\}$*, where* $m \geq 2$ *be an even integer. Then, there exists a unique quadratic mapping* $Q : X \to Y$ *such that*

$$\| f(x) - Q(x) \| \leq \frac{\theta}{(1 - 2^{p-2})} \| x \|^p \tag{37}$$

for all $x \in X - \{0\}$.

Proof. We obtain Theorem 3 by taking

$$\phi(x_1, x_2, \ldots, x_m) := \theta \sum_{i=1}^{m} \| x_i \|^p \tag{38}$$

for all $x_1, x_2, \ldots, x_m \in X - \{0\}$ Then, we can choose $L = 2^{p-2}$ and we get the desired result.

Corollary 3. *Let* p *and* $\theta \geq 0$ *be real numbers such that* $0 < p < \frac{2}{m}$*, and let* $f : X \to Y$ *be a mapping such that*

$$\|Df(x_1, x_2, \ldots, x_m)\| \leq \theta \prod_{i=1}^{m} \| x_i \|^p \tag{39}$$

for all $x_1, x_2, \ldots, x_m \in X - \{0\}$*, where* $m \geq 2$ *be an even integer. Then, there exists a unique quadratic mapping* $Q : X \to Y$ *such that*

$$\|f(x) - Q(x)\| \leq \frac{\theta}{m(1 - 2^{mp-2})} \| x \|^{mp} \tag{40}$$

for all $x \in X - \{0\}$.

Proof. The proof follows from Theorem 3 by taking

$$\phi(x_1, x_2, \ldots, x_m) := \theta \prod_{i=1}^{m} \| x_i \|^p \tag{41}$$

for all $x_1, x_2, \ldots, x_m \in X - \{0\}$. Then, we can choose $L = 2^{mp-2}$ and we get the desired result.

Corollary 4. *Let p_i, $i = 1, 2, \ldots, m$, where $m \geq 2$ be an even integer and $\sum_{i=1}^{m} p_i < 2$, let $\theta \geq 0$ be a real number, and let $f : X \to Y$ be a mapping fulfilling*

$$\|Df(x_1, x_2, \ldots, x_m)\| \leq \theta \prod_{i=1}^{m} \| x_i \|^{p_i} \tag{42}$$

for all $x_1, x_2, \ldots, x_m \in X - \{0\}$. Then, there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{\theta}{m(1 - 2^{(\sum_{i=1}^{m} p_i)-2})} \| x \|^{(\sum_{i=1}^{m} p_i)} \tag{43}$$

for all $x \in X - \{0\}$.

Proof. We obtain Theorem 3 by taking

$$\phi(x_1, x_2, \ldots, x_m) := \theta \prod_{i=1}^{m} \| x_i \|^{p_i} \tag{44}$$

for all $x_1, x_2, \ldots, x_m \in X - \{0\}$, where $m \geq 2$ be an even integer. Then, we can choose $L = 2^{(\sum_{i=1}^{m} p_i)-2}$ and we get the desired result.

Remark 1. Let $0 < p < 2$ and $\theta \geq 0$ be real numbers, and let $f : X \to Y$ be a mapping such that

$$\|Df(x_1, x_2, \ldots, x_m)\| \leq \theta \sum_{i=1}^{m} \| x_i \|^p \tag{45}$$

for all $x_1, x_2, \ldots, x_m \in X - \{0\}$, where $m \geq 3$ be an odd integer. Then, there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{m\theta}{(m-1)(1 - 2^{p-2})} \| x \|^p \tag{46}$$

for all $x \in X - \{0\}$.

Proof. We obtain Theorem 3 by taking

$$\phi(x_1, x_2, \ldots, x_m) := \theta \sum_{i=1}^{m} \| x_i \|^p \tag{47}$$

for all $x_1, x_2, \ldots, x_m \in X - \{0\}$, where $m \geq 3$ be an odd integer. Then, we can choose $L = 2^{p-2}$ and we get the desired result.

Remark 2. Let p and $\theta \geq 0$ be real numbers such that $0 < p < \frac{2}{m}$, and let $f : X \to Y$ be a mapping such that

$$\|Df(x_1, x_2, \ldots, x_m)\| \leq \theta \prod_{i=1}^{m} \| x_i \|^p \tag{48}$$

for all $x_1, x_2, \ldots, x_m \in X - \{0\}$, where $m \geq 3$ be an odd integer. Then, there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{\theta}{(m-1)(1-2^{mp-2})} \| x \|^{mp} \tag{49}$$

for all $x \in X - \{0\}$.

Proof. We obtain Theorem 3 by taking

$$\phi(x_1, x_2, \ldots, x_m) := \theta \prod_{i=1}^{m} \| x_i \|^p \tag{50}$$

for all $x_1, x_2, \ldots, x_m \in X - \{0\}$, where $m \geq 3$ be an odd integer. Then, we can choose $L = 2^{mp-2}$ and we get the desired result.

Remark 3. Let p_i, $i = 1, 2, \ldots, m$, where $m \geq 2$ be an odd integer and $\sum_{i=1}^{m} p_i < 2$, let $\theta \geq 0$ be a real number, and let $f : X \to Y$ be a mapping such that

$$\|Df(x_1, x_2, \ldots, x_m)\| \leq \theta \prod_{i=1}^{m} \| x_i \|^{p_i} \tag{51}$$

for all $x_1, x_2, \ldots, x_m \in X - \{0\}$, where $m \geq 3$ be an odd integer. Then, there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{\theta}{(m-1)(1-2^{(\sum_{i=1}^{m} p_i)-2})} \| x \|^{(\sum_{i=1}^{m} p_i)} \tag{52}$$

for all $x \in X - \{0\}$.

Proof. We obtain Theorem 3 by taking

$$\phi(x_1, x_2, \dots, x_m) := \theta \prod_{i=1}^{m} \| x_i \|^{p_i} \tag{53}$$

for all $x_1, x_2, \dots, x_m \in X - \{0\}$, where $m \geq 3$ be an odd integer. Then, we can choose $L = 2^{(\sum_{i=1}^{m} p_i)-2}$ and we get the desired result.

Remark 4. Let $f : X \to Y$ be an even mapping with $f(0) = 0$ for which there exists a function $\phi : X^m \to [0, \infty)$, where $m \geq 2$ be an integer, such that

$$\lim_{n \to +\infty} 4^n \phi\left(\frac{1}{2^n}x_1, \frac{1}{2^n}x_2, \dots, \frac{1}{2^n}x_m\right) = 0 \tag{54}$$

for all $x_1, x_2, \dots, x_m \in X$. By similar method to the proof of Theorem 3, one can show that if there exists an $L < 1$ such that

$$\phi(x, -x, \dots, \pm x) \leq \frac{1}{4} L \phi(2x, -2x, \dots, \pm 2x)$$

for all $x \in X$, then there exists a unique quadratic mapping $Q : X \to Y$ satisfying

$$\|Df(x_1, x_2, \dots, x_m)\| \leq \phi(x_1, x_2, \dots, x_m) \tag{55}$$

and

(1)

$$\|f(x) - Q(x)\| \leq \frac{L}{m - mL} \phi(x, -x, x, \dots, x, -x), \tag{56}$$

for all $x \in X$, where m is an even integer;

(2)

$$\|f(x) - Q(x)\| \leq \frac{L}{(m-1)(1-L)} \phi(x, -x, x, \dots, -x, x) \tag{57}$$

for all $x \in X$, where m is an odd integer.

For the cases $p > 2$, $p > \frac{2}{m}$ and $\sum_{i=1}^{m} p_i > 0$, $i = 1, 2, \dots, m$, we can obtain a similar result to Corollaries 2–4 respectively. For the cases $p > 2$, $p > \frac{2}{m}$ and $\sum_{i=1}^{m} p_i > 0$, $i = 1, 2, \dots, m$, we can obtain a similar result to Remarks 1–3 respectively.

Theorem 4. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exists a function $\phi : X^m \to [0, \infty)$, where $m \geq 2$ be an integer, such that there exists an $L < 1$ such that*

$$\phi(x_1, x_2, \dots, x_m) \leq 4L\phi\left(\frac{x_1}{2}, \frac{x_2}{2}, \dots, \frac{x_m}{2}\right) \tag{58}$$

for all $x \in X$, and

$$\lim_{n\to+\infty} 4^{-n}\phi(2^n x_1, 2^n x_2, \dots, 2^n x_m) = 0 \tag{59}$$

$$\|Df(x_1, x_2, \dots, x_m)\| \leq \phi(x_1, x_2, \dots, x_m) \tag{60}$$

for all $x_1, x_2, \dots, x_m \in X$. Then, there exists a unique quadratic mapping $Q : X \to Y$ which satisfies

(1)

$$\|f(x) - Q(x)\| \leq \frac{2}{m - mL}\left\{\phi\left(\frac{x}{2}, \frac{-x}{2}, \dots, \frac{-x}{2}\right) + \phi\left(\frac{-x}{2}, \frac{x}{2}, \dots, \frac{x}{2}\right)\right\} \tag{61}$$

for all $x \in X$, where m is an even integer;

(2)

$$\|f(x)-Q(x)\| \leq \frac{2}{(m-1)(1-L)}\left\{\phi\left(\frac{x}{2}, \frac{-x}{2}, \dots, \frac{x}{2}\right)+\phi\left(\frac{-x}{2}, \frac{x}{2}, \dots, \frac{-x}{2}\right)\right\} \tag{62}$$

for all $x \in X$, where m is an odd integer.

Proof. We decompose f into the odd part and the even part by putting

$$f_o(x) = \frac{f(x) - f(-x)}{2} \text{ and } f_e(x) = \frac{f(x) + f(-x)}{2}$$

for all $x \in X$. It is clear that $f(x) = f_o(x) + f_e(x)$ for all $x \in X$. It follows from (60) that

$$\|Df_e(x_1, \dots, x_m)\| \leq \frac{1}{2}\{\phi(x_1, \dots, x_m) + \phi(-x_1, \dots, -x_m)\} \tag{63}$$

$$\|Df_o(x_1, \dots, x_m)\| \leq \frac{1}{2}\{\phi(x_1, \dots, x_m) + \phi(-x_1, \dots, -x_m)\} \tag{64}$$

for all $x_1, \dots, x_m \in X$.

We have f_e is even and satisfies (63), so by Theorem 3, there exists a quadratic mapping $Q : X \longrightarrow Y$ such that

$$\|f_e(x) - Q(x)\| \leq \frac{1}{2m(1-L)}\{\phi(x,-x,\ldots,-x) + \phi(-x,x,\ldots,x)\} \tag{65}$$

where m is even, and

$$\|f_e(x) - Q(x)\| \leq \frac{1}{2(m-1)(1-L)}\{\phi(x,-x,\ldots,-x) + \phi(-x,x,\ldots,x)\} \tag{66}$$

where m is odd. It implies that

$$\|f_e(x) - Q(x)\| \leq \frac{2L}{m(1-L)}\left\{\phi\left(\frac{x}{2},\frac{-x}{2},\ldots,\frac{-x}{2}\right) + \phi\left(\frac{-x}{2},\frac{x}{2},\ldots,\frac{x}{2}\right)\right\} \tag{67}$$

where m is even, and

$$\|f_e(x)-Q(x)\| \leq \frac{2L}{(m-1)(1-L)}\left\{\phi\left(\frac{x}{2},\frac{-x}{2},\ldots,\frac{-x}{2}\right) + \phi\left(\frac{-x}{2},\frac{x}{2},\ldots,\frac{x}{2}\right)\right\} \tag{68}$$

where m is odd, for all $x \in X$.

On the other side, letting $x_i = x$ and $x_j = -x$ in (64), where i is even and j is odd, then we get two cases as follows

Firstly, if m is even, then we have

$$\left\|\frac{m}{4} f_o(2x)\right\| \leq \frac{1}{2}\{\phi(x,-x,\ldots,-x) + \phi(-x,x,\ldots,x)\} \tag{69}$$

Therefore,

$$\|f_o(x)\| \leq \frac{2}{m}\left\{\phi\left(\frac{x}{2},\frac{-x}{2},\ldots,\frac{-x}{2}\right) + \phi\left(\frac{-x}{2},\frac{x}{2},\ldots,\frac{x}{2}\right)\right\} \tag{70}$$

for all $x \in X$.

Secondly, if m is odd, then we have

$$\|f_o(x)\| \leq \frac{2}{m-1}\left\{\phi\left(\frac{x}{2},\frac{-x}{2},\ldots,\frac{x}{2}\right) + \phi\left(\frac{-x}{2},\frac{x}{2},\ldots,\frac{-x}{2}\right)\right\} \tag{71}$$

for all $x \in X$.

From (67) and (70), we get

$$\|f(x)-Q(x)\| \leq \|f_o(x)\| + \|f_e(x)-Q(x)\|$$
$$\leq \left\{\frac{2}{m}+\frac{2L}{m(1-L)}\right\}\left\{\phi\left(\frac{x}{2},\frac{-x}{2},\ldots,\frac{-x}{2}\right)+\phi\left(\frac{-x}{2},\frac{x}{2},\ldots,\frac{x}{2}\right)\right\} \tag{72}$$

for $x \in X$. Then, we obtain the inequality (61). In the same way, from (68) and (71), we get the inequality (62).

4 Solutions and Hyers–Ulam–Rassias Stability of Eq. (3)

Throughout this section, we assume that X is a normed space and Y is a Banach space.

Let us consider the functional equation

$$f\left(\sum_{i=1}^{k} x_i + \sum_{j=1}^{h} y_j\right) = \sum_{i=1}^{k} f(x_i) + \sum_{j=1}^{h} f(y_j) + \frac{1}{2}\sum_{1\leq i<j\leq k}\{f(x_i+x_j)-f(x_i-x_j)\} + \frac{1}{2}\sum_{1\leq i<j\leq h}\{f(y_i+y_j)-f(y_i-y_j)\} + \frac{1}{2}\sum_{j=1}^{h}\sum_{i=1}^{k}\{f(x_i+y_j)-f(x_i-y_j)\} \tag{73}$$

for all $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_h \in X$, where $k \geq 2$ and $h \geq 2$ are integers.

This equation is connected with the functional equation (4) as it is shown below:

Theorem 5. *A function $f : X \longrightarrow Y$ satisfies the functional equation (73) for all $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_h \in X$, if and only if f satisfies the functional equation (4) for all $x_1, x_2, \ldots, x_m \in X$.*

Proof. Assume that $f : X \longrightarrow Y$ satisfies (73). Let $y_j = x_{j+k}$; $j = 1, 2, \ldots, h$ in (73), we get

$$f\left(\sum_{i=1}^{k} x_i + \sum_{j=1}^{h} x_{j+k}\right) = \sum_{i=1}^{k} f(x_i) + \sum_{j=1}^{h} f(x_{j+k}) + \frac{1}{2}\sum_{1\leq i<j\leq k}\{f(x_i+x_j)-f(x_i-x_j)\}$$

$$+\frac{1}{2}\sum_{1\le i<j\le h}\{f(x_{i+k}+x_{j+k})-f(x_{i+k}-x_{j+k})\}$$
$$+\frac{1}{2}\sum_{j=1}^{h}\sum_{i=1}^{k}\{f(x_i+x_{j+k})-f(x_i-x_{j+k})\} \quad (74)$$

Then,

$$f\left(\sum_{i=1}^{k}x_i+\sum_{j=k+1}^{m}x_j\right)=\sum_{i=1}^{k}f(x_i)+\sum_{j=k+1}^{m}f(x_j)$$
$$+\frac{1}{2}\sum_{1\le i<j\le k}\{f(x_i+x_j)-f(x_i-x_j)\}$$
$$+\frac{1}{2}\sum_{k+1\le i<j\le m}\{f(x_i+x_j)-f(x_i-x_j)\}$$
$$+\frac{1}{2}\sum_{j=k+1}^{m}\sum_{i=1}^{k}\{f(x_i+x_j)-f(x_i-x_j)\} \quad (75)$$

for all $x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_m \in X$. Therefore,

$$f\left(\sum_{i=1}^{k+h}x_i\right)=\sum_{i=1}^{k+h}f(x_i)+\frac{1}{2}\sum_{1\le i<j\le k+h}\{f(x_i+x_j)-f(x_i-x_j)\} \quad (76)$$

So, we get that f satisfies (4).

Assume that $f : X \longrightarrow Y$ satisfies (4). Let $1 < k < m$, then we can write the functional equation (4) as follows

$$f\left(\sum_{i=1}^{k}x_i+\sum_{j=k+1}^{m}x_j\right)=\sum_{i=1}^{k}f(x_i)+\sum_{j=k+1}^{m}f(x_j)$$
$$+\frac{1}{2}\sum_{1\le i<j\le k}\{f(x_i+x_j)-f(x_i-x_j)\}$$
$$+\frac{1}{2}\sum_{k+1\le i<j\le m}\{f(x_i+x_j)-f(x_i-x_j)\}$$
$$+\frac{1}{2}\sum_{j=k+1}^{m}\sum_{i=1}^{k}\{f(x_i+x_j)-f(x_i-x_j)\} \quad (77)$$

for all $x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_m \in X$. Then,

$$f\left(\sum_{i=1}^{k} x_i + \sum_{j=1}^{h} x_{j+k}\right) = \sum_{i=1}^{k} f(x_i) + \sum_{j=1}^{h} f(x_{j+k}) + \frac{1}{2}\sum_{1\le i<j\le k}\{f(x_i + x_j) - f(x_i - x_j)\} + \frac{1}{2}\sum_{1\le i<j\le h}\{f(x_i + x_{j+k}) - f(x_i - x_{j+k})\} + \frac{1}{2}\sum_{j=1}^{h}\sum_{i=1}^{k}\{f(x_i + x_{j+k}) - f(x_i - x_{j+k})\} \quad (78)$$

where $m = k + h$. By putting $x_{j+k} = y_j$ with $j = 1, 2, \ldots, h$ in (78), we obtain the functional equation (73).

For a given mapping $f : X \longrightarrow Y$, we use the following abbreviation $\Delta : X^{k+h} \longrightarrow Y$, where

$$\Delta f(x_1, \ldots, x_k, y_1, \ldots, y_h) := f\left(\sum_{i=1}^{k} x_i + \sum_{j=1}^{h} y_j\right) - \sum_{i=1}^{k} f(x_i) - \sum_{j=1}^{h} f(y_j) - \frac{1}{2}\sum_{1\le i<j\le k}\{f(x_i + x_j) - f(x_i - x_j)\} - \frac{1}{2}\sum_{1\le i<j\le h}\{f(y_i + y_j) - f(y_i - y_j)\} - \frac{1}{2}\sum_{j=1}^{h}\sum_{i=1}^{k}\{f(x_i + y_j) - f(x_i - y_j)\} \quad (79)$$

for all $x_1, x_2, \ldots, x_k, y_1, y_2 \ldots, y_h \in X$.

In the following theorem, we will apply the fixed point method as in [19] to prove the Hyers–Ulam–Rassias stability of the quadratic functional equation (3).

Theorem 6. *Let $f : X \longrightarrow Y$ be an even mapping with $f(0) = 0$ for which there exists a function $\varphi : X^{k+h} \longrightarrow [0, \infty)$, where $h \geq 2$ and $k \geq 2$ are integers, such that there exists an $L < 1$ such that*

$$\varphi(x, -x, \ldots, \pm x) \leq 4L\varphi\left(\frac{x}{2}, \frac{-x}{2}, \ldots, \frac{\pm x}{2}\right) \quad (80)$$

for all $x \in X$, and

$$\lim_{n\to+\infty} 4^{-n}\varphi(2^n x_1,\ldots,2^n x_k,2^n y_1,\ldots,2^n y_h)=0 \tag{81}$$

$$\|\Delta f(x_1,\ldots,x_k,y_1,\ldots,y_h)\| \le \varphi(x_1,\ldots,x_k,y_1,\ldots,y_h) \tag{82}$$

for all $x_1,x_2,\ldots,x_k,y_1,y_2,\ldots,y_h \in X$. *Then, there exists a unique quadratic mapping* $Q: X \longrightarrow Y$ *which satisfies*

(1)

$$\|f(x)-Q(x)\| \le \frac{1}{(k+h)(1-L)}\varphi(x,-x,x,\ldots,x,-x) \tag{83}$$

for all $x \in X$, *where* $(k+h)$ *is an even integer;*

(2)

$$\|f(x)-Q(x)\| \le \frac{1}{(k+h-1)(1-L)}\varphi(x,-x,x,\ldots,-x,x) \tag{84}$$

for all $x \in X$, *where* $(k+h)$ *is an odd integer.*

Proof. If $f: X \longrightarrow Y$ is an even mapping, then we study two cases as follows:

Case 1: *k+h* is even

Consider the set $S := \{g : X \to Y\}$ and introduce the generalized metric on S as follows:

$$d(g,h)=\inf\{K\in[0,\infty):\|g(x)-h(x)\|\le K\varphi(x,-x,\ldots,x,-x),\forall x\in X\}.$$

It is easy to show that (S,d) is complete (see for example [11]). Now, we consider the linear mapping $J: S \to S$ such that

$$Jg(x) := \frac{1}{4}g(2x) \tag{85}$$

for all $x \in X$. Given g,h in S, let $K \in [0,\infty)$ be an arbitrary constant with $d(g,h) \le K$, that is

$$\begin{aligned}\| Jg(x)-Jh(x) \| &= \frac{1}{4}\| g(2x)-h(2x) \| \le \frac{1}{4}K\varphi(2x,-2x,\ldots,2x,-2x)\\ &\le KL\varphi(x,-x,\ldots,x,-x)\end{aligned}$$

for all $x \in X$. Hence we see that $d(Jg,Jh) \le Ld(g,h)$, for any $g,h \in S$. So J is a strictly contractive operator.

For i odd and j even, we let $x_i = x$ and $x_j = -x$ in (82), so by using the evenness of f we get

$$\left\|\frac{(k+h)}{4}f(2x)-(k+h)f(x)\right\|\leq\varphi(x,-x,\ldots,x,-x)$$

for all $x\in X$ and we obtain,

$$\left\|f(x)-\frac{1}{4}f(2x)\right\|\leq\frac{1}{k+h}\varphi(x,-x,\ldots,x,-x) \tag{86}$$

for all $x\in X$, that is

$$d(f,Jf)\leq\frac{1}{k+h}<\infty \tag{87}$$

The rest of the proof of this case is similar to the proof of Theorem 3.

Case 2: *k+h* is odd

Let us consider the set $S:=\{g:X\to Y\}$ and introduce the generalized metric on S as follows:

$$d(g,h)=\inf\{K\in[0,\infty):\|g(x)-h(x)\|\leq K\varphi(x,-x,\ldots,-x,x),\forall x\in X\}.$$

It is easy to show that (S,d) is complete (see for example [11]). Now, we consider the linear mapping $J:S\to S$ such that

$$Jg(x):=\frac{1}{4}g(2x) \tag{88}$$

for all $x\in X$. Given $g,h\in S$, let $K\in[0,\infty)$ be an arbitrary constant with $d(g,h)\leq K$, that is

$$\begin{aligned}\parallel Jg(x)-Jh(x)\parallel&=\frac{1}{4}\parallel g(2x)-h(2x)\parallel\leq\frac{1}{4}K\varphi(2x,-2x,\ldots,-2x,2x)\\&\leq KL\varphi(x,-x,\ldots,-x,x)\end{aligned}$$

for all $x\in X$. Hence we see that $(Jg,Jh)\leq Ld(g,h)$ for any $g,h\in S$. So J is a strictly contractive operator.

For i odd and j even, we let $x_i=x$ and $x_j=-x$ in (82), so by using the evenness of f we get

$$\left\|\frac{(k+h-1)}{4}f(2x)-(k+h-1)f(x)\right\|\leq\varphi(x,-x,\ldots,-x,x)$$

for all $x\in X$ and we obtain,

$$\left\|f(x)-\frac{1}{4}f(2x)\right\|\leq\frac{1}{k+h-1}\varphi(x,-x,\ldots,-x,x) \tag{89}$$

for all $x \in X$, that is

$$d(f, Jf) \leq \frac{1}{k+h-1} < \infty \tag{90}$$

$\forall f \in S$. The rest of this proof is similar to the proof of Theorem 3.

Theorem 7. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exists a function $\phi : X^m \to [0, \infty)$, where $m \geq 2$ be an integer, such that*

$$\|Df(x_1, x_2, \dots, x_m)\| \leq \phi(x_1, x_2, \dots, x_m) \tag{91}$$

$$\lim_{n \to +\infty} m^{-2n} \phi(m^n x_1, m^n x_2, \dots, m^n x_m) = 0 \tag{92}$$

for all $x_1, x_2, \dots, x_m \in X$. Let $0 < L < 1$ be a constant such that the mapping

$$x \mapsto \psi(x) := \phi(x, \dots, x) + \frac{m(m-1)}{2} \phi(x, 0, \dots, 0)$$

satisfying $\psi(x) \leq m^2 L \psi(\frac{x}{m})$ for all $x \in X$. Then, there exists a unique quadratic mapping $Q : X \to Y$ which satisfies

$$\|f(x) - Q(x)\| \leq \frac{1}{m^2(1-L)} \psi(x) \tag{93}$$

for all $x \in X$.

Proof. Consider the set $S := \{g : X \to Y\}$ and introduce the generalized metric on S as follows:

$$d(g, h) = \inf \{K \in [0, \infty) : \|g(x) - h(x)\| \leq K\psi(x), \forall x \in X\}.$$

It is easy to show that (S, d) is complete (see for example [11]). Now, we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{m^2} g(mx) \tag{94}$$

for all $x \in X$. Given $g, h \in S$, let $K \in [0, \infty)$ be an arbitrary constant with $d(g, h) \leq K$, that is

$$\begin{aligned} \| Jg(x) - Jh(x) \| &= \frac{1}{m^2} \| g(mx) - h(mx) \| \leq \frac{1}{m^2} K\psi(mx) \\ &\leq KL\psi(x) \end{aligned}$$

for all $x \in X$. Hence we see that $(Jg, Jh) \leq Ld(g, h)$ for any $g, h \in X$. From (91), we get

$$\left\| Df(x, \dots, x) - \frac{m(m-1)}{2} Df(x, 0, \dots, 0) \right\| \leq \phi(x, \dots, x) + \frac{m(m-1)}{2} \phi(x, 0, \dots, 0)$$

for all $x \in X$ and we obtain,

$$\| f(mx) - m^2 f(x) \| \leq \psi(x) \tag{95}$$

for all $x \in X$, that is

$$\| f(x) - \frac{1}{m^2} f(mx) \| \leq \frac{1}{m^2} \psi(x)$$

$$\| f(x) - Jf(x) \| \leq \frac{1}{m^2} \psi(x)$$

Hence we see that

$$d(f, Jf) \leq \frac{1}{m^2} < \infty. \tag{96}$$

The rest of the proof is similar to the proof of Theorem 3.

Corollary 5. *Let $0 < p < 2$ and $\theta \geq 0$ be real numbers, and let $f : X \to Y$ be a mapping such that*

$$\|Df(x_1, x_2, \dots, x_m)\| \leq \theta \sum_{i=1}^{m} \| x_i \|^p \tag{97}$$

for all $x_1, x_2, \dots, x_m \in X - \{0\}$, where $m \geq 2$ be an integer. Then, there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{(m+1)}{2(1 - m^{p-2})} \theta \| x \|^p \tag{98}$$

for all $x \in X - \{0\}$.

Proof. We obtain Theorem 7 by taking

$$\phi(x_1, x_2, \dots, x_m) := \theta \sum_{i=1}^{m} \| x_i \|^p \tag{99}$$

for all $x_1, x_2, \dots, x_m \in X - \{0\}$, where $m \geq 2$ be an integer. Then, we can choose $L = m^{p-2}$ and we get the desired result.

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Aspects of Global Analysis of Circle-Valued Mappings

Dorin Andrica, Dana Mangra, and Cornel Pintea

Abstract We deal with the minimum number of critical points of circular functions with respect to two different classes of functions. The first one is the whole class of smooth circular functions and, in this case, the minimum number is the so called *circular φ-category* of the involved manifold. The second class consists of all smooth circular Morse functions, and the minimum number is the so called *circular Morse–Smale characteristic* of the manifold. The investigations we perform here for the two circular concepts are being studied in relation with their real counterparts. In this respect, we first evaluate the circular φ-category of several particular manifolds. In Sect. 5, of more survey flavor, we deal with the computation of the circular Morse–Smale characteristic of closed surfaces. Section 6 provides an upper bound for the Morse–Smale characteristic in terms of a new characteristic derived from the family of circular Morse functions having both a critical point of index 0 and a critical point of index n. The minimum number of critical points for real or circle valued Morse functions on a closed orientable surface is the minimum characteristic number of suitable embeddings of the surface in $\mathbb{R}^3$ with respect to some involutive distributions. In the last section we obtain a lower and an upper bound for the minimum characteristic number of the embedded closed surfaces in the first Heisenberg group with respect to its noninvolutive horizontal distribution.

Keywords Circular function • φ-Category • Circular φ-category • Covering map • Fundamental group • Compact surface • Projective space • Morse function • Morse–Smale characteristic • Circular Morse–Smale characteristic • Riemann surface • Poincaré–Hopf Theorem • The first Heisenberg group • Characteristic point

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1 Introduction

Let M^m, N^n be two smooth manifolds and let $\mathscr{F} \subseteq C^\infty(M, N)$ be a family of smooth mappings $M \to N$. The $\varphi_{\mathscr{F}}$*-category* of the pair (M, N) is defined by

$$\varphi_{\mathscr{F}}(M, N) = \min\{\mu(f) : f \in \mathscr{F}\},$$

where $\mu(f) = \text{card}C(f)$ and $C(f)$ is the critical set of f. It is clear that $0 \leq \varphi_{\mathscr{F}}(M, N) \leq +\infty$ and we have $\varphi_{\mathscr{F}}(M, N) = 0$ if and only if the family $\mathscr{F}$ contains immersions (if $m < n$), submersions (if $m > n$), or local diffeomorphisms (if $m = n$). Several topological and geometrical properties involving the $\varphi_{\mathscr{F}}$-category of a pair (M, N) of manifolds are provided in [8]. For more details and examples of pairs of manifolds with finite or infinite φ-category we refer to the reader to [3–5, 25, 26] or [18, 45–50], respectively.

In this paper we deal with two examples of circular categories. In Sect. 2 we briefly review some results involving the real φ-category. The circular φ-category of a manifold M is defined in Sect. 3 as the φ-category of the pair (M, S^1) corresponding to the family $C^\infty(M, S^1)$. Taking into account the inequality $\varphi_{S^1}(M) \leq \varphi(M)$, emphasized by the relations (7), one of the main goals of this section is to provide classes of manifolds M satisfying the equality $\varphi_{S^1}(M) = \varphi(M)$. Section 4 is devoted to the study of some similar aspects for the circular Morse–Smale characteristic $\gamma_{S^1}(M)$ introduced by the first two authors in [6] as the φ-category of the pair (M, S^1) corresponding to the family $\mathfrak{F}(M, S^1)$ of all circular Morse functions defined on M. Taking into account the inequality $\gamma_{S^1}(M) \leq \gamma(M)$, labeled by (9), in this section we point out some classes of manifolds M with the property $\gamma_{S^1}(M) = \gamma(M)$. These results are obtained assuming some hypotheses on the topology of M related to the lifting condition

$$\text{Hom}(\pi(M), \mathbb{Z}) = 0 \tag{1}$$

of circular functions to real valued functions via the universal covering $\exp : \mathbb{R} \to S^1$, $\exp(x) = e^{ix}$. The computation of $\gamma_{S^1}(M)$ for manifolds which are not subject to such a requirement seems to be quite a challenging problem. The closed connected surfaces are examples of manifolds in this respect and their circular Morse–Smale category is computed in Sect. 5. Indeed we prove there that

$$\gamma_{S^1}(S) + \chi(S) = 0, \tag{2}$$

for every closed connected surface S, except for the unit sphere S^2 and the projective plane $\mathbb{RP}^2$ which satisfy the lifting property (1) anyway. We present two different proofs for the formula (2), based on our papers [10] and [11]. Section 6 is dealing with the estimation of the circular Morse–Smale characteristic of a connected sum of manifolds. In Theorem 2 we obtain an upper bound in terms of a new φ-category of the pair (X, S^1), denoted by $\gamma_{S^1}^{0,n}(X)$, corresponding to the family $\mathfrak{F}_{0,n}(X, S^1)$ of all smooth Morse functions defined on the n-dimensional manifold X, having both a critical point of index 0 and a critical point of index n. While the computation of $\gamma_{S^1}^{0,n}(X)$ might be a difficult problem, we only observe here that it is equal to $\gamma_{S^1}(X)$ when the lifting property $\mathrm{Hom}(\pi(X), \mathbb{Z}) = 0$ holds. This observation allows us to provide some upper bounds for $\gamma_{S^1}^{0,n}$ of some connected sums.

The minimum number of critical points for circular Morse functions on the closed orientable surface Σ is realized by the restriction of the function

$$f : \mathbb{R}^3 \setminus Oz \to \mathbb{R},\ f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2}}(x, y, 0)$$

to a certain embedding of Σ in $\mathbb{R}^3 \setminus Oz$. The critical points of this restriction are the characteristic points of the embedded copy of Σ with respect to the tangent distribution of the foliation through the fibers of f, which is obviously involutive. The last section provides a lower and an upper bound for the minimum characteristic number of a surface $S \subset \mathbb{R}^3$ with respect to the noninvolutive horizontal distribution of the first Heisenberg group $\mathbb{H}^1$.

2 The Real φ-Category of a Manifold

Consider the particular case when the target manifold N is the real line $\mathbb{R}$ and $\mathscr{F}$ is the family $\mathscr{F}(M) = C^\infty(M, \mathbb{R})$ of all smooth real functions on M. In this situation $\varphi_{\mathscr{F}}(M, \mathbb{R})$ represents the φ-category (or the real functional category) of M and it is denoted by $\varphi(M)$. More precisely, $\varphi(M)$ is defined by

$$\varphi(M) = \min\{\mu(f) : f \in \mathscr{C}^\infty(M)\}, \tag{3}$$

where $\mu(f)$ denotes the number of the critical points of f. This number was intensively studied by Takens [53] for the class of closed manifolds, i.e. compact and without boundary. There are other notations in the literature for $\varphi(M)$ such as $F.(M)$ in [53] and $\mathrm{Crit}(M)$ in the monograph [19], where it is called the *criticality* of M. In this case the following general inequalities, proved by Takens [53], hold

$$\mathrm{cat}(M) \le \varphi(M) \le \dim(M) + 1, \tag{4}$$

where $\mathrm{cat}(M)$ is the Lusternik–Schnirelmann category of M, i.e. the smallest number of open contractible subsets of M which are needed to cover M. The right-hand side inequality shows that $\varphi(M)$ is finite. If M is a closed manifold, then $\mathrm{cat}(M) \geq 2$ since M is not contractible in this case and we may complete the inequalities (4) to

$$2 \leq \mathrm{cat}(M) \leq \varphi(M) \leq \dim(M) + 1, \tag{5}$$

whenever M is closed.

Remark 1. If M, N are smooth manifolds, then

$$\varphi(M \times N) \leq \min\{\dim(M) + \dim(N) + 1, \varphi(M)\varphi(N)\}.$$

Indeed, the inequality $\varphi(M \times N) \leq \dim(M) + \dim(N) + 1$ follows from (4). The inequality $\varphi(M \times N) \leq \varphi(M)\varphi(N)$ follows by taking into account that

$$C(f \oplus g) = C(f) \times C(g), f \in \mathscr{C}^\infty(M), g \in \mathscr{C}^\infty(N),$$

where $(f \oplus g)(x, y) = f(x) + g(y)$ for all $x \in M$ and $y \in N$ [2, p. 131].

3 The Circular φ-Category of a Manifold

The systematic study of the smooth circular functions defined on a manifold was initiated by Pitcher [51, 52], in order to extend in this context the Morse theory for real-valued functions.

The *circular φ-category* (or the circular functional category) of the manifold M is defined by

$$\varphi_{S^1}(M) = \min\{\mu(f) : f \in \mathscr{C}^\infty(M, S^1)\}, \tag{6}$$

where S^1 is the unit circle. Clearly, it is the $\varphi_{\mathscr{F}}$-category of the pair (M, S^1) with respect to the family $\mathscr{F} = C^\infty(M, S^1)$.

Notice that we have the inequality $\varphi_{S^1}(M) \leq \varphi(M)$. Indeed, considering a function $f \in \mathscr{C}^\infty(M)$ with $\mu(f) = \varphi(M)$, then the function $\tilde{f} = \exp \circ f$, where $\exp : \mathbb{R} \to S^1$ is the universal covering of the circle S^1, satisfies $C(\tilde{f}) = C(f)$. Therefore, we obtain $\varphi_{S^1}(M) \leq \mu(\tilde{f}) = \mu(f) = \varphi(M)$ and the property follows. Combining this inequality with the right inequality of (4), it follows

$$\varphi_{S^1}(M) \leq \varphi(M) \leq m + 1. \tag{7}$$

The main purpose of this section is to point out some classes of closed manifolds for which the equality $\varphi_{S^1}(M) = \varphi(M)$ holds.

3.1 The Coverings of Circular Functions and the Lifting Property

In this section we relay on the lifting properties of the covering maps to obtain information on the size of critical sets of circular functions. The properties of covering maps $p : E \longrightarrow M$, we have in mind, are:

1. The group homomorphism $p_* : \pi(E) \longrightarrow \pi(M)$, induced by the projection p at the level of fundamental groups, is one-to-one.
2. The cardinality of the inverse images $p^{-1}(y)$ is independent of $y \in M$ whenever E is connected and it is equal to the index $[\pi(M) : \mathrm{Im}(p_*)]$.
3. For every subgroup H of the fundamental group $\pi(M)$ of M, there exists a covering map $q : E_H \to M$ such that $q_*(\pi(E_H)) = H$.
4. A necessary and sufficient condition for a continuous map $f : X \to M$ to be lifted to a map $\bar{f} : X \longrightarrow E$ is the inclusion $f_*(\pi(X)) \subseteq \bar{f}_*(\pi(X))$. In other words, there is a map $\bar{f} : X \longrightarrow E$ such that $p \circ \bar{f} = f$ if and only if the relation $f_*(\pi(X)) \subseteq \bar{f}_*(\pi(X))$ holds. Note that one of the two implications is obvious.

These properties were intensively used in the previous papers [45, 49] to point out maps with large critical sets and large sets of critical values. Recall that the circular functions on a compact manifold whose fundamental group is a torsion group are rather real valued functions as they all can be lifted to the real line through the exponential covering map $\exp : \mathbb{R} \longrightarrow S^1$, due to the lifting property $\mathrm{Hom}(\pi(M), \mathbb{Z}) = 0$. More precisely we have:

Remark 2. Let M be a connected smooth manifold. If $\mathrm{Hom}\,(\pi(M),\mathbb{Z}) = 0$, then every circular map $f : M \longrightarrow S^1$ can be lifted to a map $\tilde{f} : M \longrightarrow \mathbb{R}$ through the exponential covering map $\exp : \mathbb{R} \longrightarrow S^1$. Indeed, since $f_* = 0$ and $\exp_* = 0$ the existence of a lifting map $\tilde{f} : M \longrightarrow \mathbb{R}$ which factors as $f = \exp \circ \tilde{f}$ follows from property (4) in the above list. The manifolds M with torsion fundamental group is a class of topological spaces satisfying the lifting property $\mathrm{Hom}\,(\pi(M), \mathbb{Z}) = 0$. However, there are manifolds whose fundamental groups are not torsion groups and yet the lifting property $\mathrm{Hom}\,(\pi(M), \mathbb{Z}) = 0$ is satisfied. Such examples are provided below by Corollary 2.

Besides the manifolds having torsion fundamental groups, the group homomorphisms of connected sums of such manifolds are still trivial whenever the terms of connected sums, alongside the connected sums themselves, have dimension three or higher.

Proposition 1. *If* $(G_1, \cdot), \ldots, (G_r, \cdot), (H, \cdot)$ *are groups and*

$$f : G_1 * \cdots * G_r \longrightarrow H$$

is a given group homomorphism, then

$$\operatorname{Im}(f) \subseteq \langle \operatorname{Im}(f \circ i_1) \cup \cdots \cup \operatorname{Im}(f \circ i_r) \rangle,$$

where $i_k : G_k \longrightarrow G_1 * \cdots * G_r$, $k = 1, \ldots, r$ *are the natural embeddings. In particular* $\operatorname{Hom}(G_1 * \cdots * G_r, H) = 0$ *whenever* $G_1, \ldots, G_r$ *are torsion groups and* H *is torsion free.*

Proof. Every element of the free product $G_1 * \cdots * G_r$ has the form $i_{k_1}(g_1) \cdots i_{k_s}(g_s)$, where $g_\alpha \in G_{k_\alpha}$ for $\alpha = 1, \ldots, s$ and $i_{k_\beta} \neq i_{k_{\beta+1}}$ for $\beta = 1, \ldots, s-1$. Therefore $f(i_{k_1}(g_1) \cdots i_{k_s}(g_s)) = f(i_{k_1}(g_1)) \cdots f(i_{k_s}(g_s)) = f(i_{k_1}(g_1)) \cdots f(i_{k_s}(g_s)) = (f \circ i_{k_1})(g_1) \cdots (f \circ i_{k_s})(g_s)) \in \langle \operatorname{Im}(f \circ i_1) \cup \cdots \cup \operatorname{Im}(f \circ i_r) \rangle$. If $G_1, \ldots, G_r$ are torsion groups and H is torsion free, then obviously $\operatorname{Hom}(G_j, H) = 0$ for all $j = 1, \ldots, r$, which shows that $\operatorname{Im}(f \circ i_j) = 0$ for every $j = 1, \ldots, r$. □

Corollary 1. *If* $(G_1, \cdot), \ldots, (G_r, \cdot)$ *are groups and* $f : G_1 * \cdots * G_r \to \mathbb{Z}$ *is a given group homomorphism, then* $\operatorname{Im}(f) = \gcd(m_{i_1}, \ldots, m_{i_s})\mathbb{Z}$, *where* $m_{i_1}, \ldots, m_{i_s}$ *are generators of the nontrivial groups* $\operatorname{Im}(f \circ i_1), \ldots, \operatorname{Im}(f \circ i_s)$, *i.e.* $\operatorname{Im}(f \circ i_j) = m_j\mathbb{Z}$, *for* $j = 1, \ldots, r$ *and* $m_{i_1}, \ldots, m_{i_s} \neq 0$. *If* $m_1 = \cdots = m_r = 0$, *i.e.* $f \circ i_1 = \cdots f \circ i_r = 0$, *then* $\operatorname{Im}(f) = 0$. *In particular* $\operatorname{Hom}(G_1 * \cdots * G_r, \mathbb{Z}) = 0$ *whenever* $G_1, \ldots, G_r$ *are torsion groups.*

Proof. According to Proposition 1 we have,

$$\begin{aligned} \operatorname{Im}(f) &\subseteq \langle \operatorname{Im}(f \circ i_1) \cup \cdots \cup \operatorname{Im}(f \circ i_r) \rangle \\ &= m_1\mathbb{Z} + \cdots + m_r\mathbb{Z} \\ &= m_{i_1}\mathbb{Z} + \cdots + m_{i_s}\mathbb{Z} = \gcd(m_{i_1}, \ldots, m_{i_s})\mathbb{Z}. \end{aligned}$$

□

Corollary 2. *If* $M_1^n, \ldots, M_r^n, n \geq 3$, *are connected n-dimensional manifolds such that* $\pi(M_1), \ldots, \pi(M_r)$ *are torsion groups, then the connected sum* $M_1 \# \cdots \# M_r$ *satisfies the lifting property, i.e.* $\operatorname{Hom}(\pi(M_1 \# \cdots \# M_r), \mathbb{Z}) = 0$. *In particular, if each of* $M_1^n, \ldots, M_r^n, n \geq 3$ *is either a real projective space or a lens space, then* $M_1 \# \cdots \# M_r$ *satisfies the lifting property.*

Proof. The first statement follows from Corollary 1 taking into account the isomorphism $\pi(M_1 \# \cdots \# M_r) \cong \pi(M_1) * \cdots * \pi(M_r)$. For the second statement we just need to observe that the fundamental groups $\pi(M_1), \ldots, \pi(M_r)$ are torsion groups as they are actually finite. □

Remark 3. The requirement $n \geq 3$ in Corollary 1 is essential. Indeed, the fundamental group of the compact non-orientable surface $P_g = \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$ of non-orientable genus $g \geq 3$ (i.e. the connected sum of g copies of projective planes) admit circular functions inducing nontrivial group of homomorphisms at the level of fundamental groups, although the terms of the connected sum have torsion fundamental groups. Such examples will appear as circular Morse functions, both on the closed orientable surface $\Sigma_g := T^2\#\cdots\#T^2$ of genus $g \geq 2$ and on the closed non-orientable surface P_g of non-orientable genus $g \geq 2$, with all critical points of index one. This phenomenon is not possible for real valued Morse functions.

Proposition 2. *Let M be a compact smooth manifold with Abelian fundamental group. Every continuous circular function $f : M \longrightarrow S^1$ which cannot be lifted to any real valued function via the exponential covering* $\exp : \mathbb{R} \to S^1$ *can be covered by a circular function $\bar{f} : \bar{M} \longrightarrow S^1$ such that $\pi(\bar{M})$ is torsion free and the induced group homomorphism $\bar{f}_* : \pi(\bar{M}) \longrightarrow \pi(S^1) \cong \mathbb{Z}$ is onto. More precisely, there are some covering maps $\bar{p} : \bar{M} \longrightarrow M$ and $q : S^1 \longrightarrow S^1$ which, besides the already mentioned properties, make commutative the following diagrams*

$$\begin{array}{ccc} \bar{M} & \xrightarrow{\bar{f}} & S^1 \\ \bar{p}\downarrow & & \downarrow q \\ M & \xrightarrow{f} & S^1 \end{array} \qquad \begin{array}{ccc} \pi(\bar{M}) & \xrightarrow{\bar{f}_*} & \pi(S^1) \cong \mathbb{Z} \\ \bar{p}_*\downarrow & & \downarrow q_* \\ \pi(M) & \xrightarrow{f_*} & \pi(S^1) \cong \mathbb{Z}. \end{array}$$

Proof. Since $\pi(M)$ is Abelian, it follows that $\pi(M)$ is isomorphic to $H_1(M,\mathbb{Z})$, as we generally have $H_1(M,\mathbb{Z}) \cong \pi(M)\big/[\pi(M),\pi(M)]$. On the other hand the homology groups of a compact manifold are all finitely generated and their torsion parts are direct summands therefore. In particular $\pi(M) \cong t(\pi(M)) \oplus K$ for some torsion-free subgroup K of $\pi(M)$, where $t(\pi(M))$ stands for the torsion subgroup of $\pi(M)$. Let us consider a covering map $\bar{p} : \bar{M} \longrightarrow M$ with the property that $\bar{p}_*\left(\pi(\bar{M})\right) = K$. Note that $\pi(\bar{M})$ is torsion free and $\mathrm{Im}(f_*) = \mathrm{Im}(f_* \circ \bar{p}_*)$. Indeed, we have $f_*(\pi(M)) = f_*(\pi(M) \oplus K) = f_*(\pi(M)) + f_*(K) = 0 + f_*\left(\bar{p}_*\left(\pi(\bar{M})\right)\right)$. Let n be the nonnegative integer such that $\mathrm{Im}(f_*) = \mathrm{Im}(f_* \circ \bar{p}_*) = n\mathbb{Z}$. Since f cannot be lifted to any real valued function via the exponential covering $\exp : \mathbb{R} \longrightarrow S^1$, it follows that $n \neq 0$. Recall that the function $q : S^1 \longrightarrow S^1$, $q(z) = z^n$ is an n-sheeted covering function and $q_*(\mathbb{Z}) = n\mathbb{Z} = \mathrm{Im}(f_*) = \mathrm{Im}(\bar{f} \circ p)_*$. This shows that $f \circ p$ can be lifted to a map $\bar{f} : \bar{M} \longrightarrow S^1$, i.e. $q \circ \bar{f} = f \circ \bar{p}$.

We next assume that $\bar{f}_*(\pi(M)) = m\mathbb{Z}$. Thus, we obtain successively

$$\begin{aligned} n\mathbb{Z} &= (f_* \circ \bar{p}_*)(\pi(M)) = (q_* \circ \bar{f}_*)(\pi(M)) \\ &= q_*\left(\bar{f}_*(\pi(M))\right) = q_*(m\mathbb{Z}) = mq_*(\mathbb{Z}) = mn\mathbb{Z}. \end{aligned}$$

Therefore $n = nm$, i.e. $m = 1$, which shows that $\bar{f}_*(\pi(M)) = \mathbb{Z}$. □

3.2 Manifolds Satisfying $\varphi_{S^1}(M) = \varphi(M)$

Let us first observe that the inequality $\varphi_{S^1}(M) \leq \varphi(M)$ ensured by (7) can be strict. Indeed, the m-dimensional torus $T^m = S^1 \times \cdots \times S^1$ (m times) has, according to [2, Example 3.6.16], the φ-category $\varphi(T^m) = m + 1$. On the other hand, every projection $T^m \to S^1$ is a trivial differentiable fibration, hence it has no critical points, implying $\varphi_{S^1}(T^m) = 0$. This example is part of the following more general observation.

Remark 4. For a closed manifold M we have $\varphi_{S^1}(M) = 0$ if and only if there is a differentiable fibration $M \to S^1$. Indeed, the existence of a differentiable fibration $M \to S^1$ ensures the equality $\varphi_{S^1}(M) = 0$, as the fibration itself has no critical points at all. Conversely, the equality $\varphi_{S^1}(M) = 0$ ensures the existence of a submersion $M \to S^1$, which is also proper, as its inverse images of the compact sets in S^1 are obviously compact. Thus, by the Ehresmann's fibration theorem (see [22] for the original reference, or [20, p. 15]) one can conclude that our submersion is actually a locally trivial fibration.

Proposition 3. *Let M be a connected smooth manifold. If M satisfies the lifting property* $\mathrm{Hom}\,(\pi(M), \mathbb{Z}) = 0$, *then $\varphi_{S^1}(M) = \varphi(M)$. In particular $\varphi_{S^1}(M) = \varphi(M)$ whenever the fundamental group of M is a torsion group.*

Proof. Indeed, in this case every smooth circle valued function $f : M \longrightarrow S^1$ can be lifted to a smooth real valued function $\tilde{f} : M \longrightarrow \mathbb{R}$, i.e. $\exp \circ \tilde{f} = f$. Since the universal cover $\exp : \mathbb{R} \longrightarrow S^1$ is a local diffeomorphism, it follows that $C(f) = C(\tilde{f}) \geq \varphi(M)$ for every smooth function $f : M \longrightarrow S^1$. This shows that the inequality $\varphi_{S^1}(M) \geq \varphi(M)$ holds, which combined to the general inequality (1.4), leads to the desired relation. □

Corollary 3. *If $n \geq 2$ is an integer, then $\varphi_{S^1}(S^n) = \varphi(S^n) = 2$ and $\varphi_{S^1}(\mathbb{RP}^n) = \varphi(\mathbb{RP}^n) = n + 1$.*

Proof. While the equalities $\varphi_{S^1}(S^n) = \varphi(S^n)$, $\varphi_{S^1}(\mathbb{RP}^n) = \varphi(\mathbb{RP}^n)$ follow from Proposition 3, the equality $\varphi(S^n) = 2$ is obvious. On the other hand, we have

$$\varphi(\mathbb{RP}^n) \leq \mu(f) = \mathrm{card}(C(f)) = n + 1,$$

where $f : \mathbb{RP}^n \longrightarrow \mathbb{R}$ is defined by

$$f_n([x_1, \ldots, x_{n+1}]) = \frac{x_1^2 + 2x_2^2 + \cdots + nx_n^2 + (n+1)x_{n+1}^2}{x_1^2 + x_2^2 + \cdots + x_n^2 + x_{n+1}^2}$$

whose critical set consists of the $n + 1$ critical points

$$[1, 0, \ldots, 0], [0, 1, \ldots, 0], \ldots, [0, 0, \ldots, 1] \in \mathbb{RP}^n.$$

Note that f is a Morse function and its critical points have indices $0, 1, \ldots, n$, respectively, (see, for instance, [35, p. 84, 85]). Finally, we use the inequality

$$\varphi(\mathbb{RP}^n) \geq \mathrm{cat}(\mathbb{RP}^n) = n + 1$$

where $\mathrm{cat}(\mathbb{RP}^n)$ stands for the Lusternik–Schnirelmann category of the projective space $\mathbb{RP}^n$ [41, pp. 190–192]. □

Corollary 4. *If $k, l, m_1, \ldots, m_k \geq 2$ are integers, then the following relations hold:*

1. $\varphi_{S^1}(S^{m_1} \times \cdots \times S^{m_k}) = \varphi(S^{m_1} \times \cdots \times S^{m_k}) = k + 1$.
2. $\varphi_{S^1}(\mathbb{RP}^{m_1} \times \cdots \times \mathbb{RP}^{m_k}) = \varphi(\mathbb{RP}^{m_1} \times \cdots \times \mathbb{RP}^{m_k}) \leq m_1 + m_2 + \cdots + m_k + 1$.
3. $\varphi_{S^1}(L(7,1) \times S^4) = \varphi(L(7,1) \times S^4) = \varphi_{S^1}(L(7,1) \times S^4) = \varphi(L(7,1) \times S^4) = 5$, *where $L(r,s)$ is the 3-dimensional lens space of type (r,s).*
4. $\varphi_{S^1}(\mathbb{RP}^k \times S^l) = \varphi(\mathbb{RP}^k \times S^l) \leq k + 2$.

Proof. For the equalities

$$\begin{aligned}
&\varphi_{S^1}(S^{m_1} \times \cdots \times S^{m_k}) = \varphi(S^{m_1} \times \cdots \times S^{m_k})\\
&\varphi_{S^1}(\mathbb{RP}^{m_1} \times \cdots \times \mathbb{RP}^{m_k}) = \varphi(\mathbb{RP}^{m_1} \times \cdots \times \mathbb{RP}^{m_k})\\
&\varphi_{S^1}(L(7,1) \times S^4) = \varphi(L(7,1), \varphi_{S^1}(L(7,1) \times S^4) = \varphi(L(7,1) \times S^4)\\
&\varphi_{S^1}(\mathbb{RP}^k \times S^l) = \varphi(\mathbb{RP}^k \times S^l).
\end{aligned}$$

we merely need to use Proposition 3. The proofs of the equalities

$$\begin{aligned}
&\varphi(S^{m_1} \times \cdots \times S^{m_k}) = k + 1\\
&\varphi(L(7,1) \times S^4) = \varphi(L(7,1) \times S^4) = 5
\end{aligned}$$

have been done by Gavrilă [27, Proposition 4.6, Example 4.7]. The inequality $\varphi(\mathbb{RP}^{m_1} \times \cdots \times \mathbb{RP}^{m_k}) \leq m_1 + m_2 + \cdots + m_k + 1$ follows from (4) and the estimate $\varphi(\mathbb{RP}^k \times S^l) \leq k + 2$ relies on [27, Proposition 4.19]. □

Combining Corollary 2 with Proposition 3 we obtain:

Corollary 5. *If $M_1^n, \ldots, M_r^n, n \geq 3$ are connected manifolds with torsion fundamental groups, then $\varphi_{S^1}(M_1 \# \cdots \# M_r) = \varphi(M_1 \# \cdots \# M_r)$. In particular the following equality $\varphi_{S^1}(r\mathbb{RP}^n) = \varphi(r\mathbb{RP}^n)$ holds, where $r\mathbb{RP}^n$ stands for the connected sum $\mathbb{RP}^n \# \cdots \# \mathbb{RP}^n$ of r copies of $\mathbb{RP}^n$.*

The following result is mentioned in the monograph [19, p. 221].

Lemma 1. *If M and N are closed manifolds, then the following inequality holds $\varphi(M \# N) \leq \max\{\varphi(M), \varphi(N)\}$. In particular $\varphi(X \# X) \leq \varphi(X)$ for every closed manifold X.*

Recall that P_g denotes the closed connected non-orientable surface $\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$ of non-orientable genus g and Σ_g stands for the closed connected orientable surface $T^2 \# \cdots \# T^2$ of genus g.

Corollary 6. *The following relations hold:*

1. $\varphi(\Sigma_g) = \varphi(P_g) = 3$, $g \geq 1$;
2. $2 \leq \varphi(r\mathbb{RP}^n) = \varphi_{S^1}(r\mathbb{RP}^n) \leq n + 1$, $r \geq 1$, $n \geq 3$.

Proof. (1) Let us first observe that

$$\varphi(\Sigma_g) \geq \mathrm{cat}(\Sigma_g) = 3 \text{ and } \varphi(P_g) \geq \mathrm{cat}(P_g) = 3.$$

According to Lemma 1 we obtain inductively the inequalities $\varphi(\Sigma_g) \leq \varphi(T^2)$ and $\varphi(P_g) \leq \varphi(\mathbb{RP}^2) = 3$. Thus the equality $\varphi(P_g) = 3$ is now completely proved.

In order to construct a real valued function f with three critical points on the torus T^2 we follow the idea of [51]. Consider the torus by a rectangle with matched opposite edges in which one diagonal has been drawn. Then f can be defined so that it is 0 on the edges and the diagonal and nowhere else, is positive and has an absolute nondegenerate maximum interior to one triangle, is negative and has a proper nondegenerate minimum interior to the other triangle, has a monkey saddle at the point represented by the four vertices, and no other critical points. Then, we get $\varphi(T^2) \leq 3$. On the other hand we have $3 = \mathrm{cat}(T^2) \leq \varphi(T^2)$, implying $\varphi(T^2) = 3$, and we are done.

(2) The equality $\varphi(r\mathbb{RP}^n) = \varphi_{S^1}(r\mathbb{RP}^n)$ is assured by Corollary 5 and the inequality $\varphi(r\mathbb{RP}^n) \leq \varphi(\mathbb{RP}^n) = n + 1$ by Corollary 3 and Lemma 1. □

Corollary 7. *If $k, l \geq 2$ are integers, then*

$$\varphi_{S^1}\left((S^k \times S^l)\#\cdots\#(S^k \times S^l)\right) = \varphi\left((S^k \times S^l)\#\cdots\#(S^k \times S^l)\right) = 3.$$

Proof. For the equality

$$\varphi_{S^1}\left((S^k \times S^l)\#\cdots\#(S^k \times S^l)\right) = \varphi\left((S^k \times S^l)\#\cdots\#(S^k \times S^l)\right)$$

we merely need to use Corollary 5. On the other hand, according to Lemma 1 and Corollary 4(1) it follows that $\varphi\left((S^k \times S^l)\#\cdots\#(S^k \times S^l)\right) \leq \varphi(S^k \times S^l) = 3$. Finally, we have $\varphi\left((S^k \times S^l)\#\cdots\#(S^k \times S^l)\right) \geq \mathrm{cat}\left((S^k \times S^l)\#\cdots\#(S^k \times S^l)\right) = 3$, as follows from the inequality (4) combined to [21, Theorem 5.9]. □

Let us finally mention that we do not have any example of a closed manifold M such that $\mathrm{cat}(M) < \varphi(M)$ and also the equality $\mathrm{cat}(M) = \varphi(M)$ is proved only for some isolated classes of manifolds, one example in this respect is the connected sum $(S^k \times S^l)\#\cdots\#(S^k \times S^l)$, $(k, l \geq 2)$ justified by Corollary 7. In order to understand the difficulty of the problem, assume that the equality $\mathrm{cat}(M) = \varphi(M)$ holds for every closed manifold. Let us only look to the following particular situation: $\mathrm{cat}(M) = \varphi(M) = 2$. From $\mathrm{cat}(M) = 2$ one obtains that M is a homotopy sphere. Taking into account the well-known Reeb's result, from the equality $\varphi(M) = 2$ it follows that M is a topological sphere. Therefore, the equalities $\mathrm{cat}(M) = \varphi(M) = 2$ are related to the Poincaré conjecture. Taking into account the validity of the Poincaré conjecture, proved by Perelmann [42–44], it follows for instance that for any closed manifold with $\mathrm{cat}(M) = 2$ we have $\varphi(M) = 2$ and therefore $\mathrm{cat}(M) = \varphi(M) = 2$.

Taking into account these comments, we consider that the following Reeb type problem for circular functions is interesting.

Open problem. *Characterize the closed manifolds M^m with the property $\varphi_{S^1}(M) = 1$.*

When $m = 2$, one example of such a manifold, suggested to us by L. Funar, is given by the closed orientable surface Σ_g of genus $g \geq 2$, i.e. we have $\varphi_{S^1}(\Sigma_g) = 1$. Indeed, we may construct a function with one critical point from Σ_g to S^1 by composing the projection $p : T^2 = S^1 \times S^1 \to S^1$, $p(x, y) = x$, with a map $f : \Sigma_g \to T^2$ having precisely one critical point. The existence of the map f is assured by [3] (see also [4]) as $\varphi(\Sigma_g, T^2) = 1$, and the projection p is a fibration, i.e. the critical set $C(p)$ is empty. Therefore, the composed function $p \circ f$ has at most one critical point as $C(p \circ f) \subseteq C(f)$ and $\mathrm{card}(C(f)) = 1$. This shows that $\varphi_{S^1}(\Sigma_g) \leq 1$. For the opposite inequality, assume that $\varphi_{S^1}(\Sigma_g) = 0$ and consider a fibration $g : \Sigma_g \to S^1$, whose fiber F is a compact one-dimensional manifold without boundary, i.e. a circle or a disjoint union of circles. By applying the product property of the Euler–Poincaré characteristic associated to the fibration $F \hookrightarrow \Sigma_g \xrightarrow{g} S^1$, one obtains $2 - 2g = \chi(\Sigma_g) = \chi(F)\chi(S^1) = 0$ as $\chi(S^1) = 0$, a contradiction with the initial assumption $g \geq 2$.

4 The Circular Morse–Smale Characteristic

The Morse functions on finite dimensional manifolds are important tools for topological investigations on their source spaces, especially when these spaces are compact. Indeed, a homotopical spherical structure of a compact manifold can be recovered from each Morse function on the manifold in discussion [35, 36, 39, 41].

The Morse functionals on infinite dimensional Banach spaces are also of great importance as their critical sets have various important interpretations. For example the critical points of the energy functional defined on the space of piecewise smooth closed curves which lie inside a compact manifold, which is a Morse functional, are the closed geodesics of that manifold [30].

Also, the multiple solutions of the semilinear elliptic boundary problem

$$\begin{cases} -\nabla u = \psi(x, u) & \text{in } \Omega \\ \mathrm{Tr}\, u = 0 & \text{on } \partial\Omega \end{cases}$$

can be approached by using Morse Theory, where $\Omega \subseteq \mathbb{R}^n$ ($n \geq 3$) is a bounded domain and $\mathrm{Tr} : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ is the *trace operator* as defined in [38] (see also [32]). For such applications of the Morse Theory in infinite dimensional context we refer the reader to [14, 15, 28, 54]. For other boundary value problems, studied with completely different tools, we refer the reader to [31].

The minimum number of critical points over all real-valued Morse functions defined on a manifold M, equally called the *Morse–Smale characteristic* of M, is an important tool in differential topology as, for example, it is related to the minimum number of cells in the CW-decompositions of M up to homotopy. It is also a lower bound for the total curvature of M with respect to its embeddings in Euclidean spaces. In fact, the Chern–Lashof conjecture states that the Morse–Smale characteristic of a manifold M is precisely the infimum of the total curvatures of M with respect to its embeddings in Euclidean spaces. Recall that the Chern–Lashof conjecture holds true for several manifolds [16, 17].

The Morse–Smale characteristic of a compact smooth manifold is therefore worth to be studied. More precisely, it is defined by

$$\gamma(M) = \min\{\mu(f) : f \in \mathfrak{F}(M)\},$$

where $\mathfrak{F}(M)$ denotes the set of all real-valued Morse functions defined on M and $\mu(f)$ stands for $\text{card}(C(f))$. For details, examples, properties, and concrete computations of γ we refer the reader to the monograph [2, pp. 106–129]. Clearly, $\gamma(M)$ is the $\varphi_{\mathscr{F}}$-category of the pair $(M, \mathbb{R})$ with respect to the family $\mathscr{F} = \mathfrak{F}(M)$ of real valued Morse functions on M.

The minimality of the number of cells in the CW-decompositions of M up to homotopy emphasizes the importance of $\gamma(M)$ and provides a serious reason why its computation is rather a hard problem in differential topology. Hajduk [29] has proved that it is a simply homotopy invariant of the manifold M^m, in the case $m \geq 6$. On the other hand, the existence of F-perfect Morse functions on M, for some given field F, is characterized by the equality between $\gamma(M)$ and the total F-Betti number of M, i.e. the sum of F-Betti numbers of M with respect to F [2, Theorem 4.2.3]. If (M, ω) is a symplectic manifold, then the latter $\mathbb{Z}_2$-sum associated to a coisotropic submanifold of M is a lower bound for the number of leaf-wise fixed points of suitable Hamiltonian diffeomorphisms and coisotropic submanifolds of M [55].

The formulation of the circle-valued Morse theory as a new branch of differential topology with its own problems was outlined by S. P. Novikov in 1980 (see the monographs [24] and [40]). The motivation came from a problem in hydrodynamics, where the application of the variational approach led to a multi-valued Lagrangian.

The circular version of the Morse–Smale characteristic was introduced by the first two authors in the paper [6]. If M is a manifold, then the *circular Morse–Smale characteristic* of M is defined by

$$\gamma_{S^1}(M) = \min\{\mu(f) : f \in \mathfrak{F}(M, S^1)\}, \tag{8}$$

where $\mathfrak{F}(M, S^1)$ stands for the set of all circular Morse functions $f : M \to S^1$ and $\mu(f)$ for $\text{card}(C(f))$.

Also, the number $\gamma_{S^1}(M)$ is a special case of $\varphi_{\mathscr{F}}$-category of a pair of manifolds (M, N) (see the recent expository paper [8]), where N is the circle S^1 and $\mathscr{F} = \mathfrak{F}(M, S^1)$ is the family of all circle-valued Morse functions $f : M \to S^1$.

Some properties of the circular Morse–Smale characteristic are already proved in the papers [6] and [7]. For instance, for every closed manifold (i.e., compact and without boundary) we have the inequality

$$\gamma_{S^1}(M) \leq \gamma(M). \tag{9}$$

The proof is immediate, since every Morse real valued function composed with the universal cover $\exp : \mathbb{R} \longrightarrow S^1$ produces a circle valued Morse function. This property implies that $\gamma_{S^1}(M)$ is finite whenever M is compact.

Proposition 4 ([10]). *If* $\mathrm{Hom}\,(\pi(M), \mathbb{Z}) = 0$ *for some connected smooth manifold* M*, then* $\gamma_{S^1}(M) = \gamma(M)$*. In particular* $\gamma_{S^1}(M) = \gamma(M)$ *whenever* M *is connected and simply-connected.*

Corollary 8 ([10]). *If* $n \geq 2$ *is an integer, then* $\gamma_{S^1}(S^n) = \gamma(S^n) = 2$ *and* $\gamma_{S^1}(\mathbb{RP}^n) = \gamma(\mathbb{RP}^n) = n + 1$.

Corollary 9 ([10]). *If* $m_1, \ldots, m_k \geq 2$ *are positive integers, then*

$$\gamma_{S^1}(S^{m_1} \times \cdots \times S^{m_k}) = \gamma(S^{m_1} \times \cdots \times S^{m_k}) = 2^k,$$
$$\gamma_{S^1}(\mathbb{RP}^{m_1} \times \cdots \times \mathbb{RP}^{m_k}) = \gamma(\mathbb{RP}^{m_1} \times \cdots \times \mathbb{RP}^{m_k}) = (m_1 + 1) \cdots (m_k + 1).$$

Corollary 10. *If* $M_1^n, \ldots, M_r^n$*,* $n \geq 3$ *are connected manifolds with torsion fundamental groups, then* $\gamma_{S^1}\,(M_1\# \cdots \# M_r) = \gamma\,(M_1\# \cdots \# M_r)$.

Proof. Indeed, according to Corollary 2 we obtain $\mathrm{Hom}(\pi(M_1\# \cdots \# M_r), \mathbb{Z}) = 0$, which combined with Proposition 4 shows the stated relation. □

Example 1. If $n \geq 3$, then $\gamma_{S^1}\,(r\mathbb{RP}^n) = \gamma\,(r\mathbb{RP}^n) = r(n + 1) - 2(r - 1)$, where $r\mathbb{RP}^n$ stands for the connected sum $\mathbb{RP}^n\# \cdots \#\mathbb{RP}^n$ of r copies of the projective space $\mathbb{RP}^n$. Indeed, the first equality $\gamma_{S^1}\,(r\mathbb{RP}^n) = \gamma\,(r\mathbb{RP}^n)$ follows from Corollary 2 and the second equality $\gamma\,(r\mathbb{RP}^n) = r(n+1) - 2(r-1)$ is proved in [2, p. 125].

Remark 5. The hypothesis $n \geq 3$ within Corollary 10 and Example 1 is essential. Indeed, for $r \geq 2$, we have $\gamma_{S^1}\left(r\mathbb{RP}^2\right) = r - 2 \neq r(2 + 1) - 2(r - 1)$. While the first equality $\gamma_{S^1}\left(r\mathbb{RP}^2\right) = r - 2$ will be proved in the next section in a general context, the second relation is obvious.

Another property relating the circular Morse–Smale characteristics of the total and base spaces of a finite-fold covering map is provided by the following result:

Proposition 5 ([10]). *If* $\tilde{M}$ *is a* k*-fold cover of* M*, then* $\gamma_{S^1}(\tilde{M}) \leq k \cdot \gamma_{S^1}(M)$.

The proof in the case of the real Morse–Smale characteristic was given in [1].

Now, following the paper [34], we will use the Morse–Novikov inequalities (see [24] and [40]) to get a lower bound to $\gamma_{S^1}(M)$ in terms of some invariants of manifold M. Recall that the Novikov numbers of the manifold M with respect to the cohomology class $\xi \in H^1(M)$ are $b_i^{\mathrm{Nov}}(M,\xi)$ and $q_i^{\mathrm{Nov}}(M,\xi)$, $i = 0, 1, \ldots, m$, where

$$b_i^{\mathrm{Nov}}(M,\xi) = \dim_{\mathbb{Z}((z))}(H_i^{\mathrm{Nov}}(M,\xi)/T_i^{\mathrm{Nov}}(M,\xi))$$

are the Betti numbers of the Novikov homology and $q_i^{\mathrm{Nov}}(M,\xi)$ is the minimum numbers of generators of $T_i^{\mathrm{Nov}}(M,\xi)$. Here $\mathbb{Z}((z))$ is the Novikov ring and

$$T_i^{\mathrm{Nov}}(M,\xi) = \{x \in H_i^{\mathrm{Nov}}(M,\xi) : ax = 0, a \neq 0 \in \mathbb{Z}((z))\}$$

is the torsion $\mathbb{Z}((z))$-submodule of $H_i^{\mathrm{Nov}}(M,\xi)$. The Morse–Novikov inequalities are

$$c_i(f) \geq b_i^{\mathrm{Nov}}(\xi) + q_i^{\mathrm{Nov}}(\xi) + q_{i-1}^{\mathrm{Nov}}(\xi),\ i = 0, 1, \ldots, m, \tag{10}$$

where $c_i(f) = \mathrm{card}(C_i(f))$ and $C_i(f) := \{p \in C(f) : \mathrm{ind}_p(f) = i\}$. For more details we refer the reader to the monographs [24] and [40]. Note that in the proof of these Morse-Novikov inequalities the theory of h-cobordism [37] is essentially used.

Let $f : M \to S^1$ be a circle-valued Morse function, and let $f^* : H^1(S^1) \to H^1(M)$ be the induced homomorphism in cohomology. Denote

$$F^1(M) = \{f^*(1) : f \in \mathfrak{F}(M, S^1)\} \subseteq H^1(M).$$

Proposition 6. *The following inequality holds:*

$$\gamma_{S^1}(M) \geq \min\{b^{\mathrm{Nov}}(\xi) + q_m^{\mathrm{Nov}}(\xi) + 2\sum_{i=0}^{m-1} q_i^{\mathrm{Nov}}(\xi) : \xi \in F^1(M)\},$$

where

$$b^{\mathrm{Nov}}(\xi) = \sum_{i=0}^{m} b_i^{\mathrm{Nov}}(\xi)$$

is the total Betti number of the manifold M with respect to the cohomology class $\xi \in H^1(M)$.

Proof. Let $f : M \to S^1$ be a circle-valued Morse function. By using the Morse–Novikov inequalities (10) we obtain

$$\begin{aligned}\mu(f) &= \sum_{i=0}^{m} c_i(f) \geq \sum_{i=0}^{m}(b_i^{\mathrm{Nov}}(\xi) + q_i^{\mathrm{Nov}}(\xi) + q_{i-1}^{\mathrm{Nov}}(\xi)) \\ &= b^{\mathrm{Nov}}(\xi) + q_m^{\mathrm{Nov}}(\xi) + 2\sum_{i=0}^{m-1} q_i^{\mathrm{Nov}}(\xi) \\ &\geq \min\{b^{\mathrm{Nov}}(\xi) + q_m^{\mathrm{Nov}}(\xi) + 2\sum_{i=0}^{m-1} q_i^{\mathrm{Nov}}(\xi) : \xi \in F^1(M)\}.\end{aligned}$$

Since $f : M \to S^1$ is an arbitrary circle-valued Morse function, it follows that

$$\begin{aligned}\min\{\mu_{S^1}(f) = \operatorname{card}(C_{S^1}(f)) : f \in \mathfrak{F}(M, S^1)\}\\ \geq \min\left\{b^{\mathrm{Nov}}(\xi) + q_m^{\mathrm{Nov}}(\xi) + 2\sum_{i=0}^{m-1} q_i^{\mathrm{Nov}}(\xi) : \xi \in F^1(M)\right\},\end{aligned}$$

and the proof is complete. □

Proposition 7. *For $\pi(M) \cong \mathbb{Z}$ and $m \geq 6$, the following relation holds:*

$$\gamma_{S^1}(M) = \min\left\{b^{\mathrm{Nov}}(\xi) + q_m^{\mathrm{Nov}}(\xi) + 2\sum_{i=0}^{m-1} q_i^{\mathrm{Nov}}(\xi) : \xi \in F^1(M)\right\}.$$

Proof. Let $f : M \to S^1$ be a circle-valued Morse function with minimal number of critical points. According to the result in [24], we have the following relations

$$c_i(f) = b_i^{\mathrm{Nov}}(\xi) + q_i^{\mathrm{Nov}}(\xi) + q_{i-1}^{\mathrm{Nov}}(\xi), i = 0, 1, \ldots, m$$

hence

$$\begin{aligned}\gamma_{S^1}(M) \leq \mu(f) &= \sum_{i=0}^{m} c_i(f)\\ &= \sum_{i=0}^{m}(b_i^{\mathrm{Nov}}(\xi) + q_i^{\mathrm{Nov}}(\xi) + q_{i-1}^{\mathrm{Nov}}(\xi))\\ &= b(\xi) + q_m^{\mathrm{Nov}}(\xi) + 2\sum_{i=0}^{m-1} q_i^{\mathrm{Nov}}(\xi).\end{aligned}$$

That is

$$\gamma_{S^1}(M) \leq \min\left\{b^{\mathrm{Nov}}(\xi) + q_m^{\mathrm{Nov}}(\xi) + 2\sum_{i=0}^{m-1} q_i^{\mathrm{Nov}}(\xi) : \xi \in F^1(M)\right\}.$$

Taking into account the inequality in Proposition 6 the desired result follows. □

Open problem. *With the above notations, for every closed manifold M the relation $F^1(M) = H^1(M)$ holds, i.e. the map $\mathfrak{F}(M, S^1) \to H^1(M), f \mapsto f^*(1)$ is surjective.*

5 The Circular Morse–Smale Characteristic of Compact Surfaces

The circular Morse–Smale characteristic reduces to the real Morse–Smale characteristic for manifolds M which satisfy the lifting property $\mathrm{Hom}\,(\pi(M), \mathbb{Z}) = 0$. For the opposite situation $\mathrm{Hom}\,(\pi(M), \mathbb{Z}) \neq 0$ the two characteristics might be quite different. For example, this is the case for the closed surfaces. In this section we shall determine the circular Morse–Smale characteristic of any closed surface, orientable or not. Recall that the surface Σ_g is defined to be the connected sum

$$T^2 \# T^2 \# \cdots \# T^2,$$

of g copies of the two-dimensional torus $T^2 = S^1 \times S^1$. We can extend the definition for $g = 0$ by considering $\Sigma_0 = S^2$, the two-dimensional sphere. From the well-known classification theorem of surfaces, it follows that every smooth, compact, orientable, connected surface, without boundary, is diffeomorphic to Σ_g, for some value of $g \geq 0$. Recall that the Morse–Smale characteristic was completely determined by Kuiper [33] who proved the formula $\gamma(S) + \chi(S) = 4$ for every compact connected surface S. In this section we will prove that $\gamma_{S^1}(S) + \chi(S) = 0$ for every closed surface S, except for the sphere S^2 and the projective plane $\mathbb{RP}^2$.

Producing a suitable embedding of the surface Σ_g in $\mathbb{R}^3 \setminus Oz$, where Oz stands for the z-axis $\{(x, 0, 0) \,|\, x \in \mathbb{R}\}$, and a submersion $f : \mathbb{R}^3 \setminus Oz \longrightarrow S^1$, whose restriction $f\big|_{\Sigma_g}$ is a circular Morse function with exactly $2(g-1)$ critical points, is a part of our strategy to compute the circular Morse–Smale characteristic of the surface Σ_g. In this respect we need to characterize somehow the critical points of such a restriction. In fact the suitable submersion we are looking for is

$$f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2}}(x, y, 0). \tag{11}$$

Proposition 8. *Let $\Sigma \subseteq \mathbb{R}^3$ be a regular surface and $f : \mathbb{R}^3 \longrightarrow N$ be a submersion, where N is either the real line or the circle S^1. The point $p = (x_0, y_0, z_0) \in \Sigma$ is critical for the restriction $f|_\Sigma$ if and only if the tangent plane of Σ at p is the tangent plane at p to the fiber $\mathscr{F}_p := f^{-1}(f(p))$ of the submersion (11) through p.*

Proposition 8 follows from the following more general statement:

Proposition 9 ([11]). *Let M^m, N^n, P^p, $m \geq n > p$ be differential manifolds, let $f : M \to N$ be a differential map, and $g : N \to P$ be a submersion. Then $x \in M$ is a regular point of $g \circ f$ if and only if $f \pitchfork_x \mathscr{F}_x$, where $\mathscr{F}_x$ is the fiber $g^{-1}(g(x))$ of g through x.*

Theorem 1. *The circular Morse–Smale characteristic of a closed surface* $\Sigma \neq \mathbb{RP}^2$ *is*

$$\gamma_{S^1}(\Sigma) = |\chi(\Sigma)|. \tag{12}$$

The proof of Theorem 1 for $\Sigma = S^2$ or T^2 is immediate. Indeed $\gamma_{S^1}(\Sigma_0) = \gamma_{S^1}(S^2) = \gamma(S^2) = 2$, since the two-dimensional sphere S^2 is simply-connected. Also $\gamma_{S^1}(\Sigma_1) = \gamma_{S^1}(T^2) = 0$, as the projection $T^2 = S^1 \times S^1 \to S^1$ is a submersion.

Our previous works [10] and [11] provide two different proofs for Theorem 1.

The Proof of Theorem 1 that we have provided in [10] starts with the inequality $\gamma_{S^1}(\Sigma) \geq |\chi(\Sigma)|$ for surfaces Σ with the property $b_1(\Sigma) \geq 2$ and it is based on standard Morse Theory arguments for real valued Morse functions, which involve the Morse inequalities. Indeed, we considered the restriction of an arbitrary circular Morse function $f : \Sigma \to S^1$ to the complement $\Sigma \setminus f^{-1}\big(\widehat{e^{i(t_0-\varepsilon)}e^{i(t_0+\varepsilon)}}\big)$, where $\mathrm{cl}\big(\widehat{e^{i(t_0-\varepsilon)}e^{i(t_0+\varepsilon)}}\big) \subset S^1$ is a closed arc of regular values of f. We may obviously consider $t_0 = 0$. In this situation the distribution of their indexes remains unchanged if we compose f from the left side with any rotation of the unity circle S^1. Obviously the restriction

$$\Sigma_\varepsilon := \Sigma \setminus f^{-1}\big(\widehat{e^{-i\varepsilon}e^{i\varepsilon}}\big) \to S^1 \setminus \widehat{e^{-i\varepsilon}e^{i\varepsilon}}, \;\; p \mapsto f(p) \tag{13}$$

can be treated as a real valued Morse function, since the complement of $\widehat{e^{-i\varepsilon}e^{i\varepsilon}}$ in S^1 is diffeomorphic to an interval of the real axis. Also, the critical set of f is entirely inherited by the restriction (13), which shows that f and its restriction $f|_{\Sigma_\varepsilon}$ have the same number of critical points. On the other hand, the inverse image $f^{-1}\big(\mathrm{cl}\big(\widehat{e^{-i\varepsilon}e^{i\varepsilon}}\big)\big)$ is diffeomorphic to $\Sigma_1 \times [-\varepsilon, \varepsilon]$, where Σ_1 stands for $f^{-1}(1)$, hence the restriction

$$f^{-1}\big(\mathrm{cl}\big(\widehat{e^{-i\varepsilon}e^{i\varepsilon}}\big)\big) \to \widehat{e^{-i\varepsilon}e^{i\varepsilon}}, \;\; p \mapsto f(p) \tag{14}$$

is a fibration whose critical set is empty. We only need to work with circular Morse functions $f : \Sigma \to S^1$ which cannot be lifted to any real valued Morse function via the exponential function $\exp : \mathbb{R} \to S^1$, $\exp(x) = e^{ix}$ (which is the universal cover of S^1). Indeed, the number of critical points of those circular Morse functions which can be lifted to real-valued Morse functions is either the Morse–Smale characteristic $\gamma(\Sigma)$ or exceeds it and $\gamma(\Sigma) = 4 - \chi(\Sigma) \geq |\chi(\Sigma)|$.

The number of critical points of the restriction (13) is evaluated by means of the Morse inequalities, which involve the Betti numbers, of the pair $(\Sigma_\varepsilon, \partial_+\Sigma_\varepsilon)$, where $\partial_+\Sigma_\varepsilon$ stands for the component of the boundary $\partial\Sigma_\varepsilon \simeq \Sigma_1 \times \{\pm\varepsilon\}$ of the manifold Σ_ε which corresponds to ε via the diffeomorphism

$$\partial\Sigma_\varepsilon \to \Sigma_1 \times \{\pm\varepsilon\}, \;\; q \to f(q).$$

The opposite inequality $\gamma_{S^1}(\Sigma) \geq |\chi(\Sigma)|$, for surfaces Σ with the property $b_1(\Sigma) \geq 2$, is proved by using the classical description of Σ in terms of a polygon with a suitable even number of edges and suitable pairwise identifications of its edges. The associated handle decomposition to this description corresponds to a Morse function on the surface with one critical point for each handle. In our case there is a Morse function on Σ with $\gamma(\Sigma) = b_1(\Sigma) + 2$ critical points. Due to the condition $b_1(\Sigma) \geq 2$, we can also find a non-separating circle in Σ representing a primitive homology class in $H_1(\Sigma, \mathbb{Z})$ along which we cut the surface Σ to get a new surface Σ' with two circular boundary components $C, -C$ and $\chi(\Sigma) = \chi(\Sigma')$. Cap the components C and $-C$ of the boundary of Σ' with two disks $D_\pm$ to obtain a closed surface S satisfying the relation

$$\chi(S) = \chi(\Sigma') + 2 = \chi(\Sigma) + 2.$$

In particular, we have

$$b_1(S) = b_1(\Sigma) - 2.$$

We now choose a Morse function $h : S \longrightarrow \mathbb{R}$ which has minimal number of critical points equal to $b_1(S) + 2$. This function has a unique maximum point p_+ and a unique minimum point p_- which can be placed at the centers of $D_\pm$ and no other critical points in $D_\pm$. The restriction of h to Σ' has $b_1(S) = b_1(\Sigma) - 2 = |\chi(\Sigma)|$ critical points. A circular Morse function on Σ with $|\chi(\Sigma)|$ critical points can be now constructed out of h by performing some inverse surgery operations on S.

The Proof of Theorem 1 that we have provided in [11] is considering two cases depending on the orientability of the closed connected surface. The orientable case relies on the following strategy:

1. We show that $\mu(F) := \mu_0(F) + \mu_1(F) + \mu_2(F) \geq 2(g-1)$, for every circular Morse function $F : \Sigma_g \longrightarrow S^1$, where $\mu_j(F)$ stands for the number of critical points of index j of F and $\mu(F)$ for the cardinality of the critical set $C(F)$ of F;
2. We construct a circular Morse function on Σ_g with exactly $2(g-1)$ critical points.

In order to do so, we first observe that

$$2 - 2g = \mu_0(F) - \mu_1(F) + \mu_2(F). \tag{15}$$

Indeed, by using the well-known Poincaré–Hopf Theorem, one obtains

$$2 - 2g = \chi(\Sigma_g) = \sum_{p \in C(F)} \mathrm{ind}_p(\nabla F),$$

where ∇F is the gradient vector field of F with respect to some Riemannian metric on Σ_g. To finish the proof of relation (15), we just need to observe that the index of ∇F at a critical point of index one is -1, and the index of ∇F at the critical points of index zero and two is 1. Indeed, the local behavior of F around the critical points of index one is $F = x^2 - y^2$ and its gradient behaves locally around such a point like the vector field $(x, -y)$. The degree of its normalized restriction to the circle S^1 is -1 as the normalized restriction is a diffeomorphism which reverses the orientation. Similarly, the index of ∇F at a critical point of index zero or two is 1 as the local behavior of F around such a critical point is $F = x^2 + y^2$ or $F = -x^2 - y^2$ and its gradient behaves locally around such a point like the vector field (x, y) or $(-x, -y)$, respectively. The normalized restrictions of these vector fields to the circle S^1 are diffeomorphisms preserving the orientation and their degree is therefore one. Thus, the relation (15) is now completely proved via the Poincaré–Hopf Theorem.

The second item of our strategy is based on the following:

Lemma 2 ([11]). *The surface Σ_g can be suitably embedded into $\mathbb{R}^3 \setminus Oz$ such that the restriction $f\big|_{\Sigma_g} : \Sigma_g \longrightarrow S^1$ is a circular Morse function with exactly $2(g-1)$ critical points, where $f : \mathbb{R}^3 \setminus Oz \longrightarrow S^1$ is the submersion defined by*

$$f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2}}(x, y, 0).$$

To construct the suitable embedding of Σ_g in $\mathbb{R}^3 \setminus Oz$ recall that $\Sigma_1 = T^2 = S^1 \times S^1$ is being usually identified with the surface of revolution in $\mathbb{R}^3$ obtained by rotating, around the z-axis, a circle in the plane xOz centered at a point on the x-axis. In order to avoid some possible intersections, the radius of the circle is supposed to be strictly smaller than the distance from the origin to its center. A certain embedding of the surface Σ_g in $\mathbb{R}^3$, obtained from the one of Σ_1 on which we perform some surgery, will be useful in our approach. However, the above mentioned embedding of Σ_1 in $\mathbb{R}^3$ has one circle on the top and one circle on the bottom on which the Gauss curvature vanishes. The two circles define the critical set of the standard height function $f_{\mathbf{k}}$ in the z-axis direction, restricted to the embedded copy of T^2 in $\mathbb{R}^3$. Thus, this restricted height function is not a Morse function since its critical set is not finite. In order to construct our suitable embedding of Σ_g we rather need to rotate around the z-axis a closed convex curve with a unique center of symmetry, on the x-axis, which lies in the plane xOz and has no overlaps with the z-axis. This curve is also required to contain two segments mutually symmetric with respect to the x-axis, one on the top and the other one on its bottom. These two segments define the critical set of the height function $f_{\mathbf{k}}$ restricted to the curve itself.

Consider the embedding of Σ_1, obtained by rotating such a closed convex curve, instead of a circle within the plane xOz, within the same plane (Fig. 1).

The obtained copy of Σ_1 is flat on the two annuli $\mathscr{A}$ and $\mathscr{A}'$ which lie in two horizontal parallel planes. Consider the points $p_1, \ldots p_{g-1} \in \mathscr{A}$ and the points

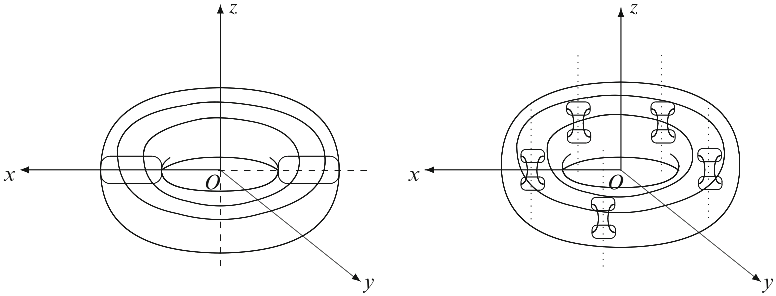

Fig. 1 An embedded copy of Σ_6 constructed out of an embedded copy of Σ_1

$q_1, \ldots, q_{g-1} \in \mathscr{A}'$ such that the lines $p_i q_i$, $i = 1, \ldots, g-1$ are vertical, i.e. parallel to the z-axis. In order to obtain a topological copy of the surface Σ_g we next remove some small open discs $D_1, \ldots, D_{g-1} \subseteq \mathscr{A}$ centered at $p_1, \ldots p_{g-1}$, and $D_1', \ldots, D_{g-1}' \subseteq \mathscr{A}'$ centered at $q_1, \ldots, q_{g-1}$ respectively. The radii of the disks D_i and D_i' are supposed to be equal. Now, consider suitable planar curves

$$\gamma_i : [0,1] \longrightarrow \mathrm{cl}\,(\mathscr{B}) \cap \pi_i,\ i = 1, \ldots, g-1$$

such that $\gamma_i(0) \in \partial D_i$ and $\gamma_i(1) \in \partial D_i'$, where $p_i q_i \cap xOy = \{(x_i, y_i, 0)\}$, π_i is the plane parallel to xOz through the point $(x_i, y_i, 0)$ (i.e. π_i has the equation $y = y_i$) and $\mathscr{B}$ is the bounded component of the complement of the embedded copy of Σ_1. The curves γ_i are chosen in such a way to complete, by their rotation around the axes $p_i q_i$, the embedded copy of $\Sigma_1 \setminus [D_1 \cup \ldots \cup D_{g-1} \cup D_1' \cup \ldots \cup D_{g-1}']$ up to a smooth embedded copy of Σ_g.

Proposition 10 ([11]). *The following equality holds* $\mu\left(f|_{\Sigma_g}\right) = 2(g-1)$.

Proposition 11 ([11]). *The restriction* $f|_{\Sigma_g}$ *is a circular Morse function, i.e. its critical points are nondegenerate. Moreover, all critical points of* $f|_{\Sigma_g}$ *have index* 1.

The Second Proof of Theorem 1 in the Case of Orientable Surfaces. For the inequality $\gamma_{S^1}(\Sigma_g) \geq 2(g-1)$ we merely need to use the relation (15) that is $2 - 2g = \mu_0(F) - \mu_1(F) + \mu_2(F) \geq -\mu_1(F)$, for every circular Morse function $F : \Sigma_g \longrightarrow S^1$. This shows that

$$2(g-1) \leq \mu_1(F) \leq \mu_0(F) + \mu_1(F) + \mu_2(F) = \mu(F),$$

for every circular Morse function $F : \Sigma_g \longrightarrow S^1$, and the inequality $2(g-1) \leq \gamma_{S^1}(\Sigma_g)$ follows. The opposite inequality is proved by the existence of the circular Morse function $f\big|_{\Sigma_g}$ which has exactly $2(g-1)$ critical points. $\square$

Remark 6. No real valued Morse function defined on a compact manifold M^m ($m \geq 2$) can merely have critical points of index one. Indeed the global minimum of such a function has index zero and its global maximum has index m. Thus, the restriction $f|_{\Sigma_g}$ cannot be lifted to any map $\tilde{f} : \Sigma_g \longrightarrow \mathbb{R}$ and the induced group homomorphism $\left(f|_{\Sigma_g}\right)_* : \pi(\Sigma_g) \longrightarrow \mathbb{Z} = \pi(S^1)$ is nontrivial therefore.

The Second Proof to Theorem 1 in the Case of Non-orientable Surfaces. In this case we use Proposition 5 in order to prove the inequality

$$\gamma_{S^1}\left(g\mathbb{RP}^2\right) \geq |\chi\left(g\mathbb{RP}^2\right)|,$$

for $g \geq 2$, where $k\mathbb{RP}^2$ stands for the connected sum $\mathbb{RP}^2\#\mathbb{RP}^2\#\cdots\#\mathbb{RP}^2$ of k copies of the projective plane. Indeed, by applying Proposition 5 to the orientable double covering

$$\Sigma_{g-1} \to g\mathbb{RP}^2$$

we obtain successively:

$$\begin{aligned}\gamma_{S^1}\left(g\mathbb{RP}^2\right) &\geq \tfrac{1}{2}\gamma_{S^1}\left(\Sigma_{g-1}\right) = \tfrac{1}{2}|\chi\left(\Sigma_{g-1}\right)| \\ &= \tfrac{1}{2}|2-2(g-1)| = |2-g| = |\chi\left(g\mathbb{RP}^2\right)|.\end{aligned}$$

For the opposite inequality we first recall that

$$f : \mathbb{RP}^2 \longrightarrow \mathbb{R},\ f([x_1, x_2, x_3]) = \frac{x_1^2 + 2x_2^2 + 3x_3^2}{x_1^2 + x_2^2 + x_3^2},$$

is a perfect Morse function with exactly three critical points of indices $0, 1, 2$, i.e. a minimum point p, a maximum point q and a saddle point s. If $\varepsilon > 0$ is small enough, then the inverse images

$$D := f^{-1}(-\infty, f_2(p) + \varepsilon),\ \ D' := f^{-1}(f(q) - \varepsilon, \infty)$$

are open disks and the inverse image $f^{-1}[f(p)+\varepsilon, f(q)-\varepsilon] = \mathbb{RP}^2 \setminus (D_1 \cup D_2)$ is a compact surface with two circular boundary components $f^{-1}(f(p)+\varepsilon)$ and $f^{-1}(f(q)-\varepsilon)$. Observe that the restriction

$$f\big|_{\mathbb{RP}^2\setminus(D\cup D')} : \mathbb{RP}^2 \setminus (D_1 \cup D_2) \longrightarrow [f(p)+\varepsilon, f(q)-\varepsilon]$$

has one critical point of index one, i.e. the saddle point s. We construct a circular Morse function on $(g+2)\mathbb{RP}^2$ with g saddle points by gluing cyclically g copies of $\mathbb{RP}^2 \setminus (D \cup D')$. This construction shows that the inequality $\gamma_{S^1}\left((g+2)\mathbb{RP}^2\right) \leq g$ holds for all $g \geq 1$. On the other hand the Klein Bottle $2\mathbb{RP}^2$ fibers over S^1 (with fiber S^1), which shows that

$$\gamma_{S^1}\left(2\mathbb{RP}^2\right) = 0 = |\chi(2\mathbb{RP}^2)|,$$

and the second proof of Theorem 1 in the non-orientable case is now complete. □

For the closed non-orientable surface of even non-orientable genus $2g + 2$ we may visualize its orientable double cover by means of the embedding of the closed orientable surface of genus $2g + 1$ in $\mathbb{R}^3 \setminus Oz$ described before. Since the genus of the embedded copy of Σ_{2g+1} is odd, we may impose the extra-requirement on the embedded image of Σ_{2g+1} to be symmetric with respect to the origin, i.e. invariant with respect to the antipodal action of $\mathbb{Z}_2$ on $\mathbb{R}^3 \setminus \{0\}$. Because the restriction of this action to Σ_{2g+1} reverses the orientation, it follows that the quotient $\Sigma_{2g+1}\big/\mathbb{Z}_2$ is a compact non-orientable surface. Obviously the projection

$$\pi : \Sigma_{2g+1} \longrightarrow \Sigma_{2g+1}\big/\mathbb{Z}_2$$

is the orientable double covering of $\Sigma_{2g+1}\big/\mathbb{Z}_2$. The reversing orientation property of the antipodal involution a follows from the reversing orientation property of the reflections σ_{xy}, σ_{xz}, and σ_{yz} with respect to the coordinate planes xOy, xOz, and yOz, respectively, and the decomposition $a = \sigma_{xy} \circ \sigma_{xz} \circ \sigma_{yz}$. Note that the three reflections commute with each other. The reversing orientation property of the reflection σ_{xy}, for example, follows by looking to the orientation behavior at a fixed point $p \in \mathrm{Fix}(\sigma_{xy}) = xOy \cap \Sigma_{2g+1}$. Since the tangent map of σ_{xy} at p reverses the orientation of the tangent space $T_p\left(\Sigma_{2g+1}\right)$, it follows that σ_{xy}, and by similar arguments each of the reflections σ_{xz} and σ_{yz}, reverses the orientation of Σ_{2g+1}. Consequently, the antipodal map $a = \sigma_{xy} \circ \sigma_{xz} \circ \sigma_{yz}$ reverses the orientation as well.

One can easily check that the non-orientable genus of $\Sigma_{2g+1}\big/\mathbb{Z}_2$ is $2g + 2$, that is $\Sigma_{2g+1}\big/\mathbb{Z}_2$ is diffeomorphic to $(2g + 2)\mathbb{RP}^2$.

Remark 7. For the inequality $\gamma_{S^1}\left((2g + 2)\mathbb{RP}^2\right) \le 2g$ we can produce a particular Morse function

$$f_0 : (2g + 2)\mathbb{RP}^2 = \Sigma_{2g+1}\big/\mathbb{Z}_2 \longrightarrow S^1$$

with precisely $2g$ critical points in the following different way. Pick the function $g_0 := f\big|_{\Sigma_{2g+1}} : \Sigma_{2g+1} \longrightarrow S^1$ considered for the proof of Theorem 1 and recall that g_0 has precisely $4g$ critical points and $4g$ critical values, i.e. $\mathrm{card}\big(g_0(C(g_0)\big)$ is also $4g$. Indeed, the restriction $g_0\big|_{C(g_0)}$ is obviously one-to-one. Due to the way we embedded Σ_{2g+1}, the critical values of g_0, alongside its critical points, are pairwise antipodal in S^1 and in Σ_{2g+1}, respectively. By considering now the covering projection $p : S^1 \longrightarrow P^1(\mathbb{R})$, $p(x) = [x] := \{-x, x\}$, one actually obtain a cyclic covering of order two $p : S^1 \longrightarrow S^1$, as $P^1(\mathbb{R})$ is diffeomorphic to S^1. The composed function $p \circ g_0$ is a circular Morse function with $2g$ critical values, each of whose inverse image consists of two critical points. Thus $\mu(p \circ g_0) = \text{card}\big(C(p \circ g_0)\big) = 4g$. In fact $\pi^{-1}(\pi(x)) = \{-x, x\} \subseteq (p \circ g_0)^{-1}(x)$, for every $x \in \Sigma_{2g+1}$. This shows that the restriction g_0 factors to a Morse function $f_0 : \Sigma_{2g+1} \big/ \mathbb{Z}_2 \longrightarrow S^1$ such that we have $p \circ g_0 = f_0 \circ \pi$. Let us now observe that $\pi^{-1}(C(f_0)) = C(p \circ g_0)$ and therefore $\mu(p \circ g_0) = \text{card}\big(C(p \circ g_0)\big) = 2\text{card}\big(C(f_0)\big) = 2\mu(f_0)$, i.e. $\mu(f_0) = \text{card}\big(C(f_0)\big) = \frac{1}{2}\text{card}\big(C(p \circ g_0)\big) = \frac{1}{2}\mu(p \circ g_0) = 2g$.

6 The Circular Morse–Smale Characteristic of Connected Sums

In this section we plan to provide an upper bound for the circular Morse–Smale characteristic of an arbitrary connected sum $X\#Y$ in terms of some circular characteristic of X and Y by using similar arguments to those in the proof of [2, Theorem 4.2.21], initially stated and proved in [9]. In this respect we consider the family $\mathfrak{F}_{0,n}(X, S^1)$ of all smooth circular Morse functions on the closed n-dimensional manifold X which admit both a critical point of index 0 and a critical point of index n. In the case of real-valued Morse functions on X, this is always the case, i.e. $\mathfrak{F}_{0,n}(X, \mathbb{R}) = \mathfrak{F}(X)$, where $\mathfrak{F}(X)$ stands for the collection of all smooth real valued Morse functions. Recall however that the function constructed in Proposition 11 is a circular Morse function on the surface Σ_g without critical points of index zero and index two. This shows that, generally, the inclusion $\mathfrak{F}_{0,n}(X, S^1) \subset \mathfrak{F}(X, S^1)$ is strict. In this section we shall merely consider the family $\mathfrak{F}_{0,n}(X, S^1)$ of circular Morse functions on X. Note that the family $\mathfrak{F}_{0,n}(X, S^1)$ is quite rich as it contains the family of functions

$$\{\exp \circ f \; : \; f \in \mathfrak{F}(X)\}.$$

In the case $\text{Hom}(\pi(X), \mathbb{Z}) = 0$, the following equalities hold

$$\mathfrak{F}_{0,n}(X, S^1) = \mathfrak{F}(X, S^1) = \exp \circ \mathfrak{F}(X). \tag{16}$$

Figure 2 suggests an example of a circular Morse function on the surface Σ_6 which belongs to $\mathfrak{F}_{0,2}(\Sigma_6, S^1)$ and cannot be lifted to any real valued Morse function.

If X is a given manifold, we define the *Morse–Smale characteristic* of X related to the family $\mathfrak{F}_{0,n}(X, S^1)$ by

$$\gamma_{S^1}^{0,n}(X) = \min\{\mu(f) \; : \; f \in \mathfrak{F}_{0,n}(X, S^1)\},$$

where $\mu(f)$ stands for card $(C(f))$.

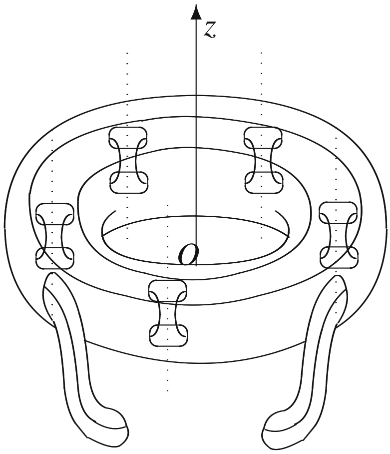

Fig. 2 An embedded copy of Σ_6 in $\mathbb{R}^3$ on which the restriction of f is in $\mathfrak{F}_{0,2}(\Sigma_6, S^1)$ and cannot be lifted to any real valued Morse function

Remark 8. If X is a smooth manifold, then $\gamma_{S^1}(X) \leq \gamma_{S^1}^{0,n}(X)$, as $\mathfrak{F}_{0,n}(X, S^1) \subseteq \mathfrak{F}(X, S^1)$. If $\operatorname{Hom}(\pi(X), \mathbb{Z}) = 0$, then $\gamma_{S^1}^{0,n}(X) = \gamma_{S^1}(X) = \gamma(X)$, as $\mathfrak{F}_{0,n}(X, S^1) = \mathfrak{F}(X, S^1) = \exp \circ \mathfrak{F}(X)$ in this case.

Theorem 2. *If X and Y are closed smooth n-manifolds, then the following inequality holds true:*

$$\gamma_{S^1}(X\#Y) \leq \gamma_{S^1}^{0,n}(X) + \gamma_{S^1}^{0,n}(Y) - 2.$$

Proof. Following the idea in the paper [9] (see also Theorem 4.2.21 in the monograph [2]), consider the circular Morse functions $f \in \mathfrak{F}_{0,n}(X, S^1)$ and $g \in \mathfrak{F}_{0,n}(Y, S^1)$ such that $\gamma_{S^1}^{0,n}(X) = \operatorname{card}(C(f)) = \mu(f)$ and $\gamma_{S^1}^{0,n}(Y) = \operatorname{card}(C(g)) = \mu(g)$. We also consider a critical point of index zero $p \in C(f)$ and a critical point $q \in C(g)$ of index n. By composing the two functions on their left-hand sides with suitable rotations of the circle S^1 we obtain new circular Morse function with the same number of critical points having the same distributions of indexes, still denoted by f and g, such that, for small enough $\varepsilon > 0$, the following requirements hold:

1. The restriction $\exp_\varepsilon := \exp|_{(-\varepsilon,\varepsilon)} : (-\varepsilon, \varepsilon) \to \exp(-\varepsilon, \varepsilon) \subset S^1$ of the exponential function $\exp : \mathbb{R} \to S^1$, $\exp(x) = e^{ix}$ is a diffeomorphism;
2. $f(p) = \exp(-\frac{\varepsilon}{2})$ and $g(q) = \exp(\frac{\varepsilon}{2})$, i.e. $e^{i\frac{\varepsilon}{2}} f(p) = 1 = e^{-i\frac{\varepsilon}{2}} g(q)$;
3. $f(U), g(V) \subseteq \exp(-\varepsilon, \varepsilon)$;
4. $\exp\left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right] = f(U) \cap g(V)$,

where (U, φ) is a Morse chart at $p \in X$ and (V, ψ) is a Morse chart at $q \in Y$, i.e.

$$\left(\exp_\varepsilon^{-1} \circ f|_U \circ \varphi^{-1}\right)(x_1, \dots, x_n) = -\frac{\varepsilon}{2} + x_1^2 + \cdots + x_n^2$$
$$\left(\exp_\varepsilon^{-1} \circ g|_V \circ \psi^{-1}\right)(y_1, \dots, y_n) = \frac{\varepsilon}{2} - y_1^2 - \cdots - y_n^2.$$

Thus, the inverse images $(f|_U)^{-1}\left(\widehat{e^{-i\left(\frac{\varepsilon}{2}+\theta\right)}1}\right)$ and $(g|_V)^{-1}\left(\widehat{1e^{i\left(\frac{\varepsilon}{2}+\theta\right)}}\right)$ of the small open arcs $\widehat{e^{-i\left(\frac{\varepsilon}{2}+\theta\right)}1}$ and $\widehat{1e^{i\left(\frac{\varepsilon}{2}+\theta\right)}}$ are some open disks, for $\theta > 0$ small enough, with spherical boundaries $(f|_U)^{-1}(1)$ and $(g|_V)^{-1}(1)$, respectively. Indeed we have successively:

$$\begin{aligned}
\varphi\left((f|_U)^{-1}\left(\widehat{e^{-i\left(\frac{\varepsilon}{2}+\theta\right)}1}\right)\right) &= \varphi\left((f|_U)^{-1}\left(\exp_\varepsilon\left(-\tfrac{\varepsilon}{2}-\theta, 0\right)\right)\right) \\
&= \left(\exp_\varepsilon^{-1} \circ f|_U \circ \varphi^{-1}\right)\left(-\tfrac{\varepsilon}{2}-\theta, 0\right)\Big) \\
&= \left\{x \in \varphi(U) : -\tfrac{\varepsilon}{2} - \theta < -\frac{\varepsilon}{2} + x_1^2 + \cdots + x_n^2 < 0\right\} \\
&= \left\{x=(x_1, \dots, x_n) \in \varphi(U) \,|\, -\theta < x_1^2 + \cdots + x_n^2 < \frac{\varepsilon}{2}\right\}.
\end{aligned}$$

Thus, the inverse image $(f|_U)^{-1}\left(\widehat{e^{-i\left(\frac{\varepsilon}{2}+\theta\right)}1}\right)$ is an open disk with the $(n-1)$-dimensional spherical boundary

$$\varphi^{-1}\left(\left\{x = (x_1, \dots, x_n) \in \varphi(U) \,|\, x_1^2 + \cdots + x_n^2 = \frac{\varepsilon}{2}\right\}\right) = (f|_U)^{-1}(1).$$

One can similarly show that $(g|_V)^{-1}\left(\widehat{1e^{i\left(\frac{\varepsilon}{2}+\theta\right)}}\right)$ is an open disk whose boundary $(g|_V)^{-1}(1)$ is an $(n-1)$-dimensional sphere.

In fact, we may think of the connected sum $X\#Y$ along these boundaries $(f|_U)^{-1}(1)$ and $(g|_V)^{-1}(1)$, which are being identified within the connected sum $X\#Y$.

The circular function $f\#g : X\#Y \to S^1$ defined by

$$(f\#g)(z) = \begin{cases} f(z) & \text{if } z \in M \setminus (f|_U)^{-1}\left(\widehat{e^{-i\left(\frac{\varepsilon}{2}+\theta\right)}1}\right) \\ 1 & \text{if } z \in (f|_U)^{-1}(1) = (g|_V)^{-1}(1) \\ g(z) & \text{if } z \in N \setminus (g|_V)^{-1}\left(\widehat{1e^{i\left(\frac{\varepsilon}{2}+\theta\right)}}\right) \end{cases}$$

is a Morse function on $X\#Y$ with

$$\mu(f) + \mu(g) - 2 = \text{card}(C(f)) + \text{card}(C(g)) - 2 = \gamma_{S^1}^{0,1}(X) + \gamma_{S^1}^{0,1}(Y) - 2$$

critical points. Consequently

$$\gamma_{S^1}(X\#Y) \le \mu(f\#g) = \mu(f) + \mu(g) - 2 = \gamma_{S^1}^{0,1}(X) + \gamma_{S^1}^{0,1}(Y) - 2$$

and the proof is now complete. □

Corollary 11. *If X is a closed smooth n-manifold, then $\gamma_{S^1}(X\#X) \le 2\gamma_{S^1}^{0,n}(X)-2$.*

Corollary 12. *If m, n are natural numbers such that $m, n \ge 2$, then:*

1. $\gamma_{S^1}((S^m \times S^n)\#(S^m \times S^n)) \le 6$.
2. $\gamma_{S^1}((\mathbb{RP}^m \times \mathbb{RP}^n)\#(\mathbb{RP}^m \times \mathbb{RP}^n)) \le 2(mn + m + n)$.

Proof. We only need to apply Corollary 11 for $X = S^m \times S^n$ or $\mathbb{RP}^m \times \mathbb{RP}^n$ and the equalities

$$\gamma_{S^1}^{0,m+n}(S^m \times S^n) = \gamma_{S^1}(S^m \times S^n) = \gamma(S^m \times S^n) = 4,$$

$$\gamma_{S^1}^{0,m+n}(\mathbb{RP}^m \times \mathbb{RP}^n) = \gamma_{S^1}(\mathbb{RP}^m \times \mathbb{RP}^n) = \gamma(\mathbb{RP}^m \times \mathbb{RP}^n) = (m+1)(n+1),$$

which work as $\mathrm{Hom}\,(\pi(S^m \times S^n), \mathbb{Z}) = 0$ and $\mathrm{Hom}\,(\pi(\mathbb{RP}^m \times \mathbb{RP}^n), \mathbb{Z}) = 0$. □

In the case of torsion fundamental groups we may consider connected sums of arbitrary many manifolds. Indeed, we have the following:

Corollary 13. *If $M_1, \ldots, M_r$ are smooth n-manifolds with torsion fundamental groups, then*

$$\gamma_{S^1}(M_1\#\cdots\#M_r) \le \gamma(M_1) + \cdots + \gamma(M_r) - 2(r-1). \tag{17}$$

Proof. Taking into account that $\mathrm{Hom}\,(\pi(M_1\#\cdots\#M_s), \mathbb{Z}) = 0$ for every $s \le r$ (see Corollary 2), we only need to apply Theorem 2 inductively with respect to r and use the obvious equalities $\gamma_{S^1}(M_i) = \gamma(M_i)$ for every $i = 1, \ldots, r$. □

Remark 9. 1. If $M_1, \ldots, M_r$ are smooth n-manifolds with torsion fundamental groups, then, according to Corollary 10, $\gamma_{S^1}(M_1\#\cdots\#M_r) = \gamma(M_1\#\cdots\#M_r)$ and the inequality (17) can be rewritten as $\gamma(M_1\#\cdots\#M_r) \le \gamma(M_1) + \cdots + \gamma(M_r) - 2(r-1)$. The latter one is a direct consequence of the inequality [2, (4.2.20), p. 124].

2. Corollary 12 can be extended to the connected sums of arbitrary many products of spheres or projective spaces, as these products have torsion fundamental groups. More precisely, if $m, n \ge 2$, then the following inequalities hold:

   1. $\gamma_{S^1}(r(S^m \times S^n)) \le 2r + 2$,
   2. $\gamma_{S^1}(r(\mathbb{RP}^m \times \mathbb{RP}^n)) \le r(m+1)(n+1) - 2(r-1)$,

   where rX stands for the connected sum $X\#\cdots\#X$ of r copies of the manifold X.

7 Estimates for the Number of Characteristic Points

The critical points of the real valued height functions alongside those of some circular functions on a surface $S \subset \mathbb{R}^3$, are, according to Propositions 8 and 9, the characteristic points with respect to some involutive distributions. In this last section we show that every closed orientable surface can be embedded into the three dimensional space $\mathbb{R}^3$ in such a way to get only finitely many characteristic points with respect to the noninvolutive horizontal distribution of the first Heisenberg group $\mathbb{H}^1 = (\mathbb{R}^3, *)$. The horizontal distribution of the first Heisenberg group $\mathbb{H}^1 = (\mathbb{R}^3, *)$ is $\mathscr{H} = \text{span}(X, Y) = \left\{\mathscr{H}_p := \text{span}(X_p, Y_p)\right\}_{p\in\mathbb{H}^1}$, where $X = \partial_x + 2y_i\partial_t$ and $Y = \partial_y - 2x\partial_t$. Let us consider a C^1-differentiable surface $S \subset \mathbb{R}^3$. The *characteristic set* of S with respect to $\mathscr{H}$ is defined as

$$C(S, \mathscr{H}) = \{p \in S \ : \ T_pS = \mathscr{H}_p\}.$$

Note that the characteristic set $C(\Sigma, \mathscr{H})$ of some surface $\Sigma \subset \mathbb{R}^3$ with respect to the horizontal distribution $\mathscr{H}$ is the set of singularities of the vector field Z_Σ on Σ obtained by projecting orthogonally $X \wedge Y$ on the tangent spaces of Σ, i.e. $C(\Sigma, \mathscr{H}_1) = \text{Sing}(Z_\Sigma)$.

The *minimum characteristic number* of S relative to $\mathscr{H}$ is defined as

$$\text{mcn}(S, \mathscr{H}) := \min\{\text{card}\,(C(f(S), \mathscr{H})) \ : \ f \in \text{Embed}(S, \mathbb{R}^3)\},$$

where $\text{Embed}(S, \mathbb{R}^3)$ stands for the set of all embeddings of S into $\mathbb{R}^3$. Note that the minimum characteristic number of a surface can be defined in relation with an arbitrary distribution on the ambient space, as the characteristic set of a surface can be defined in such a situation. In fact the characteristic sets are being defined for hypersurfaces of a given manifold with respect to arbitrary codimension one distribution on that manifold. Moreover, the more general concept of *tangency set* can be defined for a k-submanifold of a given manifold with respect to an arbitrary distribution on the ambient manifold of rank k [13] (see also [12]).

Theorem 3. *If $g \geq 2$, then the following inequalities hold true*

$$2g - 2 \leq \text{mcn}(\Sigma_g, \mathscr{H}) \leq 4g - 4.$$

An argument towards the lower bound $2g - 2$ relies on the Poincaré–Hopf Theorem

$$2 - 2g = \chi\left(\Sigma_g\right) = \sum_{x\in\text{Sing}(Z_{\Sigma_g})} \text{index}_z(Z_{\Sigma_g}).$$

applied to the vector field Z_Σ, whose singularities are generally of index ± 1 [23]. For the upper bound $4g - 4$ we need to construct an embedding of Σ_g with $4g - 4$ characteristic points. In this respect we use the possibility of Σ_1 to be embedded in $\mathbb{H}^1$ as a revolution surface and construct a suitable embedding of Σ_g out of Σ_1 by performing some suitable surgery on Σ_1. The handles we plan to glue will be surfaces of revolution as well. Therefore, we are going to pay some special attention to the size of the characteristic sets of revolution surfaces which lie inside $\mathbb{H}^1$ with respect to its horizontal distribution $\mathscr{H}$.

Every revolution surface S obtained by rotating a plane curve of parametric equations $x = \alpha(v), z = v$, with $\alpha > 0$, around the vertical line $x = x_0, y = y_0$, admits a local parametrization of type

$$\begin{aligned} x &= x_0 + \alpha(v)\cos u \\ y &= y_0 + \alpha(v)\sin u\ ,\ \ u \in I, v \in J, \\ z &= v \end{aligned}$$

where I is an open interval of length 2π and J will be symmetric with respect to the origin, i.e. $J = (-a, a)$. The function α is subject to the following requirements:

$$\alpha \text{ is bounded}\ ,\ \ \alpha'' > 0 \text{ and } \lim_{v\to\pm a} \alpha'(v) = \pm\infty. \tag{18}$$

One can easily see that the point $(x(u, v), y(u, v), z(u, v)) \in S$ is a horizontal point if and only if the vectors

$$\begin{aligned} \sin u + 2\alpha(v)\alpha'(v)\cos u &= -2x_0\alpha'(v) \\ 2\alpha(v)\alpha'(v)\sin u - \cos u &= -2y_0\alpha'(v). \end{aligned}$$

Solving this system for $\sin u$ and $\cos u$, we get

$$\begin{aligned} \sin u &= -2\alpha'(v)\frac{x_0 + 2y_0\alpha(v)\alpha'(v)}{1 + 4\alpha^2(v)(\alpha'(v))^2} \\ \cos u &= -2\alpha'(v)\frac{2x_0\alpha(v)\alpha'(v) - y_0}{1 + 4\alpha^2(v)(\alpha'(v))^2}. \end{aligned} \tag{19}$$

Remark 10. No revolution surface around the z-axis has $\mathscr{H}$-tangency points, as the system of (19) has no solutions at all for $x_0 = y_0 = 0$.

The identity $\sin^2 u + \cos^2 u = 1$ leads us to the equation

$$\left(\alpha'(v)\right)^2 = \frac{1}{4\left(||(x_0, y_0)||^2 - \alpha^2(v)\right)}, \tag{20}$$

which has at least two solutions on the interval $J = (-a, a)$. Indeed, the right-hand side of (20) is bounded and $(\alpha')^2$ covers the positive real half line $[0, \infty)$ twice, once on the interval $(-a, 0]$ and once on the interval $[0, a)$. For suitable choices of the function α, (20) has precisely two solutions. Such a choice is

$$\alpha(v) = 2 - \sqrt{\frac{2 - v^2}{2}} \tag{21}$$

for $a = \sqrt{2}$ and $||(x_0, y_0)|| = 3$. Indeed, (20), for α considered in (21), becomes:

$$4v^2\sqrt{2(2 - v^2)} = -v^4 - 9v^2 + 2,$$

which has, indeed, precisely two solutions, as can be easily checked. Making use of the above observations, the proof of Theorem 3 follows immediately. For more details on this elementary proof we refer the reader to [11].

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A Remark on Some Simultaneous Functional Inequalities in Riesz Spaces

Bogdan Batko and Janusz Brzdęk

Abstract We study continuous at a point functions that take values in a Riesz space and satisfy some systems of two simultaneous functional inequalities. In this way we obtain in particular generalizations and extensions of some earlier results of Krassowska, Matkowski, Montel, and Popoviciu.

Keywords Functional inequality • Riesz space • σ-Ideal

Mathematics Subject Classification (2010): 39B72.

1 Introduction

In what follows $\mathbb{N}_0$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{R}_+$ denote, as usual the sets of nonnegative integers, positive integers, integers, rationals, reals and nonnegative reals, respectively. Moreover, let $a, b \in \mathbb{R} \setminus \{0\}$ with $ab^{-1} \notin \mathbb{Q}$ and $ab < 0$ be fixed. Montel [13] (see also [14] and [11, p. 228]) proved that a function $f : \mathbb{R} \to \mathbb{R}$, that is continuous at a point and satisfies the system of functional inequalities

$$f(x+a) \le f(x), \qquad f(x+b) \le f(x), \qquad x \in \mathbb{R}, \tag{1}$$

has to be constant. A similar (but more abstract) result for measurable functions has been proved in [2].

In [7–9] (see also [10]) the result of Montel has been generalized and extended in several ways. In particular, motivated by some problem arising in a characterization of L^p norm, Krassowska and Matkowski [8] (cf. also [7]) have proved that if $\alpha, \beta \in$

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$\mathbb{R}$ and $\alpha b \leq \beta a$, then a continuous at a point function $f : \mathbb{R} \to \mathbb{R}$ satisfies the following two functional inequalities

$$f(x+a) \leq f(x) + \alpha, \qquad f(x+b) \leq f(x) + \beta, \qquad x \in \mathbb{R}, \tag{2}$$

if and only if $\alpha b = \beta a$; moreover f has to be of the form $f(x) = cx + d$ for $x \in \mathbb{R}$, with some $c, d \in \mathbb{R}$.

In this paper we investigate the possibility to obtain results analogous to those in [2, 8, 13] for functions taking values in Riesz spaces. Moreover, we consider the system (2) in a conditional form and almost everywhere. We obtain outcomes that correspond somewhat to the results in [3, 4] and to the problem of stability of functional equations and inequalities (for some further information concerning that problem we refer to, e.g., [1, 5, 6]).

2 Preliminaries

For the readers convenience we present the definition and some basic properties of Riesz spaces (see [12]).

Definition 1 (cf. [12, Definitions 11.1 and 22.1]). We say that a real linear space L, endowed with a partial order $\leq \subset L^2$, is a *Riesz space* if $\sup\{x, y\}$ exists for all $x, y \in L$ and

$$ax + y \leq az + y, \qquad x, y, z \in X, x \leq z, a \in \mathbb{R}_+;$$

we define the absolute value of $x \in L$ by the formula $|x| := \sup\{x, -x\} \geq 0$. Next, we write $x < z$ provided $x \leq z$ and $x \neq z$.

A Riesz space L is called *Archimedean* if, for each $x \in L$, the inequality $x \leq 0$ holds whenever the set $\{nx : n \in \mathbb{N}\}$ is bounded from above.

In the following it will be assumed that L is an Archimedean Riesz space. It is easily seen that $\alpha u \leq \beta u$ for every $u \in L_+ := \{x \in L : x > 0\}$ and $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$. Moreover, given $u \in L_+$ we can define an extended (i.e., admitting the infinite value) norm $\|\cdot\|_u$ on L by

$$\|v\|_u := \inf\{\lambda \in \mathbb{R}_+ : |v| \leq \lambda u\}, \qquad v \in L,$$

where it is understood that $\inf \emptyset = +\infty$ and $0 \cdot (+\infty) = 0$.

Let us yet recall some further necessary definitions.

Definition 2. Let $E \subset \mathbb{R}$ be nonempty and let $\mathscr{I} \subset 2^{\mathbb{R}}$. We say that a property $p(x)$ ($x \in E$) holds $\mathscr{I}$-almost everywhere in E (abbreviated in the sequel to $\mathscr{I}$-a.e. in E) provided there exists a set $A \in \mathscr{I}$ such that $p(x)$ holds for all $x \in E \setminus A$.

Definition 3. $\mathscr{I} \subset 2^{\mathbb{R}}$ is a σ-ideal provided $2^A \subset \mathscr{I}$ for $A \in \mathscr{I}$ and

$$\bigcup_{n\in\mathbb{N}} A_n \in \mathscr{I}\,, \qquad \{A_n\}_{n\in\mathbb{N}} \subset \mathscr{I}\,.$$

Moreover, if $\mathscr{I} \neq 2^{\mathbb{R}}$, then we say that $\mathscr{I}$ is proper; if $\mathscr{I} \neq \{\emptyset\}$, then we say that $\mathscr{I}$ is nontrivial. Finally, $\mathscr{I}$ is translation invariant (abbreviated to t.i. in the sequel) if $x + A \in \mathscr{I}$ for $A \in \mathscr{I}$ and $x \in \mathbb{R}$.

We have the following (see [3, Propositions 2.1 and 2.2]).

Proposition 1. *Let $\mathscr{I} \subset 2^{\mathbb{R}}$ be a proper t.i. σ-ideal and let $U \subset \mathbb{R}$ be open and nonempty. Then*

$$\operatorname{int}\,[(U \setminus T) - V] \neq \emptyset\,, \qquad V \in 2^{\mathbb{R}} \setminus \mathscr{I},\ T \in \mathscr{I}\,, \tag{3}$$

where $(U \setminus T) - V = \{u - v : u \in U \setminus T, v \in V\}$.

3 The Main Result

Let us start with an auxiliary result.

Theorem 1. *Let P be a dense subset of $\mathbb{R}$, $\mathscr{J} \subset 2^{\mathbb{R}}$ be a proper t.i. σ-ideal and let E be a subset of a nontrivial interval $I \subset \mathbb{R}$ with $H := I \setminus E \in \mathscr{J}$. We assume that $v : I \to L$ satisfies*

$$v(p + x) \leq v(x)\,, \qquad x \in E \cap (E - p), p \in P\,. \tag{4}$$

If there exists $u \in L_+$ such that v is continuous at a point $x_0 \in I$, with respect to the extended norm $\|\cdot\|_u$, then $v(x) = v(x_0)$ $\mathscr{J}$-a.e. in I.

Proof. Note that (4) yields

$$v(y) \leq v(y + q)\,, \qquad y \in E \cap (E - q), q \in -P\,, \tag{5}$$

where $-P := \{-p : p \in P\}$. Since $\mathscr{J}$ is proper and t.i., we deduce that $I \notin \mathscr{J}$, whence $E \notin \mathscr{J}$.

For each $n \in \mathbb{N}$ we write

$$D_n := \left\{z \in I : \|v(z) - v(x_0)\|_u < \frac{1}{n}\right\}\,,$$

$$E_n' := \left\{z \in E : v(z) - v(x_0) < \frac{1}{n}u\right\}\,,$$

$$F_n' := \left\{z \in E : v(x_0) - v(z) < \frac{1}{n}u\right\}\,,$$

$C_n := D_n \setminus H$, $E_n := E \setminus E'_n$ and $F_n := E \setminus F'_n$. Clearly, $\operatorname{int} D_n \neq \emptyset$ for $n \in \mathbb{N}$, because v is continuous at x_0.

Suppose that there exists $k \in \mathbb{N}$ with $E_k \notin \mathscr{J}$. Then, on account of Proposition 1, there is $p \in P$ such that $-p \in \operatorname{int}(C_k - E_k)$, whence $p + c = e \in E_k \subset E$ with some $c \in C_k$ and $e \in E_k$. Hence, by (4),

$$v(e) - v(x_0) = v(p + c) - v(x_0) \leq v(c) - v(x_0) < \frac{1}{k}u\,.$$

This is a contradiction.

Next, suppose that $F_k \notin \mathscr{J}$ for some $k \in \mathbb{N}$. Then, on account of Proposition 1, there is $q \in -P$ with $-q \in \operatorname{int}(C_k - F_k)$, whence $q + c = e \in F_k \subset E$ with some $c \in C_k$ and $e \in F_k$. Hence, by (5),

$$v(x_0) - v(e) = v(x_0) - v(c + q) \leq v(x_0) - v(c) < \frac{1}{k}u\,.$$

This is a contradiction, too.

In this way we have shown that $G_k := E_k \cup F_k \in \mathscr{J}$ for $k \in \mathbb{N}$. Clearly

$$\begin{aligned} V &:= v^{-1}(L \setminus \{v(x_0)\}) = I \setminus \bigcap_{n \in \mathbb{N}} D_n \\ &= \bigcup_{n \in \mathbb{N}} I \setminus D_n \subset H \cup \bigcup_{k \in \mathbb{N}} G_k \in \mathscr{J} \end{aligned}$$

and $v(x) = v(x_0)$ for $x \in I \setminus V$. □

The next theorem is the main result of this paper.

Theorem 2. *Let I be a real infinite interval, $\mathscr{J} \subset 2^{\mathbb{R}}$ be a proper t.i. σ-ideal, L be an Archimedean Riesz space, $v : I \to L$, $a_1, a_2, \alpha_1, \alpha_2 \in \mathbb{R}$, $a_1 < 0 < a_2$, $a_1 a_2^{-1} \notin \mathbb{Q}$ and*

$$c_i := \frac{1}{a_i}\alpha_i\,, \qquad i = 1, 2\,. \tag{6}$$

If $c_1 \geq c_2$ and there exist $\omega, u \in L_+$ such that $\|\omega\|_u < \infty$, v is continuous at some point $x_0 \in I$, with respect to the extended norm $\|\cdot\|_u$, and the following two conditional inequalities

$$\text{if } a_1 + x \in I, \text{ then } v(a_1 + x) - v(x) \leq \alpha_1 \omega\,, \tag{7}$$

$$\text{if } a_2 + x \in I, \text{ then } v(a_2 + x) - v(x) \leq \alpha_2 \omega \tag{8}$$

are valid $\mathscr{J}$*-a.e. in* I*, then* $c_2 = c_1$ *and*

$$v(x) = c_1(x - x_0)\,\omega + v(x_0)\,, \qquad \mathscr{J}\text{-a.e. in } I\ . \tag{9}$$

Conversely, if $c_1 \le c_2$ *and* (9) *holds for some* $x_0 \in I$*, then* v *satisfies inequalities* (7) *and* (8) $\mathscr{J}$*-a.e. in* I*.*

Proof. Since $\mathscr{J}$ is a proper and t.i. σ-ideal, it is easily seen that we have the following property

$$\operatorname{int} T = \emptyset\,, \qquad T \in \mathscr{J}\,. \tag{10}$$

Next, there is a set $T \in \mathscr{J}$ such that conditions (7) and (8) hold for $x \in F := I \setminus T$.

Let

$$w_i(x) := v(x) - c_i x\,\omega\,, \qquad i = 1, 2, x \in I\ .$$

Clearly w_i is continuous at x_0 with respect to $\|\cdot\|_u$. Further, for every $i, j \in \{1, 2\}$, we have $\alpha_j \le c_i a_j$ and consequently

$$\begin{aligned} w_i(x + a_j) &= v(x + a_j) - c_i(x + a_j)\omega \\ &\le v(x) + \alpha_j\omega - c_i x\omega - c_i a_j\omega \\ &\le w_i(x)\,, \qquad x \in F \cap (F - a_j)\ . \end{aligned} \tag{11}$$

Let $E := I \setminus H$, where

$$H := \bigcup_{m,n\in\mathbb{Z}} (T + na_1 + ma_2) \in \mathscr{J}\ .$$

If we write $P := \{na_1 + ma_2 : n, m \in \mathbb{N}_0\}$, then

$$H + p = H\,, \qquad p \in P\ , \tag{12}$$

the set P is dense in $\mathbb{R}$ (see, e.g., [7–9]) and, in view of (11) and (12), it is easy to notice that

$$w_i(x + p) \le w_i(x)\,, \qquad x \in E \cap (E - p), p \in P, i = 1, 2\ . \tag{13}$$

Hence, on account of Theorem 1, there are $V_1, V_2 \in \mathscr{J}$ such that

$$w_i(x) = w_i(x_0)\,, \qquad x \in E \setminus V_i, i = 1, 2\ ,$$

which implies (9).

Further, observe that, by (10), we have $\operatorname{int}(H \cup V_1 \cup V_2) = \emptyset$ and

$$v(x_0) - c_i x_0 \omega = v(x) - c_i x \omega, \qquad x \in E_0 := I \setminus (H \cup V_1 \cup V_2), i = 1, 2 .$$

Hence

$$(c_1 - c_2) x\omega = (c_1 - c_2) x_0 \omega, \qquad x \in E_0 ,$$

whence we get $c_1 = c_2$.

The converse is easy to check. □

Remark 1. Let $a_1, a_2 \in \mathbb{R}$ and $\alpha_1, \alpha_2 \in (0, \infty)$. Then every function $v : I \to \mathbb{R}$ with

$$\sup_{x \in \mathbb{R}} |v(x)| \leq \frac{1}{2} \min\{\alpha_1, \alpha_2\}$$

fulfils (7) and (8) for each real interval I. This shows that some assumptions concerning a_1, a_2, c_1, c_2 are necessary in Theorem 2.

Taking $\mathscr{J} = \{\emptyset\}$ in Theorem 2 we obtain the following corollary.

Corollary 1. *Let $a_1, a_2, \alpha_1, \alpha_2 \in \mathbb{R}$ be such that $a_1 < 0 < a_2$, $a_1 a_2^{-1} \notin \mathbb{Q}$ and $c_1 \geq c_2$, where c_1, c_2 are given by (6). Let I be a real infinite interval, L be an Archimedean Riesz space, $u, \omega \in L_+$ and $\|\omega\|_u < \infty$. Then a function $v : I \to L$, that is continuous (with respect to the extended norm $\|\cdot\|_u$) at a point $x_0 \in I$, satisfies the inequalities*

$$\text{if } a_1 + x \in I, \text{ then } v(a_1 + x) - v(x) \leq \alpha_1 \omega ,$$

$$\text{if } a_2 + x \in I, \text{ then } v(a_2 + x) - v(x) \leq \alpha_2 \omega$$

if and only if $c_2 = c_1$ and

$$v(x) = c_1 (x - x_0)\, \omega + v(x_0), \qquad x \in I .$$

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Elliptic Problems on the Sierpinski Gasket

Brigitte E. Breckner and Csaba Varga

Abstract There are treated nonlinear elliptic problems defined on the Sierpinski gasket, a highly non-smooth fractal set. Even if the structure of this fractal differs considerably from that of (open) domains of Euclidean spaces, this note emphasizes that PDEs defined on it may be studied (as in the Euclidean case) by means of certain variational methods. Using such methods, and appropriate abstract multiplicity theorems, there are proved several results concerning the existence of multiple (weak) solutions of Dirichlet problems defined on the Sierpinski gasket.

Keywords Sierpinski gasket • Weak Laplacian • Dirichlet problem on the Sierpinski gasket • Weak solution • Critical point

1 Introduction

The origins of *analysis on fractals* lie in Mandelbrot's book [21] where fractals are proposed as models for different physical phenomena. Subsequently, the Laplacian on fractals, which first appeared in physics as a tool for investigating the percolation effect and various transport processes (in classical as well as in quantum mechanics), became the subject of intensive mathematical research. An overview of these researches can be found, for instance, in the introduction of Strichartz's book [37]. Here we only point out that defining the Laplacian on a general fractal implies to cope with considerable difficulties and that, over the years, several definitions have been proposed that are applicable to certain classes of fractals. For example, in the construction that goes back to Kigami (e.g., [15–18]) the Laplacian is defined as the

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limit of discrete differences on graphs approximating the fractal, a method that fits with so-called *post-critically-finite fractals*. Another approach was taken by Mosco (e.g., [23–25]), who introduced a framework for the Laplacian by taking as a starting point a Dirichlet form that reflects the self-similarities of the underlying fractal. This framework led to the very general theory of *variational fractals*.

Once a Laplacian having been defined on a fractal, one began to study elliptic (linear and nonlinear) problems on it. In the last 20 years there have been brought many contributions to this area. The papers [2–8, 10, 12, 13, 35, 36] are only a few examples in this sense.

It has turned out that well-established methods to investigate the existence and the multiplicity of solutions of PDEs defined on open domains in the Euclidean spaces can also be used in the case of PDEs on fractals. For instance, the main tools used in the papers [8, 10, 12, 13, 36] to prove the existence of at least one nontrivial solution or of multiple solutions of nonlinear elliptic equations with zero Dirichlet boundary conditions defined on fractals are, besides suitable techniques from variational calculus, certain minimax results (mountain pass theorems, saddle-point theorems), results from genus theory, and minimization procedures. A particular concern has been devoted to PDEs on the Sierpinki gasket. In this sense, we mention the pioneering paper [12] where nonlinear elliptic equations on the Sierpinski gasket in the two-dimensional Euclidean space have been treated. It should be pointed out that this paper, which was published in 2004 and had considerably influenced further investigations in the theory of PDEs on the Sierpinski gasket, was written a few years earlier, because the paper [10], published in 1999, refers to it as a preprint. The subsequent papers [2–7, 10] brought further contributions to the study of nonlinear PDEs on the Sierpinski gasket in the N dimensional Euclidean space.

This note aims, on the one hand, to describe the general framework that allows the study of PDEs on the Sierpinski gasket (see Sect. 2 below), and, on the other hand, to present a few results concerning the existence of multiple weak solutions of elliptic problems defined on the Sierpinski gasket. More exactly, in Sect. 5, which is based on [7], we use a method that goes back to Saint Raymond in order to prove the existence of infinitely many weak solutions of certain Dirichlet problems on the Sierpinski gasket. The subsequent Sect. 6, based on [4–6], is devoted to parameter-depending Dirichlet problems on the Sierpinski gasket. Our approach for proving the existence of finitely many (weak) solutions of one-, two-, respectively, three-parameter Dirichlet problems on the Sierpinski gasket is mainly based on recent abstract multiplicity theorems by Ricceri.

2 The Sierpinski Gasket

In what follows we first describe the two major constructions that lead to the Sierpinski gasket, and afterwards we introduce that space of real-valued functions on the Sierpinski gasket that corresponds to Sobolev spaces defined in the context of open subsets of Euclidean spaces. Having also suitable measure theoretical ingredients associated with the gasket, we then construct the weak Laplacian on it.

Notations. We denote by $\mathbb{N}$ the set of natural numbers $\{0, 1, 2, \dots\}$, by $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ the set of positive naturals, and by $|\cdot|$ the Euclidean norm on the spaces $\mathbb{R}^n$, $n \in \mathbb{N}^*$. The spaces $\mathbb{R}^n$ are endowed, throughout the paper, with the topology induced by $|\cdot|$. If X is a topological space and M a subset of it, then $\overline{M}$ denotes the closure of M.

2.1 *The Construction of the Sierpinski Gasket*

Let $N \geq 2$ be a natural number and let $p_1, \dots, p_N \in \mathbb{R}^{N-1}$ be so that $|p_i - p_j| = 1$ for $i \neq j$. Define, for every $i \in \{1, \dots, N\}$, the map $S_i : \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$ by

$$S_i(x) = \frac{1}{2}x + \frac{1}{2}p_i .$$

Obviously every S_i is a similarity with ratio $\frac{1}{2}$. Let $\mathscr{S} := \{S_1, \dots, S_N\}$ and denote by $S : \mathscr{P}(\mathbb{R}^{N-1}) \to \mathscr{P}(\mathbb{R}^{N-1})$ the map assigning to a subset A of $\mathbb{R}^{N-1}$ the set

$$S(A) = \bigcup_{i=1}^{N} S_i(A).$$

It is known (see, for example, Theorem 9.1 in [9]) that there is a unique nonempty compact subset V of $\mathbb{R}^{N-1}$, called the *attractor of the family* $\mathscr{S}$, such that $S(V) = V$ (that is, V is a fixed point of the map S). The set V is called the *Sierpinski gasket* (SG for short) in $\mathbb{R}^{N-1}$.

For every $m \in \mathbb{N}^*$ denote by $\mathfrak{W}_m := (\{1, \dots, N\})^m$. Every element $w \in \mathfrak{W}_m$ is called a *word* of *length* m. For $w = (w_1, \dots, w_m) \in \mathfrak{W}_m$ put $S_w := S_{w_1} \circ \cdots \circ S_{w_m}$. The equality $V = S(V)$ clearly yields

$$V = \bigcup_{w \in \mathfrak{W}_m} S_w(V). \tag{1}$$

Equation (1) is the *level m decomposition of* V, and each $S_w(V)$, $w \in \mathfrak{W}_m$, is called a *cell of level* m, or, for short, an m-cell.

Remark 1. Let $m \in \mathbb{N}^*$. It can be proved easily by induction that two distinct m-cells are either disjoint or intersect at a single point. In the latter case the cells are said to be *adjacent*.

There are two main constructions leading to the SG: an *outer* and an *inner* construction. Both of them are based on the convex hull C of the set $\{p_1, \dots, p_N\}$ and on the following property of C

$$S_i(C) \subseteq C, \text{ for all } i \in \{1, \dots, N\}. \quad (2)$$

The *outer* construction follows from the previously mentioned Theorem 9.1 in [9] which also implies that

$$V = \bigcap_{n \in \mathbb{N}} S^n(C), \quad (3)$$

where S^0 is the identity map of $\mathbb{R}^{N-1}$ and $S^{n+1} = S^n \circ S$, for all $n \in \mathbb{N}^*$.

In order to get the *inner* construction put

$$V_0 := \{p_1, \dots, p_N\}, \; V_{m+1} := S(V_m), \text{ for } m \in \mathbb{N}, \text{ and } V_* := \bigcup_{m \geq 0} V_m. \quad (4)$$

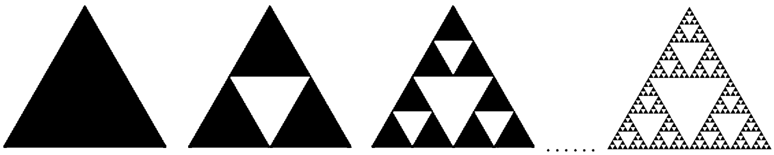
Since $p_i = S_i(p_i)$ for $i = \overline{1, N}$, the inclusion $V_0 \subseteq V_1$ holds, hence $S(V_*) = V_*$. Taking into account that the maps S_i, $i = \overline{1, N}$, are homeomorphisms, we conclude that $\overline{V_*}$ is a fixed point of S. The inclusions (2) yield that $V_m \subseteq C$ for every $m \in \mathbb{N}$, so $\overline{V_*} \subseteq C$. It follows that $\overline{V_*}$ is nonempty and compact, hence we get that

$$V = \overline{V_*}, \quad (5)$$

which yields the inner construction of the SG.

In the sequel V is considered to be endowed with the relative topology induced from the Euclidean topology on $\mathbb{R}^{N-1}$. The set V_0 is called the *intrinsic boundary* of the SG.

In the particular case $N = 2$ the SG coincides with the compact interval of $\mathbb{R}$ determined by p_1 and p_2, i.e., with C. If $N = 3$ the SG becomes the *Sierpinski triangle* (ST for short) whose construction goes back to the Polish mathematician W. Sierpinski. The ST is the subset of the plane obtained from an equilateral triangle by removing the open middle inscribed equilateral triangle of a quarter of the area, removing the corresponding open triangle from each of the three constituent triangles, and continuing this way, as it is shown in the following figure. The union of the black triangles arising in the nth step of this construction is exactly the set $S^n(C)$, $n \in \mathbb{N}$, in (3).



2.2 The Space $H_0^1(V)$

We now introduce the space $H_0^1(V)$ of certain real-valued functions on the SG that corresponds to Sobolev spaces which are defined in the context of open subsets of Euclidean spaces. The space $H_0^1(V)$ will be obtained as a subset of the space of real-valued continuous functions on V, denoted by $C(V)$. Since our interest will be mainly in functions that are zero on the intrinsic boundary of V, we consider also the set

$$C_0(V) := \{u \in C(V) \mid u|_{V_0} = 0\}.$$

Both spaces $C(V)$ and $C_0(V)$ are endowed with the usual supremum norm $||\cdot||_{\sup}$.

In order to make the presentation as clear as possible we will next describe in detail only the case $N = 3$, since the case $N \geq 4$ is a straightforward generalization of this one.

2.2.1 The Case $N = 3$

The key tool for introducing that space of functions on the ST, which plays the role of a Sobolev space in this fractal context, is the *harmonic extension procedure*, as presented in Sect. 1.3 of [37]. In order to describe this procedure, consider the sets V_m, $m \in \mathbb{N}$, defined in (4). Given $m \in \mathbb{N}$, define for every $u \colon V_m \to \mathbb{R}$

$$E_m(u) := \sum_{\substack{x,y \in V_m \\ |x-y|=2^{-m}}} (u(x) - u(y))^2. \tag{6}$$

Remark 2. The bilinear map associated with E_m is defined, for $u, v \colon V_m \to \mathbb{R}$, by

$$\mathscr{E}_m(u, v) := \sum_{\substack{x,y \in V_m \\ |x-y|=2^{-m}}} (u(x) - u(y))(v(x) - v(y)).$$

The map $\mathscr{E}_m(u, v)$ is actually an inner product on the space of real-valued functions on V_m modulo the constant functions. It follows that $\sqrt{E_m}$ is a seminorm on the space of real-valued functions on V_m.

The *harmonic extension procedure* consists in the following: Given $m \in \mathbb{N}$ and the map $u \colon V_m \to \mathbb{R}$, find a *harmonic extension* $\tilde{u} \colon V_{m+1} \to \mathbb{R}$ of u to V_{m+1}, i.e., an extension of u to V_{m+1} (hence $\tilde{u}|_{V_m} = u$) that minimizes E_{m+1} for all extensions of u to V_{m+1}. Thus, for every other extension $u' \colon V_{m+1} \to \mathbb{R}$ of u to V_{m+1}, the inequality

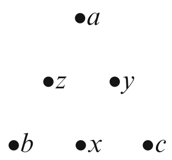
$$E_{m+1}(\tilde{u}) \leq E_{m+1}(u')$$

has to hold. We summarize in the following result the computation done in Sect. 1.3 of [37] in order to get harmonic extensions.

Theorem 1. *Let* $m \in \mathbb{N}$. *Then every* $u\colon V_m \to \mathbb{R}$ *has exactly one harmonic extension* $\tilde{u}\colon V_{m+1} \to \mathbb{R}$. *Moreover the following equality holds*

$$E_{m+1}(\tilde{u}) = \frac{3}{5} E_m(u).$$

Proof. We only sketch the proof and refer to [37] for details. The statement can be obtained using induction. The first step consists in finding the unique harmonic extension of a function $u\colon V_0 \to \mathbb{R}$. Put $a := u(p_1)$, $b := u(p_2)$, and $c := u(p_3)$. If $u'\colon V_1 \to \mathbb{R}$ is an extension of u, denote by $x := u'(S_2(p_3))$, $y := u'(S_1(p_3))$, and $z := u'(S_1(p_2))$, as shown in the figure below.



Then

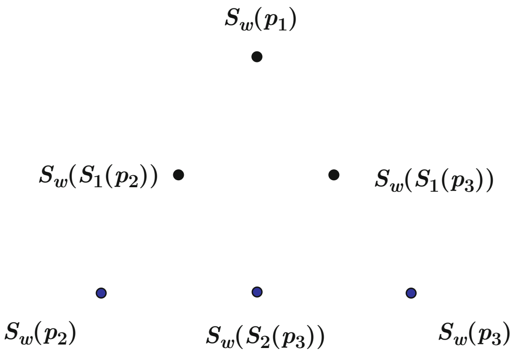
$$E_1(u') = (a-z)^2 + (z-b)^2 + (b-x)^2 + (x-c)^2 + (c-y)^2 + (y-a)^2 \\ + (z-x)^2 + (x-y)^2 + (y-z)^2.$$

To find a harmonic extension $\tilde{u}$ of u is equivalent to find a global minimum of the above quadratic function in x, y, z. One gets that there is a unique harmonic extension defined by

$$u'(S_2(p3)) = \frac{1}{5}a + \frac{2}{5}b + \frac{2}{5}c,$$
$$u'(S_1(p3)) = \frac{2}{5}a + \frac{1}{5}b + \frac{2}{5}c,$$
$$u'(S_1(p2)) = \frac{2}{5}a + \frac{2}{5}b + \frac{1}{5}c$$

and, consequently, that $E_1(\tilde{u}) = \frac{3}{5}E_0(u)$.

Consider now $m \in \mathbb{N}^*$ and $u\colon V_m \to \mathbb{R}$. If $u'\colon V_{m+1} \to \mathbb{R}$ is an extension of u, then $E_{m+1}(u')$ is the sum of contributions from each cell $S_w(V)$, $w \in \mathfrak{W}_m$. The contribution from the cell $S_w(V)$, $w \in \mathfrak{W}_m$, is obtained by considering the values of u' on the set $S_w(V_1)$, as shown in the following figure



Thus we get

$$E_{m+1}(u') = \sum_{w \in \mathfrak{W}_m} E_1(u' \circ S_w). \tag{7}$$

Observe that, for every $w \in \mathfrak{W}_m$, the cell $S_w(V)$ yields exactly three points, namely, $S_w(S_1(p_2))$, $S_w(S_1(p_3))$, and $S_w(S_2(p_3))$, for the set $V_{m+1} \setminus V_m$. Thus the problem of minimizing $E_{m+1}(u')$ can be reduced to minimize each of the 3^m terms $E_1(u' \circ S_w)$, $w \in \mathfrak{W}_m$, on the right side of the equality (7). But minimizing each of these terms is exactly a problem of the sort we have solved at the beginning of the proof. Hence we get that u has a unique harmonic extension $\tilde{u}: V_{m+1} \to \mathbb{R}$ that satisfies the equalities

$$E_1(\tilde{u} \circ S_w) = \frac{3}{5} E_0(u \circ S_w), \forall\, w \in \mathfrak{W}_m.$$

According to (7), we get

$$E_{m+1}(\tilde{u}) = \frac{3}{5} \sum_{w \in \mathfrak{W}_m} E_0(u \circ S_w) = \frac{3}{5} E_m(u),$$

which finishes the proof □

Remark 3. The proof of Theorem 1 yields in particular that harmonic extension is a linear transformation. More precisely, if $m \in \mathbb{N}$, $u, v: V_m \to \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$ then the following equality holds for the harmonic extensions of u, v, and $\alpha u + \beta v$ to V_{m+1}

$$\widetilde{(\alpha u + \beta v)} = \alpha \tilde{u} + \beta \tilde{v}.$$

Given $m \in \mathbb{N}$, we introduce now the following renormalization of the function E_m defined in (6)

$$W_m(u) = \left(\frac{3}{5}\right)^{-m} E_m(u), \text{ for every } u: V_m \to \mathbb{R}. \tag{8}$$

From Theorem 1 we now immediately derive the following result.

Corollary 1. *Let* $m \in \mathbb{N}$ *and let* $u: V_m \to \mathbb{R}$. *If* $\tilde{u}: V_{m+1} \to \mathbb{R}$ *is the harmonic extension of* u *and if* $u': V_{m+1} \to \mathbb{R}$ *is an arbitrary extension of* u *then the following relations hold*

$$W_m(u) = W_{m+1}(\tilde{u}) \leq W_{m+1}(u').$$

Recall from (4) that V_* is the union of the sets V_m, $m \in \mathbb{N}$. For a function $u: V_* \to \mathbb{R}$ consider now its restrictions $u|V_m$ to the sets V_m, $m \in \mathbb{N}$. For simplicity we denote by

$$W_m(u) := W_m\left(u|V_m\right), \forall\, m \in \mathbb{N}.$$

Corollary 1 yields then the following result.

Corollary 2. *For every* $u: V_* \to \mathbb{R}$ *the sequence* $(W_m(u))_{m\in\mathbb{N}}$ *is increasing.*

According to Corollary 2, it makes sense to define for a function $u: V_* \to \mathbb{R}$

$$W(u) := \lim_{m\to\infty} W_m(u). \tag{9}$$

Denote by

$$\operatorname{dom} W := \{u: V_* \to \mathbb{R} \mid W(u) < \infty\}.$$

Remark 4. Let $u: V_* \to \mathbb{R}$. Using the definition of W_m and Corollary 2, we get that

$$0 \leq W_m(u) \leq W(u), \forall\, m \in \mathbb{N}.$$

Thus $W(u) = 0$ if and only if u is constant.

Definition 1. Let $m \in \mathbb{N}$. A function $h: V_* \to \mathbb{R}$ is called a *harmonic function of level* m if h is obtained by specifying the values of h on V_m arbitrarily and then extending harmonically to V_k for each $k > m$. Denote by $\mathscr{H}_m$ the set of all harmonic functions of level m.

Remark 5. Let $m \in \mathbb{N}$. Remark 3 implies that $\mathscr{H}_m$ is a linear subspace of the (real) vector space of real-valued functions on V_*. Moreover, the dimension of $\mathscr{H}_m$ is card $V_m = \frac{1}{2}(3^{m+1} + 3)$. By Corollary 1 we have that

$$W(u) = W_m(u), \forall\, u \in \mathscr{H}_m,$$

thus $\mathscr{H}_m \subseteq \operatorname{dom} W$.

Remark 6. Let $m \in \mathbb{N}$. For later use we introduce the following linear subspace of $\mathscr{H}_m$

$$\mathscr{H}_m^0 := \{h \in \mathscr{H}_m \mid h|V_0 = 0\}. \tag{10}$$

It follows from Remark 5 that the dimension of $\mathscr{H}_m^0$ is $\frac{1}{2}(3^{m+1} - 3)$.

The computations on p. 19 in [37] yield the following result.

Proposition 1. *Let $u \in \operatorname{dom} W$ and $m \in \mathbb{N}$. If $x,\ y \in V_*$ belong to the same m-cell or to adjacent m-cells, then*

$$|u(x) - u(y)| \leq \frac{2r^{\frac{m}{2}}}{1 - \sqrt{r}}\sqrt{W(u)},$$

where $r = \frac{3}{5}$.

Theorem 2. *Let $u \in \operatorname{dom} W$. Then the following inequality holds*

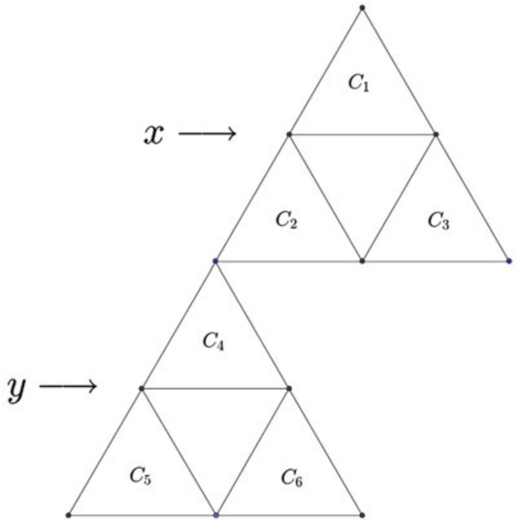
$$|u(x) - u(y)| \leq \frac{2}{r(1 - \sqrt{r})}|x - y|^{\alpha}\sqrt{W(u)},\ \forall\ x,\ y \in V_*,$$

where $r = \frac{3}{5}$ and $\alpha = \frac{\ln \frac{1}{r}}{2 \ln 2}$.

Proof. Let $x,\ y \in V_*$. Without any loss of generality we may assume that $x \neq y$. Set

$$M := \{k \in \mathbb{N}^* \mid x \text{ and } y \text{ belong to disjoint } k\text{-cells}\}.$$

Assuming that $M = \emptyset$, we get, for every $m \in \mathbb{N}^*$, that x and y belong either to the same m-cell or to adjoint m-cells. It follows that $|x-y| \leq \frac{1}{2^{m-1}}$, for all $m \in \mathbb{N}^*$, thus $x = y$, a contradiction. Hence $M \neq \emptyset$. Denote by $m := \min M$. Since two distinct 1-cells are adjacent, we conclude that $m \geq 2$. Also, due to the minimality of m, we have that $m - 1 \notin M$. Thus x and y belong to cells of level $m - 1$ with common points. We argue by contradiction to show that x and y cannot belong to the same cell of level $m - 1$. Assume that there exists $w \in \mathscr{W}_{m-1}$ such that $x,\ y \in S_w(V)$. Then each of the points x and y belongs to one of the three m-cells obtained from $S_w(V)$, namely to $S_w(S_1(V))$, $S_w(S_2(V))$, or $S_w(S_3(V))$. But every two distinct cells of these three m-cells are adjacent, contradicting the fact that $m \in M$. Hence x and y lie in adjacent cells of level $m-1$, as shown in the following figure. Denote by $C_1,\ C_2,\ C_3$ the m-cells obtained from the cell of level $m - 1$ that contains x, and by $C_4,\ C_5,\ C_6$ the m-cells obtained from the cell of level $m - 1$ that contains y.



Since x and y belong to disjoint m-cells, we consider now all possibilities that may occur: $(x, y) \in C_1 \times C_4$, $(x, y) \in C_1 \times C_5$, $(x, y) \in C_1 \times C_6$, $(x, y) \in C_2 \times C_5$, $(x, y) \in C_2 \times C_6$, $(x, y) \in C_3 \times C_4$, $(x, y) \in C_3 \times C_5$, $(x, y) \in C_3 \times C_6$. Geometric argumentations yield in all cases that for sure

$$|x - y| > \frac{1}{2^{m+1}}. \tag{11}$$

On the other hand, since x and y belong to adjacent cells of level $m-1$, Proposition 1 implies that

$$|u(x) - u(y)| \leq \frac{2r^{\frac{m-1}{2}}}{1 - \sqrt{r}} \sqrt{W(u)},$$

or, equivalently,

$$|u(x) - u(y)| \leq \frac{2r^{\frac{(m+1)}{2}}}{r(1 - \sqrt{r})} \sqrt{W(u)}. \tag{12}$$

We determine now the unique positive real α satisfying the condition

$$r^{\frac{(m+1)}{2}} = \left(\frac{1}{2^{m+1}}\right)^{\alpha} \iff \frac{m+1}{2} \ln r = \alpha(m+1) \ln \frac{1}{2} \iff \alpha = \frac{\ln \frac{1}{r}}{2 \ln 2}.$$

Since $\alpha > 0$ we thus get from (11) that

$$r^{\frac{(m+1)}{2}} = \left(\frac{1}{2^{m+1}}\right)^{\alpha} < |x - y|^{\alpha}.$$

From (12) we finally derive the inequality to be proved. $\square$

Corollary 3. *Let $u \in \text{dom } W$. Then u is uniformly continuous, thus u admits a unique continuous extension to V.*

Proof. The assertion is an immediate consequence of Theorem 2. □

According to Corollary 3, the set dom W may be viewed as a subset of $C(V)$. Define now

$$H_0^1(V) := \{u \in C_0(V) \mid u|V_* \in \text{dom } W\}.$$

If $u \in H_0^1(V)$, we write for simplicity

$$W(u) := W(u|V_*).$$

Proposition 2. *$H_0^1(V)$ is an infinite dimensional linear subspace of the real vector space $C_0(V)$.*

Proof. The constant zero function clearly belongs to $H_0^1(V)$, so $H_0^1(V) \neq \emptyset$. Let $u, v \in H_0^1(V)$. Then $u + v \in C_0(V)$ and

$$W_m(u+v) \leq 2W_m(u) + 2W_m(v), \forall\, m \in \mathbb{N},$$

where W_m is defined in (8). We obtain that $\lim_{m\to\infty} W_m(u+v) < \infty$, showing that $u + v \in H_0^1(V)$. If $t \in \mathbb{R}$ and $u \in H_0^1(V)$, then clearly $tu \in H_0^1(V)$. Thus $H_0^1(V)$ is a linear subspace of $C_0(V)$.

We have that, for every $m \in \mathbb{N}$, the continuous extensions of the functions of the space $\mathscr{H}_m^0$, introduced in (10), belong to $H_0^1(V)$. According to Remark 6, the space $H_0^1(V)$ has infinite dimension. □

Proposition 3. *Let $u, v \in H_0^1(V)$ and define for every $m \in \mathbb{N}$*

$$\mathscr{W}_m(u,v) := \left(\frac{5}{3}\right)^m \sum_{\substack{x,y\in V_m \\ |x-y|=2^{-m}}} (u(x)-u(y))(v(x)-v(y)).$$

Then the sequence $(\mathscr{W}_m(u,v))_{m\in\mathbb{N}}$ is convergent.

Proof. An easy computation yields that

$$\mathscr{W}_m(u,v) = \frac{1}{4}\left(W_m(u+v) - W_m(u-v)\right), \forall\, m \in \mathbb{N}.$$

By Proposition 2, the sequences $(W_m(u+v))_{m\in\mathbb{N}}$ and $(W_m(u-v))_{m\in\mathbb{N}}$ are convergent. Thus the assertion follows. □

According to Proposition 3 we may now define $\mathscr{W}\colon H_0^1(V) \times H_0^1(V) \to \mathbb{R}$ by

$$\mathscr{W}(u,v) = \lim_{m\to\infty} \mathscr{W}_m(u,v), \forall\, u, v \in H_0^1(V). \tag{13}$$

Theorem 3. 1° *$\mathscr{W}$ is an inner product, and the norm $||\cdot||: H_0^1(V) \to \mathbb{R}$ induced by it satisfies the equality*

$$||u|| = \sqrt{W(u)}, \forall\, u \in H_0^1(V).$$

2° *The pair $(H_0^1(V), ||\cdot||)$ is a real Hilbert space.*
3° *$H_0^1(V)$ is a dense subset of $(C_0(V), ||\cdot||_{\sup})$.*
4° *There exists a constant $c > 0$ such that every $u \in H_0^1(V)$ satisfies the inequality*

$$|u(x) - u(y)| \le c|x-y|^{\alpha}||u||, \forall\, x,\ y \in V, \tag{14}$$

where $\alpha = \frac{\ln \frac{5}{3}}{2\ln 2}$.

Proof. 1° Let $u \in H_0^1(V)$. Then, by definition,

$$\mathscr{W}_m(u,u) = W_m(u), \forall\, m \in \mathbb{N},$$

hence, according to (13),

$$\mathscr{W}(u,u) = W(u). \tag{15}$$

If $\mathscr{W}(u,u) = 0$, then, according to Remark 4, u is constant on V_*. Thus, by continuity, u is constant on V. Since u is zero on V_0, we conclude that u is the constant zero function. The other properties of an inner product follow easily. Formula (15) implies that $||\cdot||$ satisfies the asserted equality.

2° See Theorem 1.4.2 in [37].
3° See Theorem 1.4.4 in [37].
Assertion 4° follows from Theorem 2. □

Before stating the next result, we recall that a map between normed spaces is said to be *compact* if it maps bounded sets onto relatively compact sets.

Corollary 4. *There exists a constant $c > 0$ such that*

$$||u||_{\sup} \le c||u||, \forall\, u \in H_0^1(V).$$

Moreover, the embedding

$$(H_0^1(V), ||\cdot||) \hookrightarrow (C_0(V), ||\cdot||_{\sup})$$

is compact.

Proof. Let $c > 0$ be the constant in assertion 4° of Theorem 3. Let $u \in H_0^1(V)$. If we choose $y = p_1$ in (14) and keep in mind that $|x - p_1| \le 1$, for every $x \in V$, we get that $||u||_{\sup} \le c||u||$.

The compactness of the embedding follows from assertion 4° of Theorem 3 and Ascoli's Theorem. □

2.2.2 The General Case

For the case $N = 2$ we refer to Sect. 1.3 in [37] where it is treated in detail. If $N \geq 4$, then the space $H_0^1(V)$ is obtained by a straightforward generalization of the previously treated case $N = 3$. We present the main concepts and results, omitting the details.

For a function $u \in C(V)$ and for $m \in \mathbb{N}$ let

$$W_m(u) = \left(\frac{N+2}{N}\right)^m \sum_{\substack{x,y \in V_m \\ |x-y|=2^{-m}}} (u(x) - u(y))^2. \tag{16}$$

We have $W_m(u) \leq W_{m+1}(u)$ for every natural m, so we can put

$$W(u) = \lim_{m \to \infty} W_m(u). \tag{17}$$

Define now

$$H_0^1(V) := \{u \in C_0(V) \mid W(u) < \infty\}.$$

It turns out that $H_0^1(V)$ is a dense linear subspace of $(C_0(V), ||\cdot||_{\sup})$ of infinite dimension. We now endow $H_0^1(V)$ with the norm

$$||u|| = \sqrt{W(u)}. \tag{18}$$

In fact, there is an inner product defining this norm: For $u, v \in H_0^1(V)$ and $m \in \mathbb{N}$ let

$$\mathscr{W}_m(u, v) = \left(\frac{N+2}{N}\right)^m \sum_{\substack{x,y \in V_m \\ |x-y|=2^{-m}}} (u(x) - u(y))(v(x) - v(y)).$$

Put

$$\mathscr{W}(u, v) = \lim_{m \to \infty} \mathscr{W}_m(u, v). \tag{19}$$

Then $\mathscr{W}(u, v) \in \mathbb{R}$, and $H_0^1(V)$, equipped with the inner product $\mathscr{W}$ (which obviously induces the norm $||\cdot||$), becomes a real Hilbert space.

Moreover, there exists a real number $c > 0$ such that

$$||u||_{\sup} \le c||u||, \text{ for every } u \in H_0^1(V), \tag{20}$$

and the embedding

$$(H_0^1(V), ||\cdot||) \hookrightarrow (C_0(V), ||\cdot||_{\sup}) \tag{21}$$

is compact.

We now prove the analogue in the case of $H_0^1(V)$ of a property stated in [22] for Sobolev spaces.

Lemma 1. *Let* $h\colon \mathbb{R} \to \mathbb{R}$ *be a Lipschitz map with Lipschitz constant* $L \ge 0$ *and such that* $h(0) = 0$. *Then, for every* $u \in H_0^1(V)$, *the function* $h \circ u$ *belongs to* $H_0^1(V)$ *and* $||h \circ u|| \le L \cdot ||u||$.

Proof. It is clear that $h \circ u \in C_0(V)$. For every $m \in \mathbb{N}$ we have, by (16) and the Lipschitz property of h, that

$$W_m(h \circ u) \le L^2 \cdot W_m(u).$$

Hence $W(h \circ u) \le L^2 \cdot W(u)$, according to (17). Thus $h \circ u \in H_0^1(V)$ and $||h \circ u|| \le L \cdot ||u||$.

2.3 The Weak Laplacian on the SG

The idea for introducing the weak Laplacian on the SG goes back to [19]. Afterwards this notion was adopted in [10] and [12]. The advantage of the weak Laplacian is that it is obtained by means of methods of functional analysis. The main tool in this approach is a measure naturally associated with the SG.

2.3.1 A Measure Associated with the SG

Observe first that the family $\mathcal{S}$ of similarities, defined in Sect. 2.1, satisfies the open set condition (see p. 129 in [9]) with the interior of C. (Note that the interior of C is nonempty since the points $p_1, \ldots, p_N$ are affinely independent.) Applying Theorem 9.3 of [9], we then get, on the one hand, that the Hausdorff dimension d of V satisfies the equality

$$\sum_{i=1}^{N} \left(\frac{1}{2}\right)^d = 1,$$

hence $d = \frac{\ln N}{\ln 2}$, and, on the other hand, that $0 < \mathscr{H}^d(V) < \infty$, where $\mathscr{H}^d$ is the d-dimensional Hausdorff measure on $\mathbb{R}^{N-1}$. Let μ be the normalized restriction of $\mathscr{H}^d$ to the subsets of V, so $\mu(V) = 1$. In the sequel, if $p \geq 1$ is a real number, then the Lebesgue space $L^p(V, \mu)$ is considered to be equipped with the usual $||\cdot||_p$ norm.

We summarize in the next result the main properties of μ.

Proposition 4. *The measure μ has the following properties:*

1° *Every Borel set in V is μ-measurable.*
2° *The set $C(V)$ is dense in $L^2(V, \mu)$.*
3° *μ satisfies the outer regularity condition for Borel sets, i.e., the following equality holds for every Borel set $A \subseteq V$*

$$\mu(A) = \inf\{\mu(O) \mid A \subseteq O \subseteq V,\ O \text{ open in} V\}.$$

4° *The set $C_0(V)$ is dense in $L^2(V, \mu)$,*
5° *The support of μ coincides with V, i.e.,*

$$\mu(B) > 0, \text{ for every nonempty open subset } B \text{ of } V. \tag{22}$$

Proof. Assertion 1° follows, for instance, from Theorems 16 and 19 in [32].

2° Further results in [32] (e.g., Theorems 3, 20, 21, 23) yield that μ satisfies all assumptions required in Theorem 3.14 of [33] for ensuring that $C(V)$ is dense in $L^2(V, \mu)$.

Assertion 3° results from the already mentioned results (e.g., Theorems 20 and 21) in [32].

4° According to 2°, it suffices to show that $C_0(V)$ is dense in $C(V)$, when these spaces are endowed with the norm $||\cdot||_2$ of $L^2(V, \mu)$. The outer regularity condition for Borel sets mentioned at 3° implies, due to the fact that $\mu(V) < \infty$, the following inner regularity condition for Borel sets: for every Borel set $A \subseteq V$

$$\mu(A) = \sup\{\mu(F) \mid F \subseteq A,\ F \text{ closed in } V\}.$$

Applying this inner regularity condition to the set $V \setminus V_0$, we get that, for every $n \in \mathbb{N}^*$, there exists a closed set $K_n \subseteq V \setminus V_0$ such that

$$\mu(K_n) > \mu(V \setminus V_0) - \frac{1}{n^2}.$$

Since $\mu(V_0) = 0$ (by the definition of the Hausdorff measure), we obtain that

$$\mu(V \setminus K_n) < \frac{1}{n^2}.$$

By Urysohn's Lemma there exists, for $n \in \mathbb{N}^*$, a continuous function $u_n : V \to [0, 1]$ such that

$$u_n(x) = 1, \forall\, x \in V_0, \text{ and } u_n(x) = 0, \forall\, x \in K_n.$$

Thus

$$||u_n||_2^2 = \int_{V \setminus K_n} u_n^2 \mathrm{d}\mu \leq \mu(V \setminus K_n) < \frac{1}{n^2}.$$

Consider now an arbitrary $f \in C(V)$ and denote by $f_n := (1 - u_n)f$, for every $n \in \mathbb{N}^*$. Then $f_n \in C_0(V)$ and

$$||f - f_n||_2^2 = \int_V u_n^2 f^2 \mathrm{d}\mu \leq ||f||_{\sup}^2 ||u_n||_2^2 \leq \frac{||f||_{\sup}^2}{n^2}, \forall\, n \in \mathbb{N}^*.$$

It follows that the sequence (f_n) converges in the $|| \cdot ||_2$ norm to f.

5° Let B be a nonempty open subset of V and fix an arbitrary element $x \in B$. Then (see 3.1 (iii) in [14]) the equality $F(V) = V$ yields the existence of a function $\phi : \mathbb{N}^* \to \{1, \ldots, N\}$ such that x is the unique element in the intersection of the members of the following sequence of sets

$$V \supseteq V_{i_1} \supseteq V_{i_1 i_2} \supseteq \cdots \supseteq V_{i_1 i_2 \ldots i_n} \supseteq \cdots,$$

where $V_{i_1 \ldots i_n} := (S_{\phi(1)} \circ \cdots \circ S_{\phi(n)})(V)$ for every $n \in \mathbb{N}^*$. Assuming that

$$V_{i_1 \ldots i_n} \setminus B \neq \emptyset, \text{ for every } n \in \mathbb{N}^*,$$

there exists an element $x_n \in V_{i_1 \ldots i_n} \setminus B$ for every $n \in \mathbb{N}^*$. Since

$$|x_n - x| \leq \operatorname{diam} V_{i_1 \ldots i_n} = \left(\frac{1}{2}\right)^n \operatorname{diam} V, \text{ for all } n \in \mathbb{N}^*,$$

the sequence (x_n) converges to x. Thus there is an index n_0 with $x_n \in B$ for all $n \geq n_0$, a contradiction. We conclude that there is $n \in \mathbb{N}^*$ such that

$$V_{i_1 \ldots i_n} \subseteq B.$$

It follows that $\mu(V_{i_1 \ldots i_n}) \leq \mu(B)$. On the other hand, by the scaling property of the Hausdorff measure (see 2.1 in [9]), we have that

$$\mu(V_{i_1\ldots i_n}) = \left(\frac{1}{2}\right)^{nd} \cdot \mu(V) > 0,$$

so $\mu(B) > 0$.

Remark 7. Maintain the notations introduced in Sect. 2.2 for the inner product $\mathscr{W}$ defined on $H_0^1(V)$ and for the norm $||\cdot||$ induced by it. Recall that $H_0^1(V)$ is dense in $(C_0(V), ||\cdot||_{\sup})$ and that $(H_0^1(V), ||\cdot||)$ is a (real) Hilbert space. Since the embedding

$$(C_0(V), ||\cdot||_{\sup}) \hookrightarrow (L^2(V,\mu), ||\cdot||_2)$$

is continuous, assertion 4° of Proposition 4 yields that $H_0^1(V)$ is dense in $L^2(V,\mu)$. We thus conclude from Lemma 1 that $\mathscr{W}$ is a Dirichlet form on $L^2(V,\mu)$.

2.3.2 The General Framework for Defining the Weak Laplacian on the SG

Let $(X, \langle\cdot,\cdot\rangle_X)$ and $(Y, \langle\cdot,\cdot\rangle_Y)$ be real Hilbert spaces such that $Y \subseteq X$. Denote by $||\cdot||_X$, respectively, $||\cdot||_Y$ the norms induced by the inner products on X, respectively, Y. Moreover assume that Y is dense in X and that the inclusion

$$i\colon (Y, ||\cdot||_Y) \hookrightarrow (X, ||\cdot||_X)$$

is compact. Thus this inclusion is also continuous (being linear). Let $c > 0$ be so that

$$||y||_X \le c||y||_Y, \forall\ y \in Y. \tag{23}$$

It follows that, for every $x \in X$, the functional $\langle\cdot,x\rangle_X : (Y, ||\cdot||_Y) \to \mathbb{R}$ is linear and continuous. Note that by the Schwarz inequality and by (23) we have that

$$|\langle y,x\rangle_X| \le ||y||_X \cdot ||x||_X \le c||x||_X \cdot ||y||_Y, \forall\ y \in Y. \tag{24}$$

Using the Riesz representation theorem, there exists, for every $x \in X$, a unique element $\phi(x) \in Y$ such that

$$\langle y,x\rangle_X = \langle y,\phi(x)\rangle_Y\ , \forall\ y \in Y. \tag{25}$$

The map $\phi\colon (X, ||\cdot||_X) \to (Y, ||\cdot||_Y)$ is linear and, by (24), it satisfies the inequality

$$||\phi(x)||_Y \le c||x||_X, \forall\ x \in X.$$

Thus ϕ is continuous. Denote by $D := \phi(X)$. Assuming that D is not dense in Y, there exists $\bar{y} \in Y \setminus \{0\}$ such that $\langle y,\bar{y}\rangle_Y = 0$ for all $y \in D$. It follows that $\langle\phi(x),\bar{y}\rangle_Y = 0$, for all $x \in X$, thus, by (25), $\langle\bar{y},x\rangle_X = 0$, for all $x \in X$. We conclude that $\langle\bar{y},\bar{y}\rangle_X = 0$, so $\bar{y} = 0$, a contradiction. Thus D is dense in $(Y, ||\cdot||_Y)$.

Moreover, ϕ is injective, since $\phi(x_1) = \phi(x_2)$, for $x_1, x_2 \in X$, yields, by (25), that $\langle y, x_1\rangle_X = \langle y, x_2\rangle_X$, for every $y \in Y$, thus $x_1 = x_2$ by the density of Y in X.

Let $\psi: (X, ||\cdot||_X) \to (X, ||\cdot||_X)$ be the composition $\psi = i \circ \phi$. It follows from the above considerations that ψ is injective, linear, continuous, and that its image $\psi(X) = D$ is dense in $(X, ||\cdot||_X)$. Furthermore, since the inclusion i is compact, ψ is also compact. Using (25), the following equalities hold for every $u, v \in X$

$$\langle \psi(u), v\rangle_X = \langle \phi(u), \phi(v)\rangle_Y = \langle \phi(v), \phi(u)\rangle_Y = \langle \psi(v), u\rangle_X\,,$$

hence ψ is symmetric. We conclude that the inverse $\psi^{-1}: D \to X$ is self-adjoint and that, according to (25), it satisfies the equality

$$\langle y, \bar{x}\rangle_Y = \left\langle y, \psi^{-1}(\bar{x})\right\rangle_X\,, \forall\ (\bar{x}, y) \in D \times Y. \tag{26}$$

Remark 8. Assuming that X has infinite dimension, Theorem 19.B of [39], applied to $\psi: (X, ||\cdot||_X) \to (X, ||\cdot||_X)$, yields that X has a complete countable orthonormal system consisting of eigenvectors of ψ. Thus there exists a countable and dense subset M of X. It follows that $\phi(M)$ is a countable and dense subset of $(Y, ||\cdot||_Y)$, hence $(Y, ||\cdot||_Y)$ is separable.

We get now the weak Laplacian on the SG, by considering the particular case

$$(X, \langle\cdot,\cdot\rangle_X) = (L^2(V, \mu), \langle\cdot,\cdot\rangle_2),$$

where $\langle\cdot,\cdot\rangle_2$ is the inner product that induces the norm $||\cdot||_2$ on $L^2(V, \mu)$, and

$$(Y, \langle\cdot,\cdot\rangle_Y) = (H_0^1(V), \mathscr{W}),$$

where $\mathscr{W}$ is the inner product defined in (19). Recall that $||\cdot||$ stands for the norm induced by $\mathscr{W}$. We know from (21) that the embedding

$$(H_0^1(V), ||\cdot||) \hookrightarrow (C_0(V), ||\cdot||_{\sup})$$

is compact. In addition, the embedding

$$(C_0(V), ||\cdot||_{\sup}) \hookrightarrow (L^2(V, \mu), ||\cdot||_2)$$

is continuous, thus the embedding

$$(H_0^1(V), ||\cdot||) \hookrightarrow (L^2(V, \mu), ||\cdot||_2)$$

is compact. We recall from Remark 7 that $H_0^1(V)$ is dense in $L^2(V, \mu)$.

The map $\phi: (L^2(V, \mu), ||\cdot||_2) \to (H_0^1(V), ||\cdot||)$, defined according to (25), satisfies in this case the equality

$$\int_V uv \mathrm{d}\mu = \mathscr{W}(u, \phi(v)), \forall\ (u, v) \in H_0^1(V) \times L^2(V, \mu).$$

The set $D := \phi(L^2(V, \mu))$ is dense both in $(H_0^1(V), ||\cdot||)$ and in $(L^2(V, \mu), ||\cdot||_2)$. Let $\Delta := -\psi^{-1}: D \to L^2(V, \mu)$. Then Δ is linear, self-adjoint, and satisfies, by (26), the equality

$$-\mathscr{W}(u, v) = \int_V \Delta(u) \cdot v \mathrm{d}\mu, \forall\ (u, v) \in D \times H_0^1(V). \tag{27}$$

The operator Δ is called the *weak Laplacian* on the SG.

Remark 9. Since $L^2(V, \mu)$ has infinite dimension, Remark 8 implies that the Hilbert space $(H_0^1(V), ||\cdot||)$ is separable.

3 Some Basic Facts About Derivatives

The results of this section will be used in the next one for determining the energy functional of certain elliptic problems defined on the SG. For the beginning we recall a few basic notions.

Definition 2. Let E be a real Banach space, E^* its dual (endowed with the usual norm), and let $T: E \to \mathbb{R}$ be a functional.

We say that T is *Fréchet differentiable at $u \in E$* if there exists a continuous linear functional $T'(u): E \to \mathbb{R}$, called the *Fréchet differential of T at u*, such that

$$\lim_{v \to 0} \frac{|T(u+v) - T(u) - T'(u)(v)|}{||v||} = 0.$$

The functional T is *Fréchet differentiable (on E)* if T is Fréchet differentiable at every point $u \in E$. In this case the map $T': E \to E^*$, assigning to each point $u \in E$ the Fréchet differential of T at u, is called the *Fréchet derivative* of T on E. If $T': E \to E^*$ is continuous, then T is called a *C^1-functional.*

We next establish a few straightforward results in a general setting. In the subsequent part of the paper the term *differentiable* means *Fréchet differentiable* and *derivative* means *Fréchet derivative*.

Let $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ be real normed spaces such that $Y \subseteq X$ and such that the inclusion

$$i: (Y, ||\cdot||_Y) \hookrightarrow (X, ||\cdot||_X) \tag{28}$$

is continuous. Define $i^*: X^* \to Y^*$ by $i^*(x^*) = x^* \circ i$. The continuity of i implies that of i^*. The following result is now an immediate consequence of the chain rule.

Proposition 5. *Let $y \in Y$ and assume that the map $L: X \to \mathbb{R}$ is differentiable at y. Then the restriction $\ell: Y \to \mathbb{R}$ of L to Y is differentiable at y and $\ell'(y) = i^*(L'(y))$.*

Corollary 5. *Let $L: X \to \mathbb{R}$ be differentiable on X. Then the restriction $\ell: Y \to \mathbb{R}$ of L to Y is differentiable on Y and its derivative is given by $\ell' = i^* \circ L' \circ i$.*

Corollary 6. *Let $L: X \to \mathbb{R}$ be a C^1-functional. Then the restriction $\ell: Y \to \mathbb{R}$ of L to Y is also C^1.*

Proof. We know from Corollary 5 that $\ell' = i^* \circ L' \circ i$. The continuity of i^*, L', and i implies that of ℓ'. Hence ℓ is C^1. □

Corollary 7. *Assume that the inclusion from (28) is compact. If $L: X \to \mathbb{R}$ is a C^1-functional, then the restriction $\ell: Y \to \mathbb{R}$ of L to Y is a C^1-functional with compact derivative.*

Proof. That ℓ is C^1 follows from Corollary 6. The compactness of ℓ' is a consequence of Corollary 5, since the composition of a continuous map ($i^* \circ L'$ in our case) with a compact one (i in our case) is compact. □

We will apply the above results, by taking $(X, ||\cdot||_X) = (C_0(V), ||\cdot||_{\sup})$ and $(Y, ||\cdot||_Y) = (H_0^1(V), ||\cdot||)$.

Proposition 6. *Let $\gamma \in L^1(V, \mu)$, let $F: \mathbb{R} \to \mathbb{R}$ be a C^1-function with derivative $f: \mathbb{R} \to \mathbb{R}$, and define the map $L: C_0(V) \to \mathbb{R}$ by*

$$L(u) = \int_V \gamma(x) F(u(x)) \mathrm{d}\mu. \tag{29}$$

Then L is a C^1-functional and its differential at a point $u \in C_0(V)$ is given by

$$L'(u)(v) = \int_V \gamma(x) f(u(x)) v(x) \mathrm{d}\mu, \forall\, v \in C_0(V). \tag{30}$$

Proof. Let $u \in C_0(V)$ be arbitrary and denote by $a := ||u||_{\sup}$. The functional

$$v \in C_0(V) \mapsto \int_V \gamma(x) f(u(x)) v(x) \mathrm{d}\mu \in \mathbb{R}$$

is obviously linear and continuous. Consider an arbitrary $\varepsilon > 0$. Since the restriction of f to $[-a-1, a+1]$ is uniformly continuous, there exists $\delta \in]0,1[$ such that

$$|f(s) - f(t)| < \frac{\varepsilon}{||\gamma||_1 + 1}, \forall\, s, t \in [-a-1, a+1] \text{ with } |s-t| < \delta. \tag{31}$$

Pick $v \in C_0(V)$ so that $||v||_{\sup} < \delta$, and let $x \in V$ be arbitrary. According to the mean value theorem, there exists $t_x \in [0,1]$ such that

$$F(u(x) + v(x)) - F(u(x)) = f(u(x) + t_x v(x)) v(x).$$

Then

$$F(u(x)+v(x))-F(u(x))-f(u(x))v(x)=(f(u(x)+t_x v(x))-f(u(x)))\,v(x).$$

Since $u(x)$ and $u(x)+t_x v(x)$ belong to $[-a-1, a+1]$ and since $|t_x v(x)|<\delta$, inequality (31) yields that

$$|F(u(x)+v(x))-F(u(x))-f(u(x))v(x)| \le \frac{\varepsilon|v(x)|}{||\gamma||_1+1} \le \frac{\varepsilon||v||_{\sup}}{||\gamma||_1+1}.$$

Since

$$\left|L(u+v)-L(u)-\int_V \gamma(x) f(u(x))v(x)\mathrm{d}\mu\right|$$
$$\le \int_V |\gamma(x)\,(F(u(x)+v(x))-F(u(x))-f(u(x))v(x))|\,\mathrm{d}\mu,$$

we thus get

$$\left|L(u+v)-L(u)-\int_V \gamma(x) f(u(x))v(x)\mathrm{d}\mu\right| \le \frac{\varepsilon||v||_{\sup}||\gamma||_1}{||\gamma||_1+1}.$$

We conclude that

$$\frac{\left|L(u+v)-L(u)-\int_V \gamma(x) f(u(x))v(x)\mathrm{d}\mu\right|}{||v||_{\sup}} < \varepsilon, \forall v \in C_0(V)\setminus\{0\} \text{ with } ||v||_{\sup}<\delta.$$

It follows that L is differentiable at u and that its differential at this point is given by formula (30).

We now prove that the derivative $L'\colon C_0(V) \to (C_0(V))^*$ is continuous. To this end let $u \in C_0(V)$ be arbitrary and denote, as before, by $a := ||u||_{\sup}$. Consider an arbitrary $\varepsilon > 0$ and let $\delta \in]0,1[$ be so that (31) holds. Pick $w \in C_0(V)$ such that $||w-u||_{\sup} < \delta$. For every $x \in V$ we then have that $u(x)$, $w(x) \in [-a-1, a+1]$ and $|w(x)-u(x)| < \delta$. Thus, by (31), we get for $v \in C_0(V)$ with $||v||_{\sup} \le 1$ that

$$\left|L'(w)(v)-L'(u)(v)\right| \le \int_V |\gamma(x)\,(f(w(x))-f(u(x)))\,v(x)|\,\mathrm{d}\mu \le \frac{\varepsilon||\gamma||_1}{||\gamma||_1+1} < \varepsilon.$$

Hence

$$||L'(w)-L'(u)|| \le \varepsilon, \forall\, w \in C_0(V) \text{ with } ||w-u||_{\sup} < \delta.$$

We conclude that L' is continuous, thus L is C^1. □

Corollary 8. *Let $\gamma \in L^1(V, \mu)$, let $F\colon \mathbb{R} \to \mathbb{R}$ be a C^1-function with derivative $f\colon \mathbb{R} \to \mathbb{R}$, and define the map $\ell\colon H_0^1(V) \to \mathbb{R}$ by*

$$\ell(u) = \int_V \gamma(x) F(u(x)) \mathrm{d}\mu.$$

Then ℓ is a C^1-functional with compact derivative and its differential at a point $u \in H_0^1(V)$ is given by

$$\ell'(u)(v) = \int_V \gamma(x) f(u(x)) v(x) \mathrm{d}\mu, \forall\ v \in H_0^1(V).$$

Proof. The map ℓ is the restriction to $H_0^1(V)$ of the map L defined in (29). We know from (21) that the embedding $(H_0^1(V), ||\cdot||) \hookrightarrow (C_0(V), ||\cdot||_{\sup})$ is compact. The assertion follows now from Proposition 6, Corollaries 5, and 7. □

Definition 3. If $f\colon V \times \mathbb{R} \to \mathbb{R}$ is such that the partial map $f(x, \cdot)\colon \mathbb{R} \to \mathbb{R}$ is continuous for every $x \in V$, then we call the map $F\colon V \times \mathbb{R} \to \mathbb{R}$, defined by

$$F(x, t) = \int_0^t f(x, \xi) \mathrm{d}\xi, \text{ for every } (x, t) \in V \times \mathbb{R}, \tag{32}$$

the *antiderivative of f with respect to the second variable*.

Remark 10. Using the uniform continuity of continuous maps defined on compact metric spaces, it can be proved that if $f\colon V \times \mathbb{R} \to \mathbb{R}$ is continuous, then its antiderivative with respect to the second variable is also continuous.

Involving similar arguments as in the proof of Proposition 6 (based on the mean value theorem and on the uniform continuity of continuous maps defined on compact metric spaces), one can prove the following result.

Proposition 7. *Let $f\colon V \times \mathbb{R} \to \mathbb{R}$ be continuous and let $F\colon V \times \mathbb{R} \to \mathbb{R}$ be its antiderivative with respect to the second variable. Then the map $L\colon C_0(V) \to \mathbb{R}$, given by*

$$L(u) = \int_V F(x, u(x)) \mathrm{d}\mu,$$

is a C^1-functional and its differential at a point $u \in C_0(V)$ is defined by

$$L'(u)(v) = \int_V f(x, u(x)) v(x) \mathrm{d}\mu, \forall\ v \in C_0(V).$$

Corollary 9. *Let $f\colon V \times \mathbb{R} \to \mathbb{R}$ be continuous, let $F\colon V \times \mathbb{R} \to \mathbb{R}$ be its antiderivative with respect to the second variable, and define the map $\ell\colon H_0^1(V) \to \mathbb{R}$ by*

$$\ell(u) = \int_V F(x, u(x))\mathrm{d}\mu.$$

Then ℓ is a C^1-functional with compact derivative and its differential at a point $u \in H_0^1(V)$ is given by

$$\ell'(u)(v) = \int_V f(x, u(x))v(x)\mathrm{d}\mu, \forall\ v \in H_0^1(V).$$

Proof. The statement follows from Proposition 7, using similar arguments as in the proof of Corollary 8. □

4 Dirichlet Problems on the SG

Having defined the weak Laplacian, satisfying condition (27), we proceed now to formulate nonlinear elliptic problems with zero Dirichlet boundary condition on the SG. For this denote by $\mathscr{DP}(V)$ the set of all functions $B\colon V \times \mathbb{R} \to \mathbb{R}$ having the property that, for every $u \in H_0^1(V)$, the map

$$x \in V \mapsto B(x, u(x)) \in \mathbb{R}$$

belongs to $L^1(V, \mu)$.

Given $B \in \mathscr{DP}(V)$, find appropriate functions $u\colon V \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u(x) + B(x, u(x)) = 0,\ \forall\ x \in V \setminus V_0, \\ u|_{V_0} = 0. \end{cases}$$

We call this problem a *Dirichlet problem* with zero boundary condition on the SG. We are interested in *weak solutions* of it, i.e., in functions $u \in H_0^1(V)$ with the property that

$$\mathscr{W}(u, v) + \int_V B(x, u(x))v(x)\mathrm{d}\mu = 0,\ \forall\ v \in H_0^1(V).$$

Remark 11. Assume that $B\colon V \times \mathbb{R} \to \mathbb{R}$ is continuous. Using the regularity result Lemma 2.16 of [10], it follows that every weak solution of the Dirichlet problem defined above is actually a strong solution of it (as defined in [10]). That is the reason for calling in this case weak solutions of the Dirichlet problem simply *solutions* of it.

In the present note we are mainly concerned with Dirichlet problems defined with the aid of a function $B\colon V \times \mathbb{R} \to \mathbb{R}$ such that

$$B(x, t) = f_1(x, t) + \gamma(x) f_2(t),$$

where $f_1: V \times \mathbb{R} \to \mathbb{R}$ and $f_2: \mathbb{R} \to \mathbb{R}$ are continuous, and $\gamma \in L^1(V, \mu)$. Thus in this case the Dirichlet problem becomes

$$\text{(DP)} \begin{cases} -\Delta u(x) + f_1(x, u(x)) + \gamma(x) f_2(u(x)) = 0, \ \forall\, x \in V \setminus V_0, \\ u|_{V_0} = 0. \end{cases}$$

A function $u \in H_0^1(V)$ is a weak solution of (DP) if and only if

$$\mathscr{W}(u, v) + \int_V f_1(x, u(x)) v(x) \mathrm{d}\mu + \int_V \gamma(x) f_2(u(x)) v(x) \mathrm{d}\mu = 0, \ \forall\, v \in H_0^1(V). \tag{33}$$

In order to apply variational methods for the study of the existence and the multiplicity of weak solutions of problem (DP) we have to introduce the so-called *energy functional* of this problem. For this we recall the following notions.

Definition 4. Let E be a real Banach space and let $T: E \to \mathbb{R}$ be a functional. If T is differentiable on E, then a point $u \in E$ is a *critical point of* T if $T'(u) = 0$.

Remark 12. Note that if the differentiable functional $T: E \to \mathbb{R}$ has in $u \in E$ a local extremum, then u is a critical point of T.

Definition 5. A differentiable functional $T: H_0^1(V) \to \mathbb{R}$ is called an *energy functional* of problem (DP) if it has the property that $u \in H_0^1(V)$ is a weak solution of problem (DP) if and only if u is a critical point of T.

For $F \in \mathscr{D}\mathscr{P}(V)$ we denote by $T_F: H_0^1(V) \to \mathbb{R}$ the functional defined by

$$T_F(u) = \int_V F(x, u(x)) \mathrm{d}\mu. \tag{34}$$

Coming now back to problem (DP) and keeping in mind Definition 3, denote by $F_1: V \times \mathbb{R} \to \mathbb{R}$ the antiderivative of the continuous function $f_1: V \times \mathbb{R} \to \mathbb{R}$ with respect to the second variable, and by $F_2: V \times \mathbb{R} \to \mathbb{R}$ the antiderivative of the function $(x, t) \in V \times \mathbb{R} \mapsto \gamma(x) f_2(x) \in \mathbb{R}$ with respect to the second variable. Thus

$$F_2(x, t) = \gamma(x) \int_0^t f_2(\xi) \mathrm{d}\xi, \forall\, (x, t) \in V \times \mathbb{R}.$$

The maps F_1 and F_2 belong to $\mathscr{D}\mathscr{P}(V)$. Denote the functionals T_{F_1} and T_{F_2} associated with F_1, respectively, F_2, according to (34), simply by

$$T_1 := T_{F_1} \text{ and } T_2 := T_{F_2}.$$

Furthermore define $I\colon H_0^1(V) \to \mathbb{R}$ by

$$I(u) = \frac{1}{2}||u||^2, \tag{35}$$

where, as until now, $||\cdot||$ denotes the norm on $H_0^1(V)$ induced by the inner product $\mathscr{W}$ given in (19).

Proposition 8. *The above defined maps* $I,\ T_1,\ T_2\colon H_0^1(V) \to \mathbb{R}$ *have the following properties:*

1° I *is a* C^1*-functional and its differential at an arbitrary point* $u \in H_0^1(V)$ *is given by*

$$I'(u)(v) = \mathscr{W}(u, v), \forall\ v \in H_0^1(V).$$

2° T_1 *is a* C^1*-functional with compact derivative and its differential at an arbitrary point* $u \in H_0^1(V)$ *is given by*

$$T_1'(u)(v) = \int_V f_1(x, u(x))v(x)\mathrm{d}\mu, \forall\ v \in H_0^1(V).$$

3° T_2 *is a* C^1*-functional with compact derivative and its differential at an arbitrary point* $u \in H_0^1(V)$ *is given by*

$$T_2'(u)(v) = \int_V \gamma(x) f_2(u(x))v(x)\mathrm{d}\mu, \forall\ v \in H_0^1(V).$$

4° $I + T_1 + T_2$ *is an energy functional of problem* (DP).
5° T_1 *and* T_2 *are sequentially weakly continuous.*
6° $I + T_1 + T_2$ *is sequentially weakly lower semicontinuous.*

Proof. A straightforward computation yields 1°.

While assertion 2° is a consequence of Corollary 9, assertion 3° follows from Corollary 8.

Assertion 4° follows from the previous ones and (33).

5° Since T_1 and T_2 have compact derivative, Corollary 41.9 in [38] implies that these functionals are sequentially weakly continuous.

6° Since I is continuous in the norm topology on $H_0^1(V)$ and convex, it is sequentially weakly lower semicontinuous. Using 5°, we conclude that $I + T_1 + T_2$ is sequentially weakly lower semicontinuous. □

In the next sections, we will study by means of various techniques the existence of multiple weak solutions of certain Dirichlet problems on the SG.

5 Dirichlet Problems on the SG with Infinitely Many Weak Solutions

Following [7], we present now the application of a method that goes back to Saint Raymond [34] in order to prove the existence of infinitely many weak solutions of certain Dirichlet problems on the SG. This method has been used successfully to prove, in the context of certain Sobolev spaces, the existence of infinitely many solutions of Dirichlet problems on bounded domains [34], of one-dimensional scalar field equations and systems [11], of homogeneous Neumann problems [20]. The aim of the present section is to show that the methods used in [11] can be successfully adapted to prove the existence of infinitely many (weak) solutions of Dirichlet problems on the SG.

We point out that there are also other approaches for proving the existence of infinitely many weak solutions of Dirichlet problems on the SG. For instance, in [2] and [3] one uses a general variational principle by Ricceri to prove the existence of infinitely many solutions of other classes of Dirichlet problems on the Sierpinski gasket than those treated in the previously mentioned article [7].

Let $a, g \in L^1(V, \mu)$ and let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous. We are concerned with the following Dirichlet problem on the SG

$$(P)\begin{cases} \Delta u(x) + a(x)u(x) = g(x)f(u(x)),\ \forall\, x \in V \setminus V_0, \\ u|_{V_0} = 0. \end{cases}$$

Define $F\colon \mathbb{R} \to \mathbb{R}$ by $F(t) = \int_0^t f(\xi)\mathrm{d}\xi$. According to assertion 4° of Proposition 8 the functional $T\colon H_0^1(V) \to \mathbb{R}$, defined by

$$T(u) = \frac{1}{2}||u||^2 - \frac{1}{2}\int_V a(x)u^2(x)\mathrm{d}\mu + \int_V g(x)F(u(x))\mathrm{d}\mu,\ \forall\, u \in H_0^1(V), \tag{36}$$

is an energy functional of problem (P).

Remark 13. Assume that $a \le 0$ and $g \le 0$ a.e. in V. Consider $u \in H_0^1(V)$ and $d, b \in \mathbb{R}$ such that $d \le u(x) \le b$ for every $x \in V$. According to the fact that $g \le 0$ a.e. in V, we then have

$$\int_V g(x)F(u(x))\mathrm{d}\mu \ge \max_{s\in[d,b]} F(s) \cdot \int_V g(x)\mathrm{d}\mu. \tag{37}$$

For later use we state the following relations about the functional $T\colon H_0^1(V) \to \mathbb{R}$ defined by (36): The inequalities (37) and $a \le 0$ a.e. in V imply that

$$T(u) \ge \max_{s\in[d,b]} F(s) \cdot \int_V g(x)\mathrm{d}\mu \tag{38}$$

and

$$\frac{1}{2}||u||^2 \leq T(u) - \max_{s\in[d,b]} F(s) \cdot \int_V g(x)\mathrm{d}\mu. \tag{39}$$

We also note that for every $x \in V$

$$F(u(x)) \leq |F(u(x))| = \left| \int_0^{u(x)} f(t)\mathrm{d}t \right| \leq \max_{t\in[d,b]} |f(t)| \cdot ||u||_{\sup}.$$

As above we then conclude that

$$T(u) \geq \max_{t\in[d,b]} |f(t)| \cdot ||u||_{\sup} \cdot \int_V g(x)\mathrm{d}\mu \tag{40}$$

and

$$\frac{1}{2}||u||^2 \leq T(u) - \max_{t\in[d,b]} |f(t)| \cdot ||u||_{\sup} \cdot \int_V g(x)\mathrm{d}\mu. \tag{41}$$

We recall the definition of the coercivity of a functional, respectively, the subsequent standard result concerning the existence of minimum points of sequentially weakly lower semicontinuous functionals.

Definition 6. Let X be a real normed space and let M be a nonempty subset of X. A functional $L\colon M \to \mathbb{R}$ is said to be *coercive* if for every sequence (x_n) in M such that $\lim_{n\to\infty} ||x_n|| = \infty$ it follows that $\lim_{n\to\infty} L(x_n) = \infty$.

Proposition 9. *Let X be a reflexive real Banach space, M a nonempty sequentially weakly closed subset of X, and $L\colon M \to \mathbb{R}$ a sequentially weakly lower semicontinuous and coercive functional. Then L possesses at least one minimum point.*

The following corollary of Proposition 9 is a key element in our approach.

Corollary 10. *Let a, $g \in L^1(V, \mu)$ be so that $a \leq 0$ and $g \leq 0$ a.e. in V, and let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous. Consider $d, b \in \mathbb{R}$ so that $d < 0 < b$ and put*

$$M := \{u \in H_0^1(V) \mid d \leq u(x) \leq b,\ \forall x \in V\}.$$

Then the functional $T\colon H_0^1(V) \to \mathbb{R}$ defined by (36) attains its infimum on M, thus it is bounded from below on M.

Proof. Obviously the set M is nonempty (it contains the constant 0 function) and convex. Since the inclusion (21) is continuous, M is closed in the norm topology on $H_0^1(V)$. It follows that M is also closed in the weak topology on $H_0^1(V)$, thus M is sequentially weakly closed. It follows from (39) that the restriction of T to M is coercive. Proposition 9 implies now that T attains its infimum on M. □

The main result of this section is contained in the following theorem concerning the existence of multiple weak solutions of problem (P).

Theorem 4. *Assume that the following conditions hold:*

(C1) $a \in L^1(V,\mu)$ *and* $a \leq 0$ *a.e. in* V.
(C2) $f:\mathbb{R} \to \mathbb{R}$ *is continuous such that*

(1*) *There exist two sequences* (a_k) *and* (b_k) *in* $]0,\infty[$ *with* $b_{k+1} < a_k < b_k$, $\lim_{k\to\infty} b_k = 0$ *and such that* $f(s) \leq 0$ *for every* $s \in [a_k, b_k]$.
(2*) *Either* $\sup\{s < 0 \mid f(s) > 0\} = 0$, *or there is a* $\delta > 0$ *with* $f|_{[-\delta,0]} = 0$.

(C3) $F:\mathbb{R} \to \mathbb{R}$, *defined by* $F(s) = \int_0^s f(t)\mathrm{d}t$, *is such that*

(3*) $-\infty < \liminf_{s\to 0^+} \frac{F(s)}{s^2}$,
(4*) $\limsup_{s\to 0^+} \frac{F(s)}{s^2} = \infty$.

(C4) $g: V \to \mathbb{R}$ *is continuous with* $g \leq 0$ *and such that the restriction of* g *to every open subset of* V *is not identically 0.*

Then there is a sequence (u_k) *of pairwise distinct weak solutions of problem* (P) *such that* $\lim_{k\to\infty} \|u_k\| = 0$. *In particular,* $\lim_{k\to\infty} \|u_k\|_{\sup} = 0$.

Remark 14. The conditions (C2) and (C3) of Theorem 4 concerning the nonlinear term $f:\mathbb{R} \to \mathbb{R}$ of problem (P) show that this function has an oscillating behavior at 0. This oscillating behavior of f is the key element for applying Saint Raymond's method. In Example 2 of [11] there is given the following example of a function satisfying the conditions (C2) and (C3) of the theorem: Let $0 < \alpha < 1 < \beta$ and define $f:\mathbb{R} \to \mathbb{R}$ by $f(0) = 0$ and $f(t) = |t|^\alpha \max\{0, \sin|t|^{-1}\} + |t|^\beta \min\{0, \sin|t|^{-1}\}$ for $t \neq 0$.

In what follows we assume that the conditions (C1)–(C4) in the hypotheses of Theorem 4 are satisfied.

Case 1: Suppose first that the equality $\sup\{s < 0 \mid f(s) > 0\} = 0$ in condition (2*) of (C2) holds. Then there exists a strictly increasing sequence (c_k) of negative reals such that $\lim c_k = 0$ and $f(c_k) > 0$ for every natural k. By continuity of f there exists another sequence (d_k) such that $d_k < c_k < d_{k+1}$ and $f(t) > 0$ for every $t \in [d_k, c_k]$ and every natural k.
Case 2: If we have in (2*) that there is a $\delta > 0$ with $f|_{[-\delta,0]} = 0$, then choose a strictly increasing sequence (c_k) of negative reals strictly greater than $-\delta$ such that $\lim c_k = 0$. Let (d_k) be a sequence such that $-\delta < d_0$ and $d_k < c_k < d_{k+1}$ for every natural k. Then $f(t) = 0$ for every $t \in [d_k, c_k]$ and every natural k.

In both cases, since $F(s) = \int_0^s f(t)\mathrm{d}t$ for every $s \in \mathbb{R}$, it follows that

$$F(s) \le F(c_k), \text{ for every } s \in [d_k, c_k]. \tag{42}$$

Using condition (1*) of (C2), we have that

$$F(s) \le F(a_k), \text{ for every } s \in [a_k, b_k]. \tag{43}$$

For every $k \in \mathbb{N}$ set now

$$M_k := \{u \in H_0^1(V) \mid d_k \le u(x) \le b_k, \ \forall\, x \in V\}.$$

The proof of Theorem 4 includes the following main steps contained in the next lemmas:

1. we show that the map $T\colon H_0^1(V) \to \mathbb{R}$ defined by (36) has at least one critical point in each of the sets M_k,
2. we show that there are infinitely many pairwise distinct such critical points,
3. since T is an energy functional of Problem (P), each of these critical points is a weak solution of Problem (P).

Lemma 2. *For every $k \in \mathbb{N}$ there is an element $u_k \in M_k$ such that the following conditions hold:*

(i) $T(u_k) = \inf T(M_k)$,
(ii) $c_k \le u_k(x) \le a_k$, *for every* $x \in V$.

Proof. Fix $k \in \mathbb{N}$. According to Corollary 10, there is an element $\tilde{u}_k \in M_k$ such that $T(\tilde{u}_k) = \inf T(M_k)$. Define $h\colon \mathbb{R} \to \mathbb{R}$ by

$$h(t) = \begin{cases} c_k, \ t < c_k \\ t, \ t \in [c_k, a_k] \\ a_k, \ t > a_k. \end{cases}$$

Note that $h(0) = 0$ and that h is a Lipschitz map with Lipschitz constant $L = 1$. According to Lemma 1 the map $u_k := h \circ \tilde{u}_k$ belongs to $H_0^1(V)$ and

$$||u_k|| \le ||\tilde{u}_k||. \tag{44}$$

Moreover, u_k belongs to M_k and obviously satisfies condition (ii) to be proved. We next show that (i) also holds. For this set

$$V_1 := \{x \in V \mid \tilde{u}_k(x) < c_k\}, \quad V_2 := \{x \in V \mid \tilde{u}_k(x) > a_k\}.$$

Then

$$u_k(x) = \begin{cases} c_k, \; x \in V_1 \\ \tilde{u}_k(x), \; x \in V \setminus (V_1 \cup V_2) \\ a_k, \; x \in V_2. \end{cases}$$

It follows that

$$(u_k(x))^2 \leq (\tilde{u}_k(x))^2, \text{ for every } x \in V. \tag{45}$$

Furthermore, if $x \in V_1$ then $\tilde{u}_k(x) \in [d_k, c_k[$, hence $F(\tilde{u}_k(x)) \leq F(c_k) = F(u_k(x))$, by (42). Analogously, if $x \in V_2$, then (43) yields $F(\tilde{u}_k(x)) \leq F(a_k) = F(u_k(x))$. Thus

$$F(\tilde{u}_k(x)) \leq F(u_k(x)), \text{ for every } x \in V. \tag{46}$$

The inequalities (44)–(46) imply, together with the fact that $a \leq 0$ a.e. in V and $g \leq 0$, that

$$\begin{aligned} T(\tilde{u}_k) - T(u_k) = \frac{1}{2}||\tilde{u}_k||^2 - \frac{1}{2}||u_k||^2 - \frac{1}{2}\int_V a(x)(\tilde{u}_k^2(x) - u_k^2(x))\mathrm{d}\mu \\ + \int_V g(x)(F(\tilde{u}_k(x)) - F(u_k(x)))\mathrm{d}\mu \geq 0. \end{aligned}$$

Thus $T(\tilde{u}_k) \geq T(u_k)$. Since $T(\tilde{u}_k) = \inf T(M_k)$ and since $u_k \in M_k$, we conclude that $T(u_k) = \inf T(M_k)$, thus (i) is also fulfilled. □

Lemma 3. *For every $k \in \mathbb{N}$ let $u_k \in M_k$ be a function satisfying the conditions* (i) *and* (ii) *of Lemma 2. The functional T has then in u_k a local minimum (with respect to the norm topology on $H_0^1(V)$), for every $k \in \mathbb{N}$. In particular,* (u_k) *is a sequence of weak solutions of problem* (P).

Proof. Fix $k \in \mathbb{N}$. Suppose to the contrary that u_k is not a local minimum of T. This implies the existence of a sequence (w_n) in $H_0^1(V)$ converging to u_k in the norm topology such that

$$T(w_n) < T(u_k), \text{ for every } n \in \mathbb{N}.$$

In particular, $w_n \notin M_k$, for all $n \in \mathbb{N}$. Choose a real number ε such that

$$0 < \varepsilon \leq \frac{1}{2}\min\{b_k - a_k, c_k - d_k\}.$$

In view of (20) the sequence (w_n) converges to u_k in the supremum norm topology on $C(V)$. Hence there is an index $m \in \mathbb{N}$ such that

$$||w_m - u_k||_{\sup} \leq \varepsilon.$$

For every $x \in V$ we then have according to (ii) of Lemma 2

$$w_m(x) = w_m(x) - u_k(x) + u_k(x) \leq \varepsilon + u_k(x) \leq \frac{b_k - a_k}{2} + a_k < b_k$$

and

$$w_m(x) = w_m(x) - u_k(x) + u_k(x) \geq -\varepsilon + u_k(x) \geq \frac{d_k - c_k}{2} + c_k > d_k.$$

Thus $w_m \in M_k$, a contradiction. We conclude that T has in u_k a local minimum. The last assertion of the lemma follows now from Remark 12 and the fact that T is an energy functional of problem (P). □

Lemma 4. *For every $k \in \mathbb{N}$ put $\gamma_k := \inf T(M_k)$. Then $\gamma_k < 0$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} \gamma_k = 0$.*

Proof. Lemma 1 implies that $|u| \in H_0^1(V)$ whenever $u \in H_0^1(V)$. Thus we can pick a function $u \in H_0^1(V)$ such that $u(x) \geq 0$ for every $x \in V$ and such that there is an element $x_0 \in V$ with $u(x_0) > 1$. It follows that $U := \{x \in V \mid u(x) > 1\}$ is a nonempty open subset of V. Let $h\colon \mathbb{R} \to \mathbb{R}$ be defined by $h(t) = \min\{t, 1\}$, for every $t \in \mathbb{R}$. Then $h(0) = 0$ and h is a Lipschitz map with Lipschitz constant $L = 1$. Lemma 1 yields that $v := h \circ u \in H_0^1(V)$. Moreover, $v(x) = 1$ for every $x \in U$, and $0 \leq v(x) \leq 1$ for every $x \in V$.

On the other hand, condition (3*) of (C3) implies the existence of real numbers $\rho > 0$ and m such that $\frac{F(s)}{s^2} > m$ for every $s \in]0, \rho[$. It follows that

$$F(s) \geq ms^2, \text{ for every } s \in [0, \rho[. \tag{47}$$

Condition (4*) of (C3) yields the existence of a sequence (r_n) in $]0, \rho[$ such that $\lim_{n\to\infty} r_n = 0$ and

$$\lim_{n\to\infty} \frac{F(r_n)}{r_n^2} = \infty. \tag{48}$$

We then have for every $n \in \mathbb{N}$

$$T(r_n v) = \frac{r_n^2}{2}||v||^2 - \frac{r_n^2}{2}\int_V a(x)v^2(x)\mathrm{d}\mu + F(r_n)\int_U g(x)\mathrm{d}\mu$$
$$+ \int_{V\setminus U} g(x)F(r_n v(x))\mathrm{d}\mu.$$

Using (47) and the fact that $g \leq 0$ in V, we get for every $n \in \mathbb{N}$

$$T(r_n v) \leq \frac{r_n^2}{2}||v||^2 - \frac{r_n^2}{2}\int_V a(x)v^2(x)\mathrm{d}\mu + F(r_n)\int_U g(x)\mathrm{d}\mu$$
$$+ m r_n^2 \int_{V\setminus U} g(x)v^2(x)\mathrm{d}\mu.$$

Thus

$$\frac{T(r_n v)}{r_n^2} \leq \frac{1}{2}||v||^2 - \frac{1}{2}\int_V a(x)v^2(x)\mathrm{d}\mu + \frac{F(r_n)}{r_n^2}\int_U g(x)\mathrm{d}\mu + m\int_{V\setminus U} g(x)v^2(x)\mathrm{d}\mu.$$

Condition (C4) and relation (22) imply that $\int_U g(x)\mathrm{d}\mu < 0$, so we get from (48) and the above inequality that

$$\lim_{n\to\infty} \frac{T(r_n v)}{r_n^2} = -\infty.$$

Thus there is an index n_0 such that $T(r_n v) < 0$ for every $n \geq n_0$. Fix now $k \in \mathbb{N}$. Since $\lim_{n\to\infty} ||r_n v||_{\sup} = 0$, we get an index $p \geq n_0$ such that $r_p v \in M_k$. Hence $\gamma_k \leq T(r_p v) < 0$.

Let $u_k \in M_k$ be so that $\gamma_k = T(u_k)$. Since $M_k \subseteq M_0$, relation (40) yields

$$\gamma_k = T(u_k) \geq \max_{t\in[d_0,b_0]} |f(t)| \cdot ||u_k||_{\sup} \cdot \int_V g(x)\mathrm{d}\mu,$$

hence

$$0 > \gamma_k \geq \max_{t\in[d_0,b_0]} |f(t)| \cdot \max\{b_k, |d_k|\} \cdot \int_V g(x)\mathrm{d}\mu.$$

Since $\lim_{k\to\infty} b_k = \lim_{k\to\infty} d_k = 0$, we conclude that $\lim_{k\to\infty} \gamma_k = 0$. □

Proof of Theorem 4 concluded. From Lemma 3 we know that there is a sequence (u_k) of weak solutions of problem (P) such that $\gamma_k = T(u_k)$, where $\gamma_k = \inf T(M_k)$, for every natural k. Using relation (41) and the fact that $\gamma_k \leq 0$, we obtain

$$\frac{1}{2}||u_k||^2 \leq -\max_{t\in[d_0,b_0]} |f(t)| \cdot \max\{b_k, |d_k|\} \cdot \int_V g(x)\mathrm{d}\mu.$$

Using once again that $\lim_{k\to\infty} b_k = \lim_{k\to\infty} d_k = 0$, we conclude that $\lim_{k\to\infty} ||u_k|| = 0$. Thus also $\lim_{k\to\infty} ||u_k||_{\sup} = 0$, by (20).

We know from Lemma 4 that $T(u_k) = \gamma_k < 0$, for every natural k, and that $\lim_{k\to\infty} \gamma_k = 0$. Thus we can find a subsequence (u_{k_j}) of the sequence (u_k) consisting of pairwise distinct elements. □

Remark 15. By the same method one can prove (see also [11]) an analogous result in the case the nonlinear term $f: \mathbb{R} \to \mathbb{R}$ has an oscillating behavior at ∞. In this case one obtains a sequence (u_k) of pairwise distinct weak solutions of problem (P) such that $\lim_{k\to\infty} ||u_k|| = \infty$.

6 Parameter-Depending Dirichlet Problems on the SG

This section is based on the papers [4–6] and it contains applications of abstract critical points results by Ricceri to parameter-depending Dirichlet problems defined on the SG. We mention in this context that the celebrated three-critical-point theorem obtained by Ricceri in [27] turned out to be one of the most often applied abstract multiplicity results for the study of different types of nonlinear problems of variational nature. Along the years Ricceri has obtained several refinements of his previously mentioned three-critical-point theorem. Two of these refinements will be used in this section for the study of two-, respectively, three-parameter Dirichlet problems on the SG. As far as we know, Theorem 6 below (taken from [6]) contains the first application of a Ricceri type three-critical-point theorem to nonlinear partial differential equations on fractals.

6.1 Two-Parameter Dirichlet Problems on the SG

This subsection contains some results of [6]. Let $f,\ g: V \times \mathbb{R} \to \mathbb{R}$ be continuous functions, and let $\lambda,\ \eta \in \mathbb{R}$ be parameters. Consider the following Dirichlet problem on the SG

$$(P_{\lambda,\eta}) \begin{cases} -\Delta u(x) = \lambda f(x, u(x)) + \eta g(x, u(x)),\ \forall\ x \in V \setminus V_0, \\ u|_{V_0} = 0. \end{cases}$$

Let $F: V \times \mathbb{R} \to \mathbb{R}$ be the antiderivative of f with respect to the second variable, and let $G: V \times \mathbb{R} \to \mathbb{R}$ be the antiderivative of g with respect to the second variable. Assertion 4° of Proposition 8 implies that the map $T: H_0^1(V) \to \mathbb{R}$, defined by

$$T(u) = \frac{1}{2}||u||^2 - \lambda \int_V F(x, u(x))\mathrm{d}\mu - \eta \int_V G(x, u(x))\mathrm{d}\mu,$$

is an energy functional of $(P_{\lambda,\eta})$.

The study of problem $(P_{\lambda,\eta})$ is based on the following three-critical-point theorem by Ricceri (see Theorem 2 of [28]).

Theorem 5. *Let X be a separable and reflexive real Banach space, and let $\Phi,\ J: X \to \mathbb{R}$ be functionals satisfying the following conditions:*

(i) *Φ is a coercive, sequentially weakly lower semicontinuous C^1-functional, bounded on each bounded subset of X, and whose derivative admits a continuous inverse on X^*.*
(ii) *If (u_n) is a sequence in X converging weakly to u, and if $\liminf_{n\to\infty} \Phi(u_n) \leq \Phi(u)$, then (u_n) has a subsequence converging strongly to u.*
(iii) *J is a C^1-functional with compact derivative.*
(iv) *The functional Φ has a strict local minimum u_0 with $\Phi(u_0) = J(u_0) = 0$.*
(v) *The inequality $\rho_1 < \rho_2$ holds, where*

$$\rho_1 := \max\left\{0,\ \limsup_{||u||\to\infty} \frac{J(u)}{\Phi(u)},\ \limsup_{u\to u_0} \frac{J(u)}{\Phi(u)}\right\} \text{ and } \rho_2 := \sup_{u\in\Phi^{-1}(]0,\infty[)} \frac{J(u)}{\Phi(u)}.$$

Then, for each compact interval $[\lambda_1, \lambda_2] \subset]\frac{1}{\rho_2}, \frac{1}{\rho_1}[$ (where, by convention, $\frac{1}{0} := \infty$ and $\frac{1}{\infty} := 0$), there exists a positive real number r with the following property: For every $\lambda \in [\lambda_1, \lambda_2]$ and for every C^1-functional $\Psi: X \to \mathbb{R}$ with compact derivative there exists $\delta > 0$ such that, for every $\eta \in [0, \delta]$, the equation

$$\Phi'(u) = \lambda J'(u) + \eta\Psi'(u)$$

has at least three solutions in X whose norms are less than r.

The aim of this subsection is to apply Theorem 5 to show that, under suitable assumptions and for certain values of the parameters λ and η, problem $(P_{\lambda,\eta})$ has at least three (weak) solutions. More precisely, we can state the following result.

Theorem 6. *Assume that the following hypotheses hold:*

(C1) *The function $f: V \times \mathbb{R} \to \mathbb{R}$ is continuous.*
(C2) *The antiderivative $F: V \times \mathbb{R} \to \mathbb{R}$ of f with respect to the second variable satisfies the following conditions:*

(1*) *There exist $\alpha \in [0, 2[$, $a \in L^1(V, \mu)$, and $m \geq 0$ such that*

$$F(x,t) \leq m(a(x) + |t|^\alpha), \text{ for all } (x,t) \in V \times \mathbb{R}.$$

(2*) *There exist $t_0 > 0$, $M \geq 0$ and $\beta > 2$ such that*

$$F(x,t) \leq M|t|^\beta, \text{ for all } (x,t) \in V \times [-t_0, t_0].$$

(3*) *There exists $t_1 \in \mathbb{R} \setminus \{0\}$ such that for all $x \in V$ and for all t between 0 and t_1 we have*

$$F(x,t_1) > 0 \text{ and } F(x,t) \geq 0.$$

Then there exists a real number $\Lambda \geq 0$ *such that, for each compact interval* $[\lambda_1, \lambda_2] \subset]\Lambda, \infty[$*, there exists a positive real number* r *with the following property: For every* $\lambda \in [\lambda_1, \lambda_2]$ *and every continuous function* $g: V \times \mathbb{R} \to \mathbb{R}$ *there exists* $\delta > 0$ *such that, for each* $\eta \in [0, \delta]$*, the problem* $(P_{\lambda,\eta})$ *has at least three solutions whose norms are less than* r.

Proof. Set $X := H_0^1(V)$. Then X is separable (by Remark 9) and reflexive (as a Hilbert space). Define the functionals Φ, $J: X \to \mathbb{R}$, for every $u \in X$, by

$$\Phi(u) = \frac{1}{2}||u||^2, \quad J(u) = \int_V F(x, u(x))\mathrm{d}\mu.$$

In order to apply Theorem 5, we show that the conditions (i)–(v) required in this theorem are satisfied for the above defined functionals.

Clearly condition (i) of Theorem 5 is satisfied. (Recall that, according to assertion 1° of Proposition 8, $\Phi': X \to X^*$ is defined by $\Phi'(u)(v) = \mathscr{W}(u, v)$ for every $u, v \in X$.) Condition (ii) is a consequence of the facts that X is uniformly convex and that Φ is sequentially weakly lower semicontinuous. Condition (iii) follows from assertion 2° of Proposition 8. Obviously condition (iv) holds for $u_0 = 0$.

To verify (v), observe first that assumption (1*) of (C2) implies, together with (20), that for every $u \in X \setminus \{0\}$ the following inequality holds:

$$\frac{J(u)}{\Phi(u)} \leq \frac{2m}{||u||^2} \int_V a\mathrm{d}\mu + 2mc^\alpha ||u||^{\alpha-2}.$$

Since $\alpha < 2$, we conclude that

$$\limsup_{||u|| \to \infty} \frac{J(u)}{\Phi(u)} \leq 0. \tag{49}$$

Note that if $u \in X$ is so that $||u|| \leq \frac{t_0}{c}$, then, by (20), $||u||_{\sup} \leq t_0$. It follows that $u(x) \in [-t_0, t_0]$ for every $x \in V$. Using (2*) of (C2), we thus get that for every $x \in V$

$$F(x, u(x)) \leq M|u(x)|^\beta \leq Mc^\beta ||u||^\beta.$$

Hence the following inequality holds for every $u \in X \setminus \{0\}$ with $||u|| \leq \frac{t_0}{c}$

$$\frac{J(u)}{\Phi(u)} \leq 2Mc^\beta ||u||^{\beta-2}.$$

Since $\beta > 2$, we obtain

$$\limsup_{u \to 0} \frac{J(u)}{\Phi(u)} \leq 0. \tag{50}$$

The inequalities (49) and (50) yield that

$$\rho_1 := \max\left\{0,\ \limsup_{||u||\to\infty} \frac{J(u)}{\Phi(u)},\ \limsup_{u\to u_0} \frac{J(u)}{\Phi(u)}\right\} = 0. \tag{51}$$

Without any loss of generality we may assume that the real number t_1 in condition (3*) of (C2) is positive. Lemma 1 implies that $|u| \in H_0^1(V)$ whenever $u \in H_0^1(V)$. Thus we can pick a function $u \in H_0^1(V)$ such that $u(x) \geq 0$ for every $x \in V$, and such that there is an element $x_0 \in V$ with $u(x_0) > t_1$. It follows that

$$U := \{x \in V \mid u(x) > t_1\}$$

is a nonempty open subset of V. Let $h: \mathbb{R} \to \mathbb{R}$ be defined by $h(t) = \min\{t, t_1\}$, for every $t \in \mathbb{R}$. Then $h(0) = 0$ and h is a Lipschitz map with Lipschitz constant $L = 1$. Lemma 1 yields that $u_1 := h \circ u \in H_0^1(V)$. Moreover, $u_1(x) = t_1$ for every $x \in U$, and $0 \leq u_1(x) \leq t_1$ for every $x \in V$. Then, according to condition (3*) of (C2), we obtain

$$F(x, u_1(x)) > 0,\ \text{for every } x \in U, \quad \text{and} \quad F(x, u_1(x)) \geq 0,\ \text{for every } x \in V.$$

Together with (22) we then conclude that $J(u_1) > 0$. Thus

$$\rho_2 := \sup_{u\in\Phi^{-1}(]0,\infty[)} \frac{J(u)}{\Phi(u)} > 0. \tag{52}$$

Relations (51) and (52) finally imply that assertion (v) of Theorem 5 is also fulfilled. Put $\Lambda := \frac{1}{\rho_2}$ (with the convention $\frac{1}{\infty} := 0$). Note that if $g: V \times \mathbb{R} \to \mathbb{R}$ is continuous, then the map $\Psi: X \to \mathbb{R}$, defined by

$$\Psi(u) = \int_V G(x, u(x))\mathrm{d}\mu,$$

where $G: V \times \mathbb{R} \to \mathbb{R}$ is the antiderivative of g with respect to the second variable, is, by assertion 2° of Proposition 8, a C^1-functional with compact derivative. Recall that assertion 4° of Proposition 8 yields that $\Phi - \lambda J - \eta\Psi$ is an energy functional of problem $(P_{\lambda,\eta})$. So, applying Theorem 5, we obtain the asserted conclusion.

Example 1. Let $0 < \alpha < 2 < \beta$ and define $f_1: \mathbb{R} \to \mathbb{R}$ by

$$f_1(t) = \begin{cases} |t|^{\beta-2}t, & \text{if } |t| \leq 1 \\ |t|^{\alpha-2}t, & \text{if } |t| > 1. \end{cases}$$

Then $F_1: \mathbb{R} \to \mathbb{R}$, $F_1(t) = \int_0^t f_1(\xi)\mathrm{d}\xi$, is given by

$$F_1(t) = \begin{cases} \frac{1}{\beta}|t|^\beta, \text{ if } |t| \le 1 \\ \frac{1}{\beta} - \frac{1}{\alpha} + \frac{1}{\alpha}|t|^\alpha, \text{ if } |t| > 1. \end{cases}$$

Consider a continuous map $a: V \to \mathbb{R}$ with $a(x) > 0$, for every $x \in V$, and define $f: V \times \mathbb{R} \to \mathbb{R}$ by $f(x,t) = a(x) f_1(t)$. The antiderivative of f with respect to the second variable is then the map $F: V \times \mathbb{R} \to \mathbb{R}$ given by $F(x,t) = a(x)F_1(t)$, for all $(x,t) \in V \times \mathbb{R}$. Hence F satisfies condition (C2) of Theorem 6.

6.2 One-Parameter Dirichlet Problems on the SG

The following immediate consequence of Theorem 6 gives information concerning one-parameter Dirichlet problems on the SG.

Theorem 7. *Assume that the hypotheses* (C1) *and* (C2) *of Theorem* 6 *hold true. Then there exists a real number* $\Lambda \ge 0$ *such that for every* $\lambda > \Lambda$ *the problem*

$$\begin{cases} -\Delta u(x) = \lambda f(x, u(x)), \ \forall\, x \in V \setminus V_0, \\ u|_{V_0} = 0. \end{cases} \tag{53}$$

has at least three solutions.

Remark 16. Assume that the hypotheses (C1) and (C2) of Theorem 6 hold true. Note that condition (2*) of (C2) implies in particular that $f(x,0) = 0$, for every $x \in V$. Hence the constant zero function is a solution of problem (53), for every $\lambda \in \mathbb{R}$. Thus Theorem 7 yields that, for every $\lambda > \Lambda$, problem (53) has at least two nontrivial weak solutions.

Following [4], we next establish a sort of stability result concerning problem (53): As it will follow from Theorem 9 below, if one perturbs the left side of the first equation in (53) with a linear term satisfying certain conditions, and if one slightly modifies condition (2*) of (C2) (requiring that the inequality $F(x,t) \le M|t|^\beta$ holds for every $(x,t) \in V \times \mathbb{R}$), then the same conclusion as in Theorem 7 holds. The perturbation we introduce in problem (53) will affect the functional Φ defined in the proof of Theorem 6, so that the derivative of this perturbed functional will no more have necessarily a continuous inverse. Thus we cannot apply Theorem 5 in the perturbed case. In this case we will use another three-critical-point result, namely a result due to Arcoya and Carmona. For our purposes we don't need the three-critical-point result of Arcoya and Carmona in its full generality as it is stated in Theorem 3.4 of [1], but only the following immediate consequence which is the differentiable version of it. We also point out that this differentiable version is in fact an application of the three-critical-point theorem of Pucci and Serrin in [26].

Theorem 8. *Let X be a reflexive real Banach space, $\Phi\colon X \to \mathbb{R}$ a sequentially weakly lower semicontinuous C^1-functional, $\Psi\colon X \to \mathbb{R}$ a nonconstant C^1-functional with compact derivative, and $A \subseteq \mathbb{R}$ an interval such that for every $\lambda \in A$ the functional $\Phi + \lambda\Psi$ is coercive and satisfies the Palais-Smale condition. If the real number $r \in]\inf \Psi(X), \sup \Psi(X)[$ is such that $\phi_1(r) < \phi_2(r)$ and $]\phi_1(r), \phi_2(r)[\cap A \neq \emptyset$, where*

$$\phi_1(r) = \inf\left\{ \frac{\inf_{v\in\Psi^{-1}(r)} \Phi(v) - \Phi(u)}{\Psi(u) - r} \mid u \in \Psi^{-1}(]-\infty, r[)\right\} \tag{54}$$

and

$$\phi_2(r) = \sup\left\{ \frac{\inf_{v\in\Psi^{-1}(r)} \Phi(v) - \Phi(u)}{\Psi(u) - r} \mid u \in \Psi^{-1}(]r, \infty[)\right\}, \tag{55}$$

then, for every $\lambda \in]\phi_1(r), \phi_2(r)[\cap A$, the functional $\Phi + \lambda\Psi$ has at least three critical points.

Let $b \in L^1(V, \mu)$, $f \in \mathscr{DP}(V)$, and $\lambda \in \mathbb{R}$. We consider now the following one-parameter Dirichlet problem on the SG

$$(P_\lambda) \begin{cases} -\Delta u(x) + b(x)u(x) = \lambda f(x, u(x)), \ \forall\, x \in V \setminus V_0, \\ u|_{V_0} = 0. \end{cases}$$

Note that, by (33), a function $u \in H_0^1(V)$ is a weak solution of problem (P_λ) if and only if

$$\mathscr{W}(u, v) + \int_V b(x)u(x)v(x)\mathrm{d}\mu - \lambda \int_V f(x, u(x))v(x)\mathrm{d}\mu = 0, \ \forall\, v \in H_0^1(V). \tag{56}$$

Before stating the analog of Theorem 7 for problem (P_λ) recall (for example, from [10]) that if $b \in L^1(V, \mu)\setminus\{0\}$ is such that $b \leq 0$ a.e. in V, then the eigenvalue problem

$$\begin{cases} -\Delta u(x) + \gamma b(x)u(x) = 0, \ \forall\, x \in V \setminus V_0, \\ u|_{V_0} = 0 \end{cases} \tag{57}$$

has an increasing sequence of eigenvalues $(\gamma_n)_{n\in\mathbb{N}^*}$ with $\gamma_1 > 0$ and $\lim\limits_{n\to\infty} \gamma_n = \infty$.

Theorem 9. *Assume that the following hypotheses hold:*

(C1) *The function $f\colon V \times \mathbb{R} \to \mathbb{R}$ is continuous.*
(C2) *The antiderivative $F\colon V \times \mathbb{R} \to \mathbb{R}$ of f with respect to the second variable satisfies the following conditions:*

(1*) *There exist* $\alpha \in [0, 2[$, $a \in L^1(V, \mu)$, *and* $m \geq 0$ *such that*

$$F(x, t) \leq m(a(x) + |t|^{\alpha}),\ \textit{for all}\ (x, t) \in V \times \mathbb{R}.$$

(2') *There exist* $M \geq 0$ *and* $\beta > 2$ *such that*

$$F(x, t) \leq M|t|^{\beta},\ \textit{for all}\ (x, t) \in V \times \mathbb{R}.$$

(3*) *There exists* $t_1 \in \mathbb{R} \setminus \{0\}$ *such that for all* $x \in V$ *and for all* t *between* 0 *and* t_1 *we have*

$$F(x, t_1) > 0\ \textit{and}\ F(x, t) \geq 0.$$

(C3) *The map* $b \in L^1(V, \mu)$ *is such that* $b \leq 0$ *a.e. in* V *and it satisfies one of the conditions*

(4*) $||b||_1 < \frac{1}{c^2}$, *where* c *is the positive constant in* (20), *or*
(5*) $b \neq 0$ *and* $\gamma_1 > 1$, *where* γ_1 *is the first eigenvalue of problem* (57).

Then there exists a real number $\Lambda_b \geq 0$ *such that for every real* $\lambda > \Lambda_b$ *problem* (P_λ) *has at least three weak solutions, i.e., at least two nontrivial weak solutions.*

Proof. In order to apply Theorem 8, we put $X := H_0^1(V)$, $A := [0, \infty[$, and define Φ, Ψ: $X \to \mathbb{R}$ as follows

$$\Phi(u) = \frac{1}{2}||u||^2 + \frac{1}{2}\int_V b(x)u^2(x)\mathrm{d}\mu, \quad \Psi(u) = -\int_V F(x, u(x))\mathrm{d}\mu.$$

Proposition 8 yields that $\Phi: X \to \mathbb{R}$ is a sequentially weakly lower semicontinuous C^1-functional, $\Psi: X \to \mathbb{R}$ is a C^1-functional with compact derivative, and $\Phi + \lambda\Psi$ is an energy functional of problem (P_λ) for every $\lambda \in \mathbb{R}$.

Let $u_1 \in H_0^1(V)$ be the function constructed in the proof of Theorem 6, using condition (3*) of (C2). Then $\Psi(u_1) < 0$. Since $\Psi(0) = 0$, it follows that Ψ is nonconstant.

Let $u \in H_0^1(V)$. Since $b \leq 0$ a.e. in V, we get, using also (20), that

$$\Phi(u) \geq \frac{1}{2}||u||^2 + \frac{1}{2}c^2||u||^2\int_V b(x)\mathrm{d}\mu = \frac{1}{2}||u||^2(1 - c^2||b||_1).$$

It is known (see, for example, formula (3.15) in [10]) that

$$||u||^2 \geq -\gamma_1 \int_V b(x)u^2(x)\mathrm{d}\mu,$$

so

$$\Phi(u) \geq \frac{1}{2\gamma_1}||u||^2(\gamma_1 - 1).$$

Thus both conditions (4*) and (5*) of (C3) imply the existence of a real number $d > 0$ such that

$$\Phi(u) \geq d\,||u||^2, \text{ for every } u \in H_0^1(V). \tag{58}$$

Fix an arbitrary $\lambda \in A$. According to (20), (58), and to assumption (1*) of (C2), the following inequality holds for every $u \in H_0^1(V)$ with $u \neq 0$

$$\Phi(u) + \lambda\Psi(u) \geq ||u||^2 \left(d - \frac{\lambda m}{||u||^2} \int_V a \mathrm{d}\mu - \lambda m c^{\alpha} ||u||^{\alpha-2} \right).$$

Since $\alpha < 2$, we conclude that

$$\lim_{||u|| \to \infty} (\Phi(u) + \lambda\Psi(u)) = \infty,$$

i.e., $\Phi + \lambda\Psi$ is coercive. Example 38.25 in [38] yields that $\Phi + \lambda\Psi$ satisfies the Palais-Smale condition (for this note that the derivative of the map $u \in H_0^1(V) \mapsto \frac{1}{2}||u||^2$ has a continuous inverse, and both of the maps $u \in H_0^1(V) \mapsto \frac{1}{2}\int_V b(x)u^2(x)\mathrm{d}\mu$ and $\lambda\Psi$ have compact derivatives).

Next we show that

$$\limsup_{\substack{r \to 0 \\ r<0}} \phi_1(r) \leq \phi_1(0) \tag{59}$$

and

$$\liminf_{\substack{r \to 0 \\ r<0}} \phi_2(r) = \infty, \tag{60}$$

where $\phi_1(r)$ and $\phi_2(r)$ are defined by (54) and (55), respectively. To show (59), observe that (58) and the fact that $0 \in \Psi^{-1}(0)$ yield

$$\inf_{v \in \Psi^{-1}(0)} \Phi(v) = 0,$$

thus

$$\phi_1(0) = \inf \left\{ -\frac{\Phi(u)}{\Psi(u)} \mid u \in \Psi^{-1}(]-\infty, 0[) \right\}. \tag{61}$$

Fix arbitrary $u \in \Psi^{-1}(]-\infty, 0[)$ and $r \in]\Psi(u), 0[$. Then, by the definition of $\phi_1(r)$, we get that

$$\phi_1(r) \leq \frac{\inf_{v \in \Psi^{-1}(r)} \Phi(v) - \Phi(u)}{\Psi(u) - r}.$$

Since $\inf_{v \in \Psi^{-1}(r)} \Phi(v) \geq 0$, it follows that

$$\phi_1(r) \leq -\frac{\Phi(u)}{\Psi(u) - r}, \text{ for all } r \in]\Psi(u), 0[.$$

Thus

$$\limsup_{\substack{r \to 0 \\ r<0}} \phi_1(r) \leq -\frac{\Phi(u)}{\Psi(u)}, \text{ for all } u \in \Psi^{-1}(]-\infty, 0[),$$

which implies, according to (61), the inequality (59).

In order to prove (60), consider $r \in]\inf \Psi(X), 0[$. Then $0 = \Psi(0) > r$, so, by the definition of $\phi_2(r)$,

$$\phi_2(r) \geq \frac{\inf_{v \in \Psi^{-1}(r)} \Phi(v)}{-r}.$$

We know from assertion 5° of Proposition 8 that Ψ is sequentially weakly continuous. Thus $\Psi^{-1}(r)$ is a sequentially weakly closed set. Also, by the continuity of Ψ in the norm topology, this set is nonempty. Hence, Φ being coercive (in view of (58)) and sequentially weakly lower semicontinuous, there exists, according to Proposition 9, a point $v_0 \in \Psi^{-1}(r)$ such that

$$\inf_{v \in \Psi^{-1}(r)} \Phi(v) = \Phi(v_0).$$

Thus

$$-\frac{\Phi(v_0)}{r} \leq \phi_2(r). \tag{62}$$

On the other hand, using (20) and condition (2') of (C2), we get that

$$-r = -\Psi(v_0) = \int_V F(x, v_0(x)) \mathrm{d}\mu \leq Mc^\beta ||v_0||^\beta,$$

so

$$(-r)^{\frac{2}{\beta}} \leq (Mc^\beta)^{\frac{2}{\beta}} \cdot ||v_0||^2.$$

In view of (58) we thus obtain

$$-\frac{\Phi(v_0)}{r} \geq \frac{d}{(Mc^\beta)^{\frac{2}{\beta}}} (-r)^{\frac{2}{\beta}-1}.$$

The above inequality and (62) imply that

$$\frac{d}{(Mc^\beta)^{\frac{2}{\beta}}}(-r)^{\frac{2}{\beta}-1} \le \phi_2(r), \text{ for all } r \in \,]\inf \Psi(X), 0[.$$

Since $\frac{2}{\beta} - 1 < 0$, we obtain (60).

Finally, put $\Lambda_b := \phi_1(0)$. Then $0 \le \Lambda_b < \infty$ (by (58) and (61)). Consider now a real $\lambda > \Lambda_b$. From (59) and (60) we then get that

$$\limsup_{\substack{r\to 0\\ r<0}} \phi_1(r) < \lambda < \liminf_{\substack{r\to 0\\ r<0}} \phi_2(r).$$

It follows that there exists $r \in \,]\inf \Psi(X), 0\,[$ such that $\phi_1(r) < \lambda < \phi_2(r)$. Applying now Theorem 8, we see that the map $\Phi + \lambda\Psi$ has at least three critical points. Since this map is an energy functional of problem (P_λ), we conclude that (P_λ) has at least three weak solutions, thus at least two nonzero weak solutions (since, by Remark 16, the constant zero function is a weak solution of this problem). □

Remark 17. The proof of Theorem 6 implies that the real number $\Lambda \ge 0$ in Theorem 7 can be obtained as

$$\Lambda = \frac{1}{\rho_2}, \text{ where } \rho_2 = \sup\left\{ \frac{\int_V F(x,u(x))\mathrm{d}\mu}{\frac{1}{2}||u||^2} \,\middle|\, \int_V F(x,u(x))\mathrm{d}\mu > 0 \right\},$$

with the convention $\frac{1}{\infty} := 0$. On the other hand, if one considers $b = 0$, then the proof of the above Theorem 9 yields that Λ_0 can be chosen as

$$\Lambda_0 = \inf\left\{ \frac{\frac{1}{2}||u||^2}{\int_V F(x,u(x))\mathrm{d}\mu} \,\middle|\, \int_V F(x,u(x))\mathrm{d}\mu > 0 \right\}.$$

Thus $\Lambda_0 = \Lambda$, showing that in the case $b = 0$ the approach used in the proof of Theorem 9 fits to the one used in the proof of Theorem 6.

We finish the study of one-parameter Dirichlet problems on the SG with the following result which gives sufficient conditions that ensure that problem (P_λ) has only the trivial weak solution.

Proposition 10. *Assume that the functions $b \in L^1(V,\mu)$ and $f \in \mathscr{D}\mathscr{P}(V)$ satisfy the following conditions:*

(i) *There exists a positive real d with*

$$|f(x,t)| \le d|t|, \ \textit{for all } (x,t) \in V \times \mathbb{R}.$$

(ii) $||b||_1 < \frac{1}{c^2}$, *where c is the positive constant in* (20).

Then, for every $\lambda \in \mathbb{R}$ *with* $|\lambda| < \frac{1}{d}(\frac{1}{c^2} - ||b||_1)$*, the only weak solution of problem* (P_λ) *is the trivial one.*

Proof. Let $\lambda \in \mathbb{R}$ be so that $|\lambda| < \frac{1}{d}(\frac{1}{c^2} - ||b||_1)$. Suppose, by contradiction, that $u \in H_0^1(V)$ is a nontrivial weak solution of problem (P_λ). If we put $v = u$ in relation (56) and use the assumptions (i), (ii), and the inequality (20), we obtain

$$\begin{aligned} ||u||^2 &= \lambda \int_V f(x, u(x))u(x)\mathrm{d}\mu - \int_V b(x)u^2(x)\mathrm{d}\mu \\ &\leq |\lambda| \int_V |f(x, u(x))| \cdot |u(x)|\mathrm{d}\mu + \int_V |b(x)|u^2(x)\mathrm{d}\mu \\ &\leq |\lambda| d \int_V u^2(x)\mathrm{d}\mu + \int_V |b(x)|u^2(x)\mathrm{d}\mu \\ &\leq ||u||^2 c^2(|\lambda| d + ||b||_1) < ||u||^2, \end{aligned}$$

a contradiction. So, since $f(x, 0) = 0$, for every $x \in V$, problem (P_λ) has only the trivial weak solution. □

6.3 Three-Parameter Dirichlet Problems on the SG

Let $f, g, h: V \times \mathbb{R} \to \mathbb{R}$ be continuous functions, and let $\alpha, \beta, \lambda \in \mathbb{R}$ be parameters. In the present subsection, based on [5], we are concerned with the following Dirichlet problem on the SG

$$(P_{\alpha,\beta,\lambda}) \begin{cases} -\Delta u(x) + \alpha g(x, u(x)) + \beta h(x, u(x)) + \lambda f(x, u(x)) = 0, \ \forall \ x \in V \setminus V_0, \\ u|_{V_0} = 0. \end{cases}$$

Denote by $F, G, H: V \times \mathbb{R} \to \mathbb{R}$ the antiderivatives of, respectively, f, g, h with respect to the second variable, and by $J, \Psi, \Phi: H_0^1(V) \to \mathbb{R}$ the functionals

$$J := T_G, \ \Psi := T_H, \ \Phi := T_F, \tag{63}$$

defined according to (34). Let $I: H_0^1(V) \to \mathbb{R}$ be the functional defined in (35). Assertion 4° of Proposition 8 yields that $I + \alpha J + \beta \Psi + \lambda \Phi$ is an energy functional of problem $(P_{\alpha,\beta,\lambda})$.

The main result concerning the existence of multiple solutions of problem $(P_{\alpha,\beta,\lambda})$ is based on some results by Ricceri: on the one hand, on a three-critical-point theorem contained in [29] and in [31], and, on the other hand, on a four-critical-point theorem proved in [30]. More exactly, Theorem 1 established in [29] (see also its addendum, i.e., Theorem 1 in [31]) and Theorem 1 in [30] yield the following abstract multiplicity result.

Theorem 10. *Let X be a reflexive real Banach space, $I: X \to \mathbb{R}$ a sequentially weakly lower semicontinuous and coercive C^1-functional which is bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^*, and consider $J, \Psi, \Phi: X \to \mathbb{R}$ three C^1-functionals with compact derivative satisfying the conditions:*

(C1) $\liminf_{||u||\to\infty} \frac{J(u)}{I(u)} \geq 0$,

(C2) $\limsup_{||u||\to\infty} \frac{J(u)}{I(u)} < +\infty$,

(C3) $\liminf_{||u||\to\infty} \frac{\Psi(u)}{I(u)} = -\infty$,

(C4) $\inf_{u\in X} (\Psi(u) + \lambda\Phi(u)) > -\infty$, *for all* $\lambda > 0$.

Then the following assertions hold:

(a) *For every $\alpha > 0$, there exists $\beta_\alpha \in]0, +\infty]$ with the property that, for every real number $\beta \in]0, \beta_\alpha[$, there exist $\Lambda_1^\beta, \Lambda_2^\beta$ with $0 \leq \Lambda_1^\beta < \Lambda_2^\beta \leq +\infty$ such that, for each compact interval $[a, b] \subseteq]\Lambda_1^\beta, \Lambda_2^\beta[$, there exists $r > 0$ such that, for every $\lambda \in [a, b]$, the functional $I + \alpha J + \beta\Psi + \lambda\Phi$ has at least three critical points whose norms are less than r.*

(b) *If one assumes, in addition to conditions* (C1)–(C4)*, that there exist $u_0, u_1 \in X$ such that*

(C5) *u_0 is a strict local minimum of I with $I(u_0) = J(u_0) = \Psi(u_0) = \Phi(u_0) = 0$,*

(C6) $\min\left\{\liminf_{u\to u_0} \frac{J(u)}{I(u)}, \liminf_{u\to u_0} \frac{\Phi(u)}{I(u)}\right\} \geq 0$,

(C7) $\liminf_{u\to u_0} \frac{\Psi(u)}{I(u)} > -\infty$,

(C8) $\Psi(u_1) \leq 0$, $\Phi(u_1) \leq 0$, $J(u_1) < 0$,

then, for every $\alpha > 0$ large enough, there exists $\beta'_\alpha \in]0, \beta_\alpha]$ with the property that, for every $\beta \in]0, \beta'_\alpha[$, there exists $\lambda^* > 0$ such that the functional $I + \alpha J + \beta\Psi + \lambda^*\Phi$ has at least four critical points, u_0 being one of them. Moreover, two of these four critical points (different from u_0) are actually global minima of $I + \alpha J + \beta\Psi + \lambda^*\Phi$.

From Theorem 10, we derive the following multiplicity result concerning the existence of multiple solutions of $(P_{\alpha,\beta,\lambda})$. This result was inspired by Theorem 2 in [30]. We had to adapt the methods involved in [30] in the context of the Sobolev spaces $W_0^{1,p}(\Omega)$ (with $p > 1$ and Ω a bounded domain in $\mathbb{R}^n$) to the case of the function space $H_0^1(V)$. There are several differences between these two approaches. For instance, it turned out that, especially when one looks for concrete examples of functions that belong to the space one works with, the space $H_0^1(V)$ is more difficult to be treated than Sobolev spaces. This will become clear especially in the final part of the proof.

Theorem 11. *Let $f, g, h\colon V \times \mathbb{R} \to \mathbb{R}$ be continuous functions such that their antiderivatives $F, G, H\colon V \times \mathbb{R} \to \mathbb{R}$ with respect to the second variable satisfy the following conditions:*

(1*) $\liminf_{|t|\to+\infty} \dfrac{\inf_{x\in V} G(x,t)}{t^2} \geq 0,$

(2*) $\limsup_{|t|\to+\infty} \dfrac{\sup_{x\in V} G(x,t)}{t^2} < +\infty,$

(3*) $\lim_{|t|\to+\infty} \dfrac{\sup_{x\in V} H(x,t)}{t^2} = -\infty,$

(4*) *there exists a positive real q with*

$$\liminf_{|t|\to+\infty} \frac{\inf_{x\in V} H(x,t)}{|t|^q} > -\infty \text{ and } \lim_{|t|\to+\infty} \frac{\inf_{x\in V} F(x,t)}{|t|^q} = +\infty.$$

Then the following assertions hold:

(A) *For every $\alpha > 0$, there exists $\beta_\alpha \in]0,+\infty]$ with the property that, for every real number $\beta \in]0,\beta_\alpha[$, there exist $\Lambda_1^\beta, \Lambda_2^\beta$ with $0 \leq \Lambda_1^\beta < \Lambda_2^\beta \leq +\infty$ such that, for each compact interval $[a,b] \subseteq]\Lambda_1^\beta, \Lambda_2^\beta[$, there exists $r > 0$ such that, for every $\lambda \in [a,b]$, problem $(P_{\alpha,\beta,\lambda})$ has at least three solutions whose norms are less than r.*

(B) *If one assumes, in addition to conditions (1*)–(4*), that*

(5*) $\min\left\{\liminf_{t\to 0} \dfrac{\inf_{x\in V} G(x,t)}{t^2}, \liminf_{t\to 0} \dfrac{\inf_{x\in V} F(x,t)}{t^2}\right\} \geq 0,$

(6*) $\liminf_{t\to 0} \dfrac{\inf_{x\in V} H(x,t)}{t^2} > -\infty,$

(7*) *there exist a compact set $K \subseteq V \setminus V_0$ with $\mu(K) > 0$ and a nonzero real number t^* such that*

$$\max\{F(x,t^*),\ G(x,t^*),\ H(x,t^*)\} < 0, \text{ for all } x \in K,$$

then, for every $\alpha > 0$ large enough, there exists $\beta'_\alpha \in]0,\beta_\alpha]$ with the property that, for every $\beta \in]0,\beta'_\alpha[$, there exists $\lambda^* > 0$ such that problem $(P_{\alpha,\beta,\lambda^*})$ has at least three nontrivial solutions. Moreover, two of these three solutions are global minima of the energy functional of problem $(P_{\alpha,\beta,\lambda^*})$.

Proof. Let $X := H_0^1(V)$ and consider the functionals I and J, Ψ, Φ defined in (35) and (63), respectively. According to Proposition 8, these functionals satisfy all conditions mentioned at the beginning of the statement of Theorem 10.

(A) We show that I, J, Ψ, Φ satisfy the conditions (C1)–(C4) of Theorem 10. To verify condition (C1), consider an arbitrary $\varepsilon > 0$. According to (1*), there exists a real $t_\varepsilon > 0$ such that

$$G(x,t) > -\varepsilon t^2, \text{ for every } x \in V \text{ and every } t \text{ with } |t| > t_\varepsilon.$$

Put $m_\varepsilon := \min\{G(x,t) : x \in V,\ t \in [-t_\varepsilon, t_\varepsilon]\}$. Then $m_\varepsilon \leq 0$, since $G(x,0) = 0$ for all $x \in V$. Thus

$$G(x,t) \geq -\varepsilon t^2 + m_\varepsilon, \text{ for all } (x,t) \in V \times \mathbb{R}.$$

Using (20), we get

$$J(u) \geq -\varepsilon \int_V u^2 \mathrm{d}\mu + m_\varepsilon \geq -\varepsilon c^2 ||u||^2 + m_\varepsilon, \text{ for all } u \in X,$$

which yields

$$\frac{J(u)}{I(u)} \geq -2\varepsilon c^2 + \frac{2m_\varepsilon}{||u||^2}, \text{ for all } u \in X \setminus \{0\}.$$

Hence

$$\liminf_{||u|| \to \infty} \frac{J(u)}{I(u)} \geq -2\varepsilon c^2, \text{ for all } \varepsilon > 0,$$

so (C1) holds.

In order to verify (C2), fix, according to (2^*), a positive real d such that

$$\limsup_{|t| \to +\infty} \frac{\sup_{x \in V} G(x,t)}{t^2} < d.$$

Then there exists a real $t_1 > 0$ such that

$$G(x,t) < \mathrm{d}t^2, \text{ for every } x \in V \text{ and every } t \text{ with } |t| > t_1.$$

Put $M_d := \max\{G(x,t) : x \in V,\ t \in [-t_1, t_1]\}$. Then $M_d \geq 0$, so

$$G(x,t) \leq \mathrm{d}t^2 + M_d, \text{ for all } (x,t) \in V \times \mathbb{R}.$$

Using (20), we get

$$J(u) \leq d \int_V u^2 \mathrm{d}\mu + M_d \leq dc^2 ||u||^2 + M_d, \text{ for all } u \in X.$$

This implies

$$\frac{J(u)}{I(u)} \leq 2dc^2 + \frac{2M_d}{||u||^2}, \text{ for all } u \in X \setminus \{0\},$$

hence

$$\limsup_{||u||\to\infty} \frac{J(u)}{I(u)} \le 2dc^2,$$

so (C2) holds.

Next, we proceed to verify (C3). For this, recall first from Sect. 2.3 that, by construction, the map $(-\Delta)^{-1}: L^2(V,\mu) \to L^2(V,\mu)$ is linear, symmetric, compact, and that its image D is a linear subset of X which is dense in $L^2(V,\mu)$. Hence, using assertion (b) of Proposition 19.14 in [39], we get that this (injective) operator admits at least one nonzero eigenvalue $\tilde{\lambda}$. Fix a corresponding eigenvector $\tilde{u} \in X \setminus \{0\}$. Then

$$\tilde{\lambda}||\tilde{u}||^2 = \int_V \tilde{u}^2 \mathrm{d}\mu.$$

In particular, $\tilde{\lambda} > 0$. We will show that

$$\lim_{n\to\infty} \frac{\Psi(n\tilde{u})}{I(n\tilde{u})} = -\infty. \tag{64}$$

For this, let $M > 0$ be arbitrary and put $M_1 := \frac{4M}{\tilde{\lambda}}$. In view of (3*) there exists $\xi_M > 0$ such that

$$H(x,t) \le -M_1 t^2, \text{ for every } x \in V \text{ and every } t \text{ with } |t| > \xi_M. \tag{65}$$

Put $\tilde{M} := \max\{H(x,t) : x \in V,\ t \in [-\xi_M, \xi_M]\}$. Then $\tilde{M} \ge 0$. For every $n \in \mathbb{N}^*$ let

$$A_n := \left\{ x \in V : |\tilde{u}(x)| > \frac{\xi_M}{n} \right\}.$$

Consider $n \in \mathbb{N}^*$. Then

$$\frac{\Psi(n\tilde{u})}{2I(n\tilde{u})} = \frac{1}{n^2||\tilde{u}||^2}\left(\int_{A_n} H(x,n\tilde{u}(x))\mathrm{d}\mu + \int_{V\setminus A_n} H(x,n\tilde{u}(x))\mathrm{d}\mu\right).$$

If $x \in A_n$, then $|n\tilde{u}(x)| > \xi_M$, hence $H(x,n\tilde{u}(x)) \le -M_1 n^2(\tilde{u}(x))^2$, by (65). So

$$\int_{A_n} H(x,n\tilde{u}(x))\mathrm{d}\mu \le -M_1 n^2 \int_{A_n} \tilde{u}^2 \mathrm{d}\mu.$$

If $x \in V \setminus A_n$, then $|n\tilde{u}(x)| \le \xi_M$, hence $H(x,n\tilde{u}(x)) \le \tilde{M}$. It follows that

$$\int_{V\setminus A_n} H(x,n\tilde{u}(x))\mathrm{d}\mu \le \tilde{M}.$$

Thus

$$\frac{\Psi(n\tilde{u})}{2I(n\tilde{u})} \leq -\frac{M_1}{||\tilde{u}||^2}\int_{A_n} \tilde{u}^2 \mathrm{d}\mu + \frac{\tilde{M}}{n^2||\tilde{u}||^2}. \tag{66}$$

Taking into account that $A_n \subseteq A_{n+1}$ for every nonzero natural n and that $\tilde{u}(x) = 0$ if and only if $x \notin A := \bigcup_{n\in\mathbb{N}^*} A_n$, we get that

$$\lim_{n\to\infty}\int_{A_n} \tilde{u}^2 \mathrm{d}\mu = \int_{A} \tilde{u}^2 \mathrm{d}\mu = \int_{V} \tilde{u}^2 \mathrm{d}\mu = \tilde{\lambda}||\tilde{u}||^2.$$

Fix a nonzero natural n_0 such that

$$\frac{1}{||\tilde{u}||^2}\int_{A_n} \tilde{u}^2 \mathrm{d}\mu > \frac{\tilde{\lambda}}{2}, \text{ for all } n \geq n_0,$$

so

$$-\frac{M_1}{||\tilde{u}||^2}\int_{A_n} \tilde{u}^2 \mathrm{d}\mu < -\frac{\tilde{\lambda}M_1}{2}, \text{ for all } n \geq n_0.$$

Using (66) we thus get

$$\frac{\Psi(n\tilde{u})}{2I(n\tilde{u})} < -\frac{\tilde{\lambda}M_1}{2} + \frac{\tilde{M}}{n^2||\tilde{u}||^2}, \text{ for all } n \geq n_0.$$

It follows that

$$\frac{\Psi(n\tilde{u})}{2I(n\tilde{u})} < -2M + M = -M, \text{ for all } n \geq \max\left\{n_0, \frac{1}{||\tilde{u}||}\sqrt{\frac{\tilde{M}}{M}}\right\}.$$

Thus (64) holds, which finally implies condition (C3).

Next, we are going to verify (C4). For this, fix $\lambda > 0$ and choose, according to (4*), a negative real a so that

$$\liminf_{|t|\to+\infty} \frac{\inf_{x\in V} H(x,t)}{|t|^q} > a.$$

Thus there exists a positive real d_1 with

$$H(x,t) > a|t|^q, \text{ for every } x \in V \text{ and every } t \text{ with } |t| > d_1.$$

Denoting by $m_1 := \min\{H(x,t) : x \in V,\ |t| \le d_1\}$, we get

$$H(x,t) \ge a|t|^q + m_1, \text{ for all } (x,t) \in V \times \mathbb{R},$$

which yields

$$\Psi(u) \ge a \int_V |u|^q \mathrm{d}\mu + m_1, \text{ for all } u \in X. \tag{67}$$

Pick now a (positive) real number b such that $a + \lambda b > 0$. Then, by (4*), there exists a positive real d_2 with

$$F(x,t) > b|t|^q, \text{ for every } x \in V \text{ and every } t \text{ with } |t| > d_2.$$

Denoting by $m_2 := \min\{F(x,t) - b|t|^q : x \in V,\ |t| \le d_2\}$, we get (note that $m_2 \le 0$)

$$F(x,t) \ge b|t|^q + m_2, \text{ for all } (x,t) \in V \times \mathbb{R},$$

which yields

$$\Phi(u) \ge b \int_V |u|^q \mathrm{d}\mu + m_2, \text{ for all } u \in X. \tag{68}$$

We finally get from (67) and (68) that

$$\Psi(u) + \lambda\Phi(u) \ge (a + \lambda b) \int_V |u|^q \mathrm{d}\mu + m_1 + m_2 \ge m_1 + m_2, \text{ for all } u \in X,$$

hence (C4) holds. Statement (A) follows now from assertion (a) of Theorem 10.

(B) We show that conditions (C5)–(C8) of Theorem 10 are satisfied. For this, let u_0 be the constant zero function in X. Then, clearly, condition (C5) holds. Next, we are going to verify condition (C6). Let $\varepsilon > 0$ be arbitrary. In view of condition (5*) there exists a positive real δ with

$$G(x,t) \ge -\varepsilon t^2, \text{ for every } x \in V \text{ and every } t \in [-\delta, \delta].$$

If $u \in X$ is such that $||u|| \le \frac{\delta}{c}$, then, according to (20), we have that $||u||_{\sup} \le \delta$, hence

$$G(x, u(x)) \ge -\varepsilon(u(x))^2, \text{ for every } x \in V.$$

It follows, using again (20), that

$$J(u) \ge -\varepsilon c^2 ||u||^2, \text{ for all } u \in X \text{ with } ||u|| \le \frac{\delta}{c}.$$

Thus

$$\frac{J(u)}{I(u)} \geq -2\varepsilon c^2, \text{ for all } u \in X \setminus \{0\} \text{ with } ||u|| \leq \frac{\delta}{c}.$$

It follows that $\liminf_{u \to u_0} \frac{J(u)}{I(u)} \geq 0$. One proves similarly that $\liminf_{u \to u_0} \frac{\Phi(u)}{I(u)} \geq 0$, so condition (C6) of Theorem 10 holds.

Now we are going to show that condition (C7) holds, too. For this fix, according to (6*), a real number $a < 0$ with $\liminf_{t \to 0} \frac{\inf_{x \in V} H(x,t)}{t^2} > a$. Then there exists a positive real δ such that

$$H(x,t) \geq at^2, \text{ for every } x \in V \text{ and every } t \in [-\delta, \delta].$$

If $u \in X$ is such that $||u|| \leq \frac{\delta}{c}$, then, according to (20), we have that $||u||_{\sup} \leq \delta$, hence

$$H(x, u(x)) \geq a(u(x))^2, \text{ for every } x \in V.$$

It follows, using again (20), that

$$\Psi(u) \geq ac^2||u||^2, \text{ for all } u \in X \text{ with } ||u|| \leq \frac{\delta}{c}.$$

Thus

$$\frac{\Psi(u)}{I(u)} \geq 2ac^2, \text{ for all } u \in X \setminus \{0\} \text{ with } ||u|| \leq \frac{\delta}{c},$$

which yields that $\liminf_{u \to u_0} \frac{\Psi(u)}{I(u)} > -\infty$. Hence condition (C7) of Theorem 10 is satisfied.

In order to show that (C8) holds, observe first that condition (7*) yields that

$$\eta := \max\left\{\int_K F(x,t^*)\mathrm{d}\mu, \int_K G(x,t^*)\mathrm{d}\mu, \int_K H(x,t^*)\mathrm{d}\mu\right\} < 0. \tag{69}$$

Fix a real number θ with $\eta < \theta < 0$. Put

$$M^* := \max\{F(x,t),\ G(x,t),\ H(x,t) : x \in V,\ t \in [-|t^*|, |t^*|]\}. \tag{70}$$

Then $M^* \geq 0$. Fix ε such that $0 < \varepsilon \leq \frac{\theta - \eta}{M^*+1}$. According to assertion 3° of Proposition 4, there exists an open set $O \subseteq V$ with $K \subseteq O$ and $\mu(O) < \mu(K) + \varepsilon$, so (since μ is finite)

$$\mu(O \setminus K) < \varepsilon. \tag{71}$$

Since the restrictions of F, G, H to the compact metric space $V \times [-|t^*|, |t^*|]$ are uniformly continuous, there exists a positive real $\delta^* \leq |t^*|$ such that

$$|F(x,t)| \leq \varepsilon,\ |G(x,t)| \leq \varepsilon,\ |H(x,t)| \leq \varepsilon,\ \text{for all } (x,t) \in V \times [-\delta^*, \delta^*]. \quad (72)$$

Let $\delta > 0$ be such that $\delta \leq \frac{\delta^*}{\delta^* + |t^*|}$. Then $\delta \leq \frac{1}{2}$ and

$$\frac{|t^*| \cdot \delta}{1-\delta} \leq \delta^*. \quad (73)$$

By Urysohn's Lemma, there exists a continuous function $\phi: V \to [0,1]$ such that $\phi(x) = 1$, for $x \in K$, and $\phi(x) = 0$, for $x \in (V \setminus O) \cup V_0$. Since X is dense in the space $(C_0(V), ||\cdot||_{\sup})$, we can find a function $u \in X$ with $||\phi - u||_{\sup} \leq \delta$. Thus $-\delta + \phi(x) \leq u(x) \leq \delta + \phi(x)$, for every $x \in V$. These yield the following inequalities

$$-\delta \leq u(x) \leq \delta + 1,\ \text{for all } x \in V, \quad (74)$$

$$-\delta + 1 \leq u(x) \leq \delta + 1,\ \text{for all } x \in K, \quad (75)$$

$$-\delta \leq u(x) \leq \delta,\ \text{for all } x \in V \setminus O. \quad (76)$$

Define $\ell: \mathbb{R} \to \mathbb{R}$ by $\ell(t) := \min\{t, 1-\delta\}$. Since ℓ is a Lipschitz map with $\ell(0) = 0$, Lemma 1 yields that $\tilde{u} := \ell \circ u \in X$. Using the definition of $\tilde{u}$, the fact that $\delta \leq 1-\delta$, and the relations (74)–(76), we get

$$-(1-\delta) \leq \tilde{u}(x) \leq 1-\delta,\ \text{for all } x \in V, \quad (77)$$

$$\tilde{u}(x) = 1-\delta,\ \text{for all } x \in K, \quad (78)$$

$$-\delta \leq \tilde{u}(x) \leq \delta,\ \text{for all } x \in V \setminus O. \quad (79)$$

Put now $u_1 := \frac{t^*}{1-\delta}\tilde{u} \in X$. Then the relations (77)–(79), respectively, yield, according to (73), that

$$|u_1(x)| \leq |t^*|,\ \text{for all } x \in V, \quad (80)$$

$$u_1(x) = t^*,\ \text{for all } x \in K, \quad (81)$$

$$|u_1(x)| \leq \delta^*,\ \text{for all } x \in V \setminus O. \quad (82)$$

From (80) and the definition of M^* in (70), we get

$$F(x, u_1(x)) \leq M^*,\ \text{for all } x \in V,$$

so, using (71), we obtain

$$\int_{O\setminus K} F(x, u_1(x))\mathrm{d}\mu \leq \int_{O\setminus K} M^*\mathrm{d}\mu \leq \varepsilon M^*. \tag{83}$$

The relations (72) and (82) imply

$$|F(x, u_1(x)| \leq \varepsilon, \text{ for all } x \in V \setminus O,$$

so

$$\int_{V\setminus O} F(x, u_1(x))\mathrm{d}\mu \leq \int_{V\setminus O} |F(x, u_1(x))|\mathrm{d}\mu \leq \varepsilon\mu(V \setminus O) \leq \varepsilon. \tag{84}$$

Finally, we conclude from (69), (81), (83), (84), and the choice of θ and ε that

$$\begin{aligned}\Phi(u_1) = \int_V F(x, u_1(x))\mathrm{d}\mu &= \int_K F(x, t^*)\mathrm{d}\mu + \int_{O\setminus K} F(x, u_1(x))\mathrm{d}\mu \\ &+ \int_{V\setminus O} F(x, u_1(x))\mathrm{d}\mu \leq \eta + \varepsilon M^* + \varepsilon \leq \eta + \theta - \eta = \theta < 0.\end{aligned}$$

One obtains similarly that $\Psi(u_1) < 0$ and $J(u_1) < 0$. Thus condition (C8) of Theorem 10 is satisfied. By applying assertion (b) of this theorem, we finish the proof. □

Example 2. Fix real numbers $t^* \in \mathbb{R} \setminus \{0\}$, $p \leq 2$, $q > 2$, $q_1 \geq 2$, $r_1 > 0$, $r_2 > q$, $s > 0$. Consider a differentiable function $F_1 : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned}&F_1(t) = |t|^{r_1}, \text{ for } |t| \text{ sufficiently small,}\\ &F_1(t) = |t|^{r_2}, \text{ for } |t| \text{ sufficiently large,}\\ &F_1(t^*) < 0.\end{aligned}$$

Let $\gamma_1 : V \to \mathbb{R}$ be a continuous map with $\gamma_1(x) > 0$, for all $x \in V$. Define the map $F : V \times \mathbb{R} \to \mathbb{R}$ by $F(x, t) := \gamma_1(x)F_1(t)$.

Consider a differentiable function $G_1 : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned}&G_1(t) = |t|^{s}, \text{ for } |t| \text{ sufficiently small,}\\ &G_1(t) = |t|^{p}, \text{ for } |t| \text{ sufficiently large,}\\ &G_1(t^*) < 0.\end{aligned}$$

Let $\gamma_2 : V \to \mathbb{R}$ be a continuous map with $\gamma_2(x) > 0$, for all $x \in V$. Define the map $G : V \times \mathbb{R} \to \mathbb{R}$ by $G(x, t) := \gamma_2(x)G_1(t)$.

Consider a differentiable function $H_1: \mathbb{R} \to \mathbb{R}$ such that

$$H_1(t) = |t|^{q_1}, \text{ for } |t| \text{ sufficiently small,}$$
$$H_1(t) = |t|^{q}, \text{ for } |t| \text{ sufficiently large,}$$
$$H_1(t^*) > 0.$$

Let $\gamma_3: V \to \mathbb{R}$ be a continuous map with $\gamma_3(x) < 0$, for all $x \in V$. Define the map $H: V \times \mathbb{R} \to \mathbb{R}$ by $H(x,t) := \gamma_3(x) H_1(t)$.

A straightforward computation yields that the functions F, G, H satisfy the conditions (1*)–(6*) of Theorem 11. Moreover, $F(x,t^*) < 0$, $G(x,t^*) < 0$, $H(x,t^*) < 0$, for all $x \in V$. In order to find a compact set $K \subseteq V \setminus V_0$ with $\mu(K) > 0$, fix $m \in \mathbb{N}$ with $m \geq 2$ and consider the level m decomposition (1) of V. If $w = (w_1, \ldots, w_m) \in W_m$ is so that there exist $i, j \in \{1, \ldots, m\}$ with $w_i \neq w_j$, then it can be proved easily that $S_w(V) \cap V_0 = \emptyset$. Put $K := S_w(V)$. The scaling property 2.1 in [9] implies now $\mu(K) > 0$. Hence condition (7*) of Theorem 11 is satisfied, too.

7 Conclusions

By treating nonlinear elliptic problems on a fractal, the results of this note complement the theory of nonlinear PDEs defined on (open) domains of Euclidean spaces. The symmetric structure of the fractal we worked on (namely, the Sierpinski gasket in the N dimensional Euclidean space), facilitated considerably our study since it allows to introduce in a quite natural manner, from a functional analytical point of view, a Laplace operator (in the literature named as weak Laplacian) on it. The major ingredient for defining the weak Laplacian on the Sierpinski gasket is an energy form that leads to a Hilbert space of continuous real-valued functions of finite energy, defined on the gasket. Due to the geometry of this fractal, there is a Sobolev-like inequality on the Sierpinski gasket, which yields that the previously mentioned Hilbert space can be compactly embedded in a space of continuous real-valued functions, endowed with the usual supremum norm. This compact embedding is actually the central element which allows to investigate PDEs on this fractal using variational methods. We have presented how one may combine these methods with certain abstract multiplicity theorems in order to get results concerning the existence of multiple (weak) solutions of certain nonlinear elliptic problems defined on the Sierpinski gasket. Finally, it is worthy of mention that the Sierpinski gasket is of particular interest in fractal theory since it is typical for the more general class of post-critically-finite fractals. Thus, the understanding and dealing with the phenomena in the case of PDEs on the Sierpinski gasket is the first step to consider such equations in the more general setting of post-critically-finite fractals.

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Initial Value Problems in Linear Integral Operator Equations

L.P. Castro, M.M. Rodrigues, and S. Saitoh

Abstract For some general linear integral operator equations, we investigate consequent initial value problems by using the theory of reproducing kernels. A new method is proposed which—in particular—generates a new field among initial value problems, linear integral operators, eigenfunctions and values, integral transforms and reproducing kernels. In particular, examples are worked out for the integral equations of Lalesco–Picard, Dixon, and Tricomi types.

Keywords Integral transform • Reproducing kernel • Isometric mapping • Inversion formula • Initial value problem • Eigenfunction • Eigenvalue • Fourier integral transform • Inverse problem • Lalesco–Picard equation • Dixon equation • Tricomi equation

Mathematics Subject Classification (2010): Primary 45C05; Secondary 32A30, 42A38, 45A05, 45D05, 45E05, 45P05, 46E22, 47A05.

1 Introduction

Despite the fact that initial value problems for differential equations, and consequent integral equations, have already a long and rich history, there is still the need in different cases to find out additional and more suitable spaces where their

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solutions can be interpreted and used in an appropriated way. Sometimes, finding new frameworks for those solutions leads even to the discovery of completely new solutions which could not be reached in a somehow more "classical" and known setting. This is just one of the reasons why it is still well appropriate to continue to perform research in such a classical field. Moreover, additional knowledge about the solutions is also very welcome—even in cases where we know already a great variety of solutions. This is the case of the study of different kinds of stability which is, e.g., highly relevant when we would like to apply numerical methods. Within this spirit, we would like to consider initial value problems in linear integral operator equations by using reproducing kernel Hilbert space machinery [1, 12–14].

Having those general goals in mind, in [4] the authors proposed a general method for the existence and construction of the solution of the following initial problem

$$(\partial_t + L_x)u_f(t,x) = 0, \quad t > 0, \tag{1}$$

satisfying the initial value condition

$$u_f(0,x) = f(x), \tag{2}$$

for some general linear operator L_x on a certain function space, and on some domain, by using the theory of reproducing kernels.

Here, we consider a general linear integral equation

$$I_x u(x) = 0 \tag{3}$$

and we assume that the eigenfunctions L_ν and values ν are known; that is,

$$I_x L_\nu(x) = \nu L_\nu(x). \tag{4}$$

Then, note that the functions

$$\exp(-\nu t)L_\nu(x) \tag{5}$$

are the solutions of the operator equation

$$(\partial_t + I_x)\, u(t,x) = 0. \tag{6}$$

In order to consider a fully general sum, in the case that ν are positive reals, we shall consider the kernel form, for a nonnegative continuous function ρ,

$$\mathscr{K}_t(x,y;\rho) = \int_0^{+\infty} \exp\{-\nu t\} L_\nu(x)\overline{L_\nu(y)}\rho(\nu)d\nu. \tag{7}$$

Of course, in here, we are considering the integral with absolutely convergence for the kernel form.

The fully general solutions of (6) may be represented in the integral form

$$u(t,x) = \int_0^{+\infty} \exp\{-\nu t\} L_\nu(x) F(\nu)\rho(\nu) d\nu, \tag{8}$$

for the functions F satisfying

$$\int_0^{+\infty} \exp\{-\nu t\} |F(\nu)|^2 \rho(\nu) d\nu < \infty. \tag{9}$$

Then, the solution $u(t,x)$ of (6) satisfying the initial condition

$$u(0,x) = F(x) \tag{10}$$

will be obtained by $t \to +0$ in (8) with a natural meaning. However, this point will be very delicate and we will need to consider some deep and beautiful structure. Here, (7) is a reproducing kernel and in order to analyze the logic above, we will need the theory of reproducing kernels within an essential (and beautiful) way. Indeed, in order to construct natural solutions of (1)–(2), we will need a new framework and function space.

2 Preliminaries on Linear Mappings and Inversions

In order to analyze the integral transform (8) and in view to set the basic background for our purpose, we will need the essence of the theory of reproducing kernels.

We are interested in the integral transforms (8) in the framework of Hilbert spaces. Of course, we are interested in the characterization of the image functions, the consequent isometric identity like the Parseval identity and the inversion formula, basically. For these general and fundamental problems, we have a unified and fundamental method and concept in the general situation. Namely, following [12–14], we shall recall a general theory for linear mappings in the framework of Hilbert spaces.

Let $\mathcal{H}$ be a Hilbert (possibly finite-dimensional) space. Let E be an abstract set and $\mathbf{h}$ be a Hilbert $\mathcal{H}$-valued function on E. Then, we shall consider the linear transform

$$f(x) = (\mathbf{f}, \mathbf{h}(x))_{\mathcal{H}}, \quad \mathbf{f} \in \mathcal{H}, \tag{11}$$

from $\mathcal{H}$ into the linear space $\mathcal{F}(E)$ comprising all the complex valued-functions on E. In order to investigate the linear mapping (11), we form a positive definite quadratic form function $K(x,y)$ on $E \times E$ defined by

$$K(x,y) = (\mathbf{h}(y), \mathbf{h}(x))_{\mathcal{H}} \quad \text{on} \quad E \times E.$$

A complex-valued function $k : E \times E \to \mathbb{C}$ is called a *positive definite quadratic form function* on the set E, or shortly, *positive definite function*, when it satisfies the property that, for an arbitrary function $X : E \to \mathbb{C}$ and any finite subset F of E,

$$\sum_{x,y \in F} \overline{X(x)} X(y) k(x, y) \geq 0. \tag{12}$$

By the fundamental theorem, we know that for any positive definite quadratic form function K, there exists a uniquely determined reproducing kernel Hilbert space admitting the reproducing property.

Then, we obtain the following fundamental result.

Proposition 1 (cf. [12–14]).

(A) The range of the linear mapping (11) by $\mathscr{H}$ is characterized as the reproducing kernel Hilbert space $H_K(E)$ admitting the reproducing kernel $K(x, y)$ whose characterization is given by the two properties: (i) $K(\cdot, y) \in H_K(E)$ for any $y \in E$ and, (ii) for any $f \in H_K(E)$ and for any $x \in E$, $(f(\cdot), K(\cdot.x))_{H_K(E)} = f(x)$.

(B) It holds

$$\|f\|_{H_K(E)} \leq \|\mathbf{f}\|_{\mathscr{H}}.$$

Here, for any member f of $H_K(E)$ there exists a uniquely determined $\mathbf{f}^* \in \mathscr{H}$ satisfying

$$f(x) = (\mathbf{f}^*, \mathbf{h}(x))_{\mathscr{H}} \quad \text{on } E$$

and

$$\|f\|_{H_K(E)} = \|\mathbf{f}^*\|_{\mathscr{H}}.$$

(C) We have the inversion formula in (11) in the form

$$f \mapsto \mathbf{f}^* \tag{13}$$

in (B) by using the reproducing kernel Hilbert space $H_K(E)$.

However, in general, this formula (13) is not obvious. Consequently, case by case, we need different arguments to analyse it. See [13] and [14] for the details and applications. Recently, however, we obtained a very general inversion formula, based on the so-called *Aveiro Discretization Method in Mathematics* (cf. [2]), by using the ultimate realization of reproducing kernel Hilbert spaces that is introduced simply in the last section. In this paper, however, in order to give prototype examples with analytical nature, we shall consider the following global inversion formula in the general situation with natural assumptions.

Here we consider a concrete case of Proposition 1. In order to derive a general inversion formula widely applicable in analysis, we assume that $\mathscr{H} = L^2(I, dm)$ and that $H_K(E)$ is a closed subspace of $L^2(E, d\mu)$. Furthermore, below we assume that $(I, \mathscr{I}, dm)$ and $(E, \mathscr{E}, d\mu)$ are both σ-finite measure spaces and that

$$H_K(E) \hookrightarrow L^2(E, d\mu). \tag{14}$$

Suppose that we are given a measurable function $h : I \times E \to \mathbb{C}$ satisfying $h_y = h(\cdot, y) \in L^2(I, dm)$ for all $y \in E$. Let us set

$$K(x, y) \equiv \langle h_y, h_x \rangle_{L^2(I,dm)}. \tag{15}$$

As we have established in Proposition 1, we have

$$H_K(E) \equiv \{ f \in \mathscr{F}(E) \ : \ f(x) = \langle F, h_x \rangle_{L^2(I,dm)} \text{ for some } F \in \mathscr{H} \}. \tag{16}$$

Let us now define

$$L : \mathscr{H} \to H_K(E) (\hookrightarrow L^2(E, d\mu)) \tag{17}$$

by

$$LF(x) \equiv \langle F, h_x \rangle_{L^2(I,dm)} = \int_I F(\lambda) \overline{h(\lambda, x)} \, dm(\lambda), \quad x \in E, \tag{18}$$

for $F \in \mathscr{H} = L^2(I, dm)$, keeping in mind (14). Observe that $LF \in H_K(E)$.

The next result will serve to the inversion formula.

Proposition 2 (cf. [13]). *Assume that $\{E_N\}_{N=1}^{\infty}$ is an increasing sequence of measurable subsets in E such that*

$$\bigcup_{N=1}^{\infty} E_N = E \tag{19}$$

and that

$$\int_{I \times E_N} |h(\lambda, x)|^2 \, dm(\lambda) \, d\mu(x) < \infty \tag{20}$$

for all $N \in \mathbb{N}$. Then we have

$$L^* f(\lambda) \left(= \lim_{N \to \infty} (L^*[\chi_{E_N} f])(\lambda) \right) = \lim_{N \to \infty} \int_{E_N} f(x) h(\lambda, x) \, d\mu(x) \tag{21}$$

for all $f \in L^2(I, d\mu)$ *in the topology of* $\mathscr{H} = L^2(I, dm)$. *Here,* $L^* f$ *is the adjoint operator of* L*, but it represents the inversion with the minimum norm for* $f \in H_K(E)$.

Moreover, in this Proposition 2, we see that—in a very natural way—the inversion formula may be given in the strong convergence in the space $\mathscr{H} = L^2(I, dm)$.

3 Main Result

Following the general theory in Sect. 2, we shall now build our results. Without loss of generality, we will assume that ν is on the positive real line.

Then, we form the reproducing kernel

$$\mathscr{K}(x, y; \rho) = \int_0^{+\infty} L_\nu(x)\overline{L_\nu(y)}\rho(\nu)d\nu, \quad t > 0, \tag{22}$$

and consider the reproducing kernel Hilbert space $H_{\mathscr{K}(\rho)}(\mathbb{R}^+)$ admitting the kernel $\mathscr{K}(x, y; \rho)$. Here, we assume that the kernel form converges in the absolute sense. In particular, note that

$$\mathscr{K}_t(x, y; \rho) \in H_{\mathscr{K}(\rho)}(\mathbb{R}^+), \quad y > 0.$$

Then, we obtain the main theorem in this paper:

Theorem 1. *For any member* $f \in H_{\mathscr{K}(\rho)}(\mathbb{R}^+)$*, the solution* $u_f(t, x)$ *of the initial value problem, for* $t > 0$*,*

$$(\partial_t + I_x)u_f(t, x) = 0, \tag{23}$$

satisfying the initial value condition

$$u_f(0, x) = f(x), \tag{24}$$

exists and it is given by

$$u_f(t, x) = (f(\cdot), \mathscr{K}_t(\cdot, x; \rho))_{H_{\mathscr{K}(\rho)}(\mathbb{R}^+)}. \tag{25}$$

Here, the meaning of the initial value (24) is given by

$$\begin{aligned} \lim_{t\to+0} u_f(t, x) &= \lim_{t\to+0} (f(\cdot), \mathscr{K}_t(\cdot, x; \rho))_{H_{\mathscr{K}(\rho)}(\mathbb{R}^+)} \\ &= (f(\cdot), \mathscr{K}(\cdot, x; \rho))_{H_{\mathscr{K}(\rho)}(\mathbb{R}^+)} \\ &= f(x), \end{aligned} \tag{26}$$

whose existence is ensured and the limit is the uniformly convergence on any subset of $\mathbb{R}^+$ *such that* $\mathscr{K}(x, x; \rho)$ *is bounded.*

The uniqueness property of the initial value problem is depending on the completeness of the family of functions

$$\{\mathscr{K}_t(\cdot, x; \rho) : x \in \mathbb{R}^+\} \tag{27}$$

in $H_{\mathscr{K}(\rho)}(\mathbb{R}^+)$.

Before starting with the proof of the theorem some remarks are in order. In our theorem, the complete property of the solutions $u_f(t, x)$ of (23) and (24) satisfying the initial value f may be derived by the reproducing kernel Hilbert space admitting the kernel

$$k(x, t; y, \tau; \rho) := (\mathscr{K}_\tau(\cdot, y; \rho), \mathscr{K}_t(\cdot, x; \rho))_{H_{\mathscr{K}(\rho)}(\mathbb{R}^+)}. \tag{28}$$

In our method, we see that the existence problem of the initial value problem is based on the eigenfunctions and we are constructing the desired solution satisfying the desired initial condition. For a larger knowledge for the eigenfunctions we can consider a more general initial value problem.

Furthermore, by considering the linear mapping of (25) with various situations, we will be able to obtain various inverse problems looking for the initial values f from the various output data of $u_f(t, x)$.

Proof (of Theorem 1). In first place, note that the kernel $\mathscr{K}_t(x, y; \rho)$ satisfies the operator equation (23) for any fixed y, because the functions

$$\exp\{-\nu t\} L_\nu(x)$$

satisfy the operator equation and it is the summation. Similarly, the function $u_f(t, x)$ defined by (25) is the solution of the operator equation (23).

Secondly, in order to see the initial value problem, we note the important general property

$$\mathscr{K}_t(x, y; \rho) \ll \mathscr{K}(x, y; \rho); \tag{29}$$

that is, $\mathscr{K}(x, y; \rho) - \mathscr{K}_t(x, y; \rho)$ is a positive definite quadratic form function. Moreover, we have

$$H_{\mathscr{K}_t(\rho)} \subset H_{\mathscr{K}(\rho)}(\mathbb{R}^+)$$

and for any function $f \in H_{\mathscr{K}_t(\rho)}$

$$\|f\|_{H_{\mathscr{K}(\rho)}(\mathbb{R}^+)} = \lim_{t \to +0} \|f\|_{H_{\mathscr{K}_t(\rho)}}$$

in the sense of non-decreasing norm convergence (see [1]). In order to see the crucial point in (26), note that

$$\begin{aligned}
&\|\mathscr{K}(y,x;\rho) - \mathscr{K}_t(y,x;\rho)\|^2_{H_{\mathscr{K}(\rho)}(\mathbb{R}+)} \\
&\quad = \mathscr{K}(x,x;\rho) - 2\mathscr{K}_t(x,x;\rho) + \|\mathscr{K}_t(y,x;\rho)\|^2_{H_{\mathscr{K}(\rho)}(\mathbb{R}+)} \\
&\quad \leq \mathscr{K}(x,x;\rho) - 2\mathscr{K}_t(x,x;\rho) + \|\mathscr{K}_t(y,x;\rho)\|^2_{H_{\mathscr{K}_t(\rho)}} \\
&\quad = \mathscr{K}(x,x;\rho) - \mathscr{K}_t(x,x;\rho),
\end{aligned}$$

that converses to zero as $t \to +0$. We thus obtain the desired limit property in the theorem.

Finally, the uniqueness property of the initial value problem follows from (25) easily.

4 Examples

At first, we shall consider the typical and famous Lalesco–Picard equation:

$$y(x) - \lambda \int_{-\infty}^{\infty} e^{-|x-t|} y(t)dt = 0, \quad \lambda > 0. \tag{30}$$

Then, we know the general solution

$$y(x) = C_1 \exp(x\sqrt{1-2\lambda}) + C_2 \exp(-x\sqrt{1-2\lambda}), \quad 0 < \lambda < \frac{1}{2}, \tag{31}$$

and

$$y(x) = C_1 \cos(x\sqrt{2\lambda - 1}) + C_2 \sin(x\sqrt{2\lambda - 1}), \quad \frac{1}{2} < \lambda, \tag{32}$$

(see [7, 9]).

From the results, we can consider the four eigenfunctions and eigenvalues groups; so, without loss of generality, we shall consider the case, for

$$y_\lambda(x) = \exp(-x\sqrt{1-2\lambda}) \tag{33}$$

in which we have

$$\int_{-\infty}^{\infty} e^{-|x-t|} y_\lambda(t)dt = \frac{1}{\lambda} y_\lambda(x), \quad 0 < \lambda < \frac{1}{2}. \tag{34}$$

Therefore, by a suitable weight ρ, we shall consider the reproducing kernel

$$\int_0^{1/2} \exp(-y\sqrt{1-2\lambda}) \exp(-x\sqrt{1-2\lambda})\rho(\lambda)d\lambda. \tag{35}$$

Note that we can consider many weights ρ, however, as the simplest case, we obtain the reproducing kernel

$$\begin{aligned} K(x,y) &= \int_0^{1/2} \exp(-y\sqrt{1-2\lambda}) \exp(-x\sqrt{1-2\lambda}) \frac{1}{\sqrt{1-2\lambda}} d\lambda \\ &= \frac{1}{x+y}\left(1 - e^{-x}e^{-y}\right). \end{aligned} \tag{36}$$

Now, we are interested in the integral transform

$$f(x) = \int_0^{1/2} F(\lambda) \exp(-x\sqrt{1-2\lambda}) \frac{1}{\sqrt{1-2\lambda}} d\lambda \tag{37}$$

for the functions F satisfying the conditions

$$\int_0^{1/2} |F(\lambda)|^2 \frac{1}{\sqrt{1-2\lambda}} d\lambda < \infty. \tag{38}$$

Note that for the kernel form

$$\frac{1}{z+\overline{u}}, \quad z = x + iy, \tag{39}$$

on the right half complex plane, this reproducing kernel is the Szegö kernel and for the image of the integral transform

$$f(z) = \int_0^{\infty} e^{-\lambda z} F(\lambda) d\lambda, \tag{40}$$

for the $L_2(0,\infty)$ functions $F(\lambda)$, we obtain the isometric identity

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} |f(iy)|^2 dy = \int_0^{\infty} |F(\lambda)|^2 d\lambda. \tag{41}$$

Here, $f(iy)$ means the Fatou's non-tangential boundary values of the Szegö space of analytic functions on the right-hand half complex plane.

From the relation for analytic extension

$$K(z,\overline{u}) \ll \frac{1}{z+\overline{u}} \tag{42}$$

(in the sense that the right-hand side minus the left-hand side is a positive definite quadratic form function), we see that the admissible reproducing kernel Hilbert spaces H_K and H_S have the inclusion relation as functions

$$H_K \subset H_S \tag{43}$$

and we have the norm inequality, for any $f \in H_K$,

$$\|f\|_{H_S} \leq \|f\|_{H_K}. \tag{44}$$

The space H_K is a subspace of the Szegö space H_S and so we can use the Szegö space H_S for the isometric identity and inversion formula. For extra details on these general properties, see [13]. As the conclusions, we see that the image $f(x)$ of the integral transform (37) is extensible analytically onto the right half complex plane as $f(z)$, $z = x + iy$, and we obtain the norm inequalities

$$\int_0^{1/2} |F(\lambda)|^2 \frac{1}{\sqrt{1-2\lambda}} d\lambda \leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} |f(iy)|^2 dy. \tag{45}$$

Furthermore, we obtain the inversion formula in the space satisfying (38)

$$F(\lambda) = \int_{-\infty}^{\infty} f(iy) \exp(-iy\sqrt{1-2\lambda}) dy. \tag{46}$$

For other eigenfunctions, we can obtain similar results. For other weighted functions, we can obtain more complicated results; see [13] for consequent details.

Next, as a typical example of Volterra integral equations, we shall consider, the Dixon's equation

$$y(x) - \lambda \int_0^x \frac{y(t)dt}{x+t} = f(x), \quad \lambda > 0. \tag{47}$$

For the homogeneous case of $f \equiv 0$, we know the solutions

$$y(x) = Cx^{\beta}; \quad \beta > -1, \tag{48}$$

where

$$\lambda = \frac{1}{I(\beta)}; \qquad I(\beta) = \int_0^1 \frac{\xi^{\beta} d\xi}{1+\xi}, \tag{49}$$

(cf. [9, p. 136]). Therefore, for the integral operator

$$\int_0^x \frac{y(t)dt}{x+t} \tag{50}$$

we have the eigenfunctions and eigenvalues

$$y(t) = I(\lambda)t^{\lambda}, \quad \lambda > -1. \tag{51}$$

Thus, we obtain the related reproducing kernel, for $0 < x, y < 1$, for example,

$$\begin{aligned} K(x,y) &= \int_{-1}^{\infty} x^{\lambda} y^{\lambda} d\lambda \\ &= -\frac{1}{\ln xy} \frac{1}{xy}. \end{aligned} \tag{52}$$

The property of the integral transform

$$f(x) = \int_{-1}^{\infty} x^{\lambda} F(\lambda) d\lambda \tag{53}$$

for the functions F satisfying

$$\int_{-1}^{\infty} |F(\lambda)|^2 d\lambda < \infty \tag{54}$$

will be involved. However, we see that the image f is extensible analytically onto the complex plane cutted by the half real line $(-\infty, 0]$.

By the complex conformal mapping $W = \log z$, the image $f(z) = f(e^w)$ of (53) may be discussed by the Szegö space on the strip domain

$$\left\{ \Im w < \frac{\pi}{2} \right\}.$$

At last, as a typical example of singular integral equations, we shall consider the Tricomi equation

$$y(x) - \lambda \int_0^1 \left(\frac{1}{t-x} + \frac{1}{x+t-2xt} \right) y(t) dt = f(x), \quad \lambda > 0. \tag{55}$$

For the homogeneous case of $f \equiv 0$, we know the solutions

$$y(x) = C \frac{(1-x)^{\beta}}{x^{1+\beta}}; \quad \tan \frac{\beta\pi}{2} = \lambda\pi, \quad -2 < \beta < 0 \tag{56}$$

(cf. [9, p. 769]). Therefore, for the integral operator

$$\int_0^1 \left(\frac{1}{t-x} + \frac{1}{x+t-2xt} \right) y(t) dt, \tag{57}$$

we have the eigenfunctions and eigenvalues

$$y(t) = \frac{1}{\lambda} \frac{(1-x)^{\frac{2}{\pi} \arctan \pi\lambda}}{x^{1+\frac{2}{\pi} \arctan \pi\lambda}}. \tag{58}$$

Therefore, we obtain the related reproducing kernel, for $0 < x, y < 1$ and for example, for $0 < \lambda < \infty$:

$$\begin{aligned} K(x, y) &= \int_0^\infty \frac{(1-x)^{\frac{2}{\pi} \arctan \pi\lambda}}{x^{1+\frac{2}{\pi} \arctan \pi\lambda}} \frac{(1-y)^{\frac{2}{\pi} \arctan \pi\lambda}}{y^{1+\frac{2}{\pi} \arctan \pi\lambda}} d\lambda \\ &= \frac{1}{2} \int_0^1 \frac{(1-x)^\xi}{x^{1+\xi}} \frac{(1-y)^\xi}{y^{1+\xi}} \sec^2 \frac{\pi\xi}{2} d\xi. \end{aligned} \tag{59}$$

As typical integral equations, we stated the above three integral equations. However, we can consider many and many different integral equations, and the eigenfunctions structures will be mysterious deep and we are requested to analyze their structures and the corresponding integral transforms. Furthermore, we are particularly interested in kernel form integrals. To this end, the great book [9] presents a huge range of possibilities.

5 Concrete Realization of the Reproducing Kernel Hilbert Spaces

In Sect. 4, we can consider many concrete forms of the reproducing kernels and among those we have very complicated structures of the related reproducing kernel Hilbert spaces. Even just from the point of view of the theory of reproducing kernels, their realizations will give interesting research topics that are requested separate papers.

We were able to realize the important reproducing kernel Hilbert spaces concretely and analytically. However, for many kernels their realizations will be complicated. Despite this difficulty, it is clear that the concrete forms of the reproducing kernels will be very important and interested by themselves.

Meanwhile, we are also interested in the kernel forms $\mathscr{K}_t$ and k. These calculations will create a new and large field in integral formulas.

As explained, we have to analyze and realize the corresponding reproducing kernel Hilbert spaces. However, we can also apply quite general formula by the Aveiro discretization method exposed in [2, 3]. In these papers, numerical experiments are also given based on the following result:

Proposition 3 (Ultimate Realization of Reproducing Kernel Hilbert Spaces). *In our general situation and for a uniqueness set $\{p_j\}$ for the reproducing kernel Hilbert space H_K of the set E satisfying the linearly independence of $K(\cdot, p_j)$ for any finite number of the points p_j, we obtain*

$$\|f\|_{H_K}^2 = \|\boldsymbol{f}^*\|_{\mathscr{H}}^2 = \lim_{n\to\infty} \sum_{j=1}^{n} \sum_{j'=1}^{n} f(p_j)\widetilde{a_{jj'}}\overline{f(p_{j'})}. \tag{60}$$

Here, $\widetilde{a_{jj'}}$ *is the element of the complex conjugate inverse of the positive definite Hermitian matrix formed by*

$$a_{jj'} = K(p_j, p_{j'}).$$

In this proposition, for the uniqueness set of the space, if the reproducing kernel is analytical, then, the criteria will be very simple by the *identity theorem of analytic functions*. For the Sobolev space cases, we have to consider some dense subset of E for the uniqueness set. Meanwhile, the linearly independence will be easily derived from the integral representations of the kernels.

From the great mathematicians books [5,6,8,10,11], we can find many and many concrete problems among partial differential equations, eigenfunctions, integral transforms, and reproducing kernels which are admitting the application of these results.

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Extension Operators that Preserve Geometric and Analytic Properties of Biholomorphic Mappings

Teodora Chirilă

Abstract In this survey we are concerned with certain extension operators which take a univalent function f on the unit disc U to a univalent mapping F from the Euclidean unit ball B^n in $\mathbb{C}^n$ into $\mathbb{C}^n$, with the property that $f(z_1) = F(z_1, 0)$. This subject began with the Roper–Suffridge extension operator, introduced in 1995, which has the property that if f is a convex function of U then F is a convex mapping of B^n. We consider certain generalizations of the Roper–Suffridge extension operator. We show that these operators preserve the notion of g-Loewner chains, where $g(\zeta) = (1-\zeta)/(1+(1-2\gamma)\zeta)$, $|\zeta| < 1$ and $\gamma \in (0,1)$. As a consequence, the considered operators preserve certain geometric and analytic properties, such as g-parametric representation, starlikeness of order γ, spirallikeness of type δ and order γ, almost starlikeness of order δ and type γ.

We use the method of Loewner chains to generate certain subclasses of normalized biholomorphic mappings on the Euclidean unit ball B^n in $\mathbb{C}^n$, which have interesting geometric characterizations. We obtain the characterization of g-starlike and g-spirallike mappings of type $\alpha \in (-\pi/2, \pi/2)$, as well as of g-almost starlike mappings of order $\alpha \in [0,1)$, by using g-Loewner chains. Also, we will show that, under certain assumptions, the mapping $F(z) = P(z)z$, $z \in B^n$, has g-parametric representation on B^n, where $P : B^n \to \mathbb{C}$ is a holomorphic function such that $P(0) = 1$.

Keywords Biholomorphic mapping • g-Loewner chain • g-Parametric representation • g-Starlike mapping • Loewner chain • Parametric representation • Roper–Suffridge extension operator • Spirallike mapping • Starlike mapping • Subordination

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1 Introduction and Preliminaries

Let $\mathbb{C}^n$ denote the space of n complex variables $z = (z_1, \ldots, z_n)$ with the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^{n} z_j \overline{w}_j$ and the Euclidean norm $\|z\| = \langle z, z\rangle^{1/2}$. The open ball $\{z \in \mathbb{C}^n : \|z\| < r\}$ is denoted by B_r^n and the unit ball B_1^n is denoted by B^n. In the case of one complex variable, B^1 is denoted by U.

Let $L(\mathbb{C}^n, \mathbb{C}^m)$ denote the space of linear continuous operators from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the standard operator norm, $\|A\| = \sup\{\|A(z)\| : \|z\| = 1\}$ and let I_n be the identity of $L(\mathbb{C}^n, \mathbb{C}^n)$. If Ω is a domain in $\mathbb{C}^n$, we denote by $H(\Omega)$ the set of holomorphic mappings from Ω into $\mathbb{C}^n$. Also, let $H(B^n, \mathbb{C})$ be the set of holomorphic functions from B^n into $\mathbb{C}$. If $f \in H(B^n)$, we say that f is normalized if $f(0) = 0$ and $Df(0) = I_n$. We say that $f \in H(B^n)$ is locally biholomorphic on B^n if the complex Jacobian matrix $Df(z)$ is nonsingular at each $z \in B^n$. Let $\mathscr{L}S_n$ be the set of normalized locally biholomorphic mappings on B^n. We denote by $S(B^n)$ the set of normalized biholomorphic mappings on B^n. We also denote by $S^*(B^n)$ (respectively $K(B^n)$) the subset of $S(B^n)$ consisting of starlike mappings with respect to zero (respectively convex mappings). In the case of one complex variable, we write $\mathscr{L}S_1 = \mathscr{L}S$, $S(B^1) = S$, $K(B^1) = K$ and $S^*(B^1) = S^*$.

We next consider some subclasses of $S(B^n)$.

A locally biholomorphic mapping $f : B^n \to \mathbb{C}^n$ such that $f(0) = 0$ is starlike if and only if (see [35])

$$\mathrm{Re}\,\langle [Df(z)]^{-1} f(z), z\rangle > 0,\ z \in B^n\backslash\{0\}.$$

The following notion of starlikeness of order γ was introduced by Curt and Kohr (see [8, 21]).

A mapping $f \in \mathscr{L}S_n$ is called starlike of order $\gamma \in (0, 1)$ if [8, 21]

$$\left|\frac{1}{\|z\|^2}\langle [Df(z)]^{-1} f(z), z\rangle - \frac{1}{2\gamma}\right| < \frac{1}{2\gamma},\ z \in B^n\backslash\{0\}.$$

In the case $n = 1$, we obtain the usual class of starlike functions of order γ on the unit disc U. Obviously, if f is starlike of order γ, then f is starlike, and hence biholomorphic on B^n. We denote by $S_\gamma^*(B^n)$ the set of starlike mappings of order γ. In the case of one complex variable, $S_\gamma^*(U)$ is denoted by S_γ^*. It is known that $K \subset S_{1/2}^*$ (see e.g. [11]). Also, if $f \in K(B^n)$, then $f \in S_{1/2}^*(B^n)$ [8, 21].

A closely related notion to starlikeness of order $\gamma \in (0, 1)$ is that of spirallikeness of type δ and order γ, where $\delta \in (-\pi/2, \pi/2)$. This notion was studied by Liu and Liu [24] and Chirilă [4] (cf. [19]). We first recall the definition of spirallike mappings of type δ.

A normalized locally biholomorphic mapping $f : B^n \to \mathbb{C}^n$ is said to be spirallike of type $\delta \in (-\pi/2, \pi/2)$ if and only if (see [19])

$$\mathrm{Re}[e^{-i\delta}\langle [Df(z)]^{-1}f(z), z\rangle] > 0, \quad z \in B^n \setminus \{0\}.$$

The class of spirallike mappings of type δ on B^n is denoted by $\hat{S}_\delta(B^n)$. When $n = 1$, $\hat{S}_\delta(B^1)$ is denoted by $\hat{S}_\delta$.

Definition 1. Let $f \in \mathscr{L}S_n$, $\delta \in (-\pi/2, \pi/2)$ and $\gamma \in (0,1)$. We say that f is spirallike of type δ and order γ if

$$\left| e^{-i\delta}\frac{1}{\|z\|^2}\langle [Df(z)]^{-1}f(z), z\rangle + i\sin\delta - \frac{\cos\delta}{2\gamma}\right| < \frac{\cos\delta}{2\gamma}, \; z \in B^n \setminus \{0\}. \quad (1)$$

From (1) we deduce that if f is spirallike of type δ and order γ, then f is also spirallike of type δ. Hence $f \in S(B^n)$ (see [19]).

We next present the definition of almost starlike mappings of order δ and type γ. This notion was introduced by Chirilă [5]. We first recall the definition of almost starlike mappings of order δ.

A normalized locally biholomorphic mapping $f : B^n \to \mathbb{C}^n$ is said to be almost starlike of order $\delta \in [0,1)$ if (see [37])

$$\mathrm{Re}\,\langle [Df(z)]^{-1}f(z), z\rangle > \delta\|z\|^2, \; z \in B^n \setminus \{0\}.$$

Definition 2. Let $f \in \mathscr{L}S_n$, $\delta \in [0,1)$ and $\gamma \in (0,1)$. A mapping $f \in \mathscr{L}S_n$ is almost starlike of order δ and type γ if

$$\left| \frac{1}{\|z\|^2}\langle [Df(z)]^{-1}f(z), z\rangle - \delta - \frac{1-\delta}{2\gamma}\right| < \frac{1-\delta}{2\gamma}, \; z \in B^n \setminus \{0\}. \quad (2)$$

It is obvious that if f satisfies (2), then f is almost starlike of order δ, hence $f \in S(B^n)$ (see [37]). In fact, f is also starlike on B^n.

The following class of mappings plays the role of the Carathéodory class in $\mathbb{C}^n$ (see e.g. [11, 28, 35]):

$$\mathscr{M} = \{h \in H(B^n) : h(0) = 0, Dh(0) = I_n, \; \mathrm{Re}\,\langle h(z), z\rangle > 0, z \in B^n \setminus \{0\}\}.$$

In the case $n = 1$, $h \in \mathscr{M}$ if and only if $p \in \mathscr{P}$, where $h(\zeta) = \zeta p(\zeta)$ for $\zeta \in U$, and $\mathscr{P}$ is the well known Carathéodory class

$$\mathscr{P} = \{f \in H(U) : f(0) = 1, \; \mathrm{Re} f(\zeta) > 0, \zeta \in U\}.$$

The class $\mathscr{M}$ plays an important role in the study of Loewner chains and the Loewner differential equation in several complex variables, as well as in characterizing certain classes of biholomorphic mappings on the Euclidean unit ball B^n in $\mathbb{C}^n$ (for details, see [11]).

Next, we recall the definitions of subordination and Loewner chains. For various results related to Loewner chains in $\mathbb{C}^n$, the reader may consult [11, 14, 16, 18, 28]. Recent results of the theory of Loewner chains and the generalized Loewner differential equation in one and higher dimensions are due to Bracci et al. [3], Arosio et al. [2], etc. For further details, see the survey of Abate et al. [1].

Let $f, g \in H(B^n)$. We say that f is subordinate to g (and write $f \prec g$) if there is a Schwarz mapping v (i.e. $v \in H(B^n)$ and $\|v(z)\| \leq \|z\|$, $z \in B^n$) such that $f(z) = g(v(z))$, $z \in B^n$.

Definition 3. A mapping $f : B^n \times [0, \infty) \to \mathbb{C}^n$ is called a Loewner chain if $f(\cdot, t)$ is biholomorphic on B^n, $f(0, t) = 0$, $Df(0, t) = e^t I_n$ for $t \geq 0$, and $f(\cdot, s) \prec f(\cdot, t)$ whenever $0 \leq s \leq t < \infty$.

The requirement $f(z, s) \prec f(z, t)$ is equivalent to the condition that there is a unique biholomorphic Schwarz mapping $v = v(z, s, t)$, called the transition mapping associated to $f(z, t)$ such that $f(z, s) = f(v(z, s, t), t)$ for $z \in B^n$, $t \geq s \geq 0$.

Remark 1. Certain subclasses of $S(B^n)$ can be characterized in terms of Loewner chains. In particular, $f \in H(B^n)$ is starlike if and only if $f(z, t) = e^t f(z)$ is a Loewner chain (see e.g. [35]). Moreover, f is spirallike of type $\delta \in (-\pi/2, \pi/2)$ if and only if $f(z, t) = e^{(1-ia)t} f(e^{iat} z)$ is a Loewner chain, where $a = \tan \delta$ (see [19]). Also, f is almost starlike of order $\delta \in [0, 1)$ if and only if $f(z, t) = e^{\frac{t}{1-\delta}} f(e^{\frac{\delta t}{\delta - 1}} z)$ is a Loewner chain (see [37]).

The following characterization of Loewner chains was obtained by Pfaltzgraff [28] and Graham et al. [14] and yields that the Loewner differential equation provides subordination chains.

Lemma 1. *Let $h_t(z) = h(z, t) : B^n \times [0, \infty) \to \mathbb{C}^n$ satisfy the following conditions:*

(i) $h(\cdot, t) \in \mathcal{M}$ for $t \geq 0$.
(ii) $h(z, \cdot)$ is measurable on $[0, \infty)$ for $z \in B^n$.

Let $f = f(z, t) : B^n \times [0, \infty) \to \mathbb{C}^n$ be a mapping such that $f(\cdot, t) \in H(B^n)$, $f(0, t) = 0$, $Df(0, t) = e^t I_n$ for $t \geq 0$ and $f(z, \cdot)$ is locally absolutely continuous on $[0, \infty)$ locally uniformly with respect to $z \in B^n$. Assume that

$$\frac{\partial f}{\partial t}(z, t) = Df(z, t) h(z, t), \ \ a.e. \ t \geq 0, \ \forall z \in B^n.$$

Further, assume that there exists an increasing sequence $\{t_m\}_{m \in \mathbb{N}}$ such that $t_m > 0$, $t_m \to \infty$ and $\lim_{m \to \infty} e^{-t_m} f(z, t_m) = F(z)$ locally uniformly on B^n. Then $f(z, t)$ is a Loewner chain.

Remark 2. (i) Graham et al. [16] (see also [11]) proved that if $f(z, t)$ is a Loewner chain on B^n, then $f(z, \cdot)$ is locally Lipschitz on $[0, \infty)$ locally uniformly with respect to $z \in B^n$. Also, there exists a mapping $h = h(z, t)$, which satisfies the conditions (i) and (ii) in Lemma 1, such that (see [14])

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad \text{a.e. } t \geq 0, \quad \forall z \in B^n. \tag{3}$$

(ii) The mapping $h = h(z,t)$ which occurs in the Loewner differential equation (3) is unique up to a measurable set of measure zero which is independent of $z \in B^n$, i.e. if there is another mapping $q = q(z,t)$ such that $q(\cdot,t) \in \mathscr{M}$ for a.e. $t \geq 0$, $q(z,\cdot)$ is measurable on $[0,\infty)$ for $z \in B^n$, and such that the Loewner differential equation (3) holds for $q(z,t)$, then $h(\cdot,t) = q(\cdot,t)$, a.e. $t \geq 0$ (see e.g. [2]).

We next recall the class $\mathscr{M}_g$ and the notions of g-Loewner chains and g-parametric representation (see [14]).

Definition 4. Let $g \in H(U)$ be a univalent function such that $g(0) = 1$, $g(\bar{\zeta}) = \overline{g(\zeta)}$ for $\zeta \in U$ (i.e. g has real coefficients), $\operatorname{Re} g(\zeta) > 0$ on U, and assume that g satisfies the following conditions for $r \in (0,1)$:

$$\begin{cases} \min_{|\zeta|=r} \operatorname{Re} g(\zeta) = \min\{g(r), g(-r)\} \\ \max_{|\zeta|=r} \operatorname{Re} g(\zeta) = \max\{g(r), g(-r)\} \end{cases} \tag{4}$$

We define the class $\mathscr{M}_g$, where g satisfies the assumptions of Definition 4. This class was introduced by Graham et al. [14]. The class $\mathscr{M}_g$ is given by

$$\mathscr{M}_g = \Big\{h : B^n \to \mathbb{C}^n : h \in H(B^n),\ h(0) = 0,\ Dh(0) = I_n,$$

$$\Big\langle h(z), \frac{z}{\|z\|^2}\Big\rangle \in g(U),\ z \in B^n\Big\}.$$

Note that $\langle h(z), \frac{z}{\|z\|^2}\rangle$ is understood to have the value 1 (its limiting value) when $z = 0$. Clearly, $\mathscr{M}_g \subseteq \mathscr{M}$ and if $g(\zeta) \equiv \frac{1-\zeta}{1+\zeta}$, then $\mathscr{M}_g \equiv \mathscr{M}$. Note that $\mathscr{M}_g \neq \emptyset$, since $\mathrm{id}_{B^n} \in \mathscr{M}_g$.

We next present the definitions of a g-Loewner chain and g-parametric representation (cf. [14]; compare with [16] and [32] for $g(\zeta) \equiv \frac{1-\zeta}{1+\zeta}$).

Definition 5. We say that a mapping $f = f(z,t) : B^n \times [0,\infty) \to \mathbb{C}^n$ is a g-Loewner chain if $f(z,t)$ is a Loewner chain such that $\{e^{-t} f(\cdot,t)\}_{t\geq 0}$ is a normal family on B^n and the mapping $h = h(z,t)$ which occurs in the Loewner differential equation (3) satisfies the condition $h(\cdot,t) \in \mathscr{M}_g$ for a.e. $t \geq 0$.

Remark 3. In the case of one complex variable, if $f(\zeta,t)$ is a Loewner chain, then $\{e^{-t} f(\cdot,t)\}_{t\geq 0}$ is a normal family on U, and there exists a function $p = p(\zeta,t)$ such that (see [14]) $p(\cdot,t) \in \mathscr{P}$ for $t \geq 0$, $p(\zeta,\cdot)$ is measurable on $[0,\infty)$ for $\zeta \in U$, and (see [31])

$$\frac{\partial f}{\partial t}(\zeta,t) = \zeta f'(\zeta,t)p(\zeta,t), \quad \text{a.e.} \quad t \geq 0, \quad \forall \zeta \in U. \tag{5}$$

Hence, in the case $n = 1$, a g-Loewner chain $f(\zeta, t)$ is a Loewner chain such that the function $p(\zeta, t)$ defined by (5) satisfies the condition $p(\cdot, t) \in g(U)$ for a.e. $t \geq 0$. In the case $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $|\zeta| < 1$, any Loewner chain on the unit disc is also a g-Loewner chain.

Definition 6. Let $g : U \to \mathbb{C}$ satisfy the assumptions of Definition 4. We say that a normalized mapping $f \in H(B^n)$ has g-parametric representation if there exists a g-Loewner chain $f(z, t)$ such that $f = f(\cdot, 0)$.

We denote by $S_g^0(B^n)$ the set of mappings which have g-parametric representation. When $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $\zeta \in U$, the set $S_g^0(B^n)$ reduces to the set $S^0(B^n)$ of mappings which have usual parametric representation (see [14]; cf. [32, 33] on the unit polydisc in $\mathbb{C}^n$). One of the motivations for the study of g-parametric representation and g-Loewner chains is that $S^*(B^n) \subset S^0(B^n)$ and $K(B^n) \subset S_g^0(B^n)$, where $g(\zeta) \equiv 1 - \zeta$ (see [14]).

We next present some extension operators that preserve certain geometric and analytic properties (i.e. the notions of starlikeness, spirallikeness of type δ and parametric representation).

For $n \geq 2$, let $\tilde{z} = (z_2, \dots, z_n) \in \mathbb{C}^{n-1}$ such that $z = (z_1, \tilde{z}) \in \mathbb{C}^n$.

The Roper–Suffridge extension operator provides a way of extending a locally univalent function on the unit disc U to a locally biholomorphic mapping on the Euclidean unit ball B^n in $\mathbb{C}^n$. This operator was introduced by Roper and Suffridge in 1995 [34] in order to construct convex mappings on the Euclidean unit ball B^n in $\mathbb{C}^n$ starting with a convex function on the unit disc. If $f_1, \dots, f_n$ are convex functions on the unit disc U, then $F(z) = (f_1(z_1), \dots, f_n(z_n))$, $z = (z_1, \dots, z_n) \in B^n$ is not necessary a convex mapping on the Euclidean unit ball B^n in $\mathbb{C}^n$ (see e.g. [11]).

The Roper–Suffridge extension operator $\Phi_n : \mathscr{L}S \to \mathscr{L}S_n$ is defined by [34]

$$\Phi_n(f)(z) = (f(z_1), \tilde{z}\sqrt{f'(z_1)}),\ z = (z_1, \tilde{z}) \in B^n.$$

We choose the branch of the power function such that $\sqrt{f'(z_1)}\big|_{z_1=0} = 1$.

Roper and Suffridge [34] proved the following result:

Theorem 1. *If f is a convex function on the unit disc U, then $F = \Phi_n(f)$ is a convex mapping on the Euclidean unit ball B^n in $\mathbb{C}^n$. Hence $\Phi_n(K) \subseteq K(B^n)$.*

A different proof of Theorem 1 was given by Graham and Kohr in 2000 (see [10]). Graham and Kohr [10] also proved the following result, which shows that the Roper–Suffridge extension operator preserves the notion of starlikeness.

Theorem 2. *If $f \in S^*$, then $F = \Phi_n(f) \in S^*(B^n)$. Hence $\Phi_n(S^*) \subseteq S^*(B^n)$.*

Graham et al. [13] showed that the Roper–Suffridge extension operator preserves the notion of spirallikeness of type $\delta \in (-\pi/2, \pi/2)$.

Theorem 3. *If $f \in \hat{S}_\delta$, $\delta \in (-\pi/2, \pi/2)$, then $F = \Phi_n(f) \in \hat{S}_\delta(B^n)$. Hence $\Phi_n(\hat{S}_\delta) \subseteq \hat{S}_\delta(B^n)$.*

We now give the connection between the Roper–Suffridge extension operator and the Loewner theory. Graham et al. (see [13]; see also [11]) obtained the following result, which shows that the operator Φ_n preserves the notion of parametric representation.

Theorem 4. *If $f \in S$ and $F = \Phi_n(f)$, then $F \in S^0(B^n)$. Hence $\Phi_n(S) \subseteq S^0(B^n)$.*

We are now interested in other extension operators that have similar properties to those of the Roper–Suffridge extension operator. For several generalizations of the Roper–Suffridge extension operator, see [10, 15], [11, Chap. 11], [12, 13, 22, 25, 26].

Graham et al. [13] considered the following operator

$$\Phi_{n,\alpha}(f)(z) = F(z) = (f(z_1), \tilde{z}(f'(z_1))^\alpha),\ z = (z_1, \tilde{z}) \in B^n,$$

where $\alpha \in [0, 1/2]$, and f is a locally univalent function on U, normalized by $f(0) = f'(0) - 1 = 0$. We choose the branch of the power function such that $(f'(z_1))^\alpha|_{z_1=0} = 1$. When $\alpha = 1/2$, this operator reduces to the Roper–Suffridge extension operator.

Graham et al. [13] obtained a number of extension results related to the operator $\Phi_{n,\alpha}$, $\alpha \in [0, 1/2]$.

Theorem 5. *Let $f \in \mathscr{L}S$, $\alpha \in [0, 1/2]$.*

(i) If $f \in S$, then $\Phi_{n,\alpha}(f)$ can be embedded in a Loewner chain and moreover $\Phi_{n,\alpha}(f) \in S^0(B^n)$. Hence $\Phi_{n,\alpha}(S) \subseteq S^0(B^n)$.
(ii) If $f \in S^*$, then $\Phi_{n,\alpha}(f) \in S^*(B^n)$. Hence $\Phi_{n,\alpha}(S^*) \subseteq S^*(B^n)$.
(iii) If $f \in \hat{S}_\delta$, $\delta \in (-\pi/2, \pi/2)$, then $\Phi_{n,\alpha}(f) \in \hat{S}_\delta(B^n)$. Hence $\Phi_{n,\alpha}(\hat{S}_\delta) \subseteq \hat{S}_\delta(B^n)$.

Graham et al. [13] also proved that convexity is preserved under the operator $\Phi_{n,\alpha}$ only if $\alpha = 1/2$, i.e. only in the case of the Roper–Suffridge extension operator.

Graham et al. [15] considered the operator $\Phi_{n,\alpha,\beta}$ given by

$$\Phi_{n,\alpha,\beta}(f)(z) = \left(f(z_1), \tilde{z} \left(\frac{f(z_1)}{z_1} \right)^\alpha (f'(z_1))^\beta \right),\ z = (z_1, \tilde{z}) \in B^n,$$

where $\alpha \geq 0$, $\beta \geq 0$, and f is a locally univalent function on U, normalized by $f(0) = f'(0) - 1 = 0$, and such that $f(z_1) \neq 0$ for $z_1 \in U \setminus \{0\}$. The branches of the power functions are chosen such that

$$\left(\frac{f(z_1)}{z_1} \right)^\alpha \bigg|_{z_1=0} = 1 \text{ and } (f'(z_1))^\beta|_{z_1=0} = 1.$$

Note that the operator $\Phi_{n,0,1/2}$ reduces to the Roper–Suffridge extension operator.

Graham et al. [15] obtained certain extension results related to the operator $\Phi_{n,\alpha,\beta}$, where $\alpha \in [0, 1]$, $\beta \in [0, 1/2]$ and $\alpha + \beta \leq 1$.

Theorem 6. *Let* $f \in \mathscr{L}S$, $\alpha \in [0, 1]$, $\beta \in [0, 1/2]$, $\alpha + \beta \leq 1$.

(i) If $f \in S$, *then* $\Phi_{n,\alpha,\beta}(f)$ *can be embedded in a Loewner chain and moreover* $\Phi_{n,\alpha,\beta}(f) \in S^0(B^n)$. *Hence* $\Phi_{n,\alpha,\beta}(S) \subseteq S^0(B^n)$.

(ii) If $f \in S^*$, *then* $\Phi_{n,\alpha,\beta}(f) \in S^*(B^n)$. *Hence* $\Phi_{n,\alpha,\beta}(S^*) \subseteq S^*(B^n)$.

Moreover, Graham et al. [15] showed that $\Phi_{n,\alpha,\beta}(K) \subset K(B^n)$ only if $(\alpha, \beta) = (0, 1/2)$, i.e. only in the case of the Roper–Suffridge extension operator.

The following extension operator was introduced by Muir [26]. The purpose of this operator was to provide examples of extreme points of $K(B^n)$, starting with extreme points of K (see [27]).

Definition 7. Let $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ be a homogeneous polynomial of degree 2. The Muir operator $\Phi_{n,Q} : \mathscr{L}S \to \mathscr{L}S_n$ is defined by

$$\Phi_{n,Q}(f)(z) = (f(z_1) + Q(\tilde{z})f'(z_1), \tilde{z}\sqrt{f'(z_1)}),\ z = (z_1, \tilde{z}) \in B^n.$$

We choose the branch of the power function such that $\sqrt{f'(z_1)}|_{z_1=0} = 1$.

In the case $Q \equiv 0$, the Muir operator reduces to the Roper–Suffridge extension operator.

Muir [26] proved the following result. Note that (ii) was also obtained by Kohr [22].

Theorem 7. *(i)* $\Phi_{n,Q}(K) \subseteq K(B^n)$ *if and only if* $\|Q\| \leq 1/2$.
(ii) $\Phi_{n,Q}(S^*) \subseteq S^*(B^n)$ *if and only if* $\|Q\| \leq 1/4$.

Kohr [22] proved the following result regarding the Muir operator.

Theorem 8. $\Phi_{n,Q}(S) \subseteq S^0(B^n)$, *whenever* $\|Q\| \leq 1/4$.

Another generalization of the Roper–Suffridge extension operator was given by Pfaltzgraff and Suffridge [30] in 1999. This operator provides a way of extending a locally biholomorphic mapping on the Euclidean unit ball B^n in $\mathbb{C}^n$ to a locally biholomorphic mapping on the Euclidean unit ball B^{n+1} in $\mathbb{C}^{n+1}$.

For $n \geq 1$, let $z' = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $z = (z', z_{n+1}) \in \mathbb{C}^{n+1}$.

The Pfaltzgraff–Suffridge extension operator $\Psi_n : \mathscr{L}S_n \to \mathscr{L}S_{n+1}$ is defined by (see [30])

$$\Psi_n(f)(z) = F(z) = \left(f(z'), z_{n+1}[J_f(z')]^{\frac{1}{n+1}}\right),\ z = (z', z_{n+1}) \in B^{n+1},$$

where $J_f(z') = \det Df(z')$, $z' \in B^n$. We choose the branch of the power function such that $[J_f(z')]^{\frac{1}{n+1}}\big|_{z'=0} = 1$. Then $F = \Psi_n(f) \in \mathscr{L}S_{n+1}$ whenever $f \in \mathscr{L}S_n$.

Moreover, if $f \in S(B^n)$ then $F \in S(B^{n+1})$. Note that if $n = 1$, then Ψ_1 reduces to the Roper–Suffridge extension operator Φ_2.

Pfaltzgraff and Suffridge [30] proposed the following conjecture regarding the preservation of convexity under the operator Ψ_n.

Conjecture 1. If $f \in K(B^n)$ then $\Psi_n(f) \in K(B^{n+1})$.

The operator Ψ_n was also studied by Graham et al. [17]. They obtained a partial answer to Conjecture 1 (see [17]).

Graham et al. [17] also proved the following result, which shows that the Pfaltzgraff–Suffridge extension operator preserves the notion of parametric representation.

Theorem 9. *If $f \in S^0(B^n)$, then $F = \Psi_n(f) \in S^0(B^{n+1})$. Hence $\Psi_n(S^0(B^n)) \subseteq S^0(B^{n+1})$.*

In particular, Graham et al. [17] obtained that the Pfaltzgraff–Suffridge extension operator preserves starlikeness (compare with [30]).

Corollary 1. *If $f \in S^*(B^n)$, then $F = \Psi_n(f) \in S^*(B^{n+1})$. Hence $\Psi_n(S^*(B^n)) \subseteq S^*(B^{n+1})$.*

2 g-Loewner Chains Associated with Generalized Roper–Suffridge Extension Operators

In this section we are concerned with the extension operators $\Phi_{n,\alpha}$, $\Phi_{n,\alpha,\beta}$ and $\Phi_{n,Q}$ that provide a way of extending a locally univalent function f on the unit disc U to a locally biholomorphic mapping $F \in H(B^n)$. We show that if f can be embedded as the first element of a g-Loewner chain on the unit disc, where $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$ for $|\zeta| < 1$ and $\gamma \in (0, 1)$, then $F = \Phi_{n,\alpha}(f)$ can also be embedded as the first element of a g-Loewner chain on B^n, whenever $\alpha \in [0, \frac{1}{2}]$. In particular, if f is starlike of order γ on U (resp. f is spirallike of type δ and order γ on U, where $\delta \in (-\pi/2, \pi/2)$), then $F = \Phi_{n,\alpha}(f)$ is also starlike of order γ on B^n (resp. $F = \Phi_{n,\alpha}(f)$ is spirallike of type δ and order γ on B^n). Also, if f is almost starlike of order δ and type γ on U, where $\delta \in [0, 1)$, then $F = \Phi_{n,\alpha}(f)$ is almost starlike of order δ and type γ on B^n. Similar ideas are applied in the case of the Muir extension operator $\Phi_{n,Q}$, where Q is a homogeneous polynomial of degree 2 on $\mathbb{C}^{n-1}$ such that $\|Q\| \le \frac{1-|2\gamma-1|}{8\gamma}$, $\gamma \in (0, 1)$, and in the case of the extension operator $\Phi_{n,\alpha,\beta}$.

Throughout this section we consider g-Loewner chains with $g \in H(U)$ given by

$$g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}, \quad |\zeta| < 1,$$

where $\gamma \in (0, 1)$. Then g maps the unit disc onto the open disc of centre $1/(2\gamma)$ and radius $1/(2\gamma)$. Hence, in this case the class $\mathcal{M}_g$ is given by

$$\mathcal{M}_g = \left\{ h \in H(B^n) : h(0) = 0, Dh(0) = I_n, \left| \frac{1}{\|z\|^2}\langle h(z), z\rangle - \frac{1}{2\gamma} \right| < \frac{1}{2\gamma}, z \in B^n \setminus \{0\} \right\}.$$

2.1 The Operator $\Phi_{n,\alpha}$ and g-Loewner Chains

The following result due to Chirilă [5] yields that the operator $\Phi_{n,\alpha}$ preserves the notion of g-Loewner chains for $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$, where $\gamma \in (0, 1)$. In the case $\gamma = 0$, this result was obtained by Graham et al. (see [13]).

Theorem 10. *Assume $f \in S$ can be embedded as the first element of a g-Loewner chain, where $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$, $\gamma \in (0, 1)$. Then $F = \Phi_{n,\alpha}(f)$ can be embedded as the first element of a g-Loewner chain on B^n, for $\alpha \in [0, 1/2]$.*

In view of Theorem 10, Chirilă [5] obtained the following particular cases. Corollary 2 was obtained by Graham et al. [13], in the case $\gamma = 0$.

Corollary 2. *If $f : U \to \mathbb{C}$ has g-parametric representation and $\alpha \in [0, 1/2]$, then $F = \Phi_{n,\alpha}(f) \in S_g^0(B^n)$, where $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $\zeta \in U$ and $\gamma \in (0, 1)$.*

The following result was obtained by Hamada et al. [20], in the case $\alpha = \gamma = 1/2$, and by Liu [23], in the case $\gamma \in (0, 1)$ and $\alpha \in [0, 1/2]$. Chirilă [5] gave a different proof of this result using g-Loewner chains.

Corollary 3. *If $f \in S_\gamma^*$, $\gamma \in (0, 1)$ and $\alpha \in [0, 1/2]$, then $F = \Phi_{n,\alpha}(f) \in S_\gamma^*(B^n)$. In particular, the Roper–Suffridge extension operator preserves the notion of starlikeness of order γ.*

Remark 4. Since $K \subset S_{1/2}^*$, it follows in view of Corollary 3 that $\Phi_{n,\alpha}(K) \subset S_{1/2}^*(B^n)$ for $\alpha \in [0, \frac{1}{2}]$. However, $\Phi_{n,\alpha}(K) \nsubseteq K(B^n)$ for $\alpha \neq 1/2$ (see [13]).

The following result is due to Liu and Liu [24] (see also [23]). Chirilă [5] obtained this result by using g-Loewner chains.

Corollary 4. *Let $\alpha \in [0, 1/2]$, $\delta \in (-\pi/2, \pi/2)$ and $\gamma \in (0, 1)$. Also, let $f : U \to \mathbb{C}$ be a spirallike function of type δ and order γ on U, and let $F = \Phi_{n,\alpha}(f)$. Then F is also spirallike of type δ and order γ on B^n.*

Chirilă [5] obtained the following preservation result of almost starlikeness of order δ and type γ in the case of the operator $\Phi_{n,\alpha}$ (cf. [37]).

Corollary 5. *Let $\alpha \in [0, 1/2]$, $\delta \in [0, 1)$ and $\gamma \in (0, 1)$. Also, let $f : U \to \mathbb{C}$ be an almost starlike function of order δ and type γ. Then $F = \Phi_{n,\alpha}(f)$ is almost starlike of order δ and type γ on B^n.*

2.2 The Muir Extension Operator and g-Loewner Chains

Chirilă [5] proved that the Muir extension operator $\Phi_{n,Q}$ preserves the notion of g-Loewner chains, where $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$ for $\zeta \in U$, and $\gamma \in (0, 1)$. In the case $\gamma = 0$, this result was obtained by Kohr (see [22]).

Theorem 11. *Let $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ be a homogeneous polynomial of degree 2 such that $\|Q\| \leq \frac{1-|2\gamma-1|}{8\gamma}$, where $\gamma \in (0, 1)$. Assume $f \in S$ can be embedded as the first element of a g-Loewner chain. Then $F = \Phi_{n,Q}(f)$ can be embedded as the first element of a g-Loewner chain on B^n, where $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$.*

In view of the above result, Chirilă [5] deduced that the operator $\Phi_{n,Q}$ preserves the notion of g-parametric representation and starlikeness of order γ, whenever $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$, $\gamma \in (0, 1)$ and $\|Q\| \leq \frac{1-|2\gamma-1|}{8\gamma}$.

Corollary 6. *Let $\gamma \in (0, 1)$ and let $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ be a homogeneous polynomial of degree 2 such that $\|Q\| \leq \frac{1-|2\gamma-1|}{8\gamma}$. If $f : U \to \mathbb{C}$ has g-parametric representation, then $F = \Phi_{n,Q}(f) \in S_g^0(B^n)$, where $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$.*

The following result was obtained by Wang and Liu [36]. Chirilă [5] obtained this result by using the method of g-Loewner chains.

Corollary 7. *Let $\gamma \in (0, 1)$ and let $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ be a homogeneous polynomial of degree 2 such that $\|Q\| \leq \frac{1-|2\gamma-1|}{8\gamma}$. If $f \in S_\gamma^*$, then $F = \Phi_{n,Q}(f) \in S_\gamma^*(B^n)$.*

Chirilă [5] obtained the following improvement of Theorem 11 in the case of g-Loewner chains $f(z_1, t)$ such that $f(\cdot, t)$ is convex on U for $t \geq 0$, where $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$, $\gamma \in (0, 1)$ (cf. [22] and [26]).

Proposition 1. *Let $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ be a homogeneous polynomial of degree 2 such that $\|Q\| \leq \frac{1-|2\gamma-1|}{4\gamma}$, where $\gamma \in (0, 1)$. Assume $f \in S$ can be embedded as the first element of a g-Loewner chain $f(z_1, t)$ such that $f(\cdot, t)$ is convex on U for $t \geq 0$, where $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$. Then $F = \Phi_{n,Q}(f)$ can be embedded as the first element of a g-Loewner chain on B^n for $t \geq 0$.*

2.3 The Operator $\Phi_{n,\alpha,\beta}$ and g-Loewner Chains

The following theorem due to Chirilă [4] yields that the operator $\Phi_{n,\alpha,\beta}$ preserves the notion of g-Loewner chains for $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$, where $\gamma \in (0, 1)$. This result was obtained by Graham et al. [15], in the case $\gamma = 0$. In the case $\alpha = 0$ and $\gamma \in (0, 1)$, Theorem 12 was obtained by Chirilă [5].

Theorem 12. *Assume $f \in S$ can be embedded as the first element of a g-Loewner chain, where $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$ and $\gamma \in (0,1)$. Then $F = \Phi_{n,\alpha,\beta}(f)$ can be embedded as the first element of a g-Loewner chain on B^n for $\alpha \in [0,1]$, $\beta \in [0,1/2]$, $\alpha + \beta \le 1$.*

In view of Theorem 12, Chirilă [4] obtained the following particular cases. Corollary 8 was obtained by Graham et al. [15], in the case $\gamma = 0$. Also, Corollary 8 was obtained by Chirilă [5], in the case $\alpha = 0$.

Corollary 8. *If $f : U \to \mathbb{C}$ has g-parametric representation and $\alpha \in [0,1]$, $\beta \in [0,1/2]$, $\alpha + \beta \le 1$, then $F = \Phi_{n,\alpha,\beta}(f) \in S_g^0(B^n)$, where $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $\zeta \in U$, and $\gamma \in (0,1)$.*

The following result was obtained by Hamada et al. [20], in the case $\alpha = 0$, $\beta = \gamma = 1/2$, and by Liu [23], in the case $\gamma \in (0,1)$ and $\alpha \in [0,1]$, $\beta \in [0,1/2]$, $\alpha + \beta \le 1$. If $\gamma = 0$, the result below was obtained by Graham et al. [15]. Chirilă [4] proved this result by using the method of g-Loewner chains.

Corollary 9. *If $f \in S_\gamma^*$ and $\alpha \in [0,1]$, $\beta \in [0,1/2]$, $\alpha + \beta \le 1$, then $F = \Phi_{n,\alpha,\beta}(f) \in S_\gamma^*(B^n)$, where $\gamma \in (0,1)$. In particular, the Roper–Suffridge extension operator preserves the notion of starlikeness of order γ.*

The following remark follows from Corollary 9 (see [4]).

Remark 5. Since $K \subset S_{1/2}^*$, it follows in view of Corollary 9 that $\Phi_{n,\alpha,\beta}(K) \subset S_{1/2}^*(B^n)$ for $\alpha \in [0,1]$, $\beta \in [0,1/2]$, $\alpha + \beta \le 1$. However, $\Phi_{n,\alpha,\beta}(K) \nsubseteq K(B^n)$ for $(\alpha, \beta) \neq (0, 1/2)$ (see [15]).

The following result is due to Liu and Liu [24] (see also [23]). Chirilă [4] obtained a different proof by using the method of g-Loewner chains.

Corollary 10. *Let $\alpha \in [0,1]$, $\beta \in [0,1/2]$, $\alpha + \beta \le 1$, $\delta \in (-\pi/2, \pi/2)$ and $\gamma \in (0,1)$. Also, let $f : U \to \mathbb{C}$ be a spirallike function of type δ and order γ on U, and let $F = \Phi_{n,\alpha,\beta}(f)$. Then F is also spirallike of type δ and order γ on B^n.*

2.4 Subordination and Radius Problems Associated with the Operator $\Phi_{n,\alpha,\beta}$

The following subordination preserving result under the operator $\Phi_{n,\alpha,\beta}$ was obtained by Chirilă [4]. This result was obtained by Hamada et al. [20], in the case $\alpha = 0$ and $\beta = 1/2$.

Theorem 13. *Let $f, g : U \to \mathbb{C}$ be two locally univalent functions such that $f(0) = g(0) = 0$, $f'(0) = a$ and $g'(0) = b$, where $0 < a \le b$. Assume that $f(z_1) \neq 0$ and $g(z_1) \neq 0$ for $0 < |z_1| < 1$. If $\alpha \ge 0$, $\beta \in [0,1/2]$ and $f \prec g$, then $\Phi_{n,\alpha,\beta}(f) \prec \Phi_{n,\alpha,\beta}(g)$. We choose the branches of the power functions such that*

$$[f'(z_1)]^\beta|_{z_1=0} = a^\beta, \left[\frac{f(z_1)}{z_1}\right]^\alpha\Bigg|_{z_1=0} = a^\alpha,$$

$$[g'(z_1)]^\beta|_{z_1=0} = b^\beta, \left[\frac{g(z_1)}{z_1}\right]^\alpha\Bigg|_{z_1=0} = b^\alpha.$$

For various consequences of the above result, see [4] (see also [20], in the case $\alpha = 0$ and $\beta = 1/2$).

We next consider certain radius problems associated with the operator $\Phi_{n,\alpha,\beta}$. First, we recall the concept of the radius for a certain property in a certain set (see e.g. [9, 11]).

Definition 8. Given $\mathscr{F}$ a nonempty subset of $S(B^n)$ and a property $\mathscr{P}$ which the mappings in $\mathscr{F}$ may or may not have in a ball B_r^n, the radius for the property $\mathscr{P}$ in the set $\mathscr{F}$ is denoted by $R_{\mathscr{P}}(\mathscr{F})$ and is the largest R such that every mapping in the set $\mathscr{F}$ has the property $\mathscr{P}$ in each ball B_r^n for every $r < R$.

Let $R_{S^*}(\mathscr{F})$ be the radius of starlikeness of $\mathscr{F}$, $R_K(\mathscr{F})$ the radius of convexity and $R_{\hat{S}_\delta}(\mathscr{F})$ the radius of spirallikeness of type δ of $\mathscr{F}$.

It is known that $R_K(S) = R_K(S^*) = 2 - \sqrt{3}$ and $R_{S^*}(S) = \tanh(\pi/4)$ (see e.g. [31]). Graham et al. [13] obtained the radius of starlikeness and the radius of convexity associated with $\Phi_n(S)$. Also, Graham et al. [15] obtained the radius of starlikeness associated with $\Phi_{n,\alpha,\beta}(S)$.

Chirilă [4] obtained the following result regarding the radius of spirallikeness of type δ for the set $\Phi_{n,\alpha,\beta}(S)$.

Theorem 14. *$R_{\hat{S}_\delta}(\Phi_{n,\alpha,\beta}(S)) = \tanh\left[\frac{\pi}{4} - \frac{|\delta|}{2}\right]$, for $\alpha \in [0, 1]$, $\beta \in [0, 1/2]$ such that $\alpha + \beta \le 1$ and $\delta \in (-\pi/2, \pi/2)$.*

For other radius problems associated with the extension operator $\Phi_{n,\alpha,\beta}$, see [4].

3 Subclasses of Biholomorphic Mappings Associated with g-Loewner Chains

In this section we use the method of Loewner chains to generate certain subclasses of normalized biholomorphic mappings on the Euclidean unit ball B^n in $\mathbb{C}^n$, which have interesting geometric characterizations. We present the classes of g-starlike mappings, g-spirallike mappings of type $\alpha \in (-\pi/2, \pi/2)$ and g-almost starlike mappings of order $\alpha \in [0, 1)$ on B^n and we obtain their characterization by using g-Loewner chains. We also show that, under certain assumptions, the mapping $F : B^n \to \mathbb{C}^n$ given by $F(z) = P(z)z$ has g-parametric representation on B^n, where $P : B^n \to \mathbb{C}$ is a holomorphic function such that $P(0) = 1$. Several applications are provided.

Pfaltzgraff and Suffridge [30] obtained a necessary and sufficient condition of starlikeness regarding the mapping $F(z) = P(z)z$ and gave a sharp distortion theorem for a subclass of $S^*(B^n)$. Graham et al. [14] obtained a necessary and sufficient condition of g-starlikeness for the mapping $F(z) = P(z)z$. Further results in this direction were obtained by Xu and Liu [38].

We first present the set of g-starlike mappings on B^n, where g satisfies the requirements of Definition 4. This notion was introduced by Graham et al. [14] and by Hamada and Honda [18]. This notion was also studied by Xu and Liu in the case of complex Banach spaces [38].

Definition 9. A normalized locally biholomorphic mapping $f : B^n \to \mathbb{C}^n$ is said to be *g-starlike* on B^n if

$$\frac{1}{\|z\|^2}\langle [Df(z)]^{-1} f(z), z\rangle \in g(U),\ z \in B^n \setminus \{0\}. \tag{6}$$

We denote the class of g-starlike mappings on B^n by $S_g^*(B^n)$. When $n = 1$, we denote this class by S_g^*. If $g(\zeta) = (1-\zeta)/(1+\zeta)$, then this class reduces to the class of starlike mappings on B^n.

Note that if $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$ and $\gamma \in (0, 1)$, then the class $S_g^*(B^n)$ reduces to the class of starlike mappings of order γ on B^n.

Chirilă et al. [7] proved the following compactness result for the class $S_g^*(B^n)$.

Theorem 15. *Let $g : U \to \mathbb{C}$ be a univalent function, which satisfies the requirements of Definition 4. Then the family $S_g^*(B^n)$ is compact.*

For various results regarding g-starlike mappings, such as growth and covering theorems and coefficient estimates, see [14, 18, 38].

Next we define the set of g-spirallike mappings of type $\alpha \in (-\pi/2, \pi/2)$ on B^n, where g satisfies the requirements of Definition 4. This notion was introduced by Chirilă [6].

Definition 10. A normalized locally biholomorphic mapping $f : B^n \to \mathbb{C}^n$ is said to be *g-spirallike of type $\alpha \in (-\pi/2, \pi/2)$* if

$$i\frac{\sin\alpha}{\cos\alpha} + \frac{e^{-i\alpha}}{\cos\alpha}\left\langle [Df(z)]^{-1} f(z), \frac{z}{\|z\|^2}\right\rangle \in g(U),\ z \in B^n \setminus \{0\}. \tag{7}$$

If $g(\zeta) = (1-\zeta)/(1+\zeta)$, this class becomes the class of spirallike mappings of type α on B^n and when $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$, $\gamma \in (0, 1)$, we obtain the class of spirallike mappings of type α and order γ on B^n. When $\alpha = 0$, the class of g-spirallike mappings of type 0 on B^n reduces to the set of g-starlike mappings on B^n.

Obviously, if f is g-spirallike of type α, then f is also spirallike of type α, and hence f is biholomorphic on B^n. On the other hand, the motivation for introducing the subclass of g-spirallike mappings of type α is provided by Corollary 12.

We next present the set of g-almost starlike mappings of order $\alpha \in [0, 1)$ on B^n, where g satisfies the requirements of Definition 4. This class was introduced by Chirilă [6].

Definition 11. A normalized locally biholomorphic mapping $f : B^n \to \mathbb{C}^n$ is said to be *g-almost starlike of order* $\alpha \in [0, 1)$ if

$$\frac{1}{1-\alpha}\left\langle [Df(z)]^{-1} f(z), \frac{z}{\|z\|^2}\right\rangle - \frac{\alpha}{1-\alpha} \in g(U),\ z \in B^n \setminus \{0\}. \tag{8}$$

If $g(\zeta) = (1-\zeta)/(1+\zeta)$, this class reduces to the class of almost starlike mappings of order α on B^n and when $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$, $\gamma \in (0, 1)$, we obtain the class of almost starlike mappings of order α and type γ on B^n. When $\alpha = 0$, the class of g-almost starlike mappings of order 0 on B^n reduces to the set of g-starlike mappings on B^n.

If f is g-almost starlike of order α, then f is also almost starlike of order α, and hence biholomorphic on B^n. The motivation for introducing the subclass of g-almost starlike mappings of order α on B^n is provided by Corollary 13.

3.1 Characterizations by Using g-Loewner Chains

We next present the characterizations of g-starlikeness, g-spirallikeness of type α and g-almost starlikeness of order α, in terms of g-Loewner chains. We first present the characterization of g-starlike mappings by using g-Loewner chains, where g satisfies the requirements of Definition 4. This result is due to Chirilă [6]. In the case $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $|\zeta| < 1$, we obtain the usual characterization of starlikeness in terms of Loewner chains due to Pfaltzgraff and Suffridge (see [29]).

Theorem 16. *A normalized locally biholomorphic mapping $f : B^n \to \mathbb{C}^n$ is g-starlike if and only if $f(z, t) = e^t f(z)$ is a g-Loewner chain, where g satisfies the requirements of Definition* 4.

The characterization of g-spirallike mappings of type $\alpha \in (-\pi/2, \pi/2)$ on B^n by using g-Loewner chains was obtained by Chirilă [6], where g satisfies the requirements of Definition 4 (compare with [19]). In the case $g(\zeta) = \frac{1-\zeta}{1+\zeta}$ in Theorem 17, we obtain the usual characterization of spirallikeness of type α in terms of Loewner chains due to Hamada and Kohr (see [19]).

Theorem 17. *A normalized locally biholomorphic mapping $f : B^n \to \mathbb{C}^n$ is g-spirallike of type $\alpha \in (-\pi/2, \pi/2)$ if and only if $f(z, t) = e^{(1-ia)t} f(e^{iat} z)$ is a g-Loewner chain, where $a = \tan\alpha$ and g satisfies the requirements of Definition* 4.

Chirilă [6] also obtained the characterization of g-almost starlike mappings of order $\alpha \in [0, 1)$ by using g-Loewner chains, where g satisfies the requirements of Definition 4. If we take $g(\zeta) = (1-\zeta)/(1+\zeta)$ in Theorem 18, we obtain the characterization of almost starlike mappings of order α using Loewner chains due to Xu and Liu (see [37]).

Theorem 18. *A normalized locally biholomorphic mapping $f : B^n \to \mathbb{C}^n$ is g-almost starlike of order $\alpha \in [0, 1)$ if and only if $f(z,t) = e^{\frac{1}{1-\alpha}t} f(e^{\frac{\alpha}{\alpha-1}t}z)$ is a g-Loewner chain, where g satisfies the requirements of Definition 4.*

3.2 A Subclass of Biholomorphic Mappings on B^n Generated by g-Loewner Chains

In the following theorem due to Chirilă [6] we consider conditions such that a mapping $F : B^n \to \mathbb{C}^n$ given by $F(z) = P(z)z$ belongs to $S_g^0(B^n)$, where $P \in H(B^n, \mathbb{C})$ with $P(0) = 1$.

Theorem 19. *Let $P : B^n \to \mathbb{C}$ be a holomorphic function on B^n such that $P(0) = 1$ and let $F(z) = P(z)z$, $z \in B^n$. Let $F(z,t) = P(z,t)z$, $z \in B^n$, $t \geq 0$, where $P(z,t) : B^n \times [0,\infty) \to \mathbb{C}$ satisfies the following conditions:*

(i) $P(\cdot,t) \in H(B^n, \mathbb{C})$, $P(0,t) = e^t$, $t \geq 0$, $P(\cdot,0) = P$, $P(z,t) \neq 0$, $z \in B^n$, $t \geq 0$, and $1 + \dfrac{DP(z,t)(z)}{P(z,t)} \neq 0$, for $z \in B^n$ and $t \geq 0$;
(ii) $P(z,\cdot)$ is locally Lipschitz continuous on $[0,\infty)$ locally uniformly with respect to $z \in B^n$.
(iii) $\{e^{-t}P(\cdot,t)\}_{t\geq 0}$ is a normal family on B^n.

If g satisfies the requirements of Definition 4 and

$$\frac{\dfrac{\partial P}{\partial t}(z,t)}{P(z,t)\left(1 + \dfrac{DP(z,t)(z)}{P(z,t)}\right)} \in g(U), \quad a.e.\ t \geq 0,\ \forall z \in B^n, \tag{9}$$

then $F(z,t)$ is a g-Loewner chain. Moreover, $F \in S_g^0(B^n)$.

Conversely, if $F(z,t)$ is a g-Loewner chain, then the relation (9) holds.

In view of Theorem 19, Chirilă [6] obtained the following particular cases.

Corollary 11 was obtained by Pfaltzgraff and Suffridge [30] in the case $g(\zeta) = (1-\zeta)/(1+\zeta)$ and by Graham et al. [14] in the case of functions $g : U \to \mathbb{C}$ satisfying the requirements of Definition 4. Corollary 11 was also obtained by Chirilă [6].

Corollary 11. *Let $g : U \to \mathbb{C}$ satisfy the requirements of Definition 4. Also, let $F(z) = P(z)z$, $z \in B^n$, where $P : B^n \to \mathbb{C}$ is a holomorphic function such that*

$P(0) = 1$. If $1 + \frac{DP(z)(z)}{P(z)} \in \frac{1}{g}(U)$, $z \in B^n$, then $F(z,t) = e^t P(z)z$, $z \in B^n$, $t \geq 0$, is a g-Loewner chain. Moreover, $F \in S_g^*(B^n)$.

In view of Theorem 19, we obtain the following consequence regarding g-spirallike mappings of type α on B^n (see [6]).

Corollary 12. *Let $g : U \to \mathbb{C}$ satisfy the requirements of Definition 4. Also, let $F(z) = P(z)z$, $z \in B^n$, where $P : B^n \to \mathbb{C}$ is a holomorphic function such that $P(0) = 1$ and $1 + \frac{DP(z)(z)}{P(z)} \neq 0$, $z \in B^n$. If $\frac{1+ia\frac{DP(z)(z)}{P(z)}}{1+\frac{DP(z)(z)}{P(z)}} \in g(U)$, $z \in B^n$, then $F(z,t) = e^t P(e^{iat}z)z$, $z \in B^n$, $t \geq 0$, is a g-Loewner chain, where $a = \tan\alpha$ and $\alpha \in (-\pi/2, \pi/2)$. Moreover, F is g-spirallike of type α on B^n.*

In view of Theorem 19, Chirilă [6] obtained the following consequence regarding g-almost starlike mappings of order α on B^n.

Corollary 13. *Let $g : U \to \mathbb{C}$ satisfy the requirements of Definition 4. Also, let $F(z) = P(z)z$, $z \in B^n$, where $P : B^n \to \mathbb{C}$ is a holomorphic function such that $P(0) = 1$ and $1 + \frac{DP(z)(z)}{P(z)} \neq 0$, $z \in B^n$. If $\frac{1+\frac{\alpha}{\alpha-1}\frac{DP(z)(z)}{P(z)}}{1+\frac{DP(z)(z)}{P(z)}} \in g(U)$, $z \in B^n$, $\alpha \in [0,1)$, then $F(z,t) = e^t P(e^{\frac{\alpha}{\alpha-1}t}z)z$, $z \in B^n$, $t \geq 0$, is a g-Loewner chain. Moreover, F is g-almost starlike of order α on B^n.*

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Normal Cones and Thompson Metric

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Abstract The aim of this paper is to study the basic properties of the Thompson metric d_T in the general case of a linear space X ordered by a cone K. We show that d_T has monotonicity properties which make it compatible with the linear structure. We also prove several convexity properties of d_T, and some results concerning the topology of d_T, including a brief study of the d_T-convergence of monotone sequences. It is shown that most results are true without any assumption of an Archimedean-type property for K. One considers various completeness properties and one studies the relations between them. Since d_T is defined in the context of a generic ordered linear space, with no need of an underlying topological structure, one expects to express its completeness in terms of properties of the ordering with respect to the linear structure. This is done in this paper and, to the best of our knowledge, this has not been done yet. Thompson metric d_T and order-unit (semi)norms $|\cdot|_u$ are strongly related and share important properties, as both are defined in terms of the ordered linear structure. Although d_T and $|\cdot|_u$ are only topologically (and not metrically) equivalent on K_u, we prove that the completeness is a common feature. One proves the completeness of the Thompson metric on a sequentially complete normal cone in a locally convex space. At the end of the paper, it is shown that, in the case of a Banach space, the normality of the cone is also necessary for the completeness of the Thompson metric.

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1 Introduction

In his study on the foundation of geometry, Hilbert [16] introduced a metric in the Euclidean space, known now as the Hilbert projective metric. Birkhoff [5] realized that fixed point techniques for nonexpansive mappings with respect to the Hilbert projective metric can be applied to prove the Perron–Frobenius theorem on the existence of eigenvalues and eigenvectors of non-negative square matrices and of solutions to some integral equations with positive kernel. The result on the Perron–Frobenius theorem was also found independently by Samelson [39]. Birkhoff's proof relied on some results from differential projective geometry, but Bushell [7, 8] gave new and more accessible proofs to these results by using the Hilbert metric defined on cones, revitalizing the interest for this topic (for a recent account on Birkhoff's definition of the Hilbert metric see the paper [24], and for Perron–Frobenius theory, the book [23]). A related partial metric on cones in Banach spaces was devised by Thompson [41], who proved the completeness of this metric (under the hypothesis of the normality of the cone), as well as some fixed point theorems for contractions with respect to it. It turned out that both these metrics are very useful in a variety of problems in various domains of mathematics and in applications to economy and other fields. Among these applications we mention those to fixed points for mixed monotone operators and other classes of operators on ordered vector spaces, see [9–12, 38]. Nussbaum alone, or in collaboration with other mathematicians, studied the limit sets of iterates of nonexpansive mappings with respect to Hilbert or Thompson metrics, the analog of Denjoy–Wolff theorem for iterates of holomorphic mappings, see [25–27, 31–33]. These metrics have also interesting applications to operator theory—to means for positive operators, [18, 29], and to isometries in spaces of operators on Hilbert space and in C^*-algebras, see [15, 28], and the papers quoted therein.

Good presentations of Hilbert and Thompson metrics are given in the monographs [17, 31, 32], and in the papers [1, 24, 34]. A more general approach—Hilbert and Thompson metrics on convex sets— is proposed in the papers [3] and [4].

The aim of this paper, essentially based on the Ph.D. thesis [37], is to study the basic properties of the Thompson metric d_T in the general case of a vector space X ordered by a cone K. Since d_T is defined in the context of a generic ordered vector space, with no need of an underlying topological structure, one expects to express its completeness in terms of properties of the ordering, with respect to the linear structure. This is done in the present paper and, to the best of our knowledge, this has not been done yet.

For the convenience of the reader, we survey in Sect. 2 some notions and notations which will be used throughout and list, without proofs, the most important results that are assumed to be known. Since there is no standard terminology in the theory of ordered vector spaces, the main purpose of this preliminary section is to provide a central point of reference for a unitary treatment of all of the topics in the rest of the paper. As possible we give exact references to textbooks were these results can be found, [2, 6, 13, 14, 19, 35, 40].

Section 3 is devoted to the definition and basic properties of the Thompson metric. We show that d_T has monotonicity properties which make it compatible with the linear structure. We also prove several convexity properties of d_T. We close this section with some results concerning the topology of d_T, including a brief study of the d_T-convergence of monotone sequences. Note that most of these results are true without the assumption of an Archimedean-type property for K.

We show that the Thompson metric d_T and order-unit (semi)norms $|\cdot|_u$ are strongly related and share important properties (e.g., they are topologically equivalent), as both are defined in terms of the ordered linear structure.

Section 4 is devoted to various kinds of completeness. It is shown that, although d_T and $|\cdot|_u$ are only topologically, and not metrically, equivalent, the completeness is a common feature. Also we study a special notion, called self-completeness, we prove that several completeness conditions are equivalent and that the Thompson metric on a sequentially complete normal cone K in a locally convex space X is complete.

In the last subsection we show that in the case when X is a Banach space, the completeness of K with respect to d_T is also necessary for the normality of K. This is obtained as a consequence of a more general result (Theorem 14) on the equivalence of several conditions to the completeness of K with respect to d_T.

2 Cones in Vector Spaces

2.1 Ordered Vector Spaces

A *preorder* on a set Z is a reflexive and transitive relation $\leq$ on Z. If the relation $\leq$ is also antisymmetric then it is called an *order* on Z. If any two elements in Z are comparable (i.e., at least one of the relations $x \leq y$ or $y \leq x$ holds), then one says that the order (or the preorder) $\leq$ is *total.*

Since in what follows we shall be concerned only with real vector spaces, by a "vector space" we will understand always a "real vector space."

A nonempty subset W of a vector space X is called a *wedge* if

$$\begin{aligned} &\text{(C1)} \qquad W + W \subset W, \\ &\text{(C2)} \qquad tW \subset W, \quad \text{for all } \ t \geq 0. \end{aligned} \tag{1}$$

The wedge W induces a preorder on X given by

$$x \leq_W y \iff y - x \in W. \tag{2}$$

The notation $x <_W y$ means that $x \leq_W y$ and $x \neq y$. If there is no danger of confusion the subscripts will be omitted.

This preorder is compatible with the linear structure of X, that is

$$\begin{aligned} &\text{(i)} \qquad x \le y \Longrightarrow x + z \le y + z, \quad \text{and} \\ &\text{(ii)} \qquad x \le y \Longrightarrow tx \le ty, \end{aligned} \tag{3}$$

for all $x, y, z \in X$ and $t \in \mathbb{R}_+$, where $\mathbb{R}_+ = \{t \in \mathbb{R} : t \ge 0\}$. This means that one can add inequalities

$$x \le y \text{ and } x' \le y' \Longrightarrow x + x' \le y + y',$$

and multiply by positive numbers

$$x \le y \iff tx \le ty,$$

for all $x, x', y, y' \in X$ and $t > 0$. The multiplication by negative numbers reverses the inequalities

$$\forall t < 0, \ \ (x \le y \iff tx \ge ty).$$

As a consequence of this equivalence, a subset A of X is bounded above iff the set $-A$ is bounded below. Also

$$\inf A = -\sup(-A) \qquad \sup A = -\inf(-A).$$

It is obvious that the preorder $\le_W$ is total iff $X = W \cup (-W)$.

Remark 1. It follows that in definitions (or hypotheses) we can ask only one order condition. For instance, if we ask that every bounded above subset of an ordered vector space has a supremum, then every bounded below subset will have an infimum, and consequently, every bounded subset has an infimum and a supremum. Similarly, if a linear preorder is upward directed, then it is automatically downward directed, too.

Obviously, the wedge W agrees with the set of positive elements in X,

$$W = X_+ := \{x \in X : 0 \le_W x\}.$$

Conversely, if $\le$ is a preorder on a vector space X satisfying (3) (such a preorder is called a *linear preorder*), then $W = X_+$ is a wedge in X and $\le = \le_W$. Consequently, there is a perfect correspondence between linear preorders on a vector space X and wedges in X and so any property in an ordered vector space can be formulated in terms of the preorder or of the wedge.

A *cone* K is a wedge satisfying the condition

$$\text{(C3)} \qquad K \cap (-K) = \{0\}. \tag{4}$$

This is equivalent to the fact that the induced preorder is antisymmetric,

$$x \le y \text{ and } y \le x \Longrightarrow y = x, \tag{5}$$

for all $x, y \in X$, that it is an order on X.

A pair (X, K), where K is a cone (or a wedge) in a vector space X, is called an *ordered* (resp. *preordered*) *vector space*.

An *order interval* in an ordered vector space (X, K) is a (possibly empty) set of the form

$$[x; y]_o = \{z \in X : x \le z \le y\} = (x + K) \cap (y - K),$$

for some $x, y \in X$. It is clear that an order interval $[x; y]_o$ is a convex subset of X and that

$$[x; y]_o = x + [0; y - x]_o.$$

The notation $[x; y]$ will be reserved to algebraic intervals: $[x; y] := \{(1 - t)x + ty : t \in [0; 1]\}$.

A subset A of X is called *order-convex* (or *full*, or *saturated*) if $[x; y]_o \subset A$ for every $x, y \in A$. Since the intersection of an arbitrary family of order-convex sets is order-convex, we can define the order-convex hull $[A]$ of a nonempty subset A of X as the intersection of all order-convex subsets of X containing A, i.e. the smallest order-convex subset of X containing A. It follows that

$$[A] = \bigcup \{[x; y]_o : x, y \in A\} = (A + K) \cap (A - K). \tag{6}$$

Obviously, A is order-convex iff $A = [A]$.

Remark 2. It is obvious that if $x \le y$, then $[x; y] \subset [x; y]_o$, but the reverse inclusion could not hold as the following example shows. Taking $X = \mathbb{R}^2$ with the coordinate order and $x = (0, 0),\ y = (1, 1)$, then $[x; y]_o$ equals the (full) square with the vertices $(0, 0)$, $(0, 1)$, $(1, 1)$, and $(0, 1)$, so it is larger than the segment $[x; y]$.

We mention also the following result.

Proposition 1 ([6]). *Let $(X, \le)$ be an ordered vector space. Then the order $\le$ is total iff every order-convex subset of X is convex.*

We shall consider now some algebraic-topological notions concerning the subsets of a vector space X. Let A be a subset of X.

The subset A is called:

- *balanced* if $\lambda A \subset A$ for every $|\lambda| \le 1$;
- *symmetric* if $-A = A$;
- *absolutely convex* if it is convex and balanced;
- *absorbing* if $\{t > 0 : x \in tA\} \ne \emptyset$ for every $x \in X$.

The following equivalences are immediate:

$$\begin{aligned} A \text{ is absolutely convex} &\iff \forall a,b \in A,\ \forall \alpha,\beta \in \mathbb{R},\ \text{with } |\alpha|+|\beta|=1, \quad \alpha a+\beta b \in A \\ &\iff \forall a,b \in A,\ \forall \alpha,\beta \in \mathbb{R},\ \text{with } |\alpha|+|\beta|\le 1, \quad \alpha a+\beta b \in A. \end{aligned}$$

Notice that a balanced set is symmetric and a symmetric convex set containing 0 is balanced.

The following properties are easily seen.

Proposition 2. *Let X be an ordered vector space and $A \subset X$ nonempty. Then*

1. *If A is convex, then $[A]$ is also convex.*
2. *If A is balanced, then $[A]$ is also balanced.*
3. *If A is absolutely convex, then $[A]$ is also absolutely convex.*

One says that a is an *algebraic interior* point of A if

$$\forall x \in X,\ \exists \delta > 0,\ \text{such that } \forall \lambda \in [-\delta;\delta],\ a+\lambda x \in A. \tag{7}$$

The (possibly empty) set of all interior points of A, denoted by aint(A), is called the *algebraic interior* (or the *core*) of the set A. It is obvious that if X is a TVS, then $\operatorname{int}(A) \subset \operatorname{aint}(A)$ where $\operatorname{int}(A)$ denotes the interior of the set A. In finite dimension we have equality, but the inclusion can be proper if X is infinite dimensional.

A cone K is called *solid* if $\operatorname{int}(K) \neq \emptyset$.

Remark 3. Zălinescu [45] uses the notation A^i for the algebraic interior and ${}^iA$ for the algebraic interior of A with respect to its affine hull (called the relative algebraic interior). In his definition of an algebraic interior point of A one asks that the conclusion of (7) holds only for $\lambda \in [0;\delta]$, a condition equivalent to (7).

The set A is called *lineally open* (or *algebraically open*) if $A = \operatorname{aint}(A)$, and *lineally closed* if $X \setminus A$ is lineally open. This is equivalent to the fact that any line in X meets A in a closed subset of the line. The smallest lineally closed set containing a set A is called the *lineal* (or *algebraic*) closure of A and it is denoted by acl(A). Again, if X is a TVS, then any closed subset of X is lineally closed. The subset A is called *lineally bounded* if the intersection with any line D in X is a bounded subset of D.

Remark 4. The terms "lineally open," "lineally closed," etc, are taken from Jameson [19].

Remark 5. Similar to the topological case one can prove that

$$a \in \operatorname{aint}(A),\ b \in A \text{ and } \lambda \in [0;1) \implies (1-\lambda)a+\lambda b \in \operatorname{aint}(A). \tag{8}$$

Consequently, if A is convex then aint(A) is also convex.

If K is a cone, then $\operatorname{aint}(K) \cup \{0\}$ is also a cone and

$$\mathrm{aint}(K) + K \subset \mathrm{aint}(K). \tag{9}$$

We justify only the second assertion. Let $x \in \mathrm{aint}(K)$ and $y \in K$. Then

$$x + y = 2\left(\frac{1}{2}x + \frac{1}{2}y\right) \in \mathrm{aint}(K).$$

Now we shall consider some further properties of linear orders. A linear order $\leq$ on a vector space X is called:

- *Archimedean* if for every $x, y \in X$,

$$(\forall n \in \mathbb{N},\ nx \leq y) \implies x \leq 0; \tag{10}$$

- *almost Archimedean* if for every $x, y \in X$,

$$(\forall n \in \mathbb{N},\ -y \leq nx \leq y) \implies x = 0; \tag{11}$$

The following four propositions are taken from Breckner [6] and Jameson [19]. In all of them X will be a vector space and $\leq$ a linear preorder on X given by the wedge $W = X_+$.

Proposition 3. *The following are equivalent.*

1. *The preorder $\leq$ is Archimedean.*
2. *The wedge W is lineally closed.*
3. *For every $x \in X$ and $y \in W$, $0 = \inf\{n^{-1}x : n \in \mathbb{N}\}$.*
4. *For every $x \in X$ and $y \in W$, $nx \leq y$, for all $n \in \mathbb{N}$, implies $x \leq 0$.*
5. *For every $A \subset \mathbb{R}$ and $x, y \in X$, $y \leq \lambda x$ for all $\lambda \in A$, implies $y \leq \mu x$, where $\mu = \inf A$.*

Proposition 4. *The following are equivalent.*

1. *The preorder is almost-Archimedean.*
2. $\mathrm{acl}(W)$ *is a wedge.*
3. *Every order interval in X is lineally bounded.*

A wedge W in X is called *generating* if $X = W - W$. The preorder $\leq$ is called *upward* (*downward*) *directed* if for every $x, y \in X$ there is $z \in X$ such that $x \leq z$, $y \leq z$ (respectively, $x \geq z$, $y \geq z$). If the order is linear, then these two notions are equivalent, so we can say simply that $\leq$ is directed.

Proposition 5. *The following are equivalent.*

1. *The wedge W is generating.*
2. *The order $\leq$ is directed.*
3. $\forall x \in X,\ \exists y \in W,\ x \leq y$.

Let (X, W) be a preordered vector space. An element $u \in W$ is called an *order unit* if the set $[-u; u]_o$ is absorbing. It is obvious that an order unit must be different of 0 (provided $X \neq \{0\}$).

Proposition 6. *Let $u \in W \setminus \{0\}$. The following are equivalent.*

1. *The element u is an order unit.*
2. *The order interval $[0; u]_o$ is absorbing.*
3. *The element u belongs to the algebraic interior of W.*
4. $[\mathbb{R}u] = X$.

2.2 Completeness in Ordered Vector Spaces

An ordered vector space X is called a vector lattice if any two elements $x, y \in X$ have a supremum, denoted by $x \vee y$. It follows that they have also an infimum, denoted by $x \wedge y$, and these properties extend to any finite subset of X. The ordered vector space X is called *order complete* (or *Dedekind complete*) if every bounded from above subset of X has a supremum and *order σ-complete* (or *Dedekind σ-complete*) if every bounded from above countable subset of X has a supremum. The fact that every bounded above subset of X has a supremum is equivalent to the fact that every bounded below subset of X has an infimum. Indeed, if A is bounded above, then $\sup\{y : y \text{ is a lower bound for } A\} = \inf A$.

Remark 6. An ordered vector space X is order complete iff for each pair A, B of nonempty subsets of X such that $A \leq B$ there exists $z \in X$ with $A \leq z \leq B$.

This similarity with "Dedekind cuts" in $\mathbb{R}$ justifies the term *Dedekind complete* used by some authors. Here $A \leq B$ means that $a \leq b$ for all $(a, b) \in A \times B$.

The following results give characterizations of these properties in terms of directed subsets.

Proposition 7 ([2], Theorem 1.20). *Let X be a vector lattice.*

1. *The space X is order complete iff every upward directed bounded above subset of X has a supremum (equivalently, if every bounded above monotone net has a supremum).*
2. *The space X is Dedekind σ-complete iff every upward directed bounded above countable subset of X has a supremum (equivalently, if every bounded above monotone sequence has a supremum).*

2.3 Ordered Topological Vector Spaces

In the case of an ordered topological vector space (TVS) (X, τ) some connections between order and topology hold. In the following propositions (X, τ) will be a TVS with a preorder or an order, $\leq$ generated by a wedge W, or by a cone K, respectively. We start by a simple result.

Proposition 8. *A wedge W is closed iff the inequalities are preserved by limits, meaning that for all nets $(x_i : i \in I)$, $(y_i : i \in I)$ in X,*

$$\forall i \in I,\ x_i \le y_i \ \text{ and } \ \lim_i x_i = x,\ \lim_i y_i = y \Longrightarrow\ x \le y.$$

Other results are contained in the following proposition.

Proposition 9 ([2], Lemmas 2.3 and 2.4). *Let (X, τ) be a TVS ordered by a τ-closed cone K. Then*

1. *The topology τ is Hausdorff.*
2. *The cone K is Archimedean.*
3. *The order intervals are τ-closed.*
4. *If $(x_i : i \in I)$ is an increasing net which is τ-convergent to $x \in X$, then $x = \sup_i x_i$.*
5. *Conversely, if the topology τ is Hausdorff, $\operatorname{int}(K) \ne \emptyset$ and K is Archimedean, then K is τ-closed.*

Note 1. In what follows by an ordered TVS we shall understand a TVS ordered by a closed cone. Also, in an ordered TVS (X, τ, K) we have some parallel notions—with respect to topology and with respect to order. To make distinction between them, those referring to order will have the prefix "order-", as, for instance, "order-bounded", "order-complete", etc., while for those referring to topology we shall use the prefix "τ-", or "topologically-", e.g., "τ-bounded", "τ-complete" (resp. "topologically bounded", "topologically complete"), etc.

2.4 Normal Cones in TVS and in Locally Convex Spaces

Now we introduce a very important notion in the theory of ordered vector spaces. A cone K in a TVS (X, τ) is called *normal* if there exists a neighborhood basis at 0 formed of order-convex sets.

The following characterizations are taken from [6] and [35].

Theorem 1. *Let (X, τ, K) be an ordered TVS. The following are equivalent.*

1. *The cone K is normal.*
2. *There exists a basis $\mathscr{B}$ formed of order-convex balanced 0-neighborhoods.*
3. *There exists a basis $\mathscr{B}$ formed of balanced 0-neighborhoods such that for every $B \in \mathscr{B}$, $y \in B$ and $0 \le x \le y$ implies $x \in B$.*
4. *There exists a basis $\mathscr{B}$ formed of balanced 0-neighborhoods such that for every $B \in \mathscr{B}$, $y \in B$ implies $[0; y]_o \subset B$.*
5. *There exists a basis $\mathscr{B}$ formed of balanced 0-neighborhoods and a number $\gamma > 0$ such that for every $B \in \mathscr{B}$, $[B] \subset \gamma B$.*
6. *If $(x_i : i \in I)$ and $(y_i : i \in I)$ are two nets in X such that $\forall i \in I$, $0 \le x_i \le y_i$ and $\lim_i y_i = 0$, then $\lim_i x_i = 0$.*

If further, X is a LCS, then the fact that the cone K is normal is equivalent to each of the conditions 2–5, where the term "balanced" is replaced with "absolutely convex".

Remark 7. Condition 6 can be replaced with the equivalent one:
If $(x_i : i \in I)$, $(y_i : i \in I)$, and $(z_i : i \in I)$ are nets in X such that $\forall i \in I,\ x_i \le z_i \le y_i$, and $\lim_i x_i = x = \lim_i y_i$, then $\lim_i z_i = x$.

The normality implies the fact that the order-bounded sets are bounded.

Proposition 10 ([35], Proposition 1.4). *If (X, τ) is a TVS ordered by a normal cone, then every order-bounded subset of X is τ-bounded.*

Remark 8. In the case of a normed space this condition characterizes the normality, see Theorem 3 below. Also, it is clear that a subset Z of an ordered vector space X is order-bounded iff there exist $x, y \in X$ such that $Z \subset [x; y]_o$.

The existence of a normal solid cone in a TVS makes the topology normable.

Proposition 11 ([2], p. 81, Exercise 11, and [35]). *If a Hausdorff TVS (X, τ) contains a solid τ-normal cone, then the topology τ is normable.*

In order to give characterizations of normal cones in locally convex spaces (LCS) we consider some properties of seminorms. Let $\gamma > 0$. A seminorm p on a vector space X is called:

- γ-*monotone* if $0 \le x \le y \implies p(x) \le \gamma p(y)$;
- γ-*absolutely monotone* if $-y \le x \le y \implies p(x) \le \gamma p(y)$;
- γ-*normal* if $x \le z \le y \implies p(z) \le \gamma \max\{p(x), p(y)\}$.

A 1-monotone seminorm is called *monotone*. Also a seminorm which is γ-monotone for some $\gamma > 0$ is called sometimes semi-monotone (see [13]).

These properties can be characterized in terms of the Minkowski functional attached to an absorbing subset A of a vector space X, given by

$$p_A(x) = \inf\{t > 0 : x \in tA\}, \quad (x \in X.) \tag{12}$$

It is well known that if the set A is absolutely convex and absorbing, then p_A is a seminorm on X and

$$\operatorname{aint}(A) = \{x \in X : p_A(x) < 1\} \subset A \subset \{x \in X : p_A(x) \le 1\} = \operatorname{acl}(A). \tag{13}$$

Proposition 12 ([6], Proposition 2.5.6). *Let A be an absorbing absolutely convex subset of an ordered vector space X.*

1. *If $[A] \subset \gamma A$, then the seminorm p_A is γ-normal.*
2. *If $\forall y \in A,\ [0; y] \subset \gamma A$, then the seminorm p_A is γ-monotone.*
3. *If $\forall y \in A,\ [-y; y] \subset \gamma A$, then the seminorm p_A is γ-absolutely monotone.*

Based on Theorem 1 and Proposition 12 one can give further characterizations of normal cones in LCS.

Theorem 2 ([6], [35], and [40]). *Let (X, τ) be a LCS ordered by a cone K. The following are equivalent.*

1. *The cone K is normal.*
2. *There exists $\gamma > 0$ and a family of γ-normal seminorms generating the topology τ of X.*
3. *There exists $\gamma > 0$ and a family of γ-monotone seminorms generating the topology τ of X.*
4. *There exists $\gamma > 0$ and a family of γ-absolutely monotone seminorms generating the topology τ of X.*

All the above equivalences hold also with $\gamma = 1$ in all places.

2.5 Normal Cones in Normed Spaces

We shall consider now characterizations of normality in the case of normed spaces. For a normed space $(X, \|\cdot\|)$, let $B_X = \{x \in X : \|x\| \le 1\}$ be its closed unit ball and $S_X = \{x \in X : \|x\| = 1\}$ its unit sphere.

Theorem 3 ([13] and [14]). *Let K be a cone in a normed space $(X, \|\cdot\|)$. The following are equivalent.*

1. *The cone K is normal.*
2. *There exists a monotone norm $\|\cdot\|_1$ on X equivalent to the original norm $\|\cdot\|$.*
3. *For all sequences (x_n), (y_n), (z_n) in X such that $x_n \le z_n \le y_n$, $n \in \mathbb{N}$, the conditions $\lim_n x_n = x = \lim_n y_n$ imply $\lim_n z_n = x$.*
4. *The order-convex hull $[B_X]$ of the unit ball is bounded.*
5. *The order interval $[x; y]_o$ is bounded for every $x, y \in X$.*
6. *There exists $\delta > 0$ such that $\forall x, y \in K \cap S_X$, $\|x + y\| \ge \delta$.*
7. *There exists $\gamma > 0$ such that $\forall x, y \in K$, $\|x + y\| \ge \gamma \max\{\|x\|, \|y\|\}$.*
8. *There exists $\lambda > 0$ such that $\|x\| \le \lambda \|y\|$, for all $x, y \in K$ with $x \le y$.*

We notice also the following result, which can be obtained as a consequence of a result of T. Andô on ordered LCS (see [2, Theorem 2.10]).

Proposition 13 ([2], Corollary 2.12). *Let X be a Banach space ordered by a generating cone X_+ and B_X its closed unit ball. Then $(B_X \cap X_+) - (B_X \cap X_+)$ is a neighborhood of 0.*

2.6 Completeness and Order Completeness in Ordered TVS

The following notions are inspired by Cantor's theorem on the convergence of bounded monotone sequences of real numbers.

Let X be a Banach space ordered by a cone K. The cone K is called:

- *regular* if every increasing and order-bounded sequence in X is convergent;
- *fully regular* if every increasing and norm-bounded sequence in X is convergent.

By Proposition 7 if X is a regular normed lattice, then every countable subset of X has a supremum.

These notions are related in the following way.

Theorem 4 ([14], Theorems 2.2.1 and 2.2.3). *If X is a Banach space ordered by a cone K, then*

$$K \text{ fully regular} \implies K \text{ regular} \implies K \text{ normal}.$$

If the Banach space X is reflexive, then the reverse implications hold too, i.e.-both implications become equivalences.

Some relations between completeness and order completeness in ordered TVS were obtained by Ng [30], Wong [43] (see also the book [44]). Some questions about completeness in ordered metric spaces are discussed by Turinici [42].

Let (X, τ) be a TVS ordered by a cone K. One says that the space X is

- *fundamentally σ-order complete* if every increasing τ-Cauchy sequence in X has a supremum;
- *monotonically sequentially complete* if every increasing τ-Cauchy sequence in X is convergent in (X, τ).

In the following propositions (X, τ) is a TVS ordered by a cone K.

The following result is obvious.

Proposition 14.

1. *If X is sequentially complete, then X is monotonically sequentially complete.*
2. *If X is monotonically sequentially complete, then X is fundamentally σ-order complete.*
3. *If K is normal and generating, and X is fundamentally σ-order complete, then X is monotonically sequentially complete.*

The following characterizations of these completeness conditions will be used in the study of the completeness with respect to the Thompson metric.

Proposition 15. *The following conditions are equivalent.*

1. *X is fundamentally σ-order complete.*
2. *Any decreasing Cauchy sequence in X has an infimum.*
3. *Any increasing Cauchy sequence in K has a supremum.*
4. *Any decreasing Cauchy sequence in K has an infimum.*

Proposition 16. *The following conditions are equivalent.*

1. *X is monotonically sequentially complete.*
2. *Any decreasing Cauchy sequence in X has limit.*
3. *Any increasing Cauchy sequence in K has limit.*
4. *Any decreasing Cauchy sequence in K has limit.*

Proposition 17. *If K is lineally solid, then the following conditions are equivalent.*

1. *X is fundamentally σ-order complete.*
2. *Any increasing Cauchy sequence in* $\text{aint}(K)$ *has a supremum.*
3. *Any decreasing Cauchy sequence in* $\text{aint}(K)$ *has an infimum.*

Proposition 18. *If K is lineally solid, then the following conditions are equivalent.*

1. *X is monotonically sequentially complete.*
2. *Any increasing τ-Cauchy sequence in* $\text{aint}(K)$ *has limit.*
3. *Any decreasing τ-Cauchy sequence in* $\text{aint}(K)$ *has limit.*

3 The Thompson Metric

3.1 Definition and Fundamental Properties

Let X be a vector space and K a cone in X. The relation

$$x \sim y \iff \exists \lambda, \mu > 0, \quad x \leq \lambda y \text{ and } y \leq \mu x, \tag{14}$$

is an equivalence relation in K. One says that two elements $x, y \in K$ satisfying (14) are *linked* and the equivalence classes are called *components*. The equivalence class of an element $x \in K$ will be denoted by $K(x)$.

Proposition 19. *Let X be a vector space ordered by a cone K.*

1. *$K(0) = \{0\}$ and* $\text{aint}(K)$ *is a component of K if K is lineally solid.*
2. *Every component Q of K is order-convex, convex, closed under addition and multiplication by positive scalars, that is $Q \cup \{0\}$ is an order-convex cone.*

Proof. We justify only the assertion concerning $\text{aint}\, K$, the others being trivial. If $x, y \in \text{aint}\, K$, then there exist $\alpha, \beta > 0$ such that $x + ty \in K$ for all $t \in [-\alpha, \alpha]$ and $y + sx \in K$ for all $s \in [-\beta, \beta]$. It follows $y - \beta x \in K$, i.e. $y \geq \beta x$, and $x - \alpha y \in K$, i.e., $x \geq \alpha y$. □

For two linked elements $x, y \in K$ put

$$\sigma(x, y) = \{s \geq 0 : e^{-s} x \leq y \leq e^{s} x\}, \tag{15}$$

and let

$$d_T(x,y) = \inf \sigma(x,y). \tag{16}$$

Remark 9. It is convenient to define d_T for any pair of elements in K, by setting $d_T(x,y) = \infty$ for any x, y not lying in the same component of K which, by (16), is in concordance with the usual convention $\inf \emptyset = \infty$. In this way, d_T becomes an extended (or generalized) (semi)metric (in the sense of Jung [20]) on K and, for all $x, y \in K$, $x \sim y \iff d(x,y) < \infty$. Though d_T is not a usual (semi)metric on the whole cone, we will continue to call d_T a metric. The Thompson metric is also called, by some authors, the part metric (of the cone K).

Remark 10. It is obvious that the definition of $d(x,y)$ depends only on the ordering of the vector subspace spanned by $\{x, y\}$. This ensures that if x and y are seen as elements of some vector subspace Y of X, then $d_T(x,y)$ is the same in Y as in X (assuming, of course, that Y inherits the ordering from X).

The initial approach of Thompson [41] was slightly different. He considered the set

$$\alpha(x,y) = \{\lambda \geq 1 : x \leq \lambda y\}. \tag{17}$$

and defined the distance between x and y by

$$\delta(x,y) = \ln\left(\max\{\inf \alpha(x,y), \inf \alpha(y,x)\}\right). \tag{18}$$

The following proposition shows that the relations (16) and (18) yield the same function.

Proposition 20. *For every $x, y \in K$ the following equality holds*

$$d_T(x,y) = \delta(x,y).$$

Proof. It suffices to prove the equality for two linked elements $x, y \in K$. In this case let

$$\alpha_1 = \inf \alpha(x,y), \quad \alpha_2 = \inf \alpha(y,x), \quad \text{and} \quad \alpha = \max\{\alpha_1, \alpha_2\}.$$

Put also

$$d = d_T(x,y) = \inf \sigma(x,y) \quad \text{and} \quad \delta = \delta(x,y) = \ln \alpha.$$

For $s \in \mathbb{R}$ let $\lambda = e^s$. Then the following equivalences hold

$$\begin{aligned} s \in \sigma(x,y) &\iff \lambda^{-1} x \leq y \leq \lambda x \\ &\iff x \leq \lambda y \ \wedge \ y \leq \lambda x \iff \lambda \in \alpha(x,y) \cap \alpha(y,x). \end{aligned} \tag{19}$$

Consequently $\lambda \geq \max\{\alpha_1, \alpha_2\} = \alpha$ and $s \geq \ln \alpha = \delta$, for every $s \in \sigma(x, y)$, and so

$$d = \inf \sigma(x, y) \geq \delta. \tag{20}$$

To prove the reverse inequality, suppose that $\alpha_1 \geq \alpha_2$ and let $\lambda > \alpha_1$. Then $\lambda \in \alpha(x, y) \cap \alpha(y, x)$ and the equivalences (19) show that $s = \ln \lambda \in \sigma(x, y)$, so that $\ln \lambda \geq d$. It follows

$$\delta = \inf\{\ln \lambda : \lambda > \alpha_1\} \geq d,$$

which together with (20) yields $\delta = d$. □

There is another metric defined on the components of K, namely the *Hilbert projective metric*, defined by

$$d_H(x, y) = \ln\left(\inf \alpha(x, y) \cdot \inf \alpha(y, x)\right), \tag{21}$$

for any two linked elements x, y of K.

The term projective comes from the fact that $d_H(x, y) = 0$ iff $x = \lambda y$ for some $\lambda > 0$.

The original Hilbert's definition (see [16]) of the metric was the following. Consider an open bounded convex subset Ω of the Euclidean space $\mathbb{R}^n$. For two points $x, y \in \Omega$ let ℓ_{xy} denote the straight line through x and y, and denote the points of intersection of ℓ_{xy} with the boundary $\partial\Omega$ of Ω by x', y', where x is between x' and y, and y is between x and y'. For $x \neq y$ in Ω the Hilbert distance between x and y is defined by

$$\delta_H(x, y) = \ln\left(\frac{\|x' - y\| \cdot \|y' - x\|}{\|x' - x\| \cdot \|y' - y\|}\right), \tag{22}$$

and $\delta_H(x, x) = 0$ for all $x \in \Omega$, where $\|\cdot\|$ stands for the Euclidean norm in $\mathbb{R}^n$. The metric space (Ω, δ_H) is called the Hilbert geometry on Ω. In this geometry there exists a triangle with non-colinear vertices such that the sum of the lengths of two sides equals the length of the third side. If Ω is the open unit disk, the Hilbert metric is exactly the Klein model of the hyperbolic plane.

The definition (21) of Hilbert metric on cones in vector spaces was proposed by Bushell [7] (see also [8]).

Note 2. As we shall consider only the Thompson metric, the subscript T will be omitted, that is $d(\cdot, \cdot)$ will stand always for the Thompson metric.

In the following proposition we collect some properties of the set $\sigma(x, y)$.

Proposition 21. *Let X be a vector space ordered by a cone K and x, y, z linked elements in K.*

1. *Symmetry:* $\sigma(y,x) = \sigma(x,y)$.
2. $(d(x,y);\infty) \subset \sigma(x,y) \subset [d(x,y);\infty)$. *If the cone K is Archimedean, then $d(x,y) \in \sigma(x,y)$, that is $\sigma(x,y) = [d(x,y);\infty)$.*
3. $\sigma(x,y) + \sigma(y,z) \subset \sigma(x,z)$.

Proof. 1. The symmetry follows from the definition of the set $\sigma(x,y)$.
2. The inclusion $(d(x,y);\infty) \subset \sigma(x,y)$ follows from the fact that $0 < \lambda < \mu$ and $x \geq 0$ implies $\lambda x \leq \mu x$. The second inclusion follows from the fact that no $\lambda < d(x,y)$ belongs to $\sigma(x,y)$.

   Let $d = d(x,y) = \inf \sigma(x,y)$. Since an Archimedean cone is lineally closed and $y - e^{-s}x \in K$ for every $s > d$, it follows $y - e^{-d}x \in K$. Similarly $e^d x - y \in K$, showing that $d \in \sigma(x,y)$.
3. Let $s \in \sigma(x,y)$ and $t \in \sigma(y,z)$. Then

$$e^{-s}x \leq y \leq e^s x \quad \text{and} \quad e^{-t}y \leq z \leq e^t y.$$

It follows

$$e^{-(s+t)}x \leq e^{-t}y \leq z \quad \text{and} \quad z \leq e^t y \leq e^{s+t}x,$$

which shows that $s + t \in \sigma(x,z)$. □

Now it is easy to show that the function d given by (16) is an extended semimetric.

Proposition 22. *Let X be a vector space ordered by a cone K.*

1. *The function d defined by (16) is a semimetric on each component of K.*
2. *The function d is a metric on each component of K iff the order defined by the cone K is almost Archimedean.*

Proof. 1. The fact that d is a semimetric follows from the properties of the sets $\sigma(x,y)$ mentioned in Proposition 21.
2. Suppose now that the cone K is almost Archimedean and $d(x,y) = 0$ for two linked elements $x, y \in K$. It follows

$$\begin{aligned}\forall s > 0,\ \ e^{-s}x \leq y \leq e^s x &\iff \forall s > 0,\ \ (e^{-s} - 1)x \leq y - x \leq (e^s - 1)x \\ &\iff \forall s > 0,\ \ -\frac{e^s - 1}{e^s}x \leq y - x \leq (e^s - 1)x.\end{aligned}$$

The inequality $e^{-s}(e^s - 1) \leq e^s - 1$ implies $-e^{-s}(e^s - 1)x \geq -(e^s - 1)x$. Consequently,

$$\forall s > 0,\ \ -(e^s - 1)x \leq y - x \leq (e^s - 1)x \iff \forall \lambda > 0,\ \ -\lambda x \leq y - x \leq \lambda x.$$

Taking into account that K is almost Archimedean it follows $y - x = 0$, that is $y = x$.

To prove the converse, suppose that K is not almost Archimedean. Then there exists a line $D = \{x + \mu y : \mu \in \mathbb{R}\}$, with $y \neq 0$, contained in K. If $x = 0$, then $\pm y \in K$, that would imply $y = 0$, a contradiction.

Consequently $x \neq 0$. Observe that in this case, for all $\mu \in \mathbb{R}$,

$$d(x, x + \mu y) = 0, \tag{23}$$

which shows that d is not a metric. The equality (23) is equivalent to

$$\forall s > 0, \quad e^{-s}x \leq x + \mu y \leq e^s x. \tag{24}$$

The inclusion $D \subset K$ implies $x \pm \lambda y \in K$ for all $\lambda > 0$, and so

$$-x \leq \lambda y \leq x,$$

for all $\lambda > 0$. Taking $\lambda = \mu(1 - e^{-s})^{-1}$ the first inequality from above becomes

$$-(1 - e^{-s})x \leq \mu y \iff e^{-s}x \leq x + \mu y.$$

From the second inequality one obtains

$$\mu y \leq (1 - e^{-s})x = e^{-s}(e^s - 1)x \leq (e^s - 1)x,$$

which implies

$$x + \mu y \leq e^s x,$$

showing that the inequalities (24) hold. □

Remark 11. By the triangle inequality, the equality (23) implies that $d(u, v) = 0$ for any two points u, v on D, that is

$$d(x + \lambda y, x + \mu y) = 0,$$

for all $\lambda, \mu \in \mathbb{R}$.

Example 1. If $X = \mathbb{R}^n$ and $K = \mathbb{R}^n_+$, then the components of K are $\{0\}$, $(0; \infty) \cdot e_i$, $1 \leq i \leq n$, and $\operatorname{aint}(K) = \{x \in K : x_i > 0, \ i = 1, \dots, n\}$, while $d(x, y) = \max\{|\ln x_i - \ln y_i| : 1 \leq i \leq n\}$, for any $x = (x_i)_{i=1}^n$ and $y = (y_i)_{i=1}^n$, with $x_i, y_i > 0$, $i = 1, \dots, n$.

The following proposition contains some further properties of the sets $\sigma(x, y)$ and their correspondents for the Thompson metric.

Proposition 23. *Let X be a vector space ordered by a cone K.*

1. *For $x, y \in K$ and $\lambda, \mu > 0$*

$$\text{(i)}\quad \sigma(\lambda x, \lambda y) = \sigma(x, y) \quad \textit{and so} \quad d(\lambda x, \lambda y) = d(x, y);$$

$$\text{(ii)}\quad \sigma(\lambda x, \mu x) = [\big|\ln\big(\frac{\lambda}{\mu}\big)\big|; \infty) \quad \textit{and so} \quad d(\lambda x, \mu x) = \left|\ln\left(\frac{\lambda}{\mu}\right)\right|;$$

$$\text{(iii)}\quad \textit{If}\ \ \mu x \le y \le \lambda x, \textit{for some}\ \lambda, \mu > 0, \quad \textit{then} \quad d(x, y) \le \ln\max\{\mu^{-1}, \lambda\}.$$

2. *If $\sigma(x, y) \subset \sigma(x', y')$, then $d(x, y) \ge d(x', y')$. The converse is true if the order is Archimedean. Also*

$$\max\{d(x, y), d(x', y')\} = \inf[\sigma(x, y) \cap \sigma(x', y')]. \tag{25}$$

3. *The following monotony inequalities hold*

$$\begin{aligned} &(i) \quad x \le x' \text{ and } y' \le y \implies d(x', x' + y') \le d(x, x + y);\\ &(ii) \quad x \le x' \le y' \le y \implies d(x', y') \le d(x, y). \end{aligned} \tag{26}$$

4. *For all $x, y, x', y' \in K$ and $\lambda, \mu > 0$,*

$$d(\lambda x + \mu y, \lambda x' + \mu y') \le \max\{d(x, x'), d(y, y')\}. \tag{27}$$

Proof. 1. The equalities from (i) are obvious.

To prove (ii) suppose $\lambda > \mu$. Then $e^{-s}x \le x \le \lambda\mu^{-1}x$ implies $\mu e^{-s}x \le \lambda x$, for every $s > 0$. Since

$$\lambda\mu^{-1}x \le e^s x \iff s \ge \ln\big(\lambda\mu^{-1}\big),$$

it follows $\sigma(\lambda x, \mu x) = \big[\ln\big(\lambda\mu^{-1}\big); \infty)$ and $d(\lambda x, \mu x) = \ln\big(\lambda\mu^{-1}\big)$.

To prove (iii) observe that $\mu x \le y$ is equivalent to $x \le \mu^{-1}y$, that is $\mu^{-1} \in \alpha(x, y)$, and so $\mu^{-1} \ge \inf\alpha(x, y)$. Similarly, $y \le \lambda x$ is equivalent to $\lambda \in \alpha(y, x)$, implying $\lambda \ge \inf\alpha(x, y)$. It follows

$$\ln\max\{\mu^{-1}, \lambda\} \ge \ln(\max\{\inf\alpha(x, y), \inf\alpha(y, x)\} = d(x, y).$$

2. The first implication is obvious. The converse follows from the fact that $\sigma(x, y) = [d(x, y); \infty)$ and $\sigma(x', y') = [d(x', y'); \infty)$ if K is Archimedean (Proposition 21 (2)).

The equality (25) follows from the inclusions

$$\begin{aligned} &(d(x, y); \infty) \subset \sigma(x, y) \subset [d(x, y); \infty) \quad \text{and}\\ &(d(x', y'); \infty) \subset \sigma(x', y') \subset [d(x', y'); \infty). \end{aligned}$$

3. The inequality (i) for the metric d will follow from the inclusion

$$\sigma(x, x+y) \subset \sigma(x', x'+y'). \tag{28}$$

Let $s \in \sigma(x, x+y)$, that is $s > 0$ and

$$e^{-s}x \le x + y \le e^s x.$$

Then

$$\begin{aligned} e^{-s}x' &\le x' \le x' + y' \le x' + y = x + y + (x' - x) \\ &\le e^s x + e^s(x' - x) = e^s x', \end{aligned}$$

showing that $s \in \sigma(x', x'+y')$.

The inequality (ii) follows from (i) by taking $y := y - x \ge y' - x' =: y'$.

4. By 1 (i), $d(\lambda x, \lambda x') = d(x, x')$ and $d(\mu y, \mu y') = d(y, y')$, so that it is sufficient to show that

$$d(x+y, x'+y') \le \max\{d(x, x'), d(y, y')\}. \tag{29}$$

Taking into account (25) and the assertion 2 of the proposition, the inequality (29) will be a consequence of the inclusion

$$\sigma(x, x') \cap \sigma(y, y') \subset \sigma(x+y, x'+y').$$

But, if $s \in \sigma(x, x') \cap \sigma(y, y')$, then $e^{-s}x \le x' \le e^s x$ and $e^{-s}y \le y' \le e^s y$, which by addition yield $e^{-s}(x+y) \le x' + y' \le e^s(x+y)$, that is $s \in \sigma(x + y, x' + y')$. □

Based on these properties one obtains other properties of the Thompson metric.

Theorem 5. *Let X be a vector space ordered by a cone K.*

1. *The function d is quasi-convex with respect to each of its argument, that is*

$$\begin{aligned} &d((1-t)x + ty, v) \le \max\{d(x, v), d(y, v)\} \quad \textit{and} \\ &d(u, (1-t)x + ty) \le \max\{d(u, x), d(u, y)\}, \end{aligned} \tag{30}$$

for all $x, y, u, v \in K$ and $t \in [0; 1]$.

2. *The following convexity-type inequalities hold*

$$\begin{aligned} &d((1-t)x + ty, v) \le \ln\left((1-t)e^{d(x,v)} + te^{d(y,v)}\right), \\ &d(u, (1-t)x + ty) \le \ln\left((1-t)e^{d(u,x)} + te^{d(u,y)}\right), \end{aligned} \tag{31}$$

for all $x, y, u, v \in K$ and $t \in [0; 1]$, and

$$d((1-t)x+ty,(1-s)x+sy) \le \ln\left(|s-t)e^{d(x,y)}+1-|s-t|\right), \qquad (32)$$

for all $x, y \in K,\ x \sim y,$ *and* $s, t \in [0;1]$.

Proof. 1. By (25) and Proposition 23 (1) (i),

$$d((1-t)x+ty,v) = d((1-t)x+ty,(1-t)v+tv) \\ \le \max\{d((1-t)x,(1-t)v), d(ty,tv)\} = \max\{d(x,v), d(y,v)\},$$

showing that the first inequality in (30) holds. The second one follows by the symmetry of the metric d.

2. For $s_1 \in \sigma(x,v)$ and $s_2 \in \sigma(y,v)$ put $s = \ln\left((1-t)e^{s_1}+te^{s_2}\right)$. By a straightforward calculation it follows that

$$((1-t)e^{s_1}+te^{s_2})\cdot((1-t)e^{-s_1}+te^{-s_2}) = 2t(1-t)(\cosh(s_1-s_2)-1) \ge 0,$$

which implies

$$-s \le \ln\left((1-t)e^{-s_1}+te^{-s_2}\right),$$

or, equivalently,

$$e^{-s} \le (1-t)e^{-s_1}+te^{-s_2}.$$

The above inequality and the inequalities $e^{-s_1}v \le x,\ e^{-s_2}v \le y$ imply

$$e^{-s}v \le ((1-t)e^{-s_1}+te^{-s_2})\,v \le (1-t)x+ty.$$

Similarly, the inequalities $x \le e^{s_1}v,\ y \le e^{s_2}v$, and the definition of s imply

$$(1-t)x+ty \le ((1-t)e^{-s_1}+te^{-s_2})\,v = e^s v.$$

It follows $s \in \sigma((1-t)x+ty,v)$ and so

$$d((1-t)x+ty,v) \le s = \ln\left((1-t)e^{-s_1}+te^{-s_2}\right),$$

for all $s_1 \in \sigma(x,v)$ and all $s_2 \in \sigma(y,v)$. Passing to infimum with respect to s_1 and s_2, one obtains the first inequality in (31). The second inequality follows by the symmetry of d.

It is obvious that (32) holds for $s = t$, so we have to prove it only for $s \ne t$. By symmetry it suffices to consider only the case $t > s$. Putting $z_t = (1-t)x+ty$ and $z_s = (1-s)x+sy$, it follows $z_s = (1-\frac{s}{t})x+\frac{s}{t}z_t$, so that, applying twice the inequality (31),

$$d(x,z_t) \le \ln\left(1-t+te^{d(x,y)}\right),$$

and

$$d(z_s, z_t) \le \ln\left(\left(1 - \frac{s}{t}\right)e^{d(x,z_t)} + \frac{s}{t}\right)$$
$$\le \ln\left(\left(1 - \frac{s}{t}\right)(1 - t + te^{d(x,y)}) + \frac{s}{t}\right) = \ln\left((t-s)e^{d(x,y)} + 1 - (t-s)\right).$$

□

Recall that a metric space (X, ρ) is called *metrically convex* if for every pair of distinct points $x, y \in X$ there exists a point $z \in X \setminus \{x, y\}$ such that

$$\rho(x, y) = \rho(x, z) + \rho(z, y). \tag{33}$$

The following theorem, asserting that every component of K is metrically convex with respect to the Thompson metric, is a slight extension of a result of Nussbaum [31, Proposition 1.12].

Theorem 6. *Every component of K is metrically convex with respect to the Thompson metric d. More exactly, for every pair of distinct points $x, y \in X$ and every $t \in (0; 1)$ the point*

$$z = \frac{\sinh r(1-t)}{\sinh r}x + \frac{\sinh rt}{\sinh r}y,$$

where $r = d(x, y)$, satisfies (33).

Proof. By the triangle inequality it suffices to show that

$$r = d(x, y) \ge d(x, z) + d(z, y). \tag{34}$$

If $s \in \sigma(x, y)$, that is $e^{-s}x \le y \le e^s x$, then

$$\left(\frac{\sinh r(1-t)}{\sinh r} + \frac{\sinh rt}{\sinh r}e^{-s}\right)x \le z \le \left(\frac{\sinh r(1-t)}{\sinh r} + \frac{\sinh rt}{\sinh r}e^{s}\right)x. \tag{35}$$

Putting

$$\mu(s) = \frac{\sinh r(1-t)}{\sinh r} + \frac{\sinh rt}{\sinh r}e^{-s} \quad \text{and} \quad \lambda(s) = \frac{\sinh r(1-t)}{\sinh r} + \frac{\sinh rt}{\sinh r}e^{s},$$

the inequalities (35) imply

$$d(x, z) \le \ln(\max\{\mu(s)^{-1}, \lambda(s)\}),$$

for all $s > r$. Since the functions $\mu(s)^{-1}$ and $\lambda(s)$ are both continuous on $(0; \infty)$, it follows

$$d(x,z) \le \ln(\max\{\mu(r)^{-1}, \lambda(r)\}). \tag{36}$$

Taking into account the definition of the function sinh, a direct calculation shows that $\mu(r)^{-1} = \lambda(r) = e^{rt}$, and so the inequality (36) becomes

$$d(x,z) \le rt.$$

By symmetry

$$d(z,y) \le r(1-t),$$

so that (34) holds. □

3.2 Order-Unit Seminorms

Suppose that X is a vector space ordered by a cone K. For $u \in K \setminus \{0\}$ put

$$X_u = \cup_{\lambda \ge 0} \lambda[-u;u]_o. \tag{37}$$

It is obvious that X_u is a nontrivial subspace of X $(\mathbb{R}u \subset X_u)$, and that $[-u;u]_o$ is an absorbing absolutely convex subset of X_u and so u is a unit in the ordered vector space (X_u, K_u), where K_u is the cone in X_u given by

$$K_u = K \cap X_u\,, \tag{38}$$

or, equivalently, by

$$K_u = \cup_{\lambda \ge 0} \lambda[0;u]_o\,. \tag{39}$$

The Minkowski functional

$$|x|_u = \inf\{\lambda > 0 : x \in \lambda[-u;u]_o\}, \tag{40}$$

corresponding to the set $[-u;u]_o$, is a seminorm on the space X_u and

$$|-u|_u = |u|_u = 1. \tag{41}$$

For convenience, denote by the subscript u the topological notions corresponding to the seminorm $|\cdot|_u$. Let also $B_u(x,r)$, $B_u[x,r]$ be the open, respectively closed, ball with respect to $|\cdot|_u$. For $x \in X_u$ let

$$\mathscr{M}_u(x) = \{\lambda > 0 : x \in \lambda[-u;u]_o\}, \tag{42}$$

so that

$$|x|_u = \inf \mathscr{M}_u(x).$$

Taking into account the convexity of $[-u; u]_o$ it follows that

$$(|x|_u; \infty) \subset \mathscr{M}_u(x) \subset [|x|_u; \infty), \tag{43}$$

for every $x \in X_u$.

Proposition 24. *Let $u \in K \setminus \{0\}$ and $X_u, K_u, |\cdot|_u$ as above.*

1. *If $v \in K$ is linked to u, then $X_u = X_v$, $K_u = K_v$, and the seminorms $|\cdot|_u, |\cdot|_v$ are equivalent. More exactly the following inequalities hold for all $x \in X_u$*

$$|x|_u \le |v|_u |x|_v \quad \textit{and} \quad |x|_v \le |u|_v |x|_u. \tag{44}$$

2. *The Minkowski functional $|\cdot|_u$ is a norm on X_u iff the cone K_u is almost Archimedean.*
3. *The seminorm $|\cdot|_u$ is monotone: $x, y \in X_u$ and $0 \le x \le y$ implies $|x|_u \le |y|_u$.*
4. *The cone K_u is generating and normal in X_u.*
5. *For any $x \in X_u$ and $r > 0$, $B_u(x, r) \subset x + r[-u; u]_o \subset B_u[x, r]$.*
6. *The following equalities hold:*

$$\operatorname{aint}(K_u) = K(u) = \operatorname{int}_u(K_u). \tag{45}$$

7. *The following are equivalent:*

   (i) *K_u is $|\cdot|_u$-closed;*
   (ii) *K_u is lineally closed;*
   (iii) *K_u is Archimedean.*

   In this case, $|x|_u \in \mathscr{M}_u(x)$ (that is $\mathscr{M}_u(x) = [|x|_u; \infty)$) and $B_u[0, 1] = [-u; u]_o$.

Proof. 1. If $v \sim u$, then $v \in X_u$ and $u \in X_v$ which imply $X_u = X_v$ and $K_u = K_v$. We have

$$\forall \alpha > |v|_u, \quad -\alpha u \le v \le \alpha u. \tag{46}$$

Let $x \in X_u$. If $\beta > 0$ is such that

$$-\beta v \le x \le \beta v, \tag{47}$$

then

$$\forall \alpha > |v|_u, \quad -\alpha\beta u \le x \le \alpha\beta u.$$

It follows

$$|x|_u \le \alpha\beta,$$

for all $\beta > 0$ for which (47) is satisfied and for $\alpha > |v|_u$, implying $|x|_u \le |v|_u |x|_v$. The second inequality in (44) follows by symmetry.

2. It is known that the Minkowski functional corresponding to an absorbing absolutely convex subset Z of a linear space X is a norm iff the set Z is radially bounded in X (i.e., any ray from 0 intersects Z in a bounded interval). Since a cone is almost Archimedean iff any order interval is lineally bounded (Proposition 4), the equivalence follows.
3. If $0 \le x \le y$, then $\mathscr{M}_u(y) \subset \mathscr{M}_u(x)$ and so $|y|_u = \inf \mathscr{M}_u(y) \ge \inf \mathscr{M}_u(x) = |x|_u$.
4. The fact that K_u is generating follows from definitions. The normality follows from the fact that the seminorm $|\cdot|_u$ is monotone and Theorem 3.
5. If p is a seminorm corresponding to an absorbing absolutely convex subset Z of a vector space X, then

$$B_p(0,1) \subset Z \subset B_p[0,1],$$

which in our case yield

$$B_u(0,1) \subset [-u;u]_o \subset B_u[0,1],$$

which, in their turn, imply the inclusions from 4.
6. We shall prove the inclusions

$$\operatorname{int}_u(K_u) \subset \operatorname{aint}(K_u) \subset K(u) \subset \operatorname{int}_u(K_u). \tag{48}$$

The first inclusion from above is a general property in TVS.

The inclusion $\operatorname{aint}(K_u) \subset K(u)$.

For $x \in \operatorname{aint}(K_u)$ we have to prove the existence of $\alpha, \beta > 0$ such that

$$\alpha u \le x \le \beta u.$$

Since $x \in \operatorname{aint}(K_u)$ there exists $\alpha > 0$ such that $x + tu \in K_u$ for all $t \in [-\alpha, \alpha]$ which implies $x - \alpha u \in K_u$, that is $x \ge \alpha u$.

From (39) and the fact that $x \in K_u$ follows the existence of $\beta > 0$ such that $x \in \beta[0;u]_o$, so that $x \le \beta u$.

The inclusion $K(u) \subset \operatorname{int}_u(K_u)$.

If $x \in K(u)$, then there exist $\alpha, \beta > 0$ such that $\alpha u \le x \le \beta u$. But then

$$B_u\Big(x, \frac{\alpha}{2}\Big) = x + B_u\left(0, \frac{\alpha}{2}\right) \subset x + \frac{\alpha}{2}[-u;u]_o \subset \left[\frac{\alpha}{2}u; \Big(\beta + \frac{\alpha}{2}\Big)u\right]_o \subset K_u,$$

proving that x is a $|\cdot|_u$-interior point of K_u.

7. The implication (i) $\Rightarrow$ (ii) is a general property. By Proposition 3, (ii) $\iff$ (iii). It remains to prove the implication (iii) $\Rightarrow$ (i).

Let $x \in X_u$ be a point in the $|\cdot|_u$-closure of K_u. Then for every $n \in \mathbb{N}$ there exists $x_n \in K_u$ such that $|x_n - x|_u < \frac{1}{n}$. By the definition of the seminorm $|\cdot|_u$,

$$x_n - x \in \frac{1}{n}[-u; u]_o,$$

so that

$$-x \leq -x + x_n \leq \frac{1}{n} u,$$

for all $n \in \mathbb{N}$.

By Proposition 3, this implies $-x \leq 0$, that is $x \geq 0$, which means that $x \in K_u$.

Suppose now that the cone K_u is Archimedean. For $x \in X_u \setminus \{0\}$ put $\alpha := |x|_u > 0$. Then there exists a sequence $\alpha_n \searrow \alpha$ such that $x \in \alpha_n[-u; u]_o$ for all $n \in \mathbb{N}$, so that

$$\frac{1}{\alpha_n} x + u \geq 0 \quad \text{and} \quad -\frac{1}{\alpha_n} x + u \geq 0,$$

for all $n \in \mathbb{N}$. Since the cone K_u is lineally closed, it follows

$$\frac{1}{\alpha} x + u \geq 0 \quad \text{and} \quad -\frac{1}{\alpha} x + u \geq 0,$$

which means $x \in \alpha[-u; u]_o$.

By 4, $[-u; u]_o \subset B_u[0, 1]$. If $x \in B[0, 1]$ (i.e., $|x|_u \leq 1$), then $\mathscr{M}_u(x) = [|x|_u; \infty)$, and so

$$x \in |x|_u [-u; u]_o \subset [-u; u]_o. \qquad \square$$

The above construction corresponds to the one used in LCS. For a bounded absolutely convex subset A of a locally convex space (X, τ) one considers the space X_A generated by A,

$$X_A = \cup_{\lambda > 0} \lambda A = \cup_{n=1}^{\infty} n A. \tag{49}$$

Then A is an absolutely convex absorbing subset of X_A and the attached Minkowski functional

$$p_A(x) = \inf\{\lambda > 0 : x \in \lambda A\}, \quad x \in A, \tag{50}$$

is a norm on X_A .

Theorem 7. *Let (X,τ) be a Hausdorff locally convex space and A a bounded absolutely convex subset of X.*

1. *The Minkowski functional p_A is a norm on X_A and the topology generated by p_A is finer than that induced by τ (or, in other words, the embedding of (X_A, p_A) in (X,τ) is continuous).*
2. *If, in addition, the set A is sequentially complete with respect to τ, then (X_A, p_A) is a Banach space. In particular, this is true if the space X is sequentially complete.*

In the case when (X_A, p_A) is a Banach space one says that A is a *Banach disc*. These spaces are used to prove that every locally convex space is an inductive limit of Banach spaces and that weakly bounded subsets of a sequentially complete Hausdorff LCS are strongly bounded. (A subset Y of a LCS X is called *strongly bounded* if

$$\sup\{|x^*(y)| : y \in Y, x^* \in M\} < \infty,$$

for every weakly bounded subset M of X^*).

For details concerning this topic, see the book [36, Sect. 3.2], or [21, Sect. 20.11].

In our case, the normality of K guarantees the completeness of $(X_u, |\cdot|_u)$.

Theorem 8. *Let (X,τ) be a Hausdorff LCS ordered by a closed normal cone K and $u \in K \setminus \{0\}$.*

1. *The functional $|\cdot|_u$ is a norm on X_u and the topology generated by $|\cdot|_u$ on X_u is finer than that induced by τ (or, equivalently, the embedding of $(X_u, |\cdot|_u)$ in (X,τ) is continuous).*
2. *If the space X is sequentially complete, then $(X_u, |\cdot|_u)$ is a Banach space.*
3. *If u is a unit in (X, K), then $X_u = X$. If $u \in \operatorname{int}(K)$, then the topology generated by $|\cdot|_u$ agrees with τ.*

Proof. By Theorem 2, we can suppose that the topology τ is generated by a directed family P of γ-monotone seminorms, for some $\gamma > 0$.

1. By Proposition 10, the set $[-u; u]_o$ is bounded and so $|\cdot|_u$ is a norm. We show that the embedding of $(X_u, |\cdot|_u)$ in (X, P) is continuous.

   Let $p \in P$. The inequalities $-|x|_u u \le x \le |x|_u u$ imply

   $$0 \le x + |x|_u u \le 2|x|_u u \quad \text{and} \quad 0 \le -x + |x|_u u \le 2|x|_u u,$$

   for all $x \in X_u$

   By the γ-monotonicity of the seminorm p these inequalities imply in their turn

   $$\begin{aligned} 2p(x) \le& p(x + |x|_u u) + p(x - |x|_u u) = p(x + |x|_u u) + p(-x + |x|_u\, u) \\ \le& 4\gamma |x|_u\, p(u). \end{aligned}$$

Consequently, for every $p \in P$,

$$p(x) \leq 2\gamma p(u)|x|_u, \tag{51}$$

for all $x \in X_u$, which shows that the embedding of $(X_u, |\cdot|_u)$ in (X, τ) is continuous.

2. Suppose now that (X, τ) is sequentially complete and let (x_n) be a $|\cdot|_u$-Cauchy sequence in X_u. By (51), (x_n) is p-Cauchy for every $p \in P$, so it is τ-convergent to some $x \in X$. By the Cauchy condition, for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $|x_{n+k} - x_n|_u < \varepsilon$, for all $n \geq n_0$ and all $k \in \mathbb{N}$. By the definition of the functional $|\cdot|_u$, it follows

$$-\varepsilon u \leq x_{n+k} - x_n \leq \varepsilon u,$$

for all $n \geq n_0$ and all $k \in \mathbb{N}$. Letting $k \to \infty$, one obtains

$$-\varepsilon u \leq x - x_n \leq \varepsilon u,$$

for all $n \geq n_0$, which implies $x \in X_u$ and $|x - x_n|_u \leq \varepsilon$, for all $n \geq n_0$. This shows that $x_n \xrightarrow{|\cdot|_u} x$.

3. If u is a unit in (X, K), then the order interval $[-u; u]_o$ is absorbing, and so $X = \cup_{n=1}^{\infty} n\, [-u; u]_o = X_u$.

   Suppose now that $u \in \operatorname{int}(K)$. Then u is a unit in (X, K), so that $X = X_u$ and, by 1, the topology τ_u generated by $|\cdot|_u$ is finer than τ, $\tau \subset \tau_u$.

   Since $u \in \operatorname{int}(K)$, there exists $p \in P$ and $r > 0$ such that $B_p[u, r] \subset K$. Let $x \in X$, $x \neq 0$.

   If $p(x) = 0$, then $u \pm tx \in B_p[u, r] \subset K$ for every $t > 0$, so that $-t^{-1}u \leq x \leq t^{-1}u$ for all $t > 0$, which implies $|x|_u = 0$, in contradiction to the fact that $|\cdot|_u$ is a norm on X.

   Consequently, $p(x) > 0$ and $u \pm p(x)^{-1}rx \in B_p[u, r] \subset K$, that is

$$-\frac{p(x)}{r}\, u \leq x \leq \frac{p(x)}{r}\, u,$$

and so

$$|x|_u \leq \frac{p(x)}{r}.$$

   But then, $B_p[0, r] \subset B_{|\cdot|_u}[0, 1]$, which implies $B_{|\cdot|_u}[0, 1] \in \tau$, and so $\tau_u \subset \tau$.

□

Remark 12. Incidentally, the proof of the third assertion of the above theorem gives a proof to Proposition 11.

3.3 The Topology of the Thompson Metric

We shall examine some topological properties of the Thompson extended metric d. An extended metric ρ on a set Z defines a topology in the same way as a usual one, via balls. In fact all the properties reduce to the study of metric spaces formed by the components with respect to ρ. For instance, a sequence (z_n) in (Z,ρ) converges to some $z\in Z$, iff there exist a component Q with respect to ρ and $n_0\in\mathbb{N}$ such that $z\in Q$, $x_n\in Q$ for $n\ge n_0$, and $\rho(z_n,z)\to 0$ as $n\to\infty$, that is $(z_n)_{n\ge n_0}$ converges to z in the metric space $(Q,\rho|_Q)$.

The following results are immediate consequences of the definition.

Proposition 25. *Let X be a vector space ordered by a cone K.*

1. *The following inclusions hold*

$$B_d(x,r)\subset[e^{-r}x;e^rx]_o\subset B_d[x,r]. \tag{52}$$

   If K is Archimedean, then $B_d[x,r]=[e^{-r}x;e^rx]_o$.
2. *If K is Archimedean, then the set $[x;\infty)_o:=\{z\in K: x\le z\}$ and the order interval $[x;y]_o$ are d-closed, for every $x\in K$ and $y\ge x$.*
3. *Let $x,y\in K$ with $x\le y$. Then the order interval $[x;y]_o$ is d-bounded iff $x\sim y$.*

Proof. 1. If $d(x,y)<r$, then there exists s, $d(x,y)\le s<r$, such that $y\in[e^{-s}x;e^sx]_o$. Since $[e^{-s}x;e^sx]_o\subset[e^{-r}x;e^rx]_o$, the first inclusion in (52) follows. Obviously, $y\in[e^{-r}x;e^rx]_o$ implies $d(x,y)\le r$.

Suppose that K is Archimedean and $d(x,y)=r$. Let $t_n>r$ with $t_n\searrow r$. Then $e^{-t_n}x\le y\le e^{t_n}x$ for all n. Since K is Archimedean, these inequalities imply $e^{-r}x\le y\le e^rx$.

2. Let z be in the d-closure of $[x;\infty)_o$. Let $t_n>0$, $t_n\searrow 0$. Then for every $n\in\mathbb{N}$ there exists $z_n\ge x$ such that $d(z,z_n)<t_n$, implying $x\le z_n\le e^{t_n}z$. The inequalities $x\le e^{t_n}z$ yield for $n\to\infty$, $x\le z$, that is $z\in[x;\infty)_o$.

   In a similar way one shows that $[0;x]_0$ is d-closed. But then, $[x;y]_o=[x;\infty)_o\cap[0;y]_o$ is also d-closed.
3. If $[x;y]_o$ is bounded, then $d(x,y)<\infty$, and so $x\sim y$. Conversely, if $x\sim y$, then there exist $\alpha,\beta>0$ such that $\alpha x\le y\le\beta x$. Then, $x\le z\le y$ implies $x\le z\le y\le\beta x$, and so, by Proposition 23 (1) (iii), $d(x,z)\le\ln\beta$. □

Proposition 26. *Let X be a vector space ordered by a cone K.*

1. *The multiplication by scalars $\cdot:(0;\infty)\times K\to K$ and the addition $+:K\times K\to K$ are continuous with respect to the Thompson metric.*
2. *If Q is a component of K, then the mapping $(\lambda,x,y)\mapsto(1-\lambda)x+\lambda y$ from $[0;1]\times Q^2$ to Q is continuous with respect to the Thompson metric.*

Proof. 1. Let $(\lambda_0,x_0)\in(0;\infty)\times K$. Appealing to Proposition 23 (1) it follows

$$\begin{aligned}d(\lambda x,\lambda_0x_0)\le&d(\lambda x,\lambda_0x)+d(\lambda_0x,\lambda_0x_0)\\=&|\ln\lambda-\ln\lambda_0|+d(x,x_0)\to 0,\end{aligned}$$

if $\lambda \to \lambda_0$ and $x \overset{d}{\to} x_0$.

The continuity of the addition can be obtained from (27) (with $\lambda = \mu = 1$).

2. Let $\lambda, \lambda_0 \in [0;1]$ and $x, x_0, y, y_0 \in Q$. This time we shall appeal to the inequalities (27) and (32) to write

$$\begin{aligned} &d((1-\lambda)x + \lambda y, (1-\lambda_0)x_0 + \lambda_0 y_0) \\ &\quad \le d((1-\lambda)x+\lambda y, (1-\lambda)x_0+\lambda y_0)+d((1-\lambda)x_0+\lambda y_0, (1-\lambda_0)x_0+\lambda_0 y_0) \\ &\quad \le \max\{d(x,x_0), d(y,y_0)\} + \ln\left(|\lambda-\lambda_0|e^{d(x_0,y_0)} + 1 - |\lambda-\lambda_0|\right) \to 0 \end{aligned}$$

as $\lambda \to \lambda_0$, $x \overset{d}{\to} x_0$ and $y \overset{d}{\to} y_0$. □

Corollary 1. *Every component of K is path connected with respect to the Thompson metric.*

Proof. Follows from Proposition 26 (2). If x_0, x_1 are in the same component, then $\varphi(t) = (1-t)x_0 + tx_1$, $t \in [0;1]$, is a path connecting x_0 and x_1. □

Remark 13. The equivalence classes with respect to the equivalence $\sim$ are exactly the equivalence classes considered by Jung [20] (the equivalence relation considered by Jung is $x \simeq y \iff d(x,y) < \infty$, see Remark 9). Since these classes are both open and closed, it follows that the components of K with respect to $\sim$ are, in fact, the connected components of K with respect to the Thompson (extended) metric d.

In the following proposition we give a characterization of d-convergent monotone sequences.

Proposition 27. *Let X be a vector space ordered by an Archimedean cone K.*

1. *If (x_n) is an increasing sequence in K, then (x_n) is d-convergent to an $x \in K$ iff*

$$\begin{aligned} &(i) \quad \forall n \in \mathbb{N}, \quad x_n \le x, \quad \textit{and} \\ &(ii) \quad \forall \lambda > 1,\ \exists k \in \mathbb{N}, \quad x \le \lambda x_k. \end{aligned}$$

In this case, $x = \sup_n x_n$ and there exists $k \in \mathbb{N}$ such that $x_n \in K(x)$ for all $n \ge k$.

2. *If (x_n) is a decreasing sequence in K, then (x_n) is d-convergent to an $x \in K$ iff*

$$\begin{aligned} &(i) \quad \forall n \in \mathbb{N}, \quad x \le x_n, \quad \textit{and} \\ &(ii) \quad \forall \lambda \in (0;1),\ \exists k \in \mathbb{N}, \quad x \ge \lambda x_k. \end{aligned}$$

In this case, $x = \inf_n x_n$ and there exists $k \in \mathbb{N}$ such that $x_n \in K(x)$ for all $n \ge k$.

3. *Let (X, τ) be a TVS ordered by a cone K. If (x_n) is a d-Cauchy sequence in K which is τ-convergent to $x \in K$, then $x_n \xrightarrow{d} x$.*

Proof. We shall prove only the assertion 1, the proof of 2 being similar.

Suppose that the condition (i) and (ii) hold and let $\varepsilon > 0$. Then $\lambda := e^\varepsilon > 1$, so that, by (ii), there exists $k \in \mathbb{N}$ such that $x \le \lambda x_k = e^\varepsilon x_k$. Taking into account the monotony of the sequence (x_n) it follows that

$$e^{-\varepsilon} x_n \le x_n \le x \le e^\varepsilon x_n,$$

for all $n \ge k$, which implies $d(x, x_n) \le \varepsilon$ for all $n \ge k$, that is $x_n \xrightarrow{d} x$ as $n \to \infty$.

Conversely, suppose that (x_n) is an increasing sequence in K which is d-convergent to $x \in K$. For $\lambda > 1$ put $\varepsilon := \ln \lambda > 0$. Then there exists $k \in \mathbb{N}$ such that

$$\forall n \ge k, \quad d(x, x_n) < \varepsilon,$$

which implies

$$\forall n \ge k, \quad e^{-\varepsilon} x_n \le x \le e^\varepsilon x_n.$$

By the second inequality above, $x \le \lambda x_k$, which shows that (i) holds. Since (x_n) is increasing the first inequality implies that for every $n \in \mathbb{N}$

$$e^{-\varepsilon} x_n \le x,$$

for all $\varepsilon > 0$. By Proposition 3, the cone K is lineally closed, so that the above inequality yields for $\varepsilon \searrow 0$, $x_n \le x$ for all $n \in \mathbb{N}$, that is (i) holds too.

It is clear that if $x_n \xrightarrow{d} x$, then there exists $k \in \mathbb{N}$ such that $d(x, x_n) \le 1 < \infty$, for all $n \ge k$, which implies $x_n \in K(x)$ for all $n \ge k$.

It remains to show that $x = \sup_n x_n$. Let y be an upper bound for (x_n). Then for every $n \in \mathbb{N}$,

$$x_n \le x_{n+k} \le y \iff x_{n+k} \in [x_n; y]_o, \tag{53}$$

for all $k \in \mathbb{N}$. By Proposition 25 (2) the interval $[x_n; y]_o$ is d-closed, so that, letting $k \to \infty$ in (53) it follows $x \in [x_n; y]_o$. The inequality $x \le y$ shows that $x = \sup_n x_n$.

3. It follows that (x_n) is eventually contained in a component Q of K, so we can suppose $x_n \in Q$, $n \in \mathbb{N}$. Since (x_n) is d-Cauchy, there exists n_0 such that $d(x_n, x_{n_0}) < 1$ for all $n \ge n_0$. Then $e^{-1} x_{n_0} \le x_n \le e x_{n_0}$, for all $n \ge n_0$. Letting $n \to \infty$, one obtains $e^{-1} x_{n_0} \le x \le e x_{n_0}$, which shows that $x \sim x_{n_0}$, that is $x \in Q$. Now for $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $d(x_{n+k}, x_n) < \varepsilon$ for all $n \ge n_\varepsilon$ and all $k \in \mathbb{N}$. Then for every $n \ge n_\varepsilon$, $e^{-\varepsilon} x_n \le x_{n+k} \le e^\varepsilon x_n$, for all $k \in \mathbb{N}$. Letting $k \to \infty$, one obtains $e^{-\varepsilon} x_n \le x \le e^\varepsilon x_n$, implying $d(x_n, x) \le \varepsilon$ for all $n \ge n_\varepsilon$, that is $x_n \xrightarrow{d} x$. □

3.4 The Thompson Metric and Order-Unit Seminorms

The main aim of this subsection is to show that the Thompson metric and the metric seminorms are equivalent on each component of K. We begin with some inequalities.

Proposition 28. *Let X be a vector space ordered by a cone $K, u \in K \setminus \{0\}$ and $x, y \in K(u)$. The following relations hold.*

1. $d(x,y) = \ln(\max\{|x|_y, |y|_x\})$.
2. *(i)* $d(x,y) \geq |\ln|x|_u - \ln|y|_u|$, *or, equivalently,*
   (ii) $|x|_u \leq e^{d(x,y)}|y|_u$ *and* $|y|_u \leq e^{d(x,y)}|x|_u$.
3. $e^{-d(x,u)} \leq |x|_u \leq e^{d(x,u)}$.
4. $|u|_x \leq e^{d(x,y)}|u|_y$.
5. $d(x,y) \leq \ln\left(1 + |x-y|_u \cdot \max\{|u|_x, |u|_y\}\right)$.
6. $\left(e^{d(x,y)} - 1\right) \cdot \min\{|u|_x^{-1}, |u|_y^{-1}\} \leq |x-y|_u \leq \left(2e^{d(x,y)} + e^{-d(x,y)} - 1\right) \cdot \min\{|x|_u, |y|_u\}$.
7. $\left(1 - e^{-d(x,u)}\right) \cdot \max\{|u|_x^{-1}, |u|_y^{-1}\} \leq |x-y|_u$.
8. $|x-y|_x \geq 1 - e^{-d(x,y)}$.

Proof. 1. Recalling (42), it is easy to check that

$$s \in \sigma(x,y) \iff e^s \in \mathscr{M}_x(y) \cap \mathscr{M}_y(x),$$

and so

$$d(x,y) = \ln\left(\inf\left\{\mathscr{M}_x(y) \cap \mathscr{M}_y(x)\right\}\right) = \ln(\max\{|x|_y, |y|_x\}).$$

2. By (44), $|x|_u \leq |y|_u|x|_y$. Taking into account 1, it follows

$$d(x,y) \geq \ln|x|_y \geq \ln|x|_u - \ln|y|_u.$$

By symmetry, $d(x,y) \geq \ln|y|_u - \ln|x|_u$, so that 2 (i) holds. It is obvious that (i) and (ii) are equivalent.

3. Taking $y := u$ in both the inequalities from 2 (ii), one obtains

$$|x|_u \leq e^{d(x,u)}|y|_u \quad \text{and} \quad 1 \leq e^{d(x,u)}|x|_u\,.$$

4. By (44), $|u|_x \leq |u|_y|y|_x$ and, by 3, $|y|_x \leq e^{d(x,y)}$, hence $|u|_x \leq e^{d(x,y)}|u|_y$.
5. By (44) and the triangle inequality

$$|y|_x \leq |x|_x + |x-y|_x \leq 1 + |x-y|_u|u|_x\,, \quad \text{and}$$
$$|x|_y \leq |y|_y + |x-y|_y \leq 1 + |x-y|_u|u|_y\,,$$

so that

$$\max\{|x|_y, |y|_x\} \le 1 + |x-y|_u \cdot \max\{|u|_x, |u|_y\}\,.$$

The conclusion follows from 1.

6. The inequality 6 can be rewritten as $|x-y|_u \max\{|u|_x, |u|_y\} \ge e^{d(x,y)} - 1$, so that

$$|x-y|_u \ge \left(e^{d(x,y)} - 1\right)\left[\max\{|u|_x, |u|_y\}\right]^{-1} = \left(e^{d(x,y)} - 1\right) \cdot \min\{|u|_x^{-1}, |u|_y^{-1}\}.$$

To prove the second inequality, take $s \in \sigma(x,y)$ arbitrary. Then $-(e^s-1)x \le x-y \le (1-e^{-s}s)x$, so that $0 \le x-y+(e^s-1)x \le (e^s-^{-s})x$. The monotony of $|\cdot|_u$ and the triangle inequality imply

$$|x-y|_u - (e^s-1)|x_u \le |x-y-(e^s-1)(e^s-e^{-s})x|_u \le (e^s-e^{-s})|x|_u,$$

so that $|x-y|_u \le \left(2e^s + e^{-s} - 1\right)|x|_u$. Since this holds for every $s \in \sigma(x,y)$ it follows

$$|x-y|_u \le \left(2e^{d(x,y)} + e^{-d(x,y)} - 1\right)|x|_u.$$

By interchanging the roles of x and y in the above inequality, one obtains

$$|x-y|_u \le \left(2e^{d(x,y)} + e^{-d(x,y)} - 1\right)|y|_u.$$

These two inequalities imply the second inequality in 6.

7. By 4,

$$|u|_x^{-1} \ge e^{-d(x,y)}|u|_y^{-1} \quad \text{and} \quad |u|_y^{-1} \ge e^{-d(x,y)}|u|_x^{-1},$$

so that

$$\min\left\{|u|_x^{-1}, |u|_y^{-1}\right\} \ge e^{-d(x,y)} \max\left\{|u|_x^{-1}, |u|_y^{-1}\right\}.$$

The conclusion follows by 6.

8. This can be obtained by taking $u := x$ in 7. □

Theorem 9. *Let X be a vector space ordered by a cone K and $u \in K \setminus \{0\}$. Then the Thompson metric and the u-seminorm are topologically equivalent on $K(u)$.*

Proof. We have to show that d and $|\cdot|_u$ have the same convergent sequences, that is

$$x_n \xrightarrow{d} x \iff x_n \xrightarrow{|\cdot|_u} x,$$

for any sequence (x_n) in $K(u)$ and any $x \in K(u)$. But, by Proposition 24 (1),

$$x_n \xrightarrow{|\cdot|_u} x \iff x_n \xrightarrow{|\cdot|_x} x,$$

hence we have to prove the equivalence

$$x_n \xrightarrow{d} x \iff x_n \xrightarrow{|\cdot|_x} x. \tag{54}$$

Suppose that $x_n \xrightarrow{d} x$. By Proposition 28 (6)

$$|x_n - x|_x \le 2e^{d(x_n,x)} + e^{-d(x_n,x)} - 1 \to 0 \quad \text{as} \quad n \to \infty,$$

showing that $x_n \xrightarrow{|\cdot|_x} x$.

Conversely, if $x_n \xrightarrow{|\cdot|_x} x$, then by Proposition 28 (8),

$$|x_n - x|_x \ge 1 - e^{-d(x_n,x)},$$

which implies $d(x_n, x) \to 0$. □

Remark 14. The seminorm $|\cdot|_u$ and the metric d are not metrically equivalent on X_u. Take, for instance, $U := [0; u]_o \cap K(u)$. Then $|x|_u \le 1$ for every $x \in U$. But U is not d-bounded because $e^{-n}u$ belongs to U for all $n \in \mathbb{N}$, and $d(x_n, u) = n \to \infty$ for $n \to \infty$.

Corollary 2 ([9] or [17]). *Let K be a solid normal cone in a Hausdorff LCS (X, τ). Then the topology generated by d on* int K *agrees with the restriction of τ to* int K.

Proof. Let $u \in \operatorname{int} K$. By Theorem 8, $X_u = X$ and the topology generated by $|\cdot|_u$ agrees with τ, that is $|\cdot|_u$ is a norm on X generating the topology τ. Since $K(u) = \operatorname{int} K$, Theorem 9 implies that d and $|\cdot|_u$ are topologically equivalent on $K(u)$. □

4 Completeness Properties

4.1 Self-bounded Sequences and Self-complete Sets in a Cone

Let X be a vector space ordered by a cone K. A sequence (x_n) in K is called:

- *self order-bounded from above* (or *upper self-bounded*) if for every $\lambda > 1$ there exists $k \in \mathbb{N}$ such that $x_n \le \lambda x_k$ for all $n \ge k$.
- *self order-bounded from below* (or *lower self-bounded*) if for every $\mu \in (0; 1)$ there exists $k \in \mathbb{N}$ such that $x_n \ge \mu x_k$ for all $n \ge k$.
- *self order-bounded* (or, simply, *self-bounded*) if is self order-bounded both from below and from above.

Remark 15. If the sequence (x_n) is increasing, then it is self order-bounded from above iff for every $\lambda > 1$ there exists $k \in \mathbb{N}$ such that λx_k is an upper bound for the sequence (x_n).

Similarly, if the sequence (x_n) is decreasing, then it is self order-bounded from below iff for every $\mu \in (0;1)$ there exists $k \in \mathbb{N}$ such that μx_k is an lower bound for the sequence (x_n).

The following propositions put in evidence some connections between self order bounded sequences and d-Cauchy sequences.

Proposition 29. *Let X be a vector space ordered by a cone K.*

1. *Any d-Cauchy sequence in K is self-bounded.*
2. *An increasing sequence in K is upper self-bounded iff it is d-Cauchy.*
3. *A decreasing sequence in K is lower self-bounded iff it is d-Cauchy.*

Proof. 1. Let (x_n) be a d-Cauchy sequence in K. If $\lambda > 1$, then for $\varepsilon := \ln \lambda > 0$ there exists $k \in \mathbb{N}$ such that

$$e^{-\varepsilon} x_n \le x_k \ge e^{\varepsilon} x_n \iff \lambda^{-1} x_n \le x_k \ge \lambda x_n \,,$$

for all $n \ge k$. Consequently $x_n \le \lambda x_k$, for all $n \ge k$, proving that (x_n) is upper self-bounded.

The lower self-boundedness of (x_n) is proved similarly, taking $\varepsilon' = -\ln \mu$, for $\mu \in (0;1)$.

2. It suffices to prove that an increasing upper self-bounded sequence is d-Cauchy. For $\varepsilon > 0$ let $\lambda = e^{\varepsilon}$ and $k \in \mathbb{N}$ such that $x_n \le \lambda x_k$ for all $n \ge k$. It follows that

$$e^{-\varepsilon} x_m \le x_m \le x_n \le \lambda x_k \le \lambda x_m = e^{-\varepsilon} x_m \,,$$

for all $n \ge m \ge k$. Consequently, $d(x_n, x_m) \le \varepsilon$, for all $n \ge m \ge k$, which shows that the sequence (x_n) is d-Cauchy.

The proof of 3 is similar to the proof of 2, so we omit it. □

Proposition 30. *Let X be a vector space ordered by an Archimedean cone K and (x_n) an increasing sequence in K. The following statements are equivalent.*

1. *The sequence (x_n) is d-convergent.*
2. *The sequence (x_n) is d-Cauchy and has a supremum.*
3. *The sequence (x_n) is upper self-bounded and has a supremum.*

In the affirmative case $x_n \xrightarrow{d} \sup_n x_n$.

Proof. $1 \Rightarrow 2$ Follows from Proposition 27.

$2 \Rightarrow 3$. Follows from Proposition 29 (1)

$3 \Rightarrow 1$. If $x = \sup_n x_n$, then $x_n \le x$ for all $n \in \mathbb{N}$, showing that condition (i) from Proposition 27 (1) holds. Now let $\lambda > 1$. Since (x_n) is upper self-bounded there exists $k \in \mathbb{N}$ such that $x_n \le \lambda x_k$ for all $n \ge k$, and so $x_n \le \lambda x_k$ for all

$n \in \mathbb{N}$ (because (x_n) is increasing). But then $x = \sup_n x_n \le \lambda x_k$, which shows that condition (ii) of the same proposition is also fulfilled. Consequently $x_n \xrightarrow{d} x$.

The last assertion follows by the same proposition. □

Similar equivalences, with similar proofs, hold for decreasing sequences.

Proposition 31. *Let X be a vector space ordered by an Archimedean cone K and (x_n) a decreasing sequence in K. The following statements are equivalent.*

1. *The sequence (x_n) is d-convergent.*
2. *The sequence (x_n) is d-Cauchy and has an infimum.*
3. *The sequence (x_n) is lower self-bounded and has an infimum.*

In the affirmative case $x_n \xrightarrow{d} \inf_n x_n$.

The following proposition emphasizes a kind of duality between upper and lower self-bounded sequences. If Y is a subset of an ordered set X, then one denotes by $\sup|_Y A$ ($\inf|_Y A$) the supremum (resp. infimum) in Y of a subset A of Y. This may differ from the supremum (resp. infimum) of the set A in X.

Proposition 32. *Let X be a vector space ordered by a cone K, (x_n) an increasing, upper self-bounded sequence in K, and (t_k) a decreasing sequence of real numbers, convergent to 1. Then there exists a subsequence (x_{n_k}) of (x_n) such that the following conditions are satisfied.*

1. *The sequence (y_k) given by $y_k = t_k x_{n_k}$, $k \in \mathbb{N}$, is decreasing and lower self-bounded and*

$$\forall n, k \in \mathbb{N}, \quad x_n \le y_k. \tag{55}$$

2. *If the cone K is Archimedean, x is an upper bound for (x_n) and y is a lower bound for (y_k), then $y \le x$.*
3. *If the cone K is Archimedean and (x_n) lies in a vector subspace Y of X, then the following statements are equivalent.*

   (a) *(x_n) has supremum;*
   (b) *(x_n) has supremum in Y;*
   (c) *(y_k) has an infimum;*
   (d) *(y_k) has an infimum in Y;*
   (e) *there exists $x \in K$ such that*

$$\forall n, k \in \mathbb{N}, \quad x_n \le x \le y_k. \tag{56}$$

In the affirmative case

$$\sup\{x_n : n \in \mathbb{N}\} = \sup|_Y\{x_n : n \in \mathbb{N}\} = \inf\{y_k : k \in \mathbb{N}\} = \inf|_Y\{y_k : k \in \mathbb{N}\} = x,$$

and $x_n \xrightarrow{d} x$ *and* $y_k \xrightarrow{d} x$.

Proof. 1. Since (t_k) is decreasing, $\lambda_k := t_k/t_{k+1} > 1$, $k \in \mathbb{N}$. The upper self-boundedness of the sequence (x_n) implies the existence of $n_1 \in \mathbb{N}$ such that

$$\forall n \geq n_1, \quad x_n \leq \lambda_1 x_{n_1}. \tag{57}$$

Since (x_n) is increasing, the inequalities (57) hold for all $n \in \mathbb{N}$. If $m_2 \in \mathbb{N}$ is such that $x_n \leq \lambda_2 x_{m_2}$ for all $n \in \mathbb{N}$, then $n_2 := 1 + \max\{n_1, m_2\} > n_1$ and $x_n \leq \lambda_2 x_{n_2}$ for all $n \in \mathbb{N}$. Continuing in this way one obtains a sequence of indices $n_1 < n_2 < \cdots$ such that

$$x_n \leq \lambda_k x_{n_k}, \tag{58}$$

for all $n \in \mathbb{N}$ and all $k \in \mathbb{N}$.

Let $y_k := t_k x_{n_k}$, $k \in \mathbb{N}$. Putting $n = n_{k+1}$ in (58) it follows $y_{k+1} \leq y_k$. By the same inequality

$$x_n \leq t_{k+1} x_n \leq t_k x_{n_k} = y_k,$$

for all $n, k \in \mathbb{N}$.

Let now $\mu \in (0;1)$. Since $t_k \to 1$ there exists k_0 such that $t_{k_0} < \mu^{-1}$. But then, by (55),

$$\mu y_{k_0} \leq t_{k_0}^{-1} y_{k_0} = x_{n_{k_0}} \leq y_k,$$

for all $k \in \mathbb{N}$, proving that (y_k) is lower self-bounded.

2. Suppose that $x_n \leq x$, $n \in \mathbb{N}$, and $y_k \geq y$, $n \in \mathbb{N}$. Then, for all $k \in \mathbb{N}$,

$$y \leq y_k = t_k x_{n_k} \leq t_k x\,.$$

Since K is Archimedean and $t_k \to 1$, the inequalities $y \leq t_k x$, $k \in \mathbb{N}$, yield for $k \to \infty$, $y \leq x$.

3. The implications (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d) are obvious.

Let as prove (b) $\Rightarrow$ (d). Observe first that $y_k \in Y$, $k \in \mathbb{N}$. Let $x = \sup_Y\{x_n : n \in \mathbb{N}\}$. By (55) y_k is an upper bound for (x_n), for every $k \in \mathbb{N}$, so that $x \leq y_k$, for all $k \in \mathbb{N}$. If $y \in Y$ is such that $y \leq y_k$ for all $k \in \mathbb{N}$, then, by 2, $y \leq x$, proving that $x = \inf_Y\{y_k : k \in \mathbb{N}\}$. On the way we have shown that $x_n \leq x \leq y_k$, for all $n, k \in \mathbb{N}$, that is the implication (b) $\Rightarrow$ (e) holds too.

Similar reasonings show that (d) $\Rightarrow$ (b), that is (b) $\iff$ (d). The equivalence (a) $\iff$ (c) can be proved in the same way (just let $Y := X$).

Finally, let us show that (e) $\Rightarrow$ (c). Assume that for some $x \in K$, $x_n \leq x \leq y_k$ for all $n, k \in \mathbb{N}$. Suppose that $y \in K$ is such that $y \leq y_k$ for all $k \in \mathbb{N}$. Then, by 2, these inequalities imply $y \leq x$, showing that $x = \inf_k y_k$. (Similar arguments show that $x = \sup_n y_n$, that is (e) $\Rightarrow$ (a).) The equivalence of the assertions from 3 is (over) proven.

The last assertions of the proposition follow from Propositions 30 and 31.

□

Similar results, with similar proofs, hold for decreasing lower self-bounded sequences.

Proposition 33. *Let X be a vector space ordered by a cone K, (x_n) a decreasing, lower self-bounded sequence in K, and (t_k) an increasing sequence of real numbers, convergent to 1. Then there exists a subsequence (x_{n_k}) of (x_n) such that following conditions are satisfied.*

1. *The sequence (y_k) given by $y_k = t_k x_{n_k}$, $k \in \mathbb{N}$, is increasing and upper self-bounded and*

$$\forall n, k \in \mathbb{N}, \quad x_n \geq y_k. \tag{59}$$

2. *If the cone K is Archimedean, x is a lower bound for (x_n) and y is an upper bound for (y_k), then $y \geq x$.*
3. *If the cone K is Archimedean and (x_n) lies in a vector subspace Y of X, then the following statements are equivalent.*

   (a) *(x_n) has an infimum;*
   (b) *(x_n) has an infimum in Y;*
   (c) *(y_k) has a supremum;*
   (d) *(y_k) has a supremum in Y;*
   (e) *there exists $x \in K$ such that*

$$\forall n, k \in \mathbb{N}, \quad x_n \geq x \geq y_k. \tag{60}$$

In the affirmative case

$$\inf_n x_n = \inf|_Y \{x_n : n \in \mathbb{N}\} = \sup_k y_k = \sup|_Y \{y_k : k \in \mathbb{N}\} = x,$$

and $x_n \xrightarrow{d} x$ and $y_k \xrightarrow{d} x$.

The following notions will play a crucial role in the study of completeness of the Thompson metric.

Let X be a vector space ordered by a cone K. A nonempty subset U of K is called:

- *self order-complete from above* (or *upper self-complete*) if every increasing self-bounded sequence (x_n) in U has a supremum and $\sup_n x_n \in U$.
- *self order-complete from below* (or *lower self-complete*) if every decreasing self-bounded sequence (x_n) in U has an infimum and $\inf_n x_n \in U$.
- *self order-complete* (or, simply, *self-complete*) if it is self order-complete both from below and from above.

If we do not require the supremum (resp. infimum) to be in U, then we say that U is *quasi upper* (resp. *lower*) *self-complete*.

The duality results given in Propositions 32 and 33 have the following important consequence.

Theorem 10. *Let X be a vector space ordered by an Archimedean cone K. If U is an order-convex, strictly positively homogeneous, nonempty subset of K, then all six completeness properties given in the above definitions are equivalent.*

Proof. It is a simple observation that the stated equivalences hold if we show that self-completeness is implied by each of the conditions of quasi upper self-completeness and quasi lower self-completeness.

Assume that U is quasi upper self-complete and show first that U is upper self-complete.

Let (x_n) be an increasing upper self-bounded sequence in U. By hypothesis, there exists $x := \sup_n x_n \in K$. Also there exists $k \in \mathbb{N}$ such that $x_n \leq 2x_k$ for all $n \in \mathbb{N}$. Consequently $x_k \leq x \leq 2x_k$. Since x_k and $2x_k$ belong to U and U is order-convex, $x \in U$, proving that U is upper self-complete.

Let us show now that U is lower self-complete too. Suppose that (x_n) is a decreasing lower self-bounded sequence (x_n) in U and let (t_k) be an increasing sequence of positive numbers which converges to 1 (e.g., $t_k = 1 - \frac{1}{2k}$). By Proposition 33 there exists a subsequence (x_{n_k}) of (x_n) such that the sequence $y_k := t_k x_{n_k}$, $k \in \mathbb{N}$, is increasing and upper self-bounded. Since we have shown that U is upper self-complete, there exists $x := \sup_k y_k \in U$. By the last part of the same proposition, $\inf_n x_n = x \in U$, proving that U is lower self-complete.

When U is quasi lower self-complete, the proof that U is self-complete follows the same steps as before, using Proposition 32 instead of Proposition 33. □

The following corollary shows that we can restrict to order-convex subspaces of X.

Corollary 3. *Let X be a vector space ordered by an Archimedean cone K and Y an order-convex vector subspace of X. If U is an order-convex, strictly positively homogeneous, nonempty subset of $Y \cap K$, then U is self-complete in X iff U is self-complete in Y.*

For a lineally solid cone K, the self-completeness is equivalent to the self-completeness of its algebraic interior.

Proposition 34. *Let X be a vector space ordered by an Archimedean cone K.*

1. *The cone K is self-complete iff every component of K is self-complete.*
2. *If, in addition, K is lineally solid and* $\operatorname{aint}(K)$ *is self-complete then K is self-complete.*

Proof. 1. Suppose that K is self-complete. Then any component Q of K is quasi upper self-complete. By Proposition 19, Q satisfies the hypotheses of Theorem 10, so that it is self-complete.

Conversely, suppose that every component of K is self-complete and let (x_n) be an increasing upper self-bounded sequence in K. By Proposition 29 the sequence (x_n) is d-Cauchy, so there exists $k \in \mathbb{N}$ such that $d(x_k, x_n) \leq 1 < \infty$, for all $n \geq k$, implying that the set $\{x_n : n \geq k\}$ is contained in a component Q of K. By the self-completeness of Q there exists $x := \sup\{x_n : n \geq k\} \in Q$.

Since the sequence (x_n) is increasing it follows $x = \sup_n x_n$. Consequently, K is upper self-complete and, by Theorem 10, self-complete.

2. Let (x_n) be an increasing, upper self-bounded sequence in K. Fix $x \in \operatorname{aint}(K)$. Then, by Remark 5, the sequence $y_n := x_n + x$, $n \in \mathbb{N}$, is contained in $\operatorname{aint}(K)$ and it is obviously increasing and upper self-bounded. Consequently, (y_n) has a supremum, $y := \sup_n y_n \in \operatorname{aint}(K)$. But then there exists $\sup_n x_n = y - x$. Therefore the cone K is upper self-complete and, by Theorem 10, it is self-complete. □

Remark 16. All the results proven so far can be restated into local versions, by replacing X with X_u, hence K with K_u (where $u \in K \setminus \{0\}$). In this way, we can weaken the Archimedean condition by requiring only that K_u is Archimedean. In this case, the conditions "has a supremum", respectively "has an infimum" must be understood with respect to X_u. Consequently, a subset U of K_u can be self-complete in X_u, but may be not self-complete in X (yet, by Corollary 3, this cannot happen when K is Archimedean). Note that the definition of the Thompson metric is not affected by this change (see Remark 10).

4.2 Properties of Monotone Sequences with Respect to Order-Unit Seminorms

In this subsection we shall examine the behavior of monotone sequences with respect to order-unit seminorms, considered in Sect. 3.2. The results are analogous to those established in Sect. 3.3 for the Thompson metric.

Throughout this subsection X will be a vector space ordered by a cone K, $u \in K \setminus \{0\}$, and X_u, K_u are as in Sect. 3.2. We shall assume also that the cone $K_u = X_u \cap K$ is Archimedean.

Proposition 35. *Let (x_n) be an increasing sequence in X_u and $x \in X_u$.*

1. *The sequence (x_n) is $|\cdot|_u$-Cauchy iff*

$$\forall \varepsilon > 0,\ \exists k \in \mathbb{N},\ \forall n \in \mathbb{N}, \quad x_n \le x_k + \varepsilon u. \tag{61}$$

2. *The sequence (x_n) is $|\cdot|_u$-convergent to $x \in X_u$ iff*

$$(i) \quad \forall n \in \mathbb{N}, \quad x_n \le x;$$
$$(ii) \quad \forall \varepsilon > 0,\ \exists k \in \mathbb{N}, \quad \text{such that} \quad x \le x_k + \varepsilon u.$$

In the affirmative case, $x = \sup|_{X_u}\{x_n : n \in \mathbb{N}\}$. If K is Archimedean, then $x = \sup_n x_n$.

3. *The sequence (x_n) is $|\cdot|_u$-convergent to $x \in X_u$ iff it is $|\cdot|_u$-Cauchy and has a supremum in X_u. In the affirmative case $x_n \xrightarrow{|\cdot|_u} \sup|_{X_u}\{x_n : n \in \mathbb{N}\}$.*

Proof. 1. The sequence (x_n) is $|\cdot|_u$-Cauchy iff

$$\forall \varepsilon > 0,\ \exists k \in \mathbb{N},\ \forall n \geq m \geq k, \quad -\varepsilon u \leq x_n - x_m \leq \varepsilon u. \tag{62}$$

Suppose that (x_n) is $|\cdot|_u$-Cauchy and for $\varepsilon > 0$ let k be given by the above condition. Taking $m = k$ in the right inequality, one obtains $x_n \leq x_k + \varepsilon u$ for all $n \geq k$, and so for all $n \in \mathbb{N}$.

Suppose now that (x_n) satisfies (61). For $\varepsilon > 0$ let k be chosen according to this condition. By the monotony of (x_n), $x_n - x_m \geq 0 \geq -\varepsilon u$, for all $n \geq m$, and so the left inequality in (62) is true. By (61) and the monotony of (x_n),

$$x_n \leq x_k + \varepsilon u \leq x_m + \varepsilon u,$$

for all $m \geq k$, so that the right inequality in (62) holds too.

2. We have $x_n \xrightarrow{|\cdot|_u} x$ iff

$$\forall \varepsilon > 0,\ \exists k \in \mathbb{N},\ \forall n \geq k, \quad -\varepsilon u \leq x - x_n \leq \varepsilon u.$$

The left one of the above inequalities implies $x_n \leq x + \varepsilon u$ for all $n \geq k$, and so, by the monotony of (x_n), for all $n \in \mathbb{N}$. Since K_u is Archimedean, letting $\varepsilon \searrow 0$ it follows $x_n \leq x$, for every $n \in \mathbb{N}$. The right inequality implies $x \leq x_k + \varepsilon u$.

Conversely, suppose that (i) and (ii) hold. For $\varepsilon > 0$ choose k according to (ii). Then, by the monotony of (x_n),

$$x \leq x_k + \varepsilon u \leq x_n + \varepsilon u,$$

and so $x - x_n \leq \varepsilon u$, for all $n \geq k$. By (i), $x_n \leq x \leq x + \varepsilon u$, and so $x - x_n \geq -\varepsilon u$ for all $n \in \mathbb{N}$. Consequently, $-\varepsilon u \leq x - x_n \leq \varepsilon u$ for all $n \geq k$.

By Proposition 24 (7), the cone K_u is $|\cdot|_u$-closed, and so, by Proposition 9, $x_n \xrightarrow{|\cdot|_u} x$ implies $x = \sup|_{X_u}\{x_n : n \in \mathbb{N}\}$.

Suppose now that the cone K is Archimedean and that $y \in X$ is such that $x_n \leq y$ for all $n \in \mathbb{N}$. By (ii) for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $x \leq x_k + \varepsilon u$, hence $x \leq y + \varepsilon u$. Letting $\varepsilon \searrow 0$ one obtains $x \leq y$, which proves that $x = \sup_n x_n$.

The direct implication in 3 follows from 1 and 2. Suppose that (x_n) is $|\cdot|_u$-Cauchy and let $x = \sup|_{X_u}\{x_n : n \in \mathbb{N}\}$. For $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $x_n \leq x_k + \varepsilon u$ for all $n \in \mathbb{N}$, implying $x \leq x_k + \varepsilon u$. Taking into account 2, it follows $x_n \xrightarrow{|\cdot|_u} x$. □

As before, similar results, with similar proofs, hold for decreasing sequences.

Proposition 36. *Let (x_n) be a decreasing sequence in X_u and $x \in X_u$.*

1. *The sequence (x_n) is $|\cdot|_u$-Cauchy iff*

$$\forall \varepsilon > 0,\ \exists k \in \mathbb{N},\ \forall n \in \mathbb{N}, \quad x_n \geq x_k - \varepsilon u. \tag{63}$$

2. *The sequence* (x_n) *is* $|\cdot|_u$*-convergent to* $x \in X_u$ *iff*

$$(i) \quad \forall n \in \mathbb{N}, \quad x_n \geq x;$$
$$(ii) \quad \forall \varepsilon > 0,\ \exists k \in \mathbb{N}, \quad \textit{such that} \quad x \geq x_k - \varepsilon u.$$

In the affirmative case, $x = \inf|_{X_u}\{x_n : n \in \mathbb{N}\}$. *If* K *is Archimedean, then* $x = \inf_n x_n$.

3. *The sequence* (x_n) *is* $|\cdot|_u$*-convergent to* $x \in X_u$ *iff it is* $|\cdot|_u$*-Cauchy and has an infimum in* X_u. *In the affirmative case* $x_n \xrightarrow{|\cdot|_u} \inf|_{X_u}\{x_n : n \in \mathbb{N}\}$.

Now we consider the connection with self-bounded sequences.

Proposition 37. *Let* (x_n) *be a* $|\cdot|_u$*-Cauchy sequence in* K_u.

1. *If there exist* $\alpha > 0$ *and a subsequence* (x_{n_k}) *of* (x_n) *such that* $x_{n_k} \geq \alpha u$, *then* (x_n) *is self-bounded.*
2. *If* (x_n) *is increasing and there exists* $n_0 \in \mathbb{N}$ *such that* $x_{n_0} \in K(u)$, *then* (x_n) *is self-bounded.*
3. *If* (X, τ) *is a TVS ordered by a cone* K *and* (x_n) *is a* $|\cdot|_u$*-Cauchy sequence in* X_u, τ*-convergent to some* $x \in X_u$, *then* $x_n \xrightarrow{|\cdot|_u} x$.

Proof. 1. For $\lambda > 1$ put $\varepsilon := \alpha(\lambda - 1)$. Since the sequence (x_n) is $|\cdot|_u$-Cauchy, there exists $k \in \mathbb{N}$ such that $x_n \leq x_m + \varepsilon u$ for all $n, m \geq k$. Taking $m = n_k$ and $n \geq n_k (\geq k)$, it follows

$$x_n \leq x_{n_k} + (\lambda - 1)\alpha u \leq x_{n_k} + (\lambda - 1)x_{n_k} = \lambda x_{n_k},$$

proving that (x_n) is upper self-bounded.

The fact that (x_n) is lower self-bounded can be proved in a similar way, taking $\varepsilon := \alpha(1 - \mu)$ for $0 < \mu < 1$.

2. If x_{n_0} belongs to the component $K(u)$ of K, then $x_{n_0} \sim u$, so there exists $\alpha > 0$ such that $x_{n_0} \geq \alpha u$. It follows $x_n \geq \alpha u$, for all $n \geq n_0$, and so the hypotheses of 1 are satisfied.
3. For $\varepsilon > 0$ let $n_\varepsilon \in \mathbb{N}$ be such that $|x_{n+k} - x_n|_u < \varepsilon$ for all $n \geq n_\varepsilon$ and all $k \in \mathbb{N}$. By (43) $\varepsilon \in \mathscr{M}_u(x_{n+k} - x_n)$, that is $-\varepsilon u \leq x_{n+k} - x_n \leq \varepsilon u$, for every $n \geq n_\varepsilon$ and all $k \in \mathbb{N}$. Letting $k \to \infty$, one obtains $-\varepsilon u \leq x - x_n \leq \varepsilon u$, implying $|x_n - x|_u \leq \varepsilon$, for all $n \geq n_\varepsilon$. This shows that $x_n \xrightarrow{|\cdot|_u} x$. □

4.3 The Completeness Results

The following important result shows that the completeness of the Thompson metric d on $K(u)$ and that of the u-norm on X_u are equivalent when K_u is Archimedean (by Remark 14 this result is nontrivial) and also reduces the completeness to the convergence of the monotone Cauchy sequences. We also show that the completeness of d on $K(u)$ is equivalent to several order-completeness conditions in X_u.

The notation in the following theorem is that of Sects. 3.1 and 3.2.

Theorem 11. *Let X be a vector space ordered by a cone K and let $u \in K \setminus \{0\}$ be such that K_u is Archimedean. Then the following assertions are equivalent.*

1. *$K(u)$ is d-complete.*
2. *$K(u)$ is self-complete in X_u.*
3. *K_u is self-complete in X_u.*
4. *$(X_u, |\cdot|_u)$ is fundamentally σ-order complete.*
5. *$(X_u, |\cdot|_u)$ is monotonically sequentially complete.*
6. *X_u is $|\cdot|_u$-complete.*

If, in addition, K is Archimedean, then the assertions 2 and 3 can be replaced by the stronger versions:

2′. *$K(u)$ is self-complete (in X).*
3′. *K_u is self-complete (in X).*

Proof. 1 $\Rightarrow$ 2. If (x_n) is an increasing, upper self-bounded sequence in $K(u)$, then, by Proposition 29 (2), it is d-Cauchy, so that it is d-convergent to some $x \in K(u)$, and, by Theorem 9, also $|\cdot|_u$-convergent to x. By Proposition 24 the cone K_u is $|\cdot|_u$-closed in X_u, so that, by Proposition 9 (4), $x = \sup_n x_n$. Consequently, $K(u)$ is upper self-complete in X_u. But then, by Proposition 19 and Theorem 10, self-complete in X_u.

2 $\iff$ 3. By (45), $K(u) = \operatorname{aint}(K_u)$, so that, by Proposition 34, K_u is self-complete iff $K(u)$ is self-complete.

4 $\iff$ 5. The implication 5 $\Rightarrow$ 4 is trivial and 4 $\Rightarrow$ 5 follows by Propositions 35 (3) and 36 (3).

2 $\Rightarrow$ 4. Using again the fact that $K(u) = \operatorname{aint}(K_u)$, it is sufficient to show that every increasing $|\cdot|_u$-Cauchy sequence in $K(u)$ has a supremum in X_u. Indeed, by Proposition 15 this is equivalent to the fact that the space $(X_u, |\cdot|_u)$ is fundamentally σ-order complete. By Proposition 24 (4), the cone K_u is normal and generating, so that, by Proposition 14 (3), it is monotonically sequentially complete.

But, by Proposition 37 (2), the sequence (x_n) is self-bounded so it has a supremum in X_u.

5 $\Rightarrow$ 6. Since a Cauchy sequence is convergent if has a convergent subsequence, it is sufficient to show that every sequence (x_n) in X_u satisfying

$$\forall n \in \mathbb{N}_0, \quad |x_{n+1} - x_n|_u \le \frac{1}{2^n}, \tag{64}$$

is convergent in $(X_u, |\cdot|_u)$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

The inequality (64) implies

$$-\frac{1}{2^n}u \le x_{n+1} - x_n \le \frac{1}{2^n}u, \tag{65}$$

for all $n \in \mathbb{N}_0$. Writing (65) for $0, 1, \ldots, n-1$ and adding the obtained inequalities, one obtains

$$-\left(2 - \frac{1}{2^{n-1}}\right) \le x_n - x_0 \le \left(2 - \frac{1}{2^{n-1}}\right),$$

for all $n \in \mathbb{N}$. Putting $y_n := x_n - x_0 + \left(2 - \frac{1}{2^{n-1}}\right) u$ it follows

$$0 \le y_n \le \left(4 - \frac{1}{2^{n-2}}\right) u, \quad n \in \mathbb{N},$$

which proves that $y_n \in K_u$ for all $n \in \mathbb{N}$. Also from $y_{n+1} - y_n = x_{n+1} - x_n + \frac{1}{2^n} u$ and (65), one obtains

$$0 \le y_{n+1} - y_n \le \frac{1}{2^{n-1}} u,$$

implying

$$0 \le y_{n+k} - y_n \le \left(\frac{1}{2^{n-1}} + \frac{1}{2^n} + \cdots + \frac{1}{2^{n+k-2}}\right) u < \frac{1}{2^{n-2}} u.$$

It follows that (y_n) is an increasing $|\cdot|_u$-Cauchy sequence in K_u, hence it is $|\cdot|_u$-convergent to some $y \in X_u$. But then

$$x_n = y_n + x_0 - \left(2 - \frac{1}{2^{n-1}}\right) u$$

is $|\cdot|_u$-convergent to $y + x_0 - 2u \in X_u$.
$6 \Rightarrow 1$. Again, to prove the completeness of $(K(u), d)$ it is sufficient to show that every sequence (x_n) in $K(u)$ which satisfies

$$\forall n \in \mathbb{N}_0, \quad d(x_{n+1}, x_n) \le \frac{1}{2^n}, \tag{66}$$

is convergent in $(K(u), d)$. Let $s_0 = d(x_0, u)$. Then, by the triangle inequality and (66) applied successively,

$$d(x_n, x_0) \le 1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}} < 2,$$

so that

$$d(x_n, u) \le d(x_n, x_0) + d(x_0, u) < 2 + s_0,$$

implying

$$e^{-(2+s_0)} u \le x_n \le e^{2+s_0} u \tag{67}$$

for all $n \in \mathbb{N}_0$. The inequality (66) implies

$$x_{n+1} \le e^{1/2^n} x_n,$$

and

$$x_n \le e^{1/2^n} x_{n+1},$$

so that, taking into account the second inequality in (67), one obtains the inequalities

$$x_{n+1} - x_n \le \left(e^{1/2^n} - 1\right) x_n \le \left(e^{1/2^n} - 1\right) e^{2+s_0} u,$$

and

$$x_{n+1} - x_n \ge -\left(e^{1/2^n} - 1\right) x_n \ge -\left(e^{1/2^n} - 1\right) e^{2+s_0} u,$$

which, in their turn, imply

$$|x_{n+1} - x_n|_u \le \left(e^{1/2^n} - 1\right) e^{2+s_0},$$

for all $n \in \mathbb{N}_0$. The convergence of the series $\sum_n \left(e^{1/2^n} - 1\right) e^{2+s_0}$[1] and the above inequalities imply that the sequence (x_n) is $|\cdot|_u$-Cauchy, and so it is $|\cdot|_u$-convergent to some $x \in X_u$. By Proposition 9 the order intervals in X_u are $|\cdot|_u$-closed and, by (67), $x_n \in \left[e^{-(2+s_0)} u; e^{2+s_0} u\right]_u$ it follows $x \in \left[e^{-(2+s_0)} u; e^{2+s_0} u\right]_u \subset K(u)$. Since, by Theorem 9, d and $|\cdot|_u$ are topologically equivalent on $K(u)$, it follows $x_n \xrightarrow{d} x$. □

Combining Proposition 34 and Theorem 11 one obtains the following corollaries.

Corollary 4. *If K is Archimedean, then d is complete iff K is self-complete.*

Corollary 5. *If K is Archimedean and lineally solid, then the following conditions are equivalent.*

[1]Follows from the inequality $e^{1/2^n} - 1 = \frac{1}{2^n} + \frac{1}{2!} \cdot \frac{1}{2^{2n}} + \cdots < \frac{1}{2^n}(1 + \frac{1}{2!} + \cdots) = \frac{1}{2^n}(e - 1)$.

1. *The Thompson metric d is complete.*
2. *The cone K is self-complete.*
3. *The algebraic interior* $\operatorname{aint}(K)$ *of K is self-complete.*
4. *The algebraic interior* $\operatorname{aint}(K)$ *of K is d-complete.*

4.4 The Completeness of the Thompson Metric in LCS

In this subsection we shall prove the completeness of the Thompson metric d corresponding to a normal cone K in a sequentially complete LCS X. In the case of a Banach space the completeness was proved by Thompson [41]. In the locally convex case we essentially follow [17].

Note that if (X, ρ) is an extended metric space, then the completeness of X means the completeness of every component of X. Indeed, if (x_n) is a d-Cauchy sequence, then there exits $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n_0}) \leq 1$, for all $n \geq n_0$, implying that $x_n \in Q$, for all $n \geq n_0$, where Q is the component of X containing x_{n_0}. Also if $x_n \xrightarrow{d} x$, then there exists $n_1 > n_0$ in $\mathbb{N}$ such that $d(x_n, x) \leq 1$ for all $n \geq n_1$, implying that the limit x also belongs to Q.

Theorem 12. *Let (X, τ) be a locally convex space, K a sequentially complete closed normal cone in X. Then each component of K is a complete metric space with respect to the Thompson metric d.*

By Theorem 2 one can suppose that the topology τ is generated by a family P of monotone seminorms.

We start by a lemma which is an adaptation of Lemma 2.3 (ii) in [22], proved for Banach spaces, to the locally convex case.

Lemma 1. *Let (X, τ) be a Hausdorff LCS ordered by a closed normal cone K and d the Thompson metric corresponding to K. Supposing that P is a directed family of monotone seminorms generating the topology τ, then for every $x, y \in K \setminus \{0\}$ and every $p \in P$, the following inequality holds*

$$p(x-y) \leq \left(2e^{d(x,y)} + e^{-d(x,y)} - 1\right) \cdot \min\{p(x), p(y)\}\,. \tag{68}$$

Proof. We can suppose $d(x, y) < \infty$ (i.e., $x \sim y$). By Proposition 9 the cone K is Archimedean, so that, by Proposition 21, $d(x, y) \in \sigma(x, y)$. Putting $\alpha = e^{d(x,y)}$, it follows

$$\alpha^{-1}x \leq y \leq \alpha x,$$

so that $(\alpha - 1)x \leq x - y \leq (1 - \alpha^{-1})x$, and so

$$0 \leq (x-y) + (\alpha - 1)x \leq (\alpha - \alpha^{-1})x\,.$$

Let $p \in P$. By the monotony of p the above inequalities yield

$$p(x-y)-(\alpha-1)p(x) \le p((x-y)+(\alpha-1)x) \le (\alpha-\alpha^{-1})p(x),$$

and so

$$p(x-y) \le (2\alpha-\alpha^{-1}-1)p(x).$$

Interchanging the roles of x and y one obtains,

$$p(x-y) \le (2\alpha-\alpha^{-1}-1)p(y),$$

showing that (68) holds. □

Proof of Theorem 12. Let (x_n) be a d-Cauchy sequence in a component Q of K.

Observe first that the sequence (x_n) is τ-bounded, that is p-bounded for every $p \in P$.

Indeed, if $n_0 \in \mathbb{N}$ is such that $d(x_n, x_{n_0}) \le 1$, for all $n \ge n_0$, then $x_n \le e^{d(x_n,x_{n_0})}x_{n_0} \le ex_{n_0}$, for all $n \ge n_0$. By the monotony of p, it follows $p(x_n) \le ep(x_{n_0})$ for all $n \ge n_0$ and every $p \in P$. This fact and the inequality (68) imply that (x_n) is p-Cauchy for every $p \in P$, hence it is P-convergent to some $x \in X$.

If n_0 is as above, then the inequalities $e^{-1}x_{n_0} \le x_n \le ex_{n_0}$, valid for all $n \ge n_0$, yield for $n \to \infty$, $e^{-1}x_{n_0} \le x \le ex_{n_0}$, showing that $x \sim x_{n_0}$, that is $x \in Q$.

Since (x_n) is d-Cauchy and τ-convergent to x, Proposition 27 (3) implies that $x_n \xrightarrow{d} x$, proving the completeness of (K, d). □

4.5 The Case of Banach Spaces

We have seen in the previous subsection that the normality of a cone K in a sequentially complete LCS X is a sufficient condition for the completeness of K with respect to the Thompson metric. In this subsection we show that, in the case when X is a Banach space ordered by a cone K, the completeness of d implies the normality of K. The proof will be based on the following result.

Theorem 13. *Let $(X, \|\cdot\|)$ be a Banach space ordered by a cone K and $u \in K\setminus\{0\}$. Then the following assertions are equivalent.*

1. *The Thompson metric d is complete on $K(u)$.*
2. *$(X_u, |\cdot|_u)$ is a Banach space.*
3. *The embedding of $(X_u, |\cdot|_u)$ into $(X, \|\cdot\|)$ is continuous.*
4. *The order interval $[0; x]_o$ is $\|\cdot\|$-bounded for every $x \in K(u)$.*
5. *The order interval $[0; u]_o$ is $\|\cdot\|$-bounded.*

6. *Any sequence* (x_n) *in* $K(u)$ *which is* d*-convergent to* $x \in K(u)$ *is also* $\|\cdot\|$*-convergent to* x.

Proof. The equivalence $1 \iff 2$ is in fact the equivalence $1 \iff 6$ in Theorem 11.

$2 \Rightarrow 3$. Since both $(X, \|\cdot\|)$ and $(X_u, |\cdot|_u)$ are Banach spaces, by the closed graph theorem it suffices to show that the embedding mapping $I : X_u \to X,\ I(x) = x$, has closed graph. This means that for every sequence (x_n) in X_u, $x_n \xrightarrow{|\cdot|_u} x$ and $x_n \xrightarrow{\|\cdot\|} y$ imply $y = x$. Passing to limit for $n \to \infty$ with respect to $\|\cdot\|$ in the inequalities

$$x_n - x + |x_n - x|_u u \geq 0 \quad \text{and} \quad x_n - x + |x_n - x|_u u \geq 0\,,$$

and taking into account the fact that the cone K is $\|\cdot\|$-closed, one obtains

$$y - x \geq 0 \quad \text{and} \quad x - y \geq 0,$$

that is $y = x$.
$3 \Rightarrow 4$. By the continuity of the embedding, there exists $\gamma > 0$ such that $\|x\| \leq \gamma |x|_u$ for all $x \in X_u$. By Proposition 24 the norm $|\cdot|_u$ is monotone, so that $0 \leq z \leq x$ implies $\|z\| \leq \gamma |z|_u \leq \gamma |x|_u$, for all $z \in [0; x]_o$.
The implication $4 \Rightarrow 5$ is obvious.
$5 \Rightarrow 3$. Let $\gamma > 0$ be such that $\|z\| \leq \gamma$ for every $z \in [0; u]_u$. For $x \neq 0$ in X_u, the inequalities $-|x|_u u \leq x \leq |x|_u u$, imply

$$\frac{x + |x|_u u}{2|x|_u} \in [0; u]_o,$$

so that $\|x + |x|_u u\| \leq 2\gamma |x|_u$.
Hence,

$$\|x\| - |x|_u \|u\| \leq \|x + |x|_u u\| \leq 2\gamma |x|_u,$$

and so

$$\|x\| \leq (2\gamma + \|u\|)|x|_u),$$

for all $x \in X_u$, proving the continuity of the embedding of $(X_u, |\cdot|_u)$ into $(X, \|\cdot\|)$.
$3 \Rightarrow 2$. Let (x_n) be a $|\cdot|_u$-Cauchy sequence in X_u. The continuity of the embedding implies that it is $\|\cdot\|$-Cauchy and so, $\|\cdot\|$-convergent to some $x \in X$. But then, by Proposition 37 (3), (x_n) is $|\cdot|_u$-convergent to x.
The implication $3 \Rightarrow 6$ follows by Theorem 9.
$6 \Rightarrow 3$. Let (x_n) be a sequence in X_u which is $|\cdot|_u$-convergent to $x \in X_u$. Then (x_n) is $|\cdot|_u$-bounded, so there exists $\alpha > 0$ such that $-\alpha u \leq x_n \leq \alpha u$. It follows that $y_n := x_n + (\alpha + 1)u \in [u; (2\alpha + 1)u]_o$, and so $y_n \in K(u)$, $n \in \mathbb{N}$,

and $y_n \xrightarrow{|\cdot|_u} x + (\alpha + 1)u$. By Theorem 9, $y_n \xrightarrow{d} x + (\alpha + 1)u$, so that, by hypothesis, $y_n \xrightarrow{\|\cdot\|} x + (\alpha + 1)u$. It follows $x_n \xrightarrow{\|\cdot\|} x$, proving the continuity of the embedding. □

Now we present several conditions equivalent to the completeness of the Thompson metric.

Theorem 14. *Let $(X, \|\cdot\|)$ be a Banach space ordered by a cone K. The following assertions are equivalent.*

1. *The Thompson metric d is complete.*
2. *The cone K is self-complete.*
3. *The cone K is normal.*
4. *The norm topology on K is weaker than the topology of d.*

Proof. The equivalence $1 \iff 2$ follows by Corollary 4 (remind that, by Proposition 9, the cone K is Archimedean).

$2 \iff 3$. By Proposition 34, the cone K is self-complete iff each component of K is self-complete. By Theorem 13, this happens exactly when the order interval $[0; x]_o$ is $\|\cdot\|$-bounded for every $x \in K$, which is equivalent to the fact that the order intervals $[x; y]_o$ are $\|\cdot\|$-bounded for all $x, y \in K$. By Theorem 3 this is equivalent to the normality of K.

$1 \iff 4$. By Theorem 13 the cone K is d-complete iff the norm topology on each component of K is weaker than the topology generated by d, and this is equivalent to 4. □

Remark 17. By Theorem 14 in the case of an ordered Banach space the normality of the cone is both necessary and sufficient for the completeness of the Thompson metric. The proof, relying on Theorem 13, uses the closed graph theorem and the fact that a cone in a Banach space is normal iff every order interval is norm bounded. As these results are not longer true in arbitrary LCS, we ask the following question.

Problem. *Characterize the class of LCS for which the normality of K is also necessary for the completeness of the Thompson metric (or, at least, put in evidence a reasonably large class of such spaces).*

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Functional Operators and Approximate Solutions of Functional Equations

Stefan Czerwik and Krzysztof Król

Abstract In this paper we consider the problem of approximate solutions of functional equations.

In the first part of this chapter we present the integral least squares method for functional equations.

The second part is devoted to investigations on some functional operators, useful for both theory and applications.

Finally, in the last one we present some results on convergence of a sequence of approximate solutions of a functional equation obtained by the integral least squares method as well as some estimations of the errors of approximations.

Keywords Functional equation • Functional operator • Approximation by the integral least squares method • Estimation of the error of approximation

Subject Classifications: 39B22, 39B99, 40A05, 41A30.

1 The Integral Least Squares Method

In this section we shall present the integral least squares method for the functional equations.

We shall consider the following (linear) functional equation (see [5])

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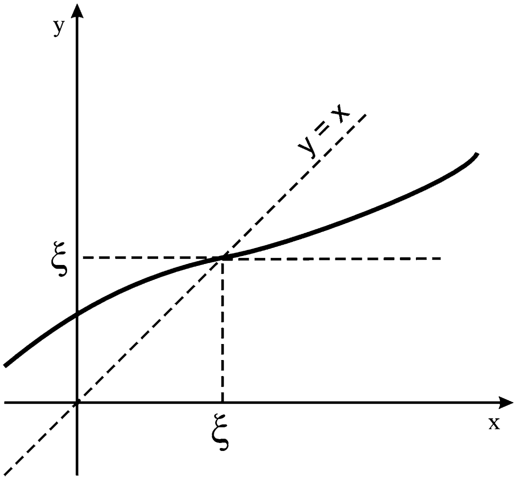

Fig. 1 Function belonging to $\mathbf{R}^{\mathbf{0}}_{\xi}[\mathbf{P}]$

$$y[f(x)] = g(x)y(x) + F(x), \tag{1}$$

where $f,\ g,\ F$ are given functions and y is an unknown function.

Denote by $\mathbf{Y}[\mathbf{P}]$ the class of all functions defined on an interval P with values in $\mathbb{R}$ (the set of all real numbers). Let $\mathbf{R}^{\mathbf{0}}_{\xi}[\mathbf{P}]$ be the class of all functions from $\mathbf{Y}[\mathbf{P}]$ which are continuous, strictly increasing on P and fulfilling for $\xi \in \overline{P}$ (the closure of P) the following conditions:

(a) $(f(x) - x)(\xi - x) > 0, \quad for\ x \in P,\ x \neq \xi,$
(b) $(f(x) - \xi)(\xi - x) < 0, \quad for\ x \in P,\ x \neq \xi.$

(see Fig. 1).

We shall use the following result.

Theorem 1 ([5]). *Let $f \in \mathbf{R}^{\mathbf{0}}_{\xi}[\mathbf{P}]$, where $\xi \in P$. Let functions $g, F \in \mathbf{Y}[\mathbf{P}]$ be continuous on P, $g(x) \neq 0$ for $x \in P$, $x \neq \xi$ and assume that*

$$|g(\xi)| > 1.$$

Then the Eq. (1) has exactly one continuous solution $y \in \mathbf{Y}[\mathbf{P}]$ on P. This solution is given by the formula

$$y(x) = -\sum_{n=0}^{\infty} \frac{F[f^n(x)]}{G_{n+1}(x)}, \quad x \in P, \tag{2}$$

where

$$G_n(x) = \prod_{i=0}^{n-1} g[f^i(x)], \qquad x \in P,\ n = 1, 2, 3, \ldots,$$

(f^k means the kth iterate of f).

Firstly we shall make some remarks.

Remark 1. In the Theorem 1 we assume that f is a strictly increasing. If f is a strictly decreasing, then we can transform this case to the former one.

In fact, $f^2 := f \circ f$ is strictly increasing function, and moreover if y satisfies the Eq. (1), then

$$\begin{aligned} y[f(f(x))] &= g(f(x))y(f(x)) + F(f(x)) \\ &= g(f(x))[g(x)y(x) + F(x)] + F(f(x)) \\ &= g(f(x))g(x)y(x) + g(f(x))F(x) + F(f(x)), \end{aligned}$$

i.e.

$$y[f^2(x)] = h(x)y(x) + K(x), \qquad x \in P,$$

where

$$h(x) = g(f(x))g(x), \qquad K(x) = g(f(x))F(x) + F(f(x)), \qquad x \in P$$

and

$$f(f(\xi)) = f(\xi) = \xi.$$

It means that y satisfies the Eq. (1) with h and K and f replaced by f^2.

Remark 2. If $\xi = b$ is the right endpoint of P, the proofs runs similarly.

Remark 3. If ξ is inside the interval P, we consider separately the subintervals $[a, \xi]$ and $[\xi, b]$.

Remark 4. If $\xi \notin P$, then the situation is different. In this case a continuous solution depends on arbitrary function on the interval $[x_0, f(x_0)]$ or $[f(x_0), x_0]$ ($x_0 \in P$). This situation is not very interesting for applications (for correctly stated problems there exists exactly one (the best) solution).

Remark 5. For f non-invertible we can also get similar results (see [5], but investigations are more complicated).

Now we present the basic result concerning the application of the integral least squares method for the functional equation (1).

Theorem 2. *Let functions $g, F\colon [a,b] \to \mathbb{R}$, $f\colon [a,b] \to [a,b]$ satisfy the assumptions of the Theorem 1 for $P = [a,b]$, $a < b$, $a, b \in \mathbb{R}$. Assume that $\Phi_j\colon [a,b] \to \mathbb{R}$,*

$j = 1, \ldots, n$, are given, continuous and linearly independent functions on interval $[a, b]$. Assume that the matrix

$$\mathbf{C} = \{C_{ij}\}_{i,j=1,\ldots,n},\ C_{ij} = \int_a^b \Psi_i(x)\Psi_j(x)\,dx,$$

where

$$\Psi_j(x) = \Phi_j[f(x)] - g(x)\Phi_j(x),\ x \in [a, b],\ j = 1, \ldots, n,$$

is positively defined. Then the continuous solution of the Eq. (1) on the interval $[a, b]$ can be approximated by the function

$$y_n(x) = \sum_{j=1}^{n} p_j \Phi_j(x), \quad x \in [a, b], \quad n \in \mathbb{N}, \tag{3}$$

where the coefficients p_j, $j = 1, \ldots, n$, are the solutions of the system of linear equations

$$\sum_{j=1}^{n} p_j \int_a^b \Psi_i(x)\Psi_j(x)\,dx = \int_a^b \Psi_i(x)F(x)\,dx, \tag{4}$$

for $i = 1, \ldots, n$.

Proof. Let's note that from the assumptions and Theorem 1 it follows that Eq. (1) has the exactly one continuous solution on I. We shall find its approximation. Define the function

$$R[y(x)] := y[f(x)] - g(x)y(x) - F(x),\ x \in P.$$

We will find for $x \in [a, b]$

$$\begin{aligned} R[y_n(x)] &= y_n[f(x)] - g(x)y_n(x) - F(x) \\ &= \sum_{j=1}^{n} p_j \Phi_j[f(x)] - g(x) \sum_{j=1}^{n} p_j \Phi_j(x) - F(x) \\ &= \sum_{j=1}^{n} p_j \Big[\Phi_j[f(x)] - g(x)\Phi_j(x) \Big] - F(x). \end{aligned}$$

Denote

$$\Psi_j(x) := \Phi_j[f(x)] - g(x)\Phi_j(x), \qquad for\ j = 1, \ldots, n,$$

then

$$R[y_n(x)] = \sum_{j=1}^{n} p_j \Psi_j(x) - F(x).$$

To find p_i, we shall calculate the minimum value of the function

$$I(\mathbf{p}) = I(p_1, \ldots, p_n) := \int_a^b \big(R[y_n(x)]\big)^2 \, dx. \tag{5}$$

To this end we take p_i such that

$$\frac{\partial I(\mathbf{p})}{\partial p_i} = 0, \qquad for\ i = 1, \ldots, n.$$

Putting $R[y_n(x)]$ to (5) we get

$$I(\mathbf{p}) = \int_a^b \left[\sum_{j=1}^{n} p_j \Psi_j(x) - F(x)\right]^2 dx,$$

and consequently

$$\frac{1}{2}\frac{\partial I(\mathbf{p})}{\partial p_i} = \int_a^b \left[\sum_{j=1}^{n} p_j \Psi_j(x) - F(x)\right]\Psi_i(x)\, dx,$$

for $i = 1, \ldots, n$.
To get the minimum of the function (5) we shall consider the (algebraic) system of linear equations

$$\sum_{j=1}^{n} p_j \int_a^b \Psi_i(x)\Psi_j(x)\, dx = \int_a^b \Psi_i(x) F(x)\, dx,$$

for $i = 1, \ldots, n$.

Calculating the Hessian of the function I, we obtain

$$\mathbf{H}_I(\mathbf{p}) = \{H_{ij}\}_{i,j=1,\ldots,n},$$

where

$$H_{ij} = \frac{\partial^2 I(\mathbf{p})}{\partial p_i \partial p_j} = 2\int_a^b \Psi_i(x)\Psi_j(x)\, dx,$$

or shortly $\mathbf{H}_I(\mathbf{p}) = 2\mathbf{C}$. This means that the Hessian is positively defined and function I has at the point $\mathbf{p} = (p_1, \ldots, p_n)$ the minimum. This ends the proof of the theorem. □

For the convenience, we can write the system of equations (4) in the form

$$\mathbf{C} \cdot \mathbf{p} = \mathbf{F},$$

where
$\mathbf{C} = \{C_{ij}\}_{i,j=1,\ldots,n}$, $\quad C_{ij} = \int_a^b \Psi_i(x)\Psi_j(x)\,dx$,
$\mathbf{p} = [p_1, \ldots, p_n]^T$,
$\mathbf{F} = [F_1, \ldots, F_n]^T$, $\quad F_i = \int_a^b \Psi_i(x)F(x)\,dx$.

Remark 6. The problems of the convergence of the sequence (3) and the estimation of the error of accuracy shall be considered in the next parts of the paper.

2 Linear Functional Operator

In functional equations (in one variable—see e.g., [5] or in several variables [2]) there occur some important functional equations which induce special functional operators very useful in many fields, especially in the theory of approximate solutions of functional equations. The theory of approximate solutions of functional equations has been systematically built mainly by the authors of this chapter (see [1, 3, 4]).

In this part we shall consider some functional equation and related functional operator.

Let's consider the following functional equation

$$y[f(x)] + \mathbf{g}(x)y(x) = F(x), \tag{6}$$

where the functions $f\colon [a,b] \to [a,b]$, $\mathbf{g}$, $F\colon [a,b] \to \mathbb{R}$, $[a,b] \subset \mathbb{R}$ are given and $y\colon [a,b] \to \mathbb{R}$ is an unknown function ($\mathbb{R}$ stands for the set of all real numbers).

We shall use the notations

$$\|f\|_1 := \sup_{x\in[a,b]} |f(x)|, \qquad \|f\|_2 := \left(\int_a^b |f(x)|^2\,dx\right)^{\frac{1}{2}}. \tag{7}$$

Denote by $C([a,b])$ the class of all continuous functions defined on the interval $[a,b]$ with values in $\mathbb{R}$.

At first we shall prove the following.

Theorem 3. *Let functions $f\colon [a,b] \to [a,b]$, $\mathbf{g}\colon [a,b] \to \mathbb{R}$ be continuous on the interval $[a,b]$ and let $A\colon C([a,b]) \to C([a,b])$ be the functional operator given by the formula*

$$A[y](x) := y[f(x)] + \mathbf{g}(x)y(x), \quad x \in [a,b], \tag{8}$$

for $y \in C([a,b])$. *Then A is linear and bounded operator. Moreover we have*

$$\|A[y]\|_1 \leq \alpha \|y\|_1, \tag{9}$$

where

$$\alpha := 1 + \sup_{x \in [a,b]} |\mathbf{g}(x)|. \tag{10}$$

Proof. We shall verify that A is a linear operator. Namely, we have for
$y_1, y_2 \in C([a,b])$
$A[y_1 + y_2](x) := y_1[f(x)] + y_2[f(x)] + \mathbf{g}(x)(y_1(x) + y_2(x)) = A[y_1](x) + A[y_2](x)$.
Further for $\lambda \in \mathbb{R}$ and $y \in C([a,b])$
$A[\lambda y](x) := \lambda y[f(x)] + \mathbf{g}(x)\lambda y(x) = \lambda A[y](x)$,
i.e. A is linear.
Now we shall prove that A is also bounded (in the norm $\|\cdot\|_1$). Really, for
$f, \mathbf{g}, y \in C([a,b])$ and $|\mathbf{g}(x)| \leq M$ for $x \in [a,b]$, one has

$$\begin{aligned}
\|A[y]\|_1 &= \sup_{x \in [a,b]} |A[y](x)| = \sup_{x \in [a,b]} |y[f(x)] + \mathbf{g}(x)y(x)| \\
&\leq \sup_{x \in [a,b]} (|y[f(x)]| + |\mathbf{g}(x)| \cdot |y(x)|) \\
&\leq \sup_{x \in [a,b]} |y[f(x)]| + \sup_{x \in [a,b]} (|\mathbf{g}(x)| \cdot |y(x)|) \\
&\leq \sup_{x \in [a,b]} |y(x)| + M \sup_{x \in [a,b]} |y(x)| \leq \|y\|_1 + M\|y\|_1 \\
&\leq (1 + M)\|y\|_1 = \alpha \|y\|_1.
\end{aligned}$$

Therefore there exists the number

$$\alpha = 1 + \sup_{x \in [a,b]} |\mathbf{g}(x)| > 0$$

such that for every $y \in C([a,b])$

$$\|A[y]\|_1 \leq \alpha \|y\|_1,$$

which means that A is a bounded operator. This completes the proof. □

In the sequel let's consider the linear homogeneous functional equation

$$y[f(x)] = g(x)y(x). \tag{11}$$

Define (see [5]) the following sequence

$$G_n(x) = \prod_{i=0}^{n-1} g[f^i(x)], \qquad x \in P,\ n = 1,2,3,\dots \quad , \tag{12}$$

where P is an subinterval of $\mathbb{R}$ and f^k denotes the kth iteration of f. We shall call this sequence as the Choczewski–Kuczma sequence.

For the Choczewski–Kuczma sequence $\{G_n\}$ the following causes can occur (see [5]):

(i) There exists the limit $G(x) = \lim_{n\to\infty} G_n(x)$, for $x \in P$. Moreover G is continuous on P and $G(x) \neq 0$ for $x \in P$.
(ii) There exists an interval $J \subset P$ such that $\lim_{n\to\infty} G_n(x) = 0$ uniformly on J.
(iii) Neither of the cases (i) and (ii) occurs.

The following theorem holds true.

Theorem 4 ([5]). *Let $f \in \mathbf{R}^{\mathbf{0}}_{\xi}[\mathbf{P}]$, where $\xi \in P$. Let $g \in \mathbf{Y}[\mathbf{P}]$ be a continuous function on P and $g(x) \neq 0$ for $x \in P$, $x \neq \xi$. In the case (iii) the function $y(x) \equiv 0$, $x \in P$ is the only continuous solution of the equation (11) in the class* $\mathbf{Y}[\mathbf{P}]$.

Moreover, we have

Corollary 1 ([5]). *Under the assumptions of the Theorem 4 and if*

$$|g(\xi)| > 1, \tag{13}$$

then the case (iii) occurs.

Now we shall prove the following theorem.

Theorem 5. *Assume that the assumptions of the Theorem 4 are satisfied and let $|g(\xi)| > 1$. Then there exists the inverse operator A^{-1} to the operator $A\colon C([a,b]) \to C([a,b])$ given by (8).*

Proof. Firstly we shall verify that A is an injection. Really, let $A[y_1] = A[y_2]$ for $y_1, y_2 \in C([a,b])$, then

$$y_1[f(x)] + \mathbf{g}(x)y_1(x) = y_2[f(x)] + \mathbf{g}(x)y_2(x), \qquad x \in [a,b],$$

or

$$(y_1 - y_2)[f(x)] + \mathbf{g}(x)(y_1(x) - y_2(x)) = 0, \qquad x \in [a,b].$$

Denote $\psi := y_1 - y_2$, so

$$\psi[f(x)] + \mathbf{g}(x)\psi(x) = 0, \qquad x \in [a,b],$$

or

$$\psi[f(x)] = -\mathbf{g}(x)\psi(x), \qquad x \in [a, b].$$

Therefore by $|-\mathbf{g}(\xi)| = |g(\xi)| > 1$, Theorem 4 and Corollary 1, $\psi(x) = 0$ for $x \in [a, b]$.
Hence $0 = \psi(x) = y_1(x) - y_2(x)$, i.e. $y_1(x) = y_2(x)$ for $x \in [a, b]$.
Finally we have proved that $A[y_1] = A[y_2]$ implies $y_1 = y_2$ which means that A is an injection and A^{-1} exists. This concludes the proof. □

Now we can prove.

Theorem 6. *Let the assumptions of the Theorem 4 be satisfied. Then the operator A^{-1}, inverse to the operator $A: C([a, b]) \to C([a, b])$ defined by the formula (8), is linear and bounded.*

Proof. In view of Theorem 5, A^{-1} exists and is a linear operator as the inverse to the linear one.

Next we shall verify that it is also bounded.

Assume that ξ is the left endpoint of $P = [a, b] = [\xi, b]$ (in other cases the proof is similar). Moreover $f(\xi) = \xi$, $|g(\xi)| > 1$ and g is continuous. Thus there exists constants $\delta > 0$ and $\Theta > 1$, such that

$$|g(x)| > \Theta > 1 \quad for\ x \in [\xi, \xi + \delta] \subset [\xi, b].$$

For $b \in P$ we can find N, such that

$$f^n(b) \in [\xi, \xi + \delta] \quad for\ n \geq N. \tag{14}$$

Then the condition (13) holds true, because $f^n(b) \to \xi$ as $n \to \infty$.
Since f is strictly increasing by (13) one gets

$$f^n(x) \in [\xi, \xi + \delta] \quad for\ n \geq N\ and\ x \in [\xi, b].$$

Denote

$$L = \inf_{x \in [\xi, b]} |g(x)| > 0.$$

This is true because g is continuous on the compact interval $P = [\xi, b]$ function without zero values.
For $x \in P$ and $n \geq N$ we get

$$\frac{1}{|G_{n+1}(x)|} = \frac{1}{|g(x) \cdot g(f(x)) \cdot \ldots \cdot g[f^n(x)]|}$$

$$= \frac{1}{|g(x) \cdot g(f(x)) \cdot \ldots \cdot g[f^{N-1}(x)] \cdot g[f^N(x)] \cdot \ldots \cdot g[f^n(x)]|}$$
$$\leq \frac{1}{L^N \Theta^{n+1-N}}.$$

Hence

$$\left|(-1)^n \frac{F[f^n(x)]}{G_{n+1}(x)}\right| \leq \frac{\|F\|_1}{L^N \Theta^{n+1-N}} \quad \textit{for } n \geq N \textit{ and } x \in [\xi, b].$$

Let's note that the formula (2) for the Eq. (6) has the form

$$y(x) = \sum_{n=0}^{\infty} (-1)^n \frac{F[f^n(x)]}{G_{n+1}(x)}, \quad x \in [a, b]. \tag{15}$$

Thus, in view of (15), for $F \in C([\xi, b]) = C([a, b])$

$$y(x) = A^{-1}[F](x) = \sum_{n=0}^{\infty} (-1)^n \frac{F[f^n(x)]}{G_{n+1}(x)}, \qquad x \in [a, b].$$

Hence we get

$$\|A^{-1}[F]\|_1 = \sup_{x \in [\xi, b]} \left|\sum_{n=0}^{\infty} (-1)^n \frac{F[f^n(x)]}{G_{n+1}(x)}\right| \leq \sum_{n=0}^{\infty} \sup_{x \in [\xi, b]} \left|(-1)^n \frac{F[f^n(x)]}{G_{n+1}(x)}\right|$$
$$\leq \sum_{n=0}^{N-1} \sup_{x \in [\xi, b]} \frac{\|F\|_1}{|G_{n+1}(x)|} + \sum_{n=N}^{\infty} \sup_{x \in [\xi, b]} \frac{\|F\|_1}{L^N \Theta^{n+1-N}}$$
$$\leq \|F\|_1 \sum_{n=0}^{N-1} \sup_{x \in [\xi, b]} \frac{1}{|G_{n+1}(x)|} + \frac{\|F\|_1}{L^N} \sum_{n=1}^{\infty} \frac{1}{\Theta^n}$$
$$\leq M_1 \|F\|_1 + M_2 \|F\|_1 \leq (M_1 + M_2) \|F\|_1 \leq K \|F\|_1.$$

Constants M_1 and M_2 exist, since the functions

$$\frac{1}{|G_{n+1}(x)|}, \ n = 0, 1, \ldots, N-1,$$

are continuous on the compact set (and with positive values) and the series is the geometric convergent one, respectively.
Finally, there exists a constant $K \geq 0$, such that for any function $F \in C([\xi, b]) = C([a, b])$

$$\|A^{-1}[F]\|_1 \leq K \|F\|_1, \tag{16}$$

where

$$K = \sum_{n=0}^{N-1} \sup_{x \in [\xi, b]} \frac{1}{|G_{n+1}(x)|} + \frac{1}{L^N(\Theta - 1)}, \tag{17}$$

which means that A^{-1} is bounded (continuous), and the proof of the theorem is completed. □

In the sequel we shall apply the following result (see [7]).

Theorem 7. *Let T be a linear operator from a normed space $(X, \|\cdot\|)$ on to a normed space $(Y, \|\cdot\|)$. Operator T is an injection and the inverse operator $T^{-1}: Y \to X$ is linear and continuous if and only if there exists a constant $m > 0$ such that*

$$\|Tx\| \geq m\|x\|, \qquad for\ x \in X.$$

We shall prove the following.

Theorem 8. *Let the assumptions of the Theorem 4 be satisfied. There exists a number $\gamma > 0$, such that for every $y \in C([a, b])$*

$$\|y\|_1 \leq \gamma \|A[y]\|_1, \tag{18}$$

where A is given by (8).
Moreover

$$\gamma = K, \tag{19}$$

(K is given by (17)).

Proof. The operator A is a linear injection and A^{-1} is a linear and continuous.
On account of the Theorem 7 the inequality (18) holds true.
We shall find the constant γ.
Let $y \in X = C([a, b])$ and let $A[y] = F$.
From (16) we get (for $K > 0$)

$$\|A[y]\|_1 = \|F\|_1 \geq \frac{1}{K}\|A^{-1}[F]\|_1 = \frac{1}{K}\|y\|_1,$$

and hence

$$\|y\|_1 \leq K\|A[y]\|_1 \quad for\ every\ y \in C([a, b]).$$

Thus $\gamma = K$, which concludes the proof. □

In the sequel we present

Theorem 9. *Let $y \in C([a,b])$ be a function with constant sign and let $f\colon [a,b] \to [a,b]$ and $\mathbf{g}\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$. Moreover, let*

$$\inf_{x\in[a,b]} \mathbf{g}(x) > 0.$$

Then there exists a constant $\delta > 0$ (independent on y), such that

$$\|y\|_2 \le \delta \|A[y]\|_2, \tag{20}$$

where

$$\delta = \left[\inf_{x\in[a,b]} \mathbf{g}(x)\right]^{-1}. \tag{21}$$

Proof. We have for y with constant sign,

$$\begin{aligned}\|A[y]\|_2^2 &= \int_a^b (y[f(x)] + \mathbf{g}(x)y(x))^2 \, dx \ge \int_a^b \mathbf{g}^2(x)y^2(x)\, dx \\ &\ge \int_a^b \left(\inf_{x\in[a,b]} \mathbf{g}(x)\right)^2 y^2(x)\, dx \ge \beta^2 \int_a^b y^2(x)\, dx = \beta^2 \|y\|_2^2,\end{aligned}$$

where $\beta = \inf_{x\in[a,b]} \mathbf{g}(x) > 0$. Hence

$$\|A[y]\|_2^2 \ge \beta^2 \|y\|_2^2$$

or

$$\|y\|_2^2 \le \frac{1}{\beta^2} \|A[y]\|_2^2,$$

i.e.

$$\|y\|_2 \le \delta \|A[y]\|_2,$$

which ends the proof. □

We also have

Theorem 10. *Let $y, \mathbf{g} \in C([a,b])$ and let $f\colon [a,b] \to [a,b]$ be continuous. Then there exists the real constant $\mu > 0$, such that*

$$\|A[y]\|_2 \le \mu \|y\|_1, \quad for\ y \in C([a,b]), \tag{22}$$

where

$$\mu \;=\; (1+\eta)\sqrt{b-a}, \qquad (23)$$
$$\eta := \sup_{x\in[a,b]} |\mathbf{g}(x)|.$$

Proof. Denote $\eta = \sup_{x\in[a,b]} |\mathbf{g}(x)|$. For any $y \in C([a,b])$ one has

$$\begin{aligned}
\|A[y]\|_2^2 &= \int_a^b \left(y[f(x)] + \mathbf{g}(x)y(x)\right)^2\, dx \\
&\le \int_a^b \left(\sup_{x\in[a,b]} |y[f(x)]| + \sup_{x\in[a,b]} |\mathbf{g}(x)y(x)|\right)^2 dx \\
&\le \left(\sup_{x\in[a,b]} |y[f(x)]| + \eta \sup_{x\in[a,b]} |y(x)|\right)^2 \cdot (b-a) \\
&\le (\|y\|_1 + \eta\|y\|_1)^2 \cdot (b-a) \\
&= (b-a)(1+\eta)^2\|y\|_1^2.
\end{aligned}$$

Therefore, for $y \in C([a,b])$

$$\|A[y]\|_2^2 \le (1+\eta)^2(b-a)\|y\|_1^2$$

or

$$\|A[y]\|_2 \le \mu\|y\|_1,$$

with

$$\mu = (1+\eta)\sqrt{b-a} > 0.$$

The proof is completed. □

The last theorem of this section is

Theorem 11. *Let $y \in C([a,b])$. Then there exists a constant $\varrho > 0$, such that*

$$\|y\|_2 \le \varrho\|y\|_1, \qquad (24)$$

where

$$\varrho = \sqrt{b-a}.$$

Proof. We have for any $y \in C([a,b])$

$$\|y\|_2^2 = \int_a^b y^2(x)\,dx \le \int_a^b (\sup_{x\in[a,b]} |y(x)|)^2\,dx \le (b-a)\|y\|_1^2.$$

This means that

$$\|y\|_2 \le \varrho\|y\|_1,$$

where

$$\varrho = \sqrt{b-a} > 0,$$

which ends the proof. □

Remark 7. The operator (8) can be considered in more general spaces.

3 On the Convergence of the Integral Least Squares Method and Estimations of the Error of Approximations

For the convenience of the reader we shall recall some ideas, which will be used later on.

Definition 1 ([6]). A system (φ_k) of elements of a normed space $(X, \|\cdot\|)$ we shall call closed, if for every $x \in X$ and every $\varepsilon > 0$ there exist a natural number n and constants (real or complex) $\alpha_1, \ldots, \alpha_n$ such that

$$\|x - y_n\| < \varepsilon,$$

where

$$y_n = \sum_{k=1}^{n} \alpha_k \varphi_k.$$

Definition 2 ([6]). Let a linear operator $A\colon (X, \|\cdot\|) \to (X, \|\cdot\|)$ be given. A system (φ_k) of elements of a normed space $(X, \|\cdot\|)$ is said to be A-closed, if for every $y \in X$ and every $\varepsilon > 0$ there exist a natural number n and constants $\beta_1, \ldots, \beta_n$, such that

$$\|A[y] - A[y_n]\| < \varepsilon,$$

where

$$y_n = \sum_{k=1}^{n} \beta_k \varphi_k.$$

The following result is true.

Corollary 2 ([6]). *If a system* (φ_k) *in* X *is closed and an operator* $A\colon X \to X$ *is linear and bounded, then the system* (φ_k) *is* A*-closed.*

Corollary 3. *The system of functions*

$$1, x, x^2, \ldots, \qquad for\ x \in [a, b], \tag{25}$$

is the closed system in $C([a, b])$ *with the Chebyshev supremum norm.*

Proof. The proof follows from the fact that the sequence of polynomials

$$\psi_n(x) = \sum_{k=1}^{n} \alpha_k \varphi_k(x), \qquad \varphi_k(x) = x^k, \qquad x \in [a, b]$$

converges uniformly to a function $\varphi \in C([a, b])$, where φ is any such function. This completes the proof. □

Now we are in a position to prove the result concerning the convergence of the sequence $\{A[y_n]\}_{n=1}^{\infty}$, where $\{y_n\}_{n=1}^{\infty}$ is a sequence of approximate solutions of the functional equation (6) obtained by the integral least squares method.

Theorem 12 (Convergence Theorem). *Let the assumptions of the Theorem 1 for* $P = [a, b]$ *be satisfied. Let, moreover,* $\{y_n\}_{n=1}^{\infty}$ *be a sequence of approximate solutions of the equation (6) obtained by the integral least squares method. Then*

$$A[y_n] \to A[y] = F\ in\ \|\cdot\|_2,\ as\ n \to \infty, \tag{26}$$

where y *is the (unique) continuous solution of the equation (6) on* P*.*

Proof. Note that, in view of Theorem 1, it follows that the Eq. (6) has exactly one continuous solution y on P.

Since the operator A given by the formula (8) is bounded, by Corollary 2 we guess that the system (25) is A-closed.

Therefore, for $\varepsilon > 0$ there exist constants $\alpha_1, \ldots, \alpha_{n_0}$, such that

$$\|A[y] - A[\psi_{n_0}]\|_1 < \frac{\varepsilon}{\varrho},$$

where

$$\psi_{n_0} = \sum_{k=1}^{n_0} \alpha_k \varphi_k, \quad \varphi_k(x) = x^k. \quad x \in [a, b].$$

Moreover, for the sequence $\{y_n\}_{n=1}^{\infty}$, obtained by the integral least squares method, we have

$$\|A[y] - A[y_{n_0}]\|_2 \le \|A[y] - A[\psi_{n_0}]\|_2,$$

because the left-hand side takes minimum value for y_{n_0}, but ψ_{n_0} is a linear combination of functions φ_n. Thus by Theorem 11, we get

$$\|A[y] - A[y_{n_0}]\|_2 \le \|A[y] - A[\psi_{n_0}]\|_2 \le \varrho \|A[y] - A[\psi_{n_0}]\|_1 < \varrho \cdot \frac{\varepsilon}{\varrho} = \varepsilon,$$

i.e.

$$\|A[y] - A[y_{n_0}]\|_2 < \varepsilon.$$

Taking into account that if n increases, then the left-hand side does not increase, thus

$$\|A[y] - A[y_n]\|_2 < \varepsilon \qquad for\ n \ge n_0.$$

This means that the condition (26) is fulfilled, which was to be shown. □

The next useful result is the following.

Theorem 13 (Convergence of a Sequence $\{y_n\}$). *Let the assumptions of the Theorem 1 for $P = [a, b]$ be satisfied and assume*

$$\inf_{x \in [a,b]} \mathbf{g}(x) > 0.$$

Let, moreover, $y \in C([a, b])$ be the (unique) continuous solution of the equation (6) and let $\{y_n\}$ be a sequence of approximate solutions of the equation (6) obtained by the integral least squares method. If $\{y - y_n\}_{n=1}^{\infty}$ is the sequence of functions with constant sign (not necessary the same), then

$$y_n \to y\ in\ \| \cdot \|_2,\ as\ n \to \infty. \tag{27}$$

Proof. By Corollary 3 it follows that the sequence of polynomials

$$\psi_n(x) = \sum_{k=1}^{n} \alpha_k \varphi_k(x), \quad \varphi_k(x) = x^k, \quad x \in [a, b], \quad \alpha_k\text{—constant},$$

is convergent to y, i.e.

$$\psi_n \to y\ in\ \| \cdot \|_1,$$

which means that for every $\varepsilon > 0$ there exists an n_0, such that

$$\|y - \psi_n\|_1 < \varepsilon \quad for\ n \ge n_0.$$

Thus by Theorem 9 we have for $n \geq n_0$

$$\|y - y_n\|_2 \leq \delta\|A[y - y_n]\|_2 \leq \delta\|A[y] - A[y_n]\|_2 \leq \delta\|A[y] - A[\psi_n]\|_2$$

(the last inequality holds true due to minimum property of the sequence $\{y_n\}$, ψ_n is just a linear combination of the functions φ_k, $k = 1, \ldots, n$).

Therefore on account of the Theorem 10, one gets for $n \geq n_0$

$$\|y - y_n\|_2 \leq \delta\|A[y] - A[\psi_n]\|_2 \leq \delta\mu\|y - \psi_n\|_1 < \delta\mu\varepsilon.$$

Since $\delta\mu$ is a constant and $\varepsilon > 0$ is arbitrary real number, this means that (27) is fulfilled, which concludes the proof. □

Remark 8. The assumptions of the Theorem 13 do not look very easy to verification.

In the last theorem we shall present a result about the estimations of the error of approximations.

Theorem 14 (Estimation Test). *Let the assumptions of the Theorem 1 for $P = [a, b]$ be satisfied. Assume that $y \in C([a, b])$ is the unique continuous solution of the functional equation (6) and y_n is its approximation obtained by the integral lest squares method. Then the following estimations hold true*

$$\|y_n - y\|_2 \leq \gamma\varrho\|A[y_n] - F\|_1 \qquad for\ \gamma > 0,\ \varrho > 0,$$
$$\|y_n - y\|_1 \leq \gamma\|A[y_n] - F\|_1 \qquad for\ \gamma > 0,$$

where

$$\varrho = \sqrt{b - a},$$
$$\gamma = K$$

(K is given by the formula (17)).

Proof. By Theorems 11 and 8 we obtain

$$\|y_n - y\|_2 \leq \varrho\|y_n - y\|_1 \leq \gamma\varrho\|A[y_n] - A[y]\|_1 \leq \gamma\varrho\|A[y_n] - F\|_1,$$

and

$$\|y_n - y\|_1 \leq \gamma\|A[y_n] - A[y]\|_1 \leq \gamma\|A[y_n] - F\|_1,$$

where

$$\varrho = \sqrt{b - a},$$
$$\gamma = K,$$

which ends the proof. □

Remark 9. The integral least squares method is very useful (in practice) because of the commutativity of the operations: integration and differentiation (for the norm $\| \cdot \|_1$ this is not the case).

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Markov-Type Inequalities with Applications in Multivariate Approximation Theory

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Abstract In this paper, we provide a brief overview of several refinements and applications of the Markov-type inequalities in various contexts.

Keywords Markov inequality • Markov-type inequalities • Polynomials with restricted zeros • Multivariate polynomials • Continuous linear extension of C^{∞} functions • Müntz polynomials • Leja–Siciak extremal function • Green function

Subject Classification MSC2010: primary 26D05, 41A17, secondary 30B10, 31C10, 32E30, 32V25, 41A21, 41A30, 41A65

1 Introduction

Let $\mathscr{P}_d(\mathbb{K})$ *be the collection of all polynomials of degree at most* d *with coefficients in the field* $\mathbb{K} = \mathbb{R}, \mathbb{C}$. An inequality of the Markov-type is an inequality of the form

$$\left\| P^{(\nu)}(x) \right\| \leq C \left\| P(x) \right\|$$

for every $P \in \mathscr{P}_d(\mathbb{K})$. Here $P^{(\nu)}(x)$ denotes the νth derivative. The best possible constant C depends on d, ν and the norm $\|\cdot\|$, and the determination and estimation of C has been the subject of numerous investigations since Andrei Markov's paper [50] (:$\nu = 1$ and $\|\cdot\|$ being the L^{∞}-norm on a bounded interval) and the paper [51]

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by his brother Vladimir Markov: ($\nu \geq 2$ and the same L^{∞}-norm). Markov's original paper dates back to 1889 and it is not readily accessible. For a modern exposition on this and other related topics we also refer to [16].

The first purpose of this paper is to give a brief overview of all the refinements, extensions and generalizations of the Markov-type inequalities. The second purpose of the paper is to summarize some basic applications of various forms of these inequalities in the context of multivariate approximation theory. A useful reference in the paper could be the book [52].

2 Markov-Type Inequalities in $\mathbb{R}$

2.1 Markov's Original Inequality and Constrained Variations

Markov's (Andrey (Andrei) Andreyevich Markov, in older works also spelled Markoff, 14 June 1856–20 July 1922) original inequality asserts that *if* $P \in \mathscr{P}_d(\mathbb{R})$ *such that* $\sup_{x\in[-1,1]} |P(x)| \leq 1$, *then*

$$\sup_{x\in[-1,1]} \left|P'(x)\right| \leq d^2.$$

Clearly Markov's result can also be equivalently stated as follows.

Theorem 1 ([31]). *For all polynomials* $P \in \mathscr{P}_d(\mathbb{R})$, *it holds*

$$\sup_{x\in[-1,1]} \left|P'(x)\right| \leq d^2 \sup_{x\in[-1,1]} |P(x)|.$$

More generally,

$$\sup_{x\in[a,b]} \left|P'(x)\right| \leq \frac{2d^2}{b-a} \sup_{x\in[a,b]} |P(x)|.$$

The prissy result of Theorem 1 for classes of polynomials under various constraints has attracted a number of authors. For example, it has been observed by Sergei Natanovich Bernstein in that *Markov's inequality for monotone polynomials is not essentially better than for arbitrary polynomials*:

Theorem 2 ([9]). *If* d *is odd, then*

$$\sup_{x\in[-1,1]} \left|P'(x)\right| \leq \left(\frac{d+1}{2}\right)^2 \sup_{x\in[-1,1]} |P(x)|$$

for all $P \in \mathscr{P}_d(\mathbb{R}) \setminus \{0\}$ *that are monotone on* $[-1,1]$.

Another direction is considered by P. Borwein and Erdélyi who proved the following Markov-type inequality for polynomials with coefficients in $\{-1,\ 0, 1\}$.

Theorem 3 ([21]). *For all polynomials $P \in \mathscr{P}_d(\mathbb{R})$ with each of their coefficients in $\{-1,\ 0, 1\}$, there are two constants $\sigma_1 > 0$ and $\sigma_2 > 0$ such that*

$$\sigma_1 \le \frac{1}{d \cdot \log(d+1)} \frac{\sup_{x\in[0,1]} \left|P'(x)\right|}{\sup_{x\in[0,1]} |P(x)|} \le \sigma_2.$$

Let me finally mention a totally different approach due to Newman and Frappier and concerning a Markov-type inequality formulation for *Müntz polynomials*.

Theorem 4 ([34, 57]). *Given any sequence $\Lambda = \left(\lambda_j\right)_{j=0,1,2,\ldots}$ of distinct real numbers, let $\mathscr{M}_d(\Lambda)$ be the linear span of Λ over $\mathbb{R}$ denoted by*

$$\begin{aligned}\mathscr{M}_d(\Lambda) &:= \operatorname{span}_{\mathbb{R}} \left\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_d}\right\} \\ &\equiv \left\{a_{\lambda_0} x^{\lambda_0} + a_{\lambda_1} x^{\lambda_1} + \cdots + a_{\lambda_d} x^{\lambda_d} : a_{\lambda_0}, a_{\lambda_1}, \ldots, a_{\lambda_d} \in \mathbb{R}\right\}\end{aligned}$$

($d = 0, 1, 2, \ldots$). Elements of $\mathscr{M}_d(\Lambda)$ are called Müntz polynomials. Every polynomial $P \in \mathscr{M}_d(\Lambda) \setminus \{0\}$ satisfies the following Markov-type inequality

$$\left(\frac{2}{3}\right) \cdot \left(\sum_{j=0}^{d} \lambda_j\right) \le \frac{\sup_{x\in[0,1]} \left|xP'(x)\right|}{\sup_{x\in[0,1]} |P(x)|} \le (8.29) \cdot \left(\sum_{j=0}^{d} \lambda_j\right).$$

2.2 Markov-Type Inequalities for the Derivatives of an Algebraic Polynomial

Having found an upper bound for $\left|P'(x)\right|$, it would be natural to go on and ask for an upper bound for $\left|P^{(k)}(x)\right|$ (where $k \le d$). Iterating Markov's theorem yields

$$\sup_{x\in[-1,1]} \left|P^{(k)}(x)\right| \le d^{2k} \cdot \sup_{x\in[-1,1]} |P(x)| \, (P \in \mathscr{P}_d(\mathbb{R})).$$

However, this inequality is not sharp. The best possible inequality was found by Markov's brother Vladimir Andreevich Markov (May 8, 1871–January 18, 1897), who proved the following.

Theorem 5 ([51]). *For all polynomials $P \in \mathscr{P}_d(\mathbb{R})$, it holds*

$$\sup_{x\in[-1,1]} \left|P^{(k)}(x)\right| \le \frac{d^2 \cdot \left(d^2 - 1^2\right) \cdot \cdots \cdot \left(d^2 - (k-1)^2\right)}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k-1)} \cdot \sup_{x\in[-1,1]} |P(x)|,$$

for every $k = 1, 2, \ldots, n$. The fraction on the right-hand side of this inequality is exactly equal to $T_d^{(k)}(1)$ where

$$T_d(x) = \cos(d \ \arccos x) = 2^{d-1} \prod_{\nu=1}^{d} \left\{ x - \cos\left(\left[\nu - \frac{1}{2} \right] \pi / d \right) \right\}$$

is the Chebyshev polynomial of the first kind.

2.3 *Refinements of the Markov-Type Inequality for Real Polynomials with Restricted Zeros*

2.3.1 The Case of Real Roots

In 1940, P. Erdös offered the following refinement of Markov's inequality.

Theorem 6 ([32]). *If* $P \in \mathscr{P}_d(\mathbb{R})$ *and* P *has all its roots in* $\mathbb{R} \setminus (-1, 1)$*, then*

$$\sup_{x \in [-1,1]} \left| P'(x) \right| \le \frac{ed}{2} \cdot \sup_{x \in [-1,1]} |P(x)| .$$

However, we cannot proceed inductively with Erdos' inequality, since some of the roots of the derivatives may be in $[-1, 1]$. With this in mind, Szabados and Varma established the following version of Erdos' inequality.

Theorem 7 ([74]). *If* $P \in \mathscr{P}_d(\mathbb{R})$ *and* P *has all its roots in* $\mathbb{R}$ *and at most one root in* $[-1, 1]$*, then*

$$\sup_{x \in [-1,1]} \left| P'(x) \right| \le C_1 \cdot d \cdot \sup_{x \in [-1,1]} |P(x)|$$

where C_1 *is independent of* d.

This, of course, yields the following inequality.

Corollary 8. *If* $P \in \mathscr{P}_d(\mathbb{R})$ *and* P *has all its roots in* $\mathbb{R} \setminus (-1, 1)$*, then*

$$\sup_{x \in [-1,1]} \left| P''(x) \right| \le C_2 \cdot d^2 \cdot \sup_{x \in [-1,1]} |P(x)|$$

where the constant C_2 *is independent of* d.

Generalizing, in 1985, P. Borwein showed the next result.

Theorem 9 ([17]). *If* $P \in \mathscr{P}_d(\mathbb{R})$ *and the polynomial* P *has at least* $d - m$ *roots in* $\mathbb{R} \setminus (-1, 1)$*, then there is a constant* C $(C \le 9)$ *so that*

$$\sup_{x\in[-1,1]}\left|P'(x)\right| \leq C\cdot d\cdot(m+1)\cdot\sup_{x\in[-1,1]}|P(x)|.$$

Remark 10. Up to the constant this result is best possible. In fact, Szabados in [73] constructed polynomials $P \in \mathscr{P}_d(\mathbb{R})$ with $d-m$ roots in $\mathbb{R}\backslash(-1,1)$ so that

$$\sup_{x\in[-1,1]}\left|P'(x)\right| \geq \frac{d\cdot m}{2}\cdot\sup_{x\in[-1,1]}|P(x)|\,(0<m\leq d).$$

We obtain immediately the following refinement of Theorem 5.

Corollary 11. *For all polynomials $P \in \mathscr{P}_d(\mathbb{R})$, having at least $n-m$ roots in $\mathbb{R}\backslash(-1,1)$, it holds*

$$\sup_{x\in[-1,1]}\left|P^{(k)}(x)\right| \leq c_m\cdot\frac{d!\cdot(m+k)}{(d-k)!\cdot m!}\cdot\sup_{x\in[-1,1]}|P(x)|,$$

for every $k=1,2,\ldots,n$. The constant $c_m \leq 9^m$ depends only on k.

2.3.2 The Case of Complex Roots

Let us now turn to the case of non-real (complex) zeros. Recall that, by Theorem 1, we have

$$\sup_{x\in[0,1]}\left|P'(x)\right| \leq 2d^2\sup_{x\in[0,1]}|P(x)|.$$

In [7], Benko and Erdélyi proved the following refinement of this Markov Inequality.

Theorem 12. *Let $\mathscr{P}_d^{(m,\,\mathbb{C})}(\mathbb{R})$ be the collection of all polynomials of degree at most d with real coefficients that have at most m distinct complex zeros. For every $P \in \mathscr{P}_d^{(m,\mathbb{C})}(\mathbb{R})$, it holds*

$$\sup_{x\in[0,1]}\left|P'(x)\right| \leq 32\cdot 8^m\cdot d\cdot\sup_{x\in[0,1]}|P(x)|.$$

2.3.3 The General Case

Independently of the location of polynomial roots, Duffin and Schaeffer provided the following nice and general uniform refinement of Markov's inequality.

Theorem 13 ([27]). *For all polynomials $P \in \mathscr{P}_d(\mathbb{R})$, it holds*

$$\sup_{x\in[-1,1]}\left|P'(x)\right| \leq d^2\cdot\max_{j=0,1,\ldots,d}|P(\cos[j\pi/d])|.$$

In this direction, but in contrast to the uniform estimate of Markov's inequality, S. N. Bernstein inequality gave nice punctual bounds for the first derivative of a polynomial.

Theorem 14 ([8]). *For every polynomial $P \in \mathscr{P}_d(\mathbb{R})$ and every $a < b$ in $\mathbb{R}$, we have*

$$\left|P'(x)\right| \leq \frac{d}{\sqrt{(x-a)(b-x)}} \cdot \sup_{y\in[a,b]} |P(y)| \; (a < x < b).$$

Apart from the above general Markov–Bernstein inequality, there is another general nice result, due to G. G. Lorentz.

Theorem 15 ([48]). *For all polynomials $P \in \mathscr{P}_d(\mathbb{R})$ having no zeros in the ellipse $\mathscr{L}_\epsilon = (x)^2 + (y/\epsilon i)^2$ with large axis $[-1, 1]$ and small axis $[-\epsilon i, \epsilon i]$ $(-1 < \epsilon < 1)$, there is an constant $c > 0$ such that*

$$\left|P'(x)\right| \leq c \cdot \min\left\{\frac{\sqrt{d}}{\epsilon\sqrt{1-x^2}}, \frac{d}{\epsilon^2}\right\} \cdot \sup_{y\in[-1,1]} |P(y)| \; (-1 < x < 1).$$

2.4 Extensions of Markov-Type Inequality

Another natural direction of interest is the extension of Markov-type inequalities to other types of compact subsets of $\mathbb{R}$. For a survey of these results see the monograph of P. Borwein and Erdélyi [20]. A number of papers study possible extensions of Markov-type inequalities to compact sets $K \subset \mathbb{R}$ when the geometry of K is known a priori (Cantor type sets, finitely many intervals, etc.; cf. Pleśniak [61] and the references therein, as well as [18, 76]).

However, most of the results concerning the extension of Markov-type inequalities in other types of compact subsets of $\mathbb{R}$ are partial cases of more general results reported in compact sets of $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$). Consequently, we would prefer not to speak at all about the extension of Markov-type inequalities in various compact subsets of the real line, and to come back when we will consider the multidimensional case.

For now, let me mention just two indicative propositions which have been proven only in the case of a real variable.

Proposition 16 ([18]). *Let $0 < a \leq 1$, and let A be a closed subset of $[0, 1]$ with Lebesgue measure $m(A) \geq 1 - a$. Then there is a constant $C > 0$ such that*

$$\sup_{x\in I} \left|P'(x)\right| \leq C \cdot d^2 \cdot \sup_{x\in A} |P(x)|$$

for every $P \in \mathscr{P}_d(\mathbb{R})$ and for every subinterval I of A with length at least a.

Proposition 17 ([31]). *Let* α *and* β *be two real numbers such that* $1 < \alpha + 1 < \beta$ *and* $5(\alpha + 1) < \beta$. *Let also*

$$K = K_{\alpha,\beta} = \bigcup_{k=1}^{\infty} \left[\frac{1}{(k+1)^{\alpha}} + \frac{\alpha}{4(k+1)^{\beta}},\ \frac{1}{k^{\alpha}} \right] \bigcup \{0\}.$$

Then there is a constant $C = C(\alpha, \beta) > 0$ *such that*

$$\sup_{x \in K} \left| P'(x) \right| \leq C \cdot d^2 \cdot \sup_{x \in K} |P(x)|$$

for every $P \in \mathscr{P}_d(\mathbb{R})$.

3 Markov-Type Inequalities in $\mathbb{R}^n$

The purpose of this section is to study Markov-type inequalities for multivariate polynomials. Thus we consider the space $\mathscr{P}_d(\mathbb{R}^n)$ of polynomials

$$P(x) = \sum_{|k| \leq d} a_k x^k$$

of n real variables and total degree $\leq d$. (As usually, $|k| = k_1 + \cdots + k_n$ and $x^k = x_1^{k_1} \ldots x_n^{k_n}$, where $k = (k_1, \ldots, k_n)$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.) In what follows $\|x\|$ denotes the Euclidean norm of $x \in \mathbb{R}^n$, $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ is the unit sphere in $\mathbb{R}^n$, while $\overline{B}^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ stands for the closed unit ball of $\mathbb{R}^n$.

We are interested in estimating $D_y P(x)$, the derivative of $P(x) \in \mathscr{P}_d(\mathbb{R}^n)$ in the direction $y \in S^{n-1}$. In particular, this leads to estimates for the magnitude of the gradient of $P(x)$ given by

$$|\mathrm{grad}\, P(x)| = \sup \left\{ \left| D_y P(x) \right| : y \in S^{n-1} \right\}.$$

Naturally, in the multivariate case the results are closely related to the geometry of the underlying set $K \subset\subset \mathbb{R}^n$ on which the uniform norm $\|P\|_K := \sup_{x \in K} |P(x)|$ of $P(x) \in \mathscr{P}_d(\mathbb{R}^n)$ is considered.

A compact subset K of $\mathbb{R}^n$ is said to ***preserve*** (or admit) ***Markov's inequality***, or simply to be ***Markov***, if there exist constants $M > 0$ and $r > 0$ such that for each polynomial $p \in \mathscr{P}_d(\mathbb{R}^n)$ we have

$$\sup_{x \in K} |\mathrm{grad}\, P(x)| \leq M \cdot d \cdot r \cdot \sup_{x \in K} |P(x)|, \text{ whenever } d \in \mathbb{N} \qquad (\mathrm{M}_n)$$

The first sharp Markov-type inequality in $\mathbb{R}^n$ was obtained by Kellogg in 1928 in the case when K is the closed unit ball $\overline{B}^n$ of $\mathbb{R}^n$.

Theorem 18 ([45]). $\overline{B}^n$ *is a Markov set; indeed, for all polynomials* $P \in \mathscr{P}_d(\mathbb{R}^n)$*, it holds*

$$\sup_{x\in\overline{B}^n} |\text{grad}\, P(x)| \leq d^2 \cdot \sup_{x\in\overline{B}^n} |P(x)|.$$

(Clearly, this inequality is sharp for every $d, n \in \mathbb{N}$.)

In 1974, Wilhelmsen gave a Markov-type estimate for an arbitrary convex body $K \in \mathbb{R}^n$. Recall that a ***convex body*** in $\mathbb{R}^n$ is a convex compact set with non-empty interior [77]. Twenty-five years later, Kroó and Révéz improved slightly Wilhelmsen's estimate by showing the following result.

Theorem 19 ([47]). *Let* $K \subset\subset \mathbb{R}^n$ *be a convex body. Denote by* $w(K)$ *the minimal distance between two parallel supporting hyperplanes for* K*. Then* K *is a Markov set; indeed, for all polynomials* $P \in \mathscr{P}_d(\mathbb{R}^n)$*, it holds*

$$\sup_{x\in K} |\text{grad}\, P(x)| \leq \frac{4d^2 - 2d}{w(K)} \cdot \sup_{x\in K} |P(x)|.$$

Wilhelmsen's inequality with a different, weaker constant was given earlier by Coatmelec [23]. Note that $w\left(\overline{B}^n\right) = 2$, i.e. for the unit ball the constant in the inequality of Theorem 19 is twice larger than in the inequality of Theorem 18. Independently Nadzhmiddinov and Subbotin proved Theorem 19 in the special case when K is a triangle in $\mathbb{R}^2$ [56]. This leads to the interesting problem of finding the exact constant in Theorem 19. Evidently, this constant must be between 2 and 4. This question was partially resolved in 1991 by Sarantopoulos who found sharp Bernstein and Markov-type inequalities in the case when K is central symmetric [66]. Recall that K is ***central symmetric*** if and only if with proper shift it is the unit ball of some norm on $\mathbb{R}^n$.

Apart from the above results, we also have some additional results reported only in the case of two real variables. First, we must notice that the constant in the estimate of Theorem 19 can be improved as follows in the special case when K is a triangle Δ in $\mathbb{R}^2$.

Theorem 21 ([47]). *Assume that* $K = \Delta \subset\subset \mathbb{R}^2$ *is a triangle with angles* $0 < \gamma \leq \beta \leq a \leq \pi/2$*. Then* K *is a Markov set; indeed, for all polynomials* $P \in \mathscr{P}_d(\mathbb{R}^n)$*, it holds*

$$\sup_{x\in\Delta} |\text{grad}\, P(x)| \leq \frac{s(\Delta)\cdot d^2}{w(K)} \cdot \sup_{x\in\Delta} |P(x)|.$$

where

$$s(\Delta) := \frac{2}{\alpha}\sqrt{\alpha^2 + \beta^2 + 2\alpha\beta\cos\gamma}.$$

Clearly, $10 \leq s(\Delta) \leq 2\sqrt{2 + 2\cos\gamma} < 4$.

We can also give Markov-type estimates on certain special irrational arcs and domains of $\mathbb{R}^2$.

Example 22 ([30]). Let $\gamma_a := \left\{(x, x^\alpha) \in \mathbb{R}^2 : 0 \le x \le 1\right\}$. For every irrational number $\alpha > 0$, there are constants $A,\ B > 1$, depending only on α, such that the tangential Markov factor

$$\mathscr{M}_d^T(\gamma_a) := \sup\left\{\|D_T P\|_{\gamma_a} : P \in \mathscr{P}_d(\mathbb{R}^n),\ \|P\|_{\gamma_a} \le 1\right\}$$

of γ_a (*T is the unit tangent to γ_a and $D_T P$ is the tangential derivative of P along* γ_a) satisfies the following Markov-type inequality

$$A^d \le \mathscr{M}_d^T(\gamma_a) \le B^d$$

for every sufficiently large d.

Example 23 ([30]). Let $\alpha > 1$ and $K_a := \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1 \text{ and } \frac{x^\alpha}{2} \le y \le 2x^\alpha\}$. There exists a constant $C > 0$, depending only on α, such that every Markov factor on K_a

$$\mathscr{M}_d(K_a) := \sup\left\{\left\|D_y P\right\|_{K_a} : P \in \mathscr{P}_d\left(\mathbb{R}^2\right),\ \|P\|_{K_a} \le 1 \text{ and } y \in S^{n-1}\right\}$$

satisfies the Markov-type inequality

$$\mathscr{M}_d(K_a) \le d^{c \log d}$$

for every sufficiently large d.

Example 24 ([36]). Let

$$K := \left\{(x, y) \in \mathbb{R}^2 :\ 0 \le x \le 1 \text{ and } 0 \le y \le x^m\right\}.$$

Then, K preserves Markov inequality; indeed, for all polynomials $P \in \mathscr{P}_d(\mathbb{R}^n)$, it holds

$$\sup_{x \in K} |\text{grad } P(x)| \le M \cdot d^{2m} \cdot \sup_{x \in K} |P(x)|.$$

This last example had inspired W. Pawlucki and W. Plésniak to investigate Markov-type inequalities in semianalytic and subanalytic sets, and more general, sets with polynomial cusps. Let us recall that a subset E of $\mathbb{R}^n$ is said to be ***semianalytic*** if for each point $x \in \mathbb{R}^n$ one can find a neighbourhood U of x and a finite number of real analytic functions $f_{i,j}$ and $g_{i,j}$ defined in U, such that

$$E \bigcap U = \bigcup_i \bigcap_j \{f_{i,j} > 0 \ and\ g_{i,j} = 0\}.$$

The projection of a semianalytic set need not be semianalytic [49]. The class of sets obtained by enlarging that of semianalytic sets to include images under the projections has been called the class of subanalytic sets. More precisely, a subset E of $\mathbb{R}^n$ is said to be ***subanalytic*** if for each point $x \in \mathbb{R}^n$ there exists an open neighbourhood U of x such that $E \backslash U$ is the projection of a bounded semianalytic subset of $\mathbb{R}^{n+m}$, where $m \geq 0$. If $n \geq 3$, the class of subanalytic sets is essentially larger than that of semianalytic sets, the classes being identical if $n \leq 2$. The union of a locally finite family and the intersection of a finite family of subanalytic sets is subanalytic. The closure, interior, boundary and complement of a subanalytic set is still subanalytic, the last property being a (non-trivial) theorem of Gabrielov.

It is clear that the set K_m of Example 24 is semianalytic, whence subanalytic. It appears that the family of (fat) subanalytic sets is a subfamily of a family of sets admitting only polynomial-type cusps.

Definition 25. A subset E of $\mathbb{R}^n$ is said to be **uniformly polynomially cuspidal** if one can choose three constants $M > 0, m \geq 1$ and $d \in \mathbb{N}$ and a mapping

$$q : \overline{E} \times [0,1] \to \overline{K}$$

such that for each $x \in \overline{E}$,

i. $q(x, 1) = x, q(x, \cdot)$ is a polynomial map of degree d and
ii. $dist(q(x, t); \mathbb{R}^n \backslash E) \geq M(1 - t)^m$ for $(x, t) \in E \times [0, 1]$.

Application of Hironaka's rectilinearisation theorem (*see* [15]) and Łojasiewicz' regular separation (*see* [49]), shows *that every bounded subanalytic subset of* $\mathbb{R}^n$ *with* $intE$ *dense in* E *is uniformly polynomially cuspidal.* The importance of uniformly polynomially cuspidal compact sets in $\mathbb{R}^n$ is explained by the following.

Theorem 26 ([58]). *If K is a uniformly polynomially cuspidal compact subset of $\mathbb{R}^n$, then there exist two constants $M > 0$ and $r > 0$ such that for each polynomial $P \in \mathscr{P}_d(\mathbb{R}^n)$, we have*

$$\sup_{x \in K} |\text{grad } P(x)| \leq M \cdot d^r \cdot \sup_{x \in K} |P(x)|.$$

Further, there are two other constants $\mathfrak{R} > 0$ and $\rho > 0$ such that for every polynomial $P \in \mathscr{P}_d(\mathbb{R}^n)$ the following inequality holds

$$\sup_{x \in K} \left| \frac{\partial P}{\partial x_j} \right| \leq \mathfrak{R} \cdot d^{\rho} \cdot \sup_{x \in K} |P(x)| \text{ whenever } j = 1, 2, \ldots, n.$$

4 Markov-Type Inequalities in $\mathbb{C}$

In the complex plane, the notion of a compact set that is uniformly polynomially cuspidal becomes trivial. Therefore we are really most interested in sets that are totally disconnected or otherwise highly irregular.

However, for completeness, we will make a short reference to the case of continuous compact subsets of the complex plane. The first sharp Markov-type inequality in $\mathbb{C}$ was obtained by S. N. Bernstein in 1928 in the case when $K = \overline{D} = \{z \in \mathbb{C}: |z| \leq 1\}$ is the closed unit disc.

Theorem 27 (Bernstein's Inequality). *For all* $P \in \mathscr{P}_d(\mathbb{C})$, *it holds*

$$\sup_{z\in\overline{D}} \left|P'(z)\right| \leq d \cdot \sup_{z\in\overline{D}} |P(z)|.$$

The result is best possible and the equality holds for $p(z) = \lambda z^n$, λ *being a complex number.*

Remark 28. The above Bernstein's inequality has an analogue for trigonometric polynomials which states that if $t(\theta) = \sum_{\nu=-d}^{d} a_\nu e^{i\nu\theta}$ is a trigonometric polynomial (possibly with complex coefficients) of degree n, such that $|t(\theta)| \leq 1$ for $0 \leq \theta < 2\pi$ then $\left|t'(\theta)\right| \leq d$ whenever $0 \leq \theta < 2\pi$. Equality holds if and only if $(\theta) = e^{i\gamma} \cos(d\theta - \alpha)$, where γ and α are arbitrary real numbers.

Remark 29 ([26]). Related to the derivative inequality of Theorem 27, Dryanov and Fournier showed the estimate

$$\sup_{|z|=1} \left|\frac{P(z) - P(\overline{z})}{z - \overline{z}}\right| \leq d \cdot \max_{j=0,1,\ldots,n} \left|\frac{P\left(e^{ij\pi/d}\right) + P\left(e^{-ij\pi/d}\right)}{2}\right| (P \in \mathscr{P}_d(\mathbb{C})).$$

Using this estimate, Dryanov and Fournier proved Duffin and Schaeffer's theorem (see Theorem 13), as well as its complex version due to Frappier, Rahman and Ruscheweyh [35]:

$$\sup_{z\in\overline{D}} \left|P'(z)\right| \leq d \cdot \max_{j=0,1,\ldots,2d-1} \left|P\left(e^{ij\pi/d}\right)\right|.$$

More generally, for any convex compact subset of the complex plane, we can prove the next result.

Theorem 30. *For any compact convex set* $K \subset\subset \mathbb{C}$ *and any* $P \in \mathscr{P}_d(\mathbb{C})$, *the following Markov-type inequality holds*

$$\sup_{z\in K} \left|P'(z)\right| \leq \frac{4}{\operatorname{diam}(K)} \cdot d^2 \sup_{z\in K} |P(z)|$$

where diam (K) *is the diameter of* K.

Generalizing even more, in 1959, Pommerenke proved the following very nice Erdos-type Markov inequality (compare with Theorem 6).

Theorem 31 ([65]). *Let $K \subset\subset \mathbb{C}$ be a connected, closed, bounded set of capacity* cap$K$. *For all $P \in \mathscr{P}_d(\mathbb{C})$ such that $\|P\|_K \leq 1$ it holds*

$$\sup_{z \in K} \left| P'(z) \right| \leq \frac{e}{2} \cdot \frac{d^2}{\operatorname{cap} K} < 1.36 \cdot \frac{d^2}{\operatorname{cap} K}.$$

Several years later, in 2007, Eremenko proved a precise version of this inequality with an arbitrary continuum K in the complex plane $\mathbb{C}$ instead of a connected compact set.

Theorem 32 ([33]). *If $P \in \mathscr{P}_d(\mathbb{C})$ is a polynomial of degree at most d with complex coefficients, then*

$$(\operatorname{cap} K) \cdot \sup_{z \in K} \left| P'(z) \right| \leq 2^{1/[\deg(P)]-1} \cdot [\deg(P)]^2 \cdot \sup_{z \in K} |P(z)|$$

whenever K is a continuum in $\mathbb{C}$ with transfinite diameter (capacity) cap$K$.

A different approach to the investigation of Markov-type polynomial inequalities in disconnected or irregular planar sets was proposed by Toókos and Totik, in 2005. Their idea was based to the Lipschitz continuity of the Green function in these sets. More specifically, they proved the following striking result.

Theorem 33 ([75]). *Let K be a compact subset of the plane such that the unbounded component Ω of $\mathbb{C} \setminus K$ is regular (with respect to the Dirichlet problem: this means that the Green function g_Ω of Ω with pole at infinity is continuous on the boundary $\partial\Omega$ of Ω). Then the following are pairwise equivalent.*

i. *Optimal Markov–Bernstein type inequality holds on K, i.e. there exists a $C > 0$ such that*

$$\sup_{z \in K} \left| P'(z) \right| \leq C \cdot d \cdot \sup_{z \in K} |P(z)|.$$

for all $P \in \mathscr{P}_d(\mathbb{C})$.

ii. *Green's function g_Ω is Lipschitz continuous, i.e. there exists a $C_1 > 0$ such that*

$$g_\Omega(z) \leq C_1 \cdot \operatorname{dist}(z, K)$$

for every $z \in \mathbb{C}$.

iii. *The equilibrium measure μ_K of K satisfies a Lipschitz type condition, i.e. there exists a $C_2 > 0$ such that*

$$\mu_K(D(z_0; \delta)) \leq C_2 \delta$$

for every $z \in K$ and $\delta > 0$. Here, $D(z_0; \delta)$ denotes the disk $\{z \in \mathbb{C} : |z_0 - z| \leq \delta\}$ centered at z_0 with radius δ.

If Ω is simply connected, then (i)–(iii) are also equivalent to

iv. *The conformal mapping Φ from Ω onto the exterior of the unit disk is Lipschitz continuous, i.e.*

$$|\Phi(z_1) - \Phi(z_2)| \leq C_3 \cdot |z_1 - z_2|, z_1,\ z_2 \in \Omega.$$

5 Markov-Type Inequalities in $\mathbb{C}^n$

It is clear that from Theorem 1 it is easily proved that for the unit cube $[-1,\ 1]^n \subset \mathbb{R}^n$ the following inequality holds

$$\sup_{z\in[-1,\ 1]^n} |D^a P| \leq d^{2|a|} \cdot \sup_{z\in[-1,\ 1]^n} |P|$$

whenever $P \in \mathscr{P}_d(\mathbb{C}^n)$ and $a \in \mathbb{N}^n$.

Throughout the sequel, $\mathbb{R}^n$ will be treated as a subset of $\mathbb{C}^n$ such that

$$\mathbb{R}^n = \left\{(z_1, \ldots, z_n) \in \mathbb{C}^n : Imz_j = 0,\ j = 1, 2, \ldots, n\right\}.$$

With this notation, a compact set $K \subset\subset \mathbb{C}^n$ is said to ***preserve*** (or admit) ***Markov's inequality*** (M_∞) or simply to be (M_∞)-***Markov***, if there exist an integer $m \geq 1$ and, for every $a \in \mathbb{N}^n$, a constant $M_a > 0$ such that for each polynomial $P \in \mathscr{P}_d(\mathbb{C}^n)$ we have

$$\sup_{z\in K} |D^a P(z)| \leq M_a \cdot d^{m\cdot|a|} \cdot \sup_{z\in K} |P(z)|.\ (M_\infty)$$

Extending this inequality in various classes of compact subsets of $\mathbb{C}^n$ was the subject of many studies which would be difficult to make an exhaustive list. Nevertheless, one can refer to the references [1, 58, 70].

It is clear that the well-known concept of a uniformly polynomially cuspidal compact subset of $\mathbb{R}^n$ (Definition 25) can be directly extended to the case of $\mathbb{C}^n$. *A compact set $K \subset\subset \mathbb{C}^n$ is said to be* ***uniformly polynomially cuspidal*** *(briefly, UPC) with parameter $m > 0$, if one can choose a constant $M > 0$ and an integer $d_0 \geq 1$ such that for any $z \in K$ there exists a polynomial application $Q_a : \mathbb{C} \to \mathbb{C}^n$ with degree $\leq d_0$ satisfying the following two conditions:*

(i) $Q_a(0) = a$ and $Q_a([0, 1]) \in K$
(ii) $\operatorname{dist}(Q_a(t), \mathbb{C}^n \backslash K) \geq M \cdot t^m$, *whenever* $t \in [0, 1]$.

The *UPC* sets are important from the pluripotential theory point of view, since they admit (pluricomplex) Green functions with nice continuity properties. To explain this, let us suppose that E is a compact subset of $\mathbb{C}^n$. We set

$$V_K(z) := \sup\left\{u(z) : u \in \mathscr{L}(\mathbb{C}^n),\ u_{|K} \leq 0\right\}, z \in \mathbb{C}^n,$$

where

$$\mathscr{L}\left(\mathbb{C}^n\right) := \left\{u \in \mathrm{PSH}\left(\mathbb{C}^n\right) : \ \sup_{z\in\mathbb{C}^n} \left|u\left(z\right) - \log\left(1+|z|\right)\right| < \infty\right\}$$

is the *Lelong class* of plurisubharmonic functions with minimal growth. The function V_K is called the ***(plurisubharmonic) extremal function*** associated with K. Its upper semicontinuous regularization V_K^* is a multidimensional counterpart of the classical ***Green function*** for $\mathbb{C}^n \backslash \hat{K}$, where $\hat{K}$ is the polynomial hull of K, since by the pluripotential theory due to E. Bedford and B.A. Taylor it is a solution of the homogeneous complex *Monge–Ampére equation*, which is reduced in the one-dimensional case to the *Laplace equation* [6]. It is known that

$$V_K(z) = \sup\left\{\frac{1}{d}\log|p\left(z\right)| : p \in \mathscr{P}_d\left(\mathbb{C}^n\right) \text{ with } d \geq 1 \text{ and } \|p\|_{|K} \leq 1\right\}, z \in \mathbb{C}^n \text{[69]}.$$

In other words,

$$V_K\left(z\right) = \log \Phi_K\left(z\right)$$

where Φ_K is ***Siciak's extremal function***.

Definition 34. The set $K \subset\subset \mathbb{C}^n$ is said to have Hölder's Continuity Property (briefly, HCP) if there exist two constants C and s such that

$$\Phi_K\left(z\right) \leq 1 + C \cdot \delta^{1/s} \text{ as } \operatorname{dist}\left(z, K\right) \leq \delta \leq 1.$$

Example 35. Let $P : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial mapping satisfying

$$\liminf_{|z|\to\infty} \frac{|P\left(z\right)|}{|z|^{\delta}} > 0$$

for some $\delta > 1$. We define the **filled-in Julia set associated with** P to be the set

$$J_P = \left\{z \in \mathbb{C}^n : \text{ the set } \{P^{\nu}\left(z\right)\}_{\nu\in\mathbb{N}_0} \text{ is bounded}\right\}$$

where P^{ν} denotes the ν-th iteration of the polynomial mapping P. Following [46], the **filled-in Julia set associated with** P is a compact, polynomially convex set satisfying HCP.

Now we can come back to the multivariate Markov inequality. In 1967, Siciak used Cauchy integral formula and proved the following property.

Theorem 36 ([68]). *If* $K \subset\subset \mathbb{C}^n$ *has HCP, then K preserves Markov's inequality* (M_∞), *with* $M_a = a!eC^{|a|}$ *and* $m = [s]$.

The importance of the class *UPC* is explained by the following

Theorem 37 ([58]). *If K is a compact UPC subset of $\mathbb{C}^n$ with parameter m, then K satisfies HCP with exponent $s = 1/2\tilde{m}$, where $\tilde{m} := k$ as $k - 1 < m \leq k$ with $k \in \mathbb{Z}$.*

There are, however, sets that are *HCP* without being *UPC*. Such Cantor-type sets were first constructed by Jonsson [44] and Siciak [71]. The problem of whether the classical Cantor ternary set has Markov's property has appeared more difficult and a positive answer was first given in 1993 by Białas and Volberg [10] who showed that this set is even *HCP*. It is worth adding that there are also Cantor-type sets which do not preserve Markov's inequality and, at the same time, they are regular with respect to the (classical) Green function [39, 60, 76].

Up to now, the problem of whether Markov's property of K implies that K is *HCP* remains open. We know only that the answer is "*yes*" for a class of one-dimensional Cantor-type sets [11, 76]. In general, we even do not know whether Markov's property of K implies the continuity of the Green function V_K or else non-pluripolarity of K. We recall that a subset K of $\mathbb{C}^n$ is said to be ***pluripolar*** if one can find a plurisubharmonic function u on $\mathbb{C}^n$ such that $K \subset \{u = -\infty\}$. However, Białas-Cież [12] proved that *any planar compact Markov set has a positive logarithmic capacity, whence it is not polar*.

Let me finally mention an (M_∞)-type inequality, due to Baran and Pleśniak.

Theorem 38 ([4]). *If K is a polynomially convex, HCP compact subset of $\mathbb{C}^n$ with HCP-exponent m and f is a nondegenerate analytic map defined in an open neighbourhood of K, with values in an algebraic subset $\mathbb{M}$ of $\mathbb{C}^n$ of dimension M, $1 \leq M \leq N$, then there exists a constant $C > 0$ such that for every polynomial $Q(z_1, \ldots, z_n)$ we have*

$$|DT(t; v)Q(f(t))| \leq C_1 d^{1/m} \|Q\|_{f(K)}$$

where $t \in K$ and $T(t, v) = D_v f(t)$ the derivative of f in direction v.

6 Markov-Type Inequalities in L^p Spaces

Markov classical inequality in Theorem 1 has been extended to the L^p-norm ($p \geq 1$) by Hille, Szegö and Tamarkin [43]. Their result reads

$$\left\| P' \right\|_p \leq C(p, d) \cdot d^2 \cdot \|P\|_p (P \in \mathscr{P}_d(\mathbb{R}))$$

where $\| \|_p$ is the usual L^p-norm on $[-1, 1]$ and $C(p, d)$ is the following bounded (by $6e^{1+(1/e)}$, whenever $p \geq 1$ and $d \geq 1$) coefficient

$$C(p,d) := \begin{cases} 2(1 + (1/d))^{d+1}, \text{ if } p = 1 \\ 2(p-1)^{(1/p)-1}(p + (1/d))(1 + p/(dp - p + 1))^{d-1+1/d}, \text{ if } p > 1. \end{cases}$$

Several years later, in 1990, Goetgheluck improved the admissible values for $C(p, d)$ [38].

During the next years, several L^p-Markov-type inequalities are obtained for many special cases. We mention two such cases. First, in 1995, P. Borwein and Erdélyi gave the following sharp L^p-Markov-type inequality on $[-1, 1]$.

Theorem 39 ([19]). *It holds*

$$\int_{-1}^{1} \left|P'(x)\right|^p dx \leq M(p) \cdot (d \cdot (k+1))^p \cdot \int_{-1}^{1} |P(x)|^p dx$$

for all real algebraic polynomials $P \in \mathscr{P}_d(\mathbb{C})$ having at most k, with $0 \leq k \leq d$, zeros (counting multiplicities) in the open unit disk of the complex plane, and for all $p > 0$, where $M(p) = c^{p+1}\left(1 + p^{-1}\right)$ with some absolute constant $c > 0$.

Next, in 2000, Erdélyi proved the following L^p-Markov-type inequality for Müntz polynomials and exponential sums on $[a, b]$.

Theorem 40 ([29]). *(Newman's Inequality in $L^p[a, b]$ for $[a, b] \subset (0, \infty)$). Let $\Lambda = \left(\lambda_j\right)_{j=0,1,2,\ldots}$ be an increasing sequence of nonnegative real numbers. Suppose $\lambda_0 = 0$ and there exists a $\delta > 0$ so that $\lambda_j \geq \delta \cdot j$ for each j. Suppose $0 < a < b$ and $1 \leq p \leq \infty$. Then there exists a constant $c(a, b, \delta)$ depending only on a, b, and δ so that*

$$\left(\int_a^b \left|P'(x)\right|^p dx\right)^{1/p} \leq c(a, b, \delta) \cdot \left(\sum_{j=0}^{d} \lambda_j\right) \cdot \left(\int_a^b |P(x)|^p dx\right)^{1/p}$$

for every $P \in \mathscr{M}_d(\Lambda)$, where $\mathscr{M}_d(\Lambda)$ is the space of Müntz polynomials, that is the linear span of λ over $\mathbb{R}$

$$\mathscr{M}_d(\Lambda) := \mathrm{span}_{\mathbb{R}}\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_d}\} \equiv \{a_{\lambda_0}x^{\lambda_0} + a_{\lambda_1}x^{\lambda_1} + \cdots + a_{\lambda_d}x^{\lambda_d} : a_{\lambda_0}, a_{\lambda_1}, \ldots, a_{\lambda_d} \in \mathbb{R}\}.$$

Generalizing, in this direction, Baouendi and Goulaouic obtained L^p-Markov's inequalities on compact subsets of $\mathbb{R}^n$ satisfying a parallelepiped property [1], while Goetgheluck obtained L^p-Markov's inequalities on compact *UPC* subsets of $\mathbb{R}^n$ [37].

Let us now turn to the more general case of a compact subset K of $\mathbb{C}^n$ and let us give general conditions for a positive measure μ on K to satisfy an L^p-Markov's inequality. *We say that (K, μ)* ***satisfies a Markov inequality for the exponent*** *$p > 0$, if for any $\alpha \in \mathbb{N}^n$ there exist a constant $M = M(p, \alpha)$ and two integers $r, m > 1$ such that for each polynomial $P \in \mathscr{P}_d(\mathbb{C}^n)$ we have*

$$\sup_{x\in K} |D^a P(z)| \leq M \cdot d^{\frac{m}{p}+r\cdot|a|} \cdot \left(\int_K |P(z)|^p d\mu\right)^{\frac{1}{p}}. \, (\mathrm{M}_p)$$

It is easily seen that

Theorem 41. *A compact set* $K \subset\subset \mathbb{C}^n$ *preserving Markov's inequality* (M_∞) *satisfies a Markov inequality* $\left(\mathrm{M}_p\right)$ *for the exponent* $p > 0$ *if the following condition is fulfilled.*

$$\sup_{x\in K} |P(z)| \leq M \cdot d^{\frac{m}{p}} \cdot \left(\int_K |P(z)|^p d\mu\right)^{\frac{1}{p}}, \text{for any } P \in \mathscr{P}_d(\mathbb{C}^n).$$

In 1993, Zeriahi gave a general density condition on the positive measure μ around any point of K, under which a condition similar to that of Theorem 40 is satisfied.

Theorem 42 ([79]). *Let* K *be a compact set in* $\mathbb{C}^n$ *preserving Markov's inequality* (M_∞). *Let also* μ *be a Borel measure on* K *satisfying the following density condition:*

there are two constants $C, \ \gamma > 0$ *and a real number* $\epsilon_0 \in (0,1)$ *such that*

$$\forall \ a \in \partial_S K \Longrightarrow \mu\left(K \bigcap B^n(a;\epsilon)\right) \geq C \cdot \epsilon^\gamma, \tag{D}$$

whenever $\epsilon \in (0,\epsilon_0)$ *where* $B^n(a;\epsilon)$ *is the open ball of* $\mathbb{C}^n$ *with centre* a *and radius* ϵ *and* $\partial_S K$ *is the Shilov boundary with respect to the uniform algebra generated by the restrictions of the analytic polynomials on* K.

Then (K,μ) *satisfies an* L^p*-Markov inequality. More specifically, there exist two constants* $M_1, \ M_2 > 0$ *and an integer* $m \geq 1$ *such that for any* $p > 0$ *it holds*

$$\sup_{x\in K} |P(z)| \leq M_1 \cdot (\mathrm{M}_2 \cdot d)^{\frac{m}{p}} \cdot \left(\int_K |P(z)|^p d\mu\right)^{\frac{1}{p}} (P \in \mathscr{P}_d(\mathbb{C}^n)).$$

Corollary 43 ([79]). *If* K *is a compact UPC subset of* $\mathbb{C}^n$ *and* λ *is the Lebesgue measure on* K, *then* (K,λ) *satisfies the following* L^p*-Markov inequality:*

$$\sup_{x\in K} |P(z)| \leq M_1 \cdot (\mathrm{M}_2 \cdot d)^{\frac{m}{p}} \cdot \left(\int_K |P(z)|^p d\lambda\right)^{\frac{1}{p}}$$

whenever $P \in \mathscr{P}_d(\mathbb{C}^n)$ *and the constants* $M_1, \ M_2, m > 0$ *are independent of* P *and* p.

7 Markov-Type Inequalities in Normed Linear Spaces

7.1 Markov-Type Inequalities in Banach Spaces

We will first recall the definitions of polynomial defined in normed linear spaces and its Fréchet and directional derivatives.

Let X and Y be real normed linear spaces. Given a positive integer d, a mapping $P : X \to Y$ is called a ***homogeneous polynomial*** of degree d if there exists a continuous symmetric d-linear mapping $F : X \times \cdots \times X \to Y$ such that $P(x) = F(x, \ldots, x)$ for all $x \in X$. In this case we write $P = \hat{F}$ and call P the homogeneous polynomial associated with F. A mapping $P : X \to Y$ is called a ***polynomial of degree at most*** **d** if

$$P = P_0 + P_1 + \ldots + P_d,$$

where $P_k : X \to Y$ is a homogeneous polynomial of degree k for $k = 1, \ldots, d$ and a constant function for $k = 0$.

Let $\mathscr{P}_d^{(Y)}(X)$ *be the collection of all polynomials* $P : X \to Y$ *of degree at most* d. Given a $x \in X$, the ***Fréchet derivative*** of the polynomial $P \in \mathscr{P}_d^{(Y)}(X)$ at x, denoted by $DP(x)$, is a continuous linear map $L : X \to Y$ such that

$$\lim_{y>0} \frac{\|P(x+y) - P(x) - L(y)\|}{\|y\|} = 0.$$

Clearly,

$$DP(x)\, y = \frac{d}{dt} P(x + ty)|_{t=0}$$

when $DP(x)$ exists.

If F is a continuous symmetric d-linear mapping, we write $F\left(x^j y^k\right)$ for

$$F\left(\underbrace{x, \ldots, x}_{j}, \underbrace{y, \ldots, y}_{k}\right).$$

Then the binomial theorem for F can be written as

$$\hat{F}(x+y) = \sum_{k=0}^{d} \binom{d}{k} F\left(x^{d-k} y^k\right).$$

It follows from the continuity of F that $D\hat{F}(x)\, y = dF\left(x^{d-1} y\right)$. Hence if $P : X \to Y$ is a polynomial of degree at most d then $DP : X \to \mathfrak{L}(X, Y)$

is a polynomial of degree at most $d-1$, where $\mathfrak{L}(X,Y)$ denotes the space of all continuous linear mappings $L : X \to Y$ with the operator norm, i.e. $\|L\| = \sup\{\|L(x)\| : \|x\| \le 1\}$.

It is an elementary fact that the k-th order Fréchet derivative $D^k P(x)$ may be identified with a continuous symmetric k-linear map. We denote the associated homogeneous polynomial of degree k by $\hat{D}^k P(x)$ and call this ***the k-th order directional derivative*** of P at x. Thus,

$$\hat{D}^k P(x)\, y = \frac{d^k}{dt^k} P(x+ty)|_{t=0} \text{ and } \frac{1}{k!}\hat{D}^k \hat{F}(x)\, y = \binom{d}{k} F\left(x^{d-k} y^k\right)$$

for all integers k with $0 \le k \le d$.

As usually, if $P \in \mathscr{P}_d^{(Y)}(X)$ is a polynomial, we define

$$\|P\| = \sup\{\|P(x)\| : \|x\| \le 1\}$$

and

$$\left\|D^k P\right\| = \sup\left\{\left\|D^k P(x)(x_1, \ldots, x_k)\right\| : \|x_1\| \le 1, \ldots, \|x_k\| \le 1\right\}$$

for $x \in X$. By an inequality of R. S. Martin (see [25], Theorem 1.7),

$$\left\|D^k P(x)\right\| \le \frac{k^k}{k!}\left\|\hat{D}^k P(x)\right\|$$

for $x \in X$. If X is a real Hilbert space, by an equality due to Banach and others (see [40], Theorem 4 or [25], Example 1.9),

$$\left\|D^k P(x)\right\| = \left\|\hat{D}^k P(x)\right\|$$

for $x \in X$. For further discussion of all these concepts, see [25] and [42].

The fundamental problem we will now consider in this section is to *find the smallest number $M = M(d,k)$ such that $\left\|\hat{D}^k P\right\| \le M \cdot \|P\|$ whenever $P \in \mathscr{P}_d^{(Y)}(X)$ is a polynomial of degree at most d and X and Y are any real normed linear spaces. We shall always assume that $Y = \mathbb{R}$ since by the Hahn–Banach theorem the numbers $M = M(d,k)$ do not change when we restrict to this case.*

We have the following crude estimates for the numbers $= M(d,k)$.

Theorem 44 ([28], p. 149). *For any two positive integers d and k with $1 \le k \le d$, it holds*

$$T_d^{(k)}(1) \le M(d,k) \le 2^{2k-1} T_d^{(k)}(1),$$

where $T_d(x)$ is the Chebyshev polynomial of the first kind:

$$T_d(x) = \cos(d\ \arccos x) = 2^{d-1}\prod_{\nu=1}^{d}\left\{x - \cos\left(\left[\nu - \frac{1}{2}\right]\pi/d\right)\right\}.$$

The following reduces the computation of the numbers $M = M(d,k)$ to the case where X is $\ell^1(\mathbb{R}^2)$, i.e. the space $\mathbb{R}^2$ with norm $\|(s,t)\| = |s| + |t|$.

Theorem 45 ([41], Lemma 9). *The number $M(d,k)$ is the supremum of the absolute values of*

$$\frac{d^k}{dt^k} q(1,t)\big|_{t=0}$$

where $q : \mathbb{R}^2 \to \mathbb{R}$ is any polynomial of degree at most d such that

$$|q(s,\ t)| \leq 1$$

whenever $\|(s,t)\| = |s| + |t| \leq 1$.

For the case of the first and highest derivative, Sarantopoulos and Muñoz have shown that the best constant in Markov's theorem is no larger for ***real*** normed linear spaces than it is for the real line.

Theorem 46 ([55, 66]). *It holds*

$$M(d,1) = d^2 \text{ and } M(d,d) = 2^{d-1}\, d!.$$

In 2002, Muñoz and Sarantopoulos (*see* [14]) have completely solved the problem of determining the Markov constants for the case of Hilbert spaces. The case of the first derivative was done in finite dimensions by Kellogg [45] and in infinite dimensions by the Harris [40].

Theorem 47 ([55]). *If the spaces X in the definition of $M = M(d,k)$ are restricted to be real Hilbert spaces, then*

$$M(d,k) = T_d^{(k)}(1)$$

whenever d and k are positive integers with $1 \leq k \leq d$.

7.2 Markov-Type Inequalities in Banach Algebras

Let $\mathbb{A} = (\mathbb{A}, \|\|)$ be a unital (complex and not necessarily commutative) Banach algebra, with unitary element e and norm $\|\|$.

For any polynomial $P \in \mathscr{P}_d(\mathbb{C})$ of the form $P(z) = \sum_{j=0}^{d} a_j z^j$ and for all $x \in \mathbb{A}$, define

$$P(x) = \sum_{j=0}^{d} a_j x^j \in \mathbb{A}$$

where

$$x^j = \begin{cases} x^k := e, \text{ for } k = 0 \\ x^k = x^{k-1} \cdot x, \text{ for } k = 1, 2, \ldots, d. \end{cases}$$

We say that $P(x)$ is ***a polynomial in*** $\mathbb{A}$ ***of degree at most*** d *with coefficients in the field* $\mathbb{C}$. The set of all polynomials $P(x)$ will be denoted by $\mathscr{P}_d(\mathbb{A})$.

Let also

$$\mathbb{A}_0 := \{x \in \mathbb{A} : \text{ there is a } d \geq 1 \text{ and a } P \in \mathscr{P}_d(\mathbb{A}) \text{ such that } P(x) = 0\}$$

$$\mathbb{A}_* := \{x \in \mathbb{A} : \text{ for any } d \geq 1 \text{ and any } P \in \mathscr{P}_d(\mathbb{A}) \text{ we have } P(x) \neq 0\}.$$

It is easily seen that

1. if $x, y \in \mathbb{A}_0$ and x and y commute, then $x + y \in \mathbb{A}_0$, $xy \in \mathbb{A}_0$ and $\lambda x \in \mathbb{A}_0$ whenever $\lambda \in \mathbb{C}$,
2. if $x \in \mathbb{A}_0$ and x is invertible in $\mathbb{A}$, then $x^{-1} \in \mathbb{A}_0$,
3. if $\mathbb{A}$ is Abelian, then $\mathbb{A}_0$ is a subalgebra of $\mathbb{A}$ and
4. if $\mathbb{A}$ has finite dimension, then $\mathbb{A} = \mathbb{A}_0$ and $\mathbb{A}_* = \emptyset$.

With the above sufficient notation, we are in position to define Markov-type inequalities in the Banach algebra $\mathbb{A}$. But, now, the approach will be notably different from the standard situation discussed in previous sections.

Definition 48. We shall say that an element $x \in \mathbb{A}_*$ has Markov's property $(\mathfrak{M})$ if there exist two positive integers M and m such that for each $d = 1, 2, \ldots$ and any polynomial $P \in \mathscr{P}_d(\mathbb{A})$ we have

$$\left\| P'(x) \right\| \leq M \cdot d^m \cdot \| P(x) \| . \quad (\mathfrak{M})$$

According to [53], to study the elements $x \in \mathbb{A}_*$ with Markov's property, we may consider Pleśniak's type conditions $(\mathfrak{B})$ and $(\mathfrak{B}^*)$:

Definition 49.

i An element $x \in \mathbb{A}_*$ satisfies Pleśniak's type condition $(\mathfrak{B})$ if there exist three positive constants c_1, c_2 and c_3 such that

$$\| P(x - \zeta e) \| \leq c_1 \cdot \| P(x) \| \quad (\mathfrak{B})$$

whenever $P \in \mathscr{P}_d(\mathbb{A})$ $(d = 1, 2, \ldots)$ and $|\zeta| \leq c_2 d^{-c_3}$.

ii Let $S \subset\subset \mathbb{C}$ be a compact set and μ be a Borel probabilistic measure on S. An element $x \in \mathbb{A}_*$ satisfies Pleśniak's type condition $(\mathfrak{B}^*)$ with respect to the pair (S, μ) if there exist three positive constants C_1, C_2 and C_3 such that

$$\left\| \int_S P\left(x - \zeta t e\right) d\mu\left(t\right) \right\| \leq C_1 \cdot \left\| P\left(x\right) \right\| \ \left(\mathfrak{B}^*\right)$$

whenever $P \in \mathscr{P}_d\left(\mathbb{A}\right)$ $(d = 1, 2, \ldots)$ and $|\zeta| \leq C_2 d^{-C_3}$.

We have the following result.

Theorem 50 ([53]).

i *Let S be a compact subset of $\mathbb{C}$ and let μ be a Borel probabilistic measure on S such that*

$$\int_S t^k d\mu\left(t\right) = 0 \textit{ for some positive integer } k \geq 1.$$

If an element $x \in \mathbb{A}_*$ satisfies Pleśniak's type condition $(\mathfrak{B}^*)$ with respect to the pair (S, μ), then x has Markov's property for the k-th derivative, that is

$$\left\| P^{(k)}\left(x\right) \right\| \leq C^k \cdot d^{kC_1} \cdot \left\| P\left(x\right) \right\| \ \left(\mathfrak{M}_k\right)$$

whenever $P \in \mathscr{P}_d\left(\mathbb{A}\right)$ $(d = 1, 2, \ldots)$. Here C is a positive constant and C_1 is the constant of condition $(\mathfrak{B}^*)$ in Definition 49.ii.

ii. *Let x be an element of $\mathbb{A}_*$ satisfying the following inequality*

$$\left\| \left(P\left(x\right)\right)^2 \right\| \geq \sigma \left\| P\left(x\right) \right\|^2 \textit{ whenever} P \in \mathscr{P}_d\left(\mathbb{A}\right) (d = 1, 2, \ldots)$$

where the constant $\sigma > 0$ is independent of the polynomial P but it can depend on the element x. Then x has Markov's property $(\mathfrak{M}_k)$.

iii. *If an element $x \in \mathbb{A}_*$ has Markov's property $(\mathfrak{M}_k)$ for some integer $k \geq 1$, then x has the following Markov's property:*

$$\left\| P'\left(x\right)^k \right\|^{\frac{1}{k}} \leq 6 \cdot C \cdot k^{C_1 - 1} \cdot d^{C_1} \in \left\| P\left(x\right) \right\| \left(P \in \mathscr{P}_d\left(\mathbb{A}\right), d = 1, 2, \ldots\right).$$

8 Six Applications in Multivariate Approximation Theory

8.1 Polynomial Approximation of $\mathbf{C}^\infty$ *Functions*

A subset K of $\mathbb{R}^n$ is $\mathbf{C}^\infty$ ***determining*** if for each function $f \in C^\infty\left(\mathbb{R}^n\right)$, condition $f = 0$ on K, implies that $D^a f = 0$ on K, for each $\alpha \in \mathbb{Z}_+^n$.

Theorem 51 ([58, 60]). *If a compact set K in $\mathbb{R}^n$ is C^∞ determining, then the following statements are equivalent:*

i. *K has Markov's property;*
ii. *K has the following property: there exist positive constants M and r such that for each polynomial every polynomial $P \in \mathscr{P}_d(\mathbb{R}^n)$ $(d = 1, 2, \ldots)$ one has*

$$|P(z)| \leq M \cdot \sup_{w \in K} |P(w)| \text{ whenever } \operatorname{dist}(z, K) \leq (1/d^r)$$

iii. (Bernstein's Theorem) *For every function $f : K \to \mathbb{R}$, if the sequence*

$$(\operatorname{dist}_K(f, \mathscr{P}_d(\mathbb{R}^n)))_{d=1,2,\ldots}$$

is rapidly decreasing, i.e. for each $s > 0$, $d^s \operatorname{dist} E(f, \mathscr{P}_d(\mathbb{R}^n)) > 0$ as $d > 1$, then f extends to a C^∞ function f in $\mathbb{R}^n$. Here,

$$\operatorname{dist}_K(f, \mathscr{P}_d(\mathbb{R}^n)) := \inf\{\|f - P\|_K : P \in \mathscr{P}_d(\mathbb{R}^n)\}.$$

8.2 Extension of $\mathbf{C^\infty}$ *Functions from Compact Sets in* $\mathbb{R}^{\mathbf{n}}$

The development that will be adopted in this *Paragraph* comes from the comprehensive presentation made by Professor W. Pleśniak on 10 November 2005 at the Seminarium Wydziau, Matematyki i Informatyki UJ, Uniwersytet Jagielloski, Poland (*see* http://minikonferencja.matinf.uj.edu.pl/assets/wyklad-plesniak.pdf, [64]).

Let K be a compact set in $\mathbb{R}^n$ and let $\hat{C}^\infty(K)$ denote the space of all functions $u : K \to \mathbb{C}$ that can be extended to C^∞ functions in the whole space $\mathbb{R}^n$. We give the space $\hat{C}^\infty(K)$ the topology $\mathbf{T_Q}$ endowed with the family of the seminorms

$$\sigma_{K,\lambda} = \inf\left\{\sup_{|a|\leq\lambda} \|D^a g\|_K : g \in C^\infty(\mathbb{R}^n),\ g/K = u\right\}.$$

$\mathbf{T_Q}$ is the quotient topology of the space $C^\infty(\mathbb{R}^n)/\mathfrak{T}(K)$, where $C^\infty(\mathbb{R}^n)$ is endowed with the natural topology determined by the seminorms $\|g\|_{K,\lambda} := \sup_{|a|\leq\lambda} \|D^a g\|_K$ and $\mathfrak{T}(K) := \{u \in C^\infty(\mathbb{R}^n) : u/K = 0\}$. Since $C^\infty(\mathbb{R}^n)$ is complete and since $\mathfrak{T}(K)$ is a closed subspace of $C^\infty(\mathbb{R}^n)$, the quotient space $C^\infty(\mathbb{R}^n)/\mathfrak{T}(K)$ is also complete, whence $\left(\hat{C}^\infty(K), \mathbf{T_Q}\right)$ is a Fréchet space.

If the set K is C^∞ determining, this space can be identified with the space of Whitney jets on K. Let us recall that a C^∞ ***Whitney jet*** on K is a vector $U = (U^\alpha)$ $(\alpha \in \mathbb{Z}^n_+)$, where each U^α is a continuous function defined on K, such that

$$|\|U\||_{K,\lambda} := \|U\|_{K,\lambda} + \sup_{|\alpha|\leq\lambda} \left\{ \frac{\left\|\left(R_x^\lambda U\right)^\alpha(y)\right\|_{\hat{E}}}{(\|x-y\|_K)^{\lambda-|\alpha|}} \right\} \quad (\lambda = 0, 1, \ldots)$$

where

$$\|U\|_{K,\lambda} := \sup_{|a|\leq\lambda}\|U^{a}\|_{K} \text{ and } \left(R_x^{\lambda}U\right)^{\alpha}(y)$$
$$:= U^{\alpha}(y) - \sum_{|\beta|\leq\lambda-|\alpha|}\frac{1}{\beta!}U^{\alpha+\beta}(x)(y-x)^{\beta}.$$

Let us denote by $\mathscr{E}(K)$ the space of all C^{∞} Whitney fields on K endowed with the topology $\mathbf{T_W}$ determined by the seminorms $|\|\||_{K,\lambda}$ $(\lambda = 0, 1, \ldots)$. It is a Fréchet space. By Whitney's Extension Theorem [78], $U \in \mathscr{E}(K)$ *if and only if there exists a* C^{∞} *function u in* $\mathbb{R}^n$ *such that for all* $\alpha \in \mathbb{Z}_{+}^{n}$, $D^{\alpha}(u/K) = U^{\alpha}$. In particular, if K is C^{∞} determining, the mapping

$$J : \hat{C}^{\infty}(K) \to \mathscr{E}(K) \ : u \longmapsto J(u) = \left(D^{\alpha}(v/K)\right)_{\alpha\in\mathbb{Z}_{+}^{n}}$$

where $v \in C^{\infty}(\mathbb{R}^n)$ and $v/K = u$, is a linear bijection of $\hat{C}^{\infty}(K)$ onto $\mathscr{E}(K)$. Since, for a cube π^n such that $K \subset int\pi^n$, the seminorms $|\|\||_{\pi^n,\lambda}$ and $\|U\|_{\pi^n,\lambda}$ are equivalent (*see* [78]), *the linear bijection* J *is a continuous mapping*, whence by Banach's theorem, *it is a linear isomorphism.*

Contrary to the case of C^k jets, for k finite, Whitney's proof does not yield a continuous linear operator extending jets from $\mathscr{E}(K)$ to functions in $C^{\infty}(\mathbb{R}^n)$. Moreover, such an operator does not in general exist, which is, e.g. the case when K is a single point. The problem of the existence of such an operator has a long history. Positive examples were first given by Mityagin [54] and Seeley [67] (case of a half-space in $\mathbb{R}^n$). Stein showed that *such an operator exists if* K *is the closure of a domain in* $\mathbb{R}^n$ *whose boundary is locally of class* 1 [72]. In 1978, Bierstone extended this result to the case of Lip α domains with $0 < \alpha < 1$ [13]. He also proved that *an extension operator exists if* K *is a fat* (i.e. $\overline{int\,K} \supset K$) *closed subanalytic subset of* $\mathbb{R}^n$. His method is essentially based on the famous *Hironaka Desingularization Theorem.*

All the above mentioned sets are *UPC* (whence they are Markov). It appears that some restrictions concerning cuspidality of K are necessary, since Tidten proved that *there exists a set* $K \subset\subset \mathbb{R}^2$ *which is not Markov and there is no continuous linear extension operator from* $\left(\hat{C}^{\infty}(K), \mathbf{T_Q}\right)$ *to the space* $C^{\infty}\left(\mathbb{R}^2\right)$. However, Pawlucki and Pleśniak showed that *if* K *is a Markov compact subset of* $\mathbb{R}^n$*, then one can easily construct a continuous linear operator extending* C^{∞} *functions on* K *to* C^{∞} *functions in* $\mathbb{R}^n$ [59]. In order to state this result, we give the space $\hat{C}^{\infty}(K)$ a topology connected with Jackson's theorem. To this end, let us put

$$\varrho_d(u) := \begin{cases} \|u\|_K, \ if \ d = -1 \\ \text{dist}_K(u, \mathscr{P}_0(\mathbb{C})), \ if \ d = 0 \\ \sup_{l\geq 1} l^k \ \text{dist}_K(u, \mathscr{P}_d(\mathbb{C})), \ if \ d \geq 1 \end{cases}$$

where

$$\text{dist}_K(u, \mathscr{P}_d(\mathbb{C})) := \inf\left\{\sup_{z\in K}|u(z) - P(z)| : P \in \mathscr{P}_d(\mathbb{C})\right\}.$$

By Jackson's theorem the functionals ϱ_d are seminorms on $\hat{C}^{\infty}(K)$. Let us denote by $\mathbf{T_J}$ the topology of $\hat{C}^{\infty}(K)$ determined by this family of seminorms. In general, it is not Fréchet. We are now in a position to state the following.

Theorem 52 ([62]). *Let K be a C^{∞} determining compact subset of $\mathbb{R}^n$. Then the following requirements are equivalent.*

- **i.** *K is Markov;*
- **ii.** *The space $\left(\hat{C}^{\infty}(K), \mathbf{T_J}\right)$ is complete;*
- **iii.** *The topologies $\mathbf{T_J}$ and $\mathbf{T_Q}$ for $\hat{C}^{\infty}(K)$ coincide;*
- **iv.** *There exists a continuous linear operator*

$$\mathfrak{L} : \left(\hat{C}^{\infty}(K), \mathbf{T_J}\right) \to C^i nfty\,(\mathbb{R}^n)$$

such that $(\mathfrak{L}u/K) = u$ for each $u \in \hat{C}^{\infty}(K)$.

Moreover, if K is Markov, such an operator can be defined by

$$\mathfrak{L}u = v_1 \ell_1 u + \sum_{d=1}^{\infty} v_d\left(\ell_{d+1}u - \ell_d u\right)$$

where $\ell_d u$ is a Lagrange interpolation polynomial of u of degree d and v_d are specially chosen cut-off functions.ę

By *Jackson's Theorem*, the topology $\mathbf{T_Q}$ is finer than the Jackson topology $\mathbf{T_J}$. Hence

Corollary 53. *If K is a Markov compact subset of $\mathbb{R}^n$, then the assignment*

$$\mathfrak{L}u = v_1 \ell_1 u + \sum_{d=1}^{\infty} v_d\left(\ell_{d+1}u - \ell_d u\right)$$

defines a continuous linear extension operator

$$\mathfrak{L} : \left(\hat{C}^{\infty}(K), \mathbf{T_Q}\right) \to C^{\infty}(\mathbb{R}^n)$$

8.3 Expansion of $\mathbf{C}^{\infty}$ and $\mathbf{A}^{\infty}$ Functions in Series of Orthogonal Polynomials

One consequence of Markov's property (M_2) for (K, μ) is that the vector space $C^{\infty}(K)$, of all complex-valued functions defined on a compact set K in $\mathbb{R}^n$ and admitting a C^{∞} extension on $\mathbb{R}^n$, has a Schauder basis consisting of orthogonal polynomials.

To see this, let $\nu : \mathbb{N} \to \mathbb{N}^n$ be a bijection with $|\nu(j)| \le |\nu(j+1)|$ for any j. If (K, μ) satisfies (M_2), the set $\{x^{\nu(j)} : j = 0, 1, 2, \ldots\}$ is linearly independent in $L^2(K, \mu)$ and, by the *Hilbert–Schmidt Orthogonalization Process*, one can construct a family $\{\phi_j : j = 0, 1, 2, \ldots\}$ of orthonormal polynomials in $L^2(K, \mu)$, such that $\deg\phi_j = |\nu(j)|$, $j = 0, 1, 2, \ldots$ For each $u \in L^2(K, \mu)$, we then write

$$\mathfrak{s}_j(u) := \int_K u\overline{\phi_j}\, d\mu (j = 0, 1, 2, \ldots).$$

Theorem 54 ([79]). *If* (K, μ) *satisfies* (M_2) *and* $u \in C^\infty(K)$*, then there holds*

$$u(z) = \sum_{j=0}^{\infty} \mathfrak{s}_j(u)\phi_j(z) \textit{ uniformly on } K.$$

The result of Theorem 54 generalizes to the context of a compact set $K \subset \mathbb{R}^n$ satisfying (M_∞). To prove this, we may first define a nuclear Fréchet topology on the space of polynomials $\mathbb{P}(\mathbb{R}^n) := \bigcup_{d=0}^{\infty} \mathscr{P}_d(\mathbb{R}^n)$, by introducing again (see Sect. 8.2) the topology $\mathbf{T_Q}$ described by the family of the seminorms $\varsigma_{K,\lambda}$ on $\hat{C}^\infty(K)$:

$$\sigma_{K,\lambda} = \inf\left\{\sup_{|a|\le\lambda} \|D^a g\|_K : g \in C^\infty(\mathbb{R}^n),\ g/K = u\right\}$$

($u \in \hat{C}^\infty(K)$, $K \subset\subset \mathbb{R}^n$, $\lambda \in \mathbb{N}$). By (M_∞), if $Q \in \mathbb{P}(\mathbb{C}^n)$ and $Q/K = 0$, then $Q \equiv 0$. From *Jackson's Theorem*, it therefore follows that for any $g \in C^\infty(\mathbb{R}^n)$, such that $g/K = 0$, the restriction of any derivative of g to K is equal to 0. This means that the injective restriction $C^\infty(K) \to \hat{C}^\infty(K)$ is continuous. Since $\hat{C}^\infty(K)$ is a nuclear Fréchet space, *Mityagin's Theorem* [54] guarantees the existence of a Hilbert space H such that the injections

$$C^\infty(K) \to H \to \hat{C}^\infty(K)$$

are continuous. Let now again $\nu : \mathbb{N} \to \mathbb{N}^n$, be a bijection with $|\nu(j)| \le |\nu(j+1)|$for any j. Since K satisfies (M_∞), the set $\{x^{\nu(j)} : j = 0, 1, 2, \ldots\}$ is linearly independent in H and hence, by the *Hilbert–Schmidt Orthogonalization Process*, one can find a system $\{\psi_j : j = 0, 1, 2, \ldots\} \subset H$ consisting of orthonormal polynomials with $\deg\psi_j = |\nu(j)|$ for any j. For each $u \in C^\infty(K)$, we then write

$$\mathfrak{t}_j(u) := \langle u \mid \psi_j \rangle_H (j = 0, 1, 2, \ldots).$$

Theorem 55 ([79]). *If* K *satisfies* (M_∞) *and* $u \in C^\infty(K)$*, then there holds*

$$u(z) = \sum_{j=0}^{\infty} \mathfrak{t}_j(u)\psi_j(z) \textit{ uniformly on } K.$$

8.4 Generalized Padé-Type Approximation to Continuous Functions

Let K be a compact subset of $\mathbb{C}^n$ ($K \neq \emptyset$).Suppose μ is a positive measure on K and assume that (K, μ) satisfies Markov's inequality (M_2). In this *Paragraph*, we will define generalized Padé-type approximants to continuous functions on K (*see* [24]).

As we have already seen in Theorem 54, there is a family $\{\phi_j : j = 0, 1, 2, \dots\}$ of orthonormal polynomials in $L^2(K, \mu)$, such that $\deg\phi_j \leq \deg\phi_{j+1}$ ($j = 0, 1, 2, \dots$) and every $u \in C^\infty(K)$ can be written as

$$u(z) = \sum_{j=0}^{\infty} \mathfrak{s}_j(u)\phi_j(z)$$

where $\mathfrak{s}_j(u) := \int_K u\overline{\phi_j}\, d\mu$ ($j = 0, 1, 2, \dots$) and the series converges uniformly on K.

In the sequel, we shall assume that $\{\phi_j : j = 0, 1, 2, \dots\}$ is a *self-summable family* in $L^2(K, \mu)$, i.e. for any $z \in K$, the sequence $\left\{\phi_j(z)\overline{\phi_j(z)} : j = 0, 1, 2, \dots\right\}$ is summable in $L^2(K, \mu)$. This means that for every $z \in K$ and every positive number ϵ there exists a finite set $J_0 = J_0(z, \epsilon)$ of indices such that

$$\left\| \sum_{j \in J} \phi_j \overline{\phi_j} \right\|_K := \left(\int_K \left| \sum_{j \in J} \phi_j \overline{\phi_j} \right|^2 d\mu \right)^{1/2} < \epsilon$$

whenever J is a finite set of indices disjoint from J_0. By this summability condition, for each $z \in K$ fixed, the function

$$\mathbb{K}_K^{(2)}(z, \cdot) : K \to \mathbb{C} \bigcup \{\infty\} : x \longmapsto \mathbb{K}_K^{(2)}(z, x) := \phi_j(z)\overline{\phi_j(x)}$$

is in $L^2(K, \mu)$.

Let now $u \in C^\infty(K)$. We introduce the linear functional

$$T_u^{(\mu)} : \overline{\Phi(\mathbb{C}^n)} \to \mathbb{C} : \overline{\phi_j(x)} \longmapsto T_u^{(\mu)}\left(\overline{\phi_j(x)}\right) := \mathfrak{s}_j(u),$$

where $\overline{\Phi(\mathbb{C}^n)}$ is the complex vector space which is spanned by all finite complex combinations of $\overline{\Phi_j}$'s. If $P(x) = \sum_{\nu=0}^{d} \beta_\nu \overline{\phi_\nu(x)} \in \overline{\Phi(\mathbb{C}^n)}$, then, from *Hölder's Inequality*, it follows that $\left| T_u^{(\mu)}(P(x)) \right| \leq \|u\|_K \|P\|_K$, and, by the Hahn–Banach Theorem, $T_u^{(\mu)}$ extends to a linear continuous functional on $L^2(K, \mu)$. For each $z \in K$ fixed, one can therefore define the number $T_u^{(\mu)}\left(\mathbb{K}_K^{(2)}(z, x)\right)$, where $T_u^{(\mu)}$

acts on the variable $x \in K$. Furthermore, by continuity, computing $u(z)$ for a fixed value of $z \in K$ is nothing else than computing $T_u^{(\mu)}\left(\mathbb{K}_K^{(2)}(z,x)\right)$.

If only a few Fourier coefficients $\tilde{c}_j(u)$ of u are known, then the function $\mathbb{K}_K^{(2)}(z,x)$ has to be replaced by a simpler expression. To do so, let us consider the $(m+1)$-dimensional complex vector space $\overline{\mathfrak{F}_{m+1}}$ spanned by the Tchebycheff system $\left\{\overline{\phi_0}, \overline{\phi_1}, \ldots, \overline{\phi_m}\right\}$, and suppose that $\overline{\mathfrak{F}_{m+1}}$ satisfies the *Haar condition* into a finite set of pair-wise distinct points $\mathscr{M}_{m+1} = \{\pi_{m,0}, \pi_{m,1}, \ldots, \pi_{m,m}\} \subset K$ with $\mathscr{M}_{m+1} \bigcap \left(\bigcup_{0 \le j \le m} \mathrm{Ker}\overline{\phi_j}\right) = \emptyset$. For any $z \in K$ there is a unique $g_m(x,z) = \sum_{j=0}^m \sigma_j^{(m)}(z)\overline{\phi_j(x)} \in \overline{\mathfrak{F}_{m+1}}$ satisfying $g_m(x,\pi_{m,k}) = \mathbb{K}_K^{(2)}(z,\pi_{m,k})$ for any $k \le m$. A necessary and sufficient condition for the existence of a unique solution

$$\left(\sigma_0^{(m)}(z), \sigma_1^{(m)}(z), \ldots, \sigma_m^{(m)}(z)\right)$$

for this linear system is that the determinant $\det\left[\overline{\phi_j(\pi_{m,k})}\right]_{k,j}$ is different from zero. Notice that this condition is equivalent to the Haar condition for $\overline{\mathfrak{F}_{m+1}}$ into the set $\mathscr{M}_{m+1}$. Then, for any $j = 0, 1, 2, \ldots, m$, there holds

$$\sigma_j^{(m)}(z) = \sum_{k=0}^m \frac{\mathbb{K}_K^{(2)}(z,\pi_{m,k})}{\overline{\phi_j(\pi_{m,k})}} \quad (Q_j)$$

Definition 56. Let $K \subset\subset \mathbb{C}^n$ and let μ be a positive measure of K, such that (K,μ) satisfies *Markov's inequality* (M_2). Assume that $\left\{\phi_j : j = 0, 1, 2, \ldots\right\}$ is a self-summable family, consisting of orthonormal polynomials in $L^2(K,\mu)$ such that $\deg\phi_j \le \deg\phi_{j+1}$ $(j = 0, 1, 2, \ldots)$. For $m \ge 0$, choose a finite set of pair-wise distinct points

$$\mathscr{M}_{m+1} = \{\pi_{m,0}, \pi_{m,1}, \ldots, \pi_{m,m}\} \subset K \backslash \left(\bigcup_{0 \le j \le m} \mathrm{Ker}\overline{\phi_j}\right)$$

so that $\det\left[\overline{\phi_j(\pi_{m,k})}\right]_{k,j} \neq 0$ and for any $k \le m$ the series $\sum_{j=0}^\infty \phi_j(\cdot)\overline{\phi_j(\pi_{m,k})}$ converges uniformly on K. Any function $(\mathrm{GPTA}/m)_u^{(\mu)}(z)$, defined by

$$T_u^{(\mu)}(g_m(x,\cdot)) : K \to \mathbb{C} : z \longmapsto (\mathrm{GPTA}/m)_u^{(\mu)}(z) := T_u^{(\mu)}(g_m(x,z))$$

is called a **generalized Padé-type approximant** to $u \in C^\infty(K)$, with generating system $\mathscr{M}_{m+1}$. If, moreover

$$\sum_{j=0 (j \neq \nu)}^m \mathfrak{s}_j(u) \sum_{k=0}^m \frac{\overline{\phi_\nu(\pi_{m,k})}}{\overline{\phi_j(\pi_{m,k})}} = 0 \text{ for every } \nu = 0, 1, 2, \ldots, m,$$

then the function $T_u^{(\mu)}(g_m(x,\cdot))$ is said to be a **Padé-type approximant** to u, with generating system $\mathscr{M}_{m+1}$. It is denoted by $(\mathrm{PTA}/m)_u^{(\mu)}(z)$.

The uniform convergence of the series $\sum_{j=0}^{\infty}\phi_j(\cdot)\overline{\phi_j(\pi_{m,k})}$ guarantees that

$$\mathbb{K}_K^{(2)}(\cdot,\pi_{m,k}) \in C^{\infty}(K)\,(k=0,1,2,\ldots,m).$$

Hence, the generalized Padé-type approximant $T_u^{(\mu)}(g_m(x,\cdot))$ is a continuous function on K. Notice that the computation of a generalized Padé-type approximant $T_u^{(\mu)}(g_m(x,\cdot))$ requires only the knowledge of the Fourier coefficients $\mathfrak{s}_0(u)$, $\mathfrak{s}_1(u),\ldots,\mathfrak{s}_m(u)$ of u and of the functions $\sigma_0^{(m)}(z)$, $\sigma_1^{(m)}(z)$, …, $\sigma_m^{(m)}(z)$ resulting from (Q_0), $(Q_1),\ldots,(Q_m)$.

Under the assumptions of *Definition 56*, we have the following result which justifies the notation Padé-type approximant.

Theorem 57 ([24]). *If $\sum_{\nu=0}^{\infty}\beta_\nu^{(m,u)}\phi_\nu(z)$ is the Fourier expansion of a Padé-type approximant $(PTA/m)_u^{(\mu)}(z)$ to $u(z)=\sum_{\nu=0}^{\infty}\mathfrak{s}_\nu(u)\phi_\nu(z)\in C^{\infty}(K)$ with respect to the family $\{\phi_\nu : \nu=0,1,2,\ldots\}$, then*

$$\beta_\nu^{(m,u)} = \mathfrak{s}_\nu(u)\,,\ \textit{for every}\ \nu=0,1,2,\ldots,m$$

Theorem 58 ([24]). *The error of a generalized Padé-type approximation equals*

$$T_u^{(\mu)}(g_m(x,z)) - u(z) = \sum_{\nu=0}^{\infty}\ \sum_{j=0(j\neq\nu)}^{m}\left[\mathfrak{s}_j(u)\sum_{k=0}^{m}\frac{\overline{\phi_\nu(\pi_{m,k})}}{\overline{\phi_j(\pi_{m,k})}}\right]\phi_\nu(z)(z\in K).$$

The error of a Padé-type approximation is

$$T_u^{(\mu)}(g_m(x,z)) - u(z) = \sum_{\nu=m+1}^{\infty}\ \sum_{j=0(j\neq\nu)}^{m}\left[\mathfrak{s}_j(u)\sum_{k=0}^{m}\frac{\overline{\phi_\nu(\pi_{m,k})}}{\overline{\phi_j(\pi_{m,k})}}\right]\phi_\nu(z)(z\in K).$$

Let us now give integral representations for the generalized Padé-type approximants. For $u\in C^{\infty}(K)$, the corresponding linear functional $T_u^{(\mu)}$ extends continuously and linearly onto the Hilbert space $L^2(K,\mu)$. By *Riez's Representation Theorem*, there exists a unique element $U\in L^2(K,\mu)$ satisfying

$$T_u^{(\mu)}(g) = \int_K g\,\overline{U}\,d\mu, \text{whenever } g\in L^2(K,\mu)\,.$$

For $g=\overline{\phi_\nu}$, we therefore obtain $T_u^{(\mu)}\left(\overline{\phi_\nu}\right)=\int_K\overline{\phi_\nu}\,\overline{U}\,d\mu=\tilde{c}_\nu(u)=\int_K u\,\overline{\phi_\nu}\,d\mu$ and, consequently

$$\int_K \left(u - \overline{U}\right) \overline{\phi_\nu}\, d\mu = 0, \text{for any } \nu = 0, 1, 2, \ldots$$

Theorem 59 ([24]). *If the family* $\{\phi_\nu : \nu = 0, 1, 2, \ldots\}$ *is complete in* $L^2(K, \mu)$*, then* the following result holds.

i. $T_u^{(\mu)}(g) = \int_K g\, u\, d\mu\ (g \in L^2(K, \mu))$.

ii. *Each generalized Padé-type approximant* $T_u^{(\mu)}(g_m(x, z))$ *to* $u \in C^\infty(K)$ *has the integral representation*

$$T_u^{(\mu)}(g_m(x, z)) = \int_K u(x)\, D_m(x, z)\, d\mu(x),$$

where $D_m(x, z)$ *is the kernel*

$$\sum_{k=0}^{m} \mathbb{K}_K^{(2)}(z, \pi_{m,k}) \sum_{j=0}^{m} \frac{\overline{\phi_j(x)}}{\overline{\phi_j(\pi_{m,k})}}.$$

Since $\mathbb{K}_K^{(2)}(\cdot, \pi_{m,k}) = \overline{\mathbb{K}_K^{(2)}(\pi_{m,k}, \cdot)} \in L^2(K, \mu)$, we also have $\int_K g(x)\ D_m(x, \cdot)\ d\mu(x) \in L^2(K, \mu)$ for any $g \in L^2(K, \mu)$. From the *Closed Graph Theorem*, it follows that the integral operator

$$S_\mu^{(m)} :\ L^2(K, \mu) \to L^2(K, \mu) : g(\cdot) \longmapsto \int_K g(x)\ D_m(x, \cdot)\ d\mu(x)$$

is continuous. Further, by *Fubini's Theorem*, its adjoint operator is given by

$$S_\mu^{(m)*} :\ L^2(K, \mu) \to L^2(K, \mu) : g(\cdot) \longmapsto \left(S_\mu^{(m)*}\right)(g)$$
$$= \int_K g(x)\ \overline{D_m(x, \cdot)}\, d\mu(x).$$

Definition 60. The restriction of $S_\mu^{(m)}$ to $C^\infty(K)$ is called a **generalized Padé-type** operator for $C^\infty(K)$. We denote $T_u^{(\mu)}(g_m(x, \cdot)) := S_\mu^{(m)}/C^\infty(K)$.

The continuity property of the operator

$$T^{(\mu)}(g_m(x, \cdot)) :\ C^\infty(K) \to C^\infty(K) : u(\cdot) \mapsto T_u^{(\mu)}(g_m(x, \cdot))$$
$$:= \int_K u(x)\, D_m(x, z)\, d\mu(x)$$

is a useful tool for the study of convergence and in this connection we have the following result.

Theorem 61 ([24]). *If the sequence* $\{u_\nu \in C^\infty(K) : \nu = 0, 1, 2, \dots\}$ *converges to* $u \in C^\infty(K)$ *with respect to the* L^2*-norm of* $L^2(K, \mu)$*, then* $\lim_{\nu\to\infty} T_{u_\nu}^{(\mu)}(g_m(x,\cdot)) = T_u^{(\mu)}(g_m(x,\cdot))$ *in* $L^2(K,\mu)$*.*

Corollary 62 ([24]). *If the series of functions* $\sum_{\nu=0}^{\infty} a_\nu u_\nu(z)$ $(a_\nu \in \mathbb{C},\ u_\nu \in C^\infty(K))$ *converges to* $u \in C^\infty(K)$ *with respect to the* L^2*-norm of* $L^2(K,\mu)$*, then* $T_u^{(\mu)}(g_m(x,\cdot)) = \sum_{\nu=0}^{\infty} a_\nu T_{u_\nu}^{(\mu)}(g_m(x,\cdot))$ *in* $L^2(K,\mu)$*.*

Until now, we have supposed that the compact set K satisfies Markov's inequality (M_2) with respect to some positive measure μ on K. We will now turn to the case where E fulfils Markov's property (M_∞). As it is pointed out in Theorem 55 of Sect. 8.3, if $K \subset\subset \mathbb{R}^n$ verifies (M_∞), then there is a Hilbert space $(H, \langle\cdot \mid \cdot\rangle_H)$ and an orthonormal system $\{\psi_j : \deg\psi_j \le \deg\psi_{j+1},\ j = 0, 1, 2, \dots\}$ in H, such that the injections

$$C^\infty(K) \to (H, \langle\cdot \mid \cdot\rangle_H) \text{ and } (H, \langle\cdot \mid \cdot\rangle_H) \to \hat{C}^\infty(K)$$

are continuous and each function $u \in C^\infty(K)$ has the Fourier expansion

$$u(z) = \sum_{j=0}^{\infty} \mathfrak{t}_j(u)\psi_j(z) \text{ uniformly on } K$$

where $\tilde{\sigma}_j(u) = \langle u \mid \psi_j\rangle_H$ $(j = 0, 1, 2, \dots)$ and where the series converges uniformly on H.

As for the (M_2)-case, we shall assume that

$$\{\psi_j : j = 0, 1, 2, \dots\}$$

is a self-summable family in $(H, \langle\cdot \mid \cdot\rangle_H)$, i.e. *for any* $z \in K$, *the sequence* $\{\psi_j(z)\overline{\psi_j} : j = 0, 1, 2, \dots\}$ *is summable in* $(H, \langle\cdot \mid \cdot\rangle_H)$, in the sense that for every $z \in K$ and every $\epsilon > 0$ there exists a finite set $J_0 = J_0(z,\epsilon)$ of indices with $\left\|\sum_{j\in J}\psi_j(z)\overline{\psi_j}\right\|_H < \epsilon$ whenever J is a finite set of indices disjoint from J_0. This summability condition implies that for each $z \in K$ fixed, the function

$$\mathbb{K}_K^{(\infty)}(z,\cdot) : K \to \mathbb{C} : x \longmapsto \mathbb{K}_K^{(\infty)}(z,x) := \sum_{j=0}^{\infty}\psi_j(z)\overline{\psi_j(x)}$$

belongs to H. Note that, for any $z \in K$ fixed, $\mathbb{K}_K^{(\infty)}(z,\cdot) \in C^\infty(K)$. Further, by the continuity of the injective map $(H, \langle\cdot \mid \cdot\rangle_H) \to \hat{C}^\infty(K)$, the series $\sum_{j=0}^{\infty}\psi_j(z)\overline{\psi_j(x)}$ converges uniformly on K to $\mathbb{K}_K^{(\infty)}(z,\cdot)$.

Let $u \in C^\infty(K)$. Define the linear functional

$$T_u : \overline{\psi(\mathbb{C}^n)} \to \mathbb{C} : \overline{\psi_j(x)} \longmapsto T_u\left(\overline{\psi_j(x)}\right) := \mathfrak{t}_j(u),$$

where $\overline{\psi(\mathbb{C}^n)}$ is the complex subspace of H, which is generated by all finite combinations of $\overline{\psi_j}$'s. If $(x) = \sum_{\nu=0}^{d} \beta_\nu \overline{\psi_\nu(x)} \in \overline{\psi(\mathbb{C}^n)}$, then from *Schwarz's Inequality*, it follows that $|T_u(P(x))| \leq \|u\|_H \|P\|_H$ and, by the Hahn–Banach Theorem, T_u extends to a continuous linear functional on H. For each $z \in K$ fixed, one can therefore define the number $T_u\left(\mathbb{K}_K^{(\infty)}(z,x)\right)$, where T_u acts on the variable $x \in K$. By continuity, *computing* $u(z)$ *for a fixed value of* z *is nothing else than computing* $T_u\left(\mathbb{K}_K^{(\infty)}(z,x)\right)$.

If only a few Fourier coefficients $\mathfrak{t}_j(u)$ of u are known or if the Fourier series expansion of u (with respect to the family $\{\psi_j : j \geq 0\}$) converges too slowly, then the function $\mathbb{K}_K^{(\infty)}(z,x)$ has to be replaced by a simpler expression.

To do so, for any $m = 0, 1, 2, \ldots$, let us consider the $(m+1)$-dimensional complex vector space $\overline{\mathfrak{G}_{m+1}}$ spanned by the Tchebycheff system $\{\overline{\psi_0}, \overline{\psi_1}, \ldots, \overline{\psi_m}\}$, and suppose that $\overline{\mathfrak{G}_{m+1}}$ satisfies the *Haar condition* into a finite set of pair-wise distinct points $\mathscr{M}_{m+1} = \{\pi_{m,0}, \pi_{m,1}, \ldots, \pi_{m,m}\} \subset K \setminus \left(\bigcup_{0 \leq j \leq m} \mathrm{Ker}\overline{\psi_j}\right)$, that is $\det\left[\overline{\psi_j(\pi_{m,k})}\right]_{k,j} \neq 0$. This is equivalent to the fact that for any $z \in K$ there is a unique $g_m(x,z) = \sum_{j=0}^{\infty} \tau_j^{(m)}(z)\overline{\psi_j(x)} \in \overline{\mathfrak{G}_{m+1}}$ satisfying $g_m(x, \pi_{m,k}) = \mathbb{K}_K^{(2)}(z, \pi_{m,k})$ for any $k \leq m$.

A necessary and sufficient condition for the existence of a unique solution $\left(\tau_0^{(m)}(z), \tau_1^{(m)}(z), \ldots, \tau_m^{(m)}(z)\right)$ for this linear system is that the determinant

$$\det\left[\overline{\psi_j(\pi_{m,k})}\right]_{k,j}$$

is different from zero. Notice that this condition is equivalent to the Haar condition for $\overline{\mathfrak{G}_{m+1}}$ into the set $\mathscr{M}_{m+1}$. Then, for any $j = 0, 1, 2, \ldots, m$ there holds

$$\tau_j^{(m)}(z) = \sum_{k=0}^{m} \frac{\mathbb{K}_K^{(\infty)}(z, \pi_{m,k})}{\overline{\psi_j(\pi_{m,k})}} \quad (P_j)$$

Definition 63. Any function $(\mathrm{GPTA}/m)_u(z)$, defined by

$$T_u(g_m(x,\cdot)) : K \to \mathbb{C} : z \longmapsto (\mathrm{GPTA}/m)_u(z) := T_u(g_m(x,z)) = \sum_{j=0}^{m} \mathfrak{t}_j(u)\, \tau_j^{(m)}(z)$$

is called a **generalized Padé-type approximant** *to* $u \in C^\infty(K)$, with generating system $\mathscr{M}_{m+1}$. If

$$\sum_{j=0(j\neq\nu)}^{m} \mathfrak{t}_j(u) \sum_{k=0}^{m} \frac{\overline{\psi_\nu(\pi_{m,k})}}{\overline{\psi_j(\pi_{m,k})}} = 0 \text{ for every } \nu = 0, 1, 2, \ldots, m,$$

then the function $T_u\left(g_m\left(x,\cdot\right)\right)$ is said to be a **Padé-type approximant** to u, with generating system $\mathscr{M}_{m+1}$.It is denoted by $\left(PTA/m\right)_u\left(z\right)$.

Obviously, the computation of a generalized Padé-type approximant $T_u\left(g_m\left(x,\cdot\right)\right)$ requires only the knowledge of the Fourier coefficients $\mathfrak{t}_0\left(u\right)$, $\mathfrak{t}_1\left(u\right)$, ..., $\mathfrak{t}_m\left(u\right)$ and of the functions $\tau_0^{(m)}\left(z\right)$, $\tau_1^{(m)}\left(z\right), \ldots, \tau_m^{(m)}\left(z\right)$ resulting from $\left(P_0\right)$, $\left(P_1\right), \ldots, \left(P_m\right)$ respectively.

Theorem 64 ([24]). *If $\sum_{\nu=0}^{\infty}\xi_\nu^{(m,u)}\psi_\nu\left(z\right)$ is the Fourier expansion of a Padé-type approximant $(\mathrm{PTA}/m)_u^{(u)}\left(z\right)$ to $u\left(z\right)=\sum_{\nu=0}^{\infty}\mathfrak{t}_\nu\left(u\right)\psi_\nu\left(z\right)\in C^\infty\left(K\right)$ with respect to the family $\{\psi_\nu:\nu=0,1,2,\ldots\}$, then*

$$\xi_\nu^{(m,u)}=\mathfrak{t}_\nu\left(u\right),\ \textit{for every}\ \nu=0,1,2,\ldots,m.$$

Theorem 65 ([24]). *The error of a generalized Padé-type approximation equals*

$$T_u\left(g_m\left(x,z\right)\right)-u\left(z\right)=\sum_{\nu=0}^{\infty}\sum_{j=0(j\neq\nu)}^{m}\left[\mathfrak{t}_j\left(u\right)\sum_{k=0}^{m}\frac{\overline{\psi_\nu\left(\pi_{m,k}\right)}}{\overline{\psi_j\left(\pi_{m,k}\right)}}-\tau_\nu\left(u\right)\right]\psi_\nu\left(z\right)(z\in K).$$

The error of a Padé-type approximation is

$$T_u\left(g_m\left(x,z\right)\right)-u\left(z\right)=\sum_{\nu=m+1}^{\infty}\sum_{j=0(j\neq\nu)}^{m}\left[\mathfrak{t}_j\left(u\right)\sum_{k=0}^{m}\frac{\overline{\psi_\nu\left(\pi_{m,k}\right)}}{\overline{\psi_j\left(\pi_{m,k}\right)}}-\tau_\nu\left(u\right)\right]\psi_\nu\left(z\right)(z\in K).$$

We can immediately obtain an answer to the convergence problem of a generalized Padé-type approximation sequence.

Theorem 66 ([24]). *Let K be a compact subset of $\mathbb{R}^n$ satisfying Markov's inequality (M_∞) and let $u\in C^\infty\left(K\right)$. Consider the intermediate Hilbert space $\left(H,\ \langle\cdot\mid\cdot\rangle_H\right)$, for which the natural injections $C^\infty\left(K\right)\to H\to\hat{C}^\infty\left(K\right)$ are continuous. Suppose $\left\{\psi_j\left(z\right):j=0,1,2,\ldots\right\}$ is a self-summable family consisting of orthonormal polynomials in H, such that $\deg\psi_j\leq\deg\psi_{j+1}$ $(j=0,1,2,\ldots)$ and assume that the function*

$$\mathbb{K}_K^{(\infty)}\left(\cdot,\cdot\right):K\to\mathbb{R}:z\mapsto\mathbb{K}_K^{(\infty)}\left(z,z\right)=\sum_{j=0}^{\infty}\left|\psi_j\left(z\right)\right|^2$$

is continuous on K. Let also $\mathscr{M}=\left(\pi_{m,k}\right)_{m\geq0,0\leq k\leq m}$ be an infinite triangular matrix, such that for any $m\geq0$

$\pi_{m,k}\neq\pi_{m,k'}$ if $k\neq k'$, $\pi_{m,k}\notin\left(\bigcup_{0\leq j\leq m}\mathrm{Ker}\overline{\psi_j}\right)$ $(k\leq m)$ and $\det\left[\overline{\psi_j\left(\pi_{m,k}\right)}\right]_{k,j}\neq0$.

If

$$\lim_{m\to\infty}\left\{\sum_{\nu=0}^{\infty}\left\|\sum_{j=0}^{m}\frac{\psi_\nu\left(\pi_{m,k}\right)}{\psi_j\left(\pi_{m,k}\right)}-\psi_\nu\right\|_H^2\right\}=0,$$

then, the corresponding generalized Padé-type approximation sequence

$$\left(T_u\left(g_m\left(x,z\right)\right)\right)_{m=0,1,2,\ldots}$$

converges to $u\left(z\right)$ *uniformly on* K.

8.5 Markov Exponents

If K is a Markov compact subset of $\mathbb{R}^n$ and $u : K \to \mathbb{C}$ admits rapid uniform approximation by polynomials on K then u extends to a C^∞ function in $\mathbb{R}^n$. In general, the extension is done at the cost of u's regularity. It is seen by the following.

Example 67 ([63]). Let $F_p = \left\{(x,y)\in\mathbb{R}^2 : x^p \le y \le 1,\ 0 < x \le 1\right\}$ and $F = \left\{(x,y)\in\mathbb{R}^2 : 0 \le x \le 1,\ -1 \le y \le 0\right\}$. Let also $u\left(x,\ y\right) = \exp(-1/x)$, if $\left(x,\ y\right) \in F_p$ and $u\left(x,\ y\right) = 0$, if $\left(x,\ y\right) \in F$. Then u is C^∞ in $intE_p$ where $E_p = F_p \bigcup F$ and all derivatives of f extend continuously to $\overline{E_p}$. Moreover, they admit the following *Gevrey* type estimates:

$$\left\|D^\alpha u\right\|_{E_p} \le C^{|\alpha|}|\alpha|^{2|\alpha|} \text{ for } \alpha\in\mathbb{Z}_+^2.$$

Since the set $\overline{E_p}$ is p-regular in the sense of Whitney, u can be extended to a C^∞ function v on $\mathbb{R}^2$ [14]. However, if $p \ge 2$, there is no open neighbourhood $\mathscr{V}$ of $\overline{E_p}$ such that the extension v could satisfy the above estimates with exponent 2 in $\mathscr{V}$, which can be easily seen by the Mean Value Theorem.

It was shown that if we know the constant r of (M_n) (see Sect. 3) then we can estimate the loss of regularity of a C^∞ extension of u [63]. This motivates the following definition of ***Markov's exponent*** of a compact set K in $\mathbb{R}^n$:

$$\mathrm{r}\left(K\right) := \inf\left\{r > 0 : K \text{ satisfies } \left(\mathrm{M}_n\right) \text{ with exponent } r\right\}$$

If K is not a Markov set, we set $\mathrm{r}\left(K\right) := \infty$. By the fact that the Chebyshev polynomials are best possible for (M_1), one can prove that if K is a compact set in $\mathbb{R}^n$ then $\mathrm{r}\left(K\right) \ge 2$. In particular, if K is a fat, convex compact subset of $\mathbb{R}^n$, then by a standard argument based on inequality (M_1), we obtain $\mathrm{r}\left(K\right) = 2$. If K is a *UPC* compact subset of $\mathbb{R}^n$ with parameter m, then by $\mathrm{r}\left(K\right) = 2m$ [2].

It appears that Markov's exponent is invariant under "good" analytic mappings. More precisely, we have the following.

Theorem 68 ([3]). *If K is a compact subset of $\mathbb{R}^n$ satisfying* (M_n) *with an exponent r, and u is an analytic mapping defined in a neighbourhood $\mathscr{U}$ of K, with values in $\mathbb{R}^n$, such that f $u(K)$ is not pluripolar (in $\mathbb{C}^n$) and $det\, d_x u \neq 0$ for each $x \in K$, then $u(K)$ also satisfies* (M_n) *with the same exponent r as that of K.*

This result is sharp in the sense that if the assumption $det\, d_x u \neq 0$ is not satisfied for all $x \in K$ then the exponent $\mathrm{r}(u(K))$ may increase [3]. Moreover, if we knew that Markov's property of K implies that K is not pluripolar, we could remove in the above theorem the assumption for $u(K)$ to be not pluripolar.

8.6 Characterization of Compact Subsets of Algebraic Varieties

Let me finally mention a beautiful result according to which the tangential Markov inequality with exponent 1 characterizes the property of a compact subset K of $\mathbb{R}^n$ to be a piece of an algebraic variety. More precisely, we have the following.

Theorem 69 ([5]). *If Let K be a compact subset of $\mathbb{R}^n$ admitting an analytic parameterization of order m $(1 \leq m \leq n)$. Then the Zariski dimension of K is m if and only if there exists a constant $C > 0$ such that*

$$\sup_{x\in K} |D_T P(x)| \leq C \cdot d \cdot \sup_{x\in K} |P(x)|$$

for every $x \in K$ and for every polynomial $P \in \mathscr{P}_d(\mathbb{R}^n)$, where $D_T P$ is a (unitary) tangential derivative of P.

Notice that if K is a smooth compact subvariety of $\mathbb{R}^n$ of dimension M, $1 \leq M \leq N$, the above theorem was earlier proved in 1995 by Bos, Levenberg, Milman and Taylor [22].

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The Number of Prime Factors Function on Shifted Primes and Normal Numbers

Jean-Marie De Koninck and Imre Kátai

Abstract In a series of papers, we have constructed large families of normal numbers using the concatenation of the values of the largest prime factor $P(n)$, as n runs through particular sequences of positive integers. A similar approach using the smallest prime factor function also allowed for the construction of normal numbers. Letting $\omega(n)$ stand for the number of distinct prime factors of the positive integer n, we show that the concatenation of the successive values of $|\omega(n) - \lfloor \log\log n \rfloor|$, as n runs through the integers $n \geq 3$, yields a normal number in any given basis $q \geq 2$. We show that the same result holds if we consider the concatenation of the successive values of $|\omega(p+1) - \lfloor \log\log(p+1) \rfloor|$, as p runs through the prime numbers.

Keywords Normal numbers • Number of distinct prime factors

Mathematics Subject Classification (2010): 11K16, 11N37, 11A41

1 Introduction

Given an integer $q \geq 2$, we say that an irrational number η is a *q-normal number* if the q-ary expansion of η is such that any preassigned sequence of length $k \geq 1$, taken within this expansion, occurs with the expected limiting frequency, namely $1/q^k$.

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In a series of papers, we have constructed large families of normal numbers using the distribution of the values of $P(n)$, the largest prime factor function (see [1–3] and [4]). Recently [5], we showed how the concatenation of the successive values of the smallest prime factor $p(n)$, as n runs through the positive integers, can also yield a normal number.

Let $\omega(n)$ stand for the number of distinct prime factors of the positive integer n. One can easily show that the concatenation of the successive values of $\omega(n)$, say by considering the real number $\xi := 0.\overline{\omega(2)}\,\overline{\omega(3)}\,\overline{\omega(4)}\,\overline{\omega(5)}\ldots$, where each $\overline{m}$ stands for the q-ary expansion of the integer m, will not yield a normal number. Indeed, since the interval $I := [e^{e^{r-1}}, e^{e^r}]$, where $r := \lfloor \log\log x \rfloor$, covers most of the interval $[1, x]$ and since $\left|\dfrac{\omega(n)}{r} - 1\right| < \dfrac{1}{r^{1/4}}$, say, with the exception of a small number of integers $n \in I$, it follows that ξ cannot be normal in basis q.

Recently, Vandehay [9] used another approach to yet create normal numbers using certain small additive functions. He considered irrational numbers formed by concatenating some of the base q digits from additive functions $f(n)$ that closely resemble the prime counting function $\Omega(n) := \sum_{p^\alpha \| n} \alpha$. More precisely, he used the concatenation of the last $\lceil y \frac{\log\log\log n}{\log q} \rceil$ digits of each $f(n)$ in succession and proved that the number thus created turns out to be normal in basis q if and only if $0 < y \le 1/2$.

In this paper, we show that the concatenation of the successive values of $|\omega(n) - \lfloor \log\log n \rfloor|$, as n runs through the integers $n \ge 3$, yields a normal number in any given basis $q \ge 2$. We show that the same result holds if we consider the concatenation of the successive values of $|\omega(p+1) - \lfloor \log\log(p+1) \rfloor|$, as p runs through the prime numbers.

2 Notation

Let $\wp$ stand for the set of all the prime numbers. The letter p, with or without subscript, will always denote a prime number. The letter c, with or without subscript, will always denote a positive constant, but not necessarily the same at each occurrence.

At times, we will use the notation $x_1 = \log x$, $x_2 = \log\log x$, $x_3 = \log\log\log x$.

Let $q \ge 2$ be a fixed integer. Given an integer $t \ge 1$, an expression of the form $i_1 i_2 \ldots i_t$, where each i_j is one of the numbers $0, 1, \ldots, q-1$, is called a *word* of length t. Given a word α, we shall write $\lambda(\alpha) = t$ to indicate that α is a *word* of length t. We shall also use the symbol Λ to denote the *empty word*.

Let $A = A_q = \{0, 1, 2, \ldots, q-1\}$. Then, A^t will stand for the set of words of length t over A, while A^* will stand for the set of all words over A regardless of their length, including the empty word Λ. Observe that the concatenation of two words $\alpha, \beta \in A^*$, written $\alpha\beta$, also belongs to A^*. Finally, given a word α and a subword β of α, we will denote by $F_\beta(\alpha)$ the number of occurrences of β in α, that is, the number of pairs of words μ_1, μ_2 such that $\mu_1 \beta \mu_2 = \alpha$.

Given a positive integer n, we write its q-ary expansion as

$$n = \varepsilon_0(n) + \varepsilon_1(n)q + \cdots + \varepsilon_t(n)q^t,$$

where $\varepsilon_i(n) \in A$ for $0 \le i \le t$ and $\varepsilon_t(n) \neq 0$. To this representation, we associate the word

$$\overline{n} = \varepsilon_0(n)\varepsilon_1(n)\ldots\varepsilon_t(n) \in A^{t+1}$$

For convenience, if $n \le 0$, we write $\overline{n} = \Lambda$.

Finally, the number of digits of such a number $\overline{n}$ will be

$$\lambda(\overline{n}) = \left\lfloor \frac{\log n}{\log q} \right\rfloor + 1.$$

Finally, given a sequence of integers $a(1), a(2), a(3), \ldots$, we will say that the concatenation of their q-ary digit expansions $\overline{a(1)}\,\overline{a(2)}\,\overline{a(3)}\ldots$, denoted by Concat$(\overline{a(n)} : n \in \mathbb{N})$, is a *normal sequence* if the number $0.\overline{a(1)}\,\overline{a(2)}\,\overline{a(3)}\ldots$ is a q-normal number.

For each integer $n \ge 2$, we let $\omega(n)$ stand for the number of distinct prime factors of n. We then introduce the arithmetic function $\delta(n) := |\omega(n) - \lfloor \log\log n \rfloor|$.

3 Main Results

Theorem 1. *Let $R \in \mathbb{Z}[x]$ be a polynomial such that $R(y) \ge 0$ for all $y \ge 0$. Let*

$$\eta = Concat(\overline{R(\delta(n))} : n = 3, 4, 5, \ldots).$$

Then, η is a normal sequence in any given basis $q \ge 2$.

Theorem 2. *Let*

$$\xi = Concat(\overline{\delta(p+1)} : p \in \wp).$$

Then, ξ is a normal sequence in any given basis $q \ge 2$.

Remark 1. We shall only provide the proof of Theorem 2, the reason being that it is somewhat harder than that of Theorem 1. Indeed, for the proof of Theorem 1, one can use the fact that

$$\pi_k(n) := \#\{\le x : \omega(n) = k\} = (1 + o(1))\frac{x}{x_1}\frac{x_2^{k-1}}{(k-1)!}$$

uniformly for $|k - x_2| \le \sqrt{x_2}\,x_3$, say, and also the Hardy–Ramanujan inequality

$$\pi_k(x) < c_1 \frac{x}{x_1} \frac{(x_2 + c_2)^{k-1}}{(k-1)!}$$

which is valid uniformly for $1 \le k \le 10x_2$ and $x \ge x_0$ (see, for instance, the book of De Koninck and Luca [6], p. 157). Hence, using these estimates, one can easily prove Theorem 1 essentially as we did to prove that Concat$(P(m) : m \in \mathbb{N})$ is a normal sequence in any given basis $q \ge 2$ (see [1]). Now, since there are no known estimate for the asymptotic behavior of $\#\{p \le x : \omega(p+1) = k\}$, we need to find another approach for the proof of Theorem 2.

Remark 2. It will be clear from our approach that if $\omega(n)$ is replaced by $\Omega(n)$ or by $\delta_2(n) := |\lfloor \log \tau(n) \rfloor - \lfloor \log\log n \rfloor|$, the same results hold.

4 Preliminary Results

For each real number $u > 0$, let $\Phi(u) := \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{u} e^{-t^2/2}\,dt$.

Lemma 1. *(a) As $x \to \infty$,*

$$\frac{1}{\pi(x)} \# \left\{ p \le x : \frac{\delta(p+1)}{\sqrt{x_2}} < u \right\} = (1 + o(1))\,(\Phi(u) - \Phi(-u))\,.$$

(b) Letting ε_x a function which tends to 0 as $x \to \infty$. Then, as $x \to \infty$,

$$\frac{1}{\pi(x)} \# \left\{ p \le x : \delta(p+1) \le \varepsilon_x \sqrt{\log\log x} \right\} \to 0.$$

Proof. For a proof of part (a), see the book of Elliott [7]. Part (b) is an immediate consequence of part (a).

Let x be a fixed large number. For each integer $n \ge 2$, we now introduce the function

$$\delta^*(n) := |\omega(n) - \lfloor \log\log x \rfloor|\,.$$

Lemma 2. *For all $x \ge 2$,*

$$\sum_{p \le x} (\delta^*(p+1))^2 \le c\pi(x)\, \log\log x.$$

Proof. For a proof, see the book of Elliott [7].

Lemma 3. *Given an arbitrary* $\kappa \in (0, 1/2)$*, then, for all* $x \geq 2$*,*

$$\#\{p \leq x : P(p+1) < x^{\kappa}\} + \#\{p \leq x : P(p+1) > x^{1-\kappa}\} \leq c\kappa\pi(x).$$

Proof. For a proof see Theorem 4.2 in the book of Halberstam and Richert [8].

Lemma 4. *Let a and b be two nonzero co-prime integers, one of which is even. Then, as* $x \to \infty$*, we have, uniformly in a and b,*

$$\#\{p \leq x : ap + b \in \wp\} \\ \leq 8 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p>2 \\ p|ab}} \frac{p-1}{p-2} \frac{x}{\log^2 x} \left(1 + O\left(\frac{\log\log x}{\log x}\right)\right).$$

Proof. This is Theorem 3.12 in the book of Halberstam and Richert [8] for the particular case $k = 1$.

Lemma 5. *Let* $M \geq 2k$*,* $\beta_1, \beta_2 \in A_q^k$*. Set* $\Delta(\alpha) = \left|F_{\beta_1}(\alpha) - F_{\beta_2}(\alpha)\right|$*. Then,*

$$\sum_{\alpha \in A_q^{M+1}} \Delta^2(\alpha) \leq cMq^M.$$

Proof. Let $\beta = b_{k-1} \dots b_0 \in A_q^k$. Consider the function $f_\beta : A_q^k \to \{0, 1\}$ defined by

$$f_\beta(u_{k-1}, \dots, u_0) = \begin{cases} 1 & \text{if } u_{k-1} \dots u_0 = \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Let $M \in \mathbb{N}$, $M \geq 2k$. Let $\alpha = \varepsilon_M \dots \varepsilon_0$ run over elements of A_q^{M+1}. It is clear that

$$\begin{aligned} A &:= \sum_{\alpha \in A_q^{M+1}} F_\beta(\alpha) \\ &= \sum_{\nu=0}^{M+1-k} \#\{\alpha \in A_q^{M+1} : \varepsilon_{\nu+k-1} \dots \varepsilon_\nu = \beta\} \\ &= (M+1-k)q^{M+1-k}. \end{aligned} \tag{1}$$

On the other hand,

$$B := \sum_{\alpha \in A_q^{M+1}} F_\beta^2(\alpha)$$

$$= \sum_{\nu_1=0}^{M+1-k} \sum_{\nu_2=0}^{M+1-k} \sum_{\varepsilon_0,\ldots,\varepsilon_M} f_\beta(\varepsilon_{\nu_1+k-1},\ldots,\varepsilon_{\nu_1}) f_\beta(\varepsilon_{\nu_2+k-1},\ldots,\varepsilon_{\nu_2})$$

$$= A + 2 \sum_{\substack{\nu_1,\nu_2=0\\ \nu_1<\nu_2}}^{M+1-k} \sum_{\varepsilon_0,\ldots,\varepsilon_M} f_\beta(\varepsilon_{\nu_1+k-1},\ldots,\varepsilon_{\nu_1}) f_\beta(\varepsilon_{\nu_2+k-1},\ldots,\varepsilon_{\nu_2})$$

$$= A + 2 \sum_{\substack{\nu_1,\nu_2=0\\ \nu_1<\nu_2\le\nu_1+k}}^{M+1-k} \sum_{\varepsilon_0,\ldots,\varepsilon_M} f_\beta(\varepsilon_{\nu_1+k-1},\ldots,\varepsilon_{\nu_1}) f_\beta(\varepsilon_{\nu_2+k-1},\ldots,\varepsilon_{\nu_2})$$

$$+ 2 \sum_{\substack{\nu_1,\nu_2=0\\ \nu_2>\nu_1+k}}^{M+1-k} \sum_{\varepsilon_0,\ldots,\varepsilon_M} f_\beta(\varepsilon_{\nu_1+k-1},\ldots,\varepsilon_{\nu_1}) f_\beta(\varepsilon_{\nu_2+k-1},\ldots,\varepsilon_{\nu_2}). \quad (2)$$

Now, on the one hand we have

$$\sum_{\substack{\nu_1,\nu_2=0\\ \nu_1<\nu_2\le\nu_1+k}}^{M+1-k} \sum_{\varepsilon_0,\ldots,\varepsilon_M} f_\beta(\varepsilon_{\nu_1+k-1},\ldots,\varepsilon_{\nu_1}) f_\beta(\varepsilon_{\nu_2+k-1},\ldots,\varepsilon_{\nu_2}) \le ckMq^{M+1-k}, \quad (3)$$

while on the other hand,

$$\sum_{\substack{\nu_1,\nu_2=0\\ \nu_2>\nu_1+k}}^{M+1-k} \sum_{\varepsilon_0,\ldots,\varepsilon_M} f_\beta(\varepsilon_{\nu_1+k-1},\ldots,\varepsilon_{\nu_1}) f_\beta(\varepsilon_{\nu_2+k-1},\ldots,\varepsilon_{\nu_2})$$

$$= \sum_{\substack{\nu_1,\nu_2=0\\ \nu_1+k<\nu_2}}^{M+1-k} q^{M+1-2k} = q^{M+1-2k}\left((M+1)^2 - O(kM)\right). \quad (4)$$

Combining (1) and (2), using estimates (3) and (4), we conclude that

$$\sum_{\alpha=\varepsilon_M\ldots\varepsilon_0} \left(F_\beta(\alpha) - \frac{M+1}{q^k}\right)^2 \le cMq^M. \quad (5)$$

Note that here we summed over those $\varepsilon_M = 0$ as well. But (5) remains true if we drop those $\varepsilon_M = 0$. This allows us to conclude that

$$\sum_{\alpha\in A_q^{M+1}} \left(F_\beta(\alpha) - \frac{M+1}{q^k}\right)^2 \le cMq^M,$$

thus completing the proof of Lemma 5.

5 Proof of Theorem 2

Let

$$\xi_x = \text{Concat}(\overline{\delta(p+1)} : p \le x).$$

Our goal is to prove that there exist two positive constants c_1 and c_2 such that

$$c_1 \le \frac{\lambda(\xi_x)}{\pi(x)\, x_3} \le c_2, \tag{6}$$

provided x is sufficiently large.

We first establish the size of $\lambda(\xi_x)$. We have

$$\lambda(\xi_x) = \sum_{\substack{p\le x\\ \delta(p+1)\ne 0}} \left\lfloor \frac{\log \delta(p+1)}{\log q} \right\rfloor + \pi(x) = \Sigma_1 + \Sigma_2 + \pi(x), \tag{7}$$

say, where the sum in Σ_1 runs over the primes $p \le x/x_2$, while that of Σ_2 runs over the primes located in the interval $J_x := (x/x_2, x]$. From this observation, it follows that

$$\Sigma_1 \le 2\pi(x/x_2)x_2 = O(\pi(x)). \tag{8}$$

On the other hand, it follows from Lemma 1 that, for each $u > 0$ there exists $c(u) > 0$ such that

$$\#\left\{ p \le x : \frac{\delta(p+1)}{\sqrt{x_2}} > u \right\} > c(u)\pi(x).$$

From this, we may conclude that

$$\Sigma_2 \ge c\pi(x)\, x_3. \tag{9}$$

Combining (8) and (9) in (7), we obtain that, if $x > x_0$, the inequality $\dfrac{\lambda(\xi_x)}{\pi(x)\, x_3} > c$ holds for some positive constant c, thereby establishing the first inequality in (6).

Now, from the definitions of the functions δ and δ^*, it is clear that

$$|\delta^*(p+1) - \delta(p+1)| \le 1 \qquad \text{for all } p \in J_x.$$

From the trivial estimate $\delta(p+1) \le c \log x$, we obtain that $\log \delta(p+1) \le x_2 + c_1$, so that

$$\Sigma_2 \le c\pi(x)\, x_3 + \sum_{\frac{\delta^*(p+1)}{\sqrt{x_2}} > 4} (\log 2)\, \delta^*(p+1) = c\pi(x)\, x_3 + \Sigma_3, \tag{10}$$

say.

From Lemma 2, we obtain that for every $A \geq 1$, we have

$$\#\left\{p \in J_x : A < \frac{\delta^*(p+1)}{\sqrt{x_2}} < 2A\right\} \leq \frac{c\pi(x)}{A^2}. \tag{11}$$

We now apply (11) successively with $A = 2^j$, $j = 2, 3, \ldots$, thus obtaining

$$\begin{aligned}
\Sigma_3 &\leq c\pi(x) \sum_{j \geq 2} \frac{\log 2^{j+2}\sqrt{x_2}}{2^{j+2}} \\
&\leq c\pi(x) \left[\frac{1}{2} x_3 \sum_{j \geq 2} \frac{1}{2^{j+2}} + c \sum_{j \geq 2} \frac{j}{2^j} \right] \\
&\leq c_1 \pi(x)\, x_3,
\end{aligned}$$

from which we may conclude that, in light of (7), (8), and (10), the right-hand side of (6) follows as well.

We will prove that, given any fixed integer $k \geq 1$, $\beta_1, \beta_2 \in A_q^k$, and setting $\Delta(\alpha) := F_{\beta_1}(\alpha) - F_{\beta_2}(\alpha)$ for each word $\alpha \in A_q^*$,

$$\lim_{x \to \infty} \frac{|\Delta(\xi_x)|}{\lambda(\xi_x)} = 0. \tag{12}$$

In order to achieve this, now that we know (from (6)) that the true size of $\lambda(\xi_x)$ is $\pi(x)\, x_3$, we essentially need to prove that $\Delta(\xi_x)$ is of smaller order than $\pi(x)\, x_3$.

Let θ_x be an arbitrary function which tends monotonically to 0 very slowly. Then consider the sets

$$\begin{aligned}
D_1 &= \{p \in \wp : p \leq x/x_2\}, \\
D_2 &= \{p \in \wp : p \leq x \text{ and } \delta(p+1) \leq \theta_x \sqrt{x_2}\}, \\
D_3 &= \{p \in \wp : p \leq x \text{ and } \delta(p+1) > \frac{1}{\theta_x}\sqrt{x_2}\},
\end{aligned}$$

and let $D = D_1 \cup D_2 \cup D_3$.

Because $\Delta(\delta(p+1)) \leq c x_3$ if $p \in D_1$ and $p \leq c x_2$, and since (11) holds for $p \in D_3$, it follows from Lemma 1 and (8) that

$$\begin{aligned}
\sum_{p \in D} |\Delta(\delta(p+1))| &\leq c x_3 \pi(x)\, (\Phi(\theta_x) - \Phi(-\theta_x)) + c\pi(x/x_2) x_2 \\
&+ \sum_{j=0}^{\infty} \#\left\{p \in J_x : \frac{\delta^*(p+1)}{\sqrt{x_2}} \in \left[\frac{2^j}{\theta_x}, \frac{2^{j+1}}{\theta_x}\right]\right\} \log\left(\sqrt{x_2} \cdot \frac{2^{j+1}}{\theta_x}\right).
\end{aligned} \tag{13}$$

Since this last sum is less than

$$\pi(x)\sum_{j\ge 0}(x_3+j+\log(1/\theta_x))\cdot\frac{\theta_x^2}{2^{2j}}\le c\,(\log(1/\theta_x)+x_3)\,\theta_x^2\pi(x),$$

it follows that (13) yields

$$\sum_{p\in D}|\Delta(\delta(p+1))|=o(\pi(x)\,x_3)\qquad(x\to\infty).\tag{14}$$

Now, from (14), we have that

$$\Delta(\xi_x)=\sum_{p\notin D}\Delta(\delta(p+1))+o(\pi(x)\,x_3)=\Sigma_A+o(\pi(x)\,x_3),\tag{15}$$

say.

From Lemma 3, we obtain, using the fact that $p\notin D_3$ (since $p\notin D$), that

$$\sum_{\substack{p\notin D\\ P(p+1)\notin[x^\kappa,x^{1-\kappa}]}}|\Delta(\delta((p+1))|\le c\pi(x)\log\left(\frac{1}{\theta_x}\sqrt{x_2}\right)\le c\pi(x)x_3,\tag{16}$$

provided that θ_x is chosen so that $1/\theta_x<x_2$, say.

Now let $K=\lfloor x_2\rfloor$ and then, for ℓ satisfying $\varepsilon_x\sqrt{K}\le|\ell|\le\frac{1}{\varepsilon_x}\sqrt{K}$, let

$$R_\kappa(\ell):=\#\{p\in J_x:P(p+1)\in(x^\kappa,x^{1-\kappa})\text{ and }\omega(p+1)=K+\ell\}.$$

Using Lemma 4, we obtain that

$$\begin{aligned}R_\kappa(\ell)&\le\#\{p\in J_x:p+1=aq,\ a<x^{1-\kappa},\ q>x^\kappa/x_2,\ \omega(a)=K+\ell-1\}\\&\le\frac{c_1}{\kappa^2}\frac{x}{\log^2 x}\sum_{\omega(a)=K+\ell-1}\frac{1}{a}\prod_{\substack{p>2\\p\mid a}}\frac{p-1}{p-2}+O\left(x^{1-\kappa}\right),\end{aligned}\tag{17}$$

where the $O(\ldots)$ term accounts for the contribution of those q such that $q^2\mid p+1$.

It follows from (17) that

$$\begin{aligned}R_\kappa(\ell)&\le\frac{c_1x}{\kappa^2\log^2 x}\left(\sum_{p\le x}\frac{1}{p}+c\right)^{K+\ell-1}\frac{1}{(K+\ell-1)!}+O\left(x^{1-\kappa}\right)\\&\le\frac{c_2x}{\kappa^2\log^2 x}\frac{(K+c)^{K+\ell-1}}{(K+\ell-1)!}.\end{aligned}\tag{18}$$

Now, observe that, if $\omega(p+1) = K + \ell$, $p \in J_x$, then $\delta(p+1) \in \{|\ell|-1, |\ell|, |\ell|+1\}$. Thus,

$$|\Sigma_A| \le \sum_{\varepsilon_x\sqrt{K}\le|\ell|\le\frac{1}{\varepsilon_x}\sqrt{K}} \Big(\Delta(|\overline{\ell}|) + \Delta(|\overline{\ell}|-1) + \Delta(|\overline{\ell}|+1)\Big)\cdot(R_\kappa(-\ell)+R_\kappa(\ell)) + c\kappa\pi(x)x_3$$
$$= \Sigma_B + c\kappa\pi(x)x_3, \tag{19}$$

say.

Using (18), we obtain that

$$\Sigma_B \le \frac{c_2 x}{\kappa^2\log^2 x}\sum_{\varepsilon_x\le\frac{\ell}{\sqrt{K}}\le\frac{1}{\varepsilon_x}}\Big(\Delta(\overline{l}) + \Delta(\overline{l}-1) + \Delta(\overline{l}+1)\Big) \times\left(\frac{(K+c)^{K+\ell-1}}{(K+\ell-1)!} + \frac{(K+c)^{K-\ell-1}}{(K-\ell-1)!}\right). \tag{20}$$

Since we can easily establish that

$$\max_{0\le\ell\le\frac{1}{\varepsilon_x\sqrt{K}}}\left(\frac{(K+c)^{K+\ell-1}}{(K+\ell-1)!} + \frac{(K+c)^{K-\ell-1}}{(K-\ell-1)!}\right) < \frac{(K+c)^{K-1}}{(K-1)!}\exp\left\{c_3\left(\frac{1}{\varepsilon_x}\right)^2\right\},$$

it follows from (20) that

$$\Sigma_B \le \frac{c_2 x}{\kappa^2\log^2 x}\exp\left\{c_3\left(\frac{1}{\varepsilon_x}\right)^2\right\}\frac{(K+c)^{K-1}}{(K-1)!}\Sigma_C, \tag{21}$$

where

$$\Sigma_C = \sum_{\varepsilon_x\le\frac{\ell}{\sqrt{K}}\le\frac{1}{\varepsilon_x}}\Big(\Delta(\overline{\ell}) + \Delta(\overline{\ell}-1) + \Delta(\overline{\ell}+1)\Big)$$
$$\le 3\sum_{\varepsilon_x\le\frac{\ell}{\sqrt{K}}\le\frac{1}{\varepsilon_x}}\Delta(\overline{\ell}) + O(x_3) = 3\Sigma_D + O(x_3), \tag{22}$$

say.

To estimate Σ_D, we will use Lemma 5. Indeed, let M_0 be the largest integer for which $q^{M_0} \le \varepsilon_x\sqrt{K}$ and let M_1 be the smallest integer for which $q^{M_1} > \frac{1}{\varepsilon_x}\sqrt{K}$. Set $\mathscr{K}_M = [q^M, q^{M+1}-1]$. With this setup, we clearly have that

$$\Sigma_D \le \sum_{M_0\le M\le M_1} T_M, \tag{23}$$

where $T_M = \sum_{\ell \in \mathscr{K}_M} \Delta(\ell)$. Now, it follows from Lemma 5 that

$$T_M \le c(q^{M+1})^{1/2}(Mq^{M-k})^{1/2} \le c\sqrt{M}q^M. \tag{24}$$

Using (24) in (23), we obtain that

$$\Sigma_D \le c\sqrt{M_1}q^{M_1}\left(1+\frac{1}{q}+\frac{1}{q^2}+\cdots\right) < \frac{c_1}{\varepsilon_x}\sqrt{K}\,\sqrt{\log K} < \frac{c_1 x_2^{1/2}\sqrt{x_3}}{\varepsilon_x}. \tag{25}$$

Gathering (21), (22), and (25), we have that

$$\Sigma_B \le \frac{cx}{\kappa^2 \log^2 x} \exp\left\{c_3\left(\frac{1}{\varepsilon_x}\right)^2\right\} \frac{(K+c)^{K-1}}{(K-1)!} \cdot \frac{\sqrt{x_2}\sqrt{x_3}}{\varepsilon_x}. \tag{26}$$

Since it is well known that $n! = n^n e^{-n}\sqrt{2\pi n}(1+O(1/n))$, it follows that by setting $\ell_K = \frac{(K+c)^{K-1}}{(K-1)!}$,

$$\begin{aligned}\log \ell_K &= (K-1)\log(K+c) - (K-1)\log\left(\frac{K-1}{e}\right) - \frac{1}{2}\log K + O(1)\\ &= (K-1)\log\frac{K+c}{K-1} - \frac{1}{2}\log K + O(1) + K - 1,\end{aligned}$$

from which it follows that

$$\ell_K \sim \frac{x_1}{\sqrt{x_2}}.$$

Using this last estimate in (26), it follows that

$$\Sigma_B \le \frac{\exp\left\{c_3/\varepsilon_x^2\right\}}{\kappa^2 \varepsilon_x}\pi(x)x_3. \tag{27}$$

Choosing $\varepsilon_x = x_5$, say, we get from (27) that

$$\limsup_{x\to\infty} \frac{\Sigma_B}{\lambda(\xi_x)} = 0. \tag{28}$$

Combining (28), (19), and (15), we obtain that

$$\limsup_{x\to\infty} \frac{\Delta(\xi_x)}{\lambda(\xi_x)} \le c\kappa. \tag{29}$$

Since κ can be taken arbitrarily small, we may finally conclude that (12) holds, thus completing the proof of Theorem 1.

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Imbedding Inequalities for Composition of Green's and Potential Operators

Shusen Ding and Yuming Xing

Abstract In this paper, we prove both local and global imbedding inequalities with L^{φ}-norms for the composition of the potential operator and Green's operator applied to differential forms.

Keywords Imbedding • Differential forms • Green's operator • Potential operator • A-harmonic equations

1 Introduction

Let P be the potential operator and G be Green's operator applied to differential forms. Assume that $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ is a Young function satisfying certain conditions. In this paper, we prove some inequalities with L^{φ}-norms for $G \circ P$ and the related operators that are applied to differential forms. Our main results are the local L^{φ}-imbedding inequality and the global L^{φ}-imbedding inequality for $G \circ P$

$$\|G(P(u)) - (G(P(u)))_{\Omega}\|_{W^{1,\varphi}(\Omega)} \leq C\|u\|_{L^{\varphi}(\Omega)},$$

where Ω is any bounded and convex domain and C is a constant independent of differential form u. These results are presented in Theorems 4 and 5, respectively. The potential operator P and Green's operator G are key operators and are widely

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studied and used in analysis, partial differential equations, physics, and potential theory [1–5].

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, B and σB be the balls with the same center and $\mathrm{diam}(\sigma B) = \sigma \mathrm{diam}(B)$ throughout this paper. We do not distinguish the balls from cubes in this paper. We use $|E|$ to denote the n-dimensional Lebesgue measure of a set $E \subseteq \mathbb{R}^n$. For a function u, the average of u over B is defined by $u_B = \frac{1}{|B|}\int_B u dx$. All integrals involved in this paper are the Lebesgue integrals. A differential 1-form $u(x)$ in $\mathbb{R}^n$ can be written as $u(x) = \sum_{i=1}^n u_i(x_1, x_2, \cdots, x_n)dx_i$, where the coefficient functions $u_i(x_1, x_2, \cdots, x_n)$, $i = 1, 2, \cdots, n$, are differentiable. A differential k-form $u(x)$ can be expressed as

$$u(x) = \sum_I u_I(x)dx_I = \sum u_{i_1 i_2 \cdots i_k}(x)dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k},$$

where $I = (i_1, i_2, \cdots, i_k)$, $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. A function $u(x_1, x_2, \cdots, x_n)$ is a 0-form. Assume that $\wedge^l = \wedge^l(\mathbb{R}^n)$ is the set of all l-forms in $\mathbb{R}^n$, $D'(\Omega, \wedge^l)$ is the space of all differential l-forms in Ω, and $L^p(\Omega, \wedge^l)$ is the l-forms $u(x) = \sum_I u_I(x)dx_I$ in Ω satisfying $\int_\Omega |u_I|^p < \infty$ for all ordered l-tuples I, $l = 1, 2, \cdots, n$. The exterior derivative is denoted by d and the Hodge star operator by $\star$. The Hodge codifferential operator $d^\star$ is given by $d^\star = (-1)^{nl+1} \star d \star$, $l = 1, 2, \cdots, n$. For $u \in D'(\Omega, \wedge^l)$ the vector-valued differential form

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right)$$

consists of differential forms $\frac{\partial u}{\partial x_i} \in D'(\Omega, \wedge^l)$, where the partial differentiation is applied to the coefficients of u. The nonlinear partial differential equation

$$d^\star A(x, du) = B(x, du) \tag{1}$$

is called non-homogeneous A-harmonic equation, where $A : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ and $B : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^{l-1}(\mathbb{R}^n)$ satisfy the conditions:

$$|A(x, \xi)| \leq a|\xi|^{p-1},\ A(x, \xi) \cdot \xi \geq |\xi|^p \ \text{ and } \ |B(x, \xi)| \leq b|\xi|^{p-1} \tag{2}$$

for almost every $x \in \Omega$ and all $\xi \in \wedge^l(\mathbb{R}^n)$. Here $a, b > 0$ are constants and $1 < p < \infty$ is a fixed exponent associated with (1). A solution to (1) is an element of the Sobolev space $W^{1,p}_{\mathrm{loc}}(\Omega, \wedge^{l-1})$ such that

$$\int_\Omega A(x, du) \cdot d\varphi + B(x, du) \cdot \varphi = 0 \tag{3}$$

for all $\varphi \in W^{1,p}_{\mathrm{loc}}(\Omega, \wedge^{l-1})$ with compact support. If u is a function (0-form) in $\mathbb{R}^n$, Eq. (1) reduces to

$$\mathrm{div}A(x, \nabla u) = B(x, \nabla u). \tag{4}$$

If the operator $B = 0$, Eq. (1) becomes $d^\star A(x, du) = 0$ which is called the (homogeneous) A-harmonic equation. See [6–10] for recent results on the A-harmonic equations and related topics.

Assume that $D \subset \mathbb{R}^n$ is a bounded, convex domain. The following operator K_y with the case $y = 0$ was first introduced by Cartan in [11]. Then, it was extended to the following general version in [12]. For each $y \in D$, there corresponds a linear operator $K_y : C^\infty(D, \wedge^l) \to C^\infty(D, \wedge^{l-1})$ defined by $(K_y u)(x; \xi_1, \cdots, \xi_{l-1}) = \int_0^1 t^{l-1} u(tx + y - ty; x - y, \xi_1, \cdots, \xi_{l-1}) dt$ and the decomposition $u = d(K_y u) + K_y(du)$. A homotopy operator $T : C^\infty(D, \wedge^l) \to C^\infty(D, \wedge^{l-1})$ is defined by averaging K_y over all points y in D $Tu = \int_D \varphi(y) K_y u dy$, where $\varphi \in C_0^\infty(D)$ is normalized by $\int_D \varphi(y) dy = 1$. For simplicity purpose, we write $\xi = (\xi_1, \cdots, \xi_{l-1})$. Then, $Tu(x; \xi) = \int_0^1 t^{l-1} \int_D \varphi(y) u(tx + y - ty; x - y, \xi) dy dt$. By substituting $z = tx + y - ty$ and $t = s/(1 + s)$, we have

$$Tu(x; \xi) = \int_D u(z, \zeta(z, x - z), \xi) dz, \tag{5}$$

where the vector function $\zeta : D \times \mathbb{R}^n \to \mathbb{R}^n$ is given by $\zeta(z, h) = h \int_0^\infty s^{l-1}(1 + s)^{n-1} \varphi(z - sh) ds$. The integral (7) defines a bounded operator $T : L^s(D, \wedge^l) \to W^{1,s}(D, \wedge^{l-1}), l = 1, 2, \cdots, n$, and the decomposition

$$u = d(Tu) + T(du) \tag{6}$$

holds for any differential form u. The l-form $u_D \in D'(D, \wedge^l)$ is defined by

$$u_D = \fint_D u(y) dy = |D|^{-1} \int_D u(y) dy, \; l = 0, \text{ and } u_D = d(Tu), \; l = 1, 2, \cdots, n, \tag{7}$$

for all $u \in L^p(D, \wedge^l)$, $1 \le p < \infty$. Also, for any differential form u, we have

$$\|\nabla(Tu)\|_{p,D} \le C|D| \|u\|_{p,D}, \text{ and } \|Tu\|_{p,D} \le C|D| diam(D) \|u\|_{p,D}. \tag{8}$$

From [13, Page 16], we know that any open subset Ω in $\mathbb{R}^n$ is the union of a sequence of cubes Q_k, whose sides are parallel to the axes, whose interiors are mutually disjoint, and whose diameters are approximately proportional to their distances from F, where F is the complement of Ω in $\mathbb{R}^n$. Specifically, (i) $\Omega = \cup_{k=1}^\infty Q_k$, (ii) $Q_j^0 \cap Q_k^0 = \phi$ if $j \ne k$, (iii) there exist two constants $c_1, c_2 > 0$ (we can take $c_1 = 1$, and $c_2 = 4$), so that

$$c_1 diam(Q_k) \le distance(Q_k, F) \le c_2 diam(Q_k). \tag{9}$$

Thus, the definition of the homotopy operator T can be generalized to any domain Ω in $\mathbb{R}^n$: For any $x \in \Omega$, $x \in Q_k$ for some k. Let T_{Q_k} be the homotopy operator defined on Q_k (each cube is bounded and convex). Thus, we can define the homotopy operator T_Ω on any domain Ω by

$$T_\Omega = \sum_{k=1}^{\infty} T_{Q_k} \chi_{Q_k(x)}. \tag{10}$$

Hui Bi extended the definition of the potential operator to the case of differential forms, see [2]. For any differential l-form $u(x)$, the potential operator P is defined by

$$Pu(x) = \sum_I \int_E K(x,y) u_I(y) dy dx_I, \tag{11}$$

where the kernel $K(x,y)$ is a nonnegative measurable function defined for $x \neq y$ and the summation is over all ordered l-tuples I. The $l = 0$ case reduces to the usual potential operator,

$$Pf(x) = \int_E K(x,y) f(y) dy, \tag{12}$$

where $f(x)$ is a function defined on $E \subset \mathbb{R}^n$. See [2] and [14] for more results about the potential operator. We say a kernel K on $\mathbb{R}^n \times \mathbb{R}^n$ satisfies the standard estimates if there exist δ, $0 < \delta \leq 1$, and constant C such that for all distinct points x and y in $\mathbb{R}^n$, and all z with $|x - z| < \frac{1}{2}|x - y|$, the kernel K satisfies (i) $K(x,y) \leq C|x-y|^{-n}$; (ii) $|K(x,y) - K(z,y)| \leq C|x-z|^{\delta}|x-y|^{-n-\delta}$; (iii) $|K(y,x) - K(y,z)| \leq C|x-z|^{\delta}|x-y|^{-n-\delta}$.

We always suppose that P is the potential operator defined in (11) with the kernel $K(x,y)$ satisfying condition (i) of the standard estimates throughout this paper. Recently, Hui Bi in [2] proved the following inequality for the potential operator.

$$\|P(u)\|_{p,E} \leq C\|u\|_{p,E}, \tag{13}$$

where $u \in D'(E, \wedge^l)$, $l = 0, 1, \ldots, n-1$, is a differential form defined in a bounded and convex domain E, and $p > 1$ is a constant.

2 Local Imbedding Inequalities

We first prove the local L^φ imbedding inequalities for $G \circ P$ applied to solutions of the non-homogeneous A-harmonic equation in a bounded domain. We will need the following definitions and standard notation. A continuously increasing function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ is called an Orlicz function. The Orlicz space

$L^{\varphi}(\Omega)$ consists of all measurable functions f on Ω such that $\int_{\Omega} \varphi\left(\frac{|f|}{\lambda}\right) dx < \infty$ for some $\lambda = \lambda(f) > 0$. $L^{\varphi}(\Omega)$ is equipped with the nonlinear Luxemburg functional

$$\|f\|_{L^{\varphi}(\Omega)} = \inf\left\{\lambda > 0 : \int_{\Omega} \varphi\left(\frac{|f|}{\lambda}\right) dx \leq 1\right\}.$$

A convex Orlicz function φ is often called a Young function. If φ is a Young function, then $\|\cdot\|_{L^{\varphi}(\Omega)}$ defines a norm in $L^{\varphi}(\Omega)$, which is called the Luxemburg norm or Orlicz norm. For any subset $E \subset \mathbb{R}^n$, we use $W^{1,\varphi}(E, \wedge^l)$ to denote the Orlicz-Sobolev space of l-forms which equals $L^{\varphi}(E, \wedge^l) \cap L_1^{\varphi}(E, \wedge^l)$ with norm

$$\|u\|_{W^{1,\varphi}(E)} = \|u\|_{W^{1,\varphi}(E,\wedge^l)} = diam(E)^{-1}\|u\|_{L^{\varphi}(E)} + \|\nabla u\|_{L^{\varphi}(E)}. \tag{14}$$

If we choose $\varphi(t) = t^p$, $p > 1$ in (14), we obtain the usual L^p norm for $W^{1,p}(E, \wedge^l)$

$$\|u\|_{W^{1,p}(E)} = \|u\|_{W^{1,p}(E,\wedge^l)} = diam(E)^{-1}\|u\|_{p,E} + \|\nabla u\|_{p,E}. \qquad ((14)')$$

Definition 1 ([15]). We say a Young function φ lies in the class $G(p, q, C)$, $1 \leq p < q < \infty$, $C \geq 1$, if (i) $1/C \leq \varphi(t^{1/p})/g(t) \leq C$ and (ii) $1/C \leq \varphi(t^{1/q})/h(t) \leq C$ for all $t > 0$, where g is a convex increasing function and h is a concave increasing function on $[0, \infty)$.

From [15], each of φ, g, and h in above definition is doubling in the sense that its values at t and $2t$ are uniformly comparable for all $t > 0$, and the consequent fact that

$$C_1 t^q \leq h^{-1}(\varphi(t)) \leq C_2 t^q, \quad C_1 t^p \leq g^{-1}(\varphi(t)) \leq C_2 t^p, \tag{15}$$

where C_1 and C_2 are constants. Also, for all $1 \leq p_1 < p < p_2$ and $\alpha \in \mathbb{R}$, the function $\varphi(t) = t^p \log_+^{\alpha} t$ belongs to $G(p_1, p_2, C)$ for some constant $C = C(p, \alpha, p_1, p_2)$. Here $\log_+(t)$ is defined by $\log_+(t) = 1$ for $t \leq e$; and $\log_+(t) = \log(t)$ for $t > e$. Particularly, if $\alpha = 0$, we see that $\varphi(t) = t^p$ lies in $G(p_1, p_2, C)$, $1 \leq p_1 < p < p_2$. We will need the following Reverse Hölder inequality.

Lemma 1 ([8]). *Let u be a solution of the non-homogeneous A-harmonic equation (1) in a domain Ω and $0 < s, t < \infty$. Then, there exists a constant C, independent of u, such that*

$$\|u\|_{s,B} \leq C|B|^{(t-s)/st}\|u\|_{t,\sigma B} \tag{16}$$

for all balls B with $\sigma B \subset \Omega$ for some $\sigma > 1$.

Using the same way in the proof of Propositions 5.15 and 5.17 in [3], we can prove that for any closed ball $\overline{B} = B \cup \partial B$, we have

$$\|dd^*G(u)\|_{s,\overline{B}} + \|d^*dG(u)\|_{s,\overline{B}} + \|dG(u)\|_{s,\overline{B}} + \|d^*G(u)\|_{s,\overline{B}} + \|G(u)\|_{s,\overline{B}} \le C(s)\|u\|_{s,\overline{B}}.$$

Note that for any Lebesgue measurable function f defined on a Lebesgue measurable set E with $|E| = 0$, we have $\int_E f dx = 0$. Thus, $\|u\|_{s,\partial B} = 0$ and $\|dd^*G(u)\|_{s,\partial B} + \|d^*dG(u)\|_{s,\partial B} + \|dG(u)\|_{s,\partial B} + \|d^*G(u)\|_{s,\partial B} + \|G(u)\|_{s,\partial B} = 0$ since $|\partial B| = 0$. Therefore, we obtain

$$\begin{aligned} &|dd^*G(u)\|_{s,B} + \|d^*dG(u)\|_{s,B} + \|dG(u)\|_{s,B} + \|d^*G(u)\|_{s,MB} + \|G(u)\|_{s,B} \\ &\quad = \|dd^*G(u)\|_{s,\overline{B}} + \|d^*dG(u)\|_{s,\overline{B}} + \|dG(u)\|_{s,\overline{B}} + \|d^*G(u)\|_{s,\overline{B}} + \|G(u)\|_{s,\overline{B}} \\ &\quad \le C(s)\|u\|_{s,\overline{B}} \\ &\quad = C(s)\|u\|_{s,B}. \end{aligned}$$

Hence, we have the following lemma.

Lemma 2. *Let u be a smooth differential form defined in M and $1 < s < \infty$. Then, there exists a positive constant $C = C(s)$, independent of u, such that*

$$\begin{aligned} &\|dd^*G(u)\|_{s,B} + \|d^*dG(u)\|_{s,B} + \|dG(u)\|_{s,B} + \|d^*G(u)\|_{s,B} + \|G(u)\|_{s,B} \\ &\quad \le C(s)\|u\|_{s,B} \end{aligned}$$

for any ball $B \subset M$.

We first prove the following local inequality for the composition $T \circ P$ with the L^φ-norm.

Theorem 1. *Let φ be a Young function in the class $G(p, q, C)$, $1 \le p < q < \infty$, $C \ge 1$, Ω be a bounded domain, G be Green's operator, and P be the potential operator defined in (11) with the kernel $K(x, y)$ satisfying condition (i) of the standard estimates. Assume that $\varphi(|u|) \in L^1_{\text{loc}}(\Omega)$ and u is a solution of the non-homogeneous A-harmonic equation (1) in Ω. Then, there exists a constant C, independent of u, such that*

$$\|G(P(u)) - (G(P(u)))_B\|_{L^\varphi(B)} \le C diam(B)\|u\|_{L^\varphi(\sigma B)} \tag{17}$$

for all balls B with $\sigma B \subset \Omega$ for some $\sigma > 1$.

Proof. Using Lemma 2 and (13), we have

$$\|dG(P(u))\|_{q,B} \le C_1\|P(u)\|_{q,B} \le C_2\|u\|_{q,B}. \tag{18}$$

From (8) and (18), we obtain

$$\begin{aligned}\|G(P(u)) - (G(P(u)))_B\|_{q,B} &= \|Td(G(P(u)))\|_{q,B} \\ &\le C_3|B|diam(B)\|dG(P(u))\|_{q,B} \\ &\le C_4|B|diam(B)\|u\|_{q,B} \qquad (19)\end{aligned}$$

for any differential form u and all balls B with $B \subset \Omega$. From Lemma 1, for any positive numbers p and q, it follows that

$$\left(\int_B |u|^q dx\right)^{1/q} \le C_5|B|^{(p-q)/pq}\left(\int_{\sigma B} |u|^p dx\right)^{1/p}, \qquad (20)$$

where σ is a constant $\sigma > 1$. Using Jensen's inequality for h^{-1}, (15), (19), (20), (i) in Definition 1, and noticing the fact that φ and h are doubling, and φ is an increasing function, we obtain

$$\begin{aligned}&\int_B \varphi\Big(|G(P(u)) - (G(P(u)))_B|\Big)dx \\ &= h\Big(h^{-1}\Big(\int_B \varphi\left(|G(P(u)) - (G(P(u)))_B|\right)dx\Big)\Big) \\ &\le h\Big(\int_B h^{-1}\Big(\varphi\left(|G(P(u)) - (G(P(u)))_B|\right)\Big)dx\Big) \\ &\le h\Big(C_6\int_B |G(P(u)) - (G(P(u)))_B|^q dx\Big) \\ &\le C_7\varphi\Big(\Big(C_6\int_B |G(P(u)) - (G(P(u)))_B|^q dx\Big)^{1/q}\Big) \\ &\le C_7\varphi\Big(C_8|B|^{1+1/n}\Big(\int_B |u|^q dx\Big)^{1/q}\Big) \\ &\le C_7\varphi\Big(C_9|B|^{1+1/n+(p-q)/pq}\Big(\int_{\sigma B} |u|^p dx\Big)^{1/p}\Big) \\ &\le C_7\varphi\Big(\Big(C_9^p|B|^{p(1+1/n)+(p-q)/q}\int_{\sigma B} |u|^p dx\Big)^{1/p}\Big) \\ &\le C_{10}g\Big(C_9^p|B|^{p(1+1/n)+(p-q)/q}\int_{\sigma B} |u|^p dx\Big) \\ &= C_{10}g\Big(\int_{\sigma B} C_9^p|B|^{p(1+1/n)+(p-q)/q}|u|^p dx\Big) \\ &\le C_{10}\int_{\sigma B} g\Big(C_9^p|B|^{p(1+1/n)+(p-q)/q}|u|^p\Big)dx \\ &\le C_{11}\int_{\sigma B} \varphi\Big(C_9|B|^{1+\frac{1}{n}+\frac{p-q}{pq}}|u|\Big)dx \qquad (21)\end{aligned}$$

Since $p \geq 1$, then $1 + \frac{1}{n} + \frac{p-q}{pq} > \frac{1}{n}$. Hence, we have $|B|^{1+\frac{1}{n}+\frac{p-q}{pq}} \leq C_{12}|B|^{\frac{1}{n}}$. Note that φ is doubling, we obtain

$$\varphi\big(C_9|B|^{1+\frac{1}{n}+\frac{p-q}{pq}}|u|\big) \leq C_{13}|B|^{\frac{1}{n}}\varphi\big(|u|\big). \tag{22}$$

Combining (21) and (22) and using $|B|^{\frac{1}{n}} = C_{14}diam(B)$ yields

$$\int_B \varphi\left(|G(P(u)) - (G(P(u)))_B|\right) dx \leq C_{15}\, diam(B) \int_{\sigma B} \varphi\left(|u|\right) dx. \tag{23}$$

Since each of φ, g, and h in Definition 1 is doubling, from (23), we have

$$\int_B \varphi\left(\frac{|G(P(u)) - (G(P(u)))_B|}{\lambda}\right) dx \leq C\, diam(B) \int_{\sigma B} \varphi\left(\frac{|u|}{\lambda}\right) dx$$

for all balls B with $\sigma B \subset \Omega$ and any constant $\lambda > 0$. From the definition of Luxemburg norm and the last inequality, we have the following inequality with the Luxemburg norm

$$\|G(P(u)) - (G(P(u)))_B\|_{L^{\varphi}(B)} \leq C\, diam(B)\|u\|_{L^{\varphi}(\sigma B)}$$

The proof of Theorem 1 has been completed.

In order to prove our main local imbedding theorem, we will need the following Theorems 2 and 3.

Theorem 2. *Let φ be a Young function in the class $G(p,q,C)$, $1 \leq p < q < \infty$, $C \geq 1$, Ω be a bounded domain, G be Green's operator, and P be the potential operator defined in* (11) *with the kernel $K(x,y)$ satisfying condition (i) of the standard estimates. Assume that $\varphi(|u|) \in L^1_{\mathrm{loc}}(\Omega)$ and u is a solution of the non-homogeneous A-harmonic equation* (1) *in Ω. Then, there exists a constant C, independent of u, such that*

$$\|Td(G(P(u)))\|_{L^{\varphi}(B)} \leq C\, diam(B)\|u\|_{L^{\varphi}(\sigma B)} \tag{24}$$

for all balls B with $\sigma B \subset \Omega$ for some $\sigma > 1$.

Proof. Using (8) and (18), we have

$$\begin{aligned}\|Td(G(P(u)))\|_{q,B} &\leq C_1|B|diam(B)\|d(G(P(u)))\|_{q,B} \\ &\leq C_2|B|diam(B)\|u\|_{q,B}\end{aligned} \tag{25}$$

for any differential form u and $q > 1$. By Lemma 1, for any positive numbers p and q, it follows that

$$\Big(\int_B |u|^q dx\Big)^{1/q} \le C_3|B|^{(p-q)/pq}\Big(\int_{\sigma B} |u|^p dx\Big)^{1/p}, \tag{26}$$

where σ is a constant $\sigma > 1$. Using Jensen's inequality for h^{-1}, (15), (26), (i) in Definition 1, and noticing the fact that φ and h are doubling, and φ is an increasing function, we obtain

$$\begin{aligned}
\int_B \varphi\Big(|Td(G(P(u)))|\Big)dx &= h\Big(h^{-1}\Big(\int_B \varphi\,(|Td(G(P(u)))|)\,dx\Big)\Big)\\
&\le h\Big(\int_B h^{-1}\Big(\varphi\,(|Td(G(P(u)))|)\Big)dx\Big)\\
&\le h\Big(C_4\int_B |Td(G(P(u)))|^q dx\Big)\\
&\le C_5\varphi\Big(\Big(C_4\int_B |Td(G(P(u)))|^q dx\Big)^{1/q}\Big)\\
&\le C_5\varphi\Big(C_6|B|diam(B)\Big(\int_B |u|^q dx\Big)^{1/q}\Big)\\
&\le C_5\varphi\Big(C_7|B|^{1+(p-q)/pq}diam(B)\Big(\int_{\sigma B} |u|^p dx\Big)^{1/p}\Big)\\
&\le C_5\varphi\Big(\Big(C_7^p|B|^{p+(p-q)/q}(diam(B))^p\int_{\sigma B} |u|^p dx\Big)^{1/p}\Big)\\
&\le C_8 g\Big(C_7^p|B|^{p+(p-q)/q}(diam(B))^p\int_{\sigma B} |u|^p dx\Big)\\
&= C_8 g\Big(\int_{\sigma B} C_7^p|B|^{p+(p-q)/q}(diam(B))^p|u|^p dx\Big)\\
&\le C_9\int_{\sigma B} g\Big(C_7^p|B|^{p+(p-q)/q}(diam(B))^p|u|^p\Big)dx\\
&\le C_{10}\int_{\sigma B}\varphi\Big(C_7|B|^{1+(p-q)/pq}diam(B)|u|\Big)dx
\end{aligned} \tag{27}$$

Since $p \ge 1$, then $1+\frac{p-q}{pq} > 0$. Hence, we have

$$|B|^{1+\frac{p-q}{pq}} = |B|^{1+1/q-1/p} \le C_{11}.$$

Note that φ is doubling, we obtain

$$\varphi\Big(C_7|B|^{1+(p-q)/pq}diam(B)|u|\Big) \le C_{12}diam(B)\varphi\big(|u|\big). \tag{28}$$

Combining (27) and (28) yields

$$\int_B \varphi\left(|Td(G(P(u)))|\right) dx \le C_{13} diam(B) \int_{\sigma B} \varphi\left(|u|\right) dx. \tag{29}$$

Since each of φ, g, and h in Definition 1 is doubling, from (29), we have

$$\int_B \varphi\left(\frac{|Td(G(P(u)))|}{\lambda}\right) dx \le C_{14} diam(B) \int_{\sigma B} \varphi\left(\frac{|u|}{\lambda}\right) dx \tag{30}$$

for all balls B with $\sigma B \subset \Omega$ and any constant $\lambda > 0$. From the definition of Luxemburg norm and (30), we have the following inequality with the Luxemburg norm

$$\|Td(G(P(u)))\|_{L^{\varphi}(B)} \le C_{15} diam(B) \|u\|_{L^{\varphi}(\sigma B)}. \tag{31}$$

The proof of Theorem 2 has been completed.

Theorem 3. *Let φ be a Young function in the class $G(p,q,C)$, $1 \le p < q < \infty$, $C \ge 1$, Ω be a bounded domain, G be Green's operator, and P be the potential operator defined in* (11) *with the kernel $K(x,y)$ satisfying condition (i) of the standard estimates. Assume that $\varphi(|u|) \in L^1_{\mathrm{loc}}(\Omega)$ and u is a solution of the non-homogeneous A-harmonic equation* (1) *in Ω. Then, there exists a constant C, independent of u, such that*

$$\|\nabla Td(G(P(u)))\|_{L^{\varphi}(B)} \le C\|u\|_{L^{\varphi}(\sigma B)} \tag{32}$$

for all balls B with $\sigma B \subset \Omega$ for some $\sigma > 1$.

Proof. Replacing u by $d(G(P(u)))$ in the first inequality in (8) and using (18), we find that

$$\|\nabla Td(G(P(u)))\|_{q,B} \le C_1|B|\|d(G(P(u)))\|_{q,B} \le C_2|B|\|u\|_{q,B} \tag{33}$$

holds for any differential form u and $q > 1$. Starting with (33) and using the similar method as we did in the proof of Theorem 2, we can obtain

$$\|\nabla Td(G(P(u)))\|_{L^{\varphi}(B)} \le C\|u\|_{L^{\varphi}(\sigma B)}. \tag{34}$$

The proof of Theorem 3 has been completed.

Now, we are ready to present and prove the main local theorem, the L^{φ}-imbedding theorem, as follows.

Theorem 4. *Let φ be a Young function in the class $G(p,q,C)$, $1 \le p < q < \infty$, $C \ge 1$, Ω be a bounded domain, G be Green's operator, and P be the potential operator defined in* (11) *with the kernel $K(x,y)$ satisfying condition (i) of the* *standard estimates. Assume that* $\varphi(|u|) \in L^1_{\text{loc}}(\Omega)$ *and u is a solution of the non-homogeneous A-harmonic equation* (1) *in* Ω*. Then, there exists a constant C, independent of u, such that*

$$\|G(P(u)) - (G(P(u)))_B\|_{W^{1,\varphi}(B,\wedge^l)} \le C\|u\|_{L^\varphi(\sigma B)} \tag{35}$$

for all balls B with $\sigma B \subset \Omega$ *for some* $\sigma > 1$.

Proof. From (14), (24) and (32), we have

$$\begin{aligned}
&\|G(P(u)) - (G(P(u)))_B\|_{W^{1,\varphi}(B,\wedge^l)} \\
&\quad = \|Td(G(P(u)))\|_{W^{1,\varphi}(B,\wedge^l)} \\
&\quad = (diam(B))^{-1}\|Td(G(P(u)))\|_{L^\varphi(B)} + \|\nabla Td(G(P(u)))\|_{L^\varphi(B)} \\
&\quad \le (diam(B))^{-1}\left(C_1 diam(B)\|u\|_{L^\varphi(\sigma_1 B)}\right) + C_2\|u\|_{L^\varphi(\sigma_2 B)} \\
&\quad \le C_1\|u\|_{L^\varphi(\sigma_1 B)} + C_2\|u\|_{L^\varphi(\sigma_2 B)} \\
&\quad \le C_3\|u\|_{L^\varphi(\sigma B)}
\end{aligned} \tag{36}$$

for all balls B with $\sigma B \subset \Omega$, where $\sigma = \max\{\sigma_1, \sigma_2\}$. The proof of Theorem 4 has been completed.

Note that if we choose $\varphi(t) = t^p \log_+^\alpha t$ or $\varphi(t) = t^p$ in Theorems 1–3 and 4, we will obtain some $L^p(\log_+^\alpha L)$-norm or L^p-norm inequalities, respectively. For example, let $\varphi(t) = t^p \log_+^\alpha t$ in Theorem 4, we have the following imbedding inequalities for $G \circ P$ with the $L^p(\log_+^\alpha L)$-norms.

Corollary 1. *Let* $\varphi(t) = t^p \log_+^\alpha t$, $p \ge 1$ *and* $\alpha \in \mathbb{R}$, *and* Ω *be a bounded domain. Assume that* $\varphi(|u|) \in L^1_{\text{loc}}(\Omega)$ *and u is a solution of the non-homogeneous A-harmonic equation* (1)*. Then, there exists a constant C, independent of u, such that*

$$\|G(P(u)) - (G(P(u)))_B\|_{W^{1,t^p \log_+^\alpha t}(B,\wedge^l)} \le C\|u\|_{L^p(\log_+^\alpha L)(\sigma B)} \tag{37}$$

for all balls B with $\sigma B \subset \Omega$, *where* $\sigma > 1$ *is a constant.*

Selecting $\varphi(t) = t^p$ in Theorem 4, we obtain the usual imbedding inequalities $G \circ P$ with the L^p-norms.

$$\|G(P(u)) - (G(P(u)))_B\|_{W^{1,p}(B,\wedge^l)} \le C\|u\|_{p,\sigma B} \tag{38}$$

for all balls B with $\sigma B \subset \Omega$, where $\sigma > 1$ is a constant. Similarly, if we choose $\varphi(t) = t^p \log_+^\alpha t$ or $\varphi(t) = t^p$ in Theorems 1–4, respectively, we will obtain the corresponding special results.

3 Global Imbedding Theorem

In this section, we prove the global L^{φ}-imbedding inequality $G \circ P$ in bounded and convex domains. We need the following Covering Lemma.

Lemma 3 ([8] Covering Lemma). *Each Ω has a modified Whitney cover of cubes $\mathcal{V} = \{Q_i\}$ such that $\cup_i Q_i = \Omega$, $\sum_{Q_i \in \mathcal{V}} \chi_{\sqrt{\frac{5}{4}}Q} \leq N\chi_{\Omega}$ and some $N > 1$, and if $Q_i \cap Q_j \neq \emptyset$, then there exists a cube R (this cube need not be a member of $\mathcal{V}$) in $Q_i \cap Q_j$ such that $Q_i \cup Q_j \subset NR$. Moreover, if Ω is δ-John, then there is a distinguished cube $Q_0 \in \mathcal{V}$ which can be connected with every cube $Q \in \mathcal{V}$ by a chain of cubes $Q_0, Q_1, \cdots, Q_k = Q$ from $\mathcal{V}$ and such that $Q \subset \rho Q_i$, $i = 0, 1, 2, \cdots, k$, for some $\rho = \rho(n, \delta)$.*

Finally, we are ready to prove another main theorem, the global imbedding inequality with the L^{φ}-norm.

Theorem 5. *Let φ be a Young function in the class $G(p, q, C)$, $1 \leq p < q < \infty$, $C \geq 1$, G be Green's, and P be the potential operator defined in (11) with the kernel $K(x, y)$ satisfying condition (i) of the standard estimates. Assume that $\varphi(|u|) \in L^1(\Omega)$ and u is a solution of the non-homogeneous A-harmonic equation (1) in Ω. Then, there exists a constant C, independent of u, such that*

$$\|G(P(u)) - (G(P(u)))_{\Omega}\|_{W^{1,\varphi}(\Omega)} \leq C\|u\|_{L^{\varphi}(\Omega)} \tag{39}$$

holds for any bounded and convex domain Ω.

Proof. From the Covering Lemma and Theorem 2, we find that

$$\begin{aligned} \|Td(G(P(u)))\|_{L^{\varphi}(\Omega)} &\leq \sum_{B \in \mathcal{V}} \|Td(G(P(u)))\|_{L^{\varphi}(B)} \\ &\leq \sum_{B \in \mathcal{V}} \left(C_1 diam(B)\|u\|_{L^{\varphi}(\sigma B)}\right) \\ &\leq C_2 diam(\Omega) N\|u\|_{L^{\varphi}(\Omega)} \\ &\leq C_3 diam(\Omega)\|u\|_{L^{\varphi}(\Omega)}, \end{aligned} \tag{40}$$

where N is a positive integer appearing in the Covering Lemma. Similarly, from the Covering Lemma and Theorem 3, we obtain

$$\begin{aligned} \|\nabla Td(G(P(u)))\|_{L^{\varphi}(\Omega)} &\leq \sum_{B \in \mathcal{V}} \|\nabla Td(G(P(u)))\|_{L^{\varphi}(B)} \\ &\leq \sum_{B \in \mathcal{V}} \left(C_4|B|\|u\|_{L^{\varphi}(\sigma B)}\right) \\ &\leq C_5 N\|u\|_{L^{\varphi}(\Omega)} \\ &\leq C_6\|u\|_{L^{\varphi}(\Omega)}. \end{aligned} \tag{41}$$

Thus, from (14), (40), and (41), we have

$$\begin{aligned}
&\|G(P(u)) - (G(P(u)))_\Omega\|_{W^{1,\varphi}(\Omega)} \\
&\quad = \|Td(G(P(u)))\|_{W^{1,\varphi}(\Omega)} \\
&\quad = (diam(\Omega))^{-1}\|Td(G(P(u)))\|_{L^\varphi(\Omega)} + \|\nabla Td(G(P(u)))\|_{L^\varphi(\Omega)} \\
&\quad \le (diam(\Omega))^{-1}(C_3 diam(\Omega)\|u\|_{L^\varphi(\Omega)}) + C_6\|u\|_{L^\varphi(\Omega)} \\
&\quad \le C_7\|u\|_{L^\varphi(\Omega)}. \qquad (42)
\end{aligned}$$

We have completed the proof of Theorem 5

From Imbedding inequality (39), we have the following global Poincaré-type inequality with L^φ-norm.

Theorem 6. *Let φ be a Young function in the class $G(p,q,C)$, $1 \le p < q < \infty$, $C \ge 1$, G be Green's, and P be the potential operator defined in (11) with the kernel $K(x,y)$ satisfying condition (i) of the standard estimates. Assume that $\varphi(|u|) \in L^1(\Omega)$ and u is a solution of the non-homogeneous A-harmonic equation (1) in Ω. Then, there exists a constant C, independent of u, such that*

$$\|G(P(u)) - (G(P(u)))_\Omega\|_{L^\varphi(\Omega)} \le C\, diam(\Omega)\|u\|_{L^\varphi(\Omega)} \qquad (43)$$

holds for any bounded and convex domain Ω.

Proof. Thus, from (14) and (39), we have

$$\begin{aligned}
&\|G(P(u)) - (G(P(u)))_\Omega\|_{L^\varphi(\Omega)} \\
&\quad \le diam(\Omega)\left((diam(\Omega))^{-1}\|G(P(u)) - (G(P(u)))_\Omega\|_{L^\varphi(\Omega)}\right) \\
&\qquad + diam(\Omega)\left(\|\nabla(G(P(u)) - (G(P(u)))_\Omega)\|_{L^\varphi(\Omega)}\right) \\
&\quad = diam(\Omega)\|G(P(u)) - (G(P(u)))_\Omega\|_{W^{1,\varphi}(\Omega)} \\
&\quad \le C\, diam(\Omega)\|u\|_{L^\varphi(\Omega)}. \qquad (44)
\end{aligned}$$

We have completed the proof of Theorem 6.

As applications of the global Poincaré-type inequality 43 with L^φ-norm, we can estimate the L^φ-norm of $G(P(u))$ in terms of the L^φ-norm of u.

Corollary 2. *Let φ be a Young function in the class $G(p,q,C)$, $1 \le p < q < \infty$, $C \ge 1$, G be Green's, and P be the potential operator defined in (11) with the kernel $K(x,y)$ satisfying condition (i) of the standard estimates. Assume that $\varphi(|u|) \in L^1(\Omega)$ and u is a solution of the non-homogeneous A-harmonic equation (1) in Ω. Then, there exists a constant C, independent of u, such that*

$$\|G(P(u))\|_{L^\varphi(\Omega)} \le C\, diam(\Omega)\|u\|_{L^\varphi(\Omega)} \qquad (45)$$

holds for any bounded and convex domain Ω.

Proof. For any solution u of the non-homogeneous A-harmonic equation (1), using the global Poincaré-type inequality (43), we have

$$\begin{aligned}\|G(P(u))\|_{L^\varphi(\Omega)} &\le \|G(P(u)) - (G(P(u)))_\Omega\|_{L^\varphi(\Omega)} + \|(G(P(u)))_\Omega\|_{L^\varphi(\Omega)}\\ &\le C_1 diam(\Omega)\|u\|_{L^\varphi(\Omega)} + \|(G(P(u)))_\Omega\|_{L^\varphi(\Omega)}. \end{aligned} \tag{46}$$

Note that for any differential form u and constant $p > 1$, $\|u_\Omega\|_{p,\Omega} \le C_2 diam(\Omega)\|u\|_{p,\Omega}$. Thus, we can also prove that

$$\|(G(P(u)))_\Omega\|_{L^\varphi(\Omega)} \le C_3 diam(\Omega)\|(G(P(u)))_\Omega\|_{L^\varphi(\Omega)} \le C_4 diam(\Omega)\|u\|_{L^\varphi(\Omega)}. \tag{47}$$

Substituting (47) into (46) yields

$$\|G(P(u))\|_{L^\varphi(\Omega)} \le C diam(\Omega)\|u\|_{L^\varphi(\Omega)}.$$

We have completed the proof of Corollary 2.

Choosing $\varphi(t) = t^p \log_+^\alpha t$ in Theorems 5, we obtain the following imbedding inequality with the $L^p(\log_+^\alpha L)$-norms.

Corollary 3. *Let $\varphi(t) = t^p \log_+^\alpha t$, $p \ge 1$, $\alpha \in \mathbb{R}$, Ω be any bounded L^φ-averaging domain, G be Green's, and P be the potential operator defined in (11) with the kernel $K(x,y)$ satisfying condition (i) of the standard estimates. Assume that $\varphi(|u|) \in L^1(\Omega)$ and u is a solution of the non-homogeneous A-harmonic equation (1) in Ω. Then, there exists a constant C, independent of u, such that*

$$\|G(P(u)) - (G(P(u)))_{B_0}\|_{W^{1,t^p \log_+^\alpha t}(\Omega)} \le C\|u\|_{L^{t^p \log_+^\alpha t}(\Omega)} \tag{48}$$

holds for any bounded and convex domain Ω.

Selecting $\varphi(t) = t^p$ in Theorem 5, we have the following imbedding inequality with L^p-norms

$$\|GP(u) - (G(P(u)))_\Omega\|_{W^{1,p}(\Omega)} \le C\|u\|_{p,\Omega} \tag{49}$$

for any bounded and convex domain Ω.

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On Approximation Properties of q-King Operators

Zoltán Finta

Abstract Based on q-integers we introduce the q-King operators which approximate each continuous function on $[0, 1]$ and preserve the functions $e_0(x) = 1$ and $e_j(x) = x^j$. We also construct a q-parametric sequence of polynomial bounded positive linear operators possessing similar properties. In both cases the rate of convergence is estimated with the aid of the modulus of continuity.

Keywords q-integers • q-Bernstein operators • q-King operators • q-derivative • positive linear operators • Korovkin type theorem • ordered normed space • modulus of continuity

1 Introduction

The Bernstein operators $B_n : C[0,1] \to C[0,1]$ are given by $(B_n f)(x) \equiv B_n(f, x)$, $n = 1, 2, \ldots$, where

$$B_n(f,x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right) \equiv \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right) \tag{1}$$

are the so-called Bernstein polynomials. These polynomials have been introduced by Bernstein [1] in 1912 and with the aid of them he gave the proof of the Weierstrass approximation theorem. Later it was found that Bernstein polynomials possess many remarkable properties, which made them an area of intensive research.

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For a systematic treatment of the theory of Bernstein polynomials until 1990s, see e.g. [8]. Nowadays, there are new papers on the subject constantly coming out and generalizations of these polynomials being studied. The aim of these generalizations is to provide appropriate tools for studying various problems of analysis. In this sense we mention the following Bernstein type operator introduced by King [7].

Definition 1. For a sequence of continuous functions $\{r_n\}$ defined on $[0, 1]$ with $0 \leq r_n(x) \leq 1$, $x \in [0, 1]$, the operators $V_n : C[0, 1] \to C[0, 1]$ are given by

$$(V_n f)(x) \equiv V_n(f, x) = \sum_{k=0}^{n} p_{n,k}(r_n(x)) f\left(\frac{k}{n}\right). \tag{2}$$

In the special case $r_n(x) = x$, $n = 1, 2, \ldots$, the expression of (2) reduces to (1). For the choice $r_n = r_n^*$, where

$$r_1^*(x) = x^2,$$
$$r_n^*(x) = -\frac{1}{2(n-1)} + \sqrt{\frac{n}{n-1}x^2 + \frac{1}{4(n-1)^2}}, \quad n = 2, 3, \ldots,$$

we have $V_n e_0 = e_0$, $V_n e_2 = e_2$ and $\lim_{n\to\infty}(V_n f)(x) = f(x)$ for each $f \in C[0, 1]$ and $x \in [0, 1]$. We have denoted by e_s, $s \geq 0$, the power function $e_s(x) = x^s$, $x \in [0, 1]$. In comparison with Bernstein operators, the King operators preserve the functions e_0 and e_2, and not the functions e_0 and e_1. Moreover, in [7] quantitative estimates and connections with summability are discussed. The quantitative estimates are obtained with the aid of the modulus of continuity of $f \in C[0, 1]$, given by

$$\omega(f, \delta) = \sup\{|f(x) - f(y)| : x, y \in [0, 1], |x - y| \leq \delta\}, \quad \delta > 0. \tag{3}$$

The development of q-calculus has led to the discovery of new Bernstein type operators involving q-integers. The first example in this direction was given by Lupaş [10] in 1987. If $q \neq 1$ then the Lupaş q-analogue of the Bernstein operators gives rational function rather than polynomial. New results concerning convergence of the Lupaş operators can be found in [12] and [2]. The so-called q-Bernstein operators were introduced by Phillips [15] in 1997 and they mean another generalization of Bernstein operators based on the q-integers. Nowadays, q-Bernstein operators form an area of an intensive research. A survey of the obtained main results and references in this area during the first decade of study can be found in [13]. It is worth mentioning that in [3] direct and converse theorems are established for the q-Bernstein operators. The direct approximation theorem is given for the second-order Ditzian–Totik modulus of smoothness, and the converse result is a theorem of Berens–Lorentz type. Further, in [4] sufficient conditions are established to insure the convergence of a sequence of positive linear operators defined on $C[0, 1]$. As applications quantitative estimates for some q-Bernstein type operators are obtained. The convergence properties of q-Bernstein operators $(0 < q < 1)$ in the complex plane were studied in [14].

To present the q-Bernstein operators we recall some notions of the q-calculus (cf. [6] and [16]). Let $q > 0$. For each non-negative integer n, the q-integers $[n]$ and the q-factorials $[n]!$ are defined by

$$[n] = \begin{cases} 1 + q + \ldots + q^{n-1}, & \text{if } n \geq 1 \\ 0, & \text{if } n = 0 \end{cases} \quad \text{and} \quad [n]! = \begin{cases} [1][2]\ldots[n], & \text{if } n \geq 1 \\ 1, & \text{if } n = 0. \end{cases}$$

For integers $0 \leq k \leq n$, the q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}.$$

If $q = 1$ then $[n] = n$, $[n]! = n!$ and $\begin{bmatrix} n \\ k \end{bmatrix} = \binom{n}{k}$.

Definition 2. For $q > 0$ the q-Bernstein operators $B_{n,q} : C[0,1] \to C[0,1]$ are given by $(B_{n,q}f)(x) \equiv B_{n,q}(f,x)$, $n = 1, 2, \ldots,$ where the q-Bernstein polynomials are defined by

$$B_{n,q}(f,x) = \sum_{k=0}^{n} p_{n,k}(q;x) f\left(\frac{[k]}{[n]}\right)$$
$$\equiv \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k (1-x)(1-qx)\ldots(1-q^{n-k-1}x) f\left(\frac{[k]}{[n]}\right) \tag{4}$$

(for $k = n$ the empty product is taken to equal 1). When $q = 1$, we recover the Bernstein polynomials: $B_{n,1}(f,x) = B_n(f,x)$. In the case $0 < q < 1$, the q-Bernstein operators are positive linear operators on $C[0,1]$ with $\|B_{n,q}\|=1$. Moreover, $B_{n,q}$ are variation-diminishing, which imply that for an increasing (decreasing) function f on $[0,1]$ we have that $B_{n,q}f$ is also increasing (decreasing) on $[0,1]$, and if f is convex (concave) on $[0,1]$ then so is $B_{n,q}f$. Furthermore,

$$B_{n,q}(e_0,x) = 1, \; B_{n,q}(e_1,x) = x \quad \text{and} \quad B_{n,q}(e_2,x) = x^2 + \frac{1}{[n]}x(1-x). \tag{5}$$

If $q = q_n$ satisfy $0 < q_n < 1$ and $q_n \to 1$ as $n \to \infty$, then $B_{n,q}(f,x)$ converges uniformly to $f(x)$ on $[0,1]$ as $n \to \infty$, and moreover

$$|B_{n,q}(f,x) - f(x)| \leq \frac{3}{2}\,\omega(f, [n]^{-1/2}). \tag{6}$$

Here we mention that $[n] \to \infty$ as $n \to \infty$. Indeed, for any fixed positive integer k, we have $[n] \geq [k] = 1 + q + \ldots + q^{k-1}$ when $n \geq k$. But $q = q_n \to 1$

as $n \to \infty$, therefore $\liminf_{n\to\infty}[n] \geq \liminf_{n\to\infty}[k] = k$. Since k has been chosen arbitrarily, it follows that $[n] \to \infty$ as $n \to \infty$. For other properties of the q-Bernstein polynomials, see [16].

Taking into account Definition 1 and Definition 2, we may introduce the q-King operators as follows.

Definition 3. For $q > 0$ and for a sequence of continuous functions $\{r_{n,q}\}$ defined on $[0,1]$ with $0 \leq r_{n,q}(x) \leq 1$, $x \in [0,1]$ and $n = 1, 2, \ldots$, the q-King operators $V_{n,q} : C[0,1] \to C[0,1]$ are defined by

$$(V_{n,q}f)(x) \equiv V_{n,q}(f,x) = \sum_{k=0}^{n} p_{n,k}(q; r_{n,q}(x)) f\left(\frac{[k]}{[n]}\right). \qquad (7)$$

Obviously $V_{n,1}(f,x) = V_n(f,x)$ and $V_{n,q}(f,x) = B_{n,q}(f, r_{n,q}(x))$.

The goal of the paper is to prove the existence of a sequence $\{V_{n,q}\}$ of type (7), $0 < q < 1$, which approximate each continuous function on $[0,1]$ such that $V_{n,q}e_0 = e_0$ and $V_{n,q}e_j = e_j$, where $j \in \{2,3,\ldots\}$ is fixed. Moreover, for $0 < q < 1$ we construct a sequence $\{L_{n,q}\}$ of polynomial bounded positive linear operators which approximate each continuous function on $[0,1]$, and $L_{n,q}e_0 = e_0$ and $L_{n,q}e_j = e_j$, respectively. In both cases the rate of convergence will be estimated by the modulus of continuity (3), obtaining quantitative estimates.

2 Auxiliary Results

In the sequel we need some lemmas.

Lemma 1. *Let $j \in \{2,3,\ldots\}$ be given, $n \geq j$ and $0 < q < 1$. Then*

$$P_{n,q,j}(y) = \sum_{k=0}^{n} p_{n,k}(q; y)\left(\frac{[k]}{[n]}\right)^j, \quad y \in [0,1],$$

is a polynomial in y of degree $\leq j$. Moreover, $P_{n,q,j}(y) = a_0 y^j + a_1 y^{j-1} + \ldots + a_{j-1} y$, $y \in [0,1]$, where $a_0, a_1, \ldots, a_{j-1}$ depend on n and q, and $a_0 = \left(1 - \frac{[1]}{[n]}\right)\left(1 - \frac{[2]}{[n]}\right)\ldots\left(1 - \frac{[j-1]}{[n]}\right)$, $a_1, \ldots, a_{j-1} > 0$, $a_0 + a_1 + \ldots + a_{j-1} = 1$.

Proof. The q-derivative of a function f, denoted by $D_q f$, is defined by

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}, \ x \neq 0 \quad \text{and} \quad D_q f(0) = \lim_{x\to 0} D_q f(x).$$

The formula for the q-derivative of a product is

$$D_q(fg)(x) = D_q f(x) g(qx) + f(x) D_q g(x) \qquad (8)$$

(for details see [6, pp. 1–3]). Hence, by (8),

$$\begin{aligned} & y(1-y)D_q p_{n,k}(q;\cdot)(y) \\ &= y(1-y)\left\{[k]\begin{bmatrix} n \\ k \end{bmatrix} y^{k-1}(1-qy)(1-q^2y)\dots(1-q^{n-k}y)\right. \\ &\quad \left. -[n-k]\begin{bmatrix} n \\ k \end{bmatrix} y^{k}(1-qy)(1-q^2y)\dots(1-q^{n-k-1}y)\right\} \\ &= [k]p_{n,k}(q;y)(1-q^{n-k}y) - y[n-k]p_{n,k}(q;y) \\ &= p_{n,k}(q;y)([k]-[n]y). \end{aligned}$$

Thus

$$\begin{aligned} D_q P_{n,q,j}(y) &= \sum_{k=0}^{n} D_q p_{n,k}(q;\cdot)(y)\left(\frac{[k]}{[n]}\right)^j = \sum_{k=0}^{n} p_{n,k}(q;y)\frac{[k]-[n]y}{y(1-y)}\left(\frac{[k]}{[n]}\right)^j \\ &= \frac{[n]}{y(1-y)}\sum_{k=0}^{n} p_{n,k}(q;y)\left(\frac{[k]}{[n]}-y\right)\left(\frac{[k]}{[n]}\right)^j \\ &= \frac{[n]}{y(1-y)}P_{n,q,j+1}(y) - \frac{[n]y}{y(1-y)}P_{n,q,j}(y). \end{aligned}$$

Hence the following relation of recurrence is obtained:

$$P_{n,q,j+1}(y) = yP_{n,q,j}(y) + \frac{1}{[n]}y(1-y)D_q P_{n,q,j}(y). \tag{9}$$

Because $P_{n,q,j}(y) = (B_{n,q}e_j)(y)$, where $B_{n,q}e_j$ is a polynomial of degree $\min\{n,j\} = j$ (see [15, p. 513]) and $D_q e_j(y) = [j]y^{j-1}$, we obtain that $D_q P_{n,q,j}$ is a polynomial in y of degree $j-1$. In conclusion, by induction, using the relation (9), we find that $P_{n,q,j}$ is a polynomial in y of degree $\le j$. Thus follows the first statement of the lemma.

We may write $P_{n,q,j}(y) = a_0y^j + a_1y^{j-1} + \ldots + a_{j-1}y + a_j$, where $a_i = a_i(n,q)$, $i = 0,1,\ldots,j$. Then $a_j = P_{n,q,j}(0) = (B_{n,q}e_j)(0) = 0$ and $a_0 + a_1 + \ldots + a_j = P_{n,q,j}(1) = (B_{n,q}e_j)(1) = 1$. Hence $a_0 + a_1 + \ldots + a_{j-1} = 1$. From the recurrence formula (9) follows by induction the expression of the coefficient a_0 and the positivity of $a_1,\ldots,a_{j-1}$, respectively. □

Lemma 2. *(a) For $j \in \{2,3,\ldots\}$ and $n \ge j$, we have*

$$0 \le 1-\left(1-\frac{[1]}{[n]}\right)\left(1-\frac{[2]}{[n]}\right)\dots\left(1-\frac{[j-1]}{[n]}\right) \le \frac{[1]+[2]+\ldots+[j-1]}{[n]}.$$

(b) For $j \in \{1,2,\ldots\}$ and $u,v \in [0,1]$, we have $(u-v)^{2j} \le (u^j-v^j)^2$.

Proof. (a) We prove it by induction. For $j = 2$ it is obvious. We shall show that if the inequality holds for j, then it holds for $j + 1$. Indeed, we have

$$\begin{aligned} 0 &\leq 1 - \left(1 - \frac{[1]}{[n]}\right)\left(1 - \frac{[2]}{[n]}\right)\cdots\left(1 - \frac{[j-1]}{[n]}\right)\left(1 - \frac{[j]}{[n]}\right) \\ &= 1 - \left(1 - \frac{[1]}{[n]}\right)\left(1 - \frac{[2]}{[n]}\right)\cdots\left(1 - \frac{[j-1]}{[n]}\right) \\ &\quad + \left(1 - \frac{[1]}{[n]}\right)\left(1 - \frac{[2]}{[n]}\right)\cdots\left(1 - \frac{[j-1]}{[n]}\right)\frac{[j]}{[n]} \\ &\leq \frac{[1] + [2] + \ldots + [j-1])}{[n]} + \frac{[j]}{[n]} = \frac{[1] + [2] + \ldots + [j]}{[n]}. \end{aligned}$$

(b) We prove it by induction. For $j = 1$ it is obvious. We shall show that if the inequality holds for j, then it holds for $j + 1$. At the same time, without loss of generality, we may suppose that $u \geq v$. Then $(u - v)^{2j+2} \leq (u^j - v^j)^2(u - v)^2 \leq (u^{j+1} - v^{j+1})^2$, because the last inequality is equivalent with $v^j(u - v) + v(u^j - v^j) \geq 0$. □

In the next lemma we establish a Korovkin type theorem.

Lemma 3. *Let $\{U_n\}$ be a sequence of positive linear operators such that $U_n : C[0,1] \to C[0,1]$, $n = 1, 2, \ldots$, and $\lim_{n\to\infty}(U_n e_s)(x) = e_s(x)$ uniformly for $x \in [0,1]$, where $s \in \{0, j/2, j\}$ and $j \in \{2, 3, \ldots\}$ is given. Then $\lim_{n\to\infty}(U_n f)(x) = f(x)$ uniformly for $x \in [0,1]$, where $f \in C[0,1]$ is arbitrary.*

Proof. Because $P_y(x) = (y^{j/2} - x^{j/2})^2 = y^j e_0(x) - 2y^{j/2}e_{j/2}(x) + e_j(x) \geq 0$ for all $x, y \in [0,1]$, and $P_y(x) = 0$ if and only if $x = y$, we obtain the validity of the lemma because of [9, p. 7, Theorem 3]. □

Before starting the next lemma we recall some notions concerning ordered normed spaces. A real linear space X is said to be *ordered linear space* if X is equipped with an order relation $\leq$ satisfying the conditions: $x, y, z \in X$, $x \leq y \Rightarrow x + z \leq y + z$; $x, y \in X$, $x \leq y$, $\alpha \geq 0$ $(\alpha \in \mathbf{R}) \Rightarrow \alpha x \leq \alpha y$. We denote by X_+ the set of all positive elements of X, i.e. $X_+ = \{x \in X : 0_X \leq x\}$. An ordered linear space X is said to be *ordered normed space* if there exists a norm $\|\cdot\|_X$ on X such that $0_X \leq x \leq y \Rightarrow \|x\|_X \leq \|y\|_X$.

Lemma 4 ([11, p. 82]). *Let X be an ordered normed space with $intX_+ \neq \emptyset$ and Y a normed subspace of X such that $Y \cap intX_+ \neq \emptyset$. If $\lambda : Y \to \mathbf{R}$ is a bounded positive linear functional, then there exists a bounded positive linear functional $\tilde{\lambda} : X \to \mathbf{R}$ such that $\tilde{\lambda}(x) = \lambda(x)$ for all $x \in Y$.*

3 Main Results

Theorem 1. *Let $j \in \{2, 3, \ldots\}$ be given, $n \geq j$ and let $q = q_n \in (0, 1)$ satisfy $q_n \to 1$ as $n \to \infty$. Then there exists a sequence $\{r_{n,q}\}$ of continuous functions on $[0, 1]$, $0 \leq r_{n,q}(x) \leq 1$ for $x \in [0, 1]$ and $n \geq j$, such that the operators $V_{n,q}$ defined by (7) satisfy the following properties:*

(i) $V_{n,q}e_0 = e_0, V_{n,q}e_j = e_j$;

(ii) $\lim_{n\to\infty} (V_{n,q} f)(x) = f(x)$ *uniformly for* $x \in [0, 1]$, *for all* $f \in C[0, 1]$;

(iii) $|(V_{n,q} f)(x)-f(x)| \leq \left\{1 + \sqrt[2j]{8\sqrt{j}([1] + [2] + \ldots + [j-1])^{1/2j} + 6j}\right\}\times \omega(f, [n]^{-1/4j^2})$ *for every* $f \in C[0, 1]$ *and* $x \in [0, 1]$.

Proof. We consider the function $\varphi_x : [0, 1] \to [0, 1]$, $\varphi_x(y) = P_{n,q,j}(y)-x^j$, where $x \in [0, 1]$ is arbitrary. We prove that the equation $\varphi_x(y) = 0$ has a unique solution $y = y(x)$. Indeed, in view of Lemma 1, if $x = 0$ then we set $y = y(0) = 0$, because $P_{n,q,j}(0) = 0$; if $x = 1$ then we set $y = y(1) = 1$, because $P_{n,q,j}(1) = 1$. Further, for $x \in (0, 1)$ we have $\varphi_x(0) \cdot \varphi_x(1) = -x^j(1-x^j) < 0$. Then there exists $y = y(x) \in (0, 1)$ such that $\varphi_x(y) = 0$. Again, in view of Lemma 1, we have $\varphi_x'(y) = ja_0y^{j-1} + (j-1)a_1y^{j-2} + \ldots + a_{j-1} > 0$ for $y \in [0, 1]$, therefore φ_x is increasing function on $[0, 1]$, so $\varphi_x(y) = 0$ has a unique solution.

Now let $r_{n,q}(x) = y(x)$, where $x \in [0, 1]$ and $n \geq j$. Then $0 \leq r_{n,q}(x) \leq 1$ for $x \in [0, 1]$ and $r_{n,q} \in C[0, 1]$, because the conditions $P_{n,q,j}(r_{n,q}(x)) = x^j$ and $P_{n,q,j}(r_{n,q}(x_0)) = x_0^j$ imply

$$\begin{aligned}
&\{r_{n,q}(x) - r_{n,q}(x_0)\}\{a_0[(r_{n,q}(x))^{j-1} + \ldots + (r_{n,q}(x_0))^{j-1}] \\
&\quad + a_1[(r_{n,q}(x))^{j-2} + \ldots + (r_{n,q}(x_0))^{j-2}] + \ldots \\
&\quad + a_{j-2}[r_{n,q}(x) + r_{n,q}(x_0)] + a_{j-1}\} \\
&= (x - x_0)(x^{j-1} + x^{j-2}x_0 + \ldots + xx_0^{j-2} + x_0^{j-1}).
\end{aligned}$$

Hence, by Lemma 1, $a_{j-1}|r_{n,q}(x) - r_{n,q}(x_0)| \leq j|x - x_0|$, which implies the continuity of $r_{n,q}$ in x_0.

For the constructed sequence $\{r_{n,q}\}$ we prove the statements (i)–(iii).

(i) In view of (7) and (5), we have $(V_{n,q}e_0)(x) = (B_{n,q}e_0)(r_{n,q}(x)) = 1 = e_0(x)$. Further, by (7) and Lemma 1, $(V_{n,q}e_j)(x) = P_{n,q,j}(r_{n,q}(x)) = x^j = e_j(x)$.

(ii) Taking into account Lemmas 1 and 2 (a), we obtain

$$\begin{aligned}
&|x^j - (r_{n,q}(x))^j| = |P_{n,q,j}(r_{n,q}(x)) - (r_{n,q}(x))^j| \\
&= |(a_0 - 1)(r_{n,q}(x))^j + a_1(r_{n,q}(x))^{j-1} + \ldots + a_{j-1}r_{n,q}(x)| \\
&= |-(a_1 + \ldots + a_{j-1})(r_{n,q}(x))^j + a_1(r_{n,q}(x))^{j-1} + \ldots + a_{j-1}r_{n,q}(x)|
\end{aligned}$$

$$= a_1[(r_{n,q}(x))^{j-1} - (r_{n,q}(x))^j] + \ldots + a_{j-1}[r_{n,q}(x) - (r_{n,q}(x))^j]$$
$$\leq a_1 + \ldots + a_{j-1} = 1 - a_0$$
$$= 1 - \left(1 - \frac{[1]}{[n]}\right)\left(1 - \frac{[2]}{[n]}\right)\ldots\left(1 - \frac{[j-1]}{[n]}\right) \leq \frac{[1]+[2]+\ldots+[j-1]}{[n]}.$$

Hence, by Lemma 2 (b), we get

$$(x - r_{n,q}(x))^{2j} \leq (x^j - (r_{n,q}(x))^j)^2 \leq \left(\frac{[1]+[2]+\ldots+[j-1]}{[n]}\right)^2.$$

In conclusion

$$|x - r_{n,q}(x)| \leq \left(\frac{[1]+[2]+\ldots+[j-1]}{[n]}\right)^{1/j}. \tag{10}$$

On the other hand, in view of (7) and (5), $x - (V_{n,q}e_1)(x) = x - (B_{n,q}e_1)(r_{n,q}(x)) = x - r_{n,q}(x)$ and $x^2 - (V_{n,q}e_2)(x) = x^2 - (B_{n,q}e_2)(r_{n,q}(x)) = x^2 - (r_{n,q}(x))^2 - \frac{1}{[n]} r_{n,q}(x)\ (1 - r_{n,q}(x))$. Hence, due to (10),

$$|(V_{n,q}e_1)(x) - e_1(x)| \leq \left(\frac{[1]+[2]+\ldots+[j-1]}{[n]}\right)^{1/j} \to 0$$

and

$$|(V_{n,q}e_2)(x) - e_2(x)| \leq 2\left(\frac{[1]+[2]+\ldots+[j-1]}{[n]}\right)^{1/j} + \frac{1}{[n]} \to 0$$

as $n \to \infty$. In conclusion, by Lemma 3 (case $j = 2$), we find that $\lim_{n\to\infty}(V_{n,q}f)(x) = f(x)$ uniformly for $x \in [0, 1]$.

(iii) Due to (5), (3), the property $\omega(f, a\delta) \leq (1 + a)\omega(f, \delta)$, $a > 0$, (5) and Hölder's inequality, we have

$$|(V_{n,q}f)(x) - f(x)| \leq \sum_{k=0}^{n} p_{n,k}(q; r_n(x)) \left| f\left(\frac{[k]}{[n]}\right) - f(x) \right|$$
$$\leq \sum_{k=0}^{n} p_{n,k}(q; r_n(x))\, \omega\left(f, \left|\frac{[k]}{[n]} - x\right|\right)$$
$$\leq \omega(f, \delta) \sum_{k=0}^{n} p_{n,k}(q; r_n(x)) \left\{1 + \delta^{-1}\left|\frac{[k]}{[n]} - x\right|\right\}$$

$$= \omega(f,\delta)\left\{1+\delta^{-1}\left(\sum_{k=0}^{n} p_{n,k}(q;r_n(x))\left(\frac{[k]}{[n]}-x\right)^{2j}\right)^{1/2j}\right\}. \quad (11)$$

But, in view of Lemma 2 (b),

$$\left(\frac{[k]}{[n]}-x\right)^{2j} \le \left(\left(\frac{[k]}{[n]}\right)^{j}-x^{j}\right)^{2}$$
$$= \left(\left(\frac{[k]}{[n]}\right)^{j/2}+x^{j/2}\right)^{2}\left(\left(\frac{[k]}{[n]}\right)^{j/2}-x^{j/2}\right)^{2} \le 4\left(\left(\frac{[k]}{[n]}\right)^{j/2}-x^{j/2}\right)^{2}.$$

Hence, by (11) and (i),

$$|(V_{n,q}f)(x)-f(x)|$$
$$\le \omega(f,\delta)\left\{1+\delta^{-1}\left(4\sum_{k=0}^{n} p_{n,k}(q;r_{n,q}(x))\left(\left(\frac{[k]}{[n]}\right)^{j/2}-x^{j/2}\right)^{2}\right)^{1/2j}\right\}$$
$$= \omega(f,\delta)\left\{1+2^{1/j}\delta^{-1}\left((V_{n,q}e_j)(x)-2x^{j/2}(V_{n,q}e_{j/2})(x)+x^{j}\right)^{1/2j}\right\}$$
$$= \omega(f,\delta)\left\{1+2^{1/j}\delta^{-1}\left(2x^{j}-2x^{j/2}(V_{n,q}e_{j/2})(x)\right)^{1/2j}\right\}$$
$$\le \omega(f,\delta)\left\{1+2^{3/2j}\delta^{-1}\left(x^{j/2}-(V_{n,q}e_{j/2})(x)\right)^{1/2j}\right\}. \quad (12)$$

On the other hand

$$|x^{j/2}-(V_{n,q}e_{j/2})(x)|$$
$$\le |x^{j/2}-(r_{n,q}(x))^{j/2}|+|(r_{n,q}(x))^{j/2}-(B_{n,q}e_{j/2})(r_{n,q}(x))|. \quad (13)$$

Again, by Lemma 2 (b) (case $j=2$), we have

$$(x^{j/2}-(r_{n,q}(x))^{j/2})^4 \le (x^{j}-(r_{n,q}(x))^{j})^2$$
$$= (x-r_{n,q}(x))^2\{x^{j-1}+x^{j-2}r_{n,q}(x)+\ldots+(r_{n,q}(x))^{j-1}\}^2$$
$$\le j^2(x-r_{n,q}(x))^2.$$

Hence, by (10),

$$|x^{j/2}-(r_{n,q}(x))^{j/2}| \le \sqrt{j}\,|x-r_{n,q}(x)|^{1/2}$$
$$\le \sqrt{j}\left(\frac{[1]+[2]+\ldots+[j-1]}{[n]}\right)^{1/2j}. \quad (14)$$

Further, due to (6) and the property $\omega(f,\delta) \le \delta\|f'\|$, where $\|\cdot\|$ is the uniform norm on $C[0,1]$, we have

$$|(r_{n,q}(x))^{j/2} - (B_{n,q}e_{j/2})(r_{n,q}(x))| \le \frac{3}{2}\omega(e_{j/2},[n]^{-1/2}) \le \frac{3}{4}j[n]^{-1/2}. \tag{15}$$

Combining (13), (14) and (15), we obtain

$$\begin{aligned}|x^{j/2} - (V_{n,q}e_{j/2})(x)| &\le \sqrt{j}\left(\frac{[1]+[2]+\ldots+[j-1]}{[n]}\right)^{1/2j} + \frac{3}{4}j[n]^{-1/2}\\ &\le \left\{\sqrt{j}\,([1]+[2]+\ldots+[j-1])^{1/2j} + \frac{3}{4}j\right\}[n]^{-1/2j}.\end{aligned}$$

Hence, by (12),

$$\begin{aligned}&|(V_{n,q}f)(x) - f(x)|\\ &\le \omega(f,\delta)\left\{1 + \delta^{-1}\sqrt[2j]{8\sqrt{j}([1]+[2]+\ldots+[j-1])^{1/2j} + 6j}\;[n]^{-1/4j^2}\right\}.\end{aligned}$$

Choosing $\delta = [n]^{-1/4j^2}$, we get the desired result. □

Theorem 2. *Let $j \in \{2,3,\ldots\}$ be given, $n \ge j$ and let $q = q_n \in (0,1)$ satisfy $q_n \to 1$ as $n \to \infty$. Then there exist polynomial bounded positive linear operators $L_{n,q} : C[0,1] \to C[0,1]$ such that*

(*i*) $L_{n,q}e_0 = e_0, L_{n,q}e_j = e_j$;
(*ii*) $\lim_{n\to\infty}(L_{n,q}f)(x) = f(x)$ *uniformly for* $x \in [0,1]$, *for all* $f \in C[0,1]$;
(*iii*) $|(L_{n,q}f)(x) - f(x)| \le \left\{1 + \sqrt[2j]{6j + 4j[j-1] + 16}\right\}\omega(f,[n]^{-1/4j})$ *for every* $f \in C[0,1]$ *and* $x \in [0,1]$.

Proof. We define the operators $L_{n,q} : C[0,1] \to C[0,1]$ by

$$(L_{n,q}f)(x) \equiv L_{n,q}(f,x) = \sum_{k=0}^{n} p_{n,k}(q;x)\,\lambda_{n,k}(f), \tag{16}$$

where the bounded positive linear functionals $\lambda_{n,k} \in C[0,1]^*$ are defined step by step as follows. We set $\lambda_{n,0}(f) = f(0)$ and $\lambda_{n,n}(f) = f(1)$ for $f \in C[0,1]$ and $n = 1,2,\ldots$ If $n \ge j$ and $k = 1,2,\ldots,n-1$, then we set

$$\begin{gathered}\lambda_{n,k}(e_0) = 1, \qquad \lambda_{n,k}(e_j) = \tfrac{[k][k-1]\ldots[k-j+1]}{[n][n-1]\ldots[n-j+1]},\\ \lambda_{n,k}(e_{j/2}) = \begin{cases}\tfrac{[k-1]}{[n]}\left(\tfrac{[k-2][k-3]\ldots[k-j+1]}{[n-2][n-3]\ldots[n-j+1]}\right)^{1/2}, & \text{if } n \ge j > 3\\ \tfrac{[k-1]}{[n]}, & \text{if } n \ge j = 2.\end{cases}\end{gathered} \tag{17}$$

Then we get

$$\lambda_{n,k}(e_j) \le \lambda_{n,k}(e_{j/2}) \le (\lambda_{n,k}(e_j))^{1/2} \le \lambda_{n,k}(e_{j/2}) + \frac{2}{[n]} \tag{18}$$

for all $n \ge j$ and $k = 1, 2, \ldots, n-1$. We justify only the last inequality, the others follow by simple computations; due to (17), we have

$$\begin{aligned}
&(\lambda_{n,k}(e_j))^{1/2} - \lambda_{n,k}(e_{j/2}) \\
&= \left(\frac{[k-1][k-2]\ldots[k-j+1]}{[n][n-2]\ldots[n-j+1]}\right)^{1/2} \left\{\sqrt{\frac{[k]}{[n-1]}} - \sqrt{\frac{[k-1]}{[n]}}\right\} \\
&\le \sqrt{\frac{[k-1]}{[n]}} \frac{\frac{[k]}{[n-1]} - \frac{[k-1]}{[n]}}{\sqrt{\frac{[k]}{n-1]}} + \sqrt{\frac{[k-1]}{[n]}}} \le \frac{[k]}{[n-1]} - \frac{[k-1]}{[n]} \\
&= \frac{[n][k] - [n-1][k-1]}{[n-1]} \frac{1}{[n]} \le \frac{2}{[n]},
\end{aligned}$$

because $[n][k]-[n-1][k-1] = (1+q[n-1])[k]-[n-1][k-1]=[k]+[n-1](q[k]-[k-1]) = [k] + [n-1]\{q - (1-q^2)[k-1]\} \le [k] + [n-1] \le 2[n-1]$ for $k = 1, 2, \ldots, n-1$ and $n \ge j > 3$; analogously

$$\begin{aligned}
(\lambda_{n,k}(e_j))^{1/2} - \lambda_{n,k}(e_{j/2}) &= \left(\frac{[k][k-1]}{[n][n-1]}\right)^{1/2} - \frac{[k-1]}{[n]} \\
&= \sqrt{\frac{[k-1]}{[n]}} \left\{\sqrt{\frac{[k]}{[n-1]}} - \sqrt{\frac{[k-1]}{[n]}}\right\} \le \frac{2}{[n]}
\end{aligned}$$

for $k = 1, 2, \ldots, n-1$ and $n \ge j = 2$.

Further, $\lambda_{n,k}$ $(k = 1, 2, \ldots, n-1)$ will be defined on the linear subspace $Y = \{\alpha e_0 + \beta e_{j/2} + \gamma e_j \,|\, \alpha, \beta, \gamma \in \mathbf{R}\}$ of the linear space $C[0, 1]$ as follows: for $P = \alpha e_0 + \beta e_{j/2} + \gamma e_j$ we set $\lambda_{n,k}(P) = \alpha\lambda_{n,k}(e_0) + \beta\lambda_{n,k}(e_{j/2}) + \gamma\lambda_{n,k}(e_j)$.

We prove that $\lambda_{n,k} \in Y^*$ $(k = 1, 2, \ldots, n-1)$ are bounded positive linear functionals. Obviously $\lambda_{n,k}$ are linear. Moreover, $\lambda_{n,k}$ are positive: if $P(x) \ge 0$ for $x \in [0, 1]$, then we distinguish the following two cases:

(a) $\gamma \ge 0$. By (18), we have $\lambda_{n,k}(P) = \alpha + \beta\lambda_{n,k}(e_{j/2}) + \gamma\lambda_{n,k}(e_j) \ge \alpha + \beta\lambda_{n,k}(e_{j/2}) + \gamma\,(\lambda_{n,k}(e_{j/2}))^2 = P((\lambda_{n,k}(e_{j/2}))^{2/j}) \ge 0$;
(b) $\gamma < 0$. By (18), we have $\lambda_{n,k}(P) = \alpha + \beta\lambda_{n,k}(e_{j/2}) + \gamma\lambda_{n,k}(e_j) \ge \alpha + \beta\lambda_{n,k}(e_{j/2}) + \gamma\lambda_{n,k}(e_{j/2}) = P(0)(1 - \lambda_{n,k}(e_{j/2})) + P(1)\lambda_{n,k}(e_{j/2}) \ge 0$.

Further, $\lambda_{n,k}$ $(k = 1, 2, \ldots, n-1)$ are bounded on Y. Indeed, the positivity of $\lambda_{n,k}$ imply for all $P \in Y$ that $|\lambda_{n,k}(P)| \le \lambda_{n,k}(|P|) \le \lambda_{n,k}(\|P\|e_0) = \|P\|\lambda_{n,k}(e_0) = \|P\|$, where $\|\cdot\|$ denotes the uniform norm on $C[0, 1]$.

Finally, we define $\lambda_{n,k}$ $(k = 1, 2, \ldots, n-1)$ on the whole space $C[0,1]$. The real linear space $C[0,1]$ is an ordered Banach space with $\|\cdot\|$ and the natural order relation: $f \leq g$ if and only if $f(x) \leq g(x)$, $x \in [0,1]$. Using the notation $C[0,1]_+ = \{f \in C[0,1] : 0_{C[0,1]} \leq f\}$, we have $\{f \in C[0,1] : \|f - e_0\| < 1\} \subset C[0,1]_+$. Thus $int\, C[0,1]_+ \neq \emptyset$ and $e_0 \in Y \cap int\, C[0,1]_+$. Now we can extend $\lambda_{n,k}$ on the whole space $C[0,1]$ as bounded positive linear functionals, because of Lemma 4.

(*i*) Obviously, by (16), $L_{n,q}$ is a polynomial operator: $L_{n,q}f$ is a polynomial of degree $\leq n$, for all $f \in C[0,1]$. By (16), (17) and (5), $(L_{n,q}e_0)(x) = (B_{n,q}e_0)(x) = e_0(x)$ and

$$(L_{n,q}e_j)(x) = \sum_{k=0}^{n} p_{n,k}(q;x)\frac{[k][k-1]\ldots[k-j+1]}{[n][n-1]\ldots[n-j+1]} = x^j B_{n-j,q}(e_0,x)$$
$$= e_j(x).$$

(*ii*) For $n \geq j \geq 3$ and $f \in C[0,1]$, we have, in view of (4) and (16), that

$$f(x) - L_{n,q}(f,x) = f(x) - B_{n,q}(f,x)$$
$$+ \sum_{k=0}^{n} p_{n,k}(q;x)\left\{ f\left(\frac{[k]}{[n]}\right) - f\left(\left(\frac{[k][k-1]\ldots[k-j+1]}{[n][n-1]\ldots[n-j+1]}\right)^{1/j}\right)\right\}$$
$$+ \sum_{k=0}^{n} p_{n,k}(q;x)\left\{ f\left(\left(\frac{[k][k-1]\ldots[k-j+1]}{[n][n-1]\ldots[n-j+1]}\right)^{1/j}\right) - \lambda_{n,k}(f)\right\}. \quad (19)$$

But (see (3))

$$\left| f\left(\frac{[k]}{[n]}\right) - f\left(\left(\frac{[k][k-1]\ldots[k-j+1]}{[n][n-1]\ldots[n-j+1]}\right)^{1/j}\right)\right| \leq \omega\left(f, \frac{[j-1]}{[n]}\right), \quad (20)$$

because the inequalities $\frac{[k]}{[n]} \geq \frac{[k-1]}{[n-1]} \geq \ldots \geq \frac{[k-j+1]}{[n-j+1]}$ imply

$$0 \leq \frac{[k]}{[n]} - \left(\frac{[k][k-1]\ldots[k-j+1]}{[n][n-1]\ldots[n-j+1]}\right)^{1/j}$$
$$\leq \frac{[k]}{[n]} - \frac{[k-j+1]}{[n-j+1]} = \frac{q^{k-j+1}[j-1][n-k]}{[n][n-j+1]} \leq \frac{[j-1]}{[n]}$$

for $n \geq k \geq j \geq 3$. For $k = 0, 1, \ldots, j-1$, we have

$$\left| f\left(\frac{[k]}{[n]}\right) - f\left(\left(\frac{[k][k-1]\ldots[k-j+1]}{[n][n-1]\ldots[n-j+1]}\right)^{1/j}\right)\right|$$

$$= \left| f\left(\frac{[k]}{[n]}\right) - f(0) \right| \le \omega\left(f, \frac{[k]}{[n]}\right) \le \omega\left(f, \frac{[j-1]}{[n]}\right). \tag{21}$$

Now combining (19), (20), (21), (6), (18) and the property $\omega(f,\delta) \le \delta\|f'\|$ (each for $f = e_{j/2}$), we find that

$$\begin{aligned}
&|e_{j/2}(x) - L_{n,q}(e_{j/2}, x)| \\
&\le |e_{j/2}(x) - B_{n,q}(e_{j/2}, x)| \\
&\quad + \sum_{k=0}^{n} p_{n,k}(q;x) \left| \left(\frac{[k]}{[n]}\right)^{j/2} - \left(\frac{[k][k-1]\ldots[k-j+1]}{[n][n-1]\ldots[n-j+1]}\right)^{1/2} \right| \\
&\quad + \sum_{k=0}^{n} p_{n,k}(q;x) \left\{(\lambda_{n,k}(e_j))^{1/2} - \lambda_{n,k}(e_{j/2})\right\} \\
&\le \frac{3}{2}\omega(e_{j/2}, [n]^{-1/2}) + \omega\left(e_{j/2}, \frac{[j-1]}{[n]}\right) + \frac{2}{[n]} \\
&\le \frac{3}{4} j\,[n]^{-1/2} + \frac{j}{2}[j-1][n]^{-1} + 2[n]^{-1} \\
&\le \left(\frac{3}{4}j + \frac{1}{2}j\,[j-1] + 2\right)[n]^{-1/2}.
\end{aligned} \tag{22}$$

On the other hand, if $n \ge j = 2$, then, by (16), (18), Hölder's inequality, (17), (i) and (5), we get

$$\begin{aligned}
L_{n,q}(e_1, x) &= \sum_{k=0}^{n} p_{n,k}(q;x)\lambda_{n,k}(e_1) \le \sum_{k=0}^{n} p_{n,k}(q;x)(\lambda_{n,k}(e_2))^{1/2} \\
&\le \left(\sum_{k=0}^{n} p_{n,k}(q;x)\lambda_{n,k}(e_2)\right)^{1/2} = (L_{n,q}(e_2, x))^{1/2} = e_1(x)
\end{aligned}$$

and

$$\begin{aligned}
L_{n,q}(e_1, x) &= \sum_{k=1}^{n-1} p_{n,k}(q;x)\frac{[k-1]}{[n]} + p_{n,n}(q;x) \\
&= \sum_{k=0}^{n} p_{n,k}(q;x)\frac{[k]}{[n]} - \sum_{k=1}^{n-1} p_{n,k}(q;x)\frac{[k]-[k-1]}{[n]} \\
&= x - \sum_{k=1}^{n-1} p_{n,k}(q;x)\frac{q^{k-1}}{[n]}
\end{aligned}$$

$$\geq x - \frac{1}{[n]} \sum_{k=1}^{n-1} p_{n,k}(q;x) \geq e_1(x) - \frac{1}{[n]},$$

respectively. Hence

$$0 \leq e_1(x) - L_{n,q}(e_1, x) \leq \frac{1}{[n]}. \tag{23}$$

Now (22), (23), (i) and Lemma 3 imply that $\lim_{n\to\infty}(L_{n,q}f)(x) = f(x)$ uniformly in $x \in [0,1]$.

(iii) Because $\lambda_{n,k} \in C[0,1]^*$ are bounded positive linear functionals and $\lambda_{n,k}(e_0) = 1$ (see (17)), we have the representations $\lambda_{n,k}(f) = \int_0^1 f(t)\,d\nu_{n,k}(t)$, where the functions $\nu_{n,k}$ are increasing on $[0,1]$ and $\int_0^1 d\nu_{n,k}(t) = 1$, $k = 0, 1, \ldots, n$. Hence, by (16), the property $\omega(f, a\delta) \leq (1+a)\omega(f,\delta)$, $a > 0$ and Hölder's inequality, we obtain

$$\begin{aligned}
&|(L_{n,q}f)(x) - f(x)| \\
&\leq \sum_{k=0}^{n} p_{n,k}(q;x)|\lambda_{n,k}(f) - f(x)| \leq \sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 |f(t) - f(x)|\,d\nu_{n,k}(t) \\
&\leq \sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 \omega(f, |t-x|)\,d\nu_{n,k}(t) \\
&\leq \omega(f,\delta) \sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 \left\{1 + \delta^{-1}|t-x|\right\} d\nu_{n,k}(t) \\
&\leq \omega(f,\delta) \left\{1 + \delta^{-1} \left(\sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 (t-x)^{2j}\,d\nu_{n,k}(t) \right)^{1/2j} \right\}.
\end{aligned} \tag{24}$$

By Lemma 2 (b), we get

$$(t-x)^{2j} \leq (t^j - x^j)^2 = (t^{j/2} + x^{j/2})^2 (t^{j/2} - x^{j/2})^2 \leq 4(t^{j/2} - x^{j/2})^2.$$

Hence, by (24) and (i),

$$\begin{aligned}
&|(L_{n,q}f)(x) - f(x)| \\
&\leq \omega(f,\delta) \left\{1 + 2^{1/j}\delta^{-1} \left(\sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 (t^{j/2} - x^{j/2})^2\,d\nu_{n,k}(t) \right)^{1/2j} \right\} \\
&= \omega(f,\delta) \{1 + 2^{1/j}\delta^{-1}
\end{aligned}$$

$$\times \left((L_{n,q}e_j)(x) - 2x^{j/2}(L_{n,q}e_{j/2})(x) + x^j(L_n e_0)(x)\right)^{1/2j}\Big\}$$
$$= \omega(f,\delta)\left\{1 + 2^{3/2j}\delta^{-1}\sqrt[2j]{x^{j/2}(x^{j/2} - (L_{n,q}e_{j/2})(x))}\right\}. \tag{25}$$

Using (22), (23) and (25), we find that

$$|(L_{n,q}f)(x) - f(x)| \le \omega(f,\delta)\left\{1 + 2^{3/2j}\sqrt[2j]{\frac{3}{4}j + \frac{1}{2}j[j-1] + 2}\,[n]^{-1/4j}\right\}.$$

Choosing $\delta = [n]^{-1/4j}$, we arrive at the desired result. □

Corollary 1. *Let $j \in \{2,3,\dots\}$ be given, $n \ge j$ and let $q \in (0,1)$ be fixed. Then the operators $L_{n,q}$ constructed in Theorem 2 verify for $j \ge 3$ that*

$$\lim_{n\to\infty}(L_{n,q}e_{j/2})(x)$$
$$= \begin{cases} \sum_{k=1}^{\infty} p_{\infty,k}(q;x)(1-q^{k-1})\sqrt{(1-q^{k-2})\dots(1-q^{k-j+1})}, & \textit{if } x \in [0,1) \\ 1, & \textit{if } x = 1 \end{cases}$$

uniformly in $x \in [0,1]$, where $p_{\infty,k}(q;x) = \dfrac{x^k}{(1-q)^k[k]!}\prod_{s=0}^{\infty}(1-q^s x)$, $k = 0,1,\dots$;

$$\lim_{n\to\infty}(L_{n,q}e_1)(x) = x - (1-q)q^{-1}(1-x)\left\{1 - \prod_{s=1}^{\infty}(1-q^s x)\right\}$$

uniformly in $x \in [0,1]$, when $j = 2$.

Proof. We introduce the notation

$$L_{\infty,q}(e_{j/2},x) = \begin{cases} \sum_{k=0}^{\infty} p_{\infty,k}(q;x)\lambda_{\infty,k}(e_{j/2}), & \text{if } 0 \le x < 1 \\ 1, & \text{if } x = 1 \end{cases} \tag{26}$$

where $\lambda_{\infty,0}(e_{j/2}) = 0$, $\lambda_{\infty,k}(e_{j/2}) = (1-q^{k-1})\sqrt{(1-q^{k-2})\dots(1-q^{k-j+1})}$, $k = 1,2,\dots$ Because of (16), (17), (26) and $\sum_{k=0}^{\infty} p_{\infty,k}(q;x) = 1$ (see [17, p. 154, (2.3)]), we have

$$L_{n,q}(e_{j/2},x) - L_{\infty,q}(e_{j/2},x) = \sum_{k=j}^{n} (\lambda_{n,k}(e_{j/2}) - \lambda_{\infty,k}(e_{j/2}))p_{n,k}(q;x)$$
$$+ \sum_{k=j}^{n} (\lambda_{\infty,k}(e_{j/2}) - 1)(p_{n,k}(q;x) - p_{\infty,k}(q;x))$$
$$- \sum_{k=n+1}^{\infty} (\lambda_{\infty,k}(e_{j/2}) - 1)p_{\infty,k}(q;x)). \tag{27}$$

Further, for $k = j, j+1, \ldots, n-1$ and $j \geq 3$, we obtain

$$|\lambda_{n,k}(e_{j/2}) - \lambda_{\infty,k}(e_{j/2})|$$
$$= \left| \frac{[k-1]}{[n]} \left(\frac{[k-2][k-3]\ldots[k-j+1]}{[n-2][n-3]\ldots[n-j+1]} \right)^{1/2} \right.$$
$$\left. - (1-q^{k-1})\sqrt{(1-q^{k-2})\ldots(1-q^{k-j+1})} \right|$$
$$= \left| \left(\frac{[k-1]}{[n]} - (1-q^{k-1}) \right) \left(\frac{[k-2][k-3]\ldots[k-j+1]}{[n-2][n-3]\ldots[n-j+1]} \right)^{1/2} \right.$$
$$+ (1-q^{k-1}) \left\{ \left(\frac{[k-2][k-3]\ldots[k-j+1]}{[n-2][n-3]\ldots[n-j+1]} \right)^{1/2} \right.$$
$$\left. \left. - \sqrt{(1-q^{k-2})\ldots(1-q^{k-j+1})} \right\} \right|$$
$$= \left| \left(\frac{[k-1]}{[n]} - (1-q^{k-1}) \right) \left(\frac{[k-2][k-3]\ldots[k-j+1]}{[n-2][n-3]\ldots[n-j+1]} \right)^{1/2} \right.$$
$$+ (1-q^{k-1}) \left\{ \left(\sqrt{\frac{[k-2]}{[n-2]}} - \sqrt{1-q^{k-2}} \right) \left(\frac{[k-3]\ldots[k-j+1]}{[n-3]\ldots[n-j+1]} \right)^{1/2} \right.$$
$$+ \sqrt{1-q^{k-2}} \left(\sqrt{\frac{[k-3]}{[n-3]}} - \sqrt{1-q^{k-3}} \right) \left(\frac{[k-4]\ldots[k-j+1]}{[n-4]\ldots[n-j+1]} \right)^{1/2} + \ldots$$
$$\left. \left. + \sqrt{(1-q^{k-2})\ldots(1-q^{k-j+1})} \left(\sqrt{\frac{[k-j+1]}{[n-j+1]}} - \sqrt{1-q^{k-j+1}} \right) \right\} \right|$$
$$\leq \left| \frac{[k-1]}{[n]} - (1-q^{k-1}) \right| + \left| \sqrt{\frac{[k-2]}{[n-2]}} - \sqrt{1-q^{k-2}} \right| + \ldots$$
$$+ \left| \sqrt{\frac{[k-j+1]}{[n-j+1]}} - \sqrt{1-q^{k-j+1}} \right|. \tag{28}$$

But

$$0 \le \frac{[k-1]}{[n]} - (1-q^{k-1}) = \frac{1-q^{k-1}}{1-q^{n-1}} - (1-q^{k-1}) = \frac{q^{n-1}(1-q^{k-1})}{1-q^{n-1}} \le q^{n-1}$$

for $k = j, j+1, \ldots, n-1$ and $j \ge 3$; further, we have

$$0 \le \sqrt{\frac{[k-i]}{[n-i]}} - \sqrt{1-q^{k-i}} = \sqrt{\frac{1-q^{k-i}}{1-q^{n-i}}} - \sqrt{1-q^{k-i}}$$
$$= \sqrt{\frac{1-q^{k-i}}{1-q^{n-i}}}(1 - \sqrt{1-q^{n-i}}) \le q^{n-i}$$

for $i = 2, 3, \ldots, j-1$. Hence, by (28), we find for $k = j, j+1, \ldots, n-1$ that

$$|\lambda_{n,k}(e_{j/2}) - \lambda_{\infty,k}(e_{j/2})| \le q^{n-1} + q^{n-2} + \ldots + q^{n-j+1} = q^{n-j+1}\frac{1-q^j}{1-q}. \quad (29)$$

In what follows we shall use the following estimate (see [17, p. 156, (2.8)]):

$$1 - \prod_{s=i}^{\infty}(1-q^s) \le \frac{q^i}{q(1-q)} \ln \frac{1}{1-q}, \qquad i = 1, 2, \ldots \quad (30)$$

Then

$$\begin{aligned} |\lambda_{n,k}(e_{j/2}) - \lambda_{\infty,k}(e_{j/2})| &= 1 - (1-q^{n-1})\sqrt{(1-q^{n-2})\ldots(1-q^{n-j+1})} \\ &\le 1 - (1-q^{n-1})(1-q^{n-2})\ldots(1-q^{n-j+1}) \\ &\le 1 - \prod_{s=n-j+1}^{\infty}(1-q^s) \\ &\le \frac{q^{n-j+1}}{q(1-q)} \ln \frac{1}{1-q}. \end{aligned} \quad (31)$$

Analogously

$$|\lambda_{\infty,k}(e_{j/2}) - 1| \le \frac{q^{k-j+1}}{q(1-q)} \ln \frac{1}{1-q} \quad (32)$$

for $k = j, j+1, \dots, n$, and

$$|\lambda_{\infty,k}(e_{j/2}) - 1| \le \frac{q^{n-j+2}}{q(1-q)} \ln \frac{1}{1-q} \tag{33}$$

for $k = n+1, n+2, \dots$

Taking into account the estimate

$$\sum_{k=0}^{n} q^k |p_{n,k}(q;x) - p_{\infty,k}(q;x)| \le \frac{2q^n}{q(1-q)} \ln \frac{1}{1-q}$$

(see [17, p. 156, (2.9)]) and the equality $\sum_{k=0}^{\infty} p_{\infty,k}(q;x) = 1$ (see [17, p. 154, (2.3)]), we find, in view of (27), (29), (31), (32) and (33), that

$$\begin{aligned}
&|L_{n,q}(e_{j/2},x) - L_{\infty,q}(e_{j/2},x)| \\
&\le \sum_{k=j}^{n} |\lambda_{n,k}(e_{j/2}) - \lambda_{\infty,k}(e_{j/2})| p_{n,k}(q;x) \\
&\quad + \sum_{k=j}^{n} |\lambda_{\infty,k}(e_{j/2}) - 1||p_{n,k}(q;x) - p_{\infty,k}(q;x)| \\
&\quad + \sum_{k=n+1}^{\infty} |\lambda_{\infty,k}(e_{j/2}) - 1| p_{\infty,k}(q;x)) \\
&\le q^{n-j+1} \frac{1-q^j}{1-q} + \frac{q^{n-j+1}}{q(1-q)} \ln \frac{1}{1-q} \\
&\quad + \frac{1}{q^j(1-q)} \ln \frac{1}{1-q} \sum_{k=0}^{n} q^k |p_{n,k}(q;x) - p_{\infty,k}(q;x)| + \frac{q^{n-j+2}}{q(1-q)} \ln \frac{1}{1-q} \\
&\le q^{n-j+1} \left\{ \frac{1-q^j}{1-q} + \frac{1+q}{q(1-q)} \ln \frac{1}{1-q} + 2 \left(\frac{1}{q(1-q)} \ln \frac{1}{1-q} \right)^2 \right\} \to 0
\end{aligned}$$

as $n \to \infty$. Besides $L_{n,q}(e_{j/2},1) = L_{\infty,q}(e_{j/2},1)$ in view of (16), (17) and (26). This means that $\lim_{n\to\infty} (L_{n,q} e_{j/2})(x) = L_{\infty,q}(e_{j/2},x)$ uniformly in $x \in [0,1]$.

For $j = 2$ we use the following result established in [5, p. 334, (3.1)]:

$$L_{n,q}(e_1,x) = \frac{1}{q}x - \frac{1}{q[n]} + \frac{1}{q[n]}(1-x)(1-qx)\dots(1-q^{n-1}x) + \left(1 - \frac{1}{q} + \frac{1}{q[n]}\right) x^n.$$

Then

$$\left| L_{n,q}(e_1,x) - x + (1-q)q^{-1}(1-x)\left\{1 - \prod_{s=0}^{\infty}(1-q^s x)\right\} \right|$$

$$= \left| -\frac{1}{q[n]} + \frac{1}{q[n]}(1-x)(1-qx)\dots(1-q^{n-1}x) + \left(1 - \frac{1}{q} + \frac{1}{q[n]}\right)x^n \right.$$

$$\left. + \frac{1-q}{q} - \frac{1-q}{q}\prod_{s=0}^{\infty}(1-q^s x) \right|$$

$$\le \left| \frac{1}{q[n]} - \frac{1-q}{q} \right| |x^n - 1| + \frac{1}{q[n]} \left| (1-x)(1-qx)\dots(1-q^{n-1}x) - \prod_{s=0}^{\infty}(1-q^s x) \right|$$

$$+ \left| \frac{1}{q[n]} - \frac{1-q}{q} \right| \prod_{s=0}^{\infty}(1-q^s x). \tag{34}$$

On the other hand

$$\left| \frac{1}{q[n]} - \frac{1-q}{q} \right| = q^{n-1}\frac{1-q}{1-q^n} \le q^{n-1}$$

and, by (30),

$$\left| (1-x)(1-qx)\dots(1-q^{n-1}x) - \prod_{s=0}^{\infty}(1-q^s x) \right|$$

$$= (1-x)(1-qx)\dots(1-q^{n-1}x)\left\{1 - \prod_{s=n}^{\infty}(1-q^s x)\right\}$$

$$\le 1 - \prod_{s=n}^{\infty}(1-q^s) \le \frac{q^n}{q(1-q)} \ln\frac{1}{1-q}.$$

Hence, due to (34),

$$\left| L_{n,q}(e_1,x) - x + (1-q)q^{-1}(1-x)\left\{1 - \prod_{s=0}^{\infty}(1-q^s x)\right\} \right|$$

$$\le 2q^{n-1} + \frac{q^{n-1}}{q(1-q)}\ln\frac{1}{1-q} \le q^{n-1}\left(2 + \frac{1}{q(1-q)}\ln\frac{1}{1-q}\right) \to 0$$

as $n \to \infty$, which completes the proof of the corollary. □

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Certain Szász-Mirakyan-Beta Operators

N.K. Govil, Vijay Gupta, and Danyal Soybaş

Abstract In the present article we discuss direct estimates of the Durrmeyer type modifications of the well-known Szász-Mirakyan operators. The present article is divided into two sections. In the first section, we mention some of the different integral modifications of the Szász-Mirakyan operators and mention their direct results which were done in ordinary, and specially in the simultaneous approximation.

In the second section for the Szász-Mirakyan-Beta operators, we find the alternate hypergeometric representation and propose their Stancu type generalization based on two parameters. We obtain the moments using confluent Hypergeometric functions. Also it is observed here that the moments are related to the Laguerre polynomials. We study direct approximation results for these Szász-Mirakyan-Beta-Stancu operators, which include a Voronovskaja-type asymptotic formula and error estimations in terms of modulus of continuity.

Keywords Direct estimates • Confluent hypergeometric function • Laguerre polynomials • Moments • Voronovskaja-type asymptotic formula • Error estimations

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1 Introduction

The Szász-Mirakyan operators [13] are given by

$$S_n(f,x) \equiv (S_n f)(x) = \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right),$$

where the Szász basis function is defined as

$$s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}. \tag{1}$$

Lots of work has been done on the Szász-Mirakyan operators in the last six decades. Some of the important results on these operators have recently been compiled by Gupta in [5]. As such Szász-Mirakyan operators are discrete operators and they are not able to approximate Lebesgue integrable functions on the positive real axis. To approximate Lebesgue integrable functions on the interval $[0,\infty)$, the two usual modifications available in the literature are due to Kantorovich and Durrmeyer. Several other summation-integral modifications of the Durrmeyer-type operators were proposed and their approximation properties have been discussed and studied by researchers in the last three decades.

The present article is divided into two sections. In the first section, below we mention some of the summation-integral type integral modifications of the Szász-Mirakyan operators and mention their direct results which were done in ordinary, and specially in the simultaneous approximation. In the second section, that contains new results, we propose the Stancu type modification of Szász-Mirakyan-Beta operators and study some direct results in simultaneous approximation. Our results include the asymptotic formula, and error estimations in terms of modulus of continuity of first and second order.

1.1 Szász-Mirakyan-Durrmeyer Operators

In the year 1985, Mazhar and Totik [11] introduced the Szász-Durrmeyer operators to approximate Lebesgue integrable functions on the interval $[0,\infty)$ as

$$\hat{S}_n(f,x) = n\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^{\infty} s_{n,k}(t) f(t)dt$$

where $s_{n,k}(x)$ is the Szász-Mirakyan basis functions defined by (1). Also, around the same time Kasana et al. [10] introduced the operators as

$$\hat{S}_{n,y}(f,x) = n\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^{\infty} s_{n,k}(t) f(t+y)dt.$$

They considered by $C_B[0,\infty)$ the class of real valued bounded and uniformly continuous functions on $[0,\infty)$ with the norm $||.||_B = \sup_{x\in[0,\infty)} |f(x)|$. The following main results were developed in [10]:

Theorem 1 ([10]). *Let f be integrable on $[0,\infty)$ and $f \in C_B[0,\infty)$. Then*

$$\left|\hat{S}_{n,y}(f,x) - f(x+y)\right| \le \left\{1 + 2\left(x + \frac{1}{n}\right)\right\} \omega(f, n^{-1/2}),$$

where $\omega(f,.)$ is the modulus of continuity of f on $[0,\infty)$.

Theorem 2 ([10]). *Let f be integrable on $[0,\infty)$ and $f' \in C_B[0,\infty)$. Then*

$$\left|\hat{S}_{n,y}(f,x) - f(x+y)\right| \le n^{-1/2}\omega(f', n^{-1/2}) \left\{2\left(x + \frac{1}{n}\right)\right\}^{1/2} \left[1 + \left\{2\left(x + \frac{1}{n}\right)\right\}^{1/2}\right] + \frac{1}{n}||f'||_B||,$$

where $\omega(f',.)$ is the modulus of continuity of f' on $[0,\infty)$.

Theorem 3 ([10]). *Let f be bounded and integrable on $[0,\infty)$ and let f'' exist at a point $x+y \in [0,\infty)$. Then*

$$\lim_{n\to\infty} \left[\hat{S}_{n,y}(f,x) - f(x+y)\right] = f'(x+y) + x f''(x+y).$$

Theorem 4 ([10]). *Let f be bounded and integrable function on $[0,\infty)$ admitting a derivative of order r at $x+y \in (0,\infty)$. Then*

$$\lim_{n\to\infty} \frac{\partial^r}{\partial x^r} \hat{S}_{n,y}(f,x) = \frac{\partial^r}{\partial x^r} f(x+y).$$

Theorem 5 ([10]). *Let f be bounded and integrable on $[0,\infty)$ and*

$$\frac{\partial^r}{\partial x^r} f(x+y) \in C[0,b).$$

Then for sufficiently large n, there holds

$$\sup_{a\le x+y\le c} \left|\frac{\partial^r}{\partial x^r} \hat{S}_{n,y}(f,x) - \frac{\partial^r}{\partial x^r} f(x+y)\right| \le \max\left\{C_1\omega(f^{(r)}, n^{-1/2}), c_2 n^{-(s-r)/2}\right\},$$

where $C_1 = C_1(r,y), C_2 = C_2(r,f,y), 0 < a < c < b, s > r$ and $\omega(f^{(r)},.)$ denotes the modulus of continuity of $f^{(r)}$ on $[0,b)$.

Theorem 6 ([10]). *Let f be integrable on $[0,\infty)$. Then*

$$\lim_{n\to\infty} \hat{S}_{n,y}(f,x) = f(x+y)$$

almost everywhere on $[0,\infty)$.

1.2 Szász-Beta Operators

In the year 1995 Gupta et al. [7] introduced another Durrmeyer type modification of the Szász-Mirakyan operators by considering the weight functions of Beta basis functions, which they called Szász-Mirakyan-Beta operators. For $x \in [0,\infty)$, the Szász-Mirakyan-Beta operators are defined as

$$B_n(f,x) = \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} b_{n,k}(t) f(t) dt, \tag{2}$$

where $s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}$, $b_{n,k}(t) = \frac{1}{B(k+1,n)}\frac{t^k}{(1+t)^{n+k+1}} = \frac{n(n+1)_k}{k!}\frac{t^k}{(1+t)^{n+k+1}}$ and the Pochhammer symbol $(n)_k$ is defined as

$$(n)_k = n(n+1)(n+2)(n+3)\ldots.(n+k-1).$$

Below, we present the alternate form of the operators (2) as

$$\begin{aligned} B_n(f,x) &= n\sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!}\int_0^{\infty}\frac{(n+1)_k}{k!}\frac{t^k}{(1+t)^{n+k+1}} f(t)dt \\ &= ne^{-nx}\int_0^{\infty}\frac{f(t)}{(1+t)^{n+1}}\sum_{k=0}^{\infty}\frac{(n+1)_k}{k!(1)_k}\left(\frac{nxt}{1+t}\right)^k dt. \end{aligned}$$

Using the confluent hypergeometric series ${}_1F_1(a;b;x) = \sum_{k=0}^{\infty}\frac{(a)_k}{(b)_k k!}x^k$, we can write

$$B_n(f,x) = ne^{-nx}\int_0^{\infty}\frac{f(t)}{(1+t)^{n+1}}\, {}_1F_1\left(n+1,1;\frac{nxt}{(1+t)}\right)dt.$$

Next, applying Kummer's first transformation

$${}_1F_1(a;b;x) = e^x\, {}_1F_1(b-a;b;-x),$$

we have

$$B_n(f,x) = n\int_0^\infty e^{\frac{nx(t-1)}{(t+1)}} \frac{f(t)}{(1+t)^{n+1}}\ {}_1F_1\left(-n;1;\frac{-nxt}{(1+t)}\right)dt, \tag{3}$$

which is the alternate form of the operators (2) in terms of hypergeometric functions.

Gupta et al. [7] denoted by $H[0,\infty)$ the class of all Lebesgue measurable functions defined on $[0,\infty)$ satisfying

$$\int_0^\infty \frac{|f(t)|dt}{(1+t)^{n+1}} < \infty,$$

for some positive integer n. This class is naturally bigger than the class of all Lebesgue integrable functions on $[0,\infty)$. The following asymptotic formula and an estimation of error in simultaneous approximation were studied in [7].

Theorem 7 ([7]). *Let $f \in H[0,\infty)$ and be bounded on every finite subinterval of $[0,\infty)$. If $f^{(r+2)}$ exists at a fixed point $x \in (0,\infty)$ and $f(y) = O(y^\alpha)$ as $y \to \infty$ for some $\alpha > 0$, then we have*

$$\lim_{n\to\infty} n[B_n^{(r)}(f,x) - f^{(r)}(x)] = \frac{r(r+1)}{2}f^{(r)}(x) + [x(1+r)+1+r]f^{(r+1)}(x) + \frac{x^2+2x}{2}f^{(r+2)}(x).$$

Theorem 8 ([7]). *Let $f \in H[0,\infty)$ be bounded on every finite subinterval of $[0,\infty)$ and $f(y) = O(y^\alpha)$ as $y \to \infty$ for some $\alpha > 0$. If $f^{(r+1)}$ exists and is continuous on $(a-\eta, b+\eta)$, $\eta > 0$, then for n sufficiently large we have*

$$||B_n^{(r)}(f,.) - f^{(r)}|| \le C_1 n^{-1/2}\left(||f^{(r)}|| + ||f^{(r+1)}||\right) + C_2 n^{-1/2}\omega(f^{(r+1)}, n^{-1/2} + O(n^{-p}),$$

for some $p > 0$ where C_1 and C_2 are constants independent of f and n, $\omega(f,\delta)$ is the modulus of continuity of f on $(a-\eta, b+\eta)$, and $||.||$ stands for the sup-norm on $[a,b]$.

1.3 Discretely Defined Szász-Beta Operators

Although there are other forms of the Szász-Beta type operators available in the literature, in this direction Gupta and Noor [6] also defined for $f \in C_\beta[0,\infty) := \{f \in C[0,\infty) : |f(t)| \le M(1+t)^\beta,\ \text{for some } M > 0,\ \ \beta > 0\}$ the following form

$$\tilde{B}_n(f,x) = \sum_{k=1}^{\infty} s_{n,k}(x) \int_0^{\infty} \tilde{b}_{n,k}(t) f(t)dt + s_{n,0}(x) f(0), \tag{4}$$

where

$$\tilde{b}_{n,k}(t) = \frac{1}{B(n+1,k)} \frac{t^{k-1}}{(1+t)^{n+k+1}}.$$

The main aim to introduce this slightly different form of Szász-Beta operators was that these form (4) reproduce constant as well as linear functions, while the form (2) reproduce only the constant functions. We may call the form (4) as genuine Szász-Beta operators. In [6], Gupta and Noor established some direct results in simultaneous approximation of this form.

Theorem 9 ([6]). *Let $f \in C_\beta[0,\infty), \beta > 0$, and $f^{(r)}$ exists at a point $x \in (0,\infty)$. Then we have*

$$\lim_{n\to\infty} \tilde{B}_n^{(r)}(f,x) = f^{(r)}(x).$$

Theorem 10 ([6]). *Let $f \in C_\beta[0,\infty), \beta > 0$, and $f^{(r+2)}$ exists at a point $x \in (0,\infty)$, then*

$$\lim_{n\to\infty} [\tilde{B}_n^{(r)}(f,x) - f^{(r)}(x)] = \frac{r(r-1)}{2} f^{(r)}(x) + (x+1)rf^{(r+1)}(x) + (x^2+x)f^{(r+2)}(x).$$

Theorem 11 ([6]). *Let $f \in C_\beta[0,\infty), \beta > 0$, and $r \le m \le (r+2)$. If $f^{(m)}$ exists and is continuous on $(a-\eta, b+\eta)$, then for n sufficiently large*

$$||\tilde{B}_n^{(r)}(f,x) - f^{(r)}(x)|| \le M_1 n^{-1} \sum_{i=r}^{m} ||f^{(i)}|| + M_2 n^{-1/2} \omega(f^{(r+1)}, n^{-1/2}) + O(n^{-2}),$$

where the constants M_1 and M_2 are independent of f and n, $\omega(f,\delta)$ is the modulus of continuity of f on $(a-\eta, b+\eta)$ and $||.||$ denotes the sup-norm in the interval $[a,b]$.

Finta et al. [2] also studied this form of Szász-Beta operators and they used iterative combinations to improve the order of approximation to estimate direct results. Without combinations, they obtained the following direct estimate:

Theorem 12 ([2]). *Let $f \in C_B[0,\infty)$. Then for every $x \in [0,\infty)$ and $n \geq 2$ there exists an absolute constant $C > 0$ such that*

$$|\tilde{B}_n(f,x) - f(x)| \leq C\omega\left(f, \sqrt{\frac{x(2+x)}{n-1}}\right).$$

1.4 Modified Szász-Beta Operators

In the recent years Dubey et al. [1] proposed some other slightly modified form of the operators (2), having weights of Beta basis functions depending on a parameter $\alpha > 0$ as

$$B_{n,\alpha}(f,x) = \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} b_{n,k,\alpha}(t) f(t) dt, \tag{5}$$

where

$$b_{n,k,\alpha}(t) = \alpha \frac{\Gamma\left(\frac{n}{\alpha}+k+1\right)}{\Gamma(k+1)\Gamma(\frac{n}{\alpha})} \frac{(\alpha t)^k}{(1+\alpha t)^{\frac{n}{\alpha}+k+1}}.$$

In [1], Dubey et al. have also obtained some direct estimates in simultaneous approximation for the operators (5).

Theorem 13 ([1]). *Let $n > \alpha(r+1) > 2\alpha$ and $f^{(i)} \in C_B[0,\infty)$ for $i \in \{0,1,2,\cdots,r\}$. Then*

$$|B^{(r)}_{n,\alpha}(f,x) - f^{(r)}(x)| \leq \left(\frac{(\frac{n}{\alpha})^r \Gamma(\frac{n}{\alpha}-r)}{\Gamma(\frac{n}{\alpha})} - 1\right) ||f^{(r)}|| + 2\frac{(\frac{n}{\alpha})^r \Gamma(\frac{n}{\alpha}-r)}{\Gamma(\frac{n}{\alpha})} \omega(f^{(r)}, \delta(n,r,x,\alpha)),$$

where

$$\delta(n,r,x,\alpha) = \left\{ \frac{n\alpha+3r\alpha^2+2\alpha^2+r^2\alpha^2}{[n-\alpha(r+1)][n-\alpha(r+2)]} x^2 + \frac{2n+\alpha r^2+6\alpha r+4\alpha}{[n-\alpha(r+1)][n-\alpha(r+2)]} x + \frac{r^2+3r+2}{[n-\alpha(r+1)][n-\alpha(r+2)]} \right\}^{1/2},$$

and $x \in [0,\infty)$.

They considered the class $f \in C_\gamma[0,\infty) := \{f \in C[0,\infty) : f(t) = O(t^\gamma), \gamma > 0$ as $t \to \infty\}$. The norm $||.|_\gamma$ on $C_\gamma[0,\infty)$ is defined as

$$||f||_\gamma = \sup_{0 \le t < \infty} |f(t)|(1+t)^{-\gamma}.$$

Theorem 14 ([1]). *Let* $f \in C_\gamma[0,\infty), \gamma > 0$, *and* $f^{(r+2)}$ *exists at a point* $x \in (0,\infty)$. *Then*

$$\lim_{n\to\infty}[B_{n,\alpha}^{(r)}(f,x) - f^{(r)}(x)] = \frac{\alpha r(r+1)}{2} f^{(r)}(x) \\ + (\alpha x+1)(r+1) f^{(r+1)}(x) + \frac{x(2+\alpha x)}{2} f^{(r+2)}(x).$$

Theorem 15 ([1]). *Let* $f \in C_\gamma[0,\infty), \gamma > 0$. *If* $f^{(r+1)}$ *exists and is continuous on* $(a-\eta, b+\eta), \eta > 0$, *then for n sufficiently large*

$$||B_{n,\alpha}^{(r)}(f,.) - f^{(r)}|| \le Cn^{-1}\left(||f^{(r)}|| + ||f^{(r+1)}||\right) \\ + Cn^{-1/2}\omega(f^{(r+1)}, n^{-1/2} + O(n^{-p}),$$

where $\omega(f^{(r+1)},\delta)$ *is the modulus of continuity of* $f^{(r+1)}$ *on* $(a-\eta, b+\eta)$ *and* $||.||$ *stands for the sup-norm on* $[a,b]$.

Theorem 16 ([1]). *Let* $f \in C_\gamma[0,\infty), \gamma > 0$ *and suppose* $0 < a < a_1 < b_1 < b < \infty$. *Then for n sufficiently large*

$$||B_{n,\alpha}^{(r)}(f,.) - f^{(r)}||_{C[a_1,b_1]} \le C\left\{\omega_2(f^{(r)}, n^{-1/2}, a, b) + n^{-1}||f||_\gamma\right\},$$

where $\omega_2(f^{(r)}, a, b) = \sup\{|\Delta_h^2 f^{(r)}(x)| : |h| \le n^{-1/2}; x, x+2h \in [a,b]\}$.

2 Szász-Beta-Stancu Operators

Also, in the year 1983, Stancu [12] generalized the classical Bernstein polynomials by considering two parameters α, β satisfying the conditions $0 \le \alpha \le \beta$. Motivated by the generalization of Bernstein polynomials recently Gupta and Yadav [8] proposed the Stancu type generalization of the Baskakov-Beta operators. Here we propose the Stancu type generalization of the Szász-Mirakyan-Beta operators (2), which for $0 \le \alpha \le \beta$ are defined as

$$B_{n,\alpha,\beta}(f,x) = n\int_0^\infty e^{\frac{nx(t-1)}{(t+1)}} \frac{1}{(1+t)^{n+1}} \, {}_1F_1\left(-n; 1; \frac{-nxt}{(1+t)}\right) f\left(\frac{nt+\alpha}{n+\beta}\right) dt. \tag{6}$$

For the special case if $\alpha = \beta = 0$ the operators (6) reduce to the well-known Szász-Mirakyan-Beta operators defined by (2), and introduced and studied in [7].

The present section deals with some of the approximation properties of the Szász-Mirakyan-Beta-Stancu (abbr. SMBS) operators. We obtain the moments of these operators in terms of hypergeometric functions. We also establish a Voronovkaja kind asymptotic formula and estimations of error in simultaneous approximation for the SMBS operators.

2.1 Moment Estimation and Auxiliary Results

In this section, we estimate moments and certain basic results:

Lemma 1. *For $n > 0$ and $r > -1$, we have*

$$B_n(t^r, x) = \frac{\Gamma(r+1)\Gamma(n-r)}{\Gamma(n)} \, {}_1F_1\left(-r; 1; -nx\right).$$

Further

$$B_n(t^r, x) = \frac{\Gamma(n-r)\Gamma(r+1)}{\Gamma(n)} L_r(-nx),$$

where $L_r(-nx)$ are the Laguerre polynomials.

Proof. Taking $f(t) = t^r$, using $\Gamma(n+k+1) = \Gamma(n+1)(n+1)_k$, and noting that $k! = (1)_k$, we have

$$\begin{aligned}
B_n(t^r, x) &= ne^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \int_0^{\infty} \frac{(n+1)_k}{k!} \frac{t^{k+r}}{(1+t)^{n+k+1}} dt \\
&= ne^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \frac{(n+1)_k}{k!} B(k+r+1, n-r) \\
&= ne^{-nx} \frac{\Gamma(r+1)\Gamma(n-r)}{\Gamma(n+1)} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \frac{(n+1)_k}{k!} \frac{(r+1)_k}{\Gamma(n+1)_k} \\
&= ne^{-nx} \frac{\Gamma(r+1)\Gamma(n-r)}{\Gamma(n+1)} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \frac{(r+1)_k}{(1)_k} \\
&= ne^{-nx} \frac{\Gamma(r+1)\Gamma(n-r)}{\Gamma(n+1)} \, {}_1F_1\left(r+1; 1; nx\right).
\end{aligned}$$

If we apply Kummer's first transformation

$${}_1F_1(a; b; x) = e^x \, {}_1F_1\left(b-a; b; -x\right)$$

we get

$$B_n(t^r, x) = \frac{\Gamma(r+1)\Gamma(n-r)}{\Gamma(n)} \;{}_1F_1\left(-r; 1; -nx\right).$$

By using the identity between confluent Hypergeometric function and the generalized Laguerre polynomials $L_n^m(x)$, we have

$$L_n^m(x) = \frac{(m+n)!}{m!n!} \,{}_1F_1\left(-n; m+1; x\right),$$

which gives

$$B_n(t^r, x) = \frac{\Gamma(n-r)\Gamma(r+1)}{\Gamma(n)} L_r(-nx),$$

and the proof is thus completed.

Remark 1. We may observe here that the Laguerre polynomials can be written as

$$L_k(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{k-j} \frac{x^j}{j!}.$$

Thus, by Lemma 1, the r-th moment can be expressed as

$$B_n(t^r, x) = \frac{\Gamma(n-r)\Gamma(r+1)}{\Gamma(n)} \sum_{j=0}^{r} \binom{r}{r-j} \frac{(nx)^j}{j!}.$$

Remark 2. From Remark 1, we easily obtain

$$B_n(t, x) = \frac{1+nx}{n-1},\; B_n(t^2, x) = \frac{n^2x^2 + 2(1+2nx)}{(n-1)(n-2)}$$

$$B_n(t-x, x) = \frac{1+x}{n-1},\; B_n((t-x)^2, x) = \frac{(n+2)x^2 + 2(n+2)x + 2}{(n-1)(n-2)}.$$

Also, by simple computation we get

$$B_n(t^r, x) = \frac{(n-r-1)!n^r}{(n-1)!} x^r + r^2 \frac{(n-r-1)!n^{r-1}}{(n-1)!} x^{r-1} + O(n^{-2}).$$

Lemma 2. *For* $0 \le \alpha \le \beta$, *we have*

$$B_{n,\alpha,\beta}(t^r, x) = x^r \frac{n^r}{(n+\beta)^r} \frac{(n-r-1)!n^r}{(n-1)!}$$

$$+x^{r-1}\left\{\frac{n^r}{(n+\beta)^r}r^2\frac{(n-r-1)!n^{r-1}}{(n-1)!}+r\alpha\frac{n^{r-1}}{(n+\beta)^r}\frac{(n-r)!n^{r-1}}{(n-1)!}\right\}$$
$$+O(n^{-2}).$$

Proof. The relation between operators (3) and (6) can be defined as

$$\begin{aligned}B_{n,\alpha,\beta}(t^r,x) &= \sum_{j=0}^{r}\binom{r}{j}\frac{n^j\alpha^{r-j}}{(n+\beta)^r}B_n(t^j,x)\\ &= \frac{n^r}{(n+\beta)^r}B_n(t^r,x)+r\alpha\frac{n^{r-1}}{(n+\beta)^r}B_n(t^{r-1},x)+\cdots\end{aligned}$$

which on using Remark 2 gives the required result.

Lemma 3 ([4]). *For $m\in\mathbb{N}\bigcup\{0\}$, if*

$$U_{n,m}(x)=\sum_{k=0}^{\infty}s_{n,k}(x)\left(\frac{k}{n}-x\right)^m,$$

then $U_{n,0}(x)=1, U_{n,1}(x)=0$ and we have the recurrence relation:

$$nU_{n,m+1}(x)=x\left[U'_{n,m}(x)+mU_{n,m-1}(x)\right].$$

Consequently, $U_{n,m}(x)=O\left(n^{-[(m+1)/2]}\right)$, where $[m]$ is integral part of m.

Lemma 4. *If we define the central moments as*

$$\begin{aligned}\mu_{n,m}(x) &= B_{n,\alpha,\beta}((t-x)^m,x)\\ &= \sum_{k=0}^{\infty}s_{n,k}(x)\int_0^{\infty}b_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt,\ \ m\in\mathbb{N},\end{aligned}$$

then for $n>m+1$, we have the following recurrence relation

$$\begin{aligned}&(n-m-1)\left(\frac{n+\beta}{n}\right)\mu_{n,m+1}(x)\\ &= x\left[\mu'_{n,m}(x)+m\mu_{n,m-1}(x)\right]\\ &+\left[(m+nx+1)+\left(\frac{n+\beta}{n}\right)\left(\frac{\alpha}{n+\beta}-x\right)(n-2m-1)\right]\mu_{n,m}(x)\\ &+\left(\frac{\alpha}{n+\beta}-x\right)\left[\left(\frac{\alpha}{n+\beta}-x\right)\left(\frac{n+\beta}{n}\right)-1\right]m\mu_{n,m-1}(x).\end{aligned}$$

From the recurrence relation, it can be easily verified that for all $x \in [0, \infty)$, *we have*

$$\mu_{n,m}(x) = O(n^{-[(m+1)/2]}).$$

Proof. From the definition of the operators (6), we obviously have $\mu_{n,0}(x) = 1$. The other moments follow from the recurrence relation. Now we prove the recurrence relation as follows:

Using the identities $xs'_{n,k}(x) = (k - nx)s_{n,k}(x)$ and $t(1+t)b'_{n,k}(t) = (k-(n+1)t)\, b_{n,k}(t)$, we have

$$x\mu'_{n,m}(x) = \sum_{k=0}^{\infty}(k-nx)s_{n,k}(x)\int_0^\infty b_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt \\ -mx\mu_{n,m-1}(x).$$

Thus

$$\begin{aligned}
& x\left[\mu'_{n,m}(x) + m\mu_{n,m-1}(x)\right] \\
&= \sum_{k=0}^{\infty} s_{n,k}(x)\int_0^\infty (k-nx)b_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt \\
&= \sum_{k=0}^{\infty} s_{n,k}(x)\int_0^\infty [\{k-(n+1)t\}+(n+1)t]\, b_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt \\
&\quad -nx\mu_{n,m}(x) \\
&= \sum_{k=0}^{\infty} s_{n,k}(x)\int_0^\infty t(1+t)b'_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt \\
&\quad +(n+1)\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^\infty b_{n,k}(t)t\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt - nx\mu_{n,m}(x).
\end{aligned}$$

Now using the identity $t = \frac{n+\beta}{n}\left[\frac{nt+\alpha}{n+\beta} - x - \left(\frac{\alpha}{n+\beta} - x\right)\right]$, we have

$$\begin{aligned}
& x\left[\mu'_{n,m}(x) + m\mu_{n,m-1}(x)\right] \\
&= \frac{n+\beta}{n}\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^\infty b'_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m+1} dt \\
&\quad -\frac{n+\beta}{n}\left(\frac{\alpha}{n+\beta}-x\right)\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^\infty b'_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt
\end{aligned}$$

$$+\left(\frac{n+\beta}{n}\right)^2\left[\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^{\infty}b'_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m+2}dt\right.$$

$$+\left(\frac{\alpha}{n+\beta}-x\right)^2\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^{\infty}b'_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m}dt$$

$$\left.-2\left(\frac{\alpha}{n+\beta}-x\right)\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^{\infty}b'_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m+1}dt\right]$$

$$+(n+1)\left(\frac{n+\beta}{n}\right)\mu_{n,m+1}(x)-(n+1)\left(\frac{n+\beta}{n}\right)\left(\frac{\alpha}{n+\beta}-x\right)\mu_{n,m}(x)$$

$$-nx\mu_{n,m}(x).$$

On integrating by parts after simple computation, we get

$$\begin{aligned}&(n-m-1)\left(\frac{n+\beta}{n}\right)\mu_{n,m+1}(x)\\ =\ & x\left[\mu'_{n,m}(x)+m\mu_{n,m-1}(x)\right]\\ &+\left[(m+nx+1)+\left(\frac{n+\beta}{n}\right)\left(\frac{\alpha}{n+\beta}-x\right)(n-2m-1)\right]\mu_{n,m}(x)\\ &+\left(\frac{\alpha}{n+\beta}-x\right)\left[\left(\frac{\alpha}{n+\beta}-x\right)\left(\frac{n+\beta}{n}\right)-1\right]m\mu_{n,m-1}(x).\end{aligned}$$

Lemma 5 ([4]). *There exist polynomials $q_{i,j,r}(x)$ on $[0,\infty)$, independent of n and k such that*

$$x^r\frac{d^r}{dx^r}s_{n,k}(x)=\sum_{\substack{2i+j\le r\\ i,j\ge 0}}n^i(k-nx)^j q_{i,j,r}(x)s_{n,k}(x).$$

Lemma 6. *If f is r times differentiable on $[0,\infty)$ such that $f^{(r-1)}(t)=O(t^\gamma)$ for some $\gamma>0$ as $t\to\infty$, then for $r=1,2,\ldots$ and $n>\gamma+r$, we have*

$$B^{(r)}_{n,\alpha,\beta}(f,x)=\left(\frac{n}{n+\beta}\right)^r\frac{(n-r-1)!n^r}{(n-1)!}\sum_{k=0}^{\infty}s_{n,k}(x)$$

$$\cdot\int_0^{\infty}b_{n-r,k+r}(t)f^{(r)}\left(\frac{nt+\alpha}{n+\beta}\right)dt.$$

Proof. By Leibniz theorem, we have

$$B^{(r)}_{n,\alpha,\beta}(f,x) = \sum_{i=0}^{r}\sum_{k=i}^{\infty}\binom{r}{i}\frac{(-1)^{r-i}n^r e^{-nx}(nx)^{k-i}}{(k-i)!}\int_0^\infty b_{n,k}(t)f\left(\frac{nt+\alpha}{n+\beta}\right)dt$$
$$= \sum_{i=0}^{r}\sum_{k=0}^{\infty}\binom{r}{i}(-1)^{r+i}n^r s_{n,k}(x)\int_0^\infty b_{n,k+i}(t)f\left(\frac{nt+\alpha}{n+\beta}\right)dt$$
$$= \sum_{k=0}^{\infty}s_{n,k}(x)\int_0^\infty(-1)^r\left(\sum_{i=0}^{r}\binom{r}{i}(-1)^i n^r b_{n,k+i}(t)\right)f\left(\frac{nt+\alpha}{n+\beta}\right)dt.$$

Again by Leibniz theorem, we have

$$b^{(r)}_{n-r,k+r}(t) = \frac{(n-1)!}{(n-r-1)!}\sum_{i=0}^{r}(-1)^i\binom{r}{i}b_{n,k+i}(t).$$

Hence

$$B^{(r)}_n(f,x) = \frac{n^r(n-r-1)!}{(n-1)!}\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^\infty(-1)^r b^{(r)}_{n-r,k+r}(t)f\left(\frac{nt+\alpha}{n+\beta}\right)dt,$$

and on integrating by parts, r times the required result follows.

By $C_\gamma[0,\infty)$, we denote the class of all continuous functions on the interval $[0,\infty)$ having growth of order $O(t^\gamma)$, $\gamma > 0$. It can be easily verified that the operators $B_{n,\alpha,\beta}(f,x)$ are well defined for $f \in C_\gamma[0,\infty)$. The norm-$||.||_\gamma$ on $C_\gamma[0,\infty)$ is defined as

$$||f||_\gamma = \sup_{x\in[0,\infty)}|f(t)|t^{-\gamma}.$$

Definition 1. Let us assume that $0 < a < a_1 < b_1 < b < \infty$. Then for sufficiently small $\eta > 0$ the Steklov mean $f_{\eta,2}$ of second order corresponding to $f \in C_\gamma[a,b]$ and $t \in I_1$ is defined as follows:

$$f_{\eta,2}(t) = \eta^{-2}\int_{-\eta/2}^{\eta/2}\int_{-\eta/2}^{\eta/2}\left(f(t)-\Delta_h^2 f(t)\right)dt_1dt_2,$$

where $h = (t_1+t_2)/2$ and Δ_h^2 is the second order forward difference operator with step length h. For $f \in C[a,b]$, $f_{\eta,2}$ satisfy the following properties [9]:

1. $f_{\eta,2}$ has continuous derivatives up to order 2 over $[a_1,b_1]$;
2. $\|f_{\eta,2}\|_{C[a_1,b_1]} \le C\,\omega_r(f,\eta,[a,b]), r = 1,2;$
3. $\|f - f_{\eta,2}\|_{C[a_1,b_1]} \le C\,\omega_2(f,\eta,[a,b]);$
4. $\|f_{\eta,2}\|_{C[a_1,b_1]} \le C\,\eta^{-2}\|f\|_{C[a,b]};$
5. $\|f_{\eta,2}\|_{C[a_1,b_1]} \le C\,\|f\|_\gamma,$

where by C we denote the certain constants independent of f and η that are different in each occurrence.

Lemma 7 ([3]). *Let* $f \in C[a,b]$*. Suppose that* $f^{(k)} \in AC[0,\infty)$ *and* $f^{(k+1)} \in C[0,\infty)$*. Then,*

$$\|f_{\eta,2k}^{(i)}\|_{C[a,b]} \leq C_i\{\|f_{\eta,2k}\|_{C[a,b]} + \|f_{\eta,2k}^{(2k)}\|_{C[a,b]}\},\ i = 1,2,\ldots,2k-1,$$

where $C_i's$ *are certain constants independent of* f*.*

2.2 Direct Estimates

In this section, we present some direct results, which include asymptotic formula and error estimations in terms of modulus of continuity in simultaneous approximation.

Theorem 17. *Let* $f \in C_\gamma[0,\infty)$ *be bounded on every finite sub-interval of* $[0,\infty)$ *admitting the derivative of order* $(r+2)$ *at a fixed point* $x \in (0,\infty)$ *and let* $f(t) = O(t^\gamma)$ *as* $t \to \infty$*, for some* $\gamma > 0$*. Then we have*

$$\lim_{n\to\infty} n\left(B_{n,\alpha,\beta}^{(r)}(f,x) - f^{(r)}(x)\right) = \frac{r(r+1-2\beta)}{2} f^{(r)}(x) + [x(1+r-\beta)+r+1+\alpha] f^{(r+1)}(x) + \frac{x(2+x)}{2} f^{(r+2)}(x).$$

Proof. By Taylor's expansion of f, we have

$$f(t) = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^{r+2},$$

where $\varepsilon(t,x) \to 0$ as $t \to x$ and $\varepsilon(t,x) = o((t-x)^\delta)$ as $t \to \infty$ for some $\delta > 0$. Now using Lemma 6, we can write

$$\begin{aligned} n\left[B_{n,\alpha,\beta}^{(r)}(f,x) - f^{(r)}(x)\right] &= n\left[\sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} B_{n,\alpha,\beta}^{(r)}((t-x)^i,x) - f^{(r)}(x)\right] \\ &\quad + nB_{n,\alpha,\beta}^{(r)}(\varepsilon(t,x)(t-x)^{r+2},x) \\ &=: J_1 + J_2. \end{aligned}$$

First to estimate J_1, we use binomial theorem as

$$\begin{aligned}
J_1 &= n\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}\sum_{j=r}^{i}\binom{i}{j}(-x)^{i-j}B_{n,\alpha,\beta}^{(r)}(t^j,x)-nf^{(r)}(x)\\
&= \frac{f^{(r)}(x)}{r!}n\left(B_{n,\alpha,\beta}^{(r)}(t^r,x)-r!\right)\\
&\quad+\frac{f^{(r+1)}(x)}{(r+1)!}n\left\{(r+1)(-x)B_{n,\alpha,\beta}^{(r)}(t^r,x)+B_{n,\alpha,\beta}^{(r)}(t^{r+1},x)\right\}\\
&\quad+\frac{f^{(r+2)}(x)}{(r+2)!}\\
&\quad\times n\left\{\frac{(r+2)(r+1)}{2}x^2B_{n,\alpha,\beta}^{(r)}(t^r,x)+(r+2)(-x)B_{n,\alpha,\beta}^{(r)}(t^{r+1},x)\right.\\
&\qquad\left.+B_{n,\alpha,\beta}^{(r)}(t^{r+2},x)\right\}
\end{aligned}$$

Next using Lemma 2, we have

$$\begin{aligned}
J_1 &= n\left[\frac{n^r}{(n+\beta)^r}\frac{(n-r-1)!n^r}{(n-1)!}-1\right]f^{(r)}(x)\\
&\quad+n\frac{f^{(r+1)}(x)}{(r+1)!}\left\{(r+1)(-x)\frac{n^r(n-r-1)!n^r}{(n+\beta)^r(n-1)!}r!\right.\\
&\quad+\frac{n^{r+1}(n-r-2)!n^{r+1}}{(n+\beta)^{r+1}(n-1)!}(r+1)!x+\frac{(r+1)^2n^{r+1}(n-r-2)!n^r}{(n+\beta)^{r+1}(n-1)!}r!\\
&\quad\left.+(r+1)\alpha\frac{n^r(n-r-1)!n^r}{(n+\beta)^{r+1}(n-1)!}r!\right\}\\
&\quad+n\frac{f^{(r+2)}(x)}{(r+2)!}\left\{\frac{(r+2)(r+1)}{2}x^2\frac{n^r}{(n+\beta)^r}\frac{(n-r-1)!n^r}{(n-1)!}r!\right.\\
&\quad-(r+2)x\left(\frac{n^{r+1}(n-r-2)!n^{r+1}}{(n+\beta)^{r+1}(n-1)!}(r+1)!x+\frac{(r+1)^2n^{r+1}(n-r-2)!n^r}{(n+\beta)^{r+1}(n-1)!}r!\right.\\
&\quad\left.+(r+1)\alpha\frac{n^r(n-r-1)!n^r}{(n+\beta)^{r+1}(n-1)!}r!\right)+\frac{n^{r+2}(n-r-3)!n^{r+2}}{(n+\beta)^{r+2}(n-1)!}\frac{(r+2)!}{2}x^2\\
&\quad+\frac{(r+2)^2n^{r+2}(n-r-3)!}{(n+\beta)^{r+2}(n-1)!}n^{r+1}(r+1)!x+(r+2)\\
&\qquad\left.\alpha\frac{n^{r+1}(n-r-2)!n^{r+1}}{(n+\beta)^{r+2}(n-1)!}(r+1)!x\right\}+O(n^{-2}).
\end{aligned}$$

In the limiting case as $n\to\infty$, we obtain the coefficients of $f^{(r)}(x)$, $f^{(r+1)}(x)$ and $f^{(r+2)}(x)$ in the above expression as $\frac{r(r+1-2\beta)}{2}$, $[x(1+r-\beta)+r+1+\alpha]$

and $\frac{x(2+x)}{2}$, respectively. Hence in order to complete the proof of the theorem it is sufficient to show that $J_2 \to 0$ as $n \to \infty$. For this note that by using Lemma 4, we have

$$|J_2| \leq n \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} n^i \frac{|q_{i,j,r}(x)|}{x^r} \sum_{k=0}^{\infty} s_{n,k}(x)|k-nx|^j \int_0^{\infty} b_{n,k}(t)|\varepsilon(t,x)| \left|\frac{nt+\alpha}{n+\beta}-x\right|^{r+2} dt.$$

Since $\varepsilon(t,x) \to 0$ as $t \to x$, hence for a given $\varepsilon > 0$ there exists a $\delta > 0$ such that $|\varepsilon(t,x)| < \varepsilon$ whenever $|t-x| < \delta$. Further, if λ is any integer $\geq \max\{\gamma, r+2\}$, then we find a constant $K > 0$ such that $|\varepsilon(t,x)| \left|\frac{nt+\alpha}{n+\beta} - x\right|^{r+2} \leq K \left|\frac{nt+\alpha}{n+\beta} - x\right|^{\gamma}$.

$$\begin{aligned}
|J_2| \leq\ & C_1 \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} n^{i+1} \sum_{k=0}^{\infty} s_{n,k}(x)|k-nx|^j \left\{ \int_{|t-x|<\delta} \varepsilon b_{n,k}(t) \left|\frac{nt+\alpha}{n+\beta} - x\right|^{r+2} dt \right. \\
& \left. + \int_{|t-x|\geq\delta} K b_{n,k}(t) \left|\frac{nt+\alpha}{n+\beta} - x\right|^{\lambda} dt \right\} \\
=:\ & J_3 + J_4.
\end{aligned}$$

Now applying Schwarz inequality for the integration and summation, we have

$$\begin{aligned}
|J_3| &\leq \varepsilon C_1 \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} n^{i+1} \sum_{k=0}^{\infty} s_{n,k}(x)|k-nx|^j \left(\int_0^{\infty} b_{n,k}(t)dt\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_0^{\infty} b_{n,k}(t) \left(\frac{nt+\alpha}{n+\beta} - x\right)^{2r+4} dt\right)^{\frac{1}{2}} \\
&\leq \varepsilon C_1 \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} n^{i+1} \left(\sum_{k=0}^{\infty} s_{n,k}(x)(k-nx)^{2j}\right)^{\frac{1}{2}} \\
&\quad \times \left(\sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} b_{n,k}(t) \left(\frac{nt+\alpha}{n+\beta} - x\right)^{2r+4} dt\right)^{\frac{1}{2}}.
\end{aligned}$$

Next, using Lemmas 3 and 4, we get

$$\begin{aligned}
|J_3| &\leq \varepsilon C_1 \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} n^{i+1} \cdot O(n^{j/2}) \cdot O(n^{-(r+2)/2}) \\
&\leq \varepsilon O(1).
\end{aligned}$$

In view of arbitrariness of ε, it follows that $J_3 = o(1)$. Again, using Schwarz inequality for the integration and summation, Lemmas 3 and 4, we have

$$\begin{aligned}|J_4| &\leq C_2 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^{i+1} \sum_{k=0}^{\infty} s_{n,k}(x)|k-nx|^j \int_{|t-x|\geq\delta} b_{n,k}(t) \left|\frac{nt+\alpha}{n+\beta}-x\right|^{\lambda} dt \\ &\leq C_2 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^{i+1} \left(\sum_{k=0}^{\infty} s_{n,k}(x)(k-nx)^{2j}\right)^{\frac{1}{2}} \\ &\quad \times \left(\sum_{k=1}^{\infty} s_{n,k}(x)\int_0^{\infty} b_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{2\lambda} dt\right)^{\frac{1}{2}} \\ &= \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^{i+1}\cdot O(n^{j/2})\cdot O(n^{-\lambda/2}) \\ &= O(n^{(r+2-\lambda)/2}) = o(1).\end{aligned}$$

Thus $J_2 \to 0$ as $n \to \infty$. Combining the estimates of J_1 and J_2, we get the desired result. This completes the proof of the theorem.

Theorem 18. *Let $f \in C_{\gamma}[0,\infty)$ for some $\gamma > 0$ and $r \leq m \leq r+2$. If $f^{(m)}$ exists and is continuous on $(a-\eta, b+\eta) \subset (0,\infty), \eta > 0$, then for n sufficiently large, we have*

$$\|B_{n,\alpha,\beta}^{(r)}(f,x)-f^{(r)}(x)\|_{C[a,b]} \leq C_1 n^{-1}\sum_{i=r}^{m}\|f^{(i)}\|_{C[a,b]}+C_2 n^{-1/2}\omega(f^{(m)},n^{-1/2}) +O(n^{-2}),$$

where C_1, C_2 are constants independent of f and n, $\omega(f,\delta)$ is the modulus of continuity of f on $(a-\eta, b+\eta)$ and $\|.\|_{C[a,b]}$ denotes the sup-*norm on* $[a,b]$.

Proof. By Taylor's expansion of f, we have

$$f(t)=\sum_{i=0}^{m}\frac{f^{(i)}(x)}{i!}(t-x)^i+\frac{f^{(m)}(\xi)-f^{(m)}(x)}{m!}(t-x)^m\chi(t)+h(t,x)(1-\chi(t)),$$

where ξ lies between t and x and $\chi(t)$ is the characteristic function on the interval $(a-\eta, b+\eta)$. Now,

$$B_{n,\alpha,\beta}^{(r)}(f,x)-f^{(r)}(x) = \left\{\sum_{i=0}^{m}\frac{f^{(i)}(x)}{i!}B_{n,\alpha,\beta}^{(r)}((t-x)^i,x)-f^{(r)}(x)\right\}$$

$$+\ B_{n,\alpha,\beta}^{(r)}\left(\frac{f^{(m)}(\xi)-f^{(m)}(x)}{m!}(t-x)^m\chi(t),x\right)$$
$$+\ B_{n,\alpha,\beta}^{(r)}\left(h(t,x)(1-\chi(t)),x\right)$$
$$=:E_1+E_2+E_3.$$

By using Lemmas 4 and 2, we have

$$E_1=\sum_{i=0}^{m}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}\frac{d^r}{dx^r}\Bigg[x^j\frac{n^j}{(n+\beta)^j}\frac{(n-j-1)!n^j}{(n-1)!}$$
$$+x^{j-1}\left\{\frac{n^j}{(n+\beta)^j}j^2\frac{(n-j-1)!n^{j-1}}{(n-1)!}\right.$$
$$\left.+j\alpha\frac{n^{j-1}}{(n+\beta)^j}\frac{(n-j)!n^{j-1}}{(n-1)!}\right\}$$
$$+x^{j-2}\left\{j(j-1)^2\alpha\frac{n^{j-1}}{(n+\beta)^j}\frac{(n-j)!n^{j-2}}{(n-1)!}\right.$$
$$\left.+\frac{j(j-1)\alpha^2}{2}\frac{n^{j-2}}{(n+\beta)^j}\frac{(n-j+1)!n^{j-2}}{(n-1)!}\right\}$$
$$+O(n^{-2})\Bigg]-f^{(r)}(x).$$

Consequently, for n sufficiently large, we get

$$\|E_1\|_{C[a,b]}\leq C_1n^{-1}\sum_{i=r}^{m}\|f^{(i)}\|_{C[a,b]}+O(n^{-2}),\text{ uniformly on }[a,b].$$

Next, we estimate E_2 as follows

$$|E_2|\leq\sum_{k=0}^{\infty}s_{n,k}^{(r)}(x)\int_0^\infty b_{n,k}(t)\left\{\left|\frac{f^{(m)}(\xi)-f^{(m)}(x)}{m!}\right|\left|\frac{nt+\alpha}{n+\beta}-x\right|^m\chi(t)\right\}dt$$
$$\leq\frac{\omega(f^{(m)},\delta)}{m!}\sum_{k=0}^{\infty}s_{n,k}^{(r)}(x)\int_0^\infty b_{n,k}(t)\left(1+\frac{\left|\frac{nt+\alpha}{n+\beta}-x\right|}{\delta}\right)\left|\frac{nt+\alpha}{n+\beta}-x\right|^m dt$$
$$\leq\frac{\omega(f^{(m)},\delta)}{m!}\sum_{k=0}^{\infty}s_{n,k}^{(r)}(x)\int_0^\infty b_{n,k}(t)\left(\left|\frac{nt+\alpha}{n+\beta}-x\right|^m+\delta^{-1}\left|\frac{nt+\alpha}{n+\beta}-x\right|^{m+1}\right)dt.$$

Therefore, by applying Lemma 5, we get

$$\sum_{k=0}^{\infty} s_{n,k}^{(r)}(x) \int_0^{\infty} b_{n,k}(t) \left| \frac{nt+\alpha}{n+\beta} - x \right|^m dt$$
$$\leq \sum_{k=0}^{\infty} \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} n^i |k-nx|^j \frac{|q_{i,j,r}(x)|}{x^r} s_{n,k}(x) \int_0^{\infty} b_{n,k}(t) \left| \frac{nt+\alpha}{n+\beta} - x \right|^m dt$$
$$\leq \left(\sup_{\substack{2i+j\leq r \\ i,j\geq 0}} \frac{|q_{i,j,r}(x)|}{x^r} \right) \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} n^i \left(\sum_{k=0}^{\infty} s_{n,k}(x)|k-nx|^j \int_0^{\infty} b_{n,k}(t) \left| \frac{nt+\alpha}{n+\beta} - x \right|^m dt \right).$$

Using Schwarz inequality for integration and summation, and using Lemmas 3 and 4, we get

$$\sum_{k=0}^{\infty} s_{n,k}(x)|k-nx|^j \int_0^{\infty} b_{n,k}(t) \left| \frac{nt+\alpha}{n+\beta} - x \right|^m dt$$
$$\leq \sum_{k=0}^{\infty} s_{n,k}(x)|k-nx|^j \left(\int_0^{\infty} b_{n,k}(t)dt \right)^{1/2} \left(\int_0^{\infty} b_{n,k}(t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^{2m} dt \right)^{1/2}$$
$$\leq \left(\sum_{k=0}^{\infty} s_{n,k}(x)(k-nx)^{2j} \right)^{\frac{1}{2}} \times \left(\sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} b_{n,k}(t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^{2m} dt \right)^{\frac{1}{2}}$$
$$= O(n^{j/2}) \cdot O(n^{-m/2}) = O(n^{(j-m)/2}), \text{ uniformly on } [a,b].$$

Thus

$$\sum_{k=0}^{\infty} s_{n,k}^{(r)}(x) \int_0^{\infty} b_{n,k}(t) \left| \frac{nt+\alpha}{n+\beta} - x \right|^m dt \tag{7}$$
$$\leq C \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} n^i O(n^{(j-m)/2}) = O(n^{(r-m)/2}), \text{ uniformly on } [a,b],$$

where $C = \sup_{\substack{2i+j\leq r \\ i,j\geq 0}} \frac{|q_{i,j,r}(x)|}{x^r}$ $\forall\ x \in [0,\infty)$. Choosing $\delta = n^{-1/2}$ and applying (7), we obtain

$$|E_2| \leq \frac{\omega(f^{(m)}, n^{-1/2})}{m!} \left[O(n^{(r-m)/2}) + n^{1/2} O(n^{(r-m-1)/2}) + O(n^{-m}) \right)$$
$$\leq C_2 n^{-(r-m)/2} \omega(f^{(m)}, n^{-1/2}).$$

Since $t \in [0, \infty) \setminus (a - \eta, b + \eta)$, we can choose δ such that $|t - x| \geq \delta$ for all $x \in [a, b]$. Thus by Lemma 5, we get

$$|E_3| \leq C \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \sum_{k=0}^{\infty} s_{n,k}(x)|k - nx|^j \int_0^{\infty} b_{n,k}(t)|h(t, x)|.$$

For $|t - x| \geq \delta$, we can find the constant M such that $|h(t, x)| \leq M \left|\frac{nt+\alpha}{n+\beta} - x\right|^{\beta}$, where β is an integer $\geq \{\gamma, m\}$. Hence using the Schwarz inequality for both integration and summation, Lemmas 3 and 4, it easily follows that $E_3 = O(n^{-s})$ for any $s > 0$, uniformly on $[a, b]$.

Combing the estimates of $E_1,\ E_2,\ E_3$, the required result is immediate.

Theorem 19. *Let $f \in C_{\gamma}[0, \infty)$ for some $\gamma > 0$ and $0 < a < a_1 < b_1 < b < \infty$. Then for n sufficiently large, we have*

$$\|B^{(r)}_{n,\alpha,\beta}(f, .) - f^{(r)}\|_{C[a_1,b_1]} \leq C_1\omega_2(f^{(r)}, n^{-1/2}, [a_1, b_1]) + C_2 n^{-k}\|f\|_{\gamma},$$

where $C_1 = C_1(r)$ and $C_2 = C_2(r, f)$.

Proof. Using the linearity property and linear approximating method viz. Steklov mean of second order (see Definition 1), we can write

$$\begin{aligned}\|B^{(r)}_{n,\alpha,\beta}(f, .) - f^{(r)}\|_{C[a_1,b_1]} &\leq \|B^{(r)}_{n,\alpha,\beta}((f - f_{\eta,2}), .)\|_{C[a_1,b_1]}\\ &\quad + \|B^{(r)}_{n,\alpha,\beta}(f, x)(f_{\eta,2}, .) - f^{(r)}_{\eta,2}\|_{C[a_1,b_1]}\\ &\quad + \|f^{(r)} - f^{(r)}_{\eta,2}\|_{C[a_1,b_1]}\\ &= S_1 + S_2 + S_3.\end{aligned}$$

Since $f^{(r)}_{\eta,2} = (f^{(r)})_{\eta,2}$, hence by property (3) of the Steklov mean, we get

$$S_3 \leq C_1\omega_2(f^{(r)}, \eta, [a, b]).$$

Next, using Theorem 17 and Lemma 7, we get

$$\begin{aligned}S_2 &\leq C_2 n^{-1} \sum_{i=r}^{2+r} \|f^{(i)}_{\eta,2}\|_{C[a,b]}\\ &\leq C_3 n^{-1}\{\|f_{\eta,2}\|_{C[a,b]} + \|f^{(2+r)}_{\eta,2}\|_{C[a,b]}\}.\end{aligned}$$

By applying properties (2) and (4) of Steklov mean, we obtain

$$S_2 \leq C_4 n^{-1}\{\|f\|_{\gamma} + \eta^{-2}\omega_2(f^{(r)}, \eta, [a, b])\}.$$

Finally, we estimate S_1 choosing a^*, b^* satisfying the condition $0 < a < a^* < a_1 < b_1 < b^* < b < \infty$. For this, let $\chi(t)$ denote the characteristic function on the interval $[a^*, b^*]$, then

$$\begin{aligned}
S_1 &\leq \|B^{(r)}_{n,\alpha,\beta}\big(\chi(t)(f(t) - f_{\eta,2}(t)), .\big)\|_{C[a_1,b_1]} \\
&\quad + \|B^{(r)}_{n,\alpha,\beta}(f,x)\big((1-\chi(t))\big(f(t) - f_{\eta,2}(t)\big), .\big)\|_{C[a_1,b_1]} \\
&= S_4 + S_5.
\end{aligned}$$

By Lemma 6, we have

$$\begin{aligned}
&B^{(r)}_{n,\alpha,\beta}(f,x)\big(\chi(t)(f(t) - f_{\eta,2}(t)), x\big) \\
&= \frac{n^r(n-r-1)!n^r}{(n+\beta)^r\,(n-1)!}\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^{\infty} b_{n-r,k+r}(t)\chi(t) \\
&\times\left[f^{(r)}\left(\frac{nt+\alpha}{n+\beta}\right) - f^{(r)}_{\eta,2}\left(\frac{nt+\alpha}{n+\beta}\right)\right]dt.
\end{aligned}$$

Hence,

$$\left\|B^{(r)}_{n,\alpha,\beta}(f,x)\big(\chi(t)(f(t) - f_{\eta,2}(t)), .\big)\right\|_{C[a_1,b_1]} \leq C_5\|f^{(r)} - f^{(r)}_{\eta,2}\|_{C[a^*,b^*]}.$$

Now for $x \in [a_1, b_1]$ and $t \in [0,\infty)\backslash[a^*, b^*]$, we choose a $\delta > 0$ satisfying $\left|\frac{nt+\alpha}{n+\beta} - x\right| \geq \delta$. By using Lemma 6 and Schwarz inequality, we have

$$\begin{aligned}
I &\equiv |B^{(r)}_{n,\alpha,\beta}(f,x)\big((1-\chi(t))\big(f(t) - f_{\eta,2}(t)\big), x\big)| \\
&\leq \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i\frac{|q_{i,j,r}(x)|}{x^r}\sum_{k=0}^{\infty} s_{n,k}(x)|k-nx|^j \\
&\quad\times\int_0^{\infty} b_{n,k}(t)\big((1-\chi(t)\big)\left|f\left(\frac{nt+\alpha}{n+\beta}\right) - f_{\eta,2}\left(\frac{nt+\alpha}{n+\beta}\right)\right|dt \\
&\leq C_6\|f\|_\gamma\left\{\delta^{-2s}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i\sum_{k=0}^{\infty} s_{n,k}(x)|k-nx|^j\left(\int_0^{\infty} b_{n,k}(t)dt\right)^{1/2}\right. \\
&\quad\left.\times\left(\int_0^{\infty} b_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta} - x\right)^{4s}dt\right)^{1/2}\right\}
\end{aligned}$$

Finally applying Lemmas 3 and 4, we get

$$
\begin{aligned}
I &\le C_6\|f\|_\gamma \delta^{-2s} \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \left\{ \sum_{k=0}^{\infty} s_{n,k}(x)(k-nx)^{2j} \right\}^{1/2} \\
&\quad \times \left(\sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} b_{n,k}(t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^{4s} dt \right)^{1/2} \\
&\le C_7\|f\|_\gamma \delta^{-2s} \sum_{\substack{2i+j\le r\\ i,j\ge 0}} O(n^{(i+j/2-s)}) \le C_7 n^{-q}\|f\|_\gamma, \ \ q = s - r/2,
\end{aligned}
$$

where the last term vanishes as $n \to \infty$. Now choosing $m > 0$ satisfying $q \ge 1$, we have

$$
I \le C_7 n^{-1}\|f\|_\gamma.
$$

Therefore by property (3) of Steklov mean, we obtain

$$
S_1 \le C_9 \omega_2(f^{(r)}, \eta, [a, b]) + C_7 n^{-1}\|f\|_\gamma.
$$

Choosing $\eta = n^{-1/2}$, the theorem follows by combining the estimates of S_1 and S_2.

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Extremal Problems and g-Loewner Chains in $\mathbb{C}^n$ and Reflexive Complex Banach Spaces

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Abstract Let X be a reflexive complex Banach space with the unit ball B. In the first part of the paper, we survey various growth and coefficient bounds for mappings in the Carathéodory family $\mathscr{M}$, which plays a key role in the study of the generalized Loewner differential equation. Then we consider recent results in the theory of Loewner chains and the generalized Loewner differential equation on the unit ball of $\mathbb{C}^n$ and reflexive complex Banach spaces. In the second part of this paper, we obtain sharp growth theorems and second coefficient bounds for mappings with g-parametric representation and we present certain particular cases of special interest. Finally, we consider extremal problems related to bounded mappings in $S_g^0(B^n)$, where B^n is the Euclidean unit ball in $\mathbb{C}^n$. To this end, we use ideas from control theory to investigate the (normalized) time-log M-reachable family $\tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)$ generated by a subset $\mathscr{M}_g$ of $\mathscr{M}$, where $M \geq 1$ and g is a univalent function on the unit disc U such that $g(0) = 1$, $\Re g(\zeta) > 0$, $|\zeta| < 1$, and which satisfies some natural conditions. We characterize this family in terms of univalent subordination chains, and we obtain certain results related to extreme points and support points associated with the compact family $\overline{\tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)}$. Also, we give some examples of mappings in $\tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)$ and obtain the sharp growth result for this family.

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1 Introduction and Preliminaries

Let X be a complex Banach space with respect to a norm $\|\cdot\|$. Let B_r be the open ball centered at zero and of radius r, and let B be the open unit ball in X. If $X = \mathbb{C}^n$, the unit ball is denoted by B^n, while the unit polydisc in $\mathbb{C}^n$ is denoted by U^n. Also, let U be the unit disc in $\mathbb{C}$. A function $g : U \to \mathbb{C}$ is said to be univalent if g is holomorphic and injective on U. Also, g is said to be convex if g is univalent and $g(U)$ is a convex domain in $\mathbb{C}$.

We denote by $L(X, Y)$ the set of continuous linear operators from X into another complex Banach space Y with the standard operator norm. The space $L(X, X)$ is denoted by $L(X)$. Let I be the identity in $L(X)$. Let Ω be a domain in X and $f : \Omega \to X$ be a mapping. We say that f is holomorphic if for each $z \in \Omega$ there exists a mapping $Df(z) \in L(X)$ such that

$$\lim_{h\to 0} \frac{\|f(z+h) - f(z) - Df(z)(h)\|}{\|h\|} = 0.$$

Let $H(\Omega)$ be the set of holomorphic mappings from Ω into X with the compact-open topology. A mapping $f \in H(\Omega)$ is said to be biholomorphic if $f(\Omega)$ is a domain, and the inverse f^{-1} exists and is holomorphic on $f(\Omega)$. A mapping $f \in H(\Omega)$ is said to be locally biholomorphic if each $z \in \Omega$ has a neighborhood V such that $f|_V$ is biholomorphic. An injective mapping in $H(\Omega)$ will be said to be univalent. A mapping $f \in H(B)$ is said to be normalized if $f(0) = 0$ and $Df(0) = I$. Let $S(B)$ be the set of normalized biholomorphic mappings of B into X. Also, let $S^*(B)$ and $K(B)$ be the subsets of $S(B)$ consisting of starlike and convex mappings, respectively. In the case $X = \mathbb{C}$, the family $S(U)$ is the usual family S of normalized univalent functions on the unit disc U. The family $S^*(U)$ is denoted by S^*.

Obviously, in the finite dimensional case $X = \mathbb{C}^n$, the notions of univalence and biholomorphy are equivalent. However, in the infinite dimensional complex Banach spaces, there exist univalent mappings which are not biholomorphic (see, e.g., [45]).

For $z \in X \setminus \{0\}$, we define

$$T(z) = \{l_z \in L(X, \mathbb{C}) : \ l_z(z) = \|z\|, \ \|l_z\| = 1\}.$$

Then $T(z) \neq \emptyset$ in view of the Hahn–Banach theorem.

It is a result of Harris [28, Theorem 1] that if $P_m : X \to X$ is a homogeneous polynomial mapping of degree m, then $\|P_m\| \leq k_m|V(P_m)|$ for $m \geq 1$, where $k_m = m^{m/(m-1)}$ for $m > 1$ and $k_1 = e$. Note that if X is a complex Hilbert space, then $k_1 = 2$ (see, e.g., [6, p. 3]). Here $|V(P_m)|$ is the numerical radius of P_m given by

$$|V(P_m)| = \sup\{|l_z(P_m(z))| : \|z\| = 1, \, l_z \in T(z)\}.$$

1.1 The Carathéodory Family

Next, we recall the Carathéodory family in $H(B)$:

$$\mathcal{M} = \{h \in H(B) : h(0) = 0, Dh(0) = I, \Re[l_z(h(z))] > 0, z \in B \setminus \{0\}, l_z \in T(z)\}.$$

If $X = \mathbb{C}$, it is clear that $f \in \mathcal{M}$ if and only if $f(z)/z \in \mathcal{P}$, where

$$\mathcal{P} = \{p \in H(U) : p(0) = 1, \Re p(z) > 0, z \in U\}$$

is the Carathéodory family on the unit disc U.

For various applications of these families in the theory of Loewner chains and biholomorphic mappings in finite and infinite dimensional complex Banach spaces, see [8, 10, 12, 13, 16, 17, 19, 23, 36, 40, 45].

The following growth and coefficient bounds for the class $\mathcal{M}$ are known. Graham et al. [13] proved the coefficient bounds using the result of [28, Theorem 1], and hence the upper growth estimate. The lower growth estimate in Theorem 1 (iii) is due to Pfaltzgraff [36] (see also [22]).

Theorem 1. *Let $h(z) = z + \sum_{m=2}^{\infty} P_m(z^m) : B^n \to \mathbb{C}^n$ be such that $h \in \mathcal{M}$, where $P_m = \frac{1}{m!}D^m h(0)$ for $m \geq 2$. Then the following relations hold:*

- (i) $|V(P_m)| \leq 2$, $m \geq 2$. *These bounds are sharp when B^n is the unit ball in $\mathbb{C}^n$ with respect to a p-norm, where $p \in [1, \infty]$.*
- (ii) $\|P_m\| \leq 2k_m$, $m \geq 2$, *where $k_m = m^{m/(m-1)}$ for $m \geq 2$.*
- (iii) $\frac{r(1-r)}{1+r} \leq \|h(z)\| \leq \frac{4r}{(1-r)^2}$, $\|z\| = r < 1$.

Recently Bracci et al. [8], Graham et al. [20] obtained the following improvement of the upper growth estimate in Theorem 1 (iii), which holds in the case of complex Banach spaces.

Proposition 1. *Let $h \in \mathcal{M}$. Then*

$$\|h(z)\| \leq r\left[1 + 8\frac{r(1 - r\ln 2)}{(1-r)^2}\right], \quad \|z\| = r < 1.$$

As a consequence of Theorem 1 (iii) and the fact that $\mathcal{M}$ is a closed family, we obtain the following compactness result (see [13]).

Corollary 1. *The family $\mathscr{M}$ is compact in the topology of $H(B^n)$.*

If $X = \mathbb{C}^n$ with respect to the maximum norm $\|\cdot\|_\infty$, the following growth and coefficient bounds for the Carathéodory family $\mathscr{M}$ hold. The upper bound in (2) was obtained by Poreda [38, Corollary 1]. The lower bound in (2) is due to Gurganus [22, Lemma 3]. The authors in [20] proved the same result as in Corollary 2 in the case of the space $X = \ell_\infty$.

Corollary 2. *Let $h : U^n \to \mathbb{C}^n$ be such that $h \in \mathscr{M}$, and let $P_m(z) = \frac{1}{m!}D^m h(0)(z^m)$, $m \geq 2$. Then*

$$\|P_m\|_\infty \leq 2, \quad m \geq 2, \tag{1}$$

$$\frac{r(1-r)}{1+r} \leq \|h(z)\|_\infty \leq \frac{r(1+r)}{1-r}, \quad \|z\|_\infty = r < 1. \tag{2}$$

These estimates are sharp.

The following family of holomorphic mappings on B is a natural refinement of the Carathéodory family $\mathscr{M}$. This family was introduced by Graham et al. in the case $X = \mathbb{C}^n$ (see [13]), and played an important role in the study of normalized biholomorphic mappings which have parametric representation on the unit ball in $\mathbb{C}^n$ (see [12, 13, 27]).

Definition 1. Let $g : U \to \mathbb{C}$ be a univalent function such that $g(0) = 1$. Also, let $h : B \to X$ be a normalized holomorphic mapping. We say that h belongs to the family $\mathscr{M}_g$ if

$$\frac{1}{\|z\|} l_z(h(z)) \in g(U), \quad z \in B \setminus \{0\}, \quad l_z \in T(z). \tag{3}$$

Remark 1. Obviously, if $\Re g(\zeta) > 0$, $\zeta \in U$, then $\mathscr{M}_g \subseteq \mathscr{M}$. Also, if $g(\zeta) = \frac{1+\zeta}{1-\zeta}$, $\zeta \in U$, then $\mathscr{M}_g = \mathscr{M}$. However, there are other choices of g which provide interesting properties of the family $\mathscr{M}_g$ (see [12, 13, 27]).

Definition 2. Let $g : U \to \mathbb{C}$ be a univalent function such that $g(0) = 1$. Also, let $f : B \to X$ be a normalized holomorphic mapping. We say that f belongs to the family $\mathscr{R}_g$ if $h \in \mathscr{M}_g$, where $h(z) = Df(z)(z)$, $z \in B$. The family $\mathscr{R}_{g_0}$ is denoted by $\mathscr{R}$, where $g_0(\zeta) = \frac{1+\zeta}{1-\zeta}$, $|\zeta| < 1$.

The following connection between the families $\mathscr{R}_g$ and $\mathscr{M}_g$ holds in the case that g is a convex function on U with $g(0) = 1$ (see [20]).

Proposition 2. *Let $g : U \to \mathbb{C}$ be a univalent function such that $g(0) = 1$. If g is convex, then $\mathscr{R}_g \subseteq \mathscr{M}_q \subseteq \mathscr{M}_g$, where $q(\zeta) = \frac{1}{\zeta}\int_0^\zeta g(t)dt$, $\zeta \in U$. In particular, $\mathscr{R} \subseteq \mathscr{M}_q \subseteq \mathscr{M}$, where $q(\zeta) = -1 - 2\frac{\log(1-\zeta)}{\zeta}$, $|\zeta| < 1$.*

The following assumption will be useful in the forthcoming sections (see [13]).

Assumption 1. *Let $g : U \to \mathbb{C}$ be a univalent function such that $g(0) = 1$, $g(\overline{\zeta}) = \overline{g(\zeta)}$ for $\zeta \in U$(so, g has real coefficients in its power series expansion), and $\Re g(\zeta) > 0$ on U. We assume that g satisfies the conditions*

$$\begin{cases} \min\limits_{|\zeta|=r} \Re g(\zeta) = \min\{g(r), g(-r)\} \\ \max\limits_{|\zeta|=r} \Re g(\zeta) = \max\{g(r), g(-r)\}, \end{cases} \tag{4}$$

for all $r \in (0, 1)$.

Remark 2. We recall that a set Ω in the complex plane is said to be Steiner symmetric with respect to the real axis if $tw + (1-t)\overline{w} \in \Omega$, for all $w \in \Omega$ and $t \in [0, 1]$. Note that the condition (4) is satisfied by all univalent functions g on U with $g(0) = 1$, $g(\overline{\zeta}) = \overline{g(\zeta)}$, $\zeta \in U$, and whose images are Steiner symmetric with respect to the real axis (see [29, Theorem 2 and p. 304]). In particular, all convex functions g on U with $g(0) = 1$ and having real coefficients in their power series expansions satisfy (4).

The following lemma will be used in the forthcoming results (see [13] for $X = \mathbb{C}^n$). The estimate (5) was obtained in [47, Lemma 3], in the case of complex Banach spaces. The proof is similar to the finite dimensional case $X = \mathbb{C}^n$. The sharpness is proved in [20].

Lemma 1. *Let g satisfy the conditions of Assumption 1. Then*

$$\begin{aligned} \|z\| \min\{g(\|z\|), g(-\|z\|)\} &\le \Re l_z(h(z)) \\ &\le \|z\| \max\{g(\|z\|), g(-\|z\|)\}, \end{aligned} \tag{5}$$

for $h \in \mathcal{M}_g$, $z \in B \setminus \{0\}$, $l_z \in T(z)$. This estimate is sharp.

Graham et al. [20] proved recently the following results related to the families $\mathcal{M}_g$ and $\mathcal{R}_g$ (compare with [13, Theorem 1.2]):

Theorem 2. *Let $g : U \to \mathbb{C}$ be a univalent function such that $g(0) = 1$. Also, let $h \in \mathcal{M}_g$ and $P_m = \frac{1}{m!} D^m h(0)$ for $m \in \mathbb{N}$. Then the following conditions hold:*

- (i) $|V(P_m)| \le (m-1)|g'(0)|$ *for* $m \ge 2$.
- (ii) $\|P_m\| \le m^{\frac{m}{m-1}}(m-1)|g'(0)|$ *for* $m \ge 2$.
- (iii) *For each $r \in (0, 1)$, there exists a constant $M = M(r, g) > 0$, which is independent of h, such that $\|h(z)\| \le M$ for $\|z\| \le r$.*

Theorem 3. *Let $g : U \to \mathbb{C}$ be a univalent function such that $g(0) = 1$. Also, let $f \in \mathcal{R}_g$ and $P_m = \frac{1}{m!} D^m f(0)$ for $m \in \mathbb{N}$. Then the following conditions hold:*

- (i) $|V(P_m)| \le \frac{m-1}{m}|g'(0)|$ *for* $m \ge 2$.
- (ii) $\|P_m\| \le m^{\frac{1}{m-1}}(m-1)|g'(0)|$ *for* $m \ge 2$.
- (iii) *For each $r \in (0, 1)$, there exists a constant $M = M(r, g) > 0$, which is independent of h, such that $\|f(z)\| \le M$ for $\|z\| \le r$.*

The following compactness result is a consequence of Theorems 2 and 3 (see [20]).

Corollary 3. *Let $g : U \to \mathbb{C}$ be a univalent function such that $g(0) = 1$. Then $\mathscr{M}_g$ and $\mathscr{R}_g$ are compact families in $H(B^n)$.*

If $X = \mathbb{C}^n$ with respect to the maximum norm $\|\cdot\|_\infty$, we obtain the following coefficient bounds for the families $\mathscr{M}_g$ and $\mathscr{R}_g$, in the case that g is not necessarily convex, but still a univalent function on U (see [20]).

Theorem 4. *Let $g : U \to \mathbb{C}$ be a univalent function such that $g(0) = 1$. Let $h \in H(U^n)$ be such that $h \in \mathscr{M}_g$, and let $P_m(z) = \frac{1}{m!}D^m h(0)(z^m)$, $m \geq 2$. Then the following relation holds:*

$$\|P_m\|_\infty \leq (m-1)|g'(0)|, \quad m \geq 2. \tag{6}$$

Theorem 5. *Let $g : U \to \mathbb{C}$ be a univalent function such that $g(0) = 1$. Let $f \in H(U^n)$ be such that $f \in \mathscr{R}_g$ and let $P_m(z) = \frac{1}{m!}D^m f(0)(z^m)$, $m \geq 2$. Then the following relation holds:*

$$\|P_m\|_\infty \leq \frac{m-1}{m}|g'(0)|, \quad m \geq 2. \tag{7}$$

1.2 Loewner Chains and the Loewner Differential Equation

We next consider the notions of subordination and Loewner chain on the unit ball B of X. Recent contributions in the Loewner theory in $\mathbb{C}^n$ and complete hyperbolic complex manifolds may be found in [1–3, 7], and [24].

For $f, g \in H(B)$, we say that f is subordinate to g (written $f \prec g$) if $f = g \circ v$ for some Schwarz mapping v (i.e., $v \in H(B)$ and $\|v(z)\| \leq \|z\|$, $z \in B$).

Definition 3. A mapping $f : B \times [0, \infty) \to X$ is called a Loewner chain if $f(\cdot, t)$ is univalent on B, $f(0, t) = 0$, $Df(0, t) = e^t I$, for all $t \geq 0$, and $f(\cdot, s) \prec f(\cdot, t)$, $0 \leq s \leq t < \infty$.

The above subordination implies the existence of a univalent Schwarz mapping $v = v(\cdot, s, t)$, called the transition mapping associated with $f(z, t)$, such that

$$f(z, s) = f(v(z, s, t), t), \quad z \in B, \quad 0 \leq s \leq t < \infty.$$

Note that the transition mapping $v(z, s, t)$ satisfies the semigroup property:

$$v(z, s, t) = v(v(z, s, u), u, t), \quad z \in B, \quad 0 \leq s \leq u \leq t < \infty. \tag{8}$$

The following existence and uniqueness result of solutions to the initial value problem (9) was obtained in [26] (see also [19]), and is a generalization to reflexive complex Banach spaces of [36, Theorem 2.1 and Lemma 2.2].

Lemma 2. *Let X be a reflexive complex Banach space and let $h = h(z,t) : B \times [0,\infty) \to X$ be a mapping which satisfies the following conditions:*

(i) $h(\cdot,t) \in \mathcal{M}$ *for* $t \geq 0$*;*
(ii) $h(z,\cdot)$ *is strongly measurable on* $[0,\infty)$ *for* $z \in B$.

Then for each $s \geq 0$ *and* $z \in B$*, the initial value problem*

$$\frac{\partial v}{\partial t} = -h(v,t) \quad a.e. \quad t \geq s, \quad v(z,s,s) = z, \tag{9}$$

has a unique solution $v = v(z,s,t)$ *such that* $v(\cdot,s,t)$ *is a univalent Schwarz mapping,* $v(z,s,\cdot)$ *is Lipschitz continuous on* $[s,\infty)$ *uniformly with respect to* $z \in \overline{B}_r$*,* $r \in (0,1)$*, and* $Dv(0,s,t) = e^{s-t}I$ *for* $t \geq s \geq 0$*. Moreover, the limit*

$$\lim_{t\to\infty} e^t v(z,s,t) = f(z,s) \tag{10}$$

exists uniformly on each closed ball $\overline{B}_r$ *for* $r \in (0,1)$ *and* $s \geq 0$*. Also,* $f(z,t)$ *is a Loewner chain. In addition, for each* $r \in (0,1)$*, there exists* $M(r) \leq \frac{r}{(1-r)^2}$ *such that*

$$\|e^{-t} f(z,t)\| \leq M(r), \quad \|z\| \leq r, \quad t \geq 0. \tag{11}$$

Definition 4. Let X be a reflexive complex Banach space. A mapping $h = h(z,t) : B \times [0,\infty) \to X$ which satisfies the assumptions (i) and (ii) of Lemma 2 is called a generating vector field (cf. [2, 7, 10]).

Remark 3. If the condition (ii) of Lemma 2 is replaced by the assumption that $h = h(z,t) : B \times [0,\infty) \to X$ is continuous on $B \times [0,\infty)$, then the conclusion of Lemma 2 is true in the case of complex Banach spaces, not necessarily reflexive, in view of [40, Lemmas 4.3–4.5, Corollary 4.3].

Remark 4. It is known that if $f(z,t)$ is a Loewner chain on $B^n \times [0,\infty)$, then $f(z,\cdot)$ is locally Lipschitz continuous on $[0,\infty)$ locally uniformly with respect to $z \in B^n$ (see [14], [12, Chap. 8]). Graham et al. [13, Theorem 1.10] proved that in the finite dimensional case $X = \mathbb{C}^n$, every Loewner chain on $B^n \times [0,\infty)$ satisfies the generalized Loewner differential equation (12).

Theorem 6. *Let* $f(z,t) : B^n \times [0,\infty) \to \mathbb{C}^n$ *be a Loewner chain. Then there exist a generating vector field* $h = h(z,t)$ *and a measurable subset* E *of* $[0,\infty)$ *of measure zero such that*

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad t \in [0,\infty) \setminus E, \quad \forall\, z \in B^n. \tag{12}$$

If, in addition, $\{e^{-t}f(\cdot,t)\}_{t\geq 0}$ *is a locally uniformly bounded family on* B^n*, then* $f(z,s)=\lim_{t\to\infty}e^t v(z,s,t)$ *locally uniformly on* B^n*, where* $v=v(z,s,t)$ *is the unique Lipschitz continuous solution on* $[s,\infty)$ *of the initial value problem*

$$\frac{\partial v}{\partial t}=-h(v,t),\quad \textit{a.e. } t\geq s,\quad \forall\, z\in B^n.$$

The notion of parametric representation on the unit ball of X was introduced in [19] (see [13] and [38], for $X=\mathbb{C}^n$).

Definition 5. Let X be a reflexive complex Banach space and let $f\in H(B)$ be a normalized mapping. We say that f has parametric representation (denote by $f\in S^0(B)$) if there exists a generating vector field $h=h(z,t):B\times[0,\infty)\to X$ such that $f(z)=\lim\limits_{t\to\infty}e^t v(z,t)$ uniformly on each closed ball $\overline{B}_r$ for $r\in(0,1)$, where $v=v(z,t)$ is the unique Lipschitz continuous solution on $[0,\infty)$ of the initial value problem

$$\frac{\partial v}{\partial t}=-h(v,t)\quad \text{a.e.}\quad t\geq 0,\quad v(z,0)=z,\quad \forall\, z\in B.$$

If, in addition, $g:U\to\mathbb{C}$ is a univalent function such that $g(0)=1$ and $\Re g(\zeta)>0$, $|\zeta|<1$, and if $h(\cdot,t)\in\mathscr{M}_g$ for a.e. $t\geq 0$, then we say that f has g-parametric representation (denote by $f\in S_g^0(B)$). Moreover, if $h(\cdot,t)\in\mathscr{R}_g$ for a.e. $t\geq 0$, then we denote that $f\in\hat{S}_g^0(B)$.

Taking into account Proposition 2, we deduce that if g is a convex function on U such that $g(0)=1$, $\Re g(\zeta)>0$, $|\zeta|<1$, then

$$\hat{S}_g^0(B)\subseteq S_q^0(B)\subseteq S_g^0(B)\subseteq S^0(B),$$

where $q(\zeta)=\frac{1}{\zeta}\int_0^\zeta g(t)dt$, $\zeta\in U$.

Definition 6. Let X be a complex Banach space (not necessarily reflexive), and let $f\in H(B)$ be a normalized mapping. We say that f has strong parametric representation (cf. [19, 39]) (and denote by $f\in\tilde{S}^0(B)$) if there exists a mapping $h=h(z,t):B\times[0,\infty)\to X$ which is continuous on $B\times[0,\infty)$, $h(\cdot,t)\in\mathscr{M}$ for $t\geq 0$, and such that $f(z)=\lim\limits_{t\to\infty}e^t v(z,t)$ uniformly on each closed ball $\overline{B}_r$ for $r\in(0,1)$, where $v=v(z,t)$ is the unique Lipschitz continuous solution on $[0,\infty)$ of the initial value problem

$$\frac{\partial v}{\partial t}=-h(v,t),\quad \forall\, t\geq 0,\quad v(z,0)=z,$$

for each $z\in B$.

If, in addition, $h(\cdot, t) \in \mathcal{M}_g$ for $t \geq 0$, where $g : U \to \mathbb{C}$ is a univalent function such that $g(0) = 1$ and $\Re g(\zeta) > 0$, $|\zeta| < 1$, then we say that f has strong g-parametric representation (and denote by $f \in \tilde{S}_g^0(B)$).

Clearly, if X is a reflexive complex Banach space, then $\tilde{S}^0(B) \subseteq S^0(B)$ and $\tilde{S}_g^0(B) \subseteq S_g^0(B)$.

Recently, the following characterization of parametric representation in terms of Loewner chains in reflexive complex Banach spaces was proved in [19] (cf. [26]; see [17] and [39], in the case $X = \mathbb{C}^n$). The second statement (ii) of Theorem 7 was obtained by Pfaltzgraff [36] in the case $X = \mathbb{C}^n$.

Theorem 7. *Let X be a reflexive complex Banach space and let $f : B \to X$ be a normalized holomorphic mapping. Then the following statements hold:*

(i) If f has parametric representation, then there exists a Loewner chain $f(z,t)$ such that $f = f(\cdot, 0)$ and the conditions (10) and (11) hold.
(ii) Conversely, let $h = h(z,t) : B \times [0,\infty) \to X$ be a generating vector field. Assume that $f(\cdot,t) \in H(B)$, $f(0,t) = 0$, $Df(0,t) = e^t I$, $t \geq 0$, $f(z,\cdot)$ is strongly locally absolutely continuous on $[0,\infty)$ for $z \in B$, and there exists a set $E \subset [0,\infty)$ of measure zero such that

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad t \in [0,\infty) \setminus E, \ \forall z \in B. \tag{13}$$

Also, assume that for each $r \in (0,1)$, there exists $M = M(r) > 0$ such that

$$\|e^{-t} f(z,t)\| \leq M(r), \quad \|z\| \leq r, \quad t \geq 0. \tag{14}$$

Then $f(z,t)$ is a Loewner chain and $f = f(\cdot,0)$ has parametric representation.

Remark 5. In the finite dimensional case $X = \mathbb{C}^n$, $f \in S^0(B^n)$ if and only if there exists a Loewner chain $f(z,t)$ such that $f = f(\cdot,0)$ and $\{e^{-t} f(\cdot,t)\}_{t \geq 0}$ is a normal family on B^n (see [13] and [14]).

Remark 6. (i) It is well known that every function $f \in S$ can be embedded as the first element of a Loewner chain. In addition, f has parametric representation, i.e. $f(z) = \lim_{t\to\infty} e^t v(z,t)$ locally uniformly on U, where $v = v(z,t)$ is the unique Lipschitz continuous solution on $[0,\infty)$ of the initial value problem

$$\frac{\partial v}{\partial t} = -vp(v,t) \text{ a.e. } t \geq 0, \quad v(z,0) = z,$$

for some choice of $p = p(z,t)$ such that $p(\cdot,t) \in \mathcal{P}$ for almost all $t \in [0,\infty)$ and $p(z,\cdot)$ is measurable on $[0,\infty)$ for $z \in U$ (see [37]). Becker [5] obtained the general form of solutions to the Loewner differential equation on the unit disc, i.e.

$$\frac{\partial f}{\partial t}(z,t) = zf'(z,t)p(z,t), \quad \text{a.e.} \quad t \geq 0, \quad \forall z \in U,$$

where $p(\cdot, t) \in \mathscr{P}$ for any fixed $t \in [0, \infty)$, and $p(z, \cdot)$ is measurable on $[0, \infty)$ for $z \in U$. In the case $n = 1$, there exists a unique normalized univalent solution $f(z,t) = e^t z + \cdots$ of the above Loewner differential equation.

(ii) In dimension $n \geq 2$, the analogous uniqueness result does not hold (see [10] and [13]). Indeed, if $f(z,t) = e^t z + \cdots$ is a Loewner chain that satisfies the Loewner differential equation (12), and if Φ is a normalized biholomorphic mapping on $\mathbb{C}^n$, not the identity mapping, then $g(z,t) = \Phi(f(z,t))$ is another Loewner chain, which satisfies the same Loewner differential equation as $f(z,t)$.

Recent work on the structure of solutions of the Loewner differential equation in $\mathbb{C}^n$ appears in [10] (see also [1, 3, 4, 17, 23, 24] and [46]).

2 Growth Theorems and Coefficient Bounds for Mappings in $S_g^0(B)$

In this section, we assume that X is a reflexive complex Banach space. Let B be the unit ball of X. We obtain growth results and second coefficient bounds for mappings with g-parametric representation on B. We also give various particular cases and consequences.

2.1 Growth Results

Using arguments similar to those in the proof of [27, Lemma 9], we obtain the following lemma.

Lemma 3. *Let X be a reflexive complex Banach space and let $g : U \to \mathbb{C}$ satisfy the conditions of Assumption 1. Also, let $h = h(z,t)$ be a generating vector field such that $h(\cdot, t) \in \mathscr{M}_g$ for a.e. $t \geq 0$. Also, let $v = v(z,s,t)$ be the solution of the initial value problem (9). Then*

$$e^s \|z\| \exp \int_{\|v(z,s,t)\|}^{\|z\|} \left[\frac{1}{\max\{g(x), g(-x)\}} - 1 \right] \frac{dx}{x} \leq e^t \|v(z,s,t)\| \tag{15}$$

$$\leq e^s \|z\| \exp \int_{\|v(z,s,t)\|}^{\|z\|} \left[\frac{1}{\min\{g(x), g(-x)\}} - 1 \right] \frac{dx}{x},$$

for $z \in B$ and $t \geq s \geq 0$.

Proof. Fix $z \in B \setminus \{0\}$ and $s \geq 0$. Let $v(t) = v(z,s,t)$ for $t \geq s$. Then $v(t) \neq 0$, $t \geq 0$. Also, let $l_z \in T(z)$. Since $v(z,s,\cdot)$ is Lipschitz continuous in $[s,\infty)$, it follows that $\|v(z,s,\cdot)\|$ is also Lipschitz continuous on $[s,\infty)$, and hence $\|v(t)\|$ is differentiable for a.e. $t \geq s$. In view of [31, Lemma 1.3], it follows that

$$\frac{d\|v(t)\|}{dt} = \Re\left[l_{v(t)}\left(\frac{\partial v}{\partial t}\right)\right], \quad \text{a.e.} \quad t \geq s, \quad \forall\, l_{v(t)} \in T(v(t)).$$

Thus, we obtain that

$$\frac{d\|v(t)\|}{dt} = -\Re[l_{v(t)}(h(v(t),t))], \quad \text{a.e.} \quad t \geq s, \quad \forall\, l_{v(t)} \in T(v(t)).$$

From (5), we have

$$-\frac{\frac{d\|v(t)\|}{dt}}{\|v(t)\| \min\{g(\|v(t)\|), g(-\|v(t)\|)\}} \geq 1$$

and

$$-\frac{\frac{d\|v(t)\|}{dt}}{\|v(t)\| \max\{g(\|v(t)\|), g(-\|v(t)\|)\}} \leq 1$$

for a.e. $t \geq s$. Integrating both sides of the above relations with respect to t, we deduce that

$$\begin{aligned}
\int_{\|v(t)\|}^{\|z\|} \frac{dx}{x \min\{g(x), g(-x)\}} &= -\int_s^t \frac{\frac{d\|v(\tau)\|}{d\tau}}{\|v(\tau)\| \min\{g(\|v(\tau)\|), g(-\|v(\tau)\|)\}} d\tau \\
&\geq \int_s^t d\tau = t - s, \\
\int_{\|v(t)\|}^{\|z\|} \frac{dx}{x \max\{g(x), g(-x)\}} &= -\int_s^t \frac{\frac{d\|v(\tau)\|}{d\tau}}{\|v(\tau)\| \max\{g(\|v(\tau)\|), g(-\|v(\tau)\|)\}} d\tau \\
&\leq \int_s^t d\tau = t - s.
\end{aligned}$$

After elementary computations, the result follows, as desired. □

Now, we are able to obtain the following sharp growth result for the family $S_g^0(B)$ (see [13, Theorem 2.2], in the case $X = \mathbb{C}^n$). Note that a similar sharp growth result as in Theorem 8 holds for the family $\tilde{S}_g^0(B)$, where B is the unit ball of a complex Banach space, not necessarily reflexive.

Theorem 8. *Let X be a reflexive complex Banach space and let $g : U \to \mathbb{C}$ satisfy the conditions of Assumption 1. Also, let $f \in S_g^0(B)$. Then*

$$\|z\| \exp \int_0^{\|z\|} \left[\frac{1}{\max\{g(x), g(-x)\}} - 1 \right] \frac{dx}{x} \le \|f(z)\|$$
$$\le \|z\| \exp \int_0^{\|z\|} \left[\frac{1}{\min\{g(x), g(-x)\}} - 1 \right] \frac{dx}{x}, \quad z \in B. \tag{16}$$

This result is sharp.

Proof. Since $f \in S_g^0(B)$, there exists a generating vector field $h = h(z,t)$ such that $h(\cdot, t) \in \mathcal{M}_g$ for a.e. $t \ge 0$, and $f(z) = \lim_{t\to\infty} e^t v(z,t)$ uniformly on each closed ball $\overline{B}_r$, $r \in (0,1)$, where $v = v(z,t)$ is the unique Lipschitz continuous solution on $[0,\infty)$ of the initial value problem

$$\frac{\partial v}{\partial t} = -h(v,t) \quad \text{a.e.} \quad t \ge 0, \quad v(z,0) = z.$$

Then

$$\lim_{t\to\infty} v(z,t) = \lim_{t\to\infty} e^{-t}(e^t v(z,t)) = \lim_{t\to\infty} e^{-t} f(z) = 0$$

uniformly on each closed ball $\overline{B}_r$, $r \in (0,1)$. Letting $s = 0$ and $t \to \infty$ in (15) and using the above relation, the result follows, as desired.

Finally, to prove sharpness of (16), it suffices to use the fact that the estimate (16) also holds and is sharp for the family $S_g^*(B)$ of normalized biholomorphic mappings f on B such that $h \in \mathcal{M}_g$, where $h(z) = [Df(z)]^{-1} f(z)$ (see [25, Theorem 3.2]). Since $S_g^*(B) \subseteq \tilde{S}_g^0(B) \subseteq S_g^0(B)$, the result follows, as desired. This completes the proof. □

In particular, if $g(\zeta) = \frac{1+\zeta}{1-\zeta}$ in Theorem 8, we obtain the following sharp growth result for the family $S^0(B)$ (see [26, Corollary 3.6]; cf. [19, Theorems 3.1 and 3.5]; see also [13] and [38], in the case of $X = \mathbb{C}^n$). Note that a similar sharp growth result as in Corollary 4 holds for the family $\tilde{S}^0(B)$ of mappings with strong parametric representation, where B is the unit ball of a complex Banach space X, not necessarily reflexive.

Corollary 4. *Let X be a reflexive complex Banach space and let $f \in S^0(B)$. Then*

$$\frac{\|z\|}{(1+\|z\|)^2} \le \|f(z)\| \le \frac{\|z\|}{(1-\|z\|)^2}, \quad z \in B.$$

This result is sharp.

Another special particular case in Theorem 8 is provided by the function $g(\zeta) = 1 + \zeta$, $|\zeta| < 1$. In this case, if $h \in H(B)$ is a normalized mapping, then $h \in \mathcal{M}_g$ if and only if

$$\left| \frac{1}{\|z\|} l_z(h(z)) - 1 \right| < 1, \quad z \in B \setminus \{0\}, \quad l_z \in T(z).$$

Let f be a starlike mapping of order $1/2$ on B (denote by $f \in S^*_{1/2}(B)$), i.e. $h \in \mathcal{M}_g$, where $h(z) = [Df(z)]^{-1} f(z)$ for $z \in B$. Hence $f(z,t) = e^t f(z)$ is a Loewner chain. Moreover, $h(z,t) = [Df(z)]^{-1} f(z)$ is a generating vector field such that $h(\cdot, t) \in \mathcal{M}_g$ for $t \geq 0$, where $g(\zeta) = 1 + \zeta$. In view of Theorem 7 and Definition 5, we deduce that $f \in S^0_g(B)$. Moreover, it is known that if $f \in K(B)$, then $f \in S^*_{1/2}(B)$ (cf. [13] and [41]; see also [12, Chap. 6]). Hence, if $g(\zeta) = 1 + \zeta$, then we deduce that (cf. [13] in the case $X = \mathbb{C}^n$)

$$K(B) \subseteq S^*_{1/2}(B) \subseteq S^0_g(B).$$

The above relation was one of the motivations for introducing the family $S^0_g(B)$ in the finite dimensional case $X = \mathbb{C}^n$ (see [13]). Note that if $g(\zeta) = 1 + \zeta$, then the following inclusion relation

$$K(B) \subseteq S^*_{1/2}(B) \subseteq \tilde{S}^0_g(B)$$

also holds in the case of complex Banach spaces not necessarily reflexive, in view of Remark 3.

The following result holds (cf. [13], in the case $X = \mathbb{C}^n$). Note that a similar sharp growth result as in Corollary 5 holds for the family $\tilde{S}^0_g(B)$ of mappings with strong g-parametric representation, where $g(\zeta) = 1 + \zeta$ and B is the unit ball of a complex Banach space X, not necessarily reflexive.

Corollary 5. *Let X be a reflexive complex Banach space and let $f \in S^0_g(B)$, where $g(\zeta) = 1 + \zeta$, $|\zeta| < 1$. Then*

$$\frac{\|z\|}{1 + \|z\|} \leq \|f(z)\| \leq \frac{\|z\|}{1 - \|z\|}, \quad z \in B.$$

This result is sharp.

2.2 Coefficient Bounds

Next, we obtain the following second coefficient bounds for mappings which can be imbedded as the first element of a Loewner chain which satisfies the conditions of

Theorem 7 (ii). We remark that the estimate (17) holds for the full family $S_g^0(B)$ in the case that B is the unit ball of $X = \mathbb{C}^n$ (see [13, Theorem 2.14]).

Theorem 9. *Let X be a reflexive complex Banach space and let $g : U \to \mathbb{C}$ be a univalent function such that $g(0) = 1$ and $\Re g(\zeta) > 0$, $|\zeta| < 1$. Also, let $h = h(z,t) : B \times [0,\infty) \to X$ be a generating vector field such that $h(\cdot,t) \in \mathcal{M}_g$ for a.e. $t \geq 0$. Assume that $f(z,t) : B \times [0,\infty) \to X$ satisfies the assumptions of Theorem 7 (ii). Let $f = f(\cdot,0)$ and $P_2(w) = \frac{1}{2}D^2 f(0)(w^2)$. Then*

$$|l_w(P_2(w))| \leq |g'(0)|, \quad \|w\| = 1, \quad l_w \in T(w). \tag{17}$$

Moreover, $\|P_2(w)\| \leq 4|g'(0)|$, $\|w\| = 1$.

Proof. It suffices to use arguments similar to those in the proof of [13, Theorem 2.14]. □

In particular, if $g(\zeta) = \frac{1+\zeta}{1-\zeta}$ in Theorem 9, we obtain the following result (see [13] and [38], in the case $X = \mathbb{C}^n$). We remark that if $X = \mathbb{C}^n$, then the estimate (18) holds for the full family $S^0(B)$ and is sharp (see [13] and [38]).

Corollary 6. *Let X be a reflexive complex Banach space and let $f(z,t) : B \times [0,\infty) \to X$ satisfy the assumptions of Theorem 7 (ii). Also, let $f = f(\cdot,0)$ and $P_2(w) = \frac{1}{2}D^2 f(0)(w^2)$. Then $f \in S^0(B)$ and*

$$|l_w(P_2(w))| \leq 2, \quad \|w\| = 1, \quad l_w \in T(w). \tag{18}$$

Moreover, $\|P_2(w)\| \leq 8$, $\|w\| = 1$.

In the case that $X = \mathbb{C}^n$ with respect to the maximum norm $\|\cdot\|_\infty$, we obtain the following improvement of Theorem 9. In the case $g(\zeta) = \frac{1+\zeta}{1-\zeta}$, the relation (19) was obtained by Poreda (see [38, Theorem 3]).

Theorem 10. *Let $g : U \to \mathbb{C}$ be a univalent function such that $g(0) = 1$ and $\Re g(\zeta) > 0$, $|\zeta| < 1$. Also, let $f \in S_g^0(U^n)$ and $P_2(w) = \frac{1}{2}D^2 f(0)(w^2)$. Then*

$$\|P_2\|_\infty \leq |g'(0)|. \tag{19}$$

If, in addition, $f \in \hat{S}_g^0(U^n)$, then

$$\|P_2\|_\infty \leq \frac{1}{2}|g'(0)|. \tag{20}$$

These estimates are sharp.

Proof. We shall use arguments similar to those in the proof of [38, Theorem 3] and [13, Theorem 2.14]. Since $f \in S_g^0(U^n) \subseteq S^0(U^n)$, there exists a Loewner chain $f(z,t)$ such that $f = f(\cdot,0)$ and $\{e^{-t} f(\cdot,t)\}_{t\geq 0}$ is a normal family on U^n. Also, there exists a generating vector field $h = h(z,t) : U^n \times [0,\infty) \to \mathbb{C}^n$ such that $h(\cdot,t) \in \mathscr{M}_g$ for a.e. $t \geq 0$, and

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad \text{a.e.} \quad t \geq 0, \quad \forall\, z \in U^n.$$

Integrating both sides of the above equality and using elementary computations, we deduce as in the proof of [38, Theorem 3] (see also the proof of [13, Theorem 2.14]) that

$$e^{-2T}D^2f(0,T)(w^2) - D^2f(0,0)(w^2) = \int_0^T e^{-t}D^2h(0,t)(w^2)dt, \tag{21}$$

for all $T > 0$ and $w \in \mathbb{C}^n$. On the other hand, since $\{e^{-t}f(\cdot,t)\}_{t\geq 0}$ is a normal family on U^n, it follows that (see [38] and [13])

$$\frac{\|z\|_\infty}{(1+\|z\|_\infty)^2} \leq \|e^{-t}f(z,t)\|_\infty \leq \frac{\|z\|_\infty}{(1-\|z\|_\infty)^2}, \quad z \in U^n, \quad t \geq 0.$$

In view of the Cauchy integral formula and the above relation, we easily deduce that $\lim_{T\to\infty} e^{-2T}D^2f(0,T)(w^2) = 0$. Finally, letting $T \to \infty$ in (21) and using the relation (6), we obtain that

$$\frac{1}{2}\|D^2f(0,0)(w^2)\|_\infty \leq \frac{1}{2}\int_0^\infty e^{-t}\|D^2h(0,t)(w^2)\|_\infty dt \leq |g'(0)|, \quad \|w\|_\infty = 1.$$

Hence, the relation (19) follows, as desired. Also, since $S_g^*(U^n) \subseteq S_g^0(U^n)$, the estimation (19) is sharp for $f \in S_g^0(U^n)$.

To deduce the relation (20), it suffices to use a similar argument as above and the relation (7). We will show the sharpness of (20). For $u \in \partial U^n$ and $l_u \in T(u)$, let

$$h(z) = \left[\int_0^1 g(l_u(tz))dt\right]z, \quad z \in U^n.$$

Then $h \in \mathscr{R}_g$. Since $\mathscr{R}_g \subseteq \mathscr{R} \subseteq \mathscr{M}$, there exists $f \in S^*(U^n)$ such that

$$[Df(z)]^{-1}f(z) = h(z) \in \mathscr{R}_g,$$

as in the proof of [17, Corollary 2.10]. Therefore, $f \in \hat{S}_g^0(U^n)$. Since h has the Taylor expansion

$$h(z) = z - \frac{1}{2}D^2f(0)(z^2) + \cdots, \quad z \in U^n,$$

we obtain

$$\|P_2(u)\|_\infty = \frac{1}{2}\|D^2h(0)(u^2)\|_\infty = \frac{1}{2}|g'(0)|.$$

This completes the proof. □

3 Extremal Problems for Reachable Families Generated by the Family $\mathcal{M}_g$

In this section, let $\mathbb{C}^n$ be the n-dimensional complex space with the Euclidean structure, and let B^n be the Euclidean unit ball in $\mathbb{C}^n$. We are concerned with extremal problems for bounded mappings with g-parametric representation on B^n.

3.1 Extreme Points and Support Points for the Family $S^0(B^n)$

We begin this section with the well-known notions of extreme points and support points for a subset of $H(B^n)$. Various results related to extreme points and support points for compact subsets of $H(B^n)$ may be found in [9,15,18,21,33,34], and [44].

Definition 7. Let $\mathcal{F}$ be a subset of $H(B^n)$.

(i) A point $f \in \mathcal{F}$ is called an *extreme point* of $\mathcal{F}$ provided $f = tg + (1-t)h$, where $t \in (0,1)$, $g, h \in \mathcal{F}$, implies $f = g = h$. In other words, $f \in \mathcal{F}$ is an extreme point of $\mathcal{F}$ if f is not a proper convex combination of two points in $\mathcal{F}$.
(ii) A point $g \in \mathcal{F}$ is called a *support point* of $\mathcal{F}$ if there exists a continuous linear functional $L : H(B^n) \to \mathbb{C}$ such that $\Re L|_{\mathcal{F}}$ is not constant and

$$\Re L(g) = \max_{h\in\mathcal{F}} \Re L(h).$$

We denote by ex $\mathcal{F}$ and supp $\mathcal{F}$ the subsets of $\mathcal{F}$ consisting of extreme points of $\mathcal{F}$ and support points of $\mathcal{F}$, respectively. It is known that if $\mathcal{F}$ is a nonempty compact subset of $H(B^n)$, then ex $\mathcal{F}$ is a nonempty subset of $\mathcal{F}$. Also, it is known that if $\mathcal{F}$ is a compact subset of $H(B^n)$ which has at least two distinct points, then supp $\mathcal{F} \neq \emptyset$.

Remark 7. In the case of one complex variable, Pell [35] and Kirwan [32] proved that if f is an extreme point of S (respectively, f is a support point of S) and if $f(z,t)$ is a Loewner chain such that $f = f(\cdot, 0)$, then $e^{-t} f(\cdot, t)$ is an extreme point of S (respectively, $e^{-t} f(\cdot, t)$ is a support point of S), for all $t \geq 0$.

Graham et al. [18] proved the following result on extreme points and support points for $S^0(B^n)$.

Theorem 11. *Let $f \in S^0(B^n)$. If $f \in \operatorname{ex} S^0(B^n)$ (respectively, $f \in \operatorname{supp} S^0(B^n)$) and $f(z,t)$ is a Loewner chain such that $f = f(\cdot,0)$ and $\{e^{-t} f(\cdot,t)\}_{t\geq 0}$ is a normal family on B^n, then $e^{-t} f(\cdot,t) \in \operatorname{ex} S^0(B^n)$ for $t \geq 0$ (respectively, there exists $t_0 > 0$ such that $e^{-t} f(\cdot,t) \in \operatorname{supp} S^0(B^n)$ for $0 \leq t < t_0$).*

Example 1. Let $n \geq 2$ and let $f : B^n \to \mathbb{C}^n$ be given by

$$f(z) = (f_1(z_1), f_2(z')), \quad z = (z_1, z') \in B^n,$$

where

$$f_1(z_1) = \frac{z_1}{(1-z_1)^2} \text{ and } f_2 \in S^0(B^{n-1}) \setminus S^*(B^{n-1}).$$

It is easy to see that $f \in S^0(B^n) \setminus S^*(B^n)$. For fixed $r \in (0,1)$, let $z_0 = (r, 0')$ and

$$L(g) = g_1(z_0), \quad \text{for } g \in H(B^n).$$

Then L is a continuous linear functional on $H(B^n)$. For $g \in S^0(B^n)$, we have

$$\Re L(g) \leq |g_1(z_0)| \leq f_1(r) = \Re L(f)$$

by Corollary 4. We also have $\mathrm{id}_{B^n} \in S^0(B^n)$ and $\Re L(\mathrm{id}_{B^n}) < f_1(r) = \Re L(f)$. Thus $f \in \operatorname{supp} S^0(B^n)$, and this gives a nontrivial example of a support point in $S^0(B^n)$.

Remark 8. (i) Let $f(z,t)$ be a Loewner chain on $B^n \times [0,\infty)$. As in the proof of Graham et al. [18, Theorem 2.1 and Proposition 2.2], it can be proved that $e^t v(\cdot,0,t) \in S^0(B^n) \setminus \operatorname{ex} S^0(B^n)$ for $t \geq 0$, where $v(z,s,t)$ is the transition mapping associated with $f(z,t)$. They also conjectured that $e^t v(\cdot,0,t) \notin \operatorname{supp} S^0(B^n)$ for $t \geq 0$ and $n \geq 2$. Recently, the above conjecture was solved by Schleissinger [44]. In view of his result, it follows that if $f \in \operatorname{supp} S^0(B^n)$ and $f(z,t)$ is a Loewner chain such that $f = f(\cdot,0)$ and $\{e^{-t} f(\cdot,t)\}_{t\geq 0}$ is a normal family on B^n, then $e^{-t} f(\cdot,t) \in \operatorname{supp} S^0(B^n)$ for $t \geq 0$ (see [44]).

(ii) Recently, Chirilă et al. [9] proved that if $g : U \to \mathbb{C}$ is a univalent function, which satisfies the conditions of Assumption 1, and if $f \in \operatorname{ex} \overline{S_g^0(B^n)}$(respectively, $f \in \operatorname{supp} \overline{S_g^0(B^n)}$), then there exists a Loewner chain $f(z,t)$ such that $f = f(\cdot,0)$, $\{e^{-t} f(\cdot,t)\}_{t\geq 0}$ is a locally uniformly bounded family on B^n, and $e^{-t} f(\cdot,t) \in \operatorname{ex} \overline{S_g^0(B^n)}$(respectively, $e^{-t} f(\cdot,t) \in \operatorname{supp} \overline{S_g^0(B^n)}$), for all $t \geq 0$.

(iii) If $g : U \to \mathbb{C}$ is a univalent function, which satisfies the conditions of Assumption 1, then the identity mapping id_{B^n} is not an extreme point (respectively, is not a support point) of $\overline{S_g^0(B^n)}$ (see [9]).

3.2 Extremal Problems for Bounded Mappings in $S_g^0(B^n)$

Next, we study bounded mappings in $S(B^n)$ which have g-parametric representation. To this end, we use ideas from control theory to obtain properties of the normalized time-log M-reachable family $\tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)$, where $M > 1$. We prove a characterization in terms of univalent subordination chains.

Let $M \in [1, \infty)$ and let

$$S_g^0(M, B^n) = \left\{ f \in S_g^0(B^n) : \|f(z)\| < M,\ z \in B^n \right\}.$$

Definition 8. Let $E \subseteq [0, \infty)$ be an interval and let $\Omega \subseteq H(B^n)$ be a normal family. A mapping $h = h(z, t) : B^n \times E \to \mathbb{C}^n$ is called a Carathéodory mapping on E with values in Ω if the following conditions hold:

(i) $h(\cdot, t) \in \Omega$ for $t \in E$.
(ii) $h(z, \cdot)$ is a measurable mapping on E for $z \in B^n$.

Let $\mathscr{C}(E, \Omega)$ be the family of all Carathéodory mappings on E with values in Ω. In terms of control theory, the mapping $h = h(z, t)$ may be called a control function and the family $\mathscr{C}(E, \Omega)$ may be called a control system in $H(B^n)$. Also, the family Ω may be called an input family (cf. [30] and [42]).

Definition 9. Let $T \in [0, \infty)$ and let $\Omega \subseteq \mathscr{M}$. Also, let $h \in \mathscr{C}([0, T], \Omega)$ and let $v = v(z, t; h)$ be the unique Lipschitz continuous solution on $[0, T]$ of the initial value problem

$$\frac{\partial v}{\partial t}(z, t) = -h(v(z, t), t) \quad \text{a.e.} \quad t \in [0, T], \quad v(z, 0) = z, \tag{22}$$

for $z \in B^n$, such that $v(\cdot, t; h)$ is a univalent Schwarz mapping and $Dv(0, t; h) = e^{-t} I$ for $t \in [0, T]$. Also let

$$\mathscr{R}_T(\mathrm{id}_{B^n}, \Omega) = \left\{ v(\cdot, T; h) : h \in \mathscr{C}([0, T], \Omega) \right\}$$

denote the family of all such solutions at $t = T$ generated by all Carathéodory mappings on $[0, T]$ with values in Ω. The family $\mathscr{R}_T(\mathrm{id}_{B^n}, \Omega)$ is called the *time-T-reachable family* of (22) (cf. [18, 43] and [42]). The set Ω is called the *input set* or *input family* (cf. [42]). Let

$$\tilde{\mathscr{R}}_T(\mathrm{id}_{B^n}, \Omega) = e^T \mathscr{R}_T(\mathrm{id}_{B^n}, \Omega) \quad \text{for} \quad T \in [0, \infty)$$

and

$$\tilde{\mathscr{R}}_\infty(\mathrm{id}_{B^n}, \Omega) = \left\{ \lim_{t \to \infty} e^t v(\cdot, t; h) : h \in \mathscr{C}([0, \infty), \Omega) \right\}.$$

The family $\tilde{\mathscr{R}}_T(\mathrm{id}_{B^n}, \Omega)$ will be called *the normalized time-T-reachable family of (22)*.

Remark 9. It is known that $\tilde{\mathscr{R}}_\infty(\mathrm{id}_U, \mathscr{M}) = S$ (see [37] and [38]). Also, if $M \in (1, \infty)$, then $\tilde{\mathscr{R}}_{\log M}(\mathrm{id}_U, \mathscr{M}) = S(M)$ (see [11] and [42, Theorem 1.48]), where $S(M) = \{f \in S : |f(z)| < M, z \in U\}$. On the other hand, $\tilde{\mathscr{R}}_\infty(\mathrm{id}_{B^n}, \mathscr{M}_g) = S_g^0(B^n)$ (see [9]). Moreover, $\tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g) \subseteq S_g^0(M, B^n)$, by Theorem 13. Obviously, if $T = 0$, then $\tilde{\mathscr{R}}_T(\mathrm{id}_{B^n}, \mathscr{M}_g) = \{\mathrm{id}_{B^n}\}$.

Let $g : U \to \mathbb{C}$ be a univalent function such that $g(0) = 1$ and $\Re g(\zeta) > 0$, $|\zeta| < 1$. In the followings, we obtain some properties of the family $\tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)$ for $M \in (1, \infty)$. First, we give two examples of mappings in the normalized reachable family $\tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)$. In the case $g(\zeta) = \frac{1+\zeta}{1-\zeta}$, see [18] (compare with [21]).

Example 2. Let $g : U \to \mathbb{C}$ be a univalent function such that $g(0) = 1$ and $\Re g(\zeta) > 0$, $|\zeta| < 1$. Also, let $M > 1$ and $F \in S_g^*(B^n)$. Let $F^M : B^n \to \mathbb{C}^n$ be given by

$$F^M(z) = MF^{-1}(M^{-1}F(z)), \quad z \in B^n. \tag{23}$$

Then $F^M \in \tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)$.

Proof. Since $F \in S_g^*(B^n)$, it follows that the mapping F^M is well defined. Also, since $F \in S_g^*(B^n)$, we deduce that $F(z,t) = e^t F(z)$ is a Loewner chain and $F(z) = F(v(z,t),t)$ for $z \in B^n$ and $t \geq 0$, where $v(z,t) = F^{-1}(e^{-t}F(z))$. Obviously,

$$Mv(z, \log M) = MF^{-1}(M^{-1}F(z)),$$

i.e. $F^M(z) = Mv(z, \log M)$, $z \in B^n$. Since

$$\frac{\partial v}{\partial t}(z,t) = -h(v(z,t)), \quad z \in B^n, \quad t \geq 0,$$

where $h(z) = [Df(z)]^{-1}f(z) \in \mathscr{M}_g$, $F^M \in \tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)$, as desired. □

Example 3. Let $g : U \to \mathbb{C}$ be a convex function such that $g(0) = 1$ and $\Re g(\zeta) > 0$, $|\zeta| < 1$. Also, let $M > 1$ and let $f_j \in \tilde{\mathscr{R}}_{\log M}(\mathrm{id}_U, \mathscr{M}_g)$ for $j = 1, \ldots, n$. If $f = (f_1, \ldots, f_n)$, then $f \in \tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)$.

Proof. Since $f_j \in \tilde{\mathscr{R}}_{\log M}(\mathrm{id}_U, \mathscr{M}_g)$, there exists a unique Lipschitz continuous solution $v_j = v_j(z_j, t)$ on $[0, \log M]$ of the initial value problem

$$\frac{\partial v_j}{\partial t}(z_j, t) = -h_j(v_j(z_j,t),t), \quad \text{a.e.} \quad t \in [0, \log M], \quad v_j(z_j, 0) = z_j, \tag{24}$$

for $z_j \in U$ such that $f_j(z_j) = Mv_j(z_j, \log M)$, $z_j \in U$, where $h_j(z_j, t)$ is a generating vector field such that $h_j(z_j, t)/z_j \in g(U)$ for a.e. $t \in [0, \log M]$ and $z_j \in U$. Now, let $h = h(z, t) : B^n \times [0, \infty) \to \mathbb{C}^n$ be given by

$$h(z, t) = (h_1(z_1, t), \ldots, h_n(z_n, t)), \quad z = (z_1, \ldots, z_n) \in B^n, \quad t \in [0, \log M].$$

Then $h(\cdot, t) \in H(B^n)$, $h(0, t) = 0$, $Dh(0, t) = I$, $t \in [0, \log M]$, and

$$\left\langle h(z, t), \frac{z}{\|z\|^2} \right\rangle = \sum_{j=1}^{n} \frac{|z_j|^2}{\|z\|^2} \cdot \frac{h_j(z_j, t)}{z_j} \in g(U),$$

for a.e. $t \in [0, \log M]$ and $z = (z_1, \ldots, z_n) \in B^n \setminus \{0\}$. Here we have used the fact that $g(U)$ is a convex domain. Hence $h(\cdot, t) \in \mathcal{M}_g$ for a.e. $t \geq 0$. Next, let $v = v(z, t) : B^n \times [0, \infty) \to \mathbb{C}^n$ be given by

$$v(z, t) = (v_1(z_1, t), \ldots, v_n(z_n, t)), \quad z = (z_1, \ldots, z_n) \in B^n, \quad t \in [0, \log M].$$

Then, in view of (24), we deduce that

$$\frac{\partial v}{\partial t}(z, t) = -h(v(z, t), t), \qquad \text{a.e.} \quad t \in [0, \log M], \quad \forall\, z \in B^n.$$

Since $f_j(\cdot) = Mv_j(\cdot, \log M) \in \tilde{\mathcal{R}}_{\log M}(\mathrm{id}_U, \mathcal{M}_g)$, $j = 1, \ldots, n$, it follows that $f(\cdot) = Mv(\cdot, \log M) \in \tilde{\mathcal{R}}_{\log M}(\mathrm{id}_{B^n}, \mathcal{M}_g)$, as desired. This completes the proof. □

Theorem 12. *Let $g : U \to \mathbb{C}$ be a univalent function such that $g(0) = 1$, $\Re g(\zeta) > 0$, and $g(\overline{\zeta}) = \overline{g(\zeta)}$, $|\zeta| < 1$. Assume that g satisfies the conditions*

$$\min_{|\zeta|=r} \Re g(\zeta) = g(r), \quad \max_{|\zeta|=r} \Re g(\zeta) = g(-r),$$

for all $r \in (0, 1)$. Let $M > 1$ and let $f \in \tilde{\mathcal{R}}_{\log M}(\mathrm{id}_{B^n}, \mathcal{M}_g)$. Then

$$-Mb^{-1}\left(\frac{1}{M}b(-\|z\|)\right) \leq \|f(z)\| \leq Mb^{-1}\left(\frac{1}{M}b(\|z\|)\right), \quad z \in B^n, \tag{25}$$

where $b \in S^*$ is defined by $b(0) = 0$, $b'(0) = 1$, and

$$\frac{\zeta b'(\zeta)}{b(\zeta)} = \frac{1}{g(\zeta)}, \quad \zeta \in U. \tag{26}$$

These estimates are sharp.

Proof. Since $f \in \tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)$, there exists $h = h(z,t) \in \mathscr{C}([0, \log M], \mathscr{M}_g)$ such that $f = Mv(\cdot, \log M)$, where $v = v(z,t)$ is the unique Lipschitz continuous solution on $[0, \log M]$ of the initial value problem (22). Now, fix $z \in B^n \setminus \{0\}$. Since $v(z, \cdot)$ is Lipschitz continuous on $[0, \log M]$ locally uniformly with respect to $z \in B^n$, it follows that $\|v(z, \cdot)\|$ is differentiable a.e. on $[0, \log M]$. After elementary computations we deduce that

$$\frac{\partial \|v(z,t)\|}{\partial t} = -\frac{1}{\|v(z,t)\|}\Re\langle h(v(z,t),t), v(z,t)\rangle, \qquad \text{a.e.} \quad t \in [0, \log M].$$

Since $h(\cdot, t) \in \mathscr{M}_g$, we obtain from Lemma 1 that

$$\|z\|^2 g(\|z\|) \le \Re\langle h(z,t), z\rangle \le \|z\|^2 g(-\|z\|), \quad z \in B^n, \quad t \in [0, \log M].$$

In view of the above relations, we obtain that

$$-g(-\|v(z,t)\|) \le \frac{1}{\|v(z,t)\|} \cdot \frac{\partial \|v(z,t)\|}{\partial t} \le -g(\|v(z,t)\|),$$

for a.e. $t \in [0, \log M]$, which is equivalent to the following:

$$\frac{1}{g(-\|v(z,t)\|)\|v(z,t)\|} \cdot \frac{\partial \|v(z,t)\|}{\partial t} \ge -1$$

and

$$\frac{1}{g(\|v(z,t)\|)\|v(z,t)\|} \cdot \frac{\partial \|v(z,t)\|}{\partial t} \le -1.$$

Integrating both sides of the above inequalities on $[0, \log M]$ and using (26) and the fact that $\|v(z, \cdot)\|$ is decreasing on $[0, \log M]$, we obtain that

$$-\log M \le \log \frac{b\left(-M^{-1}\|f(z)\|\right)}{b(-\|z\|)}, \text{ and } -\log M \ge \log \frac{b\left(M^{-1}\|f(z)\|\right)}{b(\|z\|)}.$$

Since $b(t)$ is increasing for $t \in (-1, 1)$ and $b(0) = 0$, we have

$$-Mb^{-1}\left(\frac{1}{M}b(-\|z\|)\right) \le \|f(z)\| \le Mb^{-1}\left(\frac{1}{M}b(\|z\|)\right).$$

Next, we show that the estimates (25) are sharp. Let

$$F(z) = \frac{b(z_1)}{z_1}z \quad z = (z_1, z') \in B^n.$$

Then $F \in S_g^*(B^n)$ by [25, Lemma 3.1]. Let $F^M : B^n \to \mathbb{C}^n$ be given by (23). Then $F^M \in \tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)$ by Example 2. The equalities in (25) are attained at $z = (\pm r, 0')$ for $r \in (0,1)$. This completes the proof. □

Let $f(z,t) : B^n \times [0,\infty) \to \mathbb{C}^n$ be a Loewner chain. In view of Remark 4 and Theorem 6, $f(z,\cdot)$ is locally Lipschitz continuous on $[0,\infty)$ locally uniformly with respect to $z \in B^n$, and there exists a generating vector field $h = h(z,t) : B^n \times [0,\infty) \to \mathbb{C}^n$, which is unique up to a null subset of $[0,\infty)$, such that the Loewner differential equation (12) holds (see [13, Theorem 1.10]; cf. [2]). Moreover, if $\{e^{-t}f(\cdot,t)\}_{t\geq 0}$ is a normal family on B^n, then $f(z,t)$ coincides with the canonical solution of (12).

In view of the above, we recall the notion of a g-Loewner chain (cf. [13]).

Definition 10. Let $f(z,t) : B^n \times [0,\infty) \to \mathbb{C}^n$ be a Loewner chain and let $g : U \to \mathbb{C}$ be a univalent function such that $g(0) = 1$ and $\Re g(\zeta) > 0$, $|\zeta| < 1$. We say that $f(z,t)$ is a g-Loewner chain if $\{e^{-t}f(\cdot,t)\}_{t\geq 0}$ is a normal family on B^n and $h(\cdot,t) \in \mathscr{M}_g$, for almost all $t \in [0,\infty)$, where $h = h(z,t)$ is the generating vector field given by (12).

Remark 10.

(i) A normalized mapping $f \in H(B^n)$ has g-parametric representation if and only if there exists a g-Loewner chain $f(z,t)$ such that $f = f(\cdot,0)$ (see [13]).

(ii) In view of [13, Corollary 2.3], we deduce that $S_g^0(B^n)$ is a locally uniformly bounded family, and thus $\overline{S_g^0(B^n)}$ is compact. Also, it is clear that $\overline{S_g^0(B^n)} \subseteq S^0(B^n)$.

(iii) Let $S_g^*(B^n)$ be the subset of $S^*(B^n)$ consisting of g-starlike mappings (cf. [25]). Also, let $S_g^*(B^1) = S_g^*$. Recall that $f \in S_g^*(B^n)$ if and only if f is normalized locally biholomorphic on B^n and $h \in \mathscr{M}_g$, where $h(z) = [Df(z)]^{-1}f(z)$, $z \in B^n$. Clearly, $f \in S_g^*(B^n)$ if and only if $f(z,t) = e^t f(z)$ is a g-Loewner chain (cf. [13]). Hence, it is clear that $S_g^*(B^n) \subseteq S_g^0(B^n)$.

Theorem 13. *Let $M > 1$ and let $f \in H(B^n)$ be a normalized mapping. Also, let $g : U \to \mathbb{C}$ be a univalent function such that $g(0) = 1$ and $\Re g(\zeta) > 0$, $|\zeta| < 1$. Then $f \in \tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)$ if and only if there exists a g-Loewner chain $f(z,t)$ such that $f(\cdot,0) = f$, $f(\cdot,\log M) = M\mathrm{id}_{B^n}$. Hence $\tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g) \subseteq S_g^0(M, B^n)$.*

Proof. We shall use arguments similar to those in the proofs of [18, Theorem 3.7] and [21, Theorem 4.5].

(i) First, assume that $f \in \tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)$. Let $h(z,t)$ and $v(z,t)$ satisfy the assumptions of Definition 9 for $\Omega = \mathscr{M}_g$. Also, let $w = w(z,s,t)$ be the unique Lipschitz continuous solution of the initial value problem

$$\frac{\partial w}{\partial t}(z,s,t) = -h(w(z,s,t),t), \quad \text{a.e.} \quad s \leq t \leq \log M, \quad w(z,s,s) = z,$$

for $z \in B^n$ and $0 \le s < \log M$. Then $w(\cdot, s, t)$ is a univalent Schwarz mapping and $Dw(0, s, t) = e^{s-t} I$ (cf. [17], [12, Chap. 8]). Also, $w(z, 0, t) = v(z, t)$, and thus

$$f(z) = M v(z, \log M) = M w(z, 0, \log M).$$

Now, if $F(z, s) = M w(z, s, \log M)$, $s \in [0, \log M)$, then $F(z, s)$ satisfies the subordination condition on $B^n \times [0, \log M)$ needed to be the restriction of a Loewner chain, in view of the semigroup property (8) of the Schwarz mapping $w(z, s, t)$. Also, $F(z, 0) = f(z)$ and $F(z, \log M) = Mz$ for $z \in B^n$. Next, let $f(z, s) : B^n \times [0, \infty) \to \mathbb{C}^n$ be given by

$$f(\cdot, s) = \begin{cases} F(\cdot, s), \ s \in [0, \log M), \\ e^s \mathrm{id}_{B^n}, \ s \ge \log M. \end{cases}$$

Then $f(z, s)$ is a Loewner chain such that $f(z, 0) = f(z)$ and $f(z, \log M) = Mz$ for $z \in B^n$. Also, $\{e^{-s} f(\cdot, s)\}_{s \ge 0}$ is a normal family on B^n. Indeed, if $0 \le s < \log M$, then

$$\|e^{-s} f(z, s)\| = \|e^{\log M - s} w(z, s, \log M)\| \le M e^{-s} r \le M r,$$

for $\|z\| \le r < 1$. If $s \ge \log M$, then

$$\|e^{-s} f(z, s)\| \le r, \quad \|z\| \le r < 1.$$

Thus, we obtain that $\|e^{-s} f(z, s)\| \le Mr$ for $\|z\| \le r$ and $s \ge 0$. Moreover, $f(z, s)$ is a g-Loewner chain. Indeed, we have

$$\frac{\partial f}{\partial s}(z, s) = Df(z, s) h_f(z, s), \quad \text{a.e. } s \ge 0, z \in B^n,$$

where

$$h_f(z, s) = \begin{cases} h(z, s), \ s \in [0, \log M), \\ z, \qquad\quad s \ge \log M. \end{cases}$$

Note that for $s \in [0, \log M)$, we use [14, Theorem 2.3]. Thus, $f(z, s)$ is a g-Loewner chain and $f \in S_g^0(M, B^n)$, as desired.

(ii) Conversely, assume that there exists a g-Loewner chain $f(z, t)$ such that $f(\cdot, 0) = f$, $f(\cdot, \log M) = M \mathrm{id}_{B^n}$. Let $v(z, s, t)$ be the transition mapping associated with $f(z, t)$ and let $v(z, t) = v(z, 0, t)$. Then

$$f(z) = f(v(z, \log M), \log M) = M v(z, \log M), \quad z \in B^n,$$

and hence $f = Mv(\cdot, \log M)$. On the other hand, since $f(z,t)$ is a g-Loewner chain, there exists a mapping $h = h(z,t) \in \mathscr{C}([0,\infty), \mathscr{M}_g)$ such that

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad \text{a.e.} \quad t \geq 0, \quad \forall z \in B^n.$$

In view of [17, Theorem 2.6], we deduce that $v(z,s,t)$ is the unique Lipschitz continuous solution on $[s,\infty)$ of the initial value problem

$$\frac{\partial v}{\partial t} = -h(v,t) \quad \text{a.e.} \quad t \geq s, \quad v(z,s,s) = z,$$

for all $z \in B^n$ and $s \geq 0$. Since $f(z) = Mv(z, \log M)$ for $z \in B^n$, it follows that $f \in \tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)$, as desired. This completes the proof. □

Corollary 7. *Let $M > 1$ and let $g : U \to \mathbb{C}$ be a univalent function such that $g(0) = 1$ and $\Re g(\zeta) > 0$, $|\zeta| < 1$. Then*

$$\overline{\tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)} \subset \overline{S_g^0(B^n)} \setminus \left(\mathrm{ex}\, \overline{S_g^0(B^n)} \cup \mathrm{supp}\, \overline{S_g^0(B^n)}\right). \tag{27}$$

Especially, $\tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}) \subset S^0(B^n) \setminus \left(\mathrm{ex}\, S^0(B^n) \cup \mathrm{supp}\, S^0(B^n)\right)$.

Proof. It suffices to show the inclusion (27). Let $f \in \overline{\tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)}$. By Theorem 13, $f \in \overline{S_g^0(B^n)}$. Since $f \in \overline{\tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)}$, there exists a sequence $\{f_k\}_{k\in\mathbb{N}} \subset \tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)$ such that $f_k \to f$ locally uniformly on B^n. Then by Theorem 13, there exists a g-Loewner chain $f_k(z,t)$ such that $f_k = f_k(\cdot,0)$ and $f_k(\cdot, \log M) = M\mathrm{id}_{B^n}$ for each $k \in \mathbb{N}$. Now, $\{f_k(\cdot,t)\}_{k\in\mathbb{N}}$ is a sequence of g-Loewner chains, and in view of [14, Lemma 2.8], there exists a subsequence $\{f_{k_p}(\cdot,t)\}_{p\in\mathbb{N}}$ such that $f_{k_p}(\cdot,t) \to f(\cdot,t)$ locally uniformly on B^n for each $t \geq 0$, where $f(z,t)$ is a Loewner chain and $\{e^{-t}f(\cdot,t)\}_{t\geq 0}$ is a locally uniformly bounded family in $H(B^n)$. Also, we have $f(\cdot,0) = f$ and $f(\cdot, \log M) = M\mathrm{id}_{B^n}$. If $f \in \left(\mathrm{ex}\, \overline{S_g^0(B^n)} \cup \mathrm{supp}\, \overline{S_g^0(B^n)}\right)$, then we have $e^{-t}f(\cdot,t) \in \left(\mathrm{ex}\, \overline{S_g^0(B^n)} \cup \mathrm{supp}\, \overline{S_g^0(B^n)}\right)$ for $t \geq 0$ by using arguments similar to those in the proof of [9, Theorems 3.2 and 4.3]. This implies that $\mathrm{id}_{B^n} = e^{-\log M}f(\cdot, \log M) \in \left(\mathrm{ex}\, \overline{S_g^0(B^n)} \cup \mathrm{supp}\, \overline{S_g^0(B^n)}\right)$. This is a contradiction. Thus, the inclusion (27) holds. This completes the proof. □

Next, we consider extreme points associated with the family $\overline{\tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)}$. First, we obtain the following result (cf. [21], in the case $g(\zeta) = \frac{1+\zeta}{1-\zeta}$, $|\zeta| < 1$, and [9, Lemma 3.1]).

Lemma 4. *Let $g : U \to \mathbb{C}$ be a univalent function such that $g(0) = 1$ and $\Re g(\zeta) > 0$, $|\zeta| < 1$. Let $f(z,t)$ be a g-Loewner chain on $B^n \times [0,\infty)$. Also, let $v_{s,t}(z) = v(z,s,t)$ be the transition mapping associated with $f(z,t)$ and let*

$v_t(z) = v(z,t) = v_{0,t}(z)$ *for* $z \in B^n$ *and* $t \geq 0$. *If* $r \in \tilde{\mathscr{R}}_{\log M-t}(\mathrm{id}_{B^n}, \mathscr{M}_g)$, *then* $e^t r(v(\cdot,t)) \in \tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)$ *for* $0 \leq t < \log M$.

Proof. We shall use arguments similar to those in the proof of [9, Lemma 3.1]. Fix $t \in [0, \log M)$. Since $r \in \tilde{\mathscr{R}}_{\log M-t}(\mathrm{id}_{B^n}, \mathscr{M}_g)$, there exists a g-Loewner chain $r(z,s)$ such that $r = r(\cdot,0)$ and $r(\cdot, \log M - t) = e^{\log M - t}\mathrm{id}_{B^n}$ by Theorem 13. Let $F_1 = F_1(z,s) : B^n \times [0,\infty) \to \mathbb{C}^n$ be the mapping given by

$$F_1(z,s) = \begin{cases} e^t r(v(z,s,t)), & 0 \leq s \leq t, \\ e^t r(z, s-t), & s > t. \end{cases}$$

We will show that $F_1(z,s)$ is a g-Loewner chain such that $F_1(\cdot,0) = e^t r(v(\cdot,t))$ and $F_1(\cdot, \log M) = M\mathrm{id}_{B^n}$. Then, by Theorem 13, $e^t r(v(\cdot,t)) \in \tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)$. Taking into account the proof of [18, Theorem 2.1], we deduce that $F_1(z,s)$ is a Loewner chain such that $F_1(\cdot,0) = e^t r(v(\cdot,t))$ and $F_1(\cdot, \log M) = e^t r(\cdot, \log M - t) = M\mathrm{id}_{B^n}$. Also, it is not difficult to deduce that $\{e^{-s}F_1(\cdot,s)\}_{s\geq 0}$ is a normal family on B^n, since $\{e^{-s}r(\cdot,s)\}_{s\geq 0}$ is a normal family on B^n.

Since $F_1(z,s)$ is a Loewner chain, there exist a generating vector field $h(z,s)$ and a subset E of $[0,\infty)$ of measure zero such that

$$\frac{\partial F_1}{\partial s}(z,s) = DF_1(z,s)h(z,s), \quad s \in [0,\infty) \setminus E, \quad z \in B^n. \tag{28}$$

It remains to show that $h(\cdot,s) \in \mathscr{M}_g$ for a.e. $s \in [0,\infty)$. On the other hand, since $f(z,s)$ is a g-Loewner chain, there exist a generating vector field $h_f = h_f(z,s)$ and a subset $\tilde{E}$ of $[0,\infty)$ of measure zero, such that $h_f(\cdot,s) \in \mathscr{M}_g$ for $s \in [0,\infty) \setminus \tilde{E}$, and

$$\frac{\partial f}{\partial s}(z,s) = Df(z,s)h_f(z,s), \quad s \in [0,\infty) \setminus \tilde{E}, \quad \forall\, z \in B^n.$$

Also, since $v(z,\cdot,t)$ is Lipschitz continuous on $[0,t]$ locally uniformly with respect to $z \in B^n$, there exists a subset E_t of $[0,t]$ of measure zero, such that $\frac{\partial v}{\partial s}(\cdot,s,t)$ exists and is holomorphic on B^n, for all $s \in [0,t] \setminus E_t$, and the following relation holds, in view of [14, Theorem 2.3]:

$$\frac{\partial v}{\partial s}(z,s,t) = Dv(z,s,t)h_f(z,s), \quad s \in (0,t] \setminus E_t, \quad \forall\, z \in B^n. \tag{29}$$

First, we assume that $s \in [0,t] \setminus (E \cup E_t \cup \tilde{E})$. Then $F_1(z,s) = e^t r(v(z,s,t))$ for $z \in B^n$, and hence

$$\frac{\partial F_1}{\partial s}(z,s) = e^t Dr(v(z,s,t))\frac{\partial v}{\partial s}(z,s,t) \text{ and } DF_1(z,s) = e^t Dr(v(z,s,t))Dv(z,s,t),$$

for all $z \in B^n$ and $s \in [0,t] \setminus (E \cup E_t \cup \tilde{E})$. Taking into account the above relations and (28), we easily deduce that

$$\frac{\partial v}{\partial s}(z,s,t) = Dv(z,s,t)h(z,s), \quad s \in [0,t] \setminus (E \cup E_t \cup \tilde{E}), \quad z \in B^n. \tag{30}$$

In view of the relations (29) and (30), we obtain that $h(z,s) = h_f(z,s)$, for all $s \in [0,t] \setminus (E \cup E_t \cup \tilde{E})$ and $z \in B^n$. Since $h_f(\cdot,s) \in \mathcal{M}_g$ for $s \in [0,\infty) \setminus \tilde{E}$, it follows that $h(\cdot,s) \in \mathcal{M}_g$, for $s \in [0,t] \setminus (E \cup E_t \cup \tilde{E})$, and thus $h(\cdot,s) \in \mathcal{M}_g$, a.e. $s \in [0,t]$, as desired.

Next, we assume that $s \in (t,\infty) \setminus (E \cup \hat{E}_t)$, where $\hat{E}_t = \hat{E} + t$ and $\hat{E} \subset [0,\infty)$ is a set of measure zero such that $\frac{\partial r}{\partial u}(\cdot,u)$ exists and is holomorphic in B^n for $u \in [0,\infty) \setminus \hat{E}$, and

$$\frac{\partial r}{\partial u}(z,u) = Dr(z,u)h_r(z,u), \quad u \in [0,\infty) \setminus \hat{E}, \quad z \in B^n, \tag{31}$$

where $h_r = h_r(z,u)$ is a generating vector field such that $h_r(\cdot,u) \in \mathcal{M}_g$ for $u \in [0,\infty) \setminus \hat{E}$. Since $s \in (t,\infty) \setminus (E \cup \hat{E}_t)$, it follows that $F_1(z,s) = e^t r(z,s-t)$ for $z \in B^n$. It is clear that

$$\frac{\partial F_1}{\partial s}(z,s) = e^t \frac{\partial r}{\partial s}(z,s-t) \text{ and } DF_1(z,s) = e^t Dr(z,s-t),$$

for all $s \in (t,\infty) \setminus (E \cup \hat{E}_t)$ and $z \in B^n$. In view of (28) and (31), and the above relation, it is not difficult to deduce that $h(z,s) = h_r(z,s-t)$, for all $s \in (t,\infty) \setminus (E \cup \hat{E}_t)$ and $z \in B^n$. Hence, $h(\cdot,s) \in \mathcal{M}_g$ for $s \in (t,\infty) \setminus (E \cup \hat{E}_t)$, and thus $h(\cdot,s) \in \mathcal{M}_g$ for a.e. $s \in (t,\infty)$.

In view of the above arguments, we conclude that $h(\cdot,s) \in \mathcal{M}_g$ for a.e. $s \geq 0$, as desired. This completes the proof. □

Next, we obtain some extremal results for mappings in the reachable family $\tilde{\mathcal{R}}_{\log M}(\mathrm{id}_{B^n}, \mathcal{M}_g)$ (cf. [18] and [21] in the case $g(\zeta) = \frac{1+\zeta}{1-\zeta}$; compare [42, Theorem 2.52], in the case $n = 1$). First, we obtain the following result related to extreme points for the compact family $\tilde{\mathcal{R}}_{\log M}(\mathrm{id}_{B^n}, \mathcal{M}_g)$ (cf. [9, Theorem 3.2]).

Theorem 14. *Let $g : U \to \mathbb{C}$ be a univalent function such that $g(0) = 1$ and $\Re g(\zeta) > 0$, $|\zeta| < 1$. Let $M > 1$ and let $f \in \mathrm{ex}\, \tilde{\mathcal{R}}_{\log M}(\mathrm{id}_{B^n}, \mathcal{M}_g)$. Then there exists a Loewner chain $f(z,t)$ such that $\{e^{-t} f(\cdot,t)\}_{t\geq 0}$ is a locally uniformly bounded family, $f = f(\cdot,0)$, and $e^{-t} f(\cdot,t) \in \mathrm{ex}\, \tilde{\mathcal{R}}_{\log M - t}(\mathrm{id}_{B^n}, \mathcal{M}_g)$ for $0 \leq t < \log M$.*

Proof. We shall use arguments similar to those in the proof of [9, Theorem 3.2]. Since $f \in \mathrm{ex}\, \tilde{\mathcal{R}}_{\log M}(\mathrm{id}_{B^n}, \mathcal{M}_g)$, it is clear that $f \in \tilde{\mathcal{R}}_{\log M}(\mathrm{id}_{B^n}, \mathcal{M}_g)$, and thus there exists a sequence $\{f_k\}_{k\in\mathbb{N}} \subset \tilde{\mathcal{R}}_{\log M}(\mathrm{id}_{B^n}, \mathcal{M}_g)$ such that $f_k \to f$ locally uniformly on B^n. Clearly, $f \in \tilde{\mathcal{R}}_{\log M}(\mathrm{id}_{B^n}, \mathcal{M})$, since $\tilde{\mathcal{R}}_{\log M}(\mathrm{id}_{B^n}, \mathcal{M}_g) \subseteq$

$\tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M})$ and $\tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M})$ is a compact subset of $H(B^n)$, by [21, Corollary 4.7]. Also, for each $k \in \mathbb{N}$, there exists a g-Loewner chain $f_k(z,t)$ such that $f_k = f_k(\cdot, 0)$, $f_k(\cdot, \log M) = M\,\mathrm{id}_{B^n}$ by Theorem 13. Let $v_k = v_k(z,s,t)$ be the transition mapping associated with $f_k(z,t)$. Then $\{f_k(\cdot,t)\}_{k\in\mathbb{N}}$ is a sequence of g-Loewner chains, and in view of [14, Lemma 2.8], there exists a subsequence $\{f_{k_p}(\cdot,t)\}_{p\in\mathbb{N}}$ such that $f_{k_p}(\cdot,t) \to f(\cdot,t)$ locally uniformly on B^n for each $t \geq 0$, where $f(z,t)$ is a Loewner chain and $\{e^{-t} f(\cdot,t)\}_{t\geq 0}$ is a locally uniformly bounded family in $H(B^n)$. Also, since $\{v_{k_p}(\cdot,s,t)\}$ is a sequence of univalent Schwarz mappings such that $Dv_{k_p}(0,s,t) = e^{s-t} I_n$, there exists a subsequence, again denoted by $\{v_{k_p}(\cdot,s,t)\}$, which converges locally uniformly on B^n to a univalent Schwarz mapping $v(\cdot,s,t)$ such that $Dv(0,s,t) = e^{s-t} I_n$. Taking limits through this subsequence, it is easily seen that $f_{k_p}(v_{k_p}(\cdot,s,t),t) \to f(v(\cdot,s,t),t)$ locally uniformly on B^n. Also, since $f_{k_p}(\cdot,s) \to f(\cdot,s)$ locally uniformly on B^n, and $f_{k_p}(z,s) = f_{k_p}(v_{k_p}(z,s,t),t)$, it is clear that $f(z,s) = f(v(z,s,t),t)$ for $z \in B^n$ and $t \geq s \geq 0$. Hence $v_{s,t}(z) = v(z,s,t)$ is the transition mapping associated with $f(z,t)$. Let $v(z,t) = v_{0,t}(z)$ for $z \in B^n$ and $t \geq 0$.

Next, fix $t \in [0, \log M)$. First, we prove that $e^{-t} f(\cdot,t) \in \overline{\tilde{\mathscr{R}}_{\log M - t}(\mathrm{id}_{B^n}, \mathscr{M}_g)}$. For this aim, it suffices to prove that $e^{-t} f_k(\cdot,t) \in \tilde{\mathscr{R}}_{\log M - t}(\mathrm{id}_{B^n}, \mathscr{M}_g)$ for $k \in \mathbb{N}$. Indeed, fix $k \in \mathbb{N}$, and let $L_k(z,s) = e^{-t} f_k(z, t+s)$ for $z \in B^n$ and $s \geq 0$. As in the proof of [18, Theorem 2.1], we deduce that $L_k(z,s)$ is a Loewner chain such that $\{e^{-s} L_k(\cdot,s)\}_{s\geq 0}$ is a locally uniformly bounded family in $H(B^n)$ and $L_k(\cdot,0) = e^{-t} f_k(\cdot,t)$, $L_k(\cdot, \log M - t) = e^{\log M - t}\mathrm{id}_{B^n}$. Then there exists a generating vector field $h_{L,k} = h_{L,k}(z,s)$ such that

$$\frac{\partial L_k}{\partial s}(z,s) = DL_k(z,s) h_{L,k}(z,s), \qquad \text{a.e.} \quad s \geq 0, \quad \forall\, z \in B^n. \tag{32}$$

Hence, it suffices to prove that $h_{L,k}(\cdot,s) \in \mathscr{M}_g$, a.e. $s \geq 0$. Indeed, since $f_k(z,s)$ is a g-Loewner chain, there exists a generating vector field $h_{f,k} = h_{f,k}(z,s)$ such that $h_{f,k}(\cdot,s) \in \mathscr{M}_g$ for a.e. $s \geq 0$, and

$$\frac{\partial f_k}{\partial s}(z,s) = Df_k(z,s) h_{f,k}(z,s), \qquad \text{a.e.} \quad s \geq 0, \quad \forall\, z \in B^n. \tag{33}$$

On the other hand, in view of the relation (32), we obtain that

$$\frac{\partial f_k}{\partial s}(z,t+s) = Df_k(z,t+s) h_{L,k}(z,s), \qquad \text{a.e.} \quad s \geq 0, \quad \forall\, z \in B^n.$$

Hence, the above relation and (33) yield that $h_{L,k}(z,s) = h_{f,k}(z,t+s)$, for a.e. $s \geq 0$ and for all $z \in B^n$, and thus $h_{L,k}(\cdot,s) \in \mathscr{M}_g$, a.e. $s \geq 0$, as desired.

In view of the above arguments and the fact that $f_{k_p}(\cdot,t) \to f(\cdot,t)$ locally uniformly on B^n, we deduce that $e^{-t} f(\cdot,t) \in \overline{\tilde{\mathscr{R}}_{\log M - t}(\mathrm{id}_{B^n}, \mathscr{M}_g)}$.

Next, we prove that $e^{-t}f(\cdot,t) \in \mathrm{ex}\,\overline{\tilde{\mathcal{R}}_{\log M-t}(\mathrm{id}_{B^n},\mathcal{M}_g)}$. To this end, we suppose that

$$e^{-t}f(z,t) = \lambda r(z) + (1-\lambda)q(z), \quad z \in B^n,$$

where $\lambda \in (0,1)$ and $r, q \in \overline{\tilde{\mathcal{R}}_{\log M-t}(\mathrm{id}_{B^n},\mathcal{M}_g)}$. Then

$$f(z) = f(v(z,t),t) = \lambda e^t r(v(z,t)) + (1-\lambda)e^t q(v(z,t)), \quad z \in B^n.$$

In view of Lemma 4, we deduce that

$$e^t r(v(\cdot,t)), e^t q(v(\cdot,t)) \in \overline{\tilde{\mathcal{R}}_{\log M}(\mathrm{id}_{B^n},\mathcal{M}_g)}.$$

Indeed, since $r \in \overline{\tilde{\mathcal{R}}_{\log M-t}(\mathrm{id}_{B^n},\mathcal{M}_g)}$, there exists a sequence

$$\{r_k\}_{k\in\mathbb{N}} \subset \tilde{\mathcal{R}}_{\log M-t}(\mathrm{id}_{B^n},\mathcal{M}_g)$$

such that $r_k \to r$ locally uniformly on B^n. Now, $e^t r_k(v_k(\cdot,t)) \in \tilde{\mathcal{R}}_{\log M}(\mathrm{id}_{B^n},\mathcal{M}_g)$ for $k \in \mathbb{N}$, by Lemma 4, where $v_k(\cdot,t) = v_k(\cdot,0,t)$ is defined in the first part of the proof. As in the first part of the proof, there exists a subsequence $\{v_{k_p}(\cdot,t)\}$ of $\{v_k(\cdot,t)\}$ such that $v_{k_p}(\cdot,t) \to v(\cdot,t)$ locally uniformly on B^n. Then $e^t r_{k_p}(v_{k_p}(\cdot,t)) \to e^t r(v(\cdot,t))$, and the conclusion follows, as desired. A similar argument as above applies for $e^t q(v(\cdot,t))$. On the other hand, since $f \in \mathrm{ex}\,\overline{\tilde{\mathcal{R}}_{\log M}(\mathrm{id}_{B^n},\mathcal{M}_g)}$, we must have $e^t r(v(\cdot,t)) \equiv e^t q(v(\cdot,t))$. Finally, applying the identity theorem for holomorphic mappings, we deduce that $r \equiv q$. Hence $e^{-t}f(\cdot,t) \in \mathrm{ex}\,\overline{\tilde{\mathcal{R}}_{\log M-t}(\mathrm{id}_{B^n},\mathcal{M}_g)}$, as desired. This completes the proof. □

Next, we obtain the following result related to support points for the compact family $\overline{\tilde{\mathcal{R}}_{\log M}(\mathrm{id}_{B^n},\mathcal{M}_g)}$ (cf. [21, Theorem 4.9] for $A = I_n$, and [9, Theorem 4.2]).

Theorem 15. *Let $g : U \to \mathbb{C}$ be a univalent function such that $g(0) = 1$ and $\Re g(\zeta) > 0$, $|\zeta| < 1$. Also, let $M > 1$ and let $f \in \mathrm{supp}\,\overline{\tilde{\mathcal{R}}_{\log M}(\mathrm{id}_{B^n},\mathcal{M}_g)}$. Then there exist a Loewner chain $f(z,t)$ such that $\{e^{-t}f(\cdot,t)\}_{t\geq 0}$ is a locally uniformly bounded family, $f = f(\cdot,0)$, $f(\cdot,\log M) = M\,\mathrm{id}_{B^n}$ and a constant $\varepsilon \in (0,\log M)$ such that $e^{-t}f(\cdot,t) \in \mathrm{supp}\,\overline{\tilde{\mathcal{R}}_{\log M-t}(\mathrm{id}_{B^n},\mathcal{M}_g)}$ for $0 \leq t < \varepsilon$.*

Proof. We shall use arguments similar to those in the proofs of [21, Theorem 4.9] and [9, Theorem 4.2]. Let $f(z,t)$ and $v_t(z)$ be as in Theorem 14. Since $f \in \mathrm{supp}\,\overline{\tilde{\mathcal{R}}_{\log M}(\mathrm{id}_{B^n},\mathcal{M}_g)}$, there exists a continuous linear functional L on $H(B^n)$ such that $\Re L$ is nonconstant on $\overline{\tilde{\mathcal{R}}_{\log M}(\mathrm{id}_{B^n},\mathcal{M}_g)}$ and

$$\Re L(f) = \max_{q\in\overline{\tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n},\mathscr{M}_g)}} \Re L(q). \tag{34}$$

Now fix $t \in [0, \log M)$. Let $L_t : H(B^n) \to \mathbb{C}$ be given by

$$L_t(r) = L(e^t r \circ v_t), \quad r \in H(B^n).$$

It is clear that L_t is a continuous linear functional on $H(B^n)$ and $L_t(e^{-t} f(\cdot, t)) = L(f(v(\cdot, t), t)) = L(f)$. In view of the above arguments, we deduce that

$$\Re L_t(e^{-t} f(\cdot, t)) = \Re L(f) \geq \Re L(e^t r \circ v_t) = \Re L_t(r),$$

for all $r \in \overline{\tilde{\mathscr{R}}_{\log M-t}(\mathrm{id}_{B^n}, \mathscr{M}_g)}$, i.e.

$$\Re L_t(e^{-t} f(\cdot, t)) = \max_{r\in\overline{\tilde{\mathscr{R}}_{\log M-t}(\mathrm{id}_{B^n},\mathscr{M}_g)}} \Re L_t(r).$$

Here we have used the fact that

$$e^t r \circ v_t \in \overline{\tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)} \text{ for } r \in \overline{\tilde{\mathscr{R}}_{\log M-t}(\mathrm{id}_{B^n}, \mathscr{M}_g)}$$

by Lemma 4 and the proof of Theorem 14 (cf. [9]).

Finally, we will show that there exists $\varepsilon \in (0, \log M)$ such that $\Re L_t$ is nonconstant on $\overline{\tilde{\mathscr{R}}_{\log M-t}(\mathrm{id}_{B^n}, \mathscr{M}_g)}$ for $0 \leq t < \varepsilon$. There exists $F \in \overline{\tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)}$ such that $\Re L(F) < \Re L(f)$. Then there exists a Loewner chain $F(z, t)$ such that $F(\cdot, 0) = F(\cdot)$ and $e^{-t} F(\cdot, t) \in \overline{\tilde{\mathscr{R}}_{\log M-t}(\mathrm{id}_{B^n}, \mathscr{M}_g)}$. Then

$$\Re L_t(e^{-t} F(\cdot, t)) = \Re L(F(v_t, t))$$

for $t \in [0, \log M)$. Since $L(F(v_t, t)) \to L(F)$ as $t \to 0+$, there exists $\varepsilon \in (0, \log M)$ such that

$$\Re L_t(e^{-t} F(\cdot, t)) < \Re L(f) = \Re L_t(e^{-t} f(\cdot, t)) \quad 0 \leq t < \varepsilon.$$

Hence $\Re L_t|_{\overline{\tilde{\mathscr{R}}_{\log M-t}(\mathrm{id}_{B^n},\mathscr{M}_g)}}$ is nonconstant, as desired. This completes the proof. ☐

Example 4. Let $g : U \to \mathbb{C}$ be a convex function such that $g(0) = 1$, $\Re g(\zeta) > 0$, and $g(\bar{\zeta}) = \overline{g(\zeta)}$, $|\zeta| < 1$. Assume that g satisfies the conditions

$$\min_{|\zeta|=r} \Re g(\zeta) = g(r), \quad \max_{|\zeta|=r} \Re g(\zeta) = g(-r),$$

for all $r \in (0, 1)$. Let $b \in S_g^*$ be defined by $b(0) = 0$, $b'(0) = 1$ and

$$z_1 b'(z_1)/b(z_1) = 1/g(z_1), \quad z_1 \in U.$$

Let $f_1(z_1) = Mb^{-1}(M^{-1}b(z_1))$ for $z \in U$. Then $f_1 \in \tilde{\mathscr{R}}_{\log M}(\mathrm{id}_U, \mathscr{M}_g)$ by Example 2. Let $n \geq 2$ and let $f : B^n \to \mathbb{C}^n$ be given by

$$f(z) = (f_1(z_1), f_2(z')), \quad z = (z_1, z') \in B^n,$$

where $f_2 \in \overline{\tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^{n-1}}, \mathscr{M}_g)}$. As in Example 3, $f \in \overline{\tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)}$. For fixed $r \in (0, 1)$, let $z_0 = (r, 0')$ and

$$L(g) = g_1(z_0), \quad \text{for } g \in H(B^n).$$

Then L is a continuous linear functional on $H(B^n)$. For $g \in \overline{\tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)}$, we have

$$\Re L(g) \leq |g_1(z_0)| \leq f_1(r) = \Re L(f)$$

by Theorem 12 . We also have $\mathrm{id}_{B^n} \in \overline{\tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)}$ and $\Re L(\mathrm{id}_{B^n}) < f_1(r) = \Re L(f)$. Thus $f \in \mathrm{supp}\,\overline{\tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)}$.

Taking into account Example 2, we obtain an extremal result in the family $\overline{\tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)}$ involving mappings in $S_g^*(B^n)$ (cf. [21, Theorem 4.11] for $g(\zeta) = \frac{1+\zeta}{1-\zeta}$, and [9, Corollary 4.4]; compare [42, Theorem 2.65] for $n = 1$).

Theorem 16. *Let $g : U \to \mathbb{C}$ be a univalent function such that $g(0) = 1$ and $\Re g(\zeta) > 0$, $|\zeta| < 1$. Let $\lambda : \overline{S_g^0(B^n)} \to \mathbb{R}$ be a continuous real-valued functional. Assume that $F \in S_g^*(B^n)$ provides the maximum for λ over the set $\overline{S_g^0(B^n)}$. Then for each $M > 1$, the mapping $F^M \in \tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)$ given by (23) provides the maximum on $\overline{\tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)}$ for the associated functional $\lambda^M : \overline{\tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)} \to \mathbb{R}$, given by*

$$\lambda^M(r) = \lambda(MF(r(\cdot)/M)), \quad r \in \overline{\tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)}.$$

In addition, $\lambda^M(F^M) = \lambda(F)$.

Proof. Since $F \in S_g^*(B^n)$, we deduce that $F^M \in \tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)$ by Example 2. On the other hand, if $r \in \overline{\tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)}$, then the mapping $MF(r(\cdot)/M)$ belongs to $\overline{S_g^0(B^n)}$, by an argument similar to that in the proof of [21, Theorem 4.11]. Then

$$\lambda^M(r) = \lambda(MF(r(\cdot)/M)) \leq \lambda(F) = \lambda^M(F^M), \quad r \in \overline{\tilde{\mathscr{R}}_{\log M}(\mathrm{id}_{B^n}, \mathscr{M}_g)},$$

as desired. This completes the proof. □

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Different Durrmeyer Variants of Baskakov Operators

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Abstract The present article deals with the different Durrmeyer type modifications of the well-known Baskakov. These operators came into existence almost 28 years ago when in the year 1985 the Baskakov Durrmeyer operators were introduced. After that several approximation properties of such operators were studied extensively. The present article is an attempt to present some of the results and the approximation properties of the different Durrmeyer type modifications of the classical Baskakov operators. We also give here the alternate form of some of the operators in terms of hypergeometric functions. In the last section, we present some results for mixed operators related to convergence.

Keywords Baskakov Durrmeyer operators • Simultaneous approximation • Linear combinations • Asymptotic formula • Inverse theorem • Saturation theorem • Steklov mean • Hypergeometric functions • q Baskakov Durrmeyer operators.

1 Introduction

Baskakov [6] in the year 1957 introduced a new type of operators in the general way:

Let the functions $\phi_1(x), \phi_2(x), \cdots$ possess the following properties on an interval $[0, R]$, $R > 0$:

- $\phi_n(x)$ is analytic on the interval $[0, R]$ including at the end points,
- $\phi_n(0) = 1$,
- ϕ_n is completely monotone, i.e. $(-1)^k \phi_n^{(k)}(x) \geq 0, k = 0, 1, 2, \cdots$ and $x \in [0, R]$,

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- There exists a positive integer $M(n)$ not depending on k such that

$$-\phi_n^{(k)}(x) = n\phi_{m(n)}^{k-1}(x)[1 + \alpha_{k,n}(x)], k = 1, 2, \cdots,$$

where for n sufficiently large $\alpha_{k,n}$ converges uniformly to zero.
- $$\lim_{n\to\infty} \frac{n}{m(n)} = 1.$$

From above it is observed that $-\phi_n^{(k)}(x) = n\phi_{n+c}^{k-1}(x), k = 1, 2, \cdots$ and c is a positive integer (see [8]). Then the sequence V_n of operators on $C_B[0,\infty)$ (the space of all bounded continuous functions on $[0,\infty)$) into itself is defined as

$$G_n(f, x) = \sum_{k=0}^{\infty} \frac{(-1)^k \phi_n^{(k)}(x) x^k}{k!} f(k/n), n \in \mathbb{N}. \tag{1}$$

If $c = 1$ and $\phi_n(x) = (1+x)^{-n}$, one gets from (1) the Baskakov (Lupas) operators as

$$B_n(f, x) = V_n(f, x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f(k/n) \tag{2}$$

If $c = 0$ and $\phi_n(x) = e^{-nx}$, one gets from (1) the Szász-Mirakyan operators as

$$S_n(f, x) = \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} f(k/n) \tag{3}$$

These operators are linear positive exponential type operators. These operators are discretely defined. Lot of work has been done on Baskakov operators in local and global approximation. In the year 1978 Becker [7] established the equivalence theorem for the Baskakov operators in polynomial weight spaces. It is observed that the operators V_n are not able to deal with integrable functions, in this direction the Kanorovich type integral modifications of these operators were discussed by several researchers. In the year 1985 Durrmeyer type modifications of these operators came into existence and after that several modifications were introduced and studied. In the present article we discuss different kinds of the Durrmeyer type modifications of the Baskakov operators, which include Baskakov Durrmeyer operators, Baskakov Beta operators, Discretely defined Baskakov Durrmeyer operators at zero and Baskakov-Szász type operators.

2 Baskakov Durrmeyer Operators

In the year 1985 the Durrmeyer type integral modification of these operators was first introduced in [29] so as to approximate Lebesgue integrable functions on the interval $[0,\infty)$ as

$$V_n(f,x) := (n-1)\sum_{v=0}^{\infty} b_{n,v}(x)\int_0^{\infty} b_{n,v}(t)f(t)dt \tag{4}$$

where

$$b_{n,v}(x) = \binom{n+v-1}{v}\frac{x^v}{(1+x)^{n+v}}.$$

The authors Sahai-Prasad in [29] termed the operators as modified Lupas operators. Sahai-Prasad [29] estimated an asymptotic formula and an error estimation for the operators (4) in simultaneous approximation (approximation of derivative of functions by the corresponding order derivatives of the operators).

Theorem 1 ([29]). *If f is integrable in $[0,\infty)$, admits its $(r+1)$ and $(r+2)$-th derivatives, which are bounded at a point $x \in [0,\infty)$ and $f^{(r)}(x) = O(x^{\alpha})$ (α is a positive integer ≥ 2) as $x \to \infty$, then*

$$\lim_{n\to\infty} n[V_n^{(r)}(f,x) - f^{(r)}(x)] = (r+1)(1+2x)f^{(r+1)}(x) + x(1+x)f^{(r+2)}(x).$$

Theorem 2 ([29]). *If $f \in C^{(r+1)}[0,a]$ and $\omega(f^{(r+1)},.)$ be the modulus of continuity of $f^{(r+1)}$, then for $r = 0,1,2,3,\cdots$*

$$||V_n^{(r)}(f,x) - f^{(r)}(x)||_{C[0,a]} \leq \frac{(r+1)(1+2a)}{n-r-2}||f^{(r+1)}||_{C[0,a]} + C(n,r)\left(\sqrt{\lambda}+\frac{\lambda}{2}\right)\omega(f^{(r+1)},C(n,r)),$$

where $\lambda = 2(n-1)a(1+a) + (r+1)(r+2)(1+2a)^2, C(n,r) = 1/(n-r-2)(n-r-3)$ and $||.||$ denotes the sup-norm on $[0,a]$.

Six years later Sinha–Agrawal–Gupta [31] observed that the above results obtained in [29] have some mistakes and gaps. They improved the results of [29] and termed the operators V_n as modified Baskakov operators. Actually the operators (4) are Durrmeyer type modification of Baskakov operators. Sinha–Agrawal–Gupta [31] considered the class $Ł$ of Lebesgue measurable functions f on the positive real axis as

$$Ł = \left\{ f : \int_0^\infty \frac{|f(t)|}{(1+t)^n} dt < \infty \text{ for some positive integer } n \right\}.$$

The class Ł is naturally bigger than the class of Lebesgue integrable functions on the positive real axis. The correct and improved versions of the above Theorems 1, 2 are given as follows:

Theorem 3 ([31]). *Let $f \in Ł$ be bounded on every finite subinterval of $[0, \infty)$ admitting a derivative of order $(r+2)$ at a fixed point $x \in (0, \infty)$. Let $f(t) = O(t^\alpha)$ as $t \to \infty$ for some $\alpha > 0$, then we have*

$$\lim_{n\to\infty} n[V_n^{(r)}(f, x) - f^{(r)}(x)] = r(1+r) f^{(r)}(x) + (r+1)(1+2x) f^{(r+1)}(x) + x(1+x) f^{(r+2)}(x).$$

Theorem 4 ([31]). *Let $f \in Ł$ be bounded on every finite subinterval of $[0, \infty)$ and $f(t) = O(t^\alpha)$ as $t \to \infty$ for some $\alpha > 0$. If $f^{(r+1)}$ exists and is continuous on $< a, b > \subset (0, \infty)$, where $< a, b >$ denotes an open interval containing the closed interval $[a, b]$, then for n sufficiently large*

$$||V_n^{(r)}(f, .) - f^{(r)}|| \leq C_1 \left(||f^{(r)}|| + ||f^{(r+1)}||\right) n^{-1} + C_2 n^{-1/2} \omega(f^{(r+1)}, n^{-1/2}) + O(n^{-s}),$$

for any $s > 0$, where the constants C_1 and C_2 are independent of f and n, and $\omega(f, \delta)$ is the modulus of continuity of f on $< a, b >$ and norm-$||.||$ denotes the sup-norm on $[a, b]$.

In order to make the convergence faster, Kasana–Agrawal–Gupta [27] estimated the direct, inverse and saturation theorems for the linear combinations due to May [28], which are defined as

$$V_n(f, k, x) = \sum_{j=0}^{k} C(j, k) V_{d_j n}(f, x),$$

where

$$C(j, k) = \prod_{\substack{i=0 \\ i \neq j}}^{k} \frac{d_j}{d_j - d_i}, k \neq 0; C(0, 0) = 1$$

and $d_0, d_1, \ldots.., d_k$ are arbitrary but fixed distinct positive integers. We present here the results on the Baskakov–Durrmeyer operators V_n defined on a class $C_\gamma[0, \infty)$ as

$$C_\gamma[0, \infty) \equiv \{f \in C[0, \infty) : |f(t)| \leq M t^\gamma, \gamma > 0, M > 0\}.$$

The norm-$||.||_\gamma$ on $C_\gamma[0,\infty)$ is defined as $||f||_\gamma = \sup_{t\in(0,\infty)} |f(t)|t^{-\gamma}$. The following theorem is an asymptotic formula in ordinary approximation for the linear combinations of Baskakov–Durrmeyer operators:

Theorem 5 ([27]). *Let $f \in C_\gamma[0,\infty)$ for some $\gamma > 0$. If $f^{(2k+2)}$ exists at a point $x \in (0,\infty)$, then*

$$\lim_{n\to\infty} n^{k+1}\left[V_n(f,k,x) - f(x)\right] = \sum_{j=k+1}^{2k+2} \frac{f^{(j)}(x)}{j!} Q(j,k,x) \tag{5}$$

where $Q(j,k,x)$ are certain polynomials in x of degree j. Moreover

$$Q(2k+2,k,x) = \frac{(-1)^k\{x(1+x)\}^{k+1}}{\prod_{j=0}^k d_j} \frac{(2k+2)!}{(k+1)!}$$

and

$$Q(2k+1,k,x) = \frac{(-1)^k(1+2x)\{x(1+x)\}^k}{\prod_{j=0}^k d_j} \frac{(2k+2)!}{2(k!)}.$$

Further, if $f^{(2k+1)}$ exists and is absolutely continuous over $[a,b]$ and $f^{(2k+2)} \in L_\infty[a,b]$, then for any $[c,d] \subset (a,b)$ there holds

$$||V_n(f,k,.) - f|| \le M n^{-(k+1)} \left\{||f||_\gamma + ||f^{(2k+2)}||_{L_\infty[a,b]}\right\} \tag{6}$$

The next following theorem is an error estimation in ordinary approximation for the linear combinations of Baskakov–Durrmeyer operators:

Theorem 6 ([27]). *Let $f \in C_\gamma[0,\infty)$ and $0 < a_1 < a_2 < b_2 < b_1 < \infty$. Then for n sufficiently large, there exists a constant M_k such that*

$$||V_n(f,k,.) - f||_{C[a_2,b_2]} \le M_k \left\{\omega_{2k+2}(f,n^{-1/2},a_1,b_1) + n^{-(k+1)}||f||_\gamma\right\}.$$

Following theorem is an inverse theorem in ordinary approximation for the linear combinations of Baskakov–Durrmeyer operators:

Theorem 7 ([27]). *Let $f \in C_\gamma[0,\infty)$ and $0 < \alpha < 2$. Then in the following statements $(i) \Rightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv)$ hold:*

(i) $||V_n(f,k,.) - f||_{C[a_1,b_1]} = O(n^{-(\alpha(k+1)/2)})$;
(ii) $f \in Liz(\alpha,k+1,a_2,b_2)$;
(iii) *(a) For $m < \alpha(k+1) < m+1, m = 0,1,2,\ldots\ldots.2k+1$, $f^{(m)}$ exists and belong to the class $Lip(\alpha(k+1)-m,a_2,b_2)$;*
(b) For $\alpha(k+1) = m+1, m = 0,1,2,\ldots\ldots.2k$, $f^{(m)}$ exists and belong to the class $Lip^*(1,a_2,b_2)$;
(iv) $||V_n(f,k,.) - f||_{C[a_3,b_3]} = O(n^{-(\alpha(k+1)/2)})$,

where $Liz(\alpha, k, a, b)$ *denotes the generalized Zygmund class of functions for which* $\omega_{2k}(f, h, a, b) \leq Mh^{\alpha k}$, *when* $k = 1 Liz(\alpha, 1)$ *reduces to the Zygmund class* $Lip^*\alpha$.

We present below a saturation theorem in ordinary approximation for the linear combinations of Baskakov–Durrmeyer operators:

Theorem 8 ([27]). *Let* $f \in C_\gamma[0, \infty)$ *and* $0 < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < \infty$. *Then in the following statements, the implications* $(i) \Rightarrow (ii) \Rightarrow (iii)$ *and* $(iv) \Rightarrow (v) \Rightarrow (vi)$ *hold true:*

(i) $||V_n(f, k, .) - f)||_{C[a_1,b_1]} = O(n^{-(k+1)})$;
(ii) $f^{(2k+1)} \in A.C.[a_2, b_2]$ *and* $f^{(2k+2)} \in L_\infty[a_2, b_2]$;
(iii) $||V_n(f, k, .) - f||_{C[a_3,b_3]} = O(n^{-(k+1))})$;
(iv) $||V_n(f, k, .) - f||_{C[a_1,b_1]} = o(n^{-(k+1)})$;
(v) $f \in C^{2k+2}[a_2, b_2]$ *and* $\sum_{j=k+1}^{2k+2} Q(j, k, x) f^{(j)}(x) = 0, x \in [a_2, b_2]$ *where the polynomials* $Q(j, k, x)$ *are defined in Theorem 5;*
(vi) $||V_n(f, k, .) - f||_{C[a_3,b_3]} = o(n^{-(k+1)})$.

Agrawal–Gupta–Sahai [3] also studied simultaneous approximation for the linear combinations.

The following result is an asymptotic formula in simultaneous approximation for the linear combinations.

Theorem 9 ([3]). *Let* f *be integrable on* $[0, \infty)$ *admitting* $(2k+r+2)$*-th derivative at a point* $x \in [0, \infty)$ *with* $f^{(r)}(x) = O(x^\alpha)$, *where* α *is a positive integer not less than* $2k + 2$, *as* $x \to \infty$, *then*

$$\lim_{n\to\infty} n^{k+1}[V_n^{(r)}(f, k, x) - f^{(r)}(x)] = \sum_{i=1}^{2k+2} Q(i, k, r, x) f^{(i+r)}(x)$$

and

$$\lim_{n\to\infty} n^{k+1}[V_n^{(r)}(f, k + 1, x) - f^{(r)}(x)] = 0,$$

where $Q(i, k, r, x)$ *are certain polynomials in* x *of degree at most* i. *Further if* $f^{(2k+r+2)}$ *exists and is continuous on* $< a, b >$, *then the above limits hold uniformly on* $[a, b]$. *Here* $< a, b > \subset [0, \infty)$ *denotes an open interval containing the closed interval* $[a, b]$.

The next following result is an error estimation in simultaneous approximation for the linear combinations.

Theorem 10 ([3]). *Let* $1 \leq p \leq 2k + 2$ *and* f *be integrable on* $[0, \infty)$. *If* $f^{(p+r)}$ *exists and is continuous on* $< a, b >$ *having the modulus of continuity* $\omega(f^{(p+r)}, \delta)$ *on* $< a, b >$ *and* $f^{(r)}(x) = O(x^\alpha)$, *where* α *is a positive integer* $\geq p$, *then for* n *sufficiently large, we have*

$$||V_n^{(r)}(f,k,.) - f^{(r)}|| \le \max\{C_1 n^{-p/2}\omega(f^{(p+r)}, n^{-1/2}), C_2 n^{-(k+1)}\}$$

where $C_1 = C_1(k, p, r)$ *and* $C_2 = C_2(k, p, r, f)$.

Sinha et al. [30] obtained some local error estimates in L_p-norm for the linear combinations of Baskakov–Durrmeyer operators $V_n(f, x)$ using the linear approximating technique of Steklov mean's. In the following three theorems $0 < a_1 < a_2 < b_2 < b_1 < \infty$ and $I_i = [a_i, b_i], i = 1, 2$.

Theorem 11 ([30]). *Let* $f \in L_p[0,\infty)$, $p > 1$. *If* f *has* $2k + 2$ *derivatives on* I_1 *with* $f^{(2k+1)} \in A.C.(I_1)$ *and* $f^{(2k+2)} \in L_p(I_1)$, *then for n sufficiently large*

$$||V_n(f,k,.) - f||_{L_p(I_2)} \le M_k n^{-(k+1)}\{||f^{(2k+2)}||_{L_p(I_1)} + ||f||_{L_p[0,\infty)}\},$$

where M_k *is a constant independent of* f *and* n.

Theorem 12 ([30]). *Let* $f \in L_1[0,\infty)$. *If* f *has* $2k + 1$ *derivatives on* I_1 *with* $f^{(2k)} \in A.C.(I_1)$ *and* $f^{(2k+1)} \in B.V.(I_1)$, *then for all n sufficiently large*

$$||V_n(f,k,.) - f||_{L_1(I_2)} \le M_k n^{-(k+1)}\Big\{||f^{(2k+1)}||_{B.V.(I_1)} + ||f^{(2k+1)}||_{L_1(I_2)} + ||f||_{L_1[0,\infty)}\Big\},$$

where M_k *is a constant independent of* f *and* n.

The following error estimation is in terms of higher order integral modulus of smooth, which can be obtained by using the above Theorems 11 and 12.

Theorem 13 ([30]). *Let* $f \in L_p[0,\infty)$, $p \ge 1$. *Then for n sufficiently large*

$$||V_n(f,k,.) - f||_{L_p(I_2)} \le M_k \left(\omega_{2k+2}(f, n^{-1/2}, p, I_1), n^{-(k+1)}||f||_{L_p[0,\infty)}\right),$$

where M_k *is a constant independent of* f *and* n.

Also in the year 1989 Heilmann and Muller [25] considered the general form (1) to define the Durrmeyer variant of the Baskakov operators as

$$H_n(f,x) = (n-c)\sum_{k=0}^{\infty}\frac{(-1)^k\phi_n^{(k)}(x)x^k}{k!}\int_0^\infty \frac{(-1)^k\phi_n^{(k)}(t)t^k}{k!}f(t)dt,$$

where the different cases of $\phi_n(x)$ are as given in (1). They obtained direct global result in terms of Ditzian–Totik modulus of smoothness

$$||(H_n f)^{(r)} - f^{(r)}||_p \le C\left(\omega_\varphi^2(f^{(r)}, n^{-1/2})_p + n^{-1}||f^{(r)}||_p\right),$$

where C is a constant independent of $n, \varphi(x) = \sqrt{x(1+cx)}$ and ω_φ^2 is the second order Ditzian–Totik modulus of smoothness.

3 Baskakov-Beta Operators

It was observed in [13] that by considering the weights of Beta basis functions in the integral modification of Baskakov operators, we have better approximation. These operators are defined as

$$L_n(f,x) = \sum_{k=0}^{\infty} b_{n,k}(x) \int_0^\infty v_{n,k}(t) f(t) dt, \tag{7}$$

where

$$b_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} = \frac{(n)_k}{k!} \frac{x^k}{(1+x)^{n+k}},$$

$$v_{n,k}(t) = \frac{1}{B(k+1,n)} \frac{t^k}{(1+t)^{n+k+1}} = \frac{n(n+1)_k}{k!} \frac{t^k}{(1+t)^{n+k+1}}$$

with $B(m,n) = \dfrac{(m-1)!(n-1)!}{(n+m-1)!}$. The Pochhammer symbol $(n)_k$ is defined as

$$(n)_k = n(n+1)(n+2)(n+3)\ldots.(n+k-1).$$

Motivated by the recent studies on certain Beta type operators by Ismail and Simeonov [26] in the hypergeometric form, recently Gupta–Yadav [22] proposed the Stancu variant of the Baskakov-Beta operators and represented the operators L_n in terms of hypergeometric function as

$$\begin{aligned} L_n(f,x) &= n \sum_{k=0}^{\infty} \frac{(n)_k}{k!} \frac{x^k}{(1+x)^{n+k}} \int_0^\infty \frac{(n+1)_k}{k!} \frac{t^k}{(1+t)^{n+k+1}} f(t) dt \\ &= n \int_0^\infty \frac{f(t)(1+x)}{[(1+x)(1+t)]^{n+1}} \sum_{k=0}^{\infty} \frac{(n)_k (n+1)_k}{(k!)^2} \frac{(xt)^k}{[(1+x)(1+t)]^k} dt. \end{aligned}$$

Using the hypergeometric series ${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \dfrac{(a)_k (b)_k}{(c)_k k!} x^k$, we have

$$L_n(f,x) = n \int_0^\infty \frac{f(t)(1+x)}{[(1+x)(1+t)]^{n+1}} \, {}_2F_1\left(n, n+1; 1; \frac{xt}{(1+x)(1+t)}\right) dt,$$

Now using ${}_2F_1(a,b;c;x) = {}_2F_1(b,a;c;x)$ and applying Pfaff–Kummer transformation

$$ {}_2F_1(a,b;c;x) = (1-x)^{-a}\, {}_2F_1\left(a, c-b; c; \frac{x}{x-1}\right) $$

we have

$$ L_n(f,x) = n\int_0^\infty f(t)\frac{1+x}{(1+x+t)^{n+1}}\, {}_2F_1\left(n+1, 1-n; 1; \frac{-xt}{1+x+t}\right) dt, $$

which is the alternate form of the operators L_n in terms of hypergeometric functions.

The following lemma is also represented in terms of hypergeometric function:

Lemma 1 ([22]). *For $n > 0$ and $r > -1$, we have*

$$ L_n(t^r, x) = \frac{\Gamma(n-r)\Gamma(r+1)}{\Gamma(n)}(1+x)^r\, {}_2F_1\left(1-n, -r; 1; \frac{x}{1+x}\right). $$

Proof. Taking $f(t) = t^r$, $t = (1+x)u$ and using Pfaff–Kummer transformation the right-hand side of (7) becomes

$$ n\int_0^\infty \frac{(1+x)^{r+2}u^r}{((1+x)(1+u))^{n+1}}(1+x)\sum_{k=0}^\infty \frac{(n+1)_k(1-n)_k}{(k!)^2}\frac{(-x(1+x)u)^k}{((1+x)(1+u))^k}du $$

$$ = n\sum_{k=0}^\infty \frac{(n+1)_k(1-n)_k}{(k!)^2}(-x)^k(1+x)^{r-n+1}\int_0^\infty \frac{u^{r+k}}{(1+u)^{n+k+1}}du $$

$$ = n\sum_{k=0}^\infty \frac{(n+1)_k(1-n)_k}{(k!)^2}(-x)^k(1+x)^{r-n+1}B(r+k+1, n-r) $$

$$ = n\sum_{k=0}^\infty \frac{(n+1)_k(1-n)_k}{(k!)^2}(-x)^k(1+x)^{r-n+1}\frac{\Gamma(r+k+1)\Gamma(n-r)}{\Gamma(n+k+1)}. $$

Using, $\Gamma(n+k+1) = \Gamma(n+1)(n+1)_k$, we have

$$ \begin{aligned} L_n(t^r, x) &= n\sum_{k=0}^\infty \frac{(n+1)_k(1-n)_k}{(k!)^2}(-x)^k(1+x)^{r-n+1}\frac{\Gamma(r+1)(r+1)_k\Gamma(n-r)}{\Gamma(n+1)(n+1)_k} \\ &= n(1+x)^{r-n+1}\frac{\Gamma(r+1)\Gamma(n-r)}{\Gamma(n+1)}\sum_{k=0}^\infty \frac{(r+1)_k(1-n)_k}{(k!)^2}(-x)^k \\ &= n(1+x)^{r-n+1}\frac{\Gamma(r+1)\Gamma(n-r)}{\Gamma(n+1)}\, {}_2F_1(1-n, r+1; 1; -x). \end{aligned} $$

Using ${}_2F_1(a,b;c;x)=(1-x)^{-a}\,{}_2F_1\left(a,c-b;c;\dfrac{x}{x-1}\right)$, we have

$$L_n(t^r,x)=\frac{\Gamma(n-r)\Gamma(r+1)}{\Gamma(n)}(1+x)^r\,{}_2F_1\left(1-n,-r;1;\frac{x}{1+x}\right).$$

For approximation properties of these operators, we also refer the readers to [14, 21]. In [15] Gupta, proposed the Bézier variant of the Baskakov-Beta operators and studied the rate of convergence for functions of bounded variation. Govil–Gupta [11] studied convergence in simultaneous approximation for the Bézier analogue of Baskakov-Beta operators, for ready reference we mention below the Baskakov-Beta-Bézier operators, which for $\alpha \geq 1$ are defined as

$$L_{n,\alpha}(f,x)=\sum_{k=0}^{\infty}Q_{n,k}^{(\alpha)}(x)\int_0^{\infty}b_{n,k}(t)f(t)dt,\quad n\in\mathbb{N},x\in[0,\infty)$$

where

$$Q_{n,k}^{(\alpha)}(x)=[J_{n,k}(x)]^{\alpha}-[J_{n,k+1}(x)]^{\alpha},\quad J_{n,k}(x)=\sum_{j=k}^{\infty}b_{n,j}(x).$$

As a special case when $\alpha = 1$, these operators reduce to the Baskakov-Beta operators (7).

4 Baskakov Durrmeyer Operators by Agrawal–Thamer

For $f\in C_\alpha[0,\infty)\equiv\{f\in C[0,\infty):|f(t)|\leq M(1+t)^{\alpha},\alpha>0,M>0\}$, Agrawal–Thamer [2] proposed a new type of Baskakov–Durrmeyer operator as

$$M_n(f,x)=(n-1)\sum_{k=1}^{\infty}b_{n,k}(x)\int_0^{\infty}b_{n,k}(t)f(t)\,dt+b_{n,0}(x)f(0),\tag{8}$$

where

$$b_{n,k}(x)=\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}.$$

The norm-$||.||_\alpha$ on $C_\alpha[0,\infty)$ is defined as $||f||_\alpha=\sup_{t\in[0,\infty)}|f(t)|(1+t)^{-\alpha}$. In [2], the authors have obtained point-wise convergence, asymptotic formula, and error estimation in simultaneous approximation.

Theorem 14 ([2]). *If* $r \in \mathbb{N}$, $f \in C_\alpha[0,\infty)$ *for some* $\alpha > 0$ *and* $f^{(r)}$ *exists at a point* $x \in (0,\infty)$*, then*

$$\lim_{n\to\infty} M_n^{(r)}(f,x) = f^{(r)}(x).$$

Further, if $f^{(r)}$ *exists and is continuous on* $(a-\eta, b+\eta) \subset (0,\infty), \eta > 0$*, then this limit holds uniformly in* $x \in [a,b]$.

Theorem 15 ([2]). *Let* $f \in C_\alpha[0,\infty)$ *for some* $\alpha > 0$. *If* $f^{(r+2)}$ *exists at a point* $x \in (0,\infty)$*, then*

$$\lim_{n\to\infty} n[M_n^{(r)}(f,x) - f^{(r)}(x)] = r(1+r)f^{(r)}(x) + [2(r+1)x+r]f^{(r+1)}(x) + x(1+x)f^{(r+2)}(x).$$

Further, if $f^{(r+2)}$ *exists and is continuous on* $(a-\eta, b+\eta) \subset (0,\infty), \eta > 0$*, then this limit holds uniformly in* $x \in [a,b]$.

Theorem 16 ([2]). *Let* $f \in C_\alpha[0,\infty)$ *for some* $\alpha > 0$ *and* $r \leq q \leq r+2$. *If* $f^{(q)}$ *exists and is continuous on* $(a-\eta, b+\eta) \subset (0,\infty), \eta > 0$*, then for n sufficiently large*

$$||M_n^{(r)}(f,.) - f^{(r)}|| \leq C_1 n^{-1}\left(\sum_{i=r}^{q} ||f^{(i)}||\right) + C_2 n^{-1/2}\omega(f^{(r+1)}, n^{-1/2}) + O(n^{-2}),$$

where C_1 *and* C_2 *are independent of* f *and* n*, and* $\omega(f,\delta)$ *is the modulus of continuity of* f *on* $(a-\eta, b+\eta >$ *and norm-*$||.||$ *denotes the sup-norm on* $[a,b]$.

For these operators Gupta [16] studied the rate of pointwise approximation for functions of bounded variation. To prove the rate of approximation we used some results of probability theory. He also introduced the Bézier variant of these operators.

5 Baskakov Durrmeyer Operators by Finta

For $f \in C[0,\infty)$, a new type of Baskakov–Durrmeyer operator studied by Finta in [10] is defined as

$$D_n(f,x) = \sum_{k=1}^{\infty} b_{n,k}(x)\int_0^\infty p_{n,k}(t)f(t)\,dt + b_{n,0}(x)f(0), \tag{9}$$

where

$$b_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} = \frac{(n)_k}{k!} \frac{x^k}{(1+x)^{n+k}},$$

$$p_{n,k}(x) = \frac{1}{B(k,n+1)} \frac{x^{k-1}}{(1+x)^{n+k+1}} = \frac{(n+1)_k}{(k-1)!} \frac{x^{k-1}}{(1+x)^{n+k+1}}.$$

The main advantage to consider the operators in this form is that they preserve constant and linear functions. Govil and Gupta [12] studied some approximation properties for the operators defined in (9) and estimated local results in terms of modulus of continuity. In terms of the hypergeometric form, we write the operators D_n as

$$\begin{aligned}
D_n(f,x) &= \sum_{k=1}^{\infty} \frac{(n)_k}{k!} \frac{x^k}{(1+x)^{n+k}} \int_0^{\infty} \frac{(n+1)_k}{(k-1)!} \frac{t^{k-1}}{(1+t)^{n+k+1}} f(t)dt + b_{n,0}(x) f(0) \\
&= \int_0^{\infty} \frac{f(t)x(1+x)}{[(1+x)(1+t)]^{n+2}} \sum_{k=1}^{\infty} \frac{(n)_k(n+1)_k}{(k-1)!k!} \frac{(xt)^{k-1}}{[(1+x)(1+t)]^{k-1}} dt \\
&\quad + b_{n,0}(x) f(0) \\
&= \int_0^{\infty} \frac{f(t)x(1+x)}{[(1+x)(1+t)]^{n+2}} \sum_{k=0}^{\infty} \frac{(n)_{k+1}(n+1)_{k+1}}{k!(k+1)!} \frac{(xt)^k}{[(1+x)(1+t)]^k} dt \\
&\quad + b_{n,0}(x) f(0) \\
&= n(n+1) \int_0^{\infty} \frac{f(t)x(1+x)}{[(1+x)(1+t)]^{n+2}} \sum_{k=0}^{\infty} \frac{(n+1)_k(n+2)_k}{k!(2)_k} \frac{(xt)^k}{[(1+x)(1+t)]^k} dt \\
&\quad + b_{n,0}(x) f(0),
\end{aligned}$$

where in the last equality we have used $(n)_{k+1} = n(n+1)_k$ and $(n+1)_{k+1} = (n+1)(n+2)_k$ and $(k+1)! = (2)_k$. Using the hypergeometric series ${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} x^k$, we have

$$\begin{aligned}
D_n(f,x) &= n(n+1) \int_0^{\infty} \frac{f(t)x(1+x)}{[(1+x)(1+t)]^{n+2}} \, {}_2F_1\left(n+1, n+2; 2; \frac{xt}{(1+x)(1+t)}\right) dt \\
&\quad + b_{n,0}(x) f(0),
\end{aligned}$$

Now using ${}_2F_1(a,b;c;x) = {}_2F_1(b,a;c;x)$ and applying Pfaff–Kummer transformation

$${}_2F_1(a,b;c;x) = (1-x)^{-a} \, {}_2F_1\left(a, c-b; c; \frac{x}{x-1}\right),$$

we have

$$D_n(f,x) = n(n+1)\int_0^\infty f(t)\frac{x(1+x)}{(1+x+t)^{n+2}}\ {}_2F_1\left(n+1,-n;2;\frac{-xt}{1+x+t}\right)dt$$
$$+\ b_{n,0}(x)f(0),$$

which is the alternate form of the operators L_n in terms of hypergeometric functions.

Lemma 2. *For* $n > 0$ *and* $r \geq 1$, *we have*

$$D_n(t^r,x) = \frac{\Gamma(n-r+1)\Gamma(r+1)x}{\Gamma(n)}\,{}_2F_1\left(n+1,1-r;2;-x\right).$$

Proof. Taking $f(t) = t^r$, we can write

$$\begin{aligned}
D_n(f,x) &= \sum_{k=1}^{\infty}\frac{(n)_k}{k!}\frac{x^k}{(1+x)^{n+k}}\int_0^\infty \frac{\Gamma(n+k+1)}{(k-1)!n!}\frac{t^{k+r-1}}{(1+t)^{n+k+1}}dt \\
&= \sum_{k=1}^{\infty}\frac{(n)_k}{k!}\frac{x^k}{(1+x)^{n+k}}\frac{\Gamma(n+k+1)}{(k-1)!n!}B(k+r,n-r+1) \\
&= \sum_{k=1}^{\infty}\frac{(n)_k}{k!}\frac{x^k}{(1+x)^{n+k}}\frac{\Gamma(n+k+1)}{(k-1)!n!}\frac{\Gamma(k+r)\Gamma(n-r+1)}{\Gamma(n+k+1)} \\
&= \sum_{k=1}^{\infty}\frac{(n)_k}{k!}\frac{x^k}{(1+x)^{n+k}}\frac{\Gamma(k+r)\Gamma(n-r+1)}{(k-1)!n!} \\
&= \frac{\Gamma(n-r+1)(1+x)^{-n}}{\Gamma(n+1)}\sum_{k=1}^{\infty}\frac{(n)_k}{k!}\frac{x^k}{(1+x)^k}\frac{\Gamma(k+r)}{(k-1)!} \\
&= n\frac{\Gamma(n-r+1)x(1+x)^{-(n+1)}}{\Gamma(n+1)}\sum_{k=0}^{\infty}\frac{(n+1)_k}{(2)_k}\frac{x^k}{(1+x)^k}\frac{\Gamma(k+r+1)}{k!} \\
&= n\frac{\Gamma(n-r+1)\Gamma(r+1)x(1+x)^{-(n+1)}}{\Gamma(n+1)}\sum_{k=0}^{\infty}\frac{(n+1)_k(r+1)_k}{k!(2)_k}\frac{x^k}{(1+x)^k} \\
&= n\frac{\Gamma(n-r+1)\Gamma(r+1)x(1+x)^{-(n+1)}}{\Gamma(n+1)}\ {}_2F_1\left(n+1,r+1;2;\frac{x}{1+x}\right).
\end{aligned}$$

Using ${}_2F_1(a,b;c;x) = (1-x)^{-a}\,{}_2F_1\left(a,c-b;c;\frac{x}{x-1}\right)$, we have

$$D_n(t^r,x) = \frac{\Gamma(n-r+1)\Gamma(r+1)x}{\Gamma(n)}\,{}_2F_1\left(n+1,1-r;2;-x\right).$$

Remark 1. We may remark from the above Lemma 2 that

$$D_n(1,x) = 1, D_n(t,x) = x, D_n(t^2,x) = \frac{(n+1)x^2 + 2x}{n-1}.$$

Several approximation properties of these operators were discussed in [10, 12, 23]. Recently Gupta–Verma–Agrawal [24] proposed the Stancu variant of the operators D_n and established some direct results.

6 q-Baskakov–Durrmeyer Operators

In the year 2010 Aral–Gupta [4] introduced q-Baskakov Durrmeyer operators. For every $n \in \mathbb{N}, q \in (0,\ 1)$, the q analogue of Baskakov operators is defined as

$$V_n^q(f(t), x) := [n-1]_q \sum_{k=0}^{\infty} b_{n,k}^q(x) \int_0^{\infty/A} b_{n,k}^q(t) f(t)\, d_q t, \tag{10}$$

where

$$b_{n,k}^q(x) := \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k^2}{2}} \frac{x^k}{(1+x)_q^{n+k}},$$

for $x \in [0,\ \infty)$ and for every real valued continuous and bounded function f on $[0,\ \infty)$. It can be observed that in case $q = 1$ the above operators reduce to the usual Baskakov Durrmeyer operators. Some direct results were established in [4] and [17]. For details we refer the readers to the recent book by Aral–Gupta–Agarwal [5], where notations and such results on q operators are collected.

Theorem 17 ([4]). *Let* $q \in (0,1)$ *and* $n \geq 4$. *We have*

$$|V_n^q(f,x) - f(x)| \leq C\omega_2\left(f, \frac{\delta_n(x)}{\sqrt{q^6[n-2]_q}}\right) + \omega\left(f, \frac{q^{-2}[2]_q x + q^{-1}}{[n-2]_q}\right),$$

for every $x \in [0,\infty)$ *and* $f \in C_B[0,\infty)$, *where* C *is a positive constant.*

Let $B_{x^2}[0,\infty)$ is the set of all functions defined on $[0,\infty)$ satisfying the growth condition

$$|f(x)| \leq M_f(1+x^2),$$

where M_f is a constant depending on f only. $C_{x^2}[0,\infty)$ denotes the subspace of all continuous functions belonging to $B_{x^2}[0,\infty)$. Also let $C_{x^2}^*[0,\infty)$ be the subspace of all functions $f \in C_{x^2}[0,\infty)$, for which $\lim_{x\to\infty} \frac{|f(x)|}{1+x^2}$ is finite. The norm $||.||_{x^2}$ on $C^*_{x^2}[0,\infty)$ is defined by

$$||f||_{x^2} = \sup_{x\in[0,\infty)} \frac{|f(x)|}{1+x^2}.$$

Theorem 18 ([4]). *Let* $f \in C_{x^2}[0,\infty)$, $q = q_n \in (0,1)$ *such that* $q_n \to 1$ *as* $n \to \infty$ *and* $\omega_{a+1}(f,\delta)$ *be its modulus of continuity on the finite interval* $[0, a+1] \subset [0,\infty)$, *where* $a > 0$. *Then for every* $n > 3$, *we have*

$$\|V_n^q(f) - f\|_{C[0,a]} \le \frac{K}{q^6[n-3]_q} + 2\omega_{a+1}\left(f, \sqrt{\frac{K}{q^6[n-3]_q}}\right),$$

where $K = 90M_f\left(1+a^2\right)\left(1+a+a^2\right)$.

Theorem 19 ([4]). *Let* $q = q_n$ *satisfies* $0 < q_n < 1$ *and let* $q_n \to 1$ *as* $n \to \infty$. *For each* $f \in C^*_{x^2}[0,\infty)$, *we have*

$$\lim_{n\to\infty} ||V_n^{q_n}(f) - f||_{x^2} = 0.$$

Theorem 20 ([4]). *Let* $q = q_n$ *satisfies* $0 < q_n < 1$ *and let* $q_n \to 1$ *as* $n \to \infty$. *For each* $f \in C_{x^2}[0,\infty)$ *and* $\alpha > 0$, *we have*

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{\left|V_n^{q_n}(f,x) - f(x)\right|}{(1+x^2)^{1+\alpha}} = 0.$$

Theorem 21 ([17]). *Let* $f \in C[0,\infty)$ *be a bounded function and* (q_n) *denote a sequence such that* $0 < q_n < 1$ *and* $q_n \to 1$ *as* $n \to \infty$. *Then we have for a point* $x \in (0,\infty)$

$$\lim_{n\to\infty} [n]_{q_n}\left(V_n^{q_n}(f,x) - f(x)\right) = (2x+1)\,D_{q_n}f(x) + x(1+x)D^2_{q_n}f(x).$$

7 Baskakov Szász Operators by Gupta–Srivastava

In the year 1993 Gupta–Srivastava [20] introduced the mixed Durrmeyer type operators by taking the weight functions of Szász basis function as

$$M_n(f,x) := n\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty} s_{n,k}(t)f(t)dt, \tag{11}$$

where

$$b_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}, s_{n,k}(t) = e^{-nt} \frac{(nt)^k}{k!}.$$

Gupta–Srivastava [20] estimated direct results in simultaneous approximation and established an asymptotic formula and estimation of error. In the year 2003 Gupta–Maheshwari [19] extended the studies on these operators and they proved an inverse theorem for the linear combinations of the operators M_n. Here we present alternate form of such operators and find the moments using hypergeometric representation.

We may rewrite these operators in the form of hypergeometric function as

$$\begin{aligned} M_n(f,x) &= n \sum_{k=0}^{\infty} b_{n,k}(x) \int_0^{\infty} s_{n,k}(t) f(t) dt \\ &= \frac{n}{\Gamma(n)(1+x)^n} \int_0^{\infty} e^{-nt} f(t) dt \sum_{k=1}^{\infty} \frac{\Gamma(n+k)}{(k!)^2} \left(\frac{nxt}{1+x} \right)^k . \end{aligned}$$

Using the identity $\Gamma(n+k) = \Gamma(n)(n)_k$ and $(1)_k = k!$, we have

$$\begin{aligned} M_n(f,x) &= \frac{n}{(1+x)^n} \int_0^{\infty} e^{-nt} f(t) dt \sum_{k=0}^{\infty} \frac{(n)_k}{(1)_k . k!} \left(\frac{nxt}{1+x} \right)^k \\ &= \frac{n}{(1+x)^n} \int_0^{\infty} e^{-nt} f(t) \ {}_1F_1 \left(n; 1; \frac{nxt}{1+x} \right) dt. \end{aligned}$$

To obtain the moments the authors in [20] used the standard techniques of recurrence relation. Here we give the following lemma in terms of hypergeometric representation, by which one can easily obtain the moments of higher order.

Lemma 3. *If* $e_r = t^r, r \geq 1$, *then we have*

$$M_n(e_r, x) = \frac{x\Gamma(r+1)}{n^{r-1}} \ {}_2F_1 \left(n+1, 1-r; 2; -x \right).$$

Proof. By definition (11), we have

$$\begin{aligned} M_n(e_r,x) &= n \sum_{k=0}^{\infty} b_{n,k}(x) \int_0^{\infty} s_{n,k}(t) t^r dt \\ &= \frac{n}{\Gamma(n)(1+x)^n} \sum_{k=0}^{\infty} \frac{\Gamma(n+k)}{k!} \left(\frac{x}{1+x} \right)^k \int_0^{\infty} e^{-nt} \frac{(nt)^k}{k!} t^r dt \end{aligned}$$

$$= \frac{n}{\Gamma(n)(1+x)^n} \sum_{k=0}^{\infty} \frac{\Gamma(n+k)}{k!} \left(\frac{x}{1+x}\right)^k \int_0^\infty e^{-u} \frac{u^k}{k!} \frac{u^r}{n^{r+1}} du$$

$$= \frac{1}{n^r \Gamma(n)(1+x)^n} \sum_{k=1}^{\infty} \frac{\Gamma(n+k)}{k!} \left(\frac{x}{1+x}\right)^k \frac{\Gamma(k+r+1)}{k!}$$

$$= \frac{x}{n^r \Gamma(n)(1+x)^{n+1}} \sum_{k=0}^{\infty} \frac{\Gamma(n+k+1)}{(k+1)!} \left(\frac{x}{1+x}\right)^k \frac{\Gamma(k+r+1)}{k!}$$

$$= \frac{x\Gamma(r+1)}{n^{r-1}(1+x)^{n+1}} \sum_{k=0}^{\infty} \frac{(n+1)_k (r+1)_k}{(2)_k} \left(\frac{x}{1+x}\right)^k \frac{1}{k!}$$

$$= \frac{x\Gamma(r+1)}{n^{r-1}(1+x)^{n+1}} \ {}_2F_1\left((n+1),(r+1);2;\frac{x}{1+x}\right).$$

Finally, using ${}_2F_1(a,b;c;x) = (1-x)^{-a} \ {}_2F_1\left(a, c-b; c; \frac{x}{x-1}\right)$, we get

$$M_n(e_r, x) = \frac{x\Gamma(r+1)}{n^{r-1}} \ {}_2F_1\left(n+1, 1-r; 2; -x\right).$$

8 Baskakov Szász Operators by Agrawal–Mohammad

In the year 2003, Agrawal and Mohammad [1] proposed another modification of the Baskakov operators with weights of Szász basis function. The operators considered in [1] are defined as

$$P_n(f,x) := n \sum_{k=1}^{\infty} b_{n,k}(x) \int_0^\infty s_{n,k-1}(t) f(t) dt + b_{n,0}(x) f(0) \tag{12}$$

where

$$b_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}, s_{n,k-1}(t) = e^{-nt} \frac{(nt)^{k-1}}{(k-1)!}$$

with $b_{n,0}(x) = (1+x)^{-n}$. These operators preserve constant and linear functions, which is the important property of these operators. Gupta–Gupta [18] estimated the rate of convergence of these operators for the iterative combinations in terms of higher order integral modulus of smoothness. They used the linear approximating method viz. Steklov mean to prove the main results.

We present here the alternate form of these operators in terms of hypergeometric function as

$$
\begin{aligned}
P_n(f,x) &= n\sum_{k=1}^{\infty} b_{n,k}(x)\int_0^{\infty} s_{n,k-1}(t)f(t)dt + (1+x)^{-n}f(0)\\
&= \frac{n}{\Gamma(n)(1+x)^n}\int_0^{\infty} e^{-nt}f(t)dt\sum_{k=1}^{\infty}\frac{\Gamma(n+k)}{k!}\left(\frac{x}{1+x}\right)^k\frac{(nt)^{k-1}}{(k-1)!}\\
&\quad + (1+x)^{-n}f(0)\\
&= \frac{n}{\Gamma(n)(1+x)^n}\int_0^{\infty} e^{-nt}f(t)dt\sum_{k=0}^{\infty}\frac{\Gamma(n+k+1)}{(k+1)!}\left(\frac{x}{1+x}\right)^{k+1}\frac{(nt)^{k}}{k!}\\
&\quad + (1+x)^{-n}f(0).
\end{aligned}
$$

Using the identity $\Gamma(n+k+1) = \Gamma(n+1)(n+1)_k$ and $(2)_k = (k+1)!$, we have

$$
\begin{aligned}
P_n(f,x) &= \frac{n^2x}{(1+x)^{n+1}}\int_0^{\infty} e^{-nt}f(t)dt\sum_{k=0}^{\infty}\frac{(n+1)_k}{(2)_k}\left(\frac{nxt}{1+x}\right)^k\frac{1}{k!}\\
&\quad + (1+x)^{-n}f(0)\\
&= \frac{n^2x}{(1+x)^{n+1}}\int_0^{\infty} e^{-nt}f(t)\ {}_1F_1\left(n+1;2;\frac{nxt}{1+x}\right)dt\\
&\quad + (1+x)^{-n}f(0).
\end{aligned}
$$

For these operators we establish below the moments estimation in terms of hypergeometric function, which was not done earlier.

Lemma 4. *If* $e_r = t^r, r \geq 1$, *then we have*

$$
P_n(e_r,x) = \frac{x\Gamma(r+1)}{n^{r-1}}\ {}_2F_1\left(n+1, 1-r; 2; -x\right).
$$

Proof. By definition (12), we have

$$
\begin{aligned}
P_n(e_r,x) &= n\sum_{k=1}^{\infty} b_{n,k}(x)\int_0^{\infty} s_{n,k-1}(t)t^r dt\\
&= \frac{n}{\Gamma(n)(1+x)^n}\sum_{k=1}^{\infty}\frac{\Gamma(n+k)}{k!}\left(\frac{x}{1+x}\right)^k\int_0^{\infty} e^{-nt}\frac{(nt)^{k-1}}{(k-1)!}t^r dt\\
&= \frac{n}{\Gamma(n)(1+x)^n}\sum_{k=1}^{\infty}\frac{\Gamma(n+k)}{k!}\left(\frac{x}{1+x}\right)^k\int_0^{\infty} e^{-u}\frac{u^{k-1}}{(k-1)!}\frac{u^r}{n^{r+1}}du
\end{aligned}
$$

$$= \frac{1}{n^r \Gamma(n)(1+x)^n} \sum_{k=1}^{\infty} \frac{\Gamma(n+k)}{k!} \left(\frac{x}{1+x}\right)^k \frac{\Gamma(k+r)}{(k-1)!}$$
$$= \frac{x}{n^r \Gamma(n)(1+x)^{n+1}} \sum_{k=0}^{\infty} \frac{\Gamma(n+k+1)}{(k+1)!} \left(\frac{x}{1+x}\right)^k \frac{\Gamma(k+r+1)}{k!}$$
$$= \frac{x\Gamma(r+1)}{n^{r-1}(1+x)^{n+1}} \sum_{k=0}^{\infty} \frac{(n+1)_k (r+1)_k}{(2)_k} \left(\frac{x}{1+x}\right)^k \frac{1}{k!}$$
$$= \frac{x\Gamma(r+1)}{n^{r-1}(1+x)^{n+1}} \ {}_2F_1\left((n+1),(r+1);2;\frac{x}{1+x}\right).$$

Finally, using ${}_2F_1(a,b;c;x) = (1-x)^{-a} \ {}_2F_1\left(a, c-b; c; \frac{x}{x-1}\right)$, we get

$$P_n(e_r, x) = \frac{x\Gamma(r+1)}{n^{r-1}} \ {}_2F_1\left(n+1, 1-r; 2; -x\right).$$

Remark 2. By simple computation from Lemma 4, we have

$$P_n(e_0, x) = 1, \ \ P_n(e_1, x) = x,$$
$$P_n(e_2, x) = x^2 + \frac{x(x+2)}{n}.$$

Remark 3. Let $\psi_x(t) = t - x, n \in N$ then from Remark 2, we have

$$P_n(\psi_x, x) = 0$$
$$P_n(\psi_x^2, x) = \frac{x(x+2)}{n}.$$

Further, for $r = 0, 1, 2, \ldots$, we have

$$P_n(\psi_x^r, x) = O\left(n^{-[(r+1)/2]}\right).$$

Applying Schwarz inequality, we have

$$P_n(|\psi_x^r|, x) \leq \sqrt{D_n(\psi_x^{2r}, x)} = O\left(n^{-r/2}\right).$$

Further, we can write

$$P_n(|\psi_x|, x) \leq \sqrt{\frac{x(x+2)}{n}}.$$

8.1 Convergence

Theorem 22. *Let f be a continuous function on $[0,\infty)$ for $n \to \infty$, the sequence $\{P_n(f,x)\}$ converges uniformly to $f(x)$ in $[a,b] \subset [0,\infty)$.*

Proof. For sufficiently large n, it is obvious from Remark 2 that $P_n(e_0,x)$, $P_n(e_1,x)$, and $P_n(e_2,x)$ converge uniformly on to $1,\ x$, and x^2, respectively, on every compact subset of $[0,\infty)$. Thus the required result follows from Bohman–Korovkin theorem.

By $C_B[0,\infty)$, we denote the class on real valued continuous bounded functions $f(x)$ for $x \in [0,\infty)$ with the norm

$$||f|| = \sup_{x\in[0,\infty)} |f(x)|.$$

For $f \in C_B[0,\infty)$ and $\delta > 0$ the first and second order modulus of continuity are defined as

$$\omega(f,\delta) = \sup_{0\leq h\leq\delta}\ \sup_{x\in[0,\infty)} |f(x+h) - f(x)|$$

and

$$\omega_2(f,\delta) = \sup_{0\leq h\leq\delta}\ \sup_{x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|,$$

respectively.

The Peetre's K-functional is defined as

$$K_2(f,\delta) = \inf_{g\in C_B^2[0,\infty)} \left\{||f-g|| + \delta||g''|| : g \in C_B^2[0,\infty)\right\},$$

where

$$C_B^2[0,\infty) = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}.$$

There exists a constant $C > 0$ due to [9] such that for $\delta > 0$, we have

$$K_2(f,\delta) \leq C\omega_2(f,\sqrt{\delta}). \tag{13}$$

Theorem 23. *For $x \in [0,\infty)$ and $f \in C_B[0,\infty)$, there exists a constant $C > 0$ such that*

$$|P_n(f,x) - f(x)| \leq C\omega_2\left(f, \sqrt{\frac{x(x+2)}{n}}\right).$$

Proof. Let $g \in C_B^2[0,\infty)$. By the Taylor's expansion

$$g(t) = g(x) + g'(t-x) + \int_x^t (t-u)g''(u)du.$$

and by using Remark 3, we have

$$P_n(g,x) - g(x) = \left(\int_x^t (t-u)g''(u)du, x\right).$$

Since, we have

$$\left|\int_x^t (t-u)g''(u)du\right| \leq (t-x)^2 ||g''||.$$

By Remark 3, we get

$$|P_n(g,x) - g(x)| \leq P_n((t-x)^2, x)||g''|| = \frac{x(x+2)}{n}||g''||.$$

Now by Remark 2, we have

$$|P_n(f,x)| \leq n\sum_{k=1}^{\infty} b_{n,k}(x)\int_0^\infty s_{n,k-1}(t)|f(t)|dt + b_{n,0}|f(0)| \leq ||f||.$$

Therefore

$$\begin{aligned}|P_n(f,x) - f(x)| &\leq |P_n(f-g,x) - (f-g)(x)| + |P_n(g,x) - g(x)|\\ &\leq 2||f-g|| + \frac{x(x+2)}{n}||g''||.\end{aligned}$$

Finally taking the infimum on the right-hand side over all $g \in C_B^2[0,\infty)$ and using (13), we get the desired result.

Theorem 24. *Let f be bounded and integrable on the interval $[0,\infty)$, second derivative of f exists at a fixed point $x \in [0,\infty)$, then*

$$\lim_{n\to\infty} n\,(P_n(f,x) - f(x)) = \frac{x(x+2)}{2} f''(x).$$

Proof. By the Taylor's expansion, we may write

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + r(t,x)(t-x)^2, \tag{14}$$

where $r(t,x)$ is the remainder term and $\lim_{n\to\infty} r(t,x) = 0$. Operating P_n to (14) we obtain

$$P_n(f,x) - f(x) = P_n(t-x,x)f'(x) + P_n\left((t-x)^2, x\right)\frac{f''(x)}{2}$$
$$+P_n\left(r\,(t,x)\,(t-x)^2, x\right)$$

Applying the Cauchy–Schwartz inequality, we have

$$P_n\left(r\,(t,x)\,(t-x)^2, x\right) \le \sqrt{P_n\left(r^2\,(t,x), x\right)}\sqrt{P_n\left((t-x)^4, x\right)}. \tag{15}$$

As $r^2\,(x,x) = 0$ and $r^2\,(.,x) \in C_2^*[0,\infty)$, we have from Remark 3, that

$$\lim_{n\to\infty} P_n\left(r^2\,(t,x), x\right) = r^2\,(x,x) = 0, \tag{16}$$

uniformly with respect to $x \in [0, A]$. Now from (15), (16), and Remark 2, we thus have

$$\lim_{n\to\infty} P_n\left(r\,(t,x)\,(t-x)^2, x\right) = 0.$$

Then, we obtain

$$\begin{aligned}
&\lim_{n\to\infty}(P_n(f,x) - f(x))\\
&= \lim_{n\to\infty}\left(f'(x)P_n\left((t-x), x\right) + \frac{1}{2}f''(x)P_n\left((t-x)^2, x\right) + P_n\left(r\,(t,x)\,(t-x)^2; x\right)\right)\\
&= \frac{x(x+2)}{2}f''(x).
\end{aligned}$$

Let $C^*_{x^2}[0,\infty)$ be the space of all continuous functions satisfying the condition $\lim_{x\to\infty}\frac{f(x)}{1+x^2}$ is finite and belonging to $B_{x^2}[0,\infty)$, where

$$\begin{aligned}
B_{x^2}[0,\infty) &= \{f : \text{ for every } x \in [0,\infty), |f(x)| \le M_f(1+x^2),\\
&\qquad M_f \text{ being a constant depending on } f\}.
\end{aligned}$$

The norm on $C^*_{x^2}[0, \infty)$ is $\|f\|_{x^2} = \sup_{x\in[0,\ \infty)}\dfrac{|f\ (x)|}{1+x^2}$. For weighted approximation, we give here the following result.

Theorem 25. *For each $f \in C^*_{x^2}[0, \infty)$, we have*

$$\lim_{n\to\infty}\|P_n(f) - f\|_{x^2} = 0.$$

Proof. Using Korovkin's theorem, in order to prove the theorem, it is sufficient to show that

$$\lim_{n\to\infty} \| P_n(e_\nu, x) - e_\nu \|_{x^2} = 0, \quad \nu = 0, 1, 2. \tag{17}$$

Since $P_n\,(e_0, x) = 1$ and $P_n\,(e_1, x) = x$, the above condition holds for $\nu = 0, 1$. Next, we can write

$$\left\| P_n\,(e_2, x) - x^2 \right\|_{x^2} \leq \sup_{x\in[0,\,\infty)} \frac{x^2}{1+x^2}\frac{x(x+2)}{n},$$

which implies that

$$\lim_{n\to\infty} \left\| P_n\,(e_2, x) - x^2 \right\|_{x^2} = 0.$$

Thus the result holds for $\nu = 0, 1, 2$. This completes the proof of the theorem.

8.2 Convergence on Bounded Derivative

In this section we discuss the convergence of operators P_n defined by (12), for the functions having bounded derivatives.

Rewriting the operators (12) as

$$P_n(f, x) = \int_0^\infty K_n(x, t) f(t) dt, \tag{18}$$

where K_n is the kernel function given by

$$K_n(x, t) = n \sum_{k=1}^{\infty} b_{n,k}(x) s_{n,k-1}(t) + b_{n,0}(x)\delta(t),$$

and $\delta(t)$ being the Dirac delta function.

Lemma 5. *For any fixed* $x \in (0, \infty)$*, we have*

$$\lambda_n(x, y) = \int_0^y K_n(x, t) dt \leq \frac{x(x+2)}{n(x-y)^2},\ \ 0 \leq y < x,$$

$$1 - \lambda_n(x, z) = \int_z^\infty K_n(x, t) dt \leq \frac{x(x+2)}{n(z-x)^2},\ \ x < z < \infty.$$

Let us define the class Φ_{DB} of functions as:

$$\Phi_{DB} = \left\{ f : f(x) - f(0) = \int_0^x \phi(t)dt;\ f(t) = O(t^r),\ t \to \infty \right\},$$

where ϕ is bounded on every finite subinterval of $[0, \infty)$.

For a fixed $x \in [0, \infty)$, $\lambda \geq 0$ and $f \in \Phi_{DB}$ let us define the metric form as

$$\Omega(f, \lambda) = \sup_{t \in [x-\lambda, x+\lambda] \cap [0,\infty)} |f(t) - f(x)|.$$

Theorem 26. *Let $f \in \Phi_{DB}$, $x \in (0, \infty)$ be fixed. If $\phi(x-)$ and $\phi(x+)$ exists, then for $n \geq 4$, we have*

$$\left| P_n(f, x) - f(x) - \frac{\phi(x+) - \phi(x-)}{2} \sqrt{\frac{x(x+2)}{n}} \right| \leq \frac{(2x+9)}{n} \sum_{k=1}^{[\sqrt{n}]} \Omega_x \left(\phi_x, \frac{x}{k} \right) + O(n^{-r}),$$

where

$$\phi_x(t) = \begin{cases} \phi(t) - \phi(x-), & 0 \leq t < x \\ 0, & t = x \\ \phi(t) - \phi(x+), & x < t < \infty \end{cases}.$$

Proof. By simple computation, we have

$$\begin{aligned} D_n(f, x) - f(x) \leq & \frac{\phi(x+) - \phi(x-)}{2} D_n(|t - x|, x) \\ & + \frac{\phi(x+) + \phi(x-)}{2} D_n(t - x, x) \\ & - A_{n,x}(\phi_x) + B_{n,x}(\phi_x) + C_{n,x}(\phi_x), \end{aligned} \tag{19}$$

where

$$A_{n,x}(\phi_x) = \int_0^x \left(\int_t^x \phi_x(u)du \right) d_t(\lambda_n(x, t)),$$

$$B_{n,x}(\phi_x) = \int_x^{2x} \left(\int_x^t \phi_x(u)du \right) d_t(\lambda_n(x, t)),$$

$$C_{n,x}(\phi_x) = \int_{2x}^{\infty} \left(\int_x^t \phi_x(u)du \right) d_t(\lambda_n(x, t)).$$

Integrating by parts, we have

$$A_{n,x}(\phi_x) = \int_t^x \phi_x(u)du\lambda_n(x,t)|_0^x + \int_0^x \lambda_n(x,t)\phi_x(t)dt$$
$$= \left(\int_0^{x-x/\sqrt{n}} + \int_{x-x/\sqrt{n}}^x\right)\lambda_n(x,t)\phi_x(t)dt.$$

As $\lambda_n(x,t) \leq 1$, from the monotonicity of $\Omega_x(\phi_x,\lambda)$ and the definition of $\phi_x(t)$, we have

$$\left|\int_{x-x/\sqrt{n}}^x \lambda_n(x,t)\phi_x(t)dt\right| \leq \frac{x}{\sqrt{n}}\Omega_x\left(\phi_x, \frac{x}{\sqrt{n}}\right)$$
$$\leq \frac{2x}{n}\sum_{k=1}^{[\sqrt{n}]}\Omega_x\left(\phi_x, \frac{x}{k}\right).$$

Taking $t = \frac{x}{x-u}$ and using Lemma 5, we get

$$\left|\int_0^{x-x/\sqrt{n}} \lambda_n(x,t)\phi_x(t)dt\right| \leq \frac{x(x+2)}{n}\int_0^{x-x/\sqrt{n}} \frac{\Omega_x\left(\phi_x, x-t\right)}{(x-t)^2}dt$$
$$\leq \frac{2+x}{n}\int_1^{\sqrt{n}}\Omega_x\left(\phi_x, \frac{x}{u}\right)du$$
$$\leq \frac{2+x}{n}\sum_{k=1}^{\sqrt{n}}\Omega_x\left(\phi_x, \frac{x}{k}\right).$$

Collecting above results, we get

$$|A_{n,x}(\phi_x)| \leq \frac{(3x+2)}{n}\sum_{k=1}^{\sqrt{n}}\Omega_x\left(\phi_x, \frac{x}{k}\right). \tag{20}$$

Further, we have

$$B_{n,x}(\phi_x) = \int_x^{2x}\left(\int_x^t \phi_x(u)du\right)d_t\left(\lambda_n(x,t)\right)$$
$$= -\int_x^{2x}\left(\int_x^t \phi_x(u)du\right)d_t\left(1-\lambda_n(x,t)\right)$$
$$= -\int_x^{2x}\phi_x(u)du\left(1-\lambda_n(x,2x)\right) + \int_x^{2x}\phi_x(t)\left(1-\lambda_n(x,t)\right)dt.$$

From Lemma 5, we have

$$\left|-\int_x^{2x} \phi_x(u)du\,(1-\lambda_n(x,2x))\right| \leq x\Omega_x(\phi_x,x)\frac{x(x+2)}{nx^2} = \frac{x+2}{n}\Omega_x(\phi_x,x).$$

Also, we have

$$\left|\int_x^{2x} \phi_x(t)\,(1-\lambda_n(x,t))\,dt\right| \leq \frac{x+2}{n}\sum_{k=1}^{\sqrt{n}} \Omega_x\left(\phi_x,\frac{x}{k}\right).$$

Hence, we get

$$|B_{n,x}(\phi_x)| \leq \frac{x+2}{n}\Omega_x(\phi_x,x) + \frac{x+2}{n}\sum_{k=1}^{\sqrt{n}} \Omega_x\left(\phi_x,\frac{x}{k}\right). \tag{21}$$

Assuming that there exists an integer r such that $f(t)=O(t^{2r})$ as $t\to\infty$. Finally for a certain constant $M>0$ depending only on f,x,r, we obtain

$$|C_{n,x}(\phi_x)| = Mn\sum_{k=1}^{\infty} b_{n,k}(x)\int_{2x}^{\infty} s_{n,k-1}(t)t^{2r}dt.$$

Using Remark 3 and the inequality $t\leq 2(t-x)$ for $t\geq 2x$, we get

$$\begin{aligned}|C_n(f,x)| &\leq 2^{2r}Mn\sum_{k=1}^{\infty} b_{n,k}(x)\int_0^{\infty} s_{n,k-1}(t)(t-x)^{2r}dt\\ &= 2^{2r}M\left[D_n(\psi_x^{2r},x) - b_{n,0}(-x)^{2r}\right]\\ &= O(n^{-r}) + O(n^{-s}) \text{ for any } s>0\\ &= O(n^{-r}).\end{aligned} \tag{22}$$

The proof of the theorem is completed by combining the (19), (20), (21), and (22).

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Hypergeometric Representation of Certain Summation–Integral Operators

Vijay Gupta and Themistocles M. Rassias

Abstract The general sequence of the summation-integral type operators was proposed by Srivastava and Gupta [Math. Comput. Modelling 37(12–13)(2003), 1307–1315]. In the present article we give the alternate forms of such operators in terms of hypergeometric series. We also obtain moments using hypergeometric series. Finally we obtain the rate of convergence for functions having bounded derivatives.

Keywords Linear positive operators • Srivastava-Gupta operators • Moments • Convergence

1 Introduction

About ten years ago Srivastava and Gupta [9] introduced a general sequence of linear positive operators $G_{n,c}(f,x)$ which when applied to f are defined as

$$G_{n,c}(f,x) = n \sum_{k=1}^{\infty} p_{n,k}(x,c) \int_0^{\infty} p_{n+c,k-1}(t,c)\, f(t)\, dt + p_{n,0}(x,c)\, f(0), \tag{1}$$

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where

$$p_{n,k}(x,c) = \frac{(-x)^k}{k!}\phi_{n,c}^{(k)}(x)$$

and

$$\phi_{n,c}(x) = \begin{cases} e^{-nx} & , c = 0 \\ (1+cx)^{-n/c} & , c \in \mathbb{N} := \{1,2,3,\ldots\} \ , \\ (1-x)^n & , c = -1. \end{cases}$$

Ispir and Yuksel [7] first considered the Bézier variant of these operators and estimated the rate of convergence for functions of bounded variation. They termed these operators as Srivastava–Gupta operators. Later Deo [1] and Verma and Agrawal [10] also used this terminology and they established some approximation properties of $G_{n,c}$.

The following are the special cases of the operators $G_{n,c}(f,x)$ defined in (1), which have the following forms:

1. If $c = 0$, then $p_{n,k}(x,0) = e^{-nx}\frac{(nx)^k}{k!}$ and operators become the Phillips operators $G_{n,0}(f,x)$, introduced by [8], which for $x \in [0,\infty)$ are defined by

$$G_{n,0}(f,x) = n\sum_{k=1}^{\infty} p_{n,k}(x,0)\int_0^{\infty} p_{n,k-1}(t;0)\, f(t)\, dt + e^{-nx} f(0).$$

2. If $c = 1, 2, \cdots$, one has $p_{n,k}(x,c) = \binom{\frac{n}{c}+k-1}{k}\frac{(cx)^k}{(1+cx)^{\frac{n}{c}+k}}$ and operators become the Durrmeyer type Baskakov operators $G_{n,1}(f,x)$, which was introduced by Gupta et al. in [5], and which for $x \in [0,\infty)$ are defined as

$$G_{n,1}(f,x) = n\sum_{k=1}^{\infty} p_{n,k}(x,1)\int_0^{\infty} p_{n+1,k-1}(t,1)\, f(t)\, dt + (1+cx)^{-n/c} f(0).$$

3. If $c = -1$, then $p_{n,k}(x,-1) = \binom{n}{k}x^k(1-x)^{n-k}$, and we get the Bernstein–Durrmeyer type operators $G_{n,-1}(f,x)$ introduced by Gupta and Maheshwari [6] and also studied in [3]. In this case the summation runs from 1 to n, integration from 0 to 1, and $x \in [0,1]$, and $G_{n,-1}(f,x)$ is defined as

$$G_{n,-1}(f,x) = n\sum_{k=1}^{n} p_{n,k}(x,-1)\int_0^{1} p_{n-1,k-1}(t,-1)\, f(t)\, dt + (1-x)^n f(0).$$

In the present article we give the alternate hypergeometric representations of the different cases of the operators $G_{n,c}$, and also establish moments using such representations. Finally we estimate the rate of convergence for functions having bounded derivatives.

2 Alternate Forms

It is observed that as an application of the special functions, we can write the different cases of the operators $G_{n,c}(f, x)$ in terms of Hypergeometric series. For details on Hypergeometric series, we refer the readers to [2].

The hypergeometric function is defined as

$$ {}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} . \frac{x^k}{k!}. $$

The confluent hypergeometric function is a degenerate form of the hypergeometric function ${}_2F_1(a, b; c; x)$ which arises as a solution the confluent hypergeometric differential equation is defined as

$$ {}_1F_1(a; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{x^k}{k!}, $$

where the Pochhammer symbol $(n)_k$ is defined as

$$ (n)_k = n(n + 1)(n + 2)(n + 3) \ldots .(n + k - 1). $$

Also, it is easy to observe that

$$ \lim_{b \to \infty} {}_2F_1\left(a, b; c; \frac{x}{b}\right) = {}_1F_1(a; c; x). $$

2.1 Case $c = 0$

For this case, we have

$$ G_{n,0}(f, x) = n \sum_{k=1}^{\infty} p_{n,k}(x, 0) \int_0^{\infty} p_{n,k-1}(t, 0) f(t) dt + e^{-nx} f(0), \qquad (2) $$

where the Szász basis function is given by

$$p_{n,k}(x,0) = e^{-nx}\frac{(nx)^k}{k!}.$$

The operator (2) can be written as (see also [4])

$$\begin{aligned}
G_{n,0}(f,x) &= n\int_0^\infty f(t)\sum_{k=1}^\infty e^{-nx}\frac{(nx)^k}{k!}e^{-nt}\frac{(nt)^{k-1}}{(k-1)!}dt + e^{-nx}f(0)\\
&= n^2x\int_0^\infty e^{-n(x+t)}f(t)\sum_{k=1}^\infty \frac{(n^2xt)^{k-1}}{k!(k-1)!}dt + e^{-nx}f(0)\\
&= n^2x\int_0^\infty e^{-n(x+t)}f(t)\sum_{k=0}^\infty \frac{(n^2xt)^{k}}{(2)_k k!}dt + e^{-nx}f(0)\\
&= n^2x\int_0^\infty e^{-n(x+t)}f(t)\,{}_0F_1\left(-;2;n^2xt\right)dt + e^{-nx}f(0),
\end{aligned}$$

where ${}_0F_1\left(-;2;n^2xt\right) = \Gamma(2)(n^2xt)^{-1/2}I_1(2n\sqrt{xt})$ and $I_1(2n\sqrt{xt})$ is the modified Bessel's function of first kind of index 1 which is given by

$$I_1(2n\sqrt{xt}) = \frac{1}{i}J_1(i2n\sqrt{xt}) = \frac{1}{i}\sum_{j=0}^\infty \frac{1}{j!\Gamma(j+2)}(in\sqrt{xt})^{2j+1}.$$

2.2 Case $c = 1$

For this case, we have

$$G_{n,1}(f,x) = n\sum_{k=1}^\infty p_{n,k}(x,1)\int_0^\infty p_{n+1,k-1}(t,1)f(t)dt + (1+x)^{-n}f(0) \quad (3)$$

where the Baskakov basis function is given by

$$p_{n,k}(x,1) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}} = \frac{(n)_k}{k!}\frac{x^k}{(1+x)^{n+k}}.$$

The operator (3) can be written as

$$G_{n,1}(f,x) = \sum_{k=1}^\infty \frac{(n)_k}{k!}\frac{x^k}{(1+x)^{n+k}}\int_0^\infty \frac{(n)_k}{(k-1)!}\frac{t^{k-1}}{(1+t)^{n+k}}f(t)dt + (1+x)^{-n}f(0)$$

$$= \int_0^\infty \frac{f(t)x}{[(1+x)(1+t)]^{n+1}} \sum_{k=1}^{\infty} \frac{(n)_k (n)_k}{(k-1)!k!} \frac{(xt)^{k-1}}{[(1+x)(1+t)]^{k-1}} dt$$
$$+ (1+x)^{-n} f(0)$$
$$= n^2 \int_0^\infty \frac{f(t)x}{[(1+x)(1+t)]^{n+1}} \sum_{k=0}^{\infty} \frac{(n+1)_k (n+1)_k}{(2)_k k!} \frac{(xt)^k}{[(1+x)(1+t)]^k} dt$$
$$+ (1+x)^{-n} f(0).$$

Using the hypergeometric series ${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k$, we have

$$G_{n,1}(f,x) = n^2 \int_0^\infty \frac{f(t)x}{(1+x+t)^{n+1}} \, {}_2F_1\left(n+1, n+1; 2; \frac{xt}{(1+x)(1+t)}\right) dt$$
$$+ (1+x)^{-n} f(0),$$

Now applying Pfaff-Kummer transformation

$${}_2F_1(a,b;c;x) = (1-x)^{-a} \, {}_2F_1\left(a, c-b; c; \frac{x}{x-1}\right),$$

we have

$$G_{n,1}(f,x) = n^2 \int_0^\infty \frac{f(t)x}{(1+x+t)^{n+1}} \, {}_2F_1\left(n+1, 1-n; 2; \frac{-xt}{1+x+t}\right) dt$$
$$+ (1+x)^{-n} f(0),$$

which is the alternate form of the operators (3) in terms of hypergeometric functions.

2.3 Case $c = -1$

For this case, we have

$$G_{n,-1}(f,x) = n \sum_{k=1}^{\infty} p_{n,k}(x,-1) \int_0^n p_{n-1,k-1}(t,1) f(t) dt + (1-x)^n f(0), \quad (4)$$

where the Benstein basis function is given by

$$p_{n,k}(x,-1) = \binom{n}{k} x^k (1-x)^{n-k}$$

$$= \frac{n(n-1)(n-2)\ldots..(n-k+1)}{k!} x^k (1-x)^{n-k}$$
$$= \frac{(-1)^k (-n)_k}{k!} x^k (1-x)^{n-k}.$$

The operator (4) can be written as

$$G_{n,-1}(f,x) = n \int_0^1 f(t) \sum_{k=1}^{n} \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} \frac{(n-1)!}{(k-1)!(n-k)!} t^{k-1}$$
$$\times (1-t)^{n-k} dt + (1-x)^n f(0)$$
$$= n \int_0^1 f(t) x \left[(1-x)(1-t)\right]^{n-1} \sum_{k=1}^{n} (xt)^{k-1}$$
$$\times \frac{(-n)_k (-n)_k}{k!(k-1)!} \frac{dt}{\left[(1-x)(1-t)\right]^{k-1}} + (1-x)^n f(0)$$
$$= n \int_0^1 f(t) x \left[(1-x)(1-t)\right]^{n-1} \sum_{k=0}^{n} (xt)^{k}$$
$$\times \frac{(-n)_{k+1} (-n)_{k+1}}{k!(k+1)!} \frac{dt}{\left[(1-x)(1-t)\right]^{k}} + (1-x)^n f(0).$$

Using the fact $(-n)_{k+1} = (-n)(-n+1)(-n+2)\ldots.(-n+k) = (-n).(-n+1)_k$, we have

$$G_{n,-1}(f,x) = n^2 \int_0^1 f(t) x [(1-x)(1-t)]^{n-1} \sum_{k=0}^{n-1} \frac{(-n+1)_k (-n+1)_k}{(2)_k k!}$$
$$\times \left(\frac{xt}{(1-x)(1-t)}\right)^k dt + (1-x)^n f(0)$$
$$= n^2 \int_0^1 f(t) x [(1-x)(1-t)]^{n-1} {}_2F_1$$
$$\times \left(-n+1, -n+1; 2; \frac{xt}{(1-x)(1-t)}\right) dt + (1-x)^n f(0).$$

Now applying Pfaff-Kummer transformation

$${}_2F_1(a,b;c;x) = (1-x)^{-a} {}_2F_1\left(a, c-b; c; \frac{x}{x-1}\right),$$

we have

$$G_{n,-1}(f,x) = n^2 \int_0^1 \frac{f(t)x}{(1-x-t)^{n-1}}\, {}_2F_1\left(-n+1, n+1; 2; \frac{-xt}{1-x-t}\right) dt$$
$$+ (1-x)^n f(0),$$

which is the alternate form of the operators (4) in terms of hypergeometric functions.

3 Moments

This section deals with the moment estimation for the different cases of the operators $G_{n,c}(f,x)$. Here we use the alternate form, i.e. Hypergeometric representation, which can provide easily the moments of higher order.

Lemma 1. *For $n > 0, c = -1, 0, 1, 2, \cdots$ and $r \geq 1$, we have*

$$G_{n,c}(t^r, x) = \frac{nx\Gamma(r+1)}{(n-c)(n-2c)\cdots(n-cr)}\, {}_2F_1\left(\frac{n}{c}+1, 1-r; 2; -cx\right). \quad (5)$$

Proof. We shall prove the result separately for different cases.

Case 1. By definition of operator (1) in the case $c = 0$, we have

$$\begin{aligned} G_{n,0}(t^r, x) &= n \sum_{k=1}^{\infty} e^{-nx} \frac{(nx)^k}{k!} \int_0^{\infty} t^r e^{-nt} \frac{(nt)^{k-1}}{(k-1)!} dt \\ &= n \sum_{k=1}^{\infty} e^{-nx} \frac{(nx)^k}{k!(k-1)!} \int_0^{\infty} e^{-nt} n^{k-1} t^{k+r-1} dt \\ &= n e^{-nx} \sum_{k=1}^{\infty} \frac{(nx)^k}{k!(k-1)!} \frac{\Gamma(k+r)}{n^{r+1}} \\ &= n e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^{k+1}}{(k+1)!k!} \frac{\Gamma(k+r+1)}{n^{r+1}}. \end{aligned}$$

Using $(k+1)! = (2)_k$ and $\Gamma(r+k+1) = \Gamma(r+1)(r+1)_k$, we have

$$\begin{aligned} G_{n,0}(t^r, x) &= n^2 x e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{(k+1)!k!} \frac{(r+1)_k \Gamma(r+1)}{n^{r+1}} \\ &= \frac{x\Gamma(r+1)e^{-nx}}{n^{r-1}} \sum_{k=0}^{\infty} \frac{(r+1)_k}{(2)_k k!} (nx)^k \end{aligned}$$

$$= \frac{x\Gamma(r+1)e^{-nx}}{n^{r-1}} \, {}_1F_1\left(r+1; 2; nx\right).$$

Using Kummer's transformations ${}_1F_1(a,b;x) = e^x \, {}_1F_1(b-a,b;-x)$, we have

$$G_{n,0}(t^r,x) = \frac{nx\Gamma(r+1)}{n^r} \, {}_1F_1\left(1-r; 2; -nx\right),$$

which is the limiting case of (5) when $c \to 0$.

Case 2. Taking $f(t) = t^r$ and using $\frac{n}{c}\left(\frac{n}{c}+1\right)_{k-1} = (\frac{n}{c})_k$, we have

$$\begin{aligned}
G_{n,c}(t^r,x) &= n\sum_{k=1}^{\infty} \frac{\left(\frac{n}{c}\right)_k}{k!} \frac{(cx)^k}{(1+cx)^{\frac{n}{c}+k}} \int_0^{\infty} \frac{\left(\frac{n}{c}+1\right)_{k-1}}{(k-1)!} \frac{(ct)^{k-1}t^r}{(1+ct)^{\frac{n}{c}+k}} dt \\
&= c\sum_{k=1}^{\infty} \frac{\left(\frac{n}{c}\right)_k}{k!} \frac{(cx)^k}{(1+cx)^{\frac{n}{c}+k}} \frac{\left(\frac{n}{c}\right)_k}{(k-1)!} \frac{1}{c^{r+1}} . \frac{\Gamma(k+r)\Gamma\left(\frac{n}{c}-r\right)}{\Gamma\left(\frac{n}{c}+k\right)} \\
&= \frac{\Gamma\left(\frac{n}{c}-r\right)}{c^r} \sum_{k=0}^{\infty} \frac{\left(\frac{n}{c}\right)_{k+1}}{(k+1)!} \frac{(cx)^{k+1}}{(1+cx)^{\frac{n}{c}+k+1}} \frac{\left(\frac{n}{c}\right)_{k+1}}{k!} \frac{\Gamma(k+r+1)}{\Gamma\left(\frac{n}{c}+k+1\right)} \\
&= \frac{\Gamma\left(\frac{n}{c}-r\right)}{c^r} \sum_{k=0}^{\infty} \frac{\frac{n}{c}\left(\frac{n}{c}+1\right)_k \frac{n}{c}\left(\frac{n}{c}+1\right)_k}{(2)_k k!} \\
&\quad \times \frac{\Gamma(r+1)(r+1)_k}{\Gamma\left(\frac{n}{c}+1\right)\left(\frac{n}{c}+1\right)_k} \frac{(cx)^{k+1}}{(1+cx)^{\frac{n}{c}+k+1}} \\
&= \frac{\Gamma\left(\frac{n}{c}-r\right)\Gamma(r+1)n^2x}{\Gamma\left(\frac{n}{c}+1\right)c^{r+1}(1+cx)^{\frac{n}{c}+1}} \sum_{k=0}^{\infty} \frac{\left(\frac{n}{c}+1\right)_k (r+1)_k}{(2)_k k!} \left(\frac{cx}{1+cx}\right)^k \\
&= \frac{\Gamma\left(\frac{n}{c}-r\right)\Gamma(r+1)n^2x}{\Gamma\left(\frac{n}{c}+1\right)c^{r+1}(1+cx)^{\frac{n}{c}+1}} \, {}_2F_1\left(\frac{n}{c}+1, r+1; 2; \frac{cx}{1+cx}\right).
\end{aligned}$$

Using ${}_2F_1(a,b;c;x) = (1-x)^{-a} \, {}_2F_1\left(a, c-b; c; \frac{x}{x-1}\right)$, we have

$$G_{n,c}(t^r,x) = \frac{nx\Gamma(r+1)}{(n-c)(n-2c)\cdots(n-cr)} \, {}_2F_1\left(\frac{n}{c}+1, 1-r; 2; -cx\right),$$

which proves the result when $c = 1, 2, 3, \cdots$

Case 3. By definition of operator (1) in the case $c = -1$ and using $\Gamma(r+k+1) = \Gamma(r+1)(r+1)_k$, we have

$$G_{n,-1}(t^r,x) = n\sum_{k=1}^{n} p_{n,k}(x) \int_0^1 p_{n-1,k-1}(t)t^r \, dt$$

$$
\begin{aligned}
&= n\sum_{k=1}^{n}\frac{(-1)^k(-n)_k}{k!}x^k(1-x)^{n-k}\\
&\quad\times\int_0^1\frac{(n-1)!}{(k-1)!(n-k)!}t^{k+r-1}(1-t)^{n-k}dt\\
&= n\sum_{k=1}^{n}\frac{(-1)^k(-n)_k}{k!}x^k(1-x)^{n-k}\frac{(n-1)!}{(k-1)!(n-k)!}\\
&\quad\times B(k+r,n-k+1)\\
&= n\sum_{k=1}^{n}\frac{(-1)^k(-n)_k}{k!}x^k(1-x)^{n-k}\frac{(n-1)!}{(k-1)!(n-k)!}\\
&\quad\times\frac{\Gamma(k+r)\Gamma(n-k+1)}{\Gamma(n+r+1)}\\
&= \frac{n!}{\Gamma(n+r+1)}\sum_{k=0}^{n}\frac{(-1)^{k+1}(-n)_{k+1}}{(2)_k k!}\\
&\quad\times\Gamma(r+1)(r+1)_k x^{k+1}(1-x)^{n-k-1}\\
&= \frac{\Gamma(n+1)\Gamma(r+1)}{\Gamma(n+r+1)}x(1-x)^n\sum_{k=0}^{n}\frac{(-1)^{k+1}(-n)_{k+1}(r+1)_k}{(2)_k k!}\\
&\quad\times\left(\frac{x}{1-x}\right)^k.
\end{aligned}
$$

Next $(-n)_{k+1} = (-n)(-n+1)_k$, thus

$$
\begin{aligned}
G_{n,-1}(t^r,x) &= \frac{\Gamma(n{+}1)\Gamma(r{+}1)}{\Gamma(n{+}r{+}1)}(1{-}x)^n nx\sum_{k=0}^{n}\frac{(-n+1)_k(r+1)_k}{(2)_k k!}\left(\frac{x}{x-1}\right)^k\\
&= \frac{\Gamma(r+1)\Gamma(n+1)}{\Gamma(n+r+1)}nx(1-x)^n\,{}_2F_1\left(-n+1,r+1;2;\frac{x}{x-1}\right).
\end{aligned}
$$

Using ${}_2F_1(a,b;c;x) = (1-x)^{-a}\,{}_2F_1\left(a,c-b;c;\frac{x}{x-1}\right)$, we have

$$
G_{n,-1}(t^r,x) = \frac{nx\Gamma(r+1)}{(n+1)(n+2)\cdots(n+r)}\,{}_2F_1\left(-n+1,1-r;2;x\right),
$$

which completes the proof in case $c = -1$.

Remark 1. By Lemma 1, we have the first three moments of the Phillips operators as

$$
G_{n,0}(1,x) = 1, G_{n,0}(t,x) = x, G_{n,0}(t^2,x) = x^2 + \frac{2x}{n}.
$$

We have the first three moments of the Baskakov–Durrmeyer operators as

$$G_{n,1}(1,x) = 1, G_{n,1}(t,x) = \frac{nx}{n-1}, G_{n,1}(t^2,x) = \frac{n(n+1)x^2 + 2nx}{(n-1)(n-2)}.$$

We have the first three moments of the Bernstein–Durrmeyer operators as

$$G_{n,-1}(1,x) = 1, G_{n,-1}(t,x) = \frac{nx}{n+1}, G_{n,-1}(t^2,x) = \frac{n(n-1)x^2 + 2nx}{(n+1)(n+2)}.$$

Remark 2. Let $\psi_x(t) = t - x, n \in \mathbb{N}$ then from Lemma 1, we have

$$\begin{aligned} G_{n,c}(\psi_x, x) &= \frac{cx}{n-c} \\ G_{n,c}(\psi_x^2, x) &= \frac{x(1+cx)(2n-c) + (1+3cx)cx}{(n-c)(n-2c)}. \end{aligned}$$

Further, for $r = 0, 1, 2, \ldots$, we have

$$G_{n,c}(\psi_x^r, x) = O\left(n^{-[(r+1)/2]}\right).$$

Applying Schwarz inequality, we have

$$G_{n,c}(|\psi_x^r|, x) \leq \sqrt{G_{n,c}(\psi_x^{2r}, x)} = O\left(n^{-r/2}\right).$$

In particular for certain $\lambda > 2$, we can write

$$G_{n,c}(|\psi_x|, x) \leq \sqrt{\frac{\lambda x(1+cx)}{n}}.$$

Rewriting the operators (1) as

$$G_{n,c}(f,x) = \int_0^\infty K_{n,c}(x,t) f(t) dt, \tag{6}$$

where $K_{n,c}(x,t)$ is the kernel function given by

$$K_{n,c}(x,t) = \sum_{k=1}^{\infty} p_{n,k}(x,c) p_{n+c,k-1}(t,c) + p_{n,0}(x,c)\delta(t),$$

and $\delta(t)$ being the Dirac delta function.

Lemma 2. *For certain $\lambda > 2$ and for fixed $x \in (0, \infty)$, we have*

$$\lambda_{n,c}(x, y) = \int_0^y K_{n,c}(x, t)dt \leq \frac{\lambda x(1 + cx)}{n(x - y)^2}, \;\; 0 \leq y < x,$$

$$1 - \lambda_{n,c}(x, z) = \int_z^\infty K_{n,c}(x, t)dt \leq \frac{\lambda x(1 + cx)}{n(z - x)^2}, \;\; x < z < \infty.$$

4 Convergence

In this section we discuss the convergence behavior of operators (1) in case $c \in \{0, 1, 2, \cdots\}$, for functions having bounded derivatives. Let us denote the class Φ_{DB} of the functions with growth as:

$$\Phi_{\text{DB}} = \left\{ f : f(x) - f(0) = \int_0^x \phi(t)dt; \;\; f(t) = O(t^r), \;\; t \to \infty \right\},$$

where ϕ is bounded on every finite subinterval of $[0, \infty)$. For a fixed $x \in [0, \infty)$, $\eta \geq 0$ and $f \in \Phi_{\text{DB}}$ let us define the metric form as

$$\Omega(f, \eta) = \sup_{t \in [x-\eta, x+\eta] \cap [0,\infty)} |f(t) - f(x)|.$$

Theorem 1. *Let $f \in \Phi_{\text{DB}}$, $x \in (0, \infty)$ be fixed and $c \in \{0, 1, 2, \cdots\}$. If $\phi(x-)$ and $\phi(x+)$ exists, then for certain $\lambda > 2$, we have*

$$\left| G_{n,c}(f, x) - f(x) - \frac{\phi(x+) - \phi(x-)}{2} \sqrt{\frac{\lambda x(1 + cx)}{n}} \right|$$

$$\leq \frac{\lambda[3 + x(2 + 3c)]}{n} \sum_{k=1}^{[\sqrt{n}]} \Omega_x \left(\phi_x, \frac{x}{k} \right)$$

$$+ \frac{|\phi(x+) + \phi(x-)|}{2} \cdot \frac{cx}{n - c} + O(n^{-r}),$$

where

$$\phi_x(t) = \begin{cases} \phi(t) - \phi(x-), & 0 \leq t < x \\ 0, & t = x \\ \phi(t) - \phi(x+), & x < t < \infty \end{cases}.$$

Proof. By simple computation, we have

$$G_{n,c}(f,x)-f(x)\le\frac{\phi(x+)-\phi(x-)}{2}G_{n,c}(|t-x|,x)$$
$$+\ \frac{\phi(x+)+\phi(x-)}{2}G_{n,c}(t-x,x)$$
$$-\ A_{n,x}(\phi_x)+B_{n,x}(\phi_x)+C_{n,x}(\phi_x), \qquad (7)$$

where

$$A_{n,x}(\phi_x)=\int_0^x\left(\int_t^x\phi_x(u)du\right)d_t(\lambda_{n,c}(x,t)),$$

$$B_{n,x}(\phi_x)=\int_x^{2x}\left(\int_x^t\phi_x(u)du\right)d_t(\lambda_{n,c}(x,t)),$$

$$C_{n,x}(\phi_x)=\int_{2x}^{\infty}\left(\int_x^t\phi_x(u)du\right)d_t(\lambda_{n,c}(x,t)).$$

Integrating by parts, we have

$$A_{n,x}(\phi_x)=\int_t^x\phi_x(u)du\lambda_{n,c}(x,t)|_0^x+\int_0^x\lambda_{n,c}(x,t)\phi_x(t)dt$$
$$=\left(\int_0^{x-x/\sqrt{n}}+\int_{x-x/\sqrt{n}}^x\right)\lambda_{n,c}(x,t)\phi_x(t)dt.$$

As $\lambda_{n,c}(x,t)\le 1$, from the monotonicity of $\Omega_x(\phi_x,\lambda)$ and the definition of $\phi_x(t)$, we obtain

$$\left|\int_{x-x/\sqrt{n}}^x\lambda_{n,c}(x,t)\phi_x(t)dt\right|\le\frac{x}{\sqrt{n}}\Omega_x\left(\phi_x,\frac{x}{\sqrt{n}}\right)$$
$$\le\frac{2x}{n}\sum_{k=1}^{[\sqrt{n}]}\Omega_x\left(\phi_x,\frac{x}{k}\right).$$

Taking $t=\frac{x}{x-u}$ and using Lemma 2, we get

$$\left|\int_0^{x-x/\sqrt{n}}\lambda_{n,c}(x,t)\phi_x(t)dt\right|\le\frac{\lambda x(1+cx)}{n}\int_0^{x-x/\sqrt{n}}\frac{\Omega_x\left(\phi_x,x-t\right)}{(x-t)^2}dt$$
$$\le\frac{\lambda(1+cx)}{n}\int_1^{\sqrt{n}}\Omega_x\left(\phi_x,\frac{x}{u}\right)du$$

$$\leq \frac{\lambda(1+cx)}{n} \sum_{k=1}^{\sqrt{n}} \Omega_x \left(\phi_x, \frac{x}{k}\right).$$

Collecting the above estimates, we are led to

$$|A_{n,x}(\phi_x)| \leq \frac{\lambda[1+x(1+c)]}{n} \sum_{k=1}^{\sqrt{n}} \Omega_x \left(\phi_x, \frac{x}{k}\right). \quad (8)$$

Furthermore, we derive

$$\begin{aligned} B_{n,x}(\phi_x) &= \int_x^{2x} \left(\int_x^t \phi_x(u)du \right) d_t\, (\lambda_{n,c}(x,t)) \\ &= -\int_x^{2x} \left(\int_x^t \phi_x(u)du \right) d_t\, (1-\lambda_{n,c}(x,t)) \\ &= -\int_x^{2x} \phi_x(u)du\, (1-\lambda_{n,c}(x,2x)) + \int_x^{2x} \phi_x(t)\, (1-\lambda_{n,c}(x,t))\, dt. \end{aligned}$$

Using Lemma 2, it yields

$$\begin{aligned} \left| -\int_x^{2x} \phi_x(u)du\, (1-\lambda_{n,c}(x,2x)) \right| &\leq x\Omega_x(\phi_x, x) \frac{\lambda x(1+cx)}{nx^2} \\ &= \frac{\lambda(1+cx)}{n} \Omega_x(\phi_x, x). \end{aligned}$$

Also, we have

$$\left| \int_x^{2x} \phi_x(t)\, (1-\lambda_{n,c}(x,t))\, dt \right| \leq \frac{\lambda[1+x(1+c)]}{n} \sum_{k=1}^{\sqrt{n}} \Omega_x \left(\phi_x, \frac{x}{k}\right).$$

Hence, we get

$$|B_{n,x}(\phi_x)| \leq \frac{\lambda(1+cx)}{n} \Omega_x(\phi_x, x) + \frac{\lambda[1+x(1+c)]}{n} \sum_{k=1}^{\sqrt{n}} \Omega_x \left(\phi_x, \frac{x}{k}\right). \quad (9)$$

Assuming that there exists an integer r such that $f(t) = O(t^{2r})$ as $t \to \infty$ for certain constant $M(f, x, r) > 0$, thus

$$|C_{n,x}(\phi_x)| = M \sum_{k=1}^{\infty} p_{n,k}\, (x,c) \int_{2x}^{\infty} p_{n+c,k-1}\, (t,c)\, t^{2r}\, dt.$$

Applying Remark 2 and the inequality $t \leq 2(t-x)$ for $t \geq 2x$, we obtain

$$\begin{aligned}|C_n(f,x)| &\leq 2^{2r} M \sum_{k=1}^{\infty} p_{n,k}(x,c)(x) \int_0^{\infty} p_{n+c,k-1}(t,c)(t-x)^{2r} dt \\ &= 2^{2r} M \left[G_{n,c}(\psi_x^{2r},x) - p_{n,0}(x,c)(-x)^{2r}\right] \\ &= O(n^{-r}) + O(n^{-s}) \text{ for any } s > 0 \\ &= O(n^{-r}). \end{aligned} \tag{10}$$

Finally, by combining the estimates in (7), (8), (9), and (10), we get the desired result.

Remark 3. For the case $c = -1$, the rate of convergence can be obtained analogously and we need not to take any growth in that case, we omit the details.

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On a Hybrid Fourth Moment Involving the Riemann Zeta-Function

Aleksandar Ivić and Wenguang Zhai

Abstract For each integer $1 \le j \le 6$, we provide explicit ranges for σ for which the asymptotic formula

$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4 |\zeta(\sigma+it)|^{2j}\,dt \sim T\sum_{k=0}^{4} a_{k,j}(\sigma)\log^k T$$

holds as $T \to \infty$, where $\zeta(s)$ is the Riemann zeta-function. The obtained ranges improve on an earlier result of the authors. An application to a weighted divisor problem is also given.

Keywords Riemann zeta-function • Hybrid moments • Exponent pairs • Asymptotic formula

1 Introduction

Let as usual $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ $(\Re s > 1)$ denote the Riemann zeta-function, where $s = \sigma + it$ is a complex variable. Mean values of $\zeta(s)$ in the so-called critical strip $\frac{1}{2} \le \sigma \le 1$ represent a central topic in the theory of the zeta-function (see, e.g., the monographs [10] and [9] for an extensive account). Of special interest are the

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moments on the so-called critical line $\sigma = \frac{1}{2}$. Unfortunately as of yet no bound of the form

$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^{2m} dt \ll_{\varepsilon,m} T^{1+\varepsilon} \tag{1}$$

is known to hold for any integer $m \geq 3$, while in the cases $m = 1, 2$ precise asymptotic formulas for the integrals in question are known (see op. cit.). Throughout this paper, ε denotes fixed small positive constants, not necessarily the same ones at each occurrence, while $\ll_{a,\dots}$ denotes the dependence of the $\ll$-constant on $a, \dots$.

Having in mind the difficulties of establishing (1) when $m \geq 3$, it appeared interesting to consider the following problem. For any fixed integer $j \geq 1$, let $\sigma^*_{4,j}$ $(\geq 1/2)$ denote the infimum of all σ for which the estimate

$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4 |\zeta(\sigma+it)|^{2j} dt \ll_{j,\varepsilon} T^{1+\varepsilon} \tag{2}$$

holds. The left-hand side of (2) may be called a "hybrid" moment, since it combines moments on the lines $\Re s = \frac{1}{2}$ and $\Re s = \sigma$. The problem is to estimate $\sigma^*_{4,j}$ for a given integer j. If the well-known Lindelöf hypothesis ($\zeta(1/2+it) \ll_\varepsilon |t|^\varepsilon$) is true, then $\sigma^*_{4,j} = \frac{1}{2}$ for any $j \geq 1$. However, up to now even $\sigma^*_{4,1} = \frac{1}{2}$ is out of reach by the use of existing methods. We cannot have $\sigma^*_{4,j} < \frac{1}{2}$ in view of the functional equation

$$\zeta(s) = \chi(s)\zeta(1-s), \quad \chi(s) := \frac{\Gamma(\frac{1}{2}(1-s))}{\Gamma(\frac{1}{2}s)}\pi^{s-1/2} \asymp |t|^{1/2-\sigma}.$$

In his work [11] the first author investigated the integral in (2) for the case $j = 1$ and the case $j = 2$. In particular, he proved that $\sigma^*_{4,1} \leq \frac{5}{6} = 0.8\bar{3}$, while if (k, ℓ) is an exponent pair (see, e.g., [4] or Chap. 2 of [10] for definitions) with $3k + \ell < 1$, then

$$\sigma^*_{4,2} \leq \max\left(\frac{\ell - k + 1}{2}, \frac{11k + \ell + 1}{8k + 2}\right),$$

which implies that $\sigma^*_{4,2} \leq 1953/1984 = 0.984375$. Since $\zeta(\sigma + it) \ll \log|t|$ for $\sigma \geq 1$, it is trivial that $\sigma^*_{4,j} \leq 1$ for any fixed $j \geq 1$. At the end of [11] it was stated, as an open problem, to prove the strict inequality $\sigma^*_{4,j} < 1$ for any fixed $j \geq 1$.

In [14], which is a continuation of [11], the authors proved that indeed $\sigma^*_{4,j} < 1$ holds for any fixed $j \geq 1$. In fact, if (k, ℓ) is an exponent pair with $\ell + (2j-1)k < 1$, then we showed that

$$\sigma^*_{4,j} \leq \frac{\ell + (6j-1)k}{1 + 4jk}.$$

In particular, we have $\sigma^*_{4,2} \leq \frac{37}{38} = 0.97368\cdots$.

In [14], which is a continuation of [11], we also considered the possibilities of sharpening (2) to an asymptotic formula. We showed that, for any given integer $j \ge 1$, there exists a number $\sigma_1 = \sigma_1(j)$ for which $\frac{3}{4} < \sigma_1 < 1$ such that, when $\sigma > \sigma_1$, there exists an asymptotic formula for the integral in (2) . This is

$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4 |\zeta(\sigma+it)|^{2j}\,dt \sim T\sum_{k=0}^{4} a_{k,j}(\sigma)\log^k T \quad (T\to\infty), \tag{3}$$

where all the coefficients $a_{k,j}(\sigma)$, which depend on σ and j, may be evaluated explicitly. However, in [14] we did not provide explicitly the range of σ for which (3) holds.

In this paper we shall provide some explicit values of σ for which (3) holds.

Theorem 1. *The asymptotic formula (3) holds in the following ranges:*

$$\begin{aligned}
&\sigma > \frac{4}{5} = 0.8 \ \ (j=1),\\
&\sigma > 0.904391\cdots \ \ (j=2),\\
&\sigma > 0.940001\cdots \ \ (j=3),\\
&\sigma > 0.959084\cdots \ \ (j=4),\\
&\sigma > 0.970734\cdots \ \ (j=5),\\
&\sigma > 0.978286\cdots \ \ (j=6).
\end{aligned}$$

Corollary 1. *We have*

$$\begin{aligned}
&\sigma^*_{4,1} \le \frac{4}{5} = 0.8 \ \ (j=1),\\
&\sigma^*_{4,2} \le 0.904391\cdots \ \ (j=2),\\
&\sigma^*_{4,3} \le 0.940001\cdots \ \ (j=3),\\
&\sigma^*_{4,4} \le 0.959084\cdots \ \ (j=4),\\
&\sigma^*_{4,5} \le 0.970734\cdots \ \ (j=5),\\
&\sigma^*_{4,6} \le 0.978286\cdots \ \ (j=6).
\end{aligned}$$

As an application of Theorem 1, we shall consider a weighted divisor problem. Suppose that $\ell \ge 1$ is a fixed integer and a is a fixed real number. Define the divisor function

$$d_{4,\ell}(n) = d_{4,\ell}(n;a) = \sum_{n=n_1n_2} d_4(n_1)d_\ell(n_2)n_2^{-a}, \tag{4}$$

where $d_k(n)$ denotes the number of ways n can be written as a product of k factors (so $d_k(n)$ is generated by $\zeta^k(s)$). If $a = 0$, then $d_{4,\ell}(n) \equiv d_{4+\ell}(n)$.

Henceforth we consider only the case $a > 0$. Suppose $X \geq 2$. It is expected that the summatory function $\sum_{n\leq X} d_{4,\ell}(n)$ is asymptotic to

$$X \sum_{k=0}^{3} c_{k,\ell}(a) \log^k X + X^{1-a} \sum_{k=0}^{\ell-1} c'_{k,\ell}(a) \log^k X$$

as $X \to \infty$, where the constants $c_{k,\ell}$ and $c'_{k,\ell}$ are effectively computable. More precisely, if one defines

$$E_{4,\ell}(X) := \sum_{n\leq X} d_{4,\ell}(n) - X \sum_{k=0}^{3} c_{k,\ell}(a) \log^k X - X^{1-a} \sum_{k=0}^{\ell-1} c'_{k,\ell}(a) \log^k X,$$

then we expect $E_{4,\ell}(X) = o(X)$ to hold as $X \to \infty$. Thus $E_{4,\ell}(X)$ should represent the error term in the asymptotic formula for $\sum_{n\leq X} d_{4,\ell}(n)$. It is also clear that the difficulty of the estimation of $E_{4,\ell}(x)$ increases with ℓ, and it also increases as a in (4) gets smaller.

By using (2) and the complex contour integration method, we can prove

Theorem 2. *If* $\max\left(\sigma^*_{4,j_0} - \frac{1}{2}, \frac{1}{2} - \frac{1}{\ell}\right) \leq a < \frac{1}{2}$, *then for* $\ell \geq 1$ *fixed we have*

$$E_{4,\ell}(X) \ll_\varepsilon x^{1/2+\varepsilon}, \tag{5}$$

where $j_0 = \frac{1}{2}\ell$ *if* ℓ *is even, and* $j_0 = \frac{1}{2}(\ell+1)$ *if* ℓ *is odd.*

From Theorem 2 and Corollary 1 we obtain at once

Corollary 2. *The estimate (5) holds for*

$$\frac{3}{10} < a < \frac{1}{2}, \quad (\ell = 1, 2),$$

$$0.404391\cdots < a < \frac{1}{2}, \quad (\ell = 3, 4),$$

$$0.440001\cdots < a < \frac{1}{2}, \quad (\ell = 5, 6),$$

$$0.459084\cdots < a < \frac{1}{2}, \quad (\ell = 7, 8),$$

$$0.470734\cdots < a < \frac{1}{2}, \quad (\ell = 9, 10),$$

$$0.478286\cdots < a < \frac{1}{2}, \quad (\ell = 11, 12).$$

2 The Necessary Lemmas

In order to prove our results, we require some lemmas which will be given in this section. The first lemma is the following upper bound for the fourth moment of $\zeta(\frac{1}{2}+it)$, weighted by a Dirichlet polynomial.

Lemma 2.1. *Let $a_1, a_2, \ldots, a_M$ be complex numbers. Then we have, for any $\varepsilon > 0$, $M \geq 1$ and $T \geq 1$,*

$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4 \Big|\sum_{m\leq M} a_m m^{it}\Big|^2 dt \ll_\varepsilon T^{1+\varepsilon} M(1+M^2T^{-1/2}) \max_{m\leq M} |a_m|^2. \quad (6)$$

This result is due to Watt [16]. It is founded on the earlier work of Deshouillers and Iwaniec [1], which involved the use of Kloosterman sums, but Watt's result is sharper.

We also need some results on power moments of $\zeta(s)$.

Lemma 2.2. *For any fixed $A \geq 4$, let us define $M(A)$ as*

$$M(A) = \begin{cases} \frac{A-4}{8}, & \text{if } 4 \leq A \leq 12, \\ \frac{3A-14}{22}, & \text{if } 12 \leq A \leq 178/13 = 13.6923\cdots, \\ \frac{416A-2416}{2665}, & \text{if } 178/13 \leq A \leq 20028/1313 = 15.253\cdots, \\ \frac{7A-36}{48}, & \text{if } 20028/1313 \leq A \leq 1836/101 = 18.178\cdots, \\ \frac{32(A-6)}{205}, & \text{if } A \geq 1836/101. \end{cases}$$

Then we have the estimate

$$\int_1^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^A dt \ll_\varepsilon T^{1+M(A)+\varepsilon}. \quad (7)$$

Proof. The case $4 \leq A \leq 178/13$ is contained in Theorem 8.2 of Ivić [10]. Now suppose that $A > 178/13$.

Suppose that $t_1 < t_2 < \cdots < t_R$ are real numbers which satisfy

$$|t_r| \leq T \ (r = 1, 2, \cdots, R), \ |t_s - t_r| \geq 1 \quad (1 \leq r \neq s \leq R),$$

and

$$\left|\zeta\left(\tfrac{1}{2}+it_r\right)\right| \geq V > 0 \qquad (r = 1, 2, \cdots, R).$$

The large values estimate (8.29) of Ivić [10] reads

$$R \ll TV^{-6}\log^8 T + T^{29/13}V^{-178/13}\log^{235/13} T \ll T^{29/13}V^{-178/13}\log^{235/13} T, \tag{8}$$

if we note that $\zeta(\frac{1}{2}+it) \ll_\varepsilon |t|^{32/205+\varepsilon}$ (see Huxley [6] and [7]).

We shall also use (8.33) of [10], namely

$$R \ll T^2V^{-12}\log^{16} T. \tag{9}$$

From (8) and (9) we obtain

$$\int_1^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^A dt \ll_\varepsilon \begin{cases} T^{2+\frac{3(A-12)}{22}+\varepsilon}, & \text{if } 12 \le A \le 178/13, \\ T^{\frac{29}{13}+\frac{32(A-178/13)}{205}+\varepsilon}, & \text{if } A \ge 178/13. \end{cases} \tag{10}$$

The formula (8.56) of Ivić [10] reads

$$R \ll \begin{cases} TV^{-6}\log^8 T, & \text{if } V \ge T^{11/72}\log^{5/4} T, \\ T^{15/4}V^{-24}\log^{61/2} T, & \text{if } V < T^{11/72}\log^{5/4} T. \end{cases} \tag{11}$$

From (9) and (11) we have

$$\int_1^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^A dt \ll_\varepsilon \begin{cases} T^{\max(1+\frac{32(A-12)}{205},2+\frac{7(A-12)}{48})+\varepsilon}, & \text{if } 12 \le A \le 24, \\ T^{1+\frac{32(A-6)}{205}+\varepsilon}, & \text{if } A \ge 24. \end{cases} \tag{12}$$

Now Lemma 2.2 for the case $A > 178/13$ follows from (10) and (12).

Lemma 2.3. *For* $1/2 < \sigma < 1$ *fixed we define* $m(\sigma)$ (≥ 4) *as the supremum of all numbers* m (≥ 4) *such that*

$$\int_1^T |\zeta(\sigma+it)|^m dt \ll_\varepsilon T^{1+\varepsilon} \tag{13}$$

for any $\varepsilon > 0$. *Then*

$$\begin{aligned} m(\sigma) &\ge 4/(3-4\sigma), && \tfrac{1}{2} < \sigma \le \tfrac{5}{8}, \\ m(\sigma) &\ge 10/(5-6\sigma), && \tfrac{5}{8} \le \sigma \le \tfrac{35}{54}, \\ m(\sigma) &\ge 19/(6-6\sigma), && \tfrac{35}{54} \le \sigma \le \tfrac{41}{60}, \end{aligned}$$

$$m(\sigma) \geq 2112/(859 - 948\sigma), \qquad \tfrac{41}{60} \leq \sigma \leq \tfrac{3}{4},$$

$$m(\sigma) \geq 12408/(4537 - 4890\sigma), \qquad \tfrac{3}{4} \leq \sigma \leq \tfrac{5}{6},$$

$$m(\sigma) \geq 4324/(1031 - 1044\sigma), \qquad \tfrac{5}{6} \leq \sigma \leq \tfrac{7}{8},$$

$$m(\sigma) \geq 98/(31 - 32\sigma), \qquad \tfrac{7}{8} \leq \sigma \leq 0.91591\ldots,$$

$$m(\sigma) \geq (24\sigma - 9)/(4\sigma - 1)(1 - \sigma),\ 0.91591\ldots \leq \sigma \leq 1 - \varepsilon.$$

Proof. This is Theorem 8.4 of Ivić [10]. In Ivić–Ouellet [13] some improvements have been obtained. Thus, it was shown there that $m(\sigma) \geq 258/(63 - 64\sigma)$ for $14/15 \leq \sigma \leq c_0$ and $m(\sigma) \geq (30\sigma - 12)/(4\sigma - 1)(1 - \sigma)$ for $c_0 \leq \sigma \leq 1 - \varepsilon$, where $c_0 = (171 + \sqrt{1602})/222 = 0.95056\cdots$.

Lemma 2.4. *Let $q \geq 1$ be an integer, $Q = 2^q$. Then for $|t| \geq 3$ we have*

$$\zeta\left(1 - \frac{q+2}{2^{q+2} - 2}\right) \ll |t|^{1/(2^{q+2}-2)} \log |t|. \tag{14}$$

We also have

$$\zeta(\tfrac{5}{7} + it) \ll_\varepsilon |t|^{0.07077534\cdots+\varepsilon} \qquad (|t| \geq 2). \tag{15}$$

Proof. The formula (14) is Theorem 2.12 of Graham and Kolesnik [4]. The estimate (15) is to be found on p. 66 of [4]. It improves (14) in the case when $q = 2$, when one obtains the exponent $\frac{1}{14} = 0.0714285\cdots$.

Lemma 2.5. *Suppose $\frac{1}{2} \leq \sigma_1 < \sigma_2 \leq 1$ are two real numbers such that*

$$\zeta(\sigma_j + it) \ll_\varepsilon |t|^{c(\sigma_j)+\varepsilon} \qquad (j = 1, 2),$$

then for $\sigma_1 \leq \sigma \leq \sigma_2$ we have

$$\zeta(\sigma + it) \ll_\varepsilon |t|^{c(\sigma_1)\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1} + c(\sigma_2)\frac{\sigma-\sigma_1}{\sigma_2-\sigma_1} + \varepsilon}. \tag{16}$$

Proof. This follows from the well-known Phragmén–Lindelöf principle (convexity); see, e.g., Sect. 8.2 of [10].

Lemma 2.6. *Let*

$$I(h,k) := \int_{-\infty}^{\infty} \left(\frac{h}{k}\right)^{-it} \zeta\left(\tfrac{1}{2} + \alpha + it\right) \zeta\left(\tfrac{1}{2} + \beta + it\right) \zeta\left(\tfrac{1}{2} + \gamma - it\right)$$
$$\times \zeta\left(\tfrac{1}{2} + \delta - it\right) w(t)dt,$$

where h, k are positive integers with $(h, k) = 1$, and $\alpha, \beta, \gamma, \delta$ are complex numbers $\ll 1/\log T$. Then for $hk \le T^{2/11-\varepsilon}$ we have

$$
\begin{aligned}
I(h,k) \ &= \frac{1}{\sqrt{hk}} \int_{-\infty}^{\infty} w(t) \Bigg\{ Z_{\alpha,\beta,\gamma,\delta,h,k}(0) + \left(\frac{t}{2\pi}\right)^{-\alpha-\beta-\gamma-\delta} Z_{-\gamma,-\delta,\alpha,-\beta,h,k}(0) \\
&\quad + \left(\frac{t}{2\pi}\right)^{-\alpha-\gamma} Z_{-\gamma,\beta,-\alpha,\delta,h,k}(0) + \left(\frac{t}{2\pi}\right)^{-\alpha-\delta} Z_{-\delta,\beta,-\gamma,-\alpha,h,k}(0) \\
&\quad + \left(\frac{t}{2\pi}\right)^{-\beta-\gamma} Z_{\alpha,-\gamma,-\beta,\delta,h,k}(0) + \left(\frac{t}{2\pi}\right)^{-\beta-\delta} Z_{\alpha,\delta,\gamma,-\beta,h,k}(0) \Bigg\} dt \\
&\quad + O_\varepsilon\Big(T^{3/4+\varepsilon}(hk)^{7/8}(T/T_0)^{9/4}\Big). \qquad (17)
\end{aligned}
$$

The function $Z_{\ldots}(0)$ is given in term of explicit, albeit complicated Euler products.

The formula (17) is due to Hughes and Young [5]. It is intended primarily for the asymptotic evaluation of the integral

$$
\int_0^T \left|\zeta\left(\tfrac12 + it\right)\right|^4 \left|M\left(\tfrac12 + it\right)\right|^2 dt, \qquad (18)
$$

where

$$
M(s) := \sum_{h \le T^\theta} a(h) h^{-s}
$$

is a Dirichlet polynomial of length T^θ with complex coefficients $a(h)$. The integral in (18) reduces to a sum of integrals of the type $I(h,k)$ after one develops $\left|M\left(\tfrac12 + it\right)\right|^2$ and chooses suitably the weight function $w(t)$, which is discussed below. In general, the evaluation of the integral in (18) is an important problem in analytic number theory. It was studied by Deshouillers and Iwaniec [1], Watt [16] and most recently by Motohashi [15], all of whom used powerful methods from the spectral theory of the non-Euclidean Laplacian. In [5] Hughes and Young obtained an asymptotic formula for (18) when $\theta = \frac{1}{11} - \varepsilon$. Two of the chief ingredients in their proof are an approximate functional equation for the product of four zeta values, and the so-called delta method of Duke et al. [2]. Watt's result (6) gives the expected upper bound $O_\varepsilon(T^{1+\varepsilon})$ in the range $\theta \le \frac14$, but does not produce an asymptotic formula for the integral in (18) (or (6)). At the end of [15], Motohashi comments on the value $\theta = \frac{1}{11} - \varepsilon$ of [5]. He says: "Our method should give a better result than theirs, if it is combined with works by N. Watt on this mean value."

Note that the bound $O_\varepsilon(T^{1+\varepsilon})$ for (18) with $T^\theta, \theta = \frac12$ would give the hitherto unproved sixth moment of zeta-function in the form

$$
\int_0^T \left|\zeta\left(\tfrac12 + it\right)\right|^6 dt \ \ll_\varepsilon \ T^{1+\varepsilon},
$$

which is (1) with $m = 3$.

The weight function $w(t)$ (≥ 0) which appears in the integral in (17) is a smooth function majorizing or minorizing the characteristic function of the interval $[T, 2T]$. The fact that the integrand in (18) is nonnegative makes this effective. We shall actually take two such functions: $w(t) = w_1(t)$ supported in $[T - T_0, 2T + T_0]$ such that $w_1(t) = 1$ for $t \in [T, 2T]$, and $w(t) = w_2(t)$ supported in $[T, 2T]$ such $w_2(t) = 1$ for $t \in [T - T_0, 2T - T_0]$. For an explicit construction of such a smooth function $w(t)$ see, e.g., Chap. 4 of the first author's monograph [9]. We then have, in either case, $w^{(r)}(t) \ll_r T_0^{-r}$ for all $r = 0, 1, 2, \ldots$, where T_0 is a parameter which satisfies $T^{1/2+\varepsilon} \ll T_0 \ll T$, and appears in the error term in (17).

3 Proof of Theorem 1

3.1 The Case When $j = 1$

In this subsection we shall prove Theorem 1 in the case when $j = 1$. However, we shall deal with the general case and restrict ourselves to $j = 1$ only at the end of the proof.

Suppose $T \geq 10$. It suffices to evaluate the integral

$$\int_T^{2T} \left|\zeta\left(\tfrac{1}{2} + it\right)\right|^4 |\zeta(\sigma + it)|^{2j}\, dt,$$

replace then T by $T2^{-j}$ for $j = 1, 2, \ldots$ and sum the resulting estimates. For convenience, henceforth we set $\mathscr{L} := \log T$. Let $s = \sigma + it$, $\frac{1}{2} < \sigma \leq 1$ and $T \leq t \leq 2T$. We begin with the well-known Mellin inversion integral (see, e.g., the Appendix of [10]),

$$e^{-x} = \frac{1}{2\pi i}\int_{(c)} x^{-w}\Gamma(w)\,dw \qquad (c > 0,\ x > 0), \tag{19}$$

where $\int_{(c)}$ denotes integration over the line $\Re w = c$.

Suppose $T^{1/11} \ll Y \ll T$ is a parameter to be determined later. In (19) we set $x = n/Y$, multiply by $d_j(n)n^{-s}$, and then sum over n. This gives

$$\sum_{n=1}^{\infty} d_j(n)e^{-n/Y}n^{-s} = \frac{1}{2\pi i}\int_{(2)} Y^w\zeta^j(s + w)\Gamma(w)\,dw. \tag{20}$$

Suppose σ_0 is fixed number which satisfies $\frac{1}{2} \leq \sigma_0 < \min(1, \sigma)$ and will be determined later. In (20) we shift the line of integration to $\Re w = \sigma_0 - \sigma$ and apply the residue theorem. The pole at $w = 1 - s$, which is of degree j, contributes the residue which is $\ll T^{-10}$, by Stirling's formula for $\Gamma(w)$. The pole at $w = 0$ contributes the residue $\zeta^j(s)$. Thus we have

$$\zeta^j(s) \tag{21}$$
$$= \sum_{n=1}^{\infty} d_j(n)e^{-n/Y}n^{-s} - \frac{1}{2\pi i}\int_{(\sigma_0-\sigma)} Y^w\zeta^j(s+w)\Gamma(w)dw + O(T^{-10}).$$

By the well-known elementary estimate

$$\sum_{n\le u} d_j(n) \ll u\log^{j-1}u$$

and partial summation it is easy to see that

$$\sum_{n>Y\mathscr{L}^2} d_j(n)e^{-n/Y}n^{-s} \ll T^{-10}.$$

By Stirling's formula for $\Gamma(w)$ again we have

$$\frac{1}{2\pi i}\int_{\Re w=\sigma_0-\sigma,|\Im w|>\mathscr{L}^2} Y^w\zeta^j(s+w)\Gamma(w)dw \ll T^{-10}.$$

Let $Y_1 := T^{1/11-\varepsilon}$. Inserting the above two estimates into (21) we can write

$$\zeta^j(s) = B_1(s) + B_2(s) + B_3(s) + B_4(s), \tag{22}$$

say, where

$$B_1(s) := \sum_{n\le Y_1} d_j(n)e^{-n/Y}n^{-s},$$
$$B_2(s) := \sum_{Y_1<n\le Y\mathscr{L}^2} d_j(n)e^{-n/Y}n^{-s},$$
$$B_3(s) := -\frac{1}{2\pi i}\int_{\Re w=\sigma_0-\sigma,|\Im w|\le\mathscr{L}^2} Y^w\zeta^j(s+w)\Gamma(w)dw,$$
$$B_4(s) := O(T^{-10}).$$

The partitioning in (22) is a new feature in the approach to this problem. The flexibility is present in the parameters Y and σ_0, which will allow us to use Lemmas 2.2 and 2.3, hence to connect our problem to the power moments of $|\zeta(\sigma+it)|$.

Therefore from (22) we have, since $|ab| \le \frac{1}{2}|a|^2 + \frac{1}{2}|b|^2$,

$$|\zeta(\sigma+it)|^{2j} = |B_1(\sigma+it)|^2 + \sum_{2\le k\le 4} O\left(|B_1(\sigma+it)B_k(\sigma+it)| + |B_k(\sigma+it)|^2\right).$$

Multiplying the above relation by $\left|\zeta\left(\frac{1}{2}+it\right)\right|^4$ and integrating, we obtain

$$\int_T^{2T} \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4 |\zeta(\sigma+it)|^{2j}\,dt = J_1 + \sum_{2\le k\le 4} O(J_k + J_k'), \tag{23}$$

say, where

$$J_k := \int_T^{2T} \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4 |B_k(\sigma+it)|^2 dt,$$
$$J_k' := \int_T^{2T} \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4 |B_1(\sigma+it)B_k(\sigma+it)|\,dt.$$

The main contribution to the integral in (3) will come from the integral J_1, with our choice $Y_1 = T^{1/11-\varepsilon}$. In [14], the authors evaluated the integral similar to J_1 with the help of the result of Hughes and Young (Lemma 2.6). Actually, disregarding the harmless factor $e^{-n/Y}$, the integral I_1 in (4.6) of [14] is just the integral J_1 if the parameter $Y = T^{1/(11j)-\varepsilon_1}$ therein is replaced by $Y_1 = T^{1/11-\varepsilon}$ defined above. For the sake of completeness we shall give the details of the evaluation of J_1. As a technical convenience, we consider instead of J_1 the weighted integral

$$J^* := \int_{-\infty}^{\infty} w(t) \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4 \left|\sum_{n\le Y_1} d_j(n)e^{-n/Y}n^{-\sigma-it}\right|^2 dt, \tag{24}$$

with $w(t) = w_j(t)$ $(\ge 0;\ j=1,2)$ as in the discussion following Lemma 2.6.

We note that

$$\int_{-\infty}^{\infty} w_2(t) \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4 \left|\sum_{n\le Y_1}\cdots\right|^2 dt \le J_1 \le \int_{-\infty}^{\infty} w_1(t) \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4 \left|\sum_{n\le Y_1}\cdots\right|^2 dt,$$

and we shall show that the same asymptotic formula holds for the integral with $w_1(t)$ and $w_2(t)$ above, which will show then that such a formula holds for J_1 as well. We write the square of the sum in (24) as

$$\left|\sum_{n\le Y_1} d_j(n)e^{-n/Y}n^{-\sigma-it}\right|^2 \tag{25}$$
$$= \sum_{m,n\le Y_1} d_j(m)d_j(n)e^{-m/Y}e^{-n/Y}\left(\frac{m}{n}\right)^{-it}(mn)^{-\sigma}$$
$$= \sum_{\delta\le Y_1}\delta^{-2\sigma} \sum_{h\le Y_1/\delta,\,k\le Y_1/\delta,\,(h,k)=1} d_j(\delta h)d_j(\delta k)e^{-\delta h/Y}e^{-\delta k/Y}(hk)^{-\sigma}\left(\frac{h}{k}\right)^{-it},$$

where we put $m = \delta h, n = \delta k, (h,k) = 1$. With the aid of (25) it follows that J^* reduces to the summation of integrals of the type

$$I^*(h,k) := \int_{-\infty}^{\infty} w(t) \left|\zeta\left(\tfrac{1}{2} + it\right)\right|^4 \left(\frac{h}{k}\right)^{-it} dt \qquad ((h,k) = 1).$$

We continue now the proof of Theorem 1, and we multiply (17) by

$$d_j(\delta h) d_j(\delta k) e^{-h/Y} e^{-k/Y} (hk)^{-\sigma}$$

and insert the resulting expression in (24). The error term in (17) makes a contribution which will be, since $d_j(n) \ll_\varepsilon n^\varepsilon$,

$$\begin{aligned} &\ll_\varepsilon \sum_{\delta \le Y_1} \delta^{\varepsilon - 2\sigma} \sum_{h \le Y_1/\delta, k \le Y_1/\delta} T^{3/4+\varepsilon} (hk)^{7/8-\sigma} (T/T_0)^{9/4} \\ &\ll_\varepsilon T^{3/4+\varepsilon} Y_1^{15/4 - 2\sigma} (T/T_0)^{9/4}. \end{aligned}$$

Note that $Y_1^{\frac{15}{4}-2\sigma} < Y_1^{\frac{11}{4}}$ because $\sigma > \frac{1}{2}$. Therefore we see, since $Y_1 = T^{\frac{1}{11}-\varepsilon}$, as in the discussion made in [5], that we obtain first the desired asymptotic formula, with an error term $O(T^{1-\varepsilon_1})$ for some $\varepsilon_1 > 0$, for the twisted integral J^* in (24), with $\left|\zeta\left(\frac{1}{2} + it\right)\right|^4$ replaced by

$$\zeta\left(\tfrac{1}{2} + \alpha + it\right) \zeta\left(\tfrac{1}{2} + \beta + it\right) \zeta\left(\tfrac{1}{2} + \gamma - it\right) \zeta\left(\tfrac{1}{2} + \delta - it\right).$$

Finally, if $\alpha, \beta, \gamma, \delta$ all tend to zero, we obtain the desired asymptotic formula

$$J_1 \sim T \sum_{k=0}^{4} b_{k;j}(\sigma) \log^k T \qquad (T \to \infty,), \tag{26}$$

and the coefficients $b_{k;j}(\sigma)$ depend on σ and j. It remains then to show that the contribution of J_k and J_k' in (23), for $2 \le k \le 4$, is of a lower order of magnitude than the right-hand side of (26), and Theorem 1 will follow.

We shall estimate the integral J_2 by Lemma 2.1. We split the range of summation in $B_2(s)$ into $O(\log T)$ ranges of summation of the form

$$Y_1 \le M < n \le M' \le 2M \ll Y \mathscr{L}^2.$$

Hence by Lemma 2.1 and the well-known elementary bound

$$d_j(n) \ll_\varepsilon n^\varepsilon \tag{27}$$

we have

$$\begin{aligned}
J_2 &\ll \mathscr{L} \max_{Y_1 \ll M \ll Y\mathscr{L}^2} \int_T^{2T} \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4 \left|\sum_{M<n\le M'\le 2M} d_j(n)e^{-n/Y}n^{-\sigma-it}\right|^2 dt \\
&\ll_\varepsilon \max_{Y_1 \ll M \ll Y\mathscr{L}^2} T^{1+\varepsilon}M(1+M^2T^{-1/2}) \max_{M<n\le 2M} d_j^2(n)e^{-2n/Y}n^{-2\sigma} \\
&\ll_\varepsilon \max_{Y_1 \ll M \ll Y\mathscr{L}^2} T^{1+\varepsilon}(M^{1-2\sigma}+M^{3-2\sigma}T^{-1/2}) \\
&\ll_\varepsilon \mathscr{L}^2 T^{1+3\varepsilon}Y_1^{1-2\sigma} + T^{1/2+3\varepsilon}Y^{3-2\sigma}\mathscr{L}^{8-4\sigma} \\
&\ll_\varepsilon T^{1+\varepsilon}Y_1^{1-2\sigma} + T^{1/2+\varepsilon}Y^{3-2\sigma}.
\end{aligned}$$

Since $Y_1 = T^{\frac{1}{11}-\varepsilon}$, we see that

$$J_2 \ll_\varepsilon T^{1-\varepsilon} \tag{28}$$

if

$$Y = T^{\frac{1}{6-4\sigma}-\varepsilon}, \tag{29}$$

and the condition $T^{\frac{1}{11}} \ll Y \ll T$ is seen to hold.

We turn now to the estimation of the integral J_3. From its definition we have

$$B_3(\sigma+it) \ll Y^{\sigma_0-\sigma}\int_{-\mathscr{L}^2}^{\mathscr{L}^2} |\zeta(\sigma_0+it+iv)|^j\,dv,$$

hence by using this bound and Cauchy's inequality we infer that

$$|B_3(\sigma+it)|^2 \ll Y^{2\sigma_0-2\sigma}\mathscr{L}^2\int_{-\mathscr{L}^2}^{\mathscr{L}^2} |\zeta(\sigma_0+it+iv)|^{2j}\,dv.$$

Thus by integration we have

$$J_3 \ll Y^{2\sigma_0-2\sigma}\mathscr{L}^2\int_T^{2T} \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4 \left(\int_{-\mathscr{L}^2}^{\mathscr{L}^2} |\zeta(\sigma_0+it+iv)|^{2j}\,dv\right)dt.$$

Suppose now that σ_0, besides $\frac{1}{2} \le \sigma_0 < \min(1,\sigma)$, also satisfies the condition

$$m(\sigma_0) > 2j. \tag{30}$$

Let

$$q := \frac{m(\sigma_0)}{2j}, \quad p := \frac{m(\sigma_0)}{m(\sigma_0)-2j}. \tag{31}$$

Then

$$p > 1, \; q > 1, \; \frac{1}{p} + \frac{1}{q} = 1,$$

and by Hölder's inequality for integrals we obtain

$$J_3 \ll Y^{2\sigma_0 - 2\sigma} \mathscr{L}^2 \left(\int_T^{2T} \left| \zeta\left(\tfrac{1}{2} + it\right) \right|^{4p} dt \right)^{\frac{1}{p}} \times \left(\int_T^{2T} \left(\int_{-\mathscr{L}^2}^{\mathscr{L}^2} |\zeta(\sigma_0 + it + iv)|^{2j} dv \right)^q dt \right)^{\frac{1}{q}}. \tag{32}$$

We have

$$\int_T^{2T} \left| \zeta\left(\tfrac{1}{2} + it\right) \right|^{4p} dt \;\ll_\varepsilon\; T^{1+M(4p)+\varepsilon}, \tag{33}$$

where we shall use the bounds for $M(A)$ furnished by Lemma 2.2. By Hölder's inequality again we have

$$\begin{aligned} &\left(\int_{-\mathscr{L}^2}^{\mathscr{L}^2} |\zeta(\sigma_0 + it + iv)|^{2j} dv \right)^q \\ &\ll \int_{-\mathscr{L}^2}^{\mathscr{L}^2} |\zeta(\sigma_0 + it + iv)|^{2jq} dv \times \left(\int_{-\mathscr{L}^2}^{\mathscr{L}^2} 1 dv \right)^{\frac{q}{p}} \\ &\ll \mathscr{L}^{\frac{2q}{p}} \int_{-\mathscr{L}^2}^{\mathscr{L}^2} |\zeta(\sigma_0 + it + iv)|^{2jq} dv. \end{aligned}$$

Therefore

$$\begin{aligned} &\int_T^{2T} \left(\int_{-\mathscr{L}^2}^{\mathscr{L}^2} |\zeta(\sigma_0 + it + iv)|^{2j} dv \right)^q dt \\ &\ll \mathscr{L}^{\frac{2q}{p}} \int_T^{2T} \left(\int_{-\mathscr{L}^2}^{\mathscr{L}^2} |\zeta(\sigma_0 + it + iv)|^{2jq} dv \right) dt \\ &= \mathscr{L}^{\frac{2q}{p}} \int_{-\mathscr{L}^2}^{\mathscr{L}^2} dv \int_T^{2T} |\zeta(\sigma_0 + it + iv)|^{2jq} dt \\ &= \mathscr{L}^{\frac{2q}{p}} \int_{-\mathscr{L}^2}^{\mathscr{L}^2} dv \int_T^{2T} |\zeta(\sigma_0 + it + iv)|^{m(\sigma_0)} dt \\ &\ll_\varepsilon T^{1+\varepsilon}. \end{aligned} \tag{34}$$

From (32)–(34) and (29) we obtain

$$J_3 \ll_\varepsilon Y^{2\sigma_0-2\sigma}\mathscr{L}^2\left(T^{1+M(4p)+\varepsilon}\right)^{\frac{1}{p}}\left(T^{1+\varepsilon}\mathscr{L}^{\frac{2q}{p}+2}\right)^{\frac{1}{q}} \tag{35}$$

$$\ll_\varepsilon Y^{2\sigma_0-2\sigma}T^{1+\frac{M(4p)}{p}+\varepsilon}\mathscr{L}^4 \ll_\varepsilon T^{1-\varepsilon}$$

if $T^{\frac{M(4p)}{p}+3\varepsilon} \ll Y^{2\sigma-2\sigma_0}$. With the choice (29) this condition reduces to

$$\sigma > \frac{\frac{3M(4p)}{p}+\sigma_0}{\frac{2M(4p)}{p}+1}. \tag{36}$$

To bound the integrals J_k' (see (23)) note that from (26), (28), (35) and Cauchy's inequality for integrals we obtain

$$J_k' \le J_1^{1/2}J_k^{1/2} \ll T^{1-2\varepsilon}\mathscr{L}^2 \ll T^{1-\varepsilon} \qquad (k = 2, 3). \tag{37}$$

Obviously we have

$$J_4 \ll T^{-18}, \tag{38}$$

and consequently

$$J_4' \ll T^{-16}. \tag{39}$$

From (26), (28), (35) and (37)–(39) we obtain that, if (30) and (36) hold,

$$\int_T^{2T}\left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4|\zeta(\sigma+it)|^2dt \sim T\sum_{k=0}^{4}b_{k;j}(\sigma)\log^k T \qquad (T\to\infty),$$

This implies that

$$\int_1^{T}\left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4|\zeta(\sigma+it)|^2dt \sim T\sum_{k=0}^{4}a_{k;j}(\sigma)\log^k T \qquad (T\to\infty),$$

where the $a_{k;j}$'s are constants which are easily expressible in term of the $b_{k;j}$'s.

Now we determine the permissible range of σ from (36) for the case $j = 1$. We take $\sigma_0 = \frac{5}{8}$. Lemma 2.3 gives $m(\sigma_0) = m(\frac{5}{8}) \ge 8$, so (29) holds, and $p = \frac{4}{3}$. Then (36) reduces to $\sigma > \frac{4}{5}$.

Remark 1. When $j = 2, 3, 4$, the above procedure can also give nontrivial results. Actually, when $j = 2$, we take $\sigma_0 = \frac{35}{54}$ and (36) becomes $\sigma > \frac{71}{78} = 0.91025\cdots$. When $j = 3$, we take $\sigma_0 = \frac{5}{6}$ and (36) becomes $\sigma > \frac{659}{690} = 0.95507\cdots$. When

$j = 4$, we take $\sigma_0 = \frac{7}{8}$ and (36) becomes $\sigma > \frac{221}{229} = 0.96506\cdots$. However, in Sect. 3.2 we shall give better ranges for σ in these three cases.

Remark 2. When $j > 4$, the above method does not give good results in view of the existing bounds for the functions $M(A)$ and $m(\sigma)$ defined in Lemmas 2.2 and 2.3, respectively. However, in that case it is not difficult to see that (3) holds for $\sigma > \sigma^*_{4,j}$, the infimum of numbers for which (2) holds. Thus (3) will hold for $\sigma > (\ell + (6j-1)k)/(1+4jk)$ when (k,ℓ) is an exponent pair. To see this, note first that the discussion preceding (30) yields

$$J_3 \ll Y^{2\sigma_0 - 2\sigma}\mathscr{L}^2 \max_{|v|\le\mathscr{L}^2}\int_T^{2T} \left|\zeta\left(\tfrac12 + it\right)\right|^4 |\zeta(\sigma_0 + it + iv)|^{2j}\,dt. \tag{40}$$

This is almost the same integral as the initial one, and the conclusion of Theorem 1 of our joint paper [14] holds, namely

$$\int_0^T \left|\zeta\left(\tfrac12 + it\right)\right|^4 |\zeta(\sigma + it + iv)|^{2j}\,dt \ll_{j,\varepsilon} T^{1+\varepsilon} \quad (|v| \le \mathscr{L}^2), \tag{41}$$

if

$$\sigma > \sigma_0 = \frac{\ell + (6j-1)k}{1+4jk}, \quad \ell + (2j-1)k < 1, \tag{42}$$

and (k,ℓ) is an exponent pair. With σ_0 as in (41) and $\sigma \ge \sigma_0 + \delta$ ($\delta > 0$) one has trivially $J_3 \ll T^{1-1/11}$ for δ, ε sufficiently small (since $Y^{2\sigma_0-\sigma} \ll T^{-2/11+2\delta\varepsilon}$), and we get an asymptotic formula for the initial integral in the range $\sigma > \sigma_0$ for $j > 4$.

Remark 3. We may further discuss the asymptotic formula (3). Denote by, say, $E(T;\sigma,j)$ the difference between the left and right-hand side in (3), thus $E(T;\sigma,j)$ is the error term in the asymptotic formula for our integral. Let $c(\sigma,j)$ be the infimum of numbers c such that, for a given integer $j \ge 1$,

$$E(T;\sigma,j) \ll T^c.$$

We know that $c(\sigma,j) < 1$ by [14], and it seems reasonable to expect that $c(\sigma,j) \ge \frac12$. Namely in case when $j = 0$, we have the fourth moment of $|\zeta(1/2+it)|$, and in this case a precise asymptotic formula is known, and the exponent of the error term cannot be smaller than $\frac12$ (see [12]). However, obtaining any qualitative results on $c(\sigma,j)$ will be difficult, one of the reasons being that it is hard from the method of Hughes and Young [5] to get explicit O-estimates for the error terms in their formulas.

3.2 The Case When $j \geq 2$

To deal with the case $j \geq 2$ we shall use an induction method. Namely, for each $j \geq 1$, we shall prove that there is a constant $\frac{1}{2} < c_j < 1$ such that (41) holds in the range $\sigma > c_j$. When $j = 1$, we can take $c_1 = \frac{4}{5}$ from the result in Sect. 3.1.

Let $C(\sigma) > 0$ be a function which connects $(\frac{5}{7}, 0.07077534\cdots)$ and the points (a_q, b_q) $(q \geq 3)$ with line segments, where

$$a_q := 1 - \frac{q+2}{2^{q+2}-2},\ b_q := \frac{1}{2^{q+2}-2}, \tag{43}$$

and $q = q(j)$ will be suitably chosen. We then have, in view of Lemmas 2.4 and 2.5,

$$\zeta(\sigma + it) \ll_\varepsilon |t|^{C(\sigma)+\varepsilon} \quad (\sigma \geq \tfrac{5}{7}). \tag{44}$$

Now we suppose that $j \geq 2$ and we have already defined c_l for any $1 \leq l < j$. From (40) and (44) we have

$$\begin{aligned} J_3 &\ll Y^{2\sigma_0-2\sigma}\mathscr{L}^2 \max_{|v|\leq\mathscr{L}^2}\int_T^{2T} \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4 |\zeta(\sigma_0+it+iv)|^{2+2(j-1)}\,dt \qquad (45)\\ &\ll_\varepsilon Y^{2\sigma_0-2\sigma}T^{2C(\sigma_0)+2\varepsilon}\mathscr{L}^2 \max_{|v|\leq\mathscr{L}^2}\int_T^{2T} \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4 |\zeta(\sigma_0+it+iv)|^{2(j-1)}\,dt\\ &\ll_\varepsilon Y^{2\sigma_0-2\sigma}T^{2C(\sigma_0)+1+3\varepsilon}\\ &\ll_\varepsilon T^{\left(\frac{1}{6-4\sigma}-\varepsilon\right)(2\sigma_0-2\sigma)+2C(\sigma_0)+1+3\varepsilon}\\ &\ll_\varepsilon T^{\frac{\sigma_0-\sigma}{3-2\sigma}+2C(\sigma_0)+1+3\varepsilon} \end{aligned}$$

if $\sigma_0 > c_{j-1}$.

Take $\sigma_0 = c_{j-1} + \delta$, where $\delta > 0$ is a small positive constant. When

$$\sigma > \frac{6C(c_{j-1}) + c_{j-1}}{4C(c_{j-1}) + 1}, \tag{46}$$

from (45) we have

$$J_3 \ll_\varepsilon T^{1-\varepsilon}$$

if δ, ε are sufficiently small.

We define the sequence c_j $(j \geq 1)$ as follows:

$$c_1 = \frac{4}{5}, \quad c_j := \frac{6C(c_{j-1}) + c_{j-1}}{4C(c_{j-1}) + 1} \qquad (j \geq 2). \tag{47}$$

It is easy to see that $c_j < 1$ for any j since $C(\sigma) < \frac{1}{2}(1-\sigma)$. From the above procedure and the results in Sect. 3.1 we see that the asymptotic formula (3) holds for $\sigma > c_j$ for any $j \geq 2$.

We provide now the explicit values of c_j when $j = 2, 3, 4, 5, 6$, and we remark that continuing in this fashion we could obtain the values for $j > 6$ as well.

1. The case $j = 2$: from Lemma 2.4 we have $C(\frac{5}{7}) = 0.07077534\cdots$, $C(\frac{5}{6}) = \frac{1}{30} = 0.03333333\cdots$. Thus from Lemma 2.5 we have $C(\frac{4}{5}) = 0.0438170952\cdots$. Hence
$$c_2 = \frac{6C(4/5) + 4/5}{4C(4/5) + 1} = 0.904391\cdots.$$
2. The case $j = 3$: from (3.25) we have $a_4 = \frac{28}{31} < c_2 < a_5 = \frac{119}{126}$. From Lemma 2.4 we have $C(\frac{28}{31}) = \frac{1}{62}$, $C(\frac{119}{126}) = \frac{1}{126}$. From Lemma 2.5 we get $C(c_2) = 0.01589736\cdots$. Hence
$$c_3 = \tfrac{6C(c_2)+c_2}{4C(c_2)+1} = 0.9400013\cdots.$$
3. The case $j = 4$: we have $a_4 = \frac{28}{31} < c_3 < a_5 = \frac{119}{126}$. From Lemma 2.5 we get $C(c_3) = 0.008819601\cdots$. Hence
$$c_4 = \tfrac{6C(c_3)+c_3}{4C(c_3)+1} = 0.959084\cdots.$$
4. The case $j = 5$: we have $a_5 = \frac{119}{126} < c_4 < a_6 = \frac{123}{127}$. From Lemma 2.5 we get $C(c_4) = 0.005502913\cdots$. Hence
$$c_5 = \tfrac{6C(c_4)+c_4}{4C(c_4)+1} = 0.970734\cdots.$$
5. The case $j = 6$: we have $a_6 = \frac{123}{127} < c_5 < a_7 = \frac{501}{510}$. From Lemma 2.5 we get $C(c_5) = 0.0035902\cdots$. Hence
$$c_6 = \tfrac{6C(c_5)+c_5}{4C(c_5)+1} = 0.978286\cdots.$$

Remark 4. It is not difficult to evaluate additional values of c_j. For example, we have $c_7 = 0.983536$, $c_8 = 0.987254$, $c_9 = 0.990005$, $c_{10} = 0.992046$, $c_{11} = 0.993616$. When j large, the value of c_j is close to 1.

Remark 5. The values of c_j $(j \geq 2)$ depend on the upper bound of $\zeta(\sigma + it)$. Therefore we can improve the values of c_j $(j \geq 2)$ if we have better upper bounds for $\zeta(\sigma + it)$. For example, instead of Lemma 2.4 (Th. 2.12 of Graham-Kolesnik [4]), we could use Theorem 4.2 of theirs (p. 38), which is strong for any $q \geq 1$. Then we can get small improvements for any $j \geq 2$. We also remark that we have (see (7.57) of [10])

$$\zeta(\sigma + it) \ll t^{(k+\ell-\sigma)/2} \log t \qquad \left(\sigma \geq \frac{1}{2}, \ell - k \geq \sigma\right),$$

where (k, ℓ) is an exponent pair. A judicious choice of the exponent pair (k, ℓ), especially the use of new exponent pairs due to Huxley (see, e.g., his papers [7] and [8]), would likely lead to some further small improvements.

Kevin Ford [3] proved

$$|\zeta(\sigma + it)| \leq 76.2 t^{4.45(1-\sigma)^{3/2}} \log^{2/3} t$$

for $\frac{1}{2} \leq \sigma \leq 1, t \geq 3$. This estimate is quite explicit, and best when σ is close to 1. This estimate would imply better values of c_j when j is large. There is, however, no simple procedure which yields (in closed form) the range for σ for which the asymptotic formula (3) holds, for any given j.

4 Proof of Theorem 2

In this section we shall prove Theorem 2. By the definition of the generalized divisor function $d_k(n)$ we have, for $0 \leq a < \frac{1}{2}$ and $\Re s > 1$,

$$\sum_{n=1}^{\infty} d_{4,\ell}(n) n^{-s} = \sum_{n_1, n_2 = 1}^{\infty} d_4(n_1) d_\ell(n_2) n_2^{-a} (n_1 n_2)^{-s} = \zeta^4(s) \zeta^\ell(s + a). \tag{48}$$

By using Perron's inversion formula (see, e.g., the Appendix of [10]) we have

$$\sum_{n \leq X} d_{4,\ell}(n) = \frac{1}{2\pi i} \int_{1+\varepsilon-iX}^{1+\varepsilon+iX} \zeta^4(s) \zeta^\ell(s + a) \frac{X^s}{s} ds + O_\varepsilon(X^\varepsilon), \tag{49}$$

if we note that $d_{4,\ell}(n) \ll_{\varepsilon,\ell} n^\varepsilon$. Now we put $j_0 = \frac{1}{2}\ell$ if ℓ is even, and $j_0 = \frac{1}{2}(\ell + 1)$ if ℓ is odd, and then move the line of integration in (49) to $\sigma = \frac{1}{2}$. In doing this we encounter two poles. These are $s = 1$, a pole of order four, and $s = 1 - a$ which is a pole of order ℓ. It is easy to verify that the sum of residues of the integrand in (4) is of the form (4). Thus from (5), (49) and the residue theorem we obtain

$$E_{4,\ell}(X) = I_1 + I_2 - I_3 + O_\varepsilon(X^\varepsilon), \tag{50}$$

say, where

$$I_1 := \frac{1}{2\pi i}\int_{\frac12 - iX}^{\frac12 + iX} \zeta^4(s)\zeta^\ell(s+a)\frac{X^s}{s}ds,$$
$$I_2 := \frac{1}{2\pi i}\int_{\frac12 + iX}^{1+\varepsilon+iX} \zeta^4(s)\zeta^\ell(s+a)\frac{X^s}{s}ds,$$
$$I_3 := \frac{1}{2\pi i}\int_{\frac12 - iX}^{1+\varepsilon-iX} \zeta^4(s)\zeta^\ell(s+a)\frac{X^s}{s}ds.$$

For $\zeta(s)$ we have the bounds

$$\zeta(\sigma+it) \ll \begin{cases} (2+|t|)^{\frac{1-\sigma}{3}}\log(2+|t|), & \text{if } \frac12 \le \sigma \le 1, \\ \log(2+|t|), & \text{if } 1 \le \sigma \le 2. \end{cases} \tag{51}$$

which follows from the standard bounds

$$\zeta(\tfrac12 + it) \ll |t|^{\frac16}\log|t|, \quad \zeta(\sigma+it) \ll \log|t| \quad (\sigma \ge 1, |t| \ge 2)$$

and convexity (see, e.g., (1.67) of [10]). Recalling the condition

$$\max\left(\sigma^*_{4,j_0} - \frac12, \frac12 - \frac1\ell\right) \le a < \frac12,$$

we obtain

$$\begin{aligned} &|I_2| + |I_3| \\ &\ll \int_{\frac12}^{1-a} X^{\frac{4(1-\sigma)}{3} + \frac{\ell(1-\sigma-a)}{3} - 1} X^\sigma \log^{4+\ell} X d\sigma \\ &\quad + \int_{1-a}^{1} X^{\frac{4(1-\sigma)}{3} - 1} X^\sigma \log^{4+\ell} X d\sigma + \int_1^{1+\varepsilon} X^{-1}X^\sigma \log^{4+\ell} X d\sigma \\ &\ll X^{\frac12}\log^{4+\ell} X. \end{aligned} \tag{52}$$

Now we estimate I_1. When ℓ is even, we obtain directly $I_1 \ll_\varepsilon X^{1/2+\varepsilon}$, since $j_0 = \frac12\ell$. Therefore we consider in detail the case when ℓ is odd. Let $j_1 = \ell - j_0 = \frac12(\ell-1) = j_0 - 1$. Then we have by Cauchy's inequality that

$$\begin{aligned}
&\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4 \left|\zeta\left(\tfrac{1}{2}+a+it\right)\right|^{\ell} dt \\
&\qquad = \int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4 \left|\zeta\left(\tfrac{1}{2}+a+it\right)\right|^{j_0+j_1} dt \\
&\ll \left(\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4 \left|\zeta\left(\tfrac{1}{2}+a+it\right)\right|^{2j_0} dt\right)^{1/2} \\
&\qquad \times \left(\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4 \left|\zeta\left(\tfrac{1}{2}+a+it\right)\right|^{2j_1} dt\right)^{1/2}.
\end{aligned}$$

Since $\sigma^*_{4,j_0} - \frac{1}{2} \le a$, we have $\frac{1}{2} + a \ge \sigma^*_{4,j_0}$. Hence from (2) we obtain

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4 \left|\zeta\left(\tfrac{1}{2}+a+it\right)\right|^{2j_0} dt \ll_\varepsilon T^{1+\varepsilon}.$$

Similarly we have

$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4 \left|\zeta\left(\tfrac{1}{2}+a+it\right)\right|^{2j_1} dt \ll_\varepsilon T^{1+\varepsilon}.$$

From the above three estimates we obtain

$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4 \left|\zeta\left(\tfrac{1}{2}+a+it\right)\right|^{\ell} dt \ll_\varepsilon T^{1+\varepsilon},$$

which combined with integration by parts gives

$$I_1 \ll_\varepsilon X^{\frac{1}{2}+\varepsilon}. \tag{53}$$

By combining (50), (52), and (53) we complete the proof of Theorem 2.

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On the Invertibility of Some Elliptic Operators on Manifolds with Boundary and Cylindrical Ends

Mirela Kohr and Cornel Pintea

Abstract In this paper we perform several steps towards the layer potential theory for the Brinkman system on manifolds with boundary and cylindrical ends. In addition, we refer to the Dirichlet problem for a Laplace type operator on parallelizable manifolds with cylindrical ends.

Keywords Boundary value problem • Brinkman operator • Deformation operator • Laplace type operator • Layer potential operators • Manifold with boundary and cylindrical ends • Parallelizable manifold • Pseudodifferential operator • Translation invariant operator in a neighborhood of infinity

Mathematics Subject Classification (2000): 35J25; 42B20; 46E35; 76D; 76M.

1 Introduction

The methods of layer potential theory are frequently used, due to their effectiveness, to analyze elliptic boundary value problems both on compact and noncompact manifolds. For example, the case of the Laplace operator on compact Riemannian manifolds has been intensively studied by Mitrea and Taylor [16–21] (see also [13]). Also, the cases of the Stokes and (generalized) Brinkman systems, as well as the Navier–Stokes system, have been considered by Mitrea and Taylor [16], Dindoš and Mitrea [4], and Kohr, Pintea and Wendland [10, 11]. The role of potential theory in the study of elliptic boundary value problems on Lipschitz domains in Euclidean

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setting is well emphasized by the pioneering works of Costabel [2], Fabes, Kenig, and Verchota [7], Dahlberg, Kenig and Verchota [3], Jerison and Kenig [9], Verchota [24]. Recent relevant works in this setting are due to Fabes, Mendez and Mitrea [6], Escauriaza and Mitrea [5], Hsiao and Wendland [8], Mitrea and Wright [23]. The layer potential approach for the Laplace operator on more general noncompact manifolds, i.e., manifolds with cylindrical ends, was developed by Mitrea and Nistor [15] (see also [14]). For harmonic differential forms in nonsmooth domains we refer the reader to [12].

In this paper we show the invertibility of some operators associated with the Brinkman system on manifolds with boundary and cylindrical ends. In addition, we refer to the Dirichlet problem for a Laplace type operator on parallelizable manifolds with cylindrical ends.

2 Pseudodifferential Operators

Let us denote by $OPS^{\ell}(\mathbb{R}^n) \equiv OPS^{\ell}$ the class of pseudodifferential operators $p(x, D)$ of order ℓ on $\mathbb{R}^n$. Also, by $OPS^{\ell}_{\rm cl}$ we denote the class of classical pseudodifferential operators of order ℓ on $\mathbb{R}^n$. Recall that an operator $P(x, D)$ is called a *classical pseudodifferential operator of order* ℓ if its symbol $p(x, \xi)$ has the asymptotic expansion

$$p(x, \xi) \sim p_{\ell}(x, \xi) + p_{\ell-1}(x, \xi) + \cdots , \tag{1}$$

where $p_k(x, \xi)$ is smooth in x and ξ and homogeneous of degree k in ξ (for $|\xi| \geq 1$), i.e., $p_k(x, t\xi) = t^k p_k(x, \xi)$, $\forall\, t > 0, k = \ell, \ell-1, \ldots$ The meaning of (1) is that, for each $k \geq 1$, the difference between the left-hand side and the sum of the first k terms on the right-hand side belongs to Hörmander's class $S^{\ell-k}_{1,0}$ (see, e.g., [8, Chap. 6]). We denote the class $S^{\ell-k}_{1,0}$ simply by $S^{\ell-k}$. The term $p_{\ell}(x, \xi)$ in (1) is called the *principal symbol* of $P(x, D)$ and is denoted by $\sigma^0(P; x, \xi)$. Recall that a Schwartz kernel, alongside the corresponding integral representation, can be associated with every pseudodifferential operator (see, e.g., [16, p. 186]). Let $OPS^{\ell}(\Omega)$ be the set of pseudodifferential operators of order ℓ on an open set $\Omega \subseteq \mathbb{R}^n$.

If M is a smooth n-dimensional manifold, we say that $P : C^{\infty}_0(M) \to C^{\infty}(M)$ belongs to the class $OPS^{\ell}(M)$ of pseudodifferential operators of order ℓ on M if the Schwartz kernel of P is smooth off the diagonal in $M \times M$ and if for any coordinate neighborhood U in M with $\chi : U \to \mathscr{O}$ a diffeomorphism onto an open set $\mathscr{O}$ of $\mathbb{R}^n$, the map $C^{\infty}_0(\mathscr{O}) \to C^{\infty}_0(\mathscr{O}),\ u \mapsto P(u \circ \chi) \circ \chi^{-1}$ belongs to $OPS^{\ell}(\mathscr{O})$. If E, F are vector bundles over the smooth manifold M, one can similarly define the concept of pseudodifferential operator $P : \Gamma(E) \to \Gamma(F)$ of order ℓ, where $\Gamma(E)$ and $\Gamma(F)$ are the $C^{\infty}(M)$-modules of smooth sections of E and F, respectively. We denote by $OPS^{\ell}(E, F)$ the collection of all such pseudodifferential operators of order ℓ (for more details see e.g., [25]).

3 Manifolds with Cylindrical Ends

Recall that a *manifold M with cylindrical ends* (see [15]) consists of a compact manifold M_1 with boundary $\partial M_1 \neq \emptyset$ and a decomposition

$$M := M_1 \cup (\partial M_1 \times (-\infty, 0]) \,, \tag{2}$$

in which the boundary of M_1 is identified with $\partial M_1 \times \{0\}$ and the union is considered along the boundaries of M_1 and $\partial M_1 \times (-\infty, 0]$. The Riemannian structure on M is provided by the Riemannian metric

$$g(x,t) = g_{\partial M_1}(x) + dt^2, \tag{3}$$

where $g_{\partial M_1}$ is a Riemannian metric on the boundary of M_1.

If $M = M_1 \cup (\partial M_1 \times (-\infty, 0])$ is a manifold with cylindrical ends and $s \geq 0$, consider the translation with $-s$ in the x-direction

$$\phi_s : \partial M_1 \times (-\infty, 0] \to \partial M_1 \times (-\infty, -s]. \tag{4}$$

If $s < 0$, then ϕ_s is defined as ϕ_{-s}^{-1}.

For boundary value problems in our context, the manifolds with boundary and cylindrical ends are the proper objects to be considered.

Definition 1 ([15]). Let N be a Riemannian manifold with boundary ∂N. We call N a *manifold with boundary and cylindrical ends* if there exists an open subset V of N isometric to $(-\infty, 0) \times X$, where X is a compact manifold with boundary, such that $N \setminus V$ is compact.

Lemma 1 ([15]). *Let N be a Riemannian manifold with boundary ∂N. Then N is a manifold with boundary and cylindrical ends if and only if there exists a manifold with cylindrical ends (without boundary) M with a standard decomposition*

$$M = M_1 \cup (\partial M_1 \times (-\infty, 0])$$

and containing N such that $N \cap (\partial M_1 \times (-\infty, 0]) = X \times (-\infty, 0]$ for some compact manifold with boundary $X \subset \partial M_1$.

This result has been obtained in [15, Lemma 5.2] based on the property that any metric of N is equivalent to a product metric in a small tubular neighborhood of ∂N. For a general tubular neighborhood theorem, we refer the reader to [1].

4 Translation Invariant Operators in a Neighborhood of Infinity

In this section we consider operators acting on one forms, on a manifold with cylindrical ends that are translation invariant in a neighborhood of infinity (see, e.g., [15, Definition 1.1] in the case of translation invariant in a neighborhood of infinity operators acting on functions).

Definition 2. A *translation invariant in a neighborhood of infinity operator* is a linear and continuous map $P : C_0^\infty(M, \Lambda^1 TM) \to C^\infty(M, \Lambda^1 TM)$ whose Schwartz kernel is supported in a neighborhood of the diagonal

$$\{(x, y) \in M \times M : \mathrm{dist}(x, y) < \varepsilon\}$$

for some $\varepsilon > 0$, and there exists $R > 0$ such that

$$P(\Phi_s^* \omega) = \Phi_s^*(P\omega),$$

for all $s > 0$ and all $\omega \in C_0^\infty\left(\partial M_1 \times (-\infty, -R], \Lambda^1 T\,(\partial M_1 \times (-\infty, -R])\right)$. The *space of all classical pseudodifferential operators of order m that are translation invariant in a neighborhood of infinity* is denoted by $\Psi^m_{\mathrm{inv}}(M, \Lambda^1 TM)$.

Note that the Riemannian metric (3) is translation invariant in a neighborhood of infinity in the sense that $\Phi_s^* g = g$ for sufficiently large absolute value of s. The same invariance property holds for the Levi–Civita connection (for more details see [15]).

Next, we use the notations

$$\Psi^{-\infty}_{\mathrm{inv}}(M, \Lambda^1 TM) := \bigcap_{m \in \mathbb{Z}} \Psi^m_{\mathrm{inv}}(M, \Lambda^1 TM), \;\; \Psi^{\infty}_{\mathrm{inv}}(M, \Lambda^1 TM) := \bigcup_{m \in \mathbb{Z}} \Psi^m_{\mathrm{inv}}(M, \Lambda^1 TM).$$

The following lemma can be obtained by means of similar arguments to the case of operators from $C_0^\infty(M)$ to $C^\infty(M)$ (see, e.g., [15, Lemma 1.2]).

Lemma 2. *If M is an n-dimensional manifold with cylindrical ends, then every operator P in the class $\Psi^{-n-1}_{\mathrm{inv}}(M, \Lambda^1 TM)$ induces a bounded operator on $L^2(M, \Lambda^1 TM)$.*

For each $s \in \mathbb{R}$, we denote by $H^s(M, \Lambda^1 TM)$ the Sobolev space of one forms, defined by $H^s(M, \Lambda^1 TM) := H^s(M) \otimes C^\infty(M, \Lambda^1 TM)$.

Let $\sigma_m(P) \in S^m(T^*M, \Lambda^1 TM)/S^{m-1}(T^*M, \Lambda^1 TM)$ denote the principal symbol of an operator $P \in \Psi^m_{\mathrm{inv}}(M, \Lambda^1 TM)$. Then we have the following result (see [15]).

Lemma 3. *If M is a manifold with cylindrical ends and $P \in \Psi^m_{\mathrm{inv}}(M, \Lambda^1 TM)$, then* *a)* $\Psi^r_{\mathrm{inv}}(M,\Lambda^1TM)\Psi^{r'}_{\mathrm{inv}}(M,\Lambda^1TM) \subset \Psi^{r+r'}_{\mathrm{inv}}(M,\Lambda^1TM)$ *and the principal symbol*

$$\sigma_r : \Psi^r_{\mathrm{inv}}(M,\Lambda^1TM)/\Psi^{r-1}_{\mathrm{inv}}(M,\Lambda^1TM) \to S^r(T^*M,\Lambda^1TM)/S^{r-1}(T^*M,\Lambda^1TM)$$

induces, for every $r \in \mathbb{R}$, an isomorphism onto the subspace of symbols of operators that are translation invariant in a neighborhood of infinity.

b) *Any operator $P \in \Psi^m_{\mathrm{inv}}(M,\Lambda^1TM)$ extends to a continuous operator*

$$P : H^{m'}(M,\Lambda^1TM) \to H^{m'-m}(M,\Lambda^1TM),\ \ \textit{if}\ \ m,m' \in 2\mathbb{Z}.$$

The proof of Lemma 3 follows by using similar arguments to those for [15, Lemma 1.3], in the case of a translation invariant in a neighborhood of infinity operator acting on functions.

Let us mention that any operator $P : C_0^\infty(M,\Lambda^1TM) \longrightarrow C^\infty(M,\Lambda^1TM)$ that is translation invariant in a neighborhood of infinity will be properly supported and gives rise to a pseudodifferential operator

$$\tilde{P} : C_0^\infty(\partial M_1 \times \mathbb{R},\Lambda^1T(\partial M_1 \times \mathbb{R})) \longrightarrow C_0^\infty(\partial M_1 \times \mathbb{R},\Lambda^1T(\partial M_1 \times \mathbb{R}))$$

such that

$$\tilde{P}(\omega) = \Phi^*_{-s}(P\Phi^*_s(\omega)), \tag{5}$$

where Φ_s is the translation by s on the cylinder $\partial M_1 \times \mathbb{R}$, and s is large enough for the relations $\mathrm{supp}(P\Phi^*_s(\omega)),\ \mathrm{supp}(\Phi^*_s(\omega)) \subset \partial M_1 \times (-\infty,0) \subset M$ to be satisfied. We call $\tilde{P}$ the *indicial operator* associated with P. Note that $\tilde{P}$ is well defined in the sense that the operator $\Phi^*_{-s}(P\Phi^*_s(\omega))$ does not depend on s as above. Let $\Psi^\infty_{\mathrm{inv}}(\partial M_1 \times \mathbb{R},\Lambda^1T(\partial M_1 \times \mathbb{R}))^{\mathbb{R}}$ be the class of operators contained in the space $\Psi^\infty_{\mathrm{inv}}(\partial M_1 \times \mathbb{R},\Lambda^1T(\partial M_1 \times \mathbb{R}))$ that are translation invariant with respect to the natural action of $\mathbb{R}$ on $\partial M_1 \times \mathbb{R}$.

If $T \in \Psi^\infty_{\mathrm{inv}}(\partial M_1 \times \mathbb{R},\Lambda^1T(\partial M_1 \times \mathbb{R}))^{\mathbb{R}}$ and η is a smooth function on $\partial M_1 \times \mathbb{R}$ with support in $(-\infty,-1) \times \partial M_1$, which is equal to 1 in a neighborhood of infinity, then the operator $s_0(T)$ given by

$$s_0(T) := \eta T \eta \tag{6}$$

belongs to $\Psi^{-\infty}_{\mathrm{inv}}(M,\Lambda^1TM)$.

Remark 1. If s_0 is as in (6), then for all $T \in \Psi^\infty_{\mathrm{inv}}(\partial M_1 \times \mathbb{R},\Lambda^1T(\partial M_1 \times \mathbb{R}))$ the equality $\Phi(s_0(T)) = T$ holds, where

$$\Phi : \Psi^\infty_{\mathrm{inv}}(M,\Lambda^1TM) \ni P \mapsto \tilde{P} \in \Psi^\infty(\partial M_1 \times \mathbb{R},\Lambda^1T(\partial M_1 \times \mathbb{R}))$$

is the *indicial morphism*. The range of Φ is $\Psi^\infty_{\mathrm{inv}}(\partial M_1 \times \mathbb{R},\Lambda^1T(\partial M_1 \times \mathbb{R}))^{\mathbb{R}}$.

Lemma 4. *Let $P, P_1 \in \Psi^{\infty}_{\text{inv}}(M, \Lambda^1 TM)$ and let $\rho : M \to [1, \infty)$ be a smooth function such that $\rho(y, x) := x$ on a neighborhood of infinity in $\partial M_1 \times (-\infty, 0]$. Then the following properties hold:*

(a) $\widetilde{PP_1} = \tilde{P}\tilde{P}_1$.
(b) $\text{ad}_{\rho}(P) := [\rho, P] = \rho P - P\rho \in \Psi^{\infty}_{\text{inv}}(M, \Lambda^1 TM)$.

Proof. (a) In view of the formula (5) we have

$$\widetilde{PP_1}(\omega) = \Phi^*_{-s}(PP_1\Phi^*_s(\omega)) = \Phi^*_{-s}P\Phi^*_s\left(\Phi^*_{-s}P_1\Phi^*_s(\omega)\right) = \tilde{P}\left(\widetilde{P_1}(\omega)\right).$$

(b) The proof is similar to that of [15, Lemma 1.6 (ii)].

Let us consider the one-dimensional Fourier transform

$$\mathscr{F} : L^2\big(\partial M_1 \times \mathbb{R}\big) \to L^2\big(\partial M_1 \times \mathbb{R}\big),\ \mathscr{F}(f)(y, \tau) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y, x) e^{-i\tau x} dx. \tag{7}$$

We also consider the Fourier transform of one forms on $\partial M_1 \times \mathbb{R}$ when ∂M_1 is *parallelizable*. We adopt this assumption everywhere from now on. In this case, every one form is naturally identified with an m-tuple of functions on $\partial M_1 \times \mathbb{R}$, where $m = \dim(M_1)$. The Fourier transforms of one forms is denoted in the same way as the Fourier transform of functions.

Let $\tilde{P}$ be the operator defined by (5). Since $\tilde{P}$ is translation invariant with respect to the action of $\mathbb{R}$, the resulting operator $P_1 := \mathscr{F}\tilde{P}\mathscr{F}^{-1}$ commutes with the multiplication operators in τ. Therefore, P_1 is a decomposable operator, i.e., there exist some operators

$$\hat{P}(\tau) : C^{\infty}(\partial M_1, \Lambda^1 T(\partial M_1)) \to L^2(\partial M_1, \Lambda^1 T(\partial M_1)) \tag{8}$$

defined by

$$(P_1 f)(\cdot, \tau) = \hat{P}(\tau) f(\cdot, \tau),\ \forall\ f(\cdot, \tau) \in C^{\infty}(\partial M_1, \Lambda^1 T(\partial M_1)) \subset L^2(\partial M_1, \Lambda^1 T(\partial M_1)).$$

5 The Dirichlet Problem for a Laplace Type Operator on Parallelizable Manifolds with Cylindrical Ends

Let N be a manifold with boundary and cylindrical ends. The potential analysis for such manifolds, culminating with an existence and uniqueness result for the Dirichlet problem associated with the Laplace operator, was developed by Mitrea and Nistor in [15]. Due to the triviality of the cotangent bundle in the case of a parallelizable manifold, the standard Sobolev spaces $H^r(N, \Lambda^1 TN)$ and $H^r(\partial N, \Lambda^1 TN)$,

$r > \frac{1}{2}$, are being identified with $H^r(N, \mathbb{R}^n)$ and $H^r(\partial N, \mathbb{R}^n)$, respectively, via some linear isomorphisms $\alpha_N : H^r(N, \Lambda^1 TN) \to H^r(N, \mathbb{R}^n)$, $\alpha_{\partial N} : H^r(\partial N, \Lambda^1 TN) \to H^r(\partial N, \mathbb{R}^n)$ coming from a global trivialization of the cotangent bundle of M. Consider a Riemannian metric on N and the Laplace type operator $\triangle_N$ and the trace operator $\cdot|_{\partial N}$, which makes the following diagrams commutative:

$$\begin{array}{ccc} H^r(N, \Lambda^1 TN) & \xrightarrow{\triangle_N} & H^{r-2}(N, \Lambda^1 TN) \\ \alpha_N \downarrow & & \downarrow \alpha_N \\ H^r(N, \mathbb{R}^n) & & H^{r-2}(N, \mathbb{R}^n) \\ \wr\downarrow & & \downarrow\wr \\ \big(H^r(N)\big)^n & \xrightarrow{\triangle_{\mathbb{R}^n}} & \big(H^{r-2}(N)\big)^n \end{array} \qquad \begin{array}{ccc} H^r(N, \Lambda^1 TN) & \xrightarrow{\cdot|_{\partial N}} & H^{r-\frac{1}{2}}(\partial N, \Lambda^1 TN) \\ \alpha_N \downarrow & & \downarrow \alpha_{\partial N} \\ H^r(N, \mathbb{R}^n) & & H^{r-2}(N, \mathbb{R}^n) \\ \wr\downarrow & & \downarrow\wr \\ \big(H^r(N)\big)^n & \xrightarrow{(\cdot|_{\partial N}, \dots, \cdot|_{\partial N})} & \big(H^{r-\frac{1}{2}}(\partial N)\big)^n \end{array}$$

where $\triangle_{\mathbb{R}^n}(u_1, \dots, u_n) = \big(\triangle u_1, \dots, \triangle u_n\big)$ and $\triangle = d^* d$ is the Laplace operator on N. We wonder whether the operator $\triangle_N$ coincides with the Laplace–Beltrami operator for suitable choices of the Riemannian metric on N and the global trivialization.

According to [15, Theorem 5.7] the correspondence

$$H^r(N) \ni u \mapsto \big((\triangle_N + V\mathbb{I})u, u\big|_{\partial N}\big) \in H^{r-2}(N) \oplus H^{r-1/2}(\partial N)$$

is a continuous bijection for every $r > 1/2$, where $V \geq 0$ stands for an asymptotically translation invariant smooth function. Therefore, the correspondence

$$(H^r(N))^n \ni u \mapsto \big((\triangle_{\mathbb{R}^n} + V\mathbb{I})u, u\big|_{\partial N}\big) \in (H^{r-2}(N))^n \oplus (H^{r-1/2}(\partial N))^n$$

is a bijection for every $r > 1/2$, as well. Taking into account that

$$\triangle_N + V\mathbb{I} = \alpha_N^{-1} \circ \big(\triangle_{\mathbb{R}^n} + V\mathbb{I}\big) \circ \alpha_N, \quad \cdot\big|_{\partial N} = \alpha_{\partial N}^{-1} \circ (\cdot|_{\partial N}, \dots, \cdot|_{\partial N}) \circ \alpha_{\partial N},$$

where $(u_1, \dots, u_n) = \alpha_N u$, we obtain the following:

Theorem 1. *Let N be a parallelizable manifold with boundary and cylindrical ends and $V \geq 0$ be a smooth function which is asymptotically translation invariant in a neighborhood of infinity. Then the correspondence*

$$H^r(N, \Lambda^1 TN) \ni u \mapsto \big((\triangle_N + V\mathbb{I})u, u\big|_{\partial N}\big) \in H^{r-2}(N, \Lambda^1 TN) \oplus H^{s-1/2}(\partial N, \Lambda^1 TN)$$

is a continuous bijection for every $r > 1/2$.

The proof of Theorem 1 relies on some existence and uniqueness result for the Dirichlet problem

$$\begin{cases} (\triangle + V\mathbb{I})u = 0, \ u \in H^{r+1/2}(N) \\ u\big|_{\partial N} = f \in H^r(\partial N), \end{cases} \tag{9}$$

proved by Mitrea and Nistor in [15, Theorem 5.6]. This existence and uniqueness result, as many other intermediate results, rely, especially for their existence parts, on the layer potential associated with the supramanifold M with cylindrical ends without boundary, assured by Lemma 1. The layer potential are defined by means of the inverse of the operator $\triangle_M + V\mathbb{I}$, where $\triangle_M$ is the Laplace operator on M and $V \geq 0$, $V \neq 0$ is a smooth function on M. Indeed we have:

Theorem 2 ([15]). *If M is a manifold with cylindrical ends and $V \geq 0$ is a smooth function on M that is translation invariant in a neighborhood of infinity and does not vanish at infinity, then $\triangle_M + V\mathbb{I}$ is invertible as an unbounded operator on $L^2(M)$ and $(\triangle_M + V\mathbb{I})^{-1} \in \Psi^0_{\mathrm{ai}}(M)$.*

The kernel of $(\triangle_M + V\mathbb{I})^{-1}$ restricted to the boundary ∂N gives rise to an operator

$$S := [(\triangle_M + V\mathbb{I})^{-1}]_{\partial N} \in \Psi^0_{\mathrm{ai}}(\partial N). \tag{10}$$

The *single layer potential* is defined as

$$\mathscr{S}(f) := (\triangle_M + V\mathbb{I})^{-1}(f \otimes \delta_{\partial N}), \tag{11}$$

where $f \in L^2(\partial N)$ and $f \otimes \delta_{\partial N}$ is the distribution defined, via the conditional measure on ∂N, by $\langle f \otimes \delta_{\partial N}, \varphi\rangle = \int_{\partial N} f\varphi$.

In order to define the double layer potential we first consider a unit vector field ν on M, which is normal to ∂N at every point of ∂N, and points outside N. The *double layer potential* with density $f \in L^2(\partial N)$ is defined as

$$\mathscr{D}(f) := (\triangle_M + V\mathbb{I})^{-1}(f \otimes \delta'_{\partial N}), \tag{12}$$

where $f \otimes \delta'_{\partial N}$ is the distribution defined by $\langle f \otimes \delta'_{\partial N}, \varphi\rangle = \int_{\partial N} f\partial_\nu\varphi$, via the directional derivative ∂_ν in the direction of ν. The operators S and

$$S = [(\triangle_M + V\mathbb{I})^{-1}]_{\partial N} \text{ and } K := [(\triangle_M + V\mathbb{I})^{-1}\partial^*_\nu]_{\partial N}, \tag{13}$$

relate the non-tangential limits of the single and double layer. Indeed the following statement holds and can be proved by reduction to the compact case.

Theorem 3 ([15]). *Let N be a parallelizable manifold with boundary and cylindrical ends and $V \in \Psi^0_{\mathrm{ai}}(M)$ be a smooth function, where M is a supramanifold of N assured by Lemma 1. Given $f \in L^2(\partial N)$, we have:*

$$\mathscr{S}(f)_+=\mathscr{S}(f)_-=Sf,\ \partial_\nu\mathscr{S}(f)_\pm=\left(\pm\frac{1}{2}\mathbb{I}+K^*\right)f,\ \mathscr{D}(f)_\pm=\left(\pm\frac{1}{2}\mathbb{I}+K\right)f,$$

where K^* is the transpose of K and $g_\pm$ denote the non-tangential pointwise limits of $g : M \setminus \partial N \to \mathbb{R}$. In addition, the following operators are invertible for any $s \in \mathbb{R}$:

$$-\frac{1}{2}\mathbb{I} + K : H^s(\partial N) \to H^s(\partial N),\ S : H^s(\partial N) \to H^{s+1}(\partial N).$$

Theorem 4 ([15]). *Let N be a manifold with boundary and cylindrical ends and $V \in \Psi^0_{\rm ai}(M)$ be a smooth function, where M is a supramanifold of N assured by Lemma 1. For any $r > 0$ and any $f \in H^r(\partial N)$, there exists a unique solution $u \in H^{r+1/2}(N)$ of the Dirichlet problem $(\triangle_N + V\mathbb{I})u = 0$, $u\big|_{\partial N} = f$, given by*

$$u = \mathscr{S}(S^{-1}f) = \mathscr{D}\left(-\frac{1}{2}\mathbb{I} + K\right)^{-1} f \in H^{r+1/2}(N). \tag{14}$$

The result of Theorem (4) also works in the parallelizable case.

Theorem 5. *Let N be a parallelizable manifold with boundary and cylindrical ends and $V \in \Psi^0_{\rm ai}(M)$ be a smooth function, where M is a supramanifold of N assured by Lemma 1. For any $r > 0$ and any $f = (f_1, \ldots, f_n) \in H^r(\partial N, \mathbb{R}^n)$, the Dirichlet problem $(\triangle_{\mathbb{R}^n} + V\mathbb{I})u = 0$, $u\big|_{\partial N} = f$ has a unique solution, given by any of the following integral forms*

$$\begin{aligned} u &= \left(\mathscr{S}(S^{-1}f_1), \ldots, \mathscr{S}(S^{-1}f_n)\right) \qquad (15)\\ &= \left(\mathscr{D}\left(-\frac{1}{2}\mathbb{I} + K\right)^{-1} f_1, \ldots, \mathscr{D}\left(-\frac{1}{2}\mathbb{I} + K\right)^{-1} f_n\right) \in H^{r+1/2}(N, \mathbb{R}^n). \end{aligned}$$

In addition, the Dirichlet problem

$$(\triangle_N + V\mathbb{I})w = 0,\ w\big|_{\partial N} = \alpha_N f \in H^r(\partial N, \Lambda^1 TN) \tag{16}$$

has the unique solution $w = \alpha_N^{-1}u \in H^{r+1/2}(N, \Lambda^1 TN)$, where $u \in H^{r+1/2}(N, \mathbb{R}^n)$ is given by (15).

6 Operators Between Weighted Sobolev Spaces That Are Almost Translation Invariant in a Neighborhood of Infinity

Let $T : C_0^\infty(M, \Lambda^1 TM) \to C^\infty(M, \Lambda^1 TM)$ be a linear map such that $\mathrm{ad}^k_\rho(T)$ extends to a bounded operator $\mathrm{ad}^k_\rho(T) : H^{-m}(M, \Lambda^1 TM) \to H^m(M, \Lambda^1 TM)$ for any $m \in \mathbb{Z}_+$, where we use the notations of Lemma 4. Let $\|T\|_{k,m}$ be the norm of the resulting operator $\mathrm{ad}^k_\rho(T)$.

The almost translation invariant in a neighborhood of infinity operators acting on functions has been defined in [15]. Such operators acting on one forms are defined as follows.

Definition 3. The space $\Psi_{\mathrm{ai}}^{-\infty}(M, \Lambda^1 TM)$ is defined as the closure of $\Psi_{\mathrm{inv}}^{-\infty}(M, \Lambda^1 TM)$ with respect to the countable family of seminorms

$$T \mapsto \|T\|_{k,m}, \quad T \mapsto \|\rho^{\ell}(T - s_0(\Phi(T)))\rho^{\ell}\|_{0,m}, \tag{17}$$

where $k, m/2, \ell \in \mathbb{Z}_+$. The space of *almost translation invariant in a neighborhood of infinity* operators of order m is defined by

$$\Psi_{\mathrm{ai}}^{m}(M, \Lambda^1 TM) := \Psi_{\mathrm{inv}}^{m}(M, \Lambda^1 TM) + \Psi_{\mathrm{ai}}^{-\infty}(M, \Lambda^1 TM). \tag{18}$$

We now consider the *weighted Sobolev space*

$$\rho^a H^s(M, \Lambda^1 TM) := \{\rho^a u : u \in H^s(M, \Lambda^1 TM)\},$$

endowed with the norm $\|f\|_{s,a} := \|\rho^{-a} f\|_s$.

Theorem 6. *Let M be a manifold with cylindrical ends. Let $P \in \Psi_{\mathrm{ai}}^{m}(M, \Lambda^1 TM)$ and let $\rho(y, x) = x$ on a neighborhood of infinity in $\partial M_1 \times (-\infty, 0]$. Then for $s, a \in \mathbb{R}$ the following statements hold:*

1. *P extends to a continuous operator $P : \rho^a H^s(M, \Lambda^1 TM) \to \rho^a H^{s-m}(M, \Lambda^1 TM)$.*
2. *$P : \rho^a H^s(M, \Lambda^1 TM) \to \rho^{a'} H^{s-m'}(M, \Lambda^1 TM)$ is compact for any $a' < a$, $m' > m$.*
3. *$P : \rho^a H^s(M, \Lambda^1 TM) \to \rho^a H^{s-m}(M, \Lambda^1 TM)$ is compact if and only if $\sigma_m(P) = 0$ and $\tilde{P} = 0$.*
4. *$P : \rho^a H^s(M, \Lambda^1 TM) \to \rho^a H^{s-m}(M, \Lambda^1 TM)$ is Fredholm if and only if $\sigma_m(P)$ is invertible and $P : H^s(\partial M_1 \times \mathbb{R}, \Lambda^1 T(\partial M_1 \times \mathbb{R})) \to H^{s-m}(\partial M_1 \times \mathbb{R}, \Lambda^1 T(\partial M_1 \times \mathbb{R}))$ is an isomorphism.*

Theorem 7. *Let M be a manifold with cylindrical ends. Let $T \in \Psi_{\mathrm{ai}}^{m}(M, \Lambda^1 TM)$ ($m \geq 0$) be such that its extension $T : L^2(M, \Lambda^1 TM) \to H^{-m}(M, \Lambda^1 TM)$ is invertible. In the case $m > 0$ it is also assumed that T is an elliptic operator. Then the inverse T^{-1} belongs to the class $\Psi_{\mathrm{ai}}^{-m}(M, \Lambda^1 TM)$.*

The proof of Theorem 6 can be done by using similar arguments to those for [15, Theorem 2.1] and the proof of Theorem 7 follows with arguments similar to those for [15, Theorem 2.10] (see also [14]). We omit them for the sake of brevity.

7 The Brinkman System on Manifolds with Cylindrical Ends

In this section we present the Brinkman system on manifolds with cylindrical ends.

Let $M = M_1 \cup (\partial M_1 \times (-\infty, 0])$ be a Riemannian manifold with cylindrical ends and let g be its Riemannian metric tensor such that $g = g_\partial + dx^2$ on the cylindrical end $\partial M_1 \times (-\infty, 0]$, where g_∂ is a metric on ∂M_1 and $x \in (-\infty, 0]$.

Let $d : C^\infty(M) \to C^\infty(M, \Lambda^1 TM)$, $d = \partial_j dx_j$, be the exterior derivative operator, and let $\delta : C^\infty(M, \Lambda^1 TM) \to C^\infty(M)$, $\delta = d^*$ be the exterior co-derivative operator. Denote by ∇ the Levi–Civita connection associated with the Riemannian metric tensor g of M.

If $X \in TM$, then the antisymmetric part of ∇X, defined by

$$(\nabla X)(Y, Z) = \langle \nabla_Y X, Z \rangle, \quad \forall\, Y, Z \in \mathfrak{X}(M), \tag{19}$$

is dX, i.e.,

$$\begin{aligned} dX(Y, Z) &= \langle dX, Y \wedge Z \rangle \\ &= \frac{1}{2}\big\{\langle \nabla_Y X, Z \rangle - \langle \nabla_Z X, Y \rangle\big\},\ \forall\, Y, Z \in \mathfrak{X}(M), \end{aligned}$$

where $\wedge$ denotes the exterior product of forms. The symmetric part of ∇X is Def X, the *deformation* of X, i.e.,

$$(\mathrm{Def}\, X)(Y, Z) = \frac{1}{2}\{\langle \nabla_Y X, Z \rangle + \langle \nabla_Z X, Y \rangle\}, \quad \forall\, Y, Z \in \mathfrak{X}(M), \tag{20}$$

where $\mathfrak{X}(M)$ is the $C^\infty(M)$-module of smooth vector fields on M. The coordinates of Def X are given by

$$(\mathrm{Def}\, X)_{jk} = (\mathrm{Def}\, X)(\partial_j, \partial_k) = \frac{1}{2}(X_{j;k} + X_{k;j}), \quad j, k = 1, \cdots, m, \tag{21}$$

where, for $X = X^j \partial_j$, we set $X_{k;j} = \partial_j X_k + \Gamma^l_{kj} X_l$ and Γ^l_{kj} are the Christoffel symbols of the second kind. Thus, Def X is a symmetric tensor field of type $(0, 2)$. Denoting by $S^2 T^* M$ the set of symmetric tensor fields of type $(0, 2)$, we obtain the *deformation* operator Def : $\mathfrak{X}(M) \to C^\infty(M, S^2 T^* M)$.

Note that the invariance properties of the Riemannian metric g and its associated Levi–Civita connection ∇_g assure the almost translation invariant property of the deformation operator Def in a neighborhood of infinity.

A vector field $X \in \mathfrak{X}(M)$, which satisfies Def $X = 0$ on M is called a *Killing field*. We assume that the manifold M does not have nontrivial Killing fields.

For more details on the notations in this paper we refer the reader to [10, 11].

Definition 4. An operator T on L^2_{loc} has the *unique continuation property* if $Tu = 0$ and u vanishes on some open subset then $u \equiv 0$.

Example 1. The deformation operator Def has the unique continuation property. Indeed, the lack of nontrivial Killing field on the manifold M is a stronger requirement.

Proposition 1. *Let $L \in \Psi^m_{\mathrm{ai}}(M, \Lambda^1 TM)$ $(m \geq 0)$ be an operator which has the unique continuation property and is nonnegative, i.e.,*

$$\langle L\mathbf{v}, \mathbf{v}\rangle \geq 0,\ \forall\ \mathbf{v} \in C^{\infty}(M, \Lambda^1 TM). \tag{22}$$

Also, let $P \in \Psi^0_{\text{inv}}(M, \Lambda^1 TM)$ *be a nonnegative operator, which is strictly positive on some open set* $\mathscr{O} \subset M$*, i.e.,*

$$\begin{aligned} &\langle P\mathbf{v}, \mathbf{v}\rangle \geq 0,\ \forall\ \mathbf{v} \in C^{\infty}(M, \Lambda^1 TM), \\ &\textit{For } \mathbf{v} \in C^{\infty}(\mathscr{O}, \Lambda^1 T\mathscr{O}),\ \langle P\big|_{\mathscr{O}}\mathbf{v}, \mathbf{v}\rangle = 0 \iff \mathbf{v} = 0. \end{aligned} \tag{23}$$

If $L + P : H^m(M, \Lambda^1 TM) \to L^2(M, \Lambda^1 TM)$ *is Fredholm operator of index zero, then it is invertible.*

The proof of Proposition 1 is similar to the proof of [15, Proposition 2.5].

Example 2. If $T \in \Psi^k_{\text{ai}}(M, \Lambda^1 TM)$ ($k \geq 0$) has the unique continuation property, then $L := T^*T$ satisfies the hypothesis of Proposition 1 with $m = 2k$. In particular, the operator

$$L : \mathfrak{X}(M) \to \mathfrak{X}(M),\ L := 2\text{Def}^*\text{Def} = -\triangle + d\delta - 2\text{Ric}, \tag{24}$$

satisfies the hypothesis of Proposition 1, where $\triangle := -(d\delta + \delta d)$ is the Hodge Laplacian, and Ric is the Ricci tensor.

Theorem 8. *Let* M *be a parallelizable manifold with cylindrical ends. Let* $V \geq 0$ *be a smooth function on* M *that is translation invariant in a neighborhood of infinity and does not vanish at infinity. Then the operator*

$$L_V := L + V\mathbb{I} = 2\text{Def}^*\text{Def} + V\mathbb{I}$$

is invertible as an unbounded operator on $L^2(M, \Lambda^1 TM)$ *and*

$$(L + V\mathbb{I})^{-1} \in \Psi^{-2}_{\text{ai}}(M, \Lambda^1 TM).$$

Proof. The operator L_V is nonnegative, one-to-one and hence it has the unique continuation property. In view of Proposition 1 we have to show that the operator

$$L_V : H^2(M, \Lambda^1 TM) \to L^2(M, \Lambda^1 TM)$$

is Fredholm of index zero. In this respect, by the ellipticity property of L_V and Theorem 6 (iv), the operator

$$L_V : H^2(M, \Lambda^1 TM) \to L^2(M, \Lambda^1 TM)$$

is Fredholm if and only if the operator

$$\tilde{L}_V(\tau) : H^2(\partial M_1 \times \mathbb{R}, \Lambda^1 T(\partial M_1 \times \mathbb{R})) \to L^2(\partial M_1 \times \mathbb{R}, \Lambda^1 T(\partial M_1 \times \mathbb{R}))$$

is an isomorphism. To show this result, let V_∞ be the limit at infinity of the function V. Then we have

$$\hat{L}_V(\tau) = 2(\mathrm{Def}^*\mathrm{Def})|_{\partial M_1} + (\tau^2 + V_\infty)\mathbb{I}. \tag{25}$$

Since $\tau^2 + V_\infty$ is nonnegative and does not vanish identically for any $\tau \in \mathbb{R}$, one obtains that the operator

$$\hat{L}_V(\tau) : H^2(\partial M_1, \Lambda^1 T(\partial M_1)) \to L^2(\partial M_1, \Lambda^1 T(\partial M_1)) \tag{26}$$

is one-to-one, i.e., its kernel is trivial. In addition, this operator is self-adjoint and hence its (trivial) kernel is the orthogonal complement of its range, i.e., the operator is onto. Consequently, the operator (26) is invertible for any $\tau \in \mathbb{R}$. In addition, the norm of the inverse of $\hat{L}_V(\tau)$ is bounded uniformly in τ (for details see [16]).

Remark 2. Note that the operator

$$2(\mathrm{Def}^*\mathrm{Def}) : H^2(M_1 \times \mathbb{R}, \Lambda^1 T(M_1 \times \mathbb{R})) \to L^2(M_1 \times \mathbb{R}, \Lambda^1 T(M_1 \times \mathbb{R})) \tag{27}$$

is not a Fredholm operator, although its restriction to the boundary ∂M_1

$$2(\mathrm{Def}^*\mathrm{Def})|_{\partial M_1} : H^2(\partial M_1 \times \mathbb{R}, \Lambda^1 T(\partial M_1 \times \mathbb{R})) \to L^2(\partial M_1 \times \mathbb{R}, \Lambda^1 T(\partial M_1 \times \mathbb{R}))$$

has this property. Indeed, the one forms $\omega \in H^2(\partial M_1 \times \mathbb{R}, \Lambda^1 T(\partial M_1 \times \mathbb{R}))$ depending just on the variable $t \in \mathbb{R}$, which determine an infinite dimensional subspace of the space $H^2(\partial M_1 \times \mathbb{R}, \Lambda^1 T(\partial M_1 \times \mathbb{R}))$, are all within the kernel of the operator (27).

Theorem 9. *Let M be a parallelizable manifold with cylindrical ends and let $V \geq 0$ be a smooth function on M that is translation invariant in a neighborhood of infinity and does not vanish at infinity. Then the operators*

$$\hat{B}_V(\tau) : H^2(\partial M_1, \Lambda^1 T(\partial M_1)) \times H^1_{\mathscr{F}}(\partial M_1) \to L^2(\partial M_1, \Lambda^1 T(\partial M_1)) \times L^2_{\mathscr{F}}(\partial M_1) \tag{28}$$

associated with the operator[1]

$$B_V := \begin{pmatrix} L + V\mathbb{I} & d \\ \delta & 0 \end{pmatrix} : L^2(M, \Lambda^1 TM) \times L^2_*(M) \longrightarrow L^2(M, \Lambda^1 TM) \times L^2_*(M) \tag{29}$$

are isomorphisms, where

$$L^2_*(M) := \{f \in L^2(M) : \langle f, 1 \rangle_M = 0\} \tag{30}$$

[1] See Eq. (8).

and

$$H^1_{\mathscr{F}}(\partial M_1) := \mathscr{F}^{-1}\left(H^1_*(\partial M_1)\right), \ L^2_{\mathscr{F}}(\partial M_1) := \mathscr{F}^{-1}\left(L^2_*(\partial M_1)\right). \tag{31}$$

Proof. By Proposition 1 we have to show that the operator

$$B_V : H^2(M, \Lambda^1 TM) \times H^1_*(M) \to L^2(M, \Lambda^1 TM) \times L^2_*(M) \tag{32}$$

is Fredholm of index zero. In this respect, we first recall that the ellipticity property in the sense of Agmon–Douglis–Nirenberg of B_V and Theorem 6 (iv) imply that the operator (32) is Fredholm if and only if the operator $\tilde{B}_V(\tau)$ from $H^2\left(\partial M_1 \times \mathbb{R}, \Lambda^1 T(\partial M_1 \times \mathbb{R})\right) \times H^1_*(\partial M_1 \times \mathbb{R})$ to $L^2\left(\partial M_1 \times \mathbb{R}, \Lambda^1 T(\partial M_1 \times \mathbb{R})\right) \times L^2_*(\partial M_1 \times \mathbb{R})$ is an isomorphism, for any $\tau \in \mathbb{R}$. To this purpose, recall that V_∞ is the limit at infinity of the function V and note that

$$\begin{aligned}\hat{B}_V(\tau) &= \begin{pmatrix} 2(\mathrm{Def}^*\mathrm{Def})|_{\partial M_1} + \left(\tau^2 + V_\infty\right)\mathbb{I} & \hat{d} \\ \hat{\delta} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2(\mathrm{Def}^*\mathrm{Def})|_{\partial M_1} + \tau^2\mathbb{I} & d|_{\partial M_1} \\ \delta|_{\partial M_1} & 0 \end{pmatrix} + \begin{pmatrix} V_\infty \mathbb{I} & d_{\mathrm{comp}} \\ \delta_{\mathrm{comp}} & 0 \end{pmatrix} \\ &= \hat{B}_0(\tau) + \hat{B}_{V_\infty;0}(\tau),\end{aligned}$$

where $d_{\mathrm{comp}}(\tau) f = i\tau f dt$ and $\delta_{\mathrm{comp}}(\tau) = \left(d_{\mathrm{comp}}(\tau)\right)^*$.

We first observe that the operator

$$\begin{aligned}&\hat{B}_0(\tau) : H^2(\partial M_1, \Lambda^1 T(\partial M_1)) \times H^1_{\mathscr{F}}(\partial M_1) \to L^2(\partial M_1, \Lambda^1 T(\partial M_1)) \times L^2_{\mathscr{F}}(\partial M_1), \\ &\hat{B}_0(\tau) = \begin{pmatrix} 2(\mathrm{Def}^*\mathrm{Def})|_{\partial M_1} + \tau^2\mathbb{I} & d|_{\partial M_1} \\ \delta|_{\partial M_1} & 0 \end{pmatrix}\end{aligned} \tag{33}$$

is Fredholm with index zero, as the boundary ∂M_1 is compact (see [4, 10, 22]).

In addition, the complementary operator

$$\begin{aligned}&\hat{B}_{V_\infty;0}(\tau) : H^2(\partial M_1, \Lambda^1 T(\partial M_1)) \times H^1_{\mathscr{F}}(\partial M_1) \to L^2(\partial M_1, \Lambda^1 T(\partial M_1)) \times L^2_{\mathscr{F}}(\partial M_1) \\ &\hat{B}_{V_\infty;0}(\tau) := \hat{B}_V(\tau) - \hat{B}_0(\tau)\end{aligned} \tag{34}$$

is compact as $H^2(\partial M_1, \Lambda^1 T(\partial M_1)) \times H^1_{\mathscr{F}}(\partial M_1) \hookrightarrow L^2(\partial M_1, \Lambda^1 T(\partial M_1)) \times L^2_{\mathscr{F}}(\partial M_1)$ is a compact embedding, d_{comp} is essentially a multiplication operator, and δ_{comp} is its dual. Consequently, the operator

$$\hat{B}_V(\tau) : H^2(\partial M_1, \Lambda^1 T(\partial M_1)) \times H^1_{\mathscr{F}}(\partial M_1) \to L^2(\partial M_1, \Lambda^1 T(\partial M_1)) \times L^2_{\mathscr{F}}(\partial M_1) \tag{35}$$

is Fredholm with index zero, for any $\tau \in \mathbb{R}$. We next show that this operator is one-to-one. Let $(\mathbf{u}, \pi) \in H^2(\partial M_1, \Lambda^1 T(\partial M_1)) \times H^1_{\mathscr{F}}(\partial M_1)$ be such that

$$\hat{B}_V(\tau) \begin{pmatrix} \mathbf{u} \\ \pi \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 0 \end{pmatrix} \tag{36}$$

i.e.,

$$\begin{cases} \hat{L}_V \mathbf{u} + \hat{d}\pi = \mathbf{0} \\ \hat{\delta}\mathbf{u} = 0 \end{cases} \Leftrightarrow \begin{cases} 2(\mathrm{Def}^* \mathrm{Def})|_{\partial M_1} \mathbf{u} + (\tau^2 + V_\infty)\mathbf{u} + \hat{d}\pi = \mathbf{0} \\ \hat{\delta}\mathbf{u} = 0. \end{cases}$$

Consequently, we obtain

$$\begin{aligned} 0 &= 2\langle \mathrm{Def}\, \mathbf{u}, \mathrm{Def}\, \mathbf{u} \rangle + \langle (\tau^2 + V_\infty)\mathbf{u}, \mathbf{u} \rangle + \langle \pi, \hat{\delta}\mathbf{u} \rangle \\ &= 2\langle \mathrm{Def}\, \mathbf{u}, \mathrm{Def}\, \mathbf{u} \rangle + \langle (\tau^2 + V_\infty)\mathbf{u}, \mathbf{u} \rangle, \end{aligned} \tag{37}$$

i.e., $\mathbf{u} = \mathbf{0}$. Also, one obtains $\hat{d}\pi = 0$, i.e., $\pi = 0$ as $\pi \in L^2_{\mathscr{F}}(\partial M_1)$. Hence, the operator (35) is an isomorphism.

Remark 3. The invertibility of the operator

$$B_V : H^2(M, \Lambda^1 TM) \times H^1_*(M) \to L^2(M, \Lambda^1 TM) \times L^2_*(M) \tag{38}$$

depends on the uniformly boundedness of the inverse of $\hat{B}_V(\tau)$ in τ (for details we refer the reader to [16]). The invertibility of B_V would allow us to develop a layer potential approach towards the Dirichlet problem for the Brinkman system on manifolds with cylindrical ends, via its inverse

$$B_V^{-1} = \begin{pmatrix} \mathfrak{A}_V & \mathfrak{B}_V \\ \mathfrak{C}_V & \mathfrak{D}_V \end{pmatrix} : L^2(M, \Lambda^1 TM) \times L^2_*(M) \to H^2(M, \Lambda^1 TM) \times H^1_*(M) \tag{39}$$

Indeed the role of the *single layer potential* and its corresponding pressure potential for the Brinkman system could be played by

$$\mathscr{S}(f) := \mathfrak{A}_V(f \otimes \delta_{\partial N}), \ \mathscr{P}(f) := \mathfrak{C}_V(f \otimes \delta_{\partial N}), \tag{40}$$

where $f \in L^2(\partial N, \Lambda^1 TN)$ and $f \otimes \delta_{\partial N}$ is the distribution defined, via the conditional measure on ∂N, by

$$\langle f \otimes \delta_{\partial N}, \varphi \rangle = \int_{\partial N} \langle f, \varphi \rangle.$$

Similarly, the role of the *double layer potential* and its corresponding pressure potential for the Brinkman system could be played by

$$\mathscr{D}(f) := \mathfrak{A}_V(f \otimes \delta'_{\partial N}),\ \mathscr{Q}(f) := \mathfrak{C}_V(f \otimes \delta'_{\partial N}), \tag{41}$$

where $f \in L^2(\partial N, \Lambda^1 TN)$ and $f \otimes \delta_{\partial N}$ is the distribution defined by

$$\langle f \otimes \delta'_{\partial N}, \varphi \rangle = \int_{\partial N} \langle f, \partial_\nu \varphi \rangle,$$

via the directional derivative ∂_ν in the direction of ν.

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Meaned Spaces and a General Duality Principle

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Abstract We present a new duality principle, in which we do not suppose that the range of the functions to be optimized is a subset of a linear space. The methods used in the proofs of our results are based on the notion of meaned space, which is a generalization of the notion of ordered linear space.

1 Introduction

In current mathematical literature there exists a vast variety of duality theorems regarding optimization problems. Recently, multiple important works have been published that summarize the most important results of duality theory, such as the monographies [12] (in the case of real-valued optimization) and [1] (in the case of vector-valued optimization).

The purpose of this paper is to present a general duality principle, which contains as a particular case the known duality theorems for real- and vector-valued optimization. The method used in the proof of this principle is based on the notion of meaned space, which is a triplet $(E, \mathscr{R}, *)$, where E is a nonempty set, $\mathscr{R}$ a binary relation, and $*$ a binary operation on E (called mean operation), satisfying the following conditions:

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(i) for every $f, g \in E$ with $f \mathscr{R} g$ we have $f \neq g$,
(ii) for every $f, g \in E$ with $f \mathscr{R} g$ we have $f \mathscr{R} f * g$ and $f * g \mathscr{R} g$.

It is easy to verify that ordered linear spaces are meaned spaces.

With all the generality of the obtained results, they are easy to comprehend due to the intuitive substrate of the theory, highlighted by examples in Sects. 4 and 5. Unlike the majority of papers concerning duality problems, in our paper we specify the role of each property (axiom), that is at the base of the theory. In this sense, our results can be seen as an axiomatic approach to duality theory. Our work is related, in its spirit, to the papers [10, 11], and the dissertation [6].

In the particular case when E is an open interval, our duality principle reduces to the theorem stated in [10], to which applications have been given in [11]. The results presented in Sects. 4 and 5 can be found in the first author's doctoral dissertation [6], which has not been published in a journal or book. However, some particular cases of the results of Sect. 4 were published in the articles [2,3] and [7]. The results of Sect. 5 are inspired by the pioneering work due to Gale, Kuhn, and Tucker (see [4]), and were also stated in [8].

The paper is structured as follows. In Sect. 2 we present some notions and propositions which will be used in the proof of the main result. Section 3 encompasses the main duality theorem of our paper. In Sect. 4 we present an important particular case of this theorem from the point of view of its applications in optimization theory. In Sect. 5 we show how we can apply the results of the previous section to finite dimensional linear vector (Pareto) optimization.

2 Preliminaries

Definition 1. Let E be a nonempty set, $\mathscr{R} \subseteq E \times E$ a binary relation and $* : E \times E \to E$ a binary operation on E. We say that the triplet $(E, \mathscr{R}, *)$ is a meaned space if the following conditions are satisfied:

(i) for every $f, g \in E$ with $f \mathscr{R} g$ we have $f \neq g$,
(ii) for every $f, g \in E$ with $f \mathscr{R} g$ we have $f \mathscr{R} f * g$ and $f * g \mathscr{R} g$.

Let $a = (a_1, a_2, \ldots)$ be an infinite binary string made up of symbols 0 and 1. In the following, for each $f_0, g_0 \in E$ with $f_0 \mathscr{R} g_0$ and $n \in \mathbb{N}^*$, we define

$$f_n = \begin{cases} f_{n-1}, & \text{if } a_n = 0, \\ f_{n-1} * g_{n-1}, & \text{if } a_n = 1, \end{cases}$$

and

$$g_n = \begin{cases} f_{n-1} * g_{n-1}, & \text{if } a_n = 0, \\ g_{n-1}, & \text{if } a_n = 1. \end{cases}$$

Definition 2. It is said that the meaned space $(E, \mathscr{R}, *)$ is complete if, for each $f_0, g_0 \in E$ with $f_0 \mathscr{R} g_0$ and for every $a = (a_1, a_2, \ldots)$ infinite binary string, there exists $f(a) \in E$ such that $f_0 \mathscr{R} f(a)$, $f(a) \mathscr{R} g_0$, and the sequences $(f_n)_{n \geq 0}$ and $(g_n)_{n \geq 0}$, constructed as above, satisfy the following conditions:

(α) for each $f \in E$ with $f \mathscr{R} f(a)$ there exists $n_0 \in \mathbb{N}$ such that $f \mathscr{R} f_{n_0}$;
(β) for each $g \in E$ with $f(a) \mathscr{R} g$ there exists $n_1 \in \mathbb{N}$ such that $g_{n_1} \mathscr{R} g$.

Definition 3. A complete meaned space $(E, \mathscr{R}, *)$ is called a fundamental domain if, for every $f, g \in E$, there exist $e, d \in E$ with $e \mathscr{R} f$, $e \mathscr{R} g$, $f \mathscr{R} d$, and $g \mathscr{R} d$.

Example 1. Let $E \subseteq \mathbb{R}$ be an open interval, $\mathscr{R}$ be the classic order relation "$<$" on $\mathbb{R}$, and let $*$ be the arithmetic mean on $\mathbb{R}$, that is, for each $f, g \in E$ we have $f * g = \frac{f+g}{2}$. Then $(E, \mathscr{R}, *)$ is a fundamental domain.

Example 2. Let E be the set of all closed half-spaces in $\mathbb{R}^3$, $\mathscr{R} =$ "$\subset$", and for each $f, g \in E$ we define $f * g = \frac{f+g}{2}$, in the Minkowski sense. Then $(E, \mathscr{R}, *)$ is a fundamental domain.

Example 3. Consider the Heisenberg group $(H_3(\mathbb{R}), \cdot)$, where

$$H_3(\mathbb{R}) = \{x = (a, b, c) \mid a, b, c \in \mathbb{R}\},$$

and for each $x, x' \in H_3(\mathbb{R})$, $x = (a, b, c)$, $x' = (a', b', c')$, we have

$$x \cdot x' = (a + a', b + b', c + c' + 2(a'b - ab')).$$

Let $E = H_3(\mathbb{R})$. For each $x, x' \in H_3(\mathbb{R})$, $x = (a, b, c)$, $x' = (a', b', c')$, we define

$$x * x' = \left(\frac{a + a'}{2}, \frac{b + b'}{2}, \frac{c + c'}{2} \right),$$

and $x \mathscr{R} x'$ if and only if $x^{-1} \cdot x' > 0$, where by x^{-1} we mean the inverse of x in the group $(H_3(\mathbb{R}), \cdot)$ (by "$>$" we mean the strict componentwise ordering relation in $\mathbb{R}^3$). Then $(E, \mathscr{R}, *)$ is a fundamental domain.

Observe that the relation $\mathscr{R}$ is not transitive. Indeed, let $x = (-2, -1, 0)$, $y = (1, 0, 0)$, $z = (2, 1, 0)$. We have $x \mathscr{R} y$, $y \mathscr{R} z$, but $x \overline{\mathscr{R}} z$ (where $\overline{\mathscr{R}}$ is the negation of $\mathscr{R}$).

Let $(E, \mathscr{R}, *)$ be a fundamental domain, X a nonempty set, and ρ a transitive binary relation defined on X. The negation of ρ will be denoted by $\overline{\rho}$. In the following we suppose that $w : E \to X$ is a given map.

Definition 4. We say that an element $x \in X$ is admissible if there exists at least one $f \in E$ with $x \rho w(f)$.

Let S be a fixed subset of X and consider the following properties:

P_1: For any $f, g \in E$ with $f\mathscr{R}g$ we have $w(f)\rho w(g)$.
P_2: For any $f \in E$ and any admissible element $x \in S$ there exists $g \in E$ with $g\mathscr{R}f$ and $x\overline{\rho}w(g)$.
P_3: Let $f \in E$. If for each $g \in E$ with $f\mathscr{R}g$ there exists $x' \in S$ with $x'\rho w(g)$, then there exists $x \in S$ with $x\rho w(h)$, for each $h \in E$ with $f\mathscr{R}h$.

Definition 5. Let $x \in X$ be an admissible element. We say that $f_x \in E$ is an indicator of x if the following conditions are satisfied:

1) $x\rho w(h)$ for each $h \in E$ with $f_x\mathscr{R}h$,
2) $x\overline{\rho}w(g)$ for each $g \in E$ with $g\mathscr{R}f_x$.

We will see later that an admissible element may have more than one indicator.

Let I_x be the set of all indicators of an admissible element x from X, and define

$$I_S = \bigcup_{x \in S} I_x.$$

Proposition 1. *No admissible element $x \in X$ can have two indicators f'_x and f''_x with $f'_x\mathscr{R}f''_x$.*

Proof. Suppose that there exists $x \in X$ which has two indicators f'_x and f''_x with $f'_x\mathscr{R}f''_x$. Let $g = f'_x * f''_x$, we have $g \in E$, $f'_x\mathscr{R}g$ and $g\mathscr{R}f''_x$. It follows from the definition of the indicators f'_x and f''_x that $x\rho w(g)$ and $x\overline{\rho}w(g)$, a contradiction.

Proposition 2. *Let $x \in S$ and $f \in E$. Suppose that $x\rho w(f)$ and properties P_1-P_2 hold, then there exists $f_x \in I_x$ such that either $f_x\mathscr{R}f$ or $f_x = f$.*

Proof. It follows from P_2 that there exists $g_0 \in E$ with $g_0\mathscr{R}f$ and $x\overline{\rho}w(g_0)$. Let $h_0 = f$ and define, for each $n \in \mathbb{N}^*$,

$$g_n = \begin{cases} g_{n-1}, & \text{if } x\rho w(g_{n-1} * h_{n-1}), \\ g_{n-1} * h_{n-1}, & \text{if } x\overline{\rho}w(g_{n-1} * h_{n-1}), \end{cases}$$

and

$$h_n = \begin{cases} g_{n-1} * h_{n-1}, & \text{if } x\rho w(g_{n-1} * h_{n-1}), \\ h_{n-1}, & \text{if } x\overline{\rho}w(g_{n-1} * h_{n-1}). \end{cases}$$

Let $f_x = f(a)$ from the definition of completeness, where for each $n \in \mathbb{N}^*$

$$a_n = \begin{cases} 0, \text{ if } x\rho w(g_{n-1} * h_{n-1}), \\ 1, \text{ if } x\overline{\rho}w(g_{n-1} * h_{n-1}). \end{cases}$$

We want to prove that f_x is an indicator of x.

Let $h \in E$ such that $f_x \mathscr{R} h$. It follows from the definition of completeness that there exists $n_1 \in \mathbb{N}$ such that $h_{n_1} \mathscr{R} h$. From the definition of h_{n_1} and from P_1 we have $x \rho w(h_{n_1})$ and $w(h_{n_1}) \rho w(h)$, therefore $x \rho w(h)$.

Let $g \in E$ such that $g \mathscr{R} f_x$. It follows from the definition of completeness that there exists $n_0 \in \mathbb{N}$ such that $g \mathscr{R} g_{n_0}$. From definition of g_{n_0} and from P_1 we have $x \overline{\rho} w(g)$.

To conclude the proof it is sufficient to observe that $f_x \mathscr{R} f$ or $f_x = f$.

Corollary 3. *If properties $P_1 - P_2$ hold, then each admissible element from S has at least one indicator.*

Definition 6. We say that an admissible element $x_0 \in S$ is optimal relatively to S, if there exists $f_{x_0} \in I_{x_0}$ such that, for each admissible element $x \in S$ and for each $f_x \in I_x$, we have $f_x \overline{\mathscr{R}} f_{x_0}$.

The set of elements that are optimal relatively to S is denoted by O_S.

Proposition 4. *If properties $P_1 - P_2$ hold, then an admissible element x_0 belongs to O_S if and only if there exists $f_{x_0} \in I_{x_0}$ such that, for each $f \in E$ with $f \mathscr{R} f_{x_0}$ and for each admissible element $x \in S$, we have $x \overline{\rho} w(f)$.*

Proof. Necessity: Suppose that, for each $f_{x_0} \in I_{x_0}$, there exist $f \in E$ and $x \in S$ such that $f \mathscr{R} f_{x_0}$ and $x \rho w(f)$. From Proposition 2 it follows that there exists $f_x \in I_x$ such that $f_x \mathscr{R} f_{x_0}$ or $f_x = f_{x_0}$. The second equality cannot hold, because due to $x \rho w(f)$ it would contradict the definition of f_x. Consequently, $x_0 \notin O_S$.

Sufficience: Suppose that, for each $f_{x_0} \in I_{x_0}$, there exist an admissible element $x \in S$ and there exists $f_x \in I_x$ with $f_x \mathscr{R} f_{x_0}$. Let $f = f_x * f_{x_0}$, we have $f_x \mathscr{R} f$, $f \mathscr{R} f_{x_0}$ and $x \rho w(f)$.

By the problem of the optimal element we understand the problem of studying O_S.

In the following we will formulate a problem that is, in a sense, the dual of the problem of the optimal element.

Let the nonempty set Y, the transitive relation ρ^* defined on Y, and the map $w^* : E \to Y$ be given.

Definition 7. An element $y \in Y$ is said to be admissible if there exists at least one $f \in E$ with $y \rho^* w^*(f)$.

Let S^* be a fixed subset of Y and consider the following properties:

P_1^*: For any $f, g \in E$ with $g \mathscr{R} f$ we have $w^*(f) \rho^* w^*(g)$.

P_2^*: For any $f \in E$ and any admissible element $y \in S^*$ there exists $g \in E$ with $f \mathscr{R} g$ and $y \overline{\rho^*} w^*(g)$.

P_3^*: Let $f \in E$. If for each $g \in E$ with $g \mathscr{R} f$ there exists $y' \in S^*$ with $y' \rho^* w^*(g)$, then there exists $y \in S^*$ such that $y \rho^* w^*(d)$, for each $d \in E$ with $d \mathscr{R} f$.

P_4^*: Let $f \in E$. If there exists $y \in S^*$ with $y \rho^* w^*(f)$ then, for each $g \in E$ with $g \mathscr{R} f$, there does not exist $x \in S$ such that $x \rho w(g)$.

P_5^*: Let $f \in E$. If there exist admissible elements in S or in S^*, and there does not exist $x \in S$ with $x\rho w(f)$, then for each $g \in E$ with $g\mathscr{R}f$ there exists $y \in S^*$ such that $y\rho^* w^*(g)$.

Remark 1. Properties P_1^*, P_2^* and P_3^* can be derived from properties P_1, P_2 and P_3 by replacing $\mathscr{R}$ with its dual, S with S^*, ρ with ρ^* and w with w^*.

Proposition 5. *If there exist admissible elements in S and P_1, P_2 and P_4^* hold, then P_2^* holds as well.*

Proof. Suppose that $y \in S^*$ is admissible and $f \in E$. Let $x \in S$ be an admissible element and $f_x \in I_x$. Since $(E, \mathscr{R}, *)$ is a fundamental domain, it follows that there exists $g \in E$ with $f\mathscr{R}g$ and $f_x\mathscr{R}g$. Let $d = f_x * g$, from the definition of f_x and from $f_x\mathscr{R}d$ it follows that $x\rho w(d)$. Since $d\mathscr{R}g$, it follows from P_4^* that $y\overline{\rho^*}w^*(g)$. Consequently, P_2^* holds.

Definition 8. Let $y \in Y$ be an admissible element. We say that $f^y \in E$ is an indicator of y, if the following conditions are satisfied:

1*) $y\rho^* w^*(h)$ for each $h \in E$ with $h\mathscr{R}f^y$,
2*) $y\overline{\rho^*}w^*(g)$ for each $g \in E$ with $f^y\mathscr{R}g$.

Let I^y be the set of all indicators of an admissible element y from S^*, and define

$$I^{S^*} = \bigcup_{y \in S^*} I^y.$$

From Remark 1 and from Propositions 1, 2 and 4 we may deduce the following results:

Proposition 6. *No admissible element $y \in Y$ can have two indicators f_1^y and f_2^y with $f_2^y\mathscr{R}f_1^y$.*

Proposition 7. *Let $y \in S^*$ and $f \in E$. Suppose that $y\rho^* w^*(f)$ and properties $P_1^* - P_2^*$ hold, then for each $g \in E$ with $g\mathscr{R}f$ there exists $f^y \in I^y$, such that either $g\mathscr{R}f^y$ or $g = f^y$.*

Corollary 8. *If properties $P_1^* - P_2^*$ hold, then each admissible element from S^* has at least one indicator.*

Definition 9. We say that an admissible element $y_0 \in S^*$ is optimal relatively to S^*, if there exists an $f^{y_0} \in I^{y_0}$ such that, for each admissible element $y \in S^*$ and for each $f^y \in I^y$, we have $f^{y_0}\overline{\mathscr{R}}f^y$.

The set of elements that are optimal relatively to S^* is denoted by O^{S^*}.

Proposition 9. *If properties $P_1^* - P_2^*$ hold, then an admissible element y_0 belongs to O^{S^*} if and only if there exists $f^{y_0} \in I^{y_0}$ such that, for each $f \in E$ with $f^{y_0}\mathscr{R}f$ and for each admissible element $y \in S^*$, we have $y\overline{\rho^*}w^*(f)$.*

By the dual problem of the optimal element we understand the problem of studying O^{S^*}.

3 The Duality Principle

In this section we will study the link between the problem of the optimal element and its dual.

Proposition 10. *If P_4^* holds, then for all admissible elements $x \in S$ and $y \in S^*$, and for all $f_x \in I_x$ and $f^y \in I^y$, we have $f_x \overline{\mathscr{R}} f^y$.*

Proof. Suppose that $f_x \mathscr{R} f^y$ and let $f = f_x * f^y$. From the definition of f^y and from $f \mathscr{R} f^y$ we have $y\rho^* w^*(f)$. Let $g = f_x * f$, it follows from P_4^* that $x\overline{\rho}w(g)$. On the other hand, from the definition of f_x and from $f_x \mathscr{R} g$ we have $x\rho w(g)$, a contradiction.

Proposition 11. *If P_4^* holds, $x \in S$ and $y \in S^*$ are admissible elements, and $f_x \in I_x$, $f^y \in I^y$ with $f_x = f^y$, then $x \in O_S$ and $y \in O^{S^*}$.*

Proof. If $x \notin O_S$, then there exists an admissible element $x' \in S$ and $f_{x'} \in I_{x'}$ such that $f_{x'} \mathscr{R} f_x$. It follows that $f_{x'} \mathscr{R} f^y$, which contradicts Proposition 10. Consequently, $x \in O_S$. The proof of $y \in O^{S^*}$ is similar.

Proposition 12. *If properties P_1, P_2, P_3^*, P_4^*, P_5^* hold and $x \in O_S$, then there exists $y \in O^{S^*}$ and there exist $f_x \in I_x$, $f^y \in I^y$ with $f_x = f^y$.*

Proof. From Proposition 4 it follows that there exists $f_x \in I_x$ such that for each $f \in E$ with $f \mathscr{R} f_x$ and for each admissible element $x' \in S$, we have $x'\overline{\rho}w(f)$. If $f \in E$ and $f \mathscr{R} f_x$, then it follows from P_5^* that for each $g \in E$ with $g \mathscr{R} f$ there exists $y' \in S^*$ such that $y'\rho^* w^*(g)$. We may deduce that for each $g \in E$ with $g \mathscr{R} f_x$ there exists $y' \in S^*$ such that $y'\rho^* w^*(g)$ holds (to see this, take $f = g * f_x$ in the prior statement). It follows from P_3^* that there exists $y \in S^*$ such that for each $e \in E$ with $e \mathscr{R} f_x$ we have $y\rho^* w^*(e)$.

Let $g \in E$ such that $f_x \mathscr{R} g$. If $y\rho^* w^*(g)$ holds, then from Propositions 5 and 7 it follows that there exists $f^y \in I^y$ such that $f_x \mathscr{R} f^y$ or $f_x = f^y$. However, due to Proposition 10, $f_x \mathscr{R} f^y$ cannot hold. From the definition of the indicator f^y and from $f_x \mathscr{R} g$ it follows that $y\overline{\rho^*} w^*(g)$, a contradiction. Therefore, $y\overline{\rho^*} w^*(g)$ must hold for each $g \in E$ with $f_x \mathscr{R} g$.

It follows that $f_x \in I^y$, and from Proposition 11 we have $y \in O^{S^*}$.

Proposition 13. *If properties P_3, P_1^*, P_2^*, P_4^*, P_5^* hold and $y \in O^{S^*}$, then there exists $x \in O_S$ and there exist $f_x \in I_x$, $f^y \in I^y$ with $f_x = f^y$.*

Proof. From Proposition 9 it follows that there exists $f^y \in I^y$ such that for each $g \in E$ with $f^y \mathscr{R} g$ and for each admissible element $y' \in S^*$, we have $y'\overline{\rho^*} w^*(g)$. Let $f, g \in E$ such that $f^y \mathscr{R} g$ and $g \mathscr{R} f$. It follows from P_5^* that there exists $x \in S$ with $x\rho w(f)$. We may deduce that for each $f \in E$ with $f^y \mathscr{R} f$ here exists $x' \in S$ with $x'\rho w(f)$. It follows from P_3 that there exists $x \in S$ such that for each $e \in E$ with $f^y \mathscr{R} e$, $x\rho w(e)$ holds. The proof of $f^y \in I_x$ and $x \in O_S$ is similar to the proof in case of Proposition 12.

Theorem 14.

1. *If properties $P_1,\ P_2,\ P_1^*,\ P_2^*,\ P_5^*$ hold and there are no admissible elements in S or in S^*, then $O_S = O^{S^*} = \emptyset$.*
2. *If properties $P_1,\ P_2,\ P_3,\ P_1^*,\ P_3^*,\ P_4^*$ hold and there are admissible elements in S and in S^*, then $O_S \neq \emptyset$ and $O^{S^*} \neq \emptyset$. Moreover, if $x \in O_S$ and $y \in O^{S^*}$, then for all $f_x \in I_x$ and $f^y \in I^y$, we have $f_x \overline{\mathscr{R}} f^y$.*
3. *If properties $P_1,\ P_2,\ P_3^*,\ P_4^*,\ P_5^*$ hold, then an admissible element $x \in S$ belongs to O_S if and only if there exist $y \in O^{S^*}$, $f_x \in I_x$ and $f^y \in I^y$ with $f_x = f^y$.*
4. *If properties $P_3,\ P_1^*,\ P_2^*,\ P_4^*,\ P_5^*$ hold, then an admissible element $y \in S^*$ belongs to O^{S^*} if and only if there exist $x \in O_S$, $f_x \in I_x$ and $f^y \in I^y$ with $f_x = f^y$.*

Proof. 1. It is clear that if there are no admissible elements in S and in S^*, then $O_S = O^{S^*} = \emptyset$.

Suppose that there are no optimal elements in S and that there are optimal elements in S^*. Let $y \in O^{S^*}$, $f^y \in I^y$ and $f \in E$ such that $f^y \mathscr{R} f$. It follows from the definition of the fundamental domain $(E, \mathscr{R}, *)$ that there exists $e \in E$ with $f \mathscr{R} e$. Since there are no admissible elements in S, we may deduce from P_5^* that there exists $y' \in S^*$ such that $y' \rho^* w^*(f)$. It follows from Proposition 7 that there exists $f^{y'} \in I^{y'}$ such that $f^y \mathscr{R} f^{y'}$ or $f^y = f^{y'}$. The last equality cannot hold, since $y' \rho^* w^*(f)$ and $f^y \mathscr{R} f$. Consequently, $y \notin O^{S^*}$, and therefore $O^{S^*} = \emptyset$.

Now suppose that there are no admissible elements in S^* and that there are admissible elements in S. It follows from P_5^* that for each $f \in E$ there exists $x' \in S$ such that $x' \rho w(f)$. Similarly to the first case, it can be proved that $O_S = \emptyset$.

2. Let $x \in S$ and $y \in S^*$ be admissible elements. It follows from Propositions 2 and 7 that there exist $f_x \in I_x$ and $f^y \in I^y$. Let $f \in E$ such that $f \mathscr{R} f_x$ and $f \mathscr{R} f^y$.

If we suppose that there exists $x' \in S$ with $x' \rho w(f)$, then from P_1, $f \mathscr{R} f^y$ and the transitivity of ρ we get $x' \rho w(f^y)$. Applying Proposition 2 yields that there exists $f_{x'} \in I_{x'}$ such that either $f_{x'} \mathscr{R} f^y$ or $f_{x'} = f^y$. However, due to Proposition 10, $f_{x'} \mathscr{R} f^y$ cannot hold. Furthermore, $f_{x'} = f^y$ cannot hold either, because it would yield $f \mathscr{R} f_{x'}$, and from the definition of $f_{x'}$ we get $x' \overline{\rho} w(f)$, a contradiction.

Consequently, $x' \rho w(f)$ does not hold for any $x' \in S$. Let $g \in E$ such that $f_x \mathscr{R} g$, $f_0 = f$, $g_0 = g$, and define for each $n \in \mathbb{N}^*$

$$f_n = \begin{cases} f_{n-1}, & \text{if there exists } z \in S \text{ such that } z \rho w(f_{n-1} * g_{n-1}); \\ f_{n-1} * g_{n-1}, & \text{if there does not exist } z \in S \text{ such that } z \rho w(f_{n-1} * g_{n-1}); \end{cases}$$

$$g_n = \begin{cases} f_{n-1} * g_{n-1}, & \text{if there exists } z \in S \text{ such that } z \rho w(f_{n-1} * g_{n-1}); \\ g_{n-1}, & \text{if there does not exist } z \in S \text{ such that } z \rho w(f_{n-1} * g_{n-1}). \end{cases}$$

It follows from the completeness of $(E, \mathscr{R}, *)$ that there exists $f(a) \in E$ such that $f_0 \mathscr{R} f(a)$, $f(a) \mathscr{R} g_0$, and for each $e, d \in E$ with $e \mathscr{R} f(a)$ and $f(a) \mathscr{R} d$ there exists $n_0, n_1 \in \mathbb{N}^*$ such that $e \mathscr{R} f_{n_0}$, $g_{n_1} \mathscr{R} d$. We may deduce from the definition of the sequences (f_n), (g_n) that for each $n \in \mathbb{N}$ there exists $x_n \in S$ with $x_n \rho w(g_n)$, and there does not exist $z \in S$ with $z \rho w(f_n)$. The transitivity of ρ and property P_1 imply that there exists $x_{n_0} \in S$ with $x_{n_0} \rho w(d)$, and there does not exist $z \in S$ with $z \rho w(e)$. It follows from P_3 that there exists $x_0 \in S$ with $x_0 \rho w(d)$, for each $d \in E$ with $f(a) \mathscr{R} d$. Furthermore, for each $e \in E$ with $e \mathscr{R} f(a)$, $x_0 \overline{\rho} w(e)$ holds. Consequently, $f(a)$ is an indicator of x_0.

Suppose that there exists an admissible element $z \in S$ and $f_z \in I_z$ with $f_z \mathscr{R} f(a)$. Let $e = f_z * f(a)$. It follows from $f_z \mathscr{R} e$ that $z \rho w(e)$. However, we have seen above that $e \mathscr{R} f(a)$ implies that $z \rho w(e)$ cannot hold for any $z \in S$. This contradiction proves that $x_0 \in O_S$.

The proof of $O^{S^*} \neq \emptyset$ is similar.

Statements 3 and 4 are consequences of Propositions 11, 12 and 13.

4 Duality in Vector Optimization

The following particular case of the duality theorem is of importance from the point of view of its applications in vector optimization theory.

Let $M \neq \emptyset$ and $X = 2^M$. Suppose that E is a real vector space and C is a convex pointed cone in E with a nonempty algebraic interior. We define $f * g = \frac{f+g}{2}$ for each $f, g \in E$, and we choose $\mathscr{R}$ = "$<_C$", where "$\leq_C$" is the ordering relation induced by C (that is, for each $f, g \in E$ the relation $f \leq_C g$ is equivalent to $g - f \in C$, and the relation $f <_C g$ is equivalent to $f \leq_C g$ and $f \neq g$). Similarly we may define "$\geq_C$" and "$>_C$", as usual. For the sake of simplicity, from now on we will leave the cone out of the index of the notations of these relations (using simply "$\leq$", "$<$", etc.).

Observe that $(E, \mathscr{R}, *)$ is a fundamental domain. Let $w : E \to X$, $w(f) = U(f) \cap V(f)$, where $U(f), V(f) \subset M$. We denote

$$S = \{\{x\},\ x \in M\},$$

and from now on, instead of writing $\{x\} \in S$, we will write $x \in M$. Furthermore, we choose ρ = "$\subseteq$". In this case we can apply all the notions and results stated before. For example, an element $x \in M$ is admissible if and only if there exists $f \in E$ such that $x \in U(f) \cap V(f)$.

Consider the following properties:

Q_1: For any $f, g \in E$ with $f < g$ we have

$$U(f) \subseteq U(g),\ V(f) \subseteq V(g).$$

Q_2: For any $f \in E$ and any admissible element $x \in M$ there exists $g \in E$ with $g < f$ and $x \notin U(g) \cap V(g)$.

Q_3: Let $f \in E$. If for each $g \in E$ with $f < g$ we have $U(g) \cap V(g) \neq \emptyset$, then $\bigcap_{f<d\in E} [U(d) \cap V(d)] \neq \emptyset$.

The following results are immediate:

Proposition 15. *Property Q_i implies property P_i, for each $i \in \{1, 2, 3\}$.*

Proposition 16. *Let $x \in M$ be an admissible element. Then $f_x \in E$ is an indicator of x if and only if $x \in \bigcap_{f_x<d\in E} [U(d) \cap V(d)]$ and $x \notin U(g) \cap V(g)$, for each $g \in E$ with $g < f_x$.*

It follows from Proposition 1 that two indicators of an admissible element are either equal, or uncomparable.

Let E_1 be a topological vector space over the reals, K a convex cone in E_1, that is not a linear subspace. We may consider the ordering relation induced by K, namely "$\leq_K$", and the strict ordering relation "$<_K$", defined as before, similarly for "$\geq_K$" and "$>_K$". For the sake of simplicity, from now on we will once again leave the cone out of the index of the notations of these relations (using again simply "$\leq$", "$<$", etc.). This will not cause confusion, because the cones are in different spaces.

Let M' be a set of functions that map M to E_1. We choose $Y = 2^{M'}$ and $\rho^* =$ "$\subseteq$". We denote

$$S^* = \{\{y\},\ y \in M'\},$$

and from now on, instead of writing $\{y\} \in S^*$, we will write $y \in M'$.

Let $f \in E$,

$$U^*(f) = \left\{ y \mid y \in M',\ y(x) \leq 0,\ \forall x \in \bigcap_{f<g\in E} U(g) \right\},$$

$$V^*(f) = \left\{ y \mid y \in M',\ y(x) > 0,\ \forall x \in \bigcup_{f>g\in E} V(g) \right\},$$

and $w^*(f) = U^*(f) \cap V^*(f)$.

Proposition 17. *Property P_1^* is satisfied.*

Proof. Let $f, g \in E$ with $f < g$. It follows from $\bigcap_{f<d\in E} U(d) \subseteq \bigcap_{g<d\in E} U(d)$ and $\bigcup_{f>e\in E} V(e) \subseteq \bigcup_{g>e\in E} V(e)$ it follows that $U^*(g) \subseteq U^*(f)$ and $V^*(g) \subseteq V^*(f)$, consequently $w^*(g) \subseteq w^*(f)$, thus P_1^* holds.

Proposition 18. *If property Q_1 is satisfied, then P_4^* holds.*

Proof. Let $f, g \in E$ with $g < f$, and let $y \in S^*$ be an admissible element such that $y \in w^*(f)$. It follows that $y(x) \leq 0$ for all $x \in \bigcap_{f<d\in E} U(d)$, and $y(x) > 0$ for all $x \in \bigcup_{f>e\in E} V(e)$, consequently $\left[\bigcap_{f<d\in E} U(d)\right] \cap \left[\bigcup_{f>e\in E} V(e)\right] = \emptyset$. Taking into consideration Q_1, we may deduce that $U(e) \cap V(e) = \emptyset$ for each $e \in E$ with $e < f$, and thus it holds for $e = g$ as well. Consequently, there is no $x \in S$ with $x \in w(g)$ and therefore P_4^* holds.

The next consequence of this result is due to Proposition 5.

Corollary 19. *If properties Q_1 and Q_2 are satisfied, and there exist admissible elements in M, then property P_2^* holds.*

Consider the following properties:

Q_1^*: Let $f \in E$. If $w^*(g) \neq \emptyset$ for each $g \in E$ with $g < f$, then $\bigcap_{f>e\in E} w^*(e) \neq \emptyset$.

Q_2^*: Let $f \in E$. If there exist admissible elements in M or in M', and $w(f) = \emptyset$, then $w^*(g) \neq \emptyset$ for each $g \in E$ with $g < f$.

The following results are immediate:

Proposition 20. *Property Q_1^* implies property P_3^*, and property Q_2^* implies property P_5^*.*

Proposition 21. *Let $y \in M'$ be an admissible element. We say that $f^y \in E$ is an indicator of y if and only if $y \in \bigcap_{f^y>e\in E} w^*(e)$ and $y \notin w^*(d)$, for each $d \in E$ with $f^y < d$.*

From Propositions 15–21 and from Theorem 14 we can deduce the following result:

Theorem 22. *1. If properties Q_1, Q_2, Q_2^* hold and there are no admissible elements in M or in M', then $O_S = O^{S^*} = \emptyset$.*
2. If properties Q_1, Q_2, Q_3, Q_1^* hold and there are admissible elements in M and in M', then $O_S \neq \emptyset$ and $O^{S^*} \neq \emptyset$.
3. If properties Q_1, Q_2, Q_1^*, Q_2^* hold, then an admissible element $x \in M$ belongs to O_S if and only if there exist $y \in O^{S^*}$, $f_x \in I_x$ and $f^y \in I^y$ with $f_x = f^y$.
4. If properties Q_1, Q_2, Q_3, Q_2^* hold, then an admissible element $y \in M'$ belongs to O^{S^*} if and only if there exist $x \in O_S$, $f_x \in I_x$ and $f^y \in I^y$ with $f_x = f^y$.

Example 4. In the following we will consider a generalized vector minimization problem discussed in [9].

Let Ξ be a nonempty set, P a set-valued map from Ξ to E, G a set-valued map from Ξ to E_1; E, C, E_1 and K are as defined before. If A is a nonempty subset of

E, we say that a point $a_0 \in A$ is minimal with respect to C if there is no $a \in A$ with $a_0 > a$. Denote the set of minimal points of A with respect to C by $\mathrm{Min}(A|C)$. We investigate the following problem:

$$\min P(\xi) \tag{1}$$

$$s.t.\ \xi \in \Xi,\ G(\xi) \cap -K \neq \emptyset.$$

By this we mean that if $\xi_0 \in \Xi$ with $G(\xi_0) \cap -K \neq \emptyset$, and if $a_0 \in P(\xi_0)$ satisfies $a_0 \in \mathrm{Min}(P(\Xi)|C)$, then (ξ_0, a_0) is a solution of (1), where $P(\Xi) = \bigcup_{\xi \in \Xi} P(\xi)$.

We may apply Theorem 22 in this context by choosing $M = E_1 \times E$ and defining, for each $f \in E$,

$$U(f) = U = \{(\eta, p) \mid \exists \xi \in \Xi :\ \eta \in G(\xi),\ p \in P(\xi)\},$$

$$V(f) = \{(\eta, p) \mid \eta \in -K,\ p \in P(\Xi) :\ p \leq f\}.$$

One can verify that the duality theorems presented by the authors in [9] can be included in our theory. Naturally, vector optimization can be regarded as a particular case of the optimization of set-valued functions.

Instead of going into the details of applying our theory in this general case, we will illustrate in the next section through a simple problem how one can apply Theorem 22. Observe that the methods presented in the next section can be applied (with some constraint qualifications) to more general cases as well, for example in the case of convex vector optimization in infinite dimensional spaces.

5 Linear Vector Optimization

Let a_{ij}, b_j, c_{ik} ($i = \overline{1,m}$, $j = \overline{1,n}$, $k = \overline{1,l}$, $l, m, n \in \mathbb{N}^*$) be given real numbers. Let Ξ be the set of all $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m$ with $\xi_i \geq 0$, $i = \overline{1,m}$. Suppose that H is the Euclidean space $\mathbb{R}^n$, where we define the following ordering relation: $\eta' \leq \eta''$ if and only if $\eta'_j \leq \eta''_j$, $j = \overline{1,n}$, $\eta' = (\eta'_1, \ldots, \eta'_n)$, $\eta'' = (\eta''_1, \ldots, \eta''_n)$.

Let E be the Euclidean space $\mathbb{R}^l$, where we define the following ordering relation: $f \leq g$ if and only if $f = g$ or $f_k < g_k$, $k = \overline{1,l}$, $f = (f_1, \ldots, f_l)$, $g = (g_1, \ldots, g_l)$.

Let $\eta_0 = (b_1, \ldots, b_n)$, and consider the linear maps $\phi : \Xi \to \mathbb{R}^l$ and $\psi : \Xi \to \mathbb{R}^n$ defined by

$$\phi(\xi) = \left(\sum_{i=1}^m c_{i1} \xi_i, \ldots, \sum_{i=1}^m c_{il} \xi_i \right),$$

$$\psi(\xi) = \left(\sum_{i=1}^m a_{i1} \xi_i, \ldots, \sum_{i=1}^m a_{in} \xi_i \right).$$

Our goal is to minimize the vector-valued function ϕ on the set defined by the following inequalities:

$$\begin{cases} \xi_i \geq 0, & i = \overline{1, m}, \\ \sum_{i=1}^{m} a_{ij} \xi_i \geq b_j, & j = \overline{1, n}. \end{cases} \tag{2}$$

Let $M = H \times E$ and, for each $f \in E$, define

$$U(f) = U = \{(\eta, \phi(\xi)) \mid \xi \in \Xi, \ \eta \in H, \ \eta \leq \psi(\xi)\},$$
$$V(f) = \{(\eta_0, g) \mid g \in E, \ g \leq f\},$$

and $w(f) = U \cap V(f)$.

Definition 10. We say that $\xi \in \Xi$ is permissible if $\eta_0 \leq \psi(\xi)$, i.e. inequalities (2) hold.

The following assertion is immediate:

Proposition 23. *An element $\xi \in \Xi$ is permissible if and only if $(\eta_0, \phi(\xi)) \in M$ is admissible in the sense of Sect. 2.*

If $x \in U \cap V(f)$ for some $f \in E$, then there exists a permissible element $\xi \in \Xi$ with $x = (\eta_0, \phi(\xi)) \in M$. Consequently, conditions Q_1 and Q_2 hold.
Furthermore, observe that if $\xi \in \Xi$ is permissible, then $\phi(\xi)$ is an indicator of $(\eta_0, \phi(\xi))$.

Definition 11. We say that $\xi \in \Xi$ is minimal, if there is no permissible element $\overline{\xi} \in \Xi$ with $\phi(\overline{\xi}) < \phi(\xi)$.

Observe that, for $\xi = (\xi_1, \ldots, \xi_m) \in \Xi$ and $\eta \in H$, we have $(\eta, \phi(\xi)) \in U$ if and only if there exist nonnegative numbers $\xi'_1, \ldots, \xi'_n$ such that

$$(\eta, \phi(\xi)) = \left(\sum_{i=1}^{m} a_{i1}\xi_i - \xi'_1, \ldots, \sum_{i=1}^{m} a_{in}\xi_i - \xi'_n, \sum_{i=1}^{m} c_{i1}\xi_i, \ldots, \sum_{i=1}^{m} c_{il}\xi_i \right).$$

It follows that U is actually the cone generated by $a^1, \ldots, a^m, -e^1, \ldots, -e^n$, where $a^i = (a_{i1}, \ldots, a_{in}, c_{i1}, \ldots, c_{i,l})$, $i = \overline{1, m}$, e^j is the versor of the j-th axis in $\mathbb{R}^{n+l}$. That is,

$$U = \left\{ x \in \mathbb{R}^{n+l} \mid x = \sum_{i=1}^{m} \xi_i a^i - \sum_{j=1}^{n} \xi'_j e^j; \ \xi_1, \ldots, \xi_m, \xi'_1, \ldots, \xi'_n \geq 0 \right\}.$$

Proposition 24. *Let $\xi \in \Xi$ be a permissible element, $f = (f_1, \dots, f_l) \in E$ is an indicator of $(\eta_0, \phi(\xi))$ if and only if*

$$\sum_{i=1}^{m} c_{ik}\xi_i \le f_k,\ k = \overline{1,l},$$

and there exists k_0, $1 \le k_0 \le l$, such that

$$\sum_{i=1}^{m} c_{ik_0}\xi_i = f_{k_0}.$$

Proof. Necessity: Suppose that $f = (f_1, \dots, f_l) \in E$ is an indicator of $(\eta_0, \phi(\xi))$. If there existed k_1, $1 \le k_1 \le l$, such that

$$\sum_{i=1}^{m} c_{ik_1}\xi_i > f_{k_1},$$

then for any $g = (g_1, \dots, g_l) \in E$ with $f < g$ and

$$g_{k_1} = \frac{1}{2}\left(\sum_{i=1}^{m} c_{ik_1}\xi_i + f_{k_1}\right),$$

we would have $(\eta_0, \phi(\xi)) \notin U \cap V(g)$, which contradicts the fact that f is an indicator of $(\eta_0, \phi(\xi))$.

If we had

$$\sum_{i=1}^{m} c_{ik}\xi_i < f_k,\ k = \overline{1,l},$$

then by choosing

$$g_k = \frac{1}{2}\left(\sum_{i=1}^{m} c_{ik}\xi_i + f_k\right),$$

for each $k = \overline{1,l}$, we would have $g < f$ and $(\eta_0, \phi(\xi)) \in U \cap V(g)$, which again contradicts the fact that f is an indicator of $(\eta_0, \phi(\xi))$.

Sufficience: If the said conditions hold, we have $(\eta_0, \phi(\xi)) \in U \cap V(g)$, for each $g \in E$ with $f < g$, and $(\eta_0, \phi(\xi)) \notin U \cap V(g)$, for each $g \in E$ with $g < f$, that is, f is an indicator of $(\eta_0, \phi(\xi))$.

It follows from the above result that an admissible element can have more than one indicator.

Proposition 25. *A permissible element $\xi \in \Xi$ is minimal if and only if $(\eta_0, \phi(\xi))$ is optimal.*

Proof. Suppose that ξ is not minimal. Then there exists a permissible element $\xi' = (\xi_1', \ldots, \xi_m') \in \Xi$ such that

$$\sum_{i=1}^{m} c_{ik}\xi_i' < \sum_{i=1}^{m} c_{ik}\xi_i, \; k = \overline{1, l}.$$

Let f be an indicator of $(\eta_0, \phi(\xi))$. Set $f_k' = \sum_{i=1}^{m} c_{ik}\xi_i, \; k = \overline{1, l}$ and $f' = (f_1', \ldots, f_l')$. We have from Proposition 24 that $f' < f$, consequently $(\eta_0, \phi(\xi))$ is not optimal.

If there is no permissible element $\xi' = (\xi_1', \ldots, \xi_m') \in \Xi$ with

$$\sum_{i=1}^{m} c_{ik}\xi_i' < \sum_{i=1}^{m} c_{ik}\xi_i, \; k = \overline{1, l},$$

then we may deduce from Proposition 24 that $(\eta_0, \phi(\xi))$ is optimal.

The next lemma is part of the folklore of polyhedral convex set theory (see [5]).

Lemma 1. *For any convex polyhedron $A \subset M$ and for any finite $B \subset M$, such that the convex cone K_B generated by B does not intersect A, there exists $x^* \in M^*$ such that $x^*(x) > 0$ for every $x \in A$ and $x^*(x) \leq 0$ for every $x \in K_B$ (here M^* denotes the dual space of M).*

Proposition 26. *Condition Q_3 is satisfied.*

Proof. Let $f \in E$ such that $U \cap V(g) \neq \emptyset$ for each $g \in E$ with $f < g$. Clearly

$$\bigcap_{f<d} [U \cap V(d)] \supseteq U \cap V(f).$$

Suppose that $U \cap V(f) = \emptyset$. Since U is a convex cone generated by a finite set, and $V(f)$ is a convex polyhedron, it follows from Lemma 1 that there exists a hyperplane $x^*(x) = \alpha$ that separates U from $V(f)$, and has no common points with $V(f)$ (for instance, $x^*(u) \leq \alpha < x^*(v), \; \forall u \in U, \; \forall v \in V(f)$). It follows that there exists $g \in E$ with $f < g$, such that $U \cap V(g) = \emptyset$, a contradiction.

Let M' be the set of all linear functions over $M = \mathbb{R}^{n+l}$ that have the form

$$\varphi(x) = \sum_{j=1}^{n} y_j x_j - \sum_{k=1}^{l} y_{n+k} x_{n+k},$$

where $y_1, \ldots, y_{n+l} \geq 0$, $y_{n+1} + \ldots + y_{n+l} > 0$ and $x = (x_1, \ldots, x_{n+l}) \in M$.

For each $f \in E$ we define

$$U^*(f) = U^* = \{\varphi \in M' \mid \varphi(\eta, h) \leq 0,\ \forall(\eta, h) \in U\},$$

$$V^*(f) = \left\{\varphi \in M' \mid \varphi(\eta, h) > 0,\ \forall(\eta, h) \in \bigcup_{g<f} V(g)\right\},$$

and $w^*(f) = U^* \cap V^*(f)$.

Proposition 27. *Let $f = (f_1, \dots, f_l) \in E$ and $\varphi \in M'$. The relation $\varphi \in w^*(f)$ holds if and only if the following inequalities are satisfied:*

$$\sum_{j=1}^{n} y_j a_{ij} - \sum_{k=1}^{l} y_{n+k} c_{ik} \leq 0,\ i = \overline{1, m}, \tag{3}$$

$$\sum_{j=1}^{n} y_j b_j - \sum_{k=1}^{l} y_{n+k} f_k \geq 0. \tag{4}$$

Proof. Necessity: From $\varphi \in U^*$ it follows that

$$\sum_{j=1}^{n} y_j \left(\sum_{i=1}^{m} a_{ij}\xi_i - \xi'_j\right) - \sum_{k=1}^{l} y_{n+k} \left(\sum_{i=1}^{m} c_{ik}\xi_i\right) \leq 0,$$

for every $\xi_1, \dots, \xi_m, \xi'_1, \dots, \xi'_n \geq 0$. Consequently,

$$\sum_{i=1}^{m} \left(\sum_{j=1}^{n} y_j a_{ij} - \sum_{k=1}^{l} y_{n+k} c_{ik}\right) \xi_i - \sum_{j=1}^{n} y_j \xi'_j \leq 0, \tag{5}$$

for every $\xi_1, \dots, \xi_m, \xi'_1, \dots, \xi'_n \geq 0$. It is clear that (3) holds.

The relation $\varphi \in V^*(f)$ is equivalent to

$$\sum_{j=1}^{n} y_j b_j - \sum_{k=1}^{l} y_{n+k} g_k > 0,\ \forall g < f,\ g = (g_1, \dots, g_l). \tag{6}$$

From (6) it follows that (4) holds.

Sufficience: From (3) and (4) it follows that (5) and (6) hold, and consequently $\varphi \in w^*(f)$.

Consequently, there exist admissible elements in M' if and only if the following system is compatible:

$$\begin{cases} y_1, \dots, y_{n+l} \geq 0, \\ y_{n+1} + \dots + y_{n+l} > 0, \\ -\sum_{j=1}^{n} y_j a_{ij} + \sum_{k=1}^{l} y_{n+k} c_{ik} \geq 0, \ i = \overline{1,m}. \end{cases} \tag{7}$$

Proposition 28. *Conditions Q_1^* and Q_2^* are satisfied.*

Proof. First, let us show that Q_2^* is satisfied. Let $f, g \in E$ such that $g < f$ and $U \cap V(f) = \emptyset$. It follows from Lemma 1 that there exists a hyperplane of the form

$$\varphi(x) = \sum_{j=1}^{n} y_j x_j - \sum_{k=1}^{l} y_{n+k} x_{n+k} = 0, \tag{8}$$

that strictly separates these two sets.

We have $y_j \geq 0$, $j = \overline{1,n}$, since $-e^j \in U$, $j = \overline{1,n}$. Let us prove that $y_{n+k} \geq 0$, $k = \overline{1,l}$. Suppose the contrary: if there exists k_0, $1 \leq k_0 \leq l$, with $y_{n+k_0} < 0$, then, taking into consideration that any component of x of larger order than n can be arbitrarily shrinked without leaving the set $V(f)$, there exists $x_0 = (x_1^0, \dots, x_{n+l}^0) \in V(f)$ with

$$\varphi(x_0) = \sum_{j=1}^{n} y_j b_j - \sum_{k=1}^{l} y_{n+k} x_{n+k}^0 < 0.$$

This contradicts $\varphi(x) > 0, \ \forall x \in V(f)$.

Suppose that there exist admissible elements in M. Let us prove that there exists k_0, $1 \leq k_0 \leq l$, with $-e^{n+k_0} \notin U$. Supposing the contrary, if $\overline{x} = (b_1, \dots, b_n, g_1', \dots, g_l')$ is an admissible element, then U contains the cone generated by $\overline{x}, -e^{n+1}, \dots, -e^{n+l}$, that is

$$\{x \in M \mid x = (b_1, \dots, b_n, x_{n+1}, \dots, x_{n+l}), \ x_{n+k} \leq g_k', \ k = \overline{1,l}\} \subset U.$$

This contradicts $U \cap V(f) = \emptyset$.

If $-e^{n+k_0} \notin U$, then U can be separated from $V(f)$ with a hyperplane of the form

$$\sum_{j=1}^{n} y_j x_j - \sum_{k=1}^{l} y_{n+k} x_{n+k} = 0, \tag{9}$$

where $y_j \geq 0$, $j = \overline{1, n+l}$, and $y_{n+k_0} = 1$. Consequently, $w^*(f) \neq \emptyset$, from where $w^*(g) \neq \emptyset$.

If there do not exist admissible elements in M, but there exist admissible elements in M', then clearly there exists k_0, $1 \leq k_0 \leq l$, with $-e^{n+k_0} \notin U$, from where $w^*(g) \neq \emptyset$. Therefore, Q_2^* holds.

Let $f \in E$ such that $w^*(g) \neq \emptyset$ for any $g \in E$ with $g < f$. It follows that $U \cap V(g) = \emptyset$, from where we may deduce that there exists a hyperplane of the form (8) strictly separating U and $V(f)$. It can be shown as above that $y_j \geq 0$, $j = \overline{1, n+l}$. From $w^*(g) \neq \emptyset$ for any $g \in E$ with $g < f$ it follows that there exists k_0, $1 \leq k_0 \leq l$, with $-e^{n+k_0} \notin U$. Consequently, U and $V(f)$ can be strictly separated by a hyperplane of the form (9), with $y_j \geq 0$, $j = \overline{1, n+l}$, and $y_{n+k_0} = 1$. It follows that $w^*(f) \neq \emptyset$, from where $\bigcap_{f > d} w^*(d) \neq \emptyset$. Therefore, Q_1^* holds as well.

Remark 2. The set M' is isomorphic to the set

$$\left\{ x \in M \mid x = (x_1, \ldots, x_{n+l}),\ x_j \geq 0,\ j = \overline{1, n+l},\ \sum_{k=1}^{l} x_{n+k} > 0 \right\}.$$

Therefore, when φ is defined with the formula (8), we may put $\varphi = (y_1, \ldots, y_{n+l})$.

Proposition 29. *An element $f \in E$ is an indicator of the admissible element $\varphi = (y_1, \ldots, y_{n+l})$ from M' if and only if*

$$\sum_{j=1}^{n} y_j b_j - \sum_{k=1}^{l} y_{n+k} f_k = 0. \tag{10}$$

Proof. Necessity: Let $f \in E$ be an indicator of $\varphi \in M'$, and suppose that

$$\sum_{j=1}^{n} y_j b_j - \sum_{k=1}^{l} y_{n+k} f_k < 0,$$

that is, $\varphi(\eta_0, f) < 0$. Let us consider an element $g \in E$ with $g < f$, and construct the following sequence:

$$f_n = f - \frac{1}{2^{n-1}} f + \frac{1}{2^{n-1}} g,\ n \in \mathbb{N}^*.$$

We have $f_n < f$ and $f_n < f_{n-1}$, for any $n \in \mathbb{N}^*$. Taking into account that $\lim_{n\to\infty} \varphi(\eta_0, f_n) = \varphi(\eta_0, f)$, it follows that there exists $n_0 \in \mathbb{N}^*$ such that $\varphi(\eta_0, f_{n_0}) < 0$. However, this implies $\varphi \notin V^*(f_{n_0-1})$, which contradicts Proposition 21, since $f_{n_0-1} < f$.

If we have $\sum_{j=1}^{n} y_j b_j - \sum_{k=1}^{l} y_{n+k} f_k > 0$, that is, $\varphi(\eta_0, f) > 0$, we may similarly prove that there exists $h \in E$ with $f < h$ and $\varphi \in V^*(h)$, which also contradicts Proposition 21.

Sufficience: Let $\varphi(\eta_0, f) = \sum_{j=1}^{n} y_j b_j - \sum_{k=1}^{l} y_{n+k} f_k = 0$, $g \in E$ with $g < f$, and $(\eta_0, g_1) \in \bigcap_{h<g} V(h)$, $g_1 = (g_1^1, \dots, g_l^1)$. From $y_j \geq 0$, $j = \overline{1, n+l}$, and $\sum_{k=1}^{l} y_{n+k} > 0$, we may deduce that

$$\varphi(\eta_0, g_1) = \sum_{j=1}^{n} y_j b_j - \sum_{k=1}^{l} y_{n+k} g_k^1 > \sum_{j=1}^{n} y_j b_j - \sum_{k=1}^{l} y_{n+k} f_k = 0,$$

consequently $\varphi \in V^*(g)$.

Now let $g \in E$ with $f < g$. Similarly, we have

$$\varphi(\eta_0, g) < \sum_{j=1}^{n} y_j b_j - \sum_{k=1}^{l} y_{n+k} f_k = 0,$$

and we may show, similarly to the methods used in proving the necessity, that there exists $g_1 \in E$ with $g_1 < g$ and $\varphi(\eta_0, g_1) < 0$. It follows that $\varphi \notin V^*(g)$. We may conclude from Proposition 21 that f is an indicator of φ.

Proposition 30. *The admissible element $\varphi = (y_1, \dots, y_{n+l})$ from M' is optimal if and only if there exist $f_1, \dots, f_l \in \mathbb{R}$ such that*

1. *Eq. (10) holds*
2. *there does not exist an admissible element $\varphi' = (y_1', \dots, y_{n+l}')$ and real numbers $f_1', \dots, f_l'$, such that $f_k' > f_k$, $k = \overline{1, l}$, and*

$$\sum_{j=1}^{n} y_j' b_j - \sum_{k=1}^{l} y_{n+k}' f_k' \geq 0. \tag{11}$$

Proof. Necessity: If φ is optimal and $f = (f_1, \dots, f_l)$ is an indicator of φ, it follows from Proposition 29 that (10) holds.

If we suppose that 2. does not hold, from Propositions 9 and 29, taking into account that the indicator f was chosen arbitrarily, it follows that φ is not optimal, a contradiction.

Sufficience: If 1. and 2. are satisfied, it follows from Propositions 9 and 29 that φ is optimal.

Let $\mathscr{M}$ be the set of minimal elements in Ξ and $\mathscr{O}^*$ be the set of optimal elements in M'.

Theorem 31.

1. *If one of the inequality systems (2) or (7) is incompatible, then* $\mathscr{M} = \mathscr{O}^* = \emptyset$.
2. *If both (2) and (7) are compatible, then* $\mathscr{M} \neq \emptyset$ *and* $\mathscr{O}^* \neq \emptyset$.
3. *An element* $(\xi_1, \ldots, \xi_m)$ *satisfying (2) is in* $\mathscr{M}$ *if and only if there exists* $(y_1, \ldots, y_{n+l}) \in \mathscr{O}^*$ *with*

$$\sum_{j=1}^{n} y_j b_j = \sum_{k=1}^{l} \left(\sum_{i=1}^{m} c_{ik} \xi_i \right) y_{n+k}. \tag{12}$$

4. *An element* $(y_1, \ldots, y_{n+l})$ *satisfying (7) is in* $\mathscr{O}^*$ *if and only if there exists* $(\xi_1, \ldots, \xi_m) \in \mathscr{M}$ *such that (12) holds.*

Proof. Statements 1 and 2 result from Theorem 22.

If $\xi = (\xi_1, \ldots, \xi_m) \in \mathscr{M}$, then it follows from Proposition 25 that $x = (\eta_0, \phi(\xi))$ is optimal. We may deduce from Theorem 22 and Propositions 24, 29 that there exist $f_1, \ldots, f_l \in \mathbb{R}$ and $(y_1, \ldots, y_{n+l}) \in \mathscr{O}^*$ such that (10) holds and $\sum_{i=1}^{m} c_{ik} \xi_i \leq f_k,\ k = \overline{1, l}$. It follows that

$$\sum_{j=1}^{n} y_j b_j - \sum_{k=1}^{l} \left(\sum_{i=1}^{m} c_{ik} \xi_i \right) y_{n+k} \geq 0. \tag{13}$$

If we had

$$\sum_{j=1}^{n} y_j b_j - \sum_{k=1}^{l} \left(\sum_{i=1}^{m} c_{ik} \xi_i \right) y_{n+k} > 0, \tag{14}$$

then there would exist $f' = (f'_1, \ldots, f'_l) \in E$ such that $\sum_{i=1}^{m} c_{ik} \xi_i < f'_k,\ k = \overline{1, l}$, and

$$\sum_{j=1}^{n} y_j b_j - \sum_{k=1}^{l} y_{n+k} f'_k = 0.$$

It follows from Propositions 24 and 29 that $\left(\sum_{i=1}^{m} c_{i1} \xi_i, \ldots, \sum_{i=1}^{m} c_{il} \xi_i \right)$ is an indicator of x and f' is an indicator of $(y_1, \ldots, y_{n+l})$. However, this contradicts Proposition 10.

If (12) holds, then it follows from Propositions 24 and 29 that

$$\left(\sum_{i=1}^{m} c_{i1} \xi_i, \ldots, \sum_{i=1}^{m} c_{il} \xi_i \right)$$

is an indicator of $x = (\eta_0, \phi(\xi))$ and of $\varphi = (y_1, \ldots, y_{n+l})$. It follows from Theorem 22 that x is optimal, and from Proposition 25 we get $\xi \in \mathscr{M}$.

Consequently, 3. holds.

If $\varphi = (y_1, \ldots, y_{n+l}) \in \mathscr{O}^*$, it follows from Theorem 22 that there exists an optimal element $x = (\eta_0, \phi(\xi)) \in M$ and $f = (f_1, \ldots, f_l) \in E$ such that f is an indicator of x and of φ. We may deduce from Propositions 24 and 29 that $\sum_{i=1}^{m} c_{ik}\xi_i \leq f_k,\ k = \overline{1,l}$, and (10) holds. Consequently, (13) holds as well. It can be shown, similarly to the proof of 3., that (14) cannot hold, so (12) holds.

For the converse implication, if (12) holds, then $\left(\sum_{i=1}^{m} c_{i1}\xi_i, \ldots, \sum_{i=1}^{m} c_{il}\xi_i\right)$ is an indicator of φ and of $(\eta_0, \phi(\xi))$, where $\xi = (\xi_1, \ldots, \xi_m)$. It follows from Theorem 22 that $\varphi \in \mathscr{O}^*$.

Corollary 32. *An element $(\xi_1, \ldots, \xi_m)$ satisfying (2) is in $\mathscr{M}$ if and only if there exists $(y_1, \ldots, y_{n+l})$ satisfying (7) such that*

$$\sum_{j=1}^{n} y_j a_{ij} = \sum_{k=1}^{l} c_{ik} y_{n+k},\ if\ \xi_i > 0\ (i = \overline{1,m}),$$

and

$$y_j = 0,\ if\ \sum_{i=1}^{m} a_{ij}\xi_i > b_j\ (j = \overline{1,n}).$$

Remark 3. The authors intend to revisit the optimization problem for functions that take their values in Heisenberg groups in a separate forthcoming paper.

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An AQCQ-Functional Equation in Matrix Random Normed Spaces

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Abstract In this paper, we prove the Hyers–Ulam stability of the following additive-quadratic-cubic-quartic functional equation

$$\begin{aligned} &f(x+2y)+f(x-2y) \\ &\quad = 4f(x+y)+4f(x-y)-6f(x)+f(2y)+f(-2y)-4f(y)-4f(-y) \end{aligned}$$

in matrix random normed spaces.

Keywords Hyers–Ulam stability • Matrix random normed space • Additive-quadratic-cubic-quartic functional equation

1 Introduction and Preliminaries

The abstract characterization given for linear spaces of bounded Hilbert space operators in terms of *matricially normed spaces* [26] implies that quotients, mapping spaces and various tensor products of operator spaces may again be regarded as operator spaces. Owing in part to this result, the theory of operator spaces is having an increasingly significant effect on operator algebra theory (see [8]).

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The proof given in [26] appealed to the theory of ordered operator spaces [3]. Effros and Ruan [9] showed that one can give a purely metric proof of this important theorem by using a technique of Pisier [22] and Haagerup [17] (as modified in [7]).

The stability problem of functional equations originated from a question of Ulam [30] concerning the stability of group homomorphisms. The functional equation

$$f(x+y) = f(x) + f(y)$$

is called the *Cauchy additive functional equation*. In particular, every solution of the Cauchy additive functional equation is said to be an *additive mapping*. Hyers [18] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [23] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [13] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

In 1990, Rassias [24] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Gajda [12] following the same approach as in Rassias [23] gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [12], as well as by Rassias and Šemrl [25] that one cannot prove a Rassias' type theorem when $p = 1$ (cf. the books of Czerwik [6], Hyers, Isac and Rassias [19]).

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1}$$

is related to a symmetric bi-additive mapping. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic functional equation (1) is said to be a quadratic mapping. The Hyers–Ulam stability problem for the quadratic functional equation (1) was proved by Skof for mappings $f : A \to B$, where A is a normed space and B is a Banach space (see [29]). Cholewa [4] noticed that the theorem of Skof is still true if relevant domain A is replaced by an abelian group. In [5], Czerwik proved the Hyers–Ulam stability of the functional equation (1). Grabiec [14] has generalized these results mentioned above.

In [20], Jun and Kim considered the following cubic functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x). \tag{2}$$

It is easy to show that the function $f(x) = x^3$ satisfies the functional equation (2), which is called a *cubic functional equation* and every solution of the cubic functional equation is said to be a *cubic mapping*.

In [21], Lee et al. considered the following quartic functional equation

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y). \tag{3}$$

It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (3), which is called a *quartic functional equation* and every solution of the quartic functional equation is said to be a *quartic mapping*.

In the sequel, we adopt the usual terminology, notations and conventions of the theory of random normed spaces, as in [2, 27, 28]. Throughout this paper, Δ^+ is the space of distribution functions, that is, the space of all mappings $F : \mathbf{R} \cup \{-\infty, \infty\} \to [0, 1]$ such that F is left-continuous and non-decreasing on $\mathbf{R}$, $F(0) = 0$ and $F(+\infty) = 1$. D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^- F(+\infty) = 1$, where $l^- f(x)$ denotes the left limit of the function f at the point x, that is, $l^- f(x) = \lim_{t\to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \le G$ if and only if $F(t) \le G(t)$ for all t in $\mathbf{R}$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by $\varepsilon_0(t) = 0$ if $t \le 0$ and $= 1$ if $t > 0$.

Definition 1 ([27]). A mapping $T : [0, 1]\times[0, 1] \to [0, 1]$ is a continuous triangular norm (briefly, a continuous t-norm) if T satisfies the following conditions:

(a) T is commutative and associative;
(b) T is continuous;
(c) $T(a, 1) = a$ for all $a \in [0, 1]$;
(d) $T(a, b) \le T(c, d)$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0, 1]$.

Typical examples of continuous t-norms are $T_P(a, b) = ab$, $T_M(a, b) = \min(a, b)$ and $T_L(a, b) = \max(a + b - 1, 0)$ (the Lukasiewicz t-norm). Recall (see [15, 16]) that if T is a t-norm and $\{x_n\}$ is a given sequence of numbers in [0, 1], then $T^n_{i=1}x_i$ is defined recurrently by $T^1_{i=1}x_i = x_1$ and $T^n_{i=1}x_i = T(T^{n-1}_{i=1}x_i, x_n)$ for $n \ge 2$. $T^\infty_{i=n}x_i$ is defined as $T^\infty_{i=1}x_{n+i-1}$. It is known [16] that for the Lukasiewicz t-norm the following implication holds:

$$\lim_{n\to\infty} (T_L)^\infty_{i=1}x_{n+i-1} = 1 \Longleftrightarrow \sum_{n=1}^{\infty}(1 - x_n) < \infty.$$

Definition 2 ([28]). A *random normed space* (briefly, RN-space) is a triple (X, μ, T), where X is a vector space, T is a continuous t-norm and μ is a mapping from X into D^+ such that the following conditions hold:

(RN_1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
(RN_2) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X$, $\alpha \ne 0$;
(RN_3) $\mu_{x+y}(t + s) \ge T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and all $t, s \ge 0$.

Every normed space $(X, \|.\|)$ defines a random normed space (X, μ, T_M), where

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

for all $t > 0$, and T_M is the minimum t-norm. This space is called the induced random normed space.

Definition 3. Let (X, μ, T) be an RN-space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to x in X if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x}(\epsilon) > 1 - \lambda$ whenever $n \geq N$.
(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x_m}(\epsilon) > 1 - \lambda$ whenever $n \geq m \geq N$.
(3) An RN-space (X, μ, T) is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X.

Theorem 1 ([27]). *If (X, μ, T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.*

We introduce the concept of matrix random normed space.

Definition 4. Let (X, μ) be a random normed space. Then

(1) $(X, \{\mu^{(n)}\}, T)$ is called a *matrix random normed space* if for each positive integer n, $(M_n(X), \mu^{(n)}, T)$ is a random normed space and $\mu^{(k)}_{AxB}(t) \geq \mu^{(n)}_x\left(\frac{t}{\|A\|\cdot\|B\|}\right)$ for all $t > 0$, $A \in M_{k,n}(\mathbf{R})$, $x = [x_{ij}] \in M_n(X)$ and $B \in M_{n,k}(\mathbf{R})$ with $\|A\| \cdot \|B\| \neq 0$.
(2) $(X, \{\mu^{(n)}\}, T)$ is called a *matrix random Banach space* if (X, μ) is a random Banach space and $(X, \{\mu^{(n)}\})$ is a matrix random normed space.

Example 1. Let $(X, \{\|\cdot\|_n\})$ be a matrix normed space. Let $\mu^{(n)}_x(t) := \frac{t}{t+\|x\|_n}$ for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$. Then

$$\mu^{(k)}_{AxB}(t) = \frac{t}{t + \|AxB\|_k} \geq \frac{t}{t + \|A\| \cdot \|x\|_n \cdot \|B\|} = \frac{\frac{t}{\|A\|\cdot\|B\|}}{\frac{t}{\|A\|\cdot\|B\|} + \|x\|_n}$$
$$= \mu^{(n)}_x\left(\frac{t}{\|A\| \cdot \|B\|}\right)$$

for all $t > 0$, $A \in M_{k,n}(\mathbf{R})$, $x = [x_{ij}] \in M_n(X)$ and $B \in M_{n,k}(\mathbf{R})$ with $\|A\|\cdot\|B\| \neq 0$. If $T(a, b) = \min\{a, b\}$, then $(X, \{\mu^{(n)}\}, T)$ is a matrix random normed space.

Let E, F be vector spaces. For a given mapping $h : E \to F$ and a given positive integer n, define $h_n : M_n(E) \to M_n(F)$ by

$$h_n([x_{ij}]) = [h(x_{ij})]$$

for all $[x_{ij}] \in M_n(E)$.

The aim of this paper is to investigate the Hyers–Ulam stability of the additive-quadratic-cubic-quartic functional equation

$$f(x + 2y) + f(x - 2y) = 4f(x + y) + 4f(x - y) - 6f(x) + f(2y) + f(-2y) - 4f(y) - 4f(-y) \quad (4)$$

in matrix random normed spaces.

One can easily show that an odd mapping $f : X \to Y$ satisfies (4) if and only if the odd mapping $f : X \to Y$ is an additive-cubic mapping, i.e.,

$$f(x+2y) + f(x-2y) = 4f(x+y) + 4f(x-y) - 6f(x).$$

It was shown in [11, Lemma 2.2] that $g(x) := f(2x) - 8f(x)$ and $h(x) := f(2x) - 2f(x)$ are additive and cubic, respectively, and that $f(x) = \frac{1}{6}h(x) - \frac{1}{6}g(x)$.

One can easily show that an even mapping $f : X \to Y$ satisfies (4) if and only if the even mapping $f : X \to Y$ is a quadratic-quartic mapping, i.e.,

$$f(x+2y) + f(x-2y) = 4f(x+y) + 4f(x-y) - 6f(x) + 2f(2y) - 8f(y).$$

It was shown in [10, Lemma 2.1] that $g(x) := f(2x) - 16f(x)$ and $h(x) := f(2x) - 4f(x)$ are quadratic and quartic, respectively, and that $f(x) = \frac{1}{12}h(x) - \frac{1}{12}g(x)$.

Lemma 1. *Each mapping $f : X \to Y$ satisfying (4) can be realized as the sum of an additive mapping, a quadratic mapping, a cubic mapping and a quartic mapping.*

This paper is organized as follows: In Sect. 2, we prove the Hyers–Ulam stability of the additive-quadratic-cubic-quartic functional equation (4) in matrix random normed spaces for an odd mapping case. In Sect. 3, we prove the Hyers–Ulam stability of the additive-quadratic-cubic-quartic functional equation (4) in matrix random normed spaces for an even mapping case.

Throughout this paper, assume that X is a real vector space and that $(Y, \mu^{(n)}, T)$ is a matrix random Banach space.

2 Hyers–Ulam Stability of the AQCQ-Functional Equation (4) in Matrix Random Normed Spaces: Odd Mapping Case

In this section, we prove the Hyers–Ulam stability of the AQCQ-functional equation (4) in matrix random normed spaces for an odd mapping case.

We will use the following notations:

$M_n(X)$ is the set of all $n \times n$-matrices in X;
$e_j \in M_{1,n}(\mathbf{R})$ is that j-th component is 1 and the other components are zero;
$E_{ij} \in M_n(\mathbf{R})$ is that (i, j)-component is 1 and the other components are zero;
$E_{ij} \otimes x \in M_n(X)$ is that (i, j)-component is x and the other components are zero.

Lemma 2. *Let $(X, \{\mu^{(n)}\}, T)$ be a matrix random normed space. Let $\mu^{(1)} = \mu$.*

(1) $\mu^{(n)}_{E_{kl}\otimes x}(t) = \mu_x(t)$ *for all $t > 0$ and $x \in X$.*

(2) *For all* $[x_{ij}] \in M_n(X)$ *and* $t = \sum_{i,j=1}^n t_{ij}$,

$$\mu_{x_{kl}}(t) \geq \mu_{[x_{ij}]}^{(n)}(t) \geq T^{n^2}\left(\mu_{x_{11}}(t_{11}), \mu_{x_{12}}(t_{12}), \cdots, \mu_{x_{nn}}(t_{nn})\right),$$
$$\mu_{x_{kl}}(t) \geq \mu_{[x_{ij}]}^{(n)}(t) \geq T^{n^2}\left(\mu_{x_{11}}\left(\frac{t}{n^2}\right), \mu_{x_{12}}\left(\frac{t}{n^2}\right), \cdots, \mu_{x_{nn}}\left(\frac{t}{n^2}\right)\right)$$

(3) $\lim_{n\to\infty} x_n = x$ *if and only if* $\lim_{n\to\infty} x_{ijn} = x_{ij}$ *for* $x_n = [x_{ijn}], x = [x_{ij}] \in M_k(X)$.

Proof. (1) Since $E_{kl} \otimes x = e_k^* x e_l$ and $\|e_k^*\| = \|e_l\| = 1$, $\mu_{E_{kl}\otimes x}^{(n)}(t) \geq \mu_x(t)$. Since $e_k(E_{kl} \otimes x)e_l^* = x$, $\mu_{E_{kl}\otimes x}^{(n)}(t) \leq \mu_x(t)$. So $\mu_{E_{kl}\otimes x}^{(n)}(t) = \mu_x(t)$.

(2)

$$\begin{aligned}\mu_{[x_{ij}]}^{(n)}(t) &= \mu_{\sum_{i,j=1}^n E_{ij}\otimes x_{ij}}^{(n)}(t)\\ &\geq T^{n^2}\left(\mu_{E_{11}\otimes x_{11}}^{(n)}(t_{11}), \mu_{E_{12}\otimes x_{12}}^{(n)}(t_{12}), \cdots, \mu_{E_{nn}\otimes x_{nn}}^{(n)}(t_{nn})\right)\\ &= T^{n^2}\left(\mu_{x_{11}}(t_{11}), \mu_{x_{12}}(t_{12}), \cdots, \mu_{x_{nn}}(t_{nn})\right),\end{aligned}$$

where $t = \sum_{i,j=1}^n t_{ij}$. In particular,

$$\mu_{[x_{ij}]}^{(n)}(t) \geq T^{n^2}\left(\mu_{x_{11}}\left(\frac{t}{n^2}\right), \mu_{x_{12}}\left(\frac{t}{n^2}\right), \cdots, \mu_{x_{nn}}\left(\frac{t}{n^2}\right)\right).$$

So $\mu_{x_{kl}}(t) = \mu_{e_k x e_l^*}(t) \geq \mu_x^{(n)}\left(\frac{t}{\|e_k\|\cdot\|e_l^*\|}\right) = \mu_x^{(n)}(t)$.

(3) By $\mu_{x_{kl}}(t) \geq \mu_{[x_{ij}]}^{(n)}(t) \geq T^{n^2}\left(\mu_{x_{11}}\left(\frac{t}{n^2}\right), \mu_{x_{12}}\left(\frac{t}{n^2}\right), \cdots, \mu_{x_{nn}}\left(\frac{t}{n^2}\right)\right)$, we obtain the result.

For a mapping $f : X \to Y$, define $Df : X^2 \to Y$ and $Df_n : M_n(X^2) \to M_n(Y)$ by

$$\begin{aligned}Df(a,b) &:= f(a+2b) + f(a-2b) - 4f(a+b) - 4f(a-b) + 6f(a)\\ &\quad -f(2b) - f(-2b) + 4f(b) + 4f(-b),\end{aligned}$$
$$\begin{aligned}Df_n([x_{ij}],[y_{ij}]) &:= f_n([x_{ij}]+2[y_{ij}]) + f_n([x_{ij}]-2[y_{ij}]) - 4f_n([x_{ij}]+[y_{ij}])\\ &\quad -4f_n([x_{ij}]-[y_{ij}]) + 6f_n([x_{ij}]) - f_n(2[y_{ij}]) - f_n(-2[y_{ij}]) + 4f_n([y_{ij}])\\ &\quad +4f_n(-[y_{ij}])\end{aligned}$$

for all $a, b \in X$ and all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Theorem 2. *Let $f : X \to Y$ be an odd mapping for which there is a $\rho : X^2 \to D^+$ ($\rho(a,b)$ is denoted by $\rho_{a,b}$) such that*

$$\mu_{Df(a,b)}(t) \geq \rho_{a,b}(t) \tag{5}$$

for all $a, b \in X$ and all $t > 0$. If

$$\lim_{l\to\infty} T_{k=1}^{\infty}\left(T\left(\rho_{2^{k+l-1}a,2^{k+l-1}a}\left(2^{l-3}t\right), \rho_{2^{k+l}a,2^{k+l-1}a}\left(2^{l-1}t\right)\right)\right) = 1$$

and

$$\lim_{l\to\infty} \rho_{2^l a,2^l b}(2^l t) = 1$$

for all $a, b \in X$ and all $t > 0$, then there exist a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that

$$\begin{aligned}
&\mu^{(n)}_{f_n(2[x_{ij}])-8f_n([x_{ij}])-A_n([x_{ij}])}(t)\\
&\geq T_{k=1}^{\infty}\left(T^{n^2+1}\left(\rho_{2^{k-1}x_{11},2^{k-1}x_{11}}\left(\frac{t}{8n^2}\right), \rho_{2^k x_{11},2^{k-1}x_{11}}\left(\frac{t}{2n^2}\right),\right.\right.\\
&\qquad \rho_{2^{k-1}x_{12},2^{k-1}x_{12}}\left(\frac{t}{8n^2}\right), \rho_{2^k x_{12},2^{k-1}x_{12}}\left(\frac{t}{2n^2}\right), \cdots,\\
&\qquad \left.\left.\rho_{2^{k-1}x_{nn},2^{k-1}x_{nn}}\left(\frac{t}{8n^2}\right), \rho_{2^k x_{nn},2^{k-1}x_{nn}}\left(\frac{t}{2n^2}\right)\right)\right),
\end{aligned} \tag{6}$$

$$\begin{aligned}
&\mu^{(n)}_{f_n(2[x_{ij}])-2f_n([x_{ij}])-C_n([x_{ij}])}(t)\\
&\geq T_{k=1}^{\infty}\left(T^{n^2+1}\left(\rho_{2^{k-1}x_{11},2^{k-1}x_{11}}\left(\frac{t}{8n^2}\right), \rho_{2^k x_{11},2^{k-1}x_{11}}\left(\frac{t}{2n^2}\right),\right.\right.\\
&\qquad \rho_{2^{k-1}x_{12},2^{k-1}x_{12}}\left(\frac{t}{8n^2}\right), \rho_{2^k x_{12},2^{k-1}x_{12}}\left(\frac{t}{2n^2}\right), \cdots,\\
&\qquad \left.\left.\rho_{2^{k-1}x_{nn},2^{k-1}x_{nn}}\left(\frac{t}{8n^2}\right), \rho_{2^k x_{nn},2^{k-1}x_{nn}}\left(\frac{t}{2n^2}\right)\right)\right)
\end{aligned} \tag{7}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$ and all $t > 0$.

Proof. Putting $a = b$ in (5), we get

$$\mu_{f(3b)-4f(2b)+5f(b)}(t) \geq \rho_{b,b}(t) \tag{8}$$

for all $b \in X$ and all $t > 0$. Replacing a by $2b$ in (5), we get

$$\mu_{f(4b)-4f(3b)+6f(2b)-4f(b)}(t) \geq \rho_{2b,b}(t) \tag{9}$$

for all $b \in X$ and all $t > 0$. It follows from (8) and (9) that

$$\begin{aligned}
&\mu_{f(4a)-10f(2a)+16f(a)}(t)\\
&\quad = \mu_{(4f(3a)-16f(2a)+20f(a))+(f(4a)-4f(3a)+6f(2a)-4f(a))}(t)\\
&\quad \geq T\left(\mu_{4f(3a)-16f(2a)+20f(a)}\left(\frac{t}{2}\right), \mu_{f(4a)-4f(3a)+6f(2a)-4f(a)}\left(\frac{t}{2}\right)\right)\\
&\quad \geq T\left(\rho_{a,a}\left(\frac{t}{8}\right), \rho_{2a,a}\left(\frac{t}{2}\right)\right)
\end{aligned}$$

for all $a \in X$ and all $t > 0$. Let $g : X \to Y$ be a mapping defined by $g(a) := f(2a) - 8f(a)$. Then we conclude that

$$\mu_{g(2a)-2g(a)}(t) \geq T\left(\rho_{a,a}\left(\frac{t}{8}\right), \rho_{2a,a}\left(\frac{t}{2}\right)\right)$$

for all $a \in X$ and all $t > 0$. Thus we have

$$\mu_{\frac{g(2a)}{2}-g(a)}(t) \geq T\left(\rho_{a,a}\left(\frac{t}{4}\right), \rho_{2a,a}(t)\right)$$

for all $a \in X$ and all $t > 0$. Hence

$$\mu_{\frac{g(2^{k+1}a)}{2^{k+1}}-\frac{g(2^k a)}{2^k}}(t) \geq T\left(\rho_{2^k a,2^k a}\left(2^{k-2}t\right), \rho_{2^{k+1}a,2^k a}\left(2^k t\right)\right)$$

for all $a \in X$, all $t > 0$ and all $k \in \mathbf{N}$. From $1 > \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^l}$, it follows that

$$\begin{aligned}
\mu_{\frac{g(2^l a)}{2^l}-g(a)}(t) &\geq T_{k=1}^{l}\left(\mu_{\frac{g(2^k a)}{2^k}-\frac{g(2^{k-1}a)}{2^{k-1}}}\left(\frac{t}{2^k}\right)\right)\\
&\geq T_{k=1}^{l}\left(T\left(\rho_{2^{k-1}a,2^{k-1}a}\left(\frac{t}{8}\right), \rho_{2^k a,2^{k-1}a}\left(\frac{t}{2}\right)\right)\right) \qquad (10)
\end{aligned}$$

for all $a \in X$ and all $t > 0$. In order to prove the convergence of the sequence $\left\{\frac{g(2^l a)}{2^l}\right\}$, replacing a with $2^m a$ in (10), we obtain that

$$\begin{aligned}
&\mu_{\frac{g(2^{l+m}a)}{2^{l+m}}-\frac{g(2^m a)}{2^m}}(t)\\
&\quad \geq T_{k=1}^{l}\left(T\left(\rho_{2^{k+m-1}a,2^{k+m-1}a}\left(2^{m-3}t\right), \rho_{2^{k+m}a,2^{k+m-1}a}\left(2^{m-1}t\right)\right)\right). \qquad (11)
\end{aligned}$$

Since the right-hand side of the inequality (11) tends to 1 as m and l tend to infinity, the sequence $\left\{\frac{g(2^l a)}{2^l}\right\}$ is a Cauchy sequence. Thus we may define $A(a) = \lim_{l\to\infty} \frac{g(2^l a)}{2^l}$ for all $a \in X$.

Now we show that A is an additive mapping. Replacing a and b with $2^l a$ and $2^l b$ in (5), respectively, we get

$$\mu_{\frac{Df(2^l a, 2^l b)}{2^l}}(t) \geq \rho_{2^l a, 2^l b}(2^l t).$$

Taking the limit as $l \to \infty$, we find that $A : X \to Y$ satisfies (4) for all $a, b \in X$. Since $f : X \to Y$ is odd, $A : X \to Y$ is odd. By [11, Lemma 2.2], the mapping $A : X \to Y$ is additive. Letting the limit as $l \to \infty$ in (10), we get

$$\begin{aligned} &\mu_{f(2a)-8f(a)-A(a)}(t) \\ &\geq T_{k=1}^{\infty}\left(T\left(\rho_{2^{k-1}a, 2^{k-1}a}\left(\frac{t}{8}\right), \rho_{2^k a, 2^{k-1}a}\left(\frac{t}{2}\right)\right)\right) \end{aligned} \tag{12}$$

for all $a \in X$ and all $t > 0$.

Next, we prove the uniqueness of the additive mapping $A : X \to Y$ subject to (12). Let us assume that there exists another additive mapping $L : X \to Y$ which satisfies (12). Since $A(2^l a) = 2^l A(a)$, $L(2^l a) = 2^l L(a)$ for all $a \in X$ and all $l \in \mathbf{N}$, from (12), it follows that

$$\begin{aligned} \mu_{A(a)-L(a)}(2t) &= \mu_{A(2^l a)-L(2^l a)}(2^{l+1} t) \\ &\geq T(\mu_{A(2^l a)-g(2^l a)}(2^l t), \mu_{g(2^l a)-L(2^l a)}(2^l t)) \\ &\geq T\left(T_{k=1}^{\infty}\left(T\left(\rho_{2^{l+k-1}a, 2^{l+k-1}a}\left(2^{l-3}t\right), \rho_{2^{l+k}a, 2^{l+k-1}a}\left(2^{l-1}t\right)\right)\right),\right. \\ &\qquad \left. T_{k=1}^{\infty}\left(T\left(\rho_{2^{l+k-1}a, 2^{l+k-1}a}\left(2^{l-3}t\right), \rho_{2^{l+k}a, 2^{l+k-1}a}\left(2^{l-1}t\right)\right)\right)\right) \end{aligned} \tag{13}$$

for all $a \in X$ and all $t > 0$. Letting $l \to \infty$ in (13), we conclude that $A = L$.

By Lemma 2 and (12),

$$\begin{aligned} &\mu^{(n)}_{g_n([x_{ij}])-A_n([x_{ij}])}(t) \\ &\geq T^{n^2}\left(\mu_{g(x_{11})-A(x_{11})}\left(\frac{t}{n^2}\right), \mu_{g(x_{12})-A(x_{12})}\left(\frac{t}{n^2}\right), \cdots, \mu_{g(x_{nn})-A(x_{nn})}\left(\frac{t}{n^2}\right)\right) \\ &\geq T_{k=1}^{\infty}\left(T^{n^2+1}\left(\rho_{2^{k-1}x_{11}, 2^{k-1}x_{11}}\left(\frac{t}{8n^2}\right), \rho_{2^k x_{11}, 2^{k-1}x_{11}}\left(\frac{t}{2n^2}\right),\right.\right. \\ &\quad \rho_{2^{k-1}x_{12}, 2^{k-1}x_{12}}\left(\frac{t}{8n^2}\right), \rho_{2^k x_{12}, 2^{k-1}x_{12}}\left(\frac{t}{2n^2}\right), \cdots, \\ &\quad \left.\left.\rho_{2^{k-1}x_{nn}, 2^{k-1}x_{nn}}\left(\frac{t}{8n^2}\right), \rho_{2^k x_{nn}, 2^{k-1}x_{nn}}\left(\frac{t}{2n^2}\right)\right)\right) \end{aligned}$$

for all $x = [x_{ij}] \in M_n(X)$ and all $t > 0$. Thus $A : X \to Y$ is a unique additive mapping satisfying (6).

Let $h : X \to Y$ be a mapping defined by $h(a) := f(2a) - 2f(a)$. Then we conclude that

$$\mu_{h(2a)-8h(a)}(t) \geq T\left(\rho_{a,a}\left(\frac{t}{8}\right), \rho_{2a,a}\left(\frac{t}{2}\right)\right)$$

for all $a \in X$ and all $t > 0$. Thus we have

$$\mu_{\frac{h(2a)}{8}-h(a)}(t) \geq T\left(\rho_{a,a}(t), \rho_{2a,a}(4t)\right)$$

for all $a \in X$ and all $t > 0$. Hence

$$\mu_{\frac{h(2^{k+1}a)}{8^{k+1}}-\frac{h(2^k a)}{8^k}}(t) \geq T\left(\rho_{2^k a,2^k a}\left(8^k t\right), \rho_{2^{k+1}a,2^k a}\left(4 \cdot 8^k t\right)\right)$$

for all $a \in X$, all $t > 0$ and all $k \in \mathbf{N}$. From $1 > \frac{1}{8} + \frac{1}{8^2} + \cdots + \frac{1}{8^l}$, it follows that

$$\begin{aligned}\mu_{\frac{h(2^l a)}{8^l}-h(a)}(t) &\geq T_{k=1}^{l}\left(\mu_{\frac{h(2^k a)}{8^k}-\frac{h(2^{k-1}a)}{8^{k-1}}}\left(\frac{t}{8^k}\right)\right)\\ &\geq T_{k=1}^{l}\left(T\left(\rho_{2^{k-1}a,2^{k-1}a}\left(\frac{t}{8}\right), \rho_{2^k a,2^{k-1}a}\left(\frac{t}{2}\right)\right)\right)\end{aligned} \tag{14}$$

for all $a \in X$ and all $t > 0$. In order to prove the convergence of the sequence $\left\{\frac{h(2^l a)}{8^l}\right\}$, replacing a with $2^m a$ in (14), we obtain that

$$\begin{aligned}&\mu_{\frac{h(2^{l+m}a)}{8^{l+m}}-\frac{h(2^m a)}{8^m}}(t)\\ &\geq T_{k=1}^{l}\left(T\left(\rho_{2^{k+m-1}x,2^{k+m-1}a}\left(8^{m-1}t\right), \rho_{2^{k+m}a,2^{k+m-1}a}\left(4 \cdot 8^{m-1}t\right)\right)\right).\end{aligned} \tag{15}$$

Since the right-hand side of the inequality (15) tends to 1 as m and l tend to infinity, the sequence $\left\{\frac{h(2^l a)}{8^l}\right\}$ is a Cauchy sequence. Thus we may define $C(a) = \lim_{l\to\infty} \frac{h(2^l a)}{8^l}$ for all $a \in X$.

Now we show that C is a cubic mapping. Replacing a and b with $2^l a$ and $2^l b$ in (5), respectively, we get

$$\mu_{\frac{Df(2^l a,2^l b)}{8^l}}(t) \geq \rho_{2^l a,2^l b}(8^n t) \geq \rho_{2^l a,2^l b}(2^l t).$$

Taking the limit as $l \to \infty$, we find that $C : X \to Y$ satisfies (4) for all $a, b \in X$. Since $f : X \to Y$ is odd, $C : X \to Y$ is odd. By [11, Lemma 2.2], the mapping $C : X \to Y$ is cubic. Letting the limit as $l \to \infty$ in (14), we get

$$\mu_{f(2a)-2f(a)-C(a)}(t) \geq T_{k=1}^{\infty}\left(T\left(\rho_{2^{k-1}a,2^{k-1}a}\left(\frac{t}{8}\right), \rho_{2^k a,2^{k-1}a}\left(\frac{t}{2}\right)\right)\right) \tag{16}$$

for all $a \in X$ and all $t > 0$.

Next, we prove the uniqueness of the cubic mapping $C : X \to Y$ subject to (16). Let us assume that there exists another cubic mapping $L : X \to Y$ which satisfies (16). Since $C(2^l a) = 8^l C(a)$, $L(2^l a) = 8^l L(a)$ for all $a \in X$ and all $l \in \mathbf{N}$, from (16), it follows that

$$\begin{aligned}\mu_{C(a)-L(a)}(2t) &= \mu_{C(2^l a)-L(2^l a)}(2\cdot 8^l t)\\ &\geq T(\mu_{C(2^l a)-h(2^l a)}(8^l t), \mu_{h(2^l a)-L(2^l a)}(8^l t))\\ &\geq T\left(T_{k=1}^{\infty}\left(T\left(\rho_{2^{l+k-1}a,2^{l+k-1}a}\left(8^{l-1}t\right), \rho_{2^{l+k}a,2^{l+k-1}a}\left(4\cdot 8^{l-1}t\right)\right)\right),\right.\\ &\qquad \left.T_{k=1}^{\infty}\left(T\left(\rho_{2^{l+k-1}a,2^{l+k-1}a}\left(8^{l-1}t\right), \rho_{2^{l+k}a,2^{l+k-1}a}\left(4\cdot 8^{l-1}t\right)\right)\right)\right)\\ &\geq T\left(T_{k=1}^{\infty}\left(T\left(\rho_{2^{l+k-1}a,2^{l+k-1}a}\left(2^{l-3}t\right), \rho_{2^{l+k}a,2^{l+k-1}a}\left(2^{l-1}t\right)\right)\right),\right.\\ &\qquad \left.T_{k=1}^{\infty}\left(T\left(\rho_{2^{l+k-1}a,2^{l+k-1}a}\left(2^{l-3}t\right), \rho_{2^{l+k}a,2^{l+k-1}a}\left(2^{l-1}t\right)\right)\right)\right)\end{aligned} \tag{17}$$

for all $a \in X$ and all $t > 0$. Letting $l \to \infty$ in (17), we conclude that $C = L$.

By Lemma 2 and (16),

$$\begin{aligned}&\mu^{(n)}_{h_n([x_{ij}])-C_n([x_{ij}])}(t)\\ &\geq T^{n^2}\left(\mu_{h(x_{11})-C(x_{11})}\left(\frac{t}{n^2}\right), \mu_{h(x_{12})-C(x_{12})}\left(\frac{t}{n^2}\right), \cdots, \mu_{h(x_{nn})-C(x_{nn})}\left(\frac{t}{n^2}\right)\right)\\ &\geq T_{k=1}^{\infty}\left(T^{n^2+1}\left(\rho_{2^{k-1}x_{11},2^{k-1}x_{11}}\left(\frac{t}{8n^2}\right), \rho_{2^k x_{11},2^{k-1}x_{11}}\left(\frac{t}{2n^2}\right),\right.\right.\\ &\qquad \rho_{2^{k-1}x_{12},2^{k-1}x_{12}}\left(\frac{t}{8n^2}\right), \rho_{2^k x_{12},2^{k-1}x_{12}}\left(\frac{t}{2n^2}\right), \cdots,\\ &\qquad \left.\left.\rho_{2^{k-1}x_{nn},2^{k-1}x_{nn}}\left(\frac{t}{8n^2}\right), \rho_{2^k x_{nn},2^{k-1}x_{nn}}\left(\frac{t}{2n^2}\right)\right)\right)\end{aligned}$$

for all $x = [x_{ij}] \in M_n(X)$ and all $t > 0$. Thus $C : X \to Y$ is a unique cubic mapping satisfying (7).

Similarly, one can obtain the following result.

Theorem 3. *Let $f : X \to Y$ be an odd mapping for which there is a $\rho : X^2 \to D^+$ ($\rho(a,b)$ is denoted by $\rho_{a,b}$) satisfying (5). If*

$$\lim_{l\to\infty} T_{k=1}^{\infty}\left(T\left(\rho_{\frac{a}{2^{k+l}},\frac{a}{2^{k+l}}}\left(\frac{t}{8^{l+2k}}\right), \rho_{\frac{a}{2^{k+l-1}},\frac{a}{2^{k+l}}}\left(\frac{4t}{8^{l+2k}}\right)\right)\right) = 1$$

and

$$\lim_{l\to\infty} \rho_{\frac{a}{2^l},\frac{b}{2^l}}\left(\frac{t}{8^l}\right) = 1$$

for all $a, b \in X$ and all $t > 0$, then there exist a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that

$$\begin{aligned}
&\mu^{(n)}_{f_n(2[x_{ij}])-8f_n([x_{ij}])-A_n([x_{ij}])}(t)\\
&\geq T_{k=1}^{\infty}\left(T^{n^2+1}\left(\rho_{\frac{x_{11}}{2^k},\frac{x_{11}}{2^k}}\left(\frac{t}{2^{2k+1}n^2}\right), \rho_{\frac{x_{11}}{2^{k-1}},\frac{x_{11}}{2^k}}\left(\frac{t}{2^{2k-1}n^2}\right),\right.\right.\\
&\qquad \rho_{\frac{x_{12}}{2^k},\frac{x_{12}}{2^k}}\left(\frac{t}{2^{2k+1}n^2}\right), \rho_{\frac{x_{12}}{2^{k-1}},\frac{x_{12}}{2^k}}\left(\frac{t}{2^{2k-1}n^2}\right), \cdots,\\
&\qquad \left.\left.\rho_{\frac{x_{nn}}{2^k},\frac{x_{nn}}{2^k}}\left(\frac{t}{2^{2k+1}n^2}\right), \rho_{\frac{x_{nn}}{2^{k-1}},\frac{x_{nn}}{2^k}}\left(\frac{t}{2^{2k-1}n^2}\right)\right)\right),
\end{aligned}$$

$$\begin{aligned}
&\mu^{(n)}_{f_n(2[x_{ij}])-2f_n([x_{ij}])-C_n([x_{ij}])}(t)\\
&\geq T_{k=1}^{\infty}\left(T^{n^2+1}\left(\rho_{\frac{x_{11}}{2^k},\frac{x_{11}}{2^k}}\left(\frac{t}{8^{2k}n^2}\right), \rho_{\frac{x_{11}}{2^{k-1}},\frac{x_{11}}{2^k}}\left(\frac{4t}{8^{2k}n^2}\right),\right.\right.\\
&\qquad \rho_{\frac{x_{12}}{2^k},\frac{x_{12}}{2^k}}\left(\frac{t}{8^{2k}n^2}\right), \rho_{\frac{x_{12}}{2^{k-1}},\frac{x_{12}}{2^k}}\left(\frac{4t}{8^{2k}n^2}\right), \cdots,\\
&\qquad \left.\left.\rho_{\frac{x_{nn}}{2^k},\frac{x_{nn}}{2^k}}\left(\frac{t}{8^{2k}n^2}\right), \rho_{\frac{x_{nn}}{2^{k-1}},\frac{x_{nn}}{2^k}}\left(\frac{4t}{8^{2k}n^2}\right)\right)\right)
\end{aligned}$$

for all $x = [x_{ij}] \in M_n(X)$ and all $t > 0$.

3 Hyers–Ulam Stability of the AQCQ-Functional Equation (4) in Matrix Random Normed Spaces: Odd Mapping Case

In this section, we prove the Hyers–Ulam stability of the AQCQ-functional equation (4) in matrix random normed spaces for an even mapping case.

Theorem 4. *Let $f : X \to Y$ be an even mapping for which there is a $\rho : X^2 \to D^+$ ($\rho(a,b)$ is denoted by $\rho_{a,b}$) satisfying $f(0) = 0$ and (5). If*

$$\lim_{l\to\infty} T_{k=1}^{\infty}\left(T\left(\rho_{2^{k+l-1}a,2^{k+l-1}a}\left(2\cdot 4^{l-2}t\right), \rho_{2^{k+l}a,2^{k+l-1}a}\left(2\cdot 4^{l-1}t\right)\right)\right) = 1$$

and

$$\lim_{l\to\infty}\rho_{2^l a,2^l b}(4^l t)=1$$

for all $a, b \in X$ *and all* $t > 0$*, then there exist a unique quadratic mapping* $P : X \to Y$ *and a unique quartic mapping* $Q : X \to Y$ *such that*

$$\begin{aligned}
&\mu^{(n)}_{f_n(2[x_{ij}])-16f_n([x_{ij}])-P_n([x_{ij}])}(t)\\
&\geq T_{k=1}^{\infty}\left(T^{n^2+1}\left(\rho_{2^{k-1}x_{11},2^{k-1}x_{11}}\left(\frac{t}{8n^2}\right),\rho_{2^k x_{11},2^{k-1}x_{11}}\left(\frac{t}{2n^2}\right),\right.\right.\\
&\qquad\rho_{2^{k-1}x_{12},2^{k-1}x_{12}}\left(\frac{t}{8n^2}\right),\rho_{2^k x_{12},2^{k-1}x_{12}}\left(\frac{t}{2n^2}\right),\cdots,\\
&\qquad\left.\left.\rho_{2^{k-1}x_{nn},2^{k-1}x_{nn}}\left(\frac{t}{8n^2}\right),\rho_{2^k x_{nn},2^{k-1}x_{nn}}\left(\frac{t}{2n^2}\right)\right)\right),
\end{aligned}$$

$$\begin{aligned}
&\mu^{(n)}_{f_n(2[x_{ij}])-4f_n([x_{ij}])-Q_n([x_{ij}])}(t)\\
&\geq T_{k=1}^{\infty}\left(T^{n^2+1}\left(\rho_{2^{k-1}x_{11},2^{k-1}x_{11}}\left(\frac{t}{8n^2}\right),\rho_{2^k x_{11},2^{k-1}x_{11}}\left(\frac{t}{2n^2}\right),\right.\right.\\
&\qquad\rho_{2^{k-1}x_{12},2^{k-1}x_{12}}\left(\frac{t}{8n^2}\right),\rho_{2^k x_{12},2^{k-1}x_{12}}\left(\frac{t}{2n^2}\right),\cdots,\\
&\qquad\left.\left.\rho_{2^{k-1}x_{nn},2^{k-1}x_{nn}}\left(\frac{t}{8n^2}\right),\rho_{2^k x_{nn},2^{k-1}x_{nn}}\left(\frac{t}{2n^2}\right)\right)\right)
\end{aligned}$$

for all $x = [x_{ij}] \in M_n(X)$ *and all* $t > 0$.

Proof. Putting $a = b$ in (5), we get

$$\mu_{f(3b)-6f(2b)+15f(b)}(t)\geq\rho_{b,b}(t) \tag{18}$$

for all $b \in X$ and all $t > 0$. Replacing a by $2b$ in (5), we get

$$\mu_{f(4b)-4f(3b)+4f(2b)+4f(b)}(t)\geq\rho_{2b,b}(t) \tag{19}$$

for all $b \in X$ and all $t > 0$. It follows from (18) and (19) that

$$\begin{aligned}
&\mu_{f(4a)-20f(2a)+64f(a)}(t)\\
&=\mu_{(4f(3a)-24f(2a)+60f(a))+(f(4a)-4f(3a)+4f(2a)+4f(a))}(t)\\
&\geq T\left(\mu_{4f(3a)-24f(2a)+60f(a)}\left(\frac{t}{2}\right),\mu_{f(4a)-4f(3a)+4f(2a)+4f(a)}\left(\frac{t}{2}\right)\right)\\
&\geq T\left(\rho_{a,a}\left(\frac{t}{8}\right),\rho_{2a,a}\left(\frac{t}{2}\right)\right)
\end{aligned}$$

for all $a \in X$ and all $t > 0$. Let $g : X \to Y$ be a mapping defined by $g(a) := f(2a) - 16f(a)$. Then we conclude that

$$\mu_{g(2a)-4g(a)}(t) \geq T\left(\rho_{a,a}\left(\frac{t}{8}\right), \rho_{2a,a}\left(\frac{t}{2}\right)\right)$$

for all $a \in X$ and all $t > 0$. Thus we have

$$\mu_{\frac{g(2a)}{4}-g(a)}(t) \geq T\left(\rho_{a,a}\left(\frac{t}{2}\right), \rho_{2a,a}\left(2t\right)\right)$$

for all $a \in X$ and all $t > 0$. Hence

$$\mu_{\frac{g(2^{k+1}a)}{4^{k+1}}-\frac{g(2^k a)}{4^k}}(t) \geq T\left(\rho_{2^k a,2^k a}\left(2 \cdot 4^{k-1}t\right), \rho_{2^{k+1}a,2^k a}\left(2 \cdot 4^k t\right)\right)$$

for all $a \in X$, all $t > 0$ and all $k \in \mathbf{N}$. From $1 > \frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^l}$, it follows that

$$\begin{aligned} \mu_{\frac{g(2^l a)}{4^l}-g(a)}(t) &\geq T_{k=1}^{l}\left(\mu_{\frac{g(2^k a)}{4^k}-\frac{g(2^{k-1}a)}{4^{k-1}}}\left(\frac{t}{4^k}\right)\right) \\ &\geq T_{k=1}^{l}\left(T\left(\rho_{2^{k-1}a,2^{k-1}a}\left(\frac{t}{8}\right), \rho_{2^k a,2^{k-1}a}\left(\frac{t}{2}\right)\right)\right) \end{aligned} \tag{20}$$

for all $a \in X$ and all $t > 0$. In order to prove the convergence of the sequence $\left\{\frac{g(2^l a)}{4^l}\right\}$, replacing a with $2^m a$ in (20), we obtain that

$$\begin{aligned} &\mu_{\frac{g(2^{l+m}a)}{4^{l+m}}-\frac{g(2^m a)}{4^m}}(t) \\ &\geq T_{k=1}^{l}\left(T\left(\rho_{2^{k+m-1}a,2^{k+m-1}a}\left(2 \cdot 4^{m-2}t\right), \rho_{2^{k+m}a,2^{k+m-1}a}\left(2 \cdot 4^{m-1}t\right)\right)\right). \end{aligned} \tag{21}$$

Since the right-hand side of the inequality (21) tends to 1 as m and l tend to infinity, the sequence $\left\{\frac{g(2^l a)}{4^l}\right\}$ is a Cauchy sequence. Thus we may define $P(a) = \lim_{l\to\infty} \frac{g(2^l a)}{4^l}$ for all $a \in X$.

Now we show that P is a quadratic mapping. Replacing a and b with $2^l a$ and $2^l b$ in (5), respectively, we get

$$\mu_{\frac{Df(2^l a,2^l b)}{4^l}}(t) \geq \rho_{2^l a,2^l b}(4^l t).$$

Taking the limit as $l \to \infty$, we find that $P : X \to Y$ satisfies (4) for all $a, b \in X$. Since $f : X \to Y$ is even, $P : X \to Y$ is even. By [10, Lemma 2.1], the mapping $P : X \to Y$ is quadratic. Letting the limit as $l \to \infty$ in (20), we get

$$\mu_{f(2a)-16f(a)-P(a)}(t) \geq T_{k=1}^{\infty}\left(T\left(\rho_{2^{k-1}a,2^{k-1}a}\left(\frac{t}{8}\right), \rho_{2^k a,2^{k-1}a}\left(\frac{t}{2}\right)\right)\right), \tag{22}$$

for all $a \in X$ and all $t > 0$.

Next, we prove the uniqueness of the quadratic mapping $P : X \to Y$ subject to (22). Let us assume that there exists another quadratic mapping $L : X \to Y$ which satisfies (22). Since $P(2^l a) = 4^l P(a),\ L(2^l a) = 4^l L(a)$ for all $a \in X$ and all $l \in \mathbf{N}$, from (22), it follows that

$$\begin{aligned}\mu_{P(a)-L(a)}(2t) &= \mu_{P(2^l a)-L(2^l a)}(2\cdot 4^l t)\\ &\geq T(\mu_{P(2^l a)-g(2^l a)}(4^l t), \mu_{g(2^l a)-L(2^l a)}(4^l t))\\ &\geq T\left(T_{k=1}^{\infty}\left(T\left(\rho_{2^{l+k-1}a,2^{l+k-1}a}\left(2\cdot 4^{l-2}t\right), \rho_{2^{l+k}a,2^{l+k-1}a}\left(2\cdot 4^{l-1}t\right)\right)\right),\right.\\ &\qquad \left. T_{k=1}^{\infty}\left(T\left(\rho_{2^{l+k-1}a,2^{l+k-1}a}\left(2\cdot 4^{l-2}t\right), \rho_{2^{l+k}a,2^{l+k-1}a}\left(2\cdot 4^{l-1}t\right)\right)\right)\right) \end{aligned} \tag{23}$$

for all $a \in X$ and all $t > 0$. Letting $l \to \infty$ in (23), we conclude that $P = L$.

Let $h : X \to Y$ be a mapping defined by $h(a) := f(2a) - 4f(a)$. Then we conclude that

$$\mu_{h(2a)-16h(a)}(t) \geq T\left(\rho_{a,a}\left(\frac{t}{8}\right), \rho_{2a,a}\left(\frac{t}{2}\right)\right)$$

for all $a \in X$ and all $t > 0$. Thus we have

$$\mu_{\frac{h(2a)}{16}-h(a)}(t) \geq T\left(\rho_{a,a}\left(2t\right), \rho_{2a,a}\left(8t\right)\right)$$

for all $a \in X$ and all $t > 0$. Hence

$$\mu_{\frac{h(2^{k+1}a)}{16^{k+1}}-\frac{h(2^k a)}{16^k}}(t) \geq T\left(\rho_{2^k a,2^k a}\left(2\cdot 16^k t\right), \rho_{2^{k+1}a,2^k a}\left(8\cdot 16^k t\right)\right)$$

for all $a \in X$, all $t > 0$ and all $k \in \mathbf{N}$. From $1 > \frac{1}{16} + \frac{1}{16^2} + \cdots + \frac{1}{16^l}$, it follows that

$$\begin{aligned}\mu_{\frac{h(2^l a)}{16^l}-h(a)}(t) &\geq T_{k=1}^{l}\left(\mu_{\frac{h(2^k a)}{16^k}-\frac{h(2^{k-1}a)}{16^{k-1}}}\left(\frac{t}{16^k}\right)\right)\\ &\geq T_{k=1}^{l}\left(T\left(\rho_{2^{k-1}a,2^{k-1}a}\left(\frac{t}{8}\right), \rho_{2^k a,2^{k-1}a}\left(\frac{t}{2}\right)\right)\right)\end{aligned} \tag{24}$$

for all $a \in X$ and all $t > 0$. In order to prove the convergence of the sequence $\left\{\frac{h(2^l a)}{16^l}\right\}$, replacing a with $2^m a$ in (24), we obtain that

$$\mu_{\frac{h(2^{l+m}a)}{16^{l+m}} - \frac{h(2^m a)}{16^m}}(t)$$
$$\geq T_{k=1}^{l}\left(T\left(\rho_{2^{k+m-1}a,2^{k+m-1}a}\left(2\cdot 16^{m-1}t\right), \rho_{2^{k+m}a,2^{k+m-1}a}\left(8\cdot 16^{m-1}t\right)\right)\right). \quad (25)$$

Since the right-hand side of the inequality (25) tends to 1 as m and l tend to infinity, the sequence $\left\{\frac{h(2^l a)}{16^l}\right\}$ is a Cauchy sequence. Thus we may define $Q(a) = \lim_{l\to\infty} \frac{h(2^l a)}{16^l}$ for all $a \in X$.

Now we show that Q is a quartic mapping. Replacing a and b with $2^l a$ and $2^l b$ in (5), respectively, we get

$$\mu_{\frac{Df(2^l a,2^l b)}{16^l}}(t) \geq \rho_{2^l a,2^l b}(16^l t) \geq \rho_{2^l a,2^l b}(4^l t).$$

Taking the limit as $l \to \infty$, we find that $Q : X \to Y$ satisfies (4) for all $a, b \in X$. Since $f : X \to Y$ is even, $Q : X \to Y$ is even. By [10, Lemma 2.1], the mapping $Q : X \to Y$ is quartic. Letting the limit as $l \to \infty$ in (24), we get

$$\mu_{f(2a)-4f(a)-Q(a)}(t)$$
$$\geq T_{k=1}^{\infty}\left(T\left(\rho_{2^{k-1}a,2^{k-1}a}\left(\frac{t}{8}\right), \rho_{2^k a,2^{k-1}a}\left(\frac{t}{2}\right)\right)\right) \quad (26)$$

for all $a \in X$ and all $t > 0$.

Next, we prove the uniqueness of the quartic mapping $Q : X \to Y$ subject to (26). Let us assume that there exists another quartic mapping $L : X \to Y$ which satisfies (26). Since $Q(2^l a) = 16^l Q(a)$, $L(2^l a) = 16^l L(a)$ for all $a \in X$ and all $l \in \mathbf{N}$, from (26), it follows that

$$\mu_{Q(a)-L(a)}(2t) = \mu_{Q(2^l a)-L(2^l a)}(2\cdot 16^l t)$$
$$\geq T(\mu_{Q(2^l a)-h(2^l a)}(16^l t), \mu_{h(2^l a)-L(2^l a)}(16^l t))$$
$$\geq T\left(T_{k=1}^{\infty}\left(T\left(\rho_{2^{l+k-1}a,2^{l+k-1}a}\left(2\cdot 16^{l-1}t\right), \rho_{2^{l+k}a,2^{l+k-1}a}\left(8\cdot 16^{l-1}t\right)\right)\right),\right.$$
$$\left. T_{k=1}^{\infty}\left(T\left(\rho_{2^{l+k-1}a,2^{l+k-1}a}\left(2\cdot 16^{l-1}t\right), \rho_{2^{l+k}a,2^{l+k-1}a}\left(8\cdot 16^{l-1}t\right)\right)\right)\right)$$
$$\geq T\left(T_{k=1}^{\infty}\left(T\left(\rho_{2^{l+k-1}a,2^{l+k-1}a}\left(2\cdot 4^{l-2}t\right), \rho_{2^{l+k}a,2^{l+k-1}a}\left(2\cdot 4^{l-1}t\right)\right)\right),\right.$$
$$\left. T_{k=1}^{\infty}\left(T\left(\rho_{2^{l+k-1}a,2^{l+k-1}a}\left(2\cdot 4^{l-2}t\right), \rho_{2^{l+k}a,2^{l+k-1}a}\left(2\cdot 4^{l-1}t\right)\right)\right)\right) \quad (27)$$

for all $a \in X$ and all $t > 0$. Letting $l \to \infty$ in (27), we conclude that $Q = L$.

The rest of the proof is similar to the proof of Theorem 2.

Similarly, one can obtain the following result.

Theorem 5. *Let $f : X \to Y$ be an even mapping for which there is a $\rho : X^2 \to D^+$ ($\rho(a, b)$ is denoted by $\rho_{a,b}$) satisfying $f(0) = 0$ and (5). If*

$$\lim_{l\to\infty} T_{k=1}^{\infty}\left(T\left(\rho_{\frac{a}{2^{k+l}},\frac{a}{2^{k+l}}}\left(\frac{2t}{16^{l+2k}}\right),\rho_{\frac{a}{2^{k+l-1}},\frac{a}{2^{k+l}}}\left(\frac{8t}{16^{l+2k}}\right)\right)\right)=1$$

and

$$\lim_{l\to\infty}\rho_{\frac{a}{2^l},\frac{b}{2^l}}\left(\frac{t}{16^l}\right)=1$$

for all $a, b \in X$ and all $t > 0$, then there exist a unique quadratic mapping $P : X \to Y$ and a unique quartic mapping $Q : X \to Y$ such that

$$\begin{aligned}
&\mu^{(n)}_{f_n(2[x_{ij}])-16f_n([x_{ij}])-P_n([x_{ij}])}(t)\\
&\geq T_{k=1}^{\infty}\left(T^{n^2+1}\left(\rho_{\frac{x_{11}}{2^k},\frac{x_{11}}{2^k}}\left(\frac{2t}{4^{2k+1}n^2}\right),\rho_{\frac{x_{11}}{2^{k-1}},\frac{x_{11}}{2^k}}\left(\frac{2t}{4^{2k}n^2}\right),\right.\right.\\
&\quad\rho_{\frac{x_{12}}{2^k},\frac{x_{12}}{2^k}}\left(\frac{2t}{4^{2k+1}n^2}\right),\rho_{\frac{x_{12}}{2^{k-1}},\frac{x_{12}}{2^k}}\left(\frac{2t}{4^{2k}n^2}\right),\cdots,\\
&\quad\left.\left.\rho_{\frac{x_{nn}}{2^k},\frac{x_{nn}}{2^k}}\left(\frac{2t}{4^{2k+1}n^2}\right),\rho_{\frac{x_{nn}}{2^{k-1}},\frac{x_{nn}}{2^k}}\left(\frac{2t}{4^{2k}n^2}\right)\right)\right),
\end{aligned}$$

$$\begin{aligned}
&\mu^{(n)}_{f_n(2[x_{ij}])-4f_n([x_{ij}])-Q_n([x_{ij}])}(t)\\
&\geq T_{k=1}^{\infty}\left(T^{n^2+1}\left(\rho_{\frac{x_{11}}{2^k},\frac{x_{11}}{2^k}}\left(\frac{2t}{16^{2k}n^2}\right),\rho_{\frac{x_{11}}{2^{k-1}},\frac{x_{11}}{2^k}}\left(\frac{8t}{16^{2k}n^2}\right),\right.\right.\\
&\quad\rho_{\frac{x_{12}}{2^k},\frac{x_{12}}{2^k}}\left(\frac{2t}{16^{2k}n^2}\right),\rho_{\frac{x_{12}}{2^{k-1}},\frac{x_{12}}{2^k}}\left(\frac{8t}{16^{2k}n^2}\right),\cdots,\\
&\quad\left.\left.\rho_{\frac{x_{nn}}{2^k},\frac{x_{nn}}{2^k}}\left(\frac{2t}{16^{2k}n^2}\right),\rho_{\frac{x_{nn}}{2^{k-1}},\frac{x_{nn}}{2^k}}\left(\frac{8t}{16^{2k}n^2}\right)\right)\right)
\end{aligned}$$

for all $x = [x_{ij}] \in M_n(X)$ and all $t > 0$.

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A Planar Location-Allocation Problem with Waiting Time Costs

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Abstract We study a location-allocation problem where the social planner has to locate some new facilities minimizing the social costs, i.e. the fixed costs plus the waiting time costs, taking into account that the citizens are partitioned in the region according to minimizing the capacity acquisition costs plus the distribution costs in the service regions. In order to find the optimal location of the new facilities and the optimal partition of the consumers, we consider a two-stage optimization model. Theoretical and computational aspects of the location-allocation problem are discussed for a planar region and illustrated with examples.

Keywords Bilevel optimization • Continuous facility location

1 Introduction

A distribution of citizens in an urban area (rectangular region in the plane), where a given number of services must be located, is given. Citizens are partitioned in service regions such that each facility serves the consumer demand in one of the

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service regions [1–4, 7–14]. For a fixed location of all the services, every citizen chooses the service minimizing the total cost, i.e. the capacity acquisition cost plus the distribution cost (depending on the travel distance). In our model there is a fixed cost of each service depending on its location and an additional cost due to time spent being in the queue for a service, depending on the amount of people waiting for the same service, but also on the characteristics of the service itself (for example, its dimension). The objective is to find the optimal location of the services in the urban area and the related consumers' partition. We consider a two-stage optimization model to solve this location-allocation problem. The social planner minimizes the social costs, i.e. the fixed costs plus the waiting time costs, taking into account that the citizens are partitioned in the region according to minimizing the capacity acquisition costs plus the distribution costs in the service regions.

This model has been studied in [10] in the linear city case from a theoretical and numerical point of view. In [11] the general planar case has been investigated and existence results of the solution to the bilevel problem have been proved by using optimal transport theory.

Here we consider the problem to be defined in a square of the plane: by using the results present in [11] we find the solution of the bilevel problem numerically, by means of a genetic algorithm procedure. In Sect. 2 the model is presented; in Sect. 3 computational aspects and some examples are discussed; Sect. 4 contains concluding remarks.

2 The Bilevel Problem

We consider a bounded region of the plane $\Omega \subset \mathbb{R}^2$. Each point $p = (x, y) \in \Omega$ has a given demand density D, namely an absolutely continuous probability measure where $D : \Omega \to \mathbb{R}$ is a non-negative function with unit integral

$$\int_{\Omega} D(q)dq = 1, \tag{H1}$$

with $dq = dxdy$. The problem is to locate n new facilities $p_1, \dots, p_n$, $p_i = (x_i, y_i) \in \Omega$ for any $i \in N = \{1, 2, \dots, n\}$. Facility p_i serves the consumers demand in the region $A_i \subseteq \Omega$: we have a partition of the set Ω, i.e. $\cup_{i=1}^{n} A_i = \Omega$ and $\mathring{A}_i \cap \mathring{A}_j \neq \emptyset$ for any $i = j$. Citizens are partitioned in service regions such that each facility serves the consumer demand in only one of the service regions. More precisely, for any $i \in N$, we denote by

$$\omega_i = \int_{A_i} D(q)dq,$$

the total demand within each service region A_i. Now we define for any $i \in N$ the following non-negative functions:

1. $F_i : \Omega \to \mathbb{R}$, being $F_i(p_i)$ the annualized fixed cost of facility i;
2. $a_i : \Omega \to \mathbb{R}$, being $a_i(p_i)$ the annualized variable capacity acquisition cost per unit demand;
3. $C_i : \Omega \to \mathbb{R}$, being

$$C_i(p_i) = c \int_{A_i} d^2(p_i, p) D(p) dp,$$

the distribution cost in service region A_i, $d(\cdot, \cdot)$ the Euclidean distance in $\mathbb{R}^2$ and c the distribution cost per unit distance, that we suppose to be constant in Ω;
4. $h_i : [0, 1] \to \mathbb{R}$, being $h_i(\omega_i)$ the total cost, in terms of time spent to be served, of consumers of region A_i using the service p_i.

We denote by $\mathscr{A}_n$ the set of all partitions in n sub-regions of the region Ω, $A = (A_1, \ldots, A_n) \in \mathscr{A}_n$ and $p = (p_1, \ldots, p_n) \in \Omega^n$.

Definition 1. Any tuple $< \Omega; p_1, \ldots, p_n; l, Z >$ is called a facility location situation, where Ω is a compact set in $\mathbb{R}^2$, $p_i \in \Omega$ for any $i \in N$; $l, Z : \Omega^n \times \mathscr{A}_n \to \mathbb{R}$ defined by

$$l(p, A) = \sum_{i=1}^{n} \Big[F_i(p_i) + \omega_i h_i(\omega_i) \Big], \tag{s}$$

$$Z(p, A) = \sum_{i=1}^{n} \Big[c \int_{A_i} d^2(p_i, p) D(p) dp + a_i(p_i) \int_{A_i} D(p) dp \Big]. \tag{g}$$

Given a facility location situation, the problem is to find an optimal location for the facilities $p_1, \ldots, p_n$ and also an optimal partition $A_1, \ldots, A_n$ of the consumers in the market region Ω by minimizing the costs. We distinguish the total cost in a geographical part that is given by (g) and in a social part that is given by (s).

In order to find the optimal pair given by the optimal partition of Ω and the optimal location of the facilities, we propose a bilevel approach. Given the location of the new facilities, we search the optimal partition of the consumers minimizing the geographical cost (g). Then, we optimize the social cost (s) to look for the optimal location of the facilities according to a bilevel formulation.

For a given location $p \in \Omega^n$ of the n facilities, the consumers decide which is the best facility to use: they minimize the costs given by the distribution costs, that depend on the distance from the chosen facility, plus the acquisition costs, that is the capacity acquisition cost of the facility supposed to be linear with respect to the density in the region where the chosen facility is located. We assume that $p = (p_1, \ldots, p_n)$ with $p_i \neq p_j$ for $i \neq j, i, j \in N$ and $p_i \in A_i$ for each $i \in N$. Then, the optimal partition of the consumers in the set $\mathscr{A}_n$ will be a solution to the following lower level problem $LL(p)$:

$$\min_{(A_1,\ldots,A_n) \in \mathscr{A}_n} Z(p, A). \tag{$LL(p)$}$$

Suppose that the problem $LL(p)$ has a unique solution for any $p \in \Omega^n$, called $(A_1(p), \dots, A_n(p)) = A(p)$. The function $p \to \mathscr{A}(p)$ is called the best reply function.

In a second step, the social planner proposes the best location of the n facilities in such a way that additional costs, that are social costs given by the functional $l(p, A)$, must be the lowest possible, knowing that the best partition of the customers is given by the best reply function. These additional costs are the fixed cost of each facility plus a cost due to the waiting time given by the function h_i. We have a constrained optimization problem: the optimal location of the facilities $\bar{p} \in \Omega^n$ solves the following upper level problem UL:

$$\min_{(p_1,\dots,p_n)\in\Omega^n} l(p, A(p)), \qquad (UL)$$

where, for a given location p, the optimal partition $A(p)$ of Ω is given by the unique solution of the problem $LL(p)$.

The problem UL is known as a bilevel problem, since it is a constrained optimization problem with the constraint that $A(p)$ is the solution of another optimization problem $LL(p)$, for any $p \in \Omega^n$. We solve it by using the backward induction procedure as specified in the following definition.

Definition 2. Any $\bar{p}$ that solves the problem UL is an optimal solution to the bilevel problem. In this case the *optimal pair* is $(\bar{p}, A(\bar{p}))$, where $\bar{p}$ solves the problem UL and $A(p)$ is the unique solution of the problem $LL(p)$ for each $p \in \Omega^n$.

In a Game Theory context, the solution of the upper level problem is called Stackelberg strategy and the pair solution of the bilevel problem as given in Definition 2 is called a Stackelberg equilibrium [2].

2.1 The Linear City

Let us recall the location-allocation problem in the linear city case [10]. We consider a linear region on the real line, i.e. a compact real interval Ω. Without loss of generality, we normalize it and assume $\Omega = [0, 1]$. This assumption corresponds to concrete situations as the location of a gasoline station along a highway or the location of a railway station to improve the service to the inhabitants of the region.

Let $D(p)$ be the demand density s.t. $\int_0^1 D(p)dp = 1$ where $dp = dx$. We want to locate n facilities $p_i = x_i \in [0, 1]$ for any $i = 1, \dots, n$ with $p_1 < p_2 < \dots < p_n$. A partition $A = (A_1, \dots, A_n)$ of the region $\Omega = [0, 1]$ is given by a real vector $\lambda = (\lambda_1, \dots, \lambda_{n-1})$ such that $\lambda_i \in [p_i, p_{i+1}], i = 1, .., n-1$. The partition in this case is: $A_1 = [0, \lambda_1[, \dots, A_n =]\lambda_{n-1}, 1]$. We denote $\lambda_0 = 0$ and $\lambda_n = 1$.

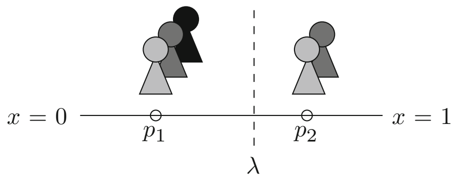

Fig. 1 Location of two facilities in the linear city [10]

A linear facility location situation is a tuple $< \Omega; p_1, \ldots, p_n; l_1, Z_1 >$, where $\Omega = [0,1]$, $p_i \in \Omega$ for any $i \in N$; $l_1, Z_1 : \Omega^n \times \mathscr{A}_n \to \mathscr{R}$ defined by

$$l_1(p,\lambda) = \sum_{i=0}^{n-1} [F_{i+1}(p_{i+1}) + \omega_{i+1} h_{i+1}(\omega_{i+1})],$$

$$Z_1(p,\lambda) = \sum_{i=0}^{n-1} \left[\omega_{i+1} a_{i+1}(p_{i+1}) + c \int_{\lambda_i}^{\lambda_{i+1}} d^2(p_{i+1}, p) D(p) dp \right],$$

where ω_i is the total demand within service region $A_i = [\lambda_{i-1}, \lambda_i]$ for any $i = 1, \ldots, n$, namely

$$\omega_i = \int_{\lambda_{i-1}}^{\lambda_i} D(p)dp.$$

Here the functions l_1 and Z_1 represent the social (s) and the geographic cost (g), respectively.

Definition 3. Any $\bar{p}$ that solves the problem

$$\min_{p \in \Omega^n} l_1(p, \lambda(p)), \qquad (UL)$$

is an optimal solution to the bilevel problem, where for each $p \in \Omega^n$, $\lambda(p)$ is the unique solution of the problem $LL(p)$ defined by:

$$\min_{\lambda \in [p_1, p_2] \times \ldots \times [p_{n-1}, p_n]} Z_1(p, \lambda).$$

In this case the *optimal pair* is $(\bar{p}, \lambda(\bar{p}))$ where $\bar{p}$ solves the problem UL and $\lambda(\bar{p})$ is the unique solution of the problem $LL(p)$.

Existence results for the bilevel problem UL, together with computational test cases can be found in [10], from which is taken the sketch reported in Fig. 1.

2.2 The Planar Region

In what follows, we shall assume that:

- $h_i(\cdot)$ is a continuous function on $[0, 1]$ for any i; (H2)
- $F_i(\cdot), a_i(\cdot)$ are continuous functions on Ω. (H3)

Now we recall the following preliminary result, which is a characterization of the sets of the optimal partition for the lower level problem.

Lemma 1 (From [11]). *Given $p \in \Omega^n$. Suppose that there exists $(A_1, \ldots, A_n)$ an optimum for the problem $LL(p)$. Then*

$$A_i = \left\{ x \in \Omega : a_i(p_i) + c|x - p_i|^2 < a_j(p_j) + c|x - p_j|^2 \qquad \forall j \neq i \right\}, \quad (1)$$

where the equalities is intended up to D-negligible sets.

Remark 1. Previous Lemma allows us to describe the shape of the optimal partition. In fact the sets A_i are polygons whose boundaries can be obtained as follows

$$x \in A_i \cap A_j \qquad \Leftrightarrow \qquad a_i(p_i) + c|x - p_i|^2 = a_j(p_j) + c|x - p_j|^2,$$

and hence $x = (x_1, x_2) \in A_i \cap A_j$ if and only if

$$x_2 = -\frac{p_i^1 - p_j^1}{p_i^2 - p_j^2} x_1 + \frac{|p_i|^2 - |p_j|^2}{2(p_i^2 - p_j^2)} + \frac{a_i(p_i) - a_j(p_j)}{2c(p_i^2 - p_j^2)},$$

in case $p_i^2 \neq p_j^2$, where $p_i = (p_i^1, p_i^2)$. If $p_i^2 = p_j^2$, $x = (x_1, x_2) \in A_i \cap A_j$ if and only if

$$x_1 = \frac{|p_i|^2 - |p_j|^2}{2(p_i^1 - p_j^1)} + \frac{a_i(p_i) - a_j(p_j)}{2c(p_i^1 - p_j^1)}.$$

The following result guarantees a unique solution to the problem $LL(p)$ for each p and the existence of the solution to the upper level problem UL.

Theorem 1 (From [11]). *Assume* (H1)–(H3). *Then, for any $p \in \Omega^n$, the problem $LL(p)$ admits a unique solution $\mathscr{A}(p)$ and there exists a solution $\bar{p}$ to the problem UL.*

Example 1 ([11]). We want to locate two new facilities $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$, in the market region $\Omega = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ where the consumers are uniformly distributed ($D(p) = 1$ for any $p \in [0, 1]^2$).

Suppose that the capacity acquisition costs are $a_1(p_1) = |p_1|^2$, $a_2(p_2) = |p_2|^2$, and the unit distribution cost is $c = 3$: for a given pair $p = (p_1, p_2) \in \Omega^2$ we consider the partition (A_1, A_2) of Ω that minimize the cost function

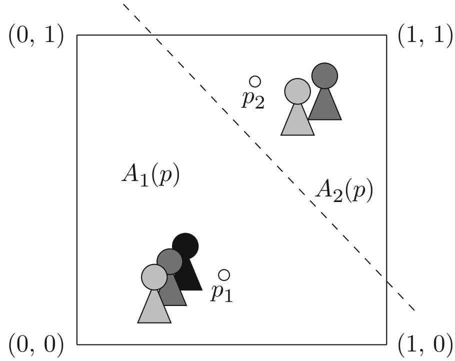

Fig. 2 Location of facilities in planar region

$Z(p_1, p_2, A_1, A_2)$ as in problem $LL(p)$. As claimed in Lemma 1, we give the expression of the optimal partition $(A_1(p), A_2(p))$. The set $A_1(p)$ is the set of the pairs $(x, y) \in [0, 1]^2$ such that

$$x_1^2 + y_1^2 + 3(x - x_1)^2 + 3(y - y_1)^2 < x_2^2 + y_2^2 + 3(x - x_2)^2 + 3(y - y_2)^2,$$

then

$$3y(y_2 - y_1) < 2[(y_2^2 - y_1^2) + (x_2^2 - x_1^2)] - 3x(x_2 - x_1).$$

For $x_1 \leq x_2$ and $y_1 < y_2$ we have:

$$A_1(p) = \begin{cases} \left\{(x, y) \in \Omega : y < \frac{2}{3}\left[y_1 + y_2 + \frac{x_2^2 - x_1^2}{y_2 - y_1}\right] - x\left(\frac{x_2 - x_1}{y_2 - y_1}\right)\right\} \text{ if } y_1 < \frac{y_2}{2} \text{ and } x_1 < \frac{x_2}{2} \\ \left\{(x, y) \in \Omega : y < y_2 - (x - x_2)\left(\frac{x_2 - x_1}{y_2 - y_1}\right)\right\} \text{ otherwise} \end{cases},$$

and if $y_1 = y_2$ with $x_1 < x_2$

$$A_1(p) = \begin{cases} \{(x, y) \in \Omega : x < \frac{2}{3}(x_1 + x_2)\} \text{ if } x_1 < \frac{x_2}{2} \\ \{(x, y) \in \Omega : x < x_2\} \text{ otherwise} \end{cases},$$

being $A_2(p) = \Omega \setminus A_1(p)$. The other cases are similar (a sketch is reported in Fig. 2).

3 Numerical Results

In this section we present some computational results to solve the location-allocation problem. Our approach is based on Genetic Algorithms (GAs), a heuristic search technique modeled on the principle of evolution with natural selection. Namely, the main idea is the reproduction of the best elements with possible crossover and mutation to improve population [5, 6].

The initial population is provided with a random seeding in the leader's strategy space. In the following examples, the bilevel algorithm has been simplified, using analytical results in Sect. 2.2. Thus for each individual (or chromosome), the optimal partition of the region is computed using expression (1) representing the best reply. A full Gauss numerical integration algorithm has been developed in order to evaluate ω_i in (s) for each region, to compute the objective function and fitness for each configuration.

The leader population is sorted under objective function criterion and a mating pool is generated. Now a second step begins and a common crossover and mutation operation on the leader population is performed. Again the follower's best reply should be computed, in the same way described above.

This is the kernel procedure of the genetic algorithm that is repeated until a terminal period is reached or an exit criterion is met.

For the algorithm validation we consider the parameters as specified in Table 1.

3.1 Test Cases

Test cases on several scenarios have been performed.

Example 2. We want to locate two new facilities $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$, in the market region $\Omega = [0,1] \times [0,1] \subset \mathbb{R}^2$ where the consumers are uniformly distributed ($D(p) = 1$ for any $p \in [0,1]^2$).

As in Example 1, we suppose that the capacity acquisition costs are $a_1(p_1) = |p_1|^2$, $a_2(p_2) = |p_2|^2$, and the unit distribution cost is $c = 3$: for a given pair

Table 1 GA details

Parameter	Value
Population size (−)	100
Crossover fraction (−)	0.90
Mutation fraction (−)	0.10
Parent sorting	Tournament between couple
Mating pool (%)	50
Elitism	No
Crossover mode	Simulated Binary Crossover (SBX)
Mutation mode	Polynomial

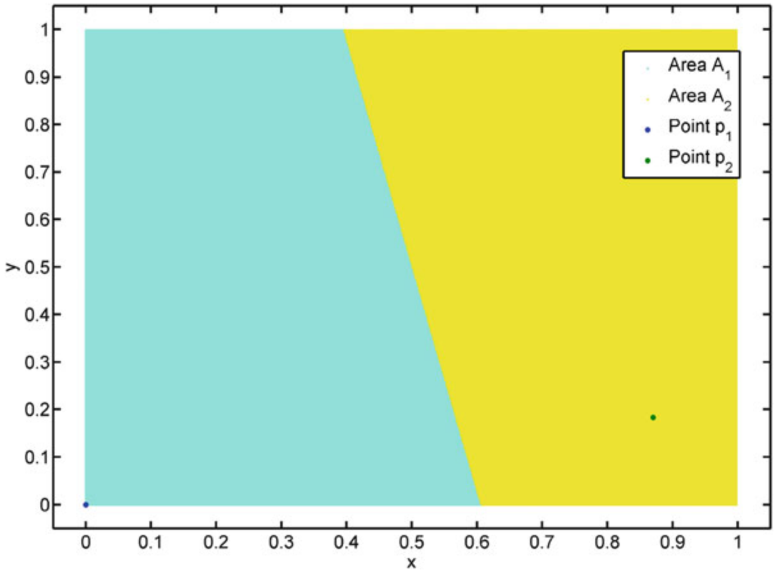

Fig. 3 Points and relative partitions for Example 2 ($\epsilon = 0$)

$p = (p_1, p_2) \in \Omega^2$ we consider the partition (A_1, A_2) of Ω that minimize the cost function $Z(p_1, p_2, A_1, A_2)$ as in problem $LL(p)$.

Fixed costs and waiting time costs are, respectively, for $\epsilon > 0$:

$$F_1(p_1) = |p_1|^2, F_2(p_2) = 0, \tag{2}$$

$$h_1(t) = (1 + \epsilon)t, h_2(t) = t, \tag{3}$$

Numerical results are summarized in Figs. 3 and 4, respectively, for $\epsilon = 0$ and $\epsilon = 1$ with relative evolution (Figs. 5 and 6) during generations of genetic optimization algorithm.

Example 3. We want to locate two new facilities $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$, in the market region $\Omega = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ where the consumers are uniformly distributed ($D(p) = 1$ for any $p \in [0, 1]^2$). Points position is constrained to stay on the x axis to validate the algorithm using the linear city results ($p_1^2 = 0$ and $p_2^2 = 0$).

Fixed costs, the acquisition costs and waiting time costs are, respectively, for $\epsilon > 0$:

$$F_1(p_1) = |p_1|^2, F_2(p_2) = |p_2|/4, \tag{4}$$

$$a_1(p_1) = |p_1|^2, a_2(p_2) = |p_2|^2, \tag{5}$$

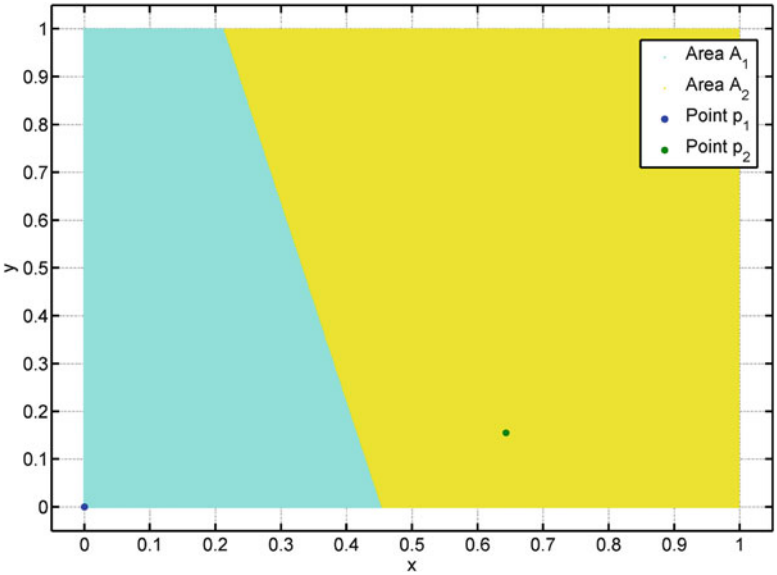

Fig. 4 Points and relative partitions for Example 2 ($\epsilon = 1$)

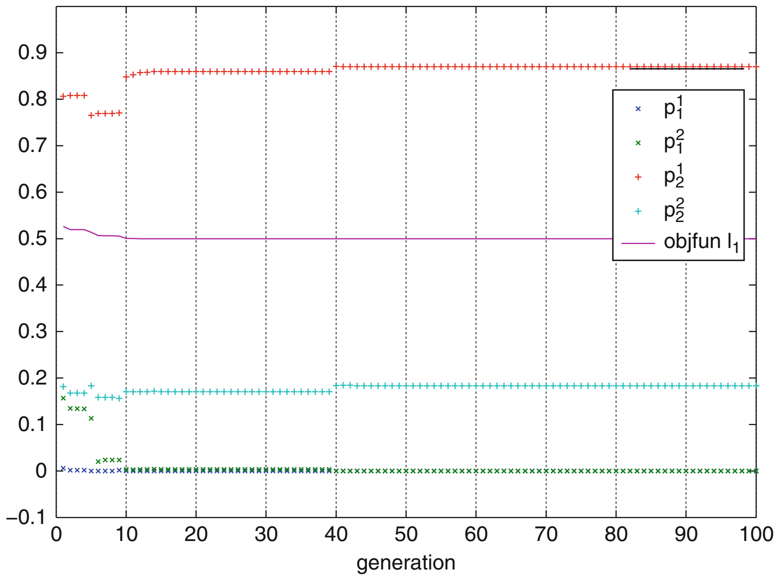

Fig. 5 Evolution of points position and objective function during algorithm generations for Example 2 ($\epsilon = 0$)

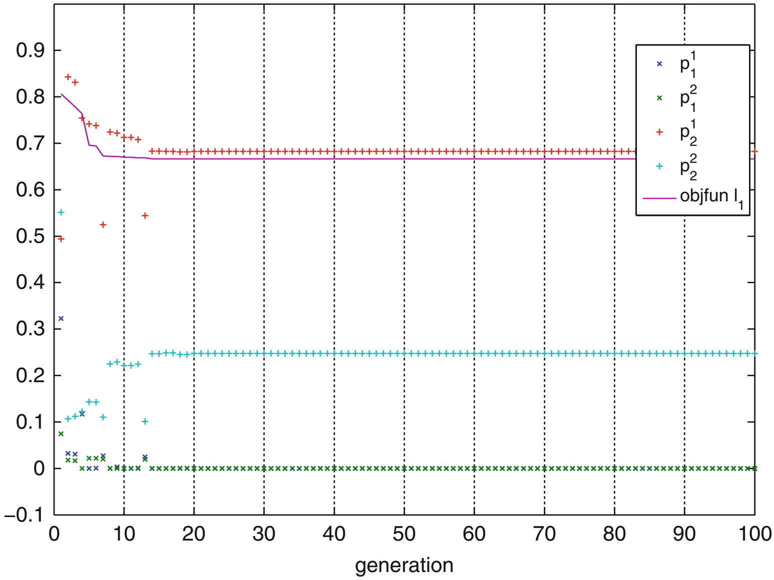

Fig. 6 Evolution of points position and objective function during algorithm generations for Example 2 ($\epsilon = 1$)

$$h_1(t) = (1+\epsilon)t, h_2(t) = t. \tag{6}$$

In [10] has been computed that for $\epsilon < 5/4$ the solution is

$$\hat{p}_1 = \left(\frac{1}{8}, 0\right), \tag{7}$$

$$\hat{p}_2 = \left(\frac{31-4\epsilon}{32(2+\epsilon)}, 0\right), \tag{8}$$

$$\hat{A}_1 = \left\{x < \frac{13}{16(2+\epsilon)}\right\}. \tag{9}$$

For $\epsilon = 1$ the analytical solution is:

$$\hat{p}_1 = (0.125, 0), \hat{p}_2 = (0.2812, 0). \tag{10}$$

Numerical results are summarized in Figs. 7 and 8, respectively, for $\epsilon = 0$ and $\epsilon = 1$. They are compliant with results obtained in the linear city problem [10].

Example 4. We want to locate five new facilities $p_i = (x_i, y_i)$, $i = 1, \ldots, 5$, in the market region $\Omega = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ where the consumers are uniformly distributed ($D(p) = 1$ for any $p \in [0, 1]^2$).

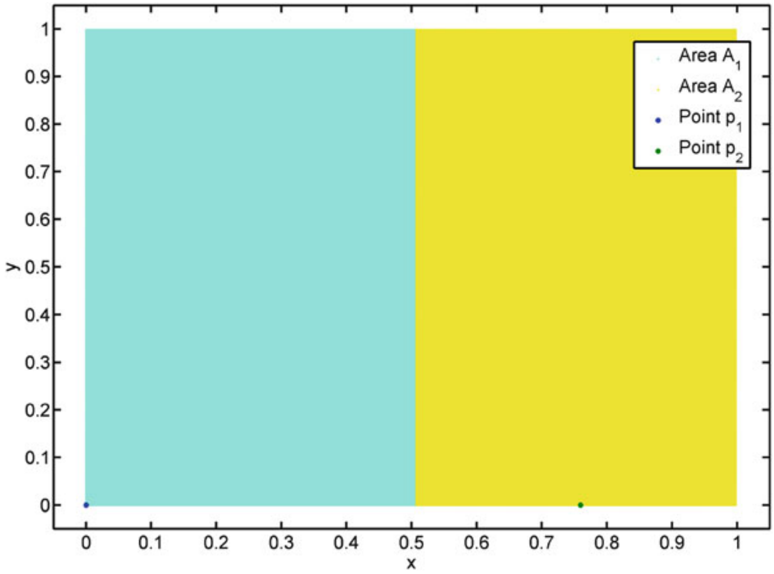

Fig. 7 Points and relative partitions for Example 3 ($\epsilon = 0$)

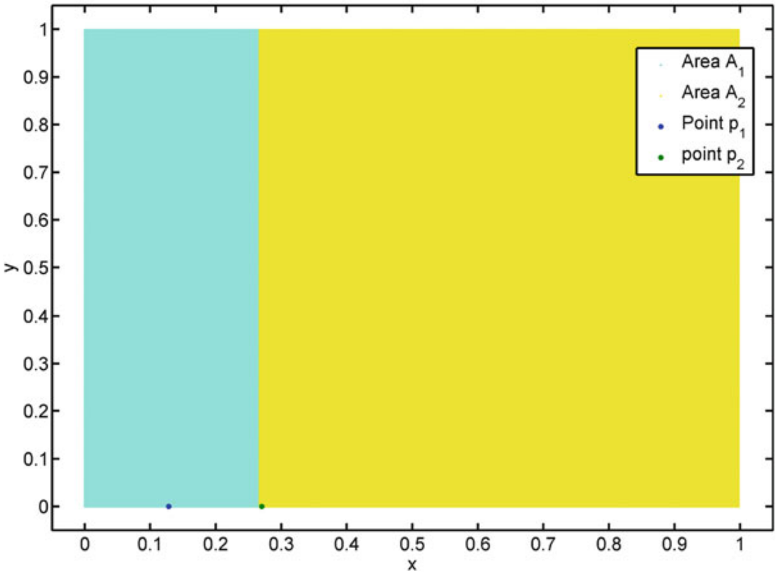

Fig. 8 Points and relative partitions for Example 3 ($\epsilon = 1$)

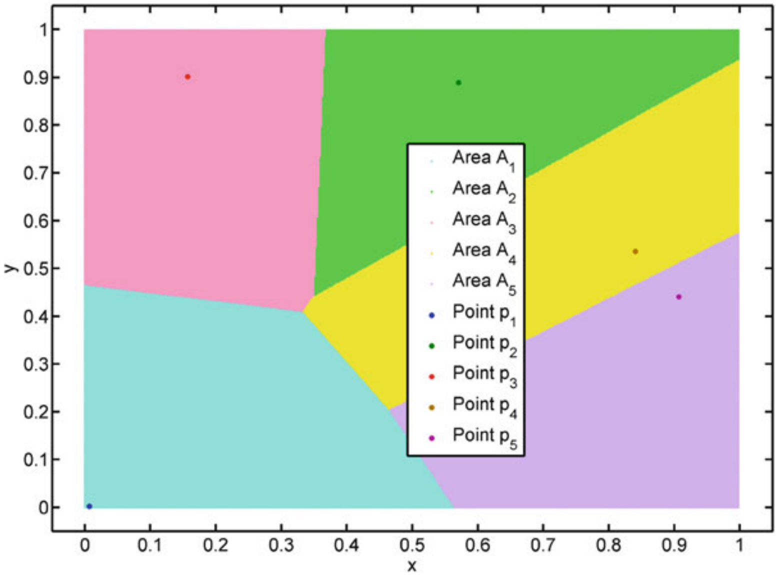

Fig. 9 Points and relative partitions for Example 4 ($\epsilon = 0$)

We suppose that the capacity acquisition costs are zero for all new facilities, i. e. $a_i(p_i) = 0, \forall i = 1, \ldots, 5$, and the unit distribution cost is $c = 3$: for a given tuple $p = (p_1, p_2, p_3, p_4, p_5) \in \Omega^5$ we consider the partition $(A_1, A_2, A_3, A_4, A_5)$ of Ω that minimize the cost function $Z(p_1, p_2, p_3, p_4, p_5, A_1, A_2, A_3, A_4, A_5)$ as in problem $LL(p)$.

Fixed costs and waiting time costs are, respectively, for $\epsilon > 0$:

$$F_1(p_1) = |p_1|^2, F_2(p_2) = F_3(p_3) = F_4(p_4) = F_5(p_5) = 0, \tag{11}$$

$$h_1(t) = (1 + \epsilon)t, h_2(t) = t. \tag{12}$$

Numerical results are summarized in Fig. 9 for $\epsilon = 0$, with relative evolution during generations of genetic optimization algorithm in Fig. 10: the GA in this example has been run with a population size equal to 200 and a number of generations equal to 100.

Example 5. As in Example 4, we want to locate five new facilities $p_i = (x_i, y_i)$, $i = 1, \ldots, 5$, in the market region $\Omega = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ where the consumers have, in this case, a Gaussian distribution ($D(p) = \exp\left(-(16(x - 0.5)^2 + 16(y - 0.5)^2\right)$ for any $p = (x, y) \in [0, 1]^2$) as in Fig. 11.

We suppose that the capacity acquisition costs are zero for all new facilities, i.e. $a_i(p_i) = 0, \forall i = 1, \ldots, 5$, and the unit distribution cost is $c = 3$: for a given tuple $p = (p_1, p_2, p_3, p_4, p_5) \in \Omega^5$ we consider the partition $(A_1, A_2, A_3, A_4, A_5)$ of Ω that minimize the cost function $Z(p_1, p_2, p_3, p_4, p_5, A_1, A_2, A_3, A_4, A_5)$ as in problem $LL(p)$.

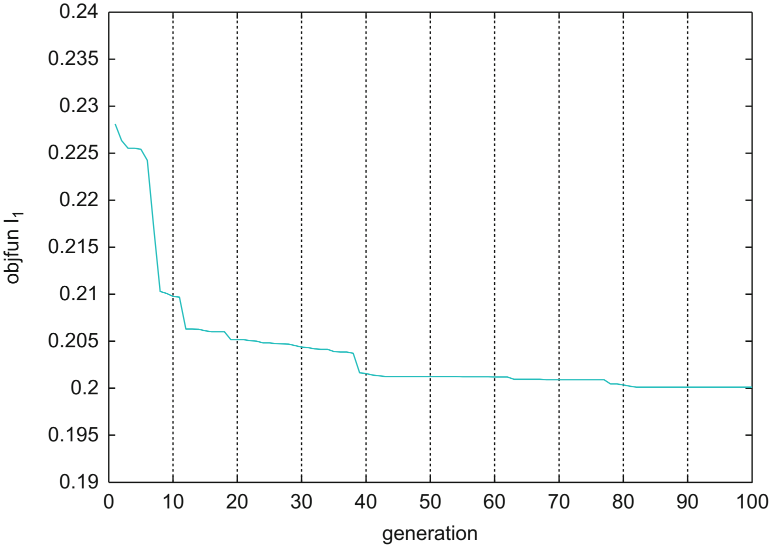

Fig. 10 Evolution objective function during algorithm generations for Example 4 ($\epsilon = 0$)

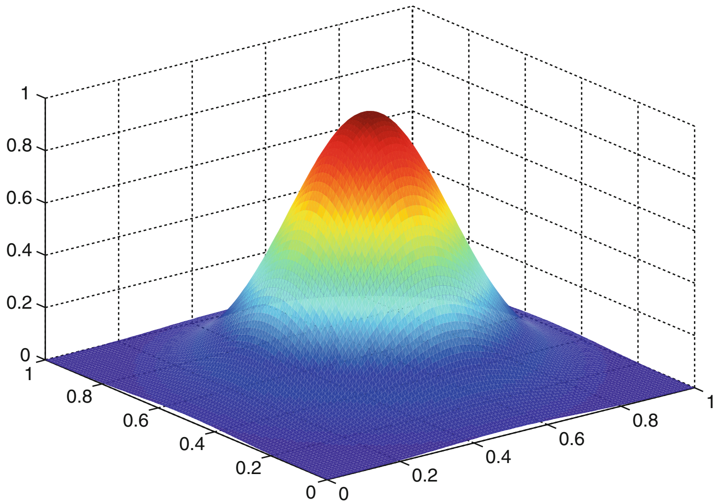

Fig. 11 Population distribution for Example 5

Fixed costs and waiting time costs are, respectively, for $\epsilon > 0$:

$$F_1(p_1) = |p_1 - (0.5, 0.5)|^2, F_2(p_2) = F_3(p_3) = F_4(p_4) = F_5(p_5) = 0, \quad (13)$$

$$h_1(t) = (1 + \epsilon)t, h_2(t) = t. \quad (14)$$

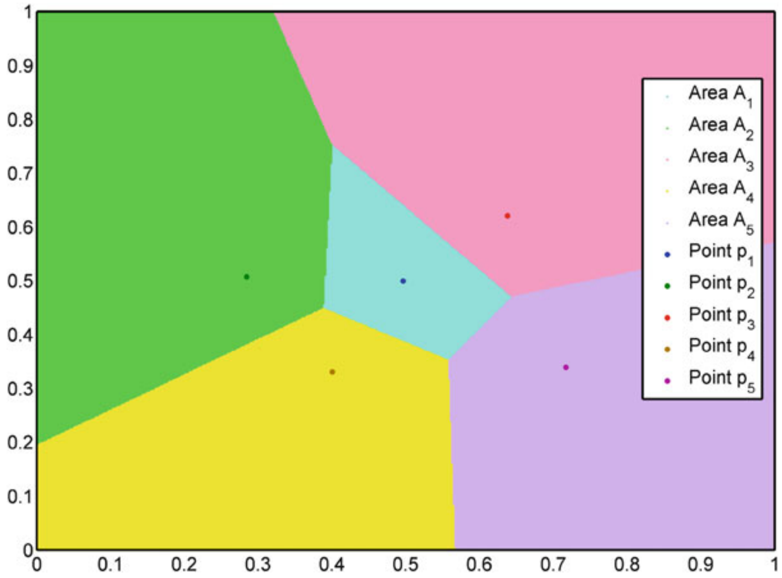

Fig. 12 Points and relative partitions for Example 5 ($\epsilon = 0$)

Numerical results are summarized in Fig. 12, for $\epsilon = 0$: as in the Example 4, the algorithm has been run with a population size equal to 200 and a number of generations equal to 100.

4 Concluding Remark

The problem studied in this work encloses many computational difficulties, mainly exploited in the planar region case. An algorithm based on sections of the elements $A_1, \ldots, A_n$ of the partitions is given in [13] for a similar problem formulated as an optimization problem not by considering several hierarchical levels and without the waiting time costs. The algorithm in [13] uses Voronoi diagrams. In this work we approached the problem in a planar region by using a genetic algorithm. We solved a bilevel problem where the solution of the lower level problem is given in Lemma 1. This allowed to use a numerical procedure dealing with an optimization problem. Several test cases have been provided and a comparison with the problem in a linear city has been discussed. The circular region case (see, for example, [12]), as well the problem with some obstacles in the planar region would be the subject of future works.

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The Stability of an Affine Type Functional Equation with the Fixed Point Alternative

M. Mursaleen and Khursheed J. Ansari

Abstract In this paper, we consider the following affine functional equation

$$f(3x+y+z)+f(x+3y+z)+f(x+y+3z)+f(x)+f(y)+f(z) = 6f(x+y+z).$$

We obtain the general solution and establish some stability results by using *direct method* as well as *the fixed point method*. Further we define the stability of the above functional equation by using *the fixed point alternative*.

Keywords Hyers-Ulam stability • Affine functional equation • Fixed point method • Alternative fixed point method

1 Introduction and Preliminaries

The concept of stability of a functional equation arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. In 1940, Ulam [31] raised a question concerning the stability of group homomorphism as follows:
Let G_1 be a group and let G_2 be a metric group with the metric $d(.,.)$. Given $\varepsilon > 0$. Does there exists a $\delta > 0$ such that if a function $f : G_1 \to G_2$ satisfies the inequality

$$d(f(xy), f(x)f(y)) < \delta \text{ for all } x, y \in G_1,$$

then there exists a homomorphism $h : G_1 \to G_2$ with

$$d(f(x), H(x)) < \varepsilon \text{ for all } x \in G_1?$$

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The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, the Ulam problem for the case of approximately additive mappings was solved by Hyers [12] under the assumption that G_2 is a Banach space. In 1978, a generalized version of the theorem of Hyers for approximately linear mapping was proved by Rassias [27]. Rassias proved the following Theorem: Let a mapping $f : E_1 \to E_2$ be such that $f(tx)$ is continuous in t for each real value of t for a fixed $x \in E_1$. Assume that there exists a constant $\varepsilon > 0$ and $p \in [0, 1)$ with

$$\| f(x + y) - f(x) - f(y) \| \leq \varepsilon(\| x \|^p + \| y \|^p) \tag{1}$$

for $x, y \in E_1$. Then there exists a unique linear mapping $T : E_1 \to E_2$ such that

$$\| f(x) - T(x) \| \leq \frac{2\varepsilon}{2 - 2^p} \| x \|^p \tag{2}$$

for $x \in E_1$.

A number of mathematicians were attracted by the result of Rassias. The stability concept that was introduced and investigated by Rassias is called the Hyers–Ulam–Rassias stability. During the last decades, stability problems of several functional equations have been extensively investigated by a number of mathematicians (c.f. [1, 8, 13, 15, 18–24, 26, 28, 29] and [30] etc.).

Several proofs in this domain of research use the direct method that was introduced by Hyers [26]. The exact solution of the functional equation is explicitly constructed as a limit of a sequence, starting from the given approximate solution. On the other hand, Baker [2] used the Banach fixed point theorem to prove Hyers–Ulam stability results for a nonlinear functional equation. In 2003, Radu [25] proposed a new method, successively developed in [4–6], to obtain the existence of the exact solutions and the error estimations, based on the fixed point alternative (see also [19]). Subsequently, these results were generalized by Mihet [17], Gavruta [10] and by Cadariu and Radu [7]. Later on, Gavruta and Gavruta introduced a new method in [11], called the weighted space method, for the generalized Hyers–Ulam stability. Mohiuddine has shown the stability of some functional equations [20, 21] via fixed point technique. For most recent work, we refer to [16]. Cadariu, Gavruta and Gavruta [3] obtained the general solution, the generalized Hyers–Ulam stability by using the direct method as well as the fixed point method for the following affine functional equation:

$$f(2x + y) + f(x + 2y) + f(x) + f(y) = 4f(x + y), \quad \text{for all } x, y \in G,$$

where $f : G \to X$, G is an abelian group and X is a normed space.

In the present paper we obtain the general solution of the following affine functional equation

$$f(3x+y+z)+f(x+3y+z)+f(x+y+3z)+f(x)+f(y)+f(z) = 6f(x+y+z) \tag{3}$$

for all $x, y \in G$, where $f : G \to X$, G is an abelian group and X is a normed space. By using the direct method, the fixed point method as well as the fixed point alternative, we provide proofs of generalized Hyers–Ulam stability results for the respective equation.

2 Solution of the Functional Equation (3)

Theorem 2.1. *A mapping $f : G \to X$, where G is an abelian group and X is a normed space, is a solution of the functional equation (3) iff it is an affine mapping (i.e., it is the summation of a constant and an additive function).*

Proof. It is obvious that any affine function f is a solution of (3).
Conversely, we consider two cases:

$Case\ 1: \quad f(0) = 0.$

If we take $y = z = -x$ in (3), then we obtain

$$f(-3x) + f(x) = 2f(-x), \forall x \in G. \tag{4}$$

By replace x by $-x$ and put $y = z = 0$ in (3), we get

$$f(-3x) = 3f(-x), \forall x \in G$$

Using this result in (4), we have $f(-x) = -f(x)$, for all $x \in G$. Hence f is an odd mapping.
Take $z = -y$ in (3), we have

$$f(3x) + f(x + 2y) + f(x - 2y) = 5f(x), \forall x \in G. \tag{5}$$

If we substitute $y = z = 0$ in (3), we get $f(3x) = 3f(x), \forall x \in G$. Using this result in (5), we obtain

$$f(x + 2y) + f(x - 2y) = 2f(x), \forall x \in G. \tag{6}$$

Put $y = x/2$ in the last equation and using $f(0) = 0$, we get the following relation

$$f(2x) = 2f(x), \forall x \in G. \tag{7}$$

If we replace x by $\frac{u+v}{2}$ and y by $\frac{u-v}{4}$ in (6), we obtain

$$f(u) + f(v) = 2f.$$

Hence f is an additive mapping.

Case 2 : General case.

Let us consider the function $g(x) := f(x) - f(0)$. It is clear that $g(0) = 0$ and $f(x) = g(x) + f(0)$.
Replacing f in (3), it results

$$g(3x + y + z) + g(x + 3y + z) + g(x + y + 3z) + g(x) + g(y) + g(z)$$
$$= 6g(x + y + z), \forall x, y \in G.$$

Taking into account that $g(0) = 0$, from Case 1, we obtain that g is an additive mapping. Hence $f(x) = g(x) + f(0)$ is an afine function.

This completes the proof.

3 Direct Method

In this section we will obtain some properties of the generalized Hyers–Ulam stability for the affine functional equation (3). For the proof, we will use the direct method.

Let $(G, +)$ be an abelian group and $(X, \|.\|)$ be a Banach space. Let a mapping $\varphi : G \times G \times G \longrightarrow [0, \infty)$ be such that

$$\Phi(x) = \sum_{k=0}^{\infty} \frac{\varphi(3^k x, 0, 0)}{3^k} < \infty, \forall x \in G \tag{8}$$

and

$$\lim_{n\to\infty} \frac{\varphi(3^n x, 3^n y, 3^n z)}{3^n} = 0, \forall x, y, z \in G. \tag{9}$$

We formulate the main result of the paper:

Theorem 3.1. *Let $f : G \longrightarrow X$, such that*

$$\|f(3x + y + z) + f(x + 3y + z) + f(x + y + 3z) + f(x) + f(y) + f(z) - 6f(x + y + z)\| \le \varphi(x, y, z) \tag{10}$$

for all $x, y, z \in G$. Then there exists a unique mapping $A : G \longrightarrow X$, which satisfies the Eq. (3) and

$$\|f(x) - A(x) - f(0)\| \le \frac{1}{3}\Phi(x), \tag{11}$$

for all $x \in G$.

Proof. For $y = z = 0$ in (10), we obtain

$$\|f(3x) - 3f(x) + 2f(0)\| \leq \varphi(x, 0, 0), \forall x \in G.$$

If we define the function $g : G \longrightarrow X$,

$$g(x) := f(x) - f(0), \tag{12}$$

we have

$$\|g(3x) - 3g(x)\| \leq \varphi(x, 0, 0), \forall x \in G$$

Thus

$$\left\| \frac{g(3x)}{3} - g(x) \right\| \leq \frac{1}{3}\varphi(x, 0, 0), \forall x \in G. \tag{13}$$

If we replace x by $3x$ in the above relation and divide it by 3, it results

$$\left\| \frac{g(3^2x)}{3^2} - \frac{g(x)}{3} \right\| \leq \frac{1}{3^2}\varphi(3x, 0, 0), \forall x \in G. \tag{14}$$

Using the triangle inequality, from (13) and (14), it follows that

$$\left\| \frac{g(3^2x)}{3^2} - g(x) \right\| \leq \frac{1}{3}\left(\varphi(x, 0, 0) + \frac{1}{3}\varphi(3x, 0, 0)\right), \forall x \in G.$$

It is easy to prove, by induction on n, that

$$\left\| \frac{g(3^n x)}{3^n} - g(x) \right\| \leq \frac{1}{3}\sum_{k=0}^{n-1} \frac{\varphi(3^k x, 0, 0)}{3^k}, \forall x \in G.$$

Now, we claim that the sequence $\left\{\frac{g(3^n x)}{3^n}\right\}$ is a Cauchy sequence. Indeed, for $n > m > 0$, we have:

$$\begin{aligned}
\left\| \frac{g(3^n x)}{3^n} - \frac{g(3^m x)}{3^m} \right\| &= \frac{1}{3^m}\left\| \frac{g(3^{n-m}.3^m x)}{3^{n-m}} - g(3^m x) \right\| \\
&\leq \frac{1}{3^m}.\frac{1}{3}\sum_{k=0}^{n-m-1} \frac{\varphi(3^{k+m}x, 0, 0)}{3^k} \\
&= \frac{1}{3}\sum_{k=0}^{n-m-1} \frac{\varphi(3^{k+m}x, 0, 0)}{3^{k+m}} \\
&= \frac{1}{3}\sum_{p=m}^{n-1} \frac{\varphi(3^p x, 0, 0)}{3^p}, \forall x \in G.
\end{aligned}$$

Taking the limit as $m \to \infty$, it results that

$$\lim_{m\to\infty} \left\| \frac{g(3^n x)}{3^n} - \frac{g(3^m x)}{3^m} \right\| = 0, \forall x \in G.$$

Since X is a Banach space, then we obtain that the sequence $\left\{\frac{g(3^n x)}{3^n}\right\}$ converges. We define

$$A(x) := \lim_{n\to\infty} \frac{g(3^n x)}{3^n},$$

for each x in G. From (12) it is clear that

$$A(x) = \lim_{n\to\infty} \frac{f(3^n x)}{3^n}, \forall x \in G. \tag{15}$$

We claim that A satisfies (3). Replace x and y by $3^n x$ and $3^n y$, respectively, in relation (10) and divide by 3^n. It follows that

$$\begin{aligned}
&\|3^{-n} f(3^n(3x + y + z)) + 3^{-n} f(3^n(x + 3y + z)) + 3^{-n} f(3^n(x + y + 3z)) \\
&\quad + 3^{-n} f(3^n x) + 3^{-n} f(3^n y) + 3^{-n} f(3^n z) - 6.3^{-n} f(3^n(x + y + z))\| \\
&\quad \le \frac{1}{3^n}\varphi(3^n x, 3^n y, 3^n z),
\end{aligned}$$

for all $x, y, z \in G$. Taking on the limit as $n \to \infty$ in the above relation and using (9) and (15), it results

$$A(3x+y+z)+A(x+3y+z)+A(x+y+3z)+A(x)+A(y)+f(z) = 6A(x+y+z).$$

In order to show that A is the unique function defined on G, with the properties (3) and (11), let $B : G \longrightarrow X$ be another affine mapping such that

$$\|f(x) - B(x) - f(0)\| \le \frac{1}{3}\Phi(x), \forall x \in G.$$

It follows that

$$A(3^n x) + A(0) = 3^n A(x), \quad B(3^n x) + B(0) = 3^n B(x),$$

for all x in G. Then

$$\|A(x) - B(x)\| = \left\| \frac{(A(3^n x) + A(0)) - (B(3^n x) + B(0))}{3^n} \right\|$$

$$\leq \left\| \frac{A(3^n x) - f(0) - f(3^n x)}{3^n} \right\| + \left\| \frac{B(3^n x) - f(0) - f(3^n x)}{3^n} \right\|$$
$$+ \left\| \frac{A(0) - B(0)}{3^n} \right\|$$
$$\leq \frac{1}{3^n}.\frac{1}{3}\Phi(3^n x) + \frac{1}{3^n}.\frac{1}{3}\Phi(3^n x) + \frac{1}{3^n}\|A(0) - B(0)\|$$
$$= \frac{1}{3^n}\Phi(3^n x) + \frac{1}{3^n}\|A(0) - B(0)\|$$
$$= \sum_{k=0}^{\infty} \frac{\varphi(3^{k+n}x, 0, 0)}{3^k.3^n} + \frac{1}{3^n}\|A(0) - B(0)\|$$
$$= \sum_{p=n}^{\infty} \frac{\varphi(3^p x, 0, 0)}{3^p} + \frac{1}{3^n}\|A(0) - B(0)\|, \forall x \in G$$

Taking the limit as $n \to \infty$ in the above relation we obtain A coincides with B. This completes the proof of the theorem. From the Theorem 3.1 we obtain the following corollary concerning the stability for the Eq. (3).

Corollary 3.2. *Let G be an abelian group and X a Banach space. Let p, q, r, ε_i be real numbers such that $\varepsilon_i > 0$ $(i = 1, 2, 3)$, $p, q, r \in [0, 1)$. Suppose that a function $f : G \longrightarrow X$ satisfies*

$$\|f(3x + y + z) + f(x + 3y + z) + f(x + y + 3z) + f(x) + f(y) + f(z)$$
$$- 6f(x + y + z)\| \leq (\varepsilon_1\|x\|^p + \varepsilon_2\|y\|^q + \varepsilon_3\|z\|^r)$$

for all $x, y, z \in G$. Then there exists a unique mapping $A : G \longrightarrow X$, which satisfies the Eq. (3) and the estimation

$$\|f(x) - A(x) - f(0)\| \leq \frac{\varepsilon_1}{3 - 3^p}\|x\|^p, \quad \forall x \in G.$$

If we take $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon$ (say), then we have the estimation

$$\|f(x) - A(x) - f(0)\| \leq \frac{\varepsilon}{3 - 3^p}\|x\|^p, \quad \forall x \in G.$$

Remark 3.3. For $p = q = r = 0$ in the above corollary, properties of stability in Hyers–Ulam sense for (3) are obtained.

Remark 3.4. In the case $p = q = r = 1$ the affine functional equation (3) is unstable.

Corollary 3.5. *Let G be an abelian group and X be a Banach space, respectively. Suppose that a function $f : G \longrightarrow X$ satisfies*

$$\|f(3x+y+z)+f(x+3y+z)+f(x+y+3z)+f(x)+f(y)+f(z)-6f(x+y+z)\| \leq \theta,$$

for all $x, y, z \in G, \theta \geq 0$ is fixed. Then there exists a unique affine mapping $A : G \longrightarrow X$, which satisfies the Eq. (3) and the estimation

$$\|f(x) - A(x) - f(0)\| \leq \frac{\theta}{2}$$

holds $\forall\ x \in G$.

4 Fixed Point Method

We consider a nonempty set G, a complete metric space (X, d) and the mappings $\Lambda : \mathscr{R}_+^G \rightarrow \mathscr{R}_+^G$ and $\mathscr{T} : X^G \longrightarrow X^G$. We remember that X^G is the space of all mappings from G into X. In the following, we suppose that Λ satisfies the condition: for every sequence $(\delta_n)_{n\in\mathscr{N}}$, with

$$\delta_n(t \longrightarrow 0)(n \rightarrow \infty), t \in G \Longrightarrow (\Lambda\delta_n)(t) \longrightarrow 0(n \rightarrow \infty),\ t \in G. \qquad (C_1)$$

Proposition 4.1. *Let G be a nonempty set, (X, d) a complete metric space and $\Lambda : \mathscr{R}_+^G \rightarrow \mathscr{R}_+^G$ a non-decreasing operator satisfying the hypothesis (C_1). If $\mathscr{T} : X^G \longrightarrow X^G$ is an operator satisfying the inequality*

$$d((\mathscr{T}\xi)(x), (\mathscr{T}\mu)(x)) \leq \Lambda(d(\xi(x), \mu(x))),\ \ \xi, \mu \in X^G,\ x \in G, \qquad (16)$$

and the functions $\varepsilon : G \longrightarrow \mathscr{R}_+$ and $g : G \longrightarrow X$ are such that

$$d((\mathscr{T}g)(x), g(x)) \leq \varepsilon(x),\ x \in G, \qquad (17)$$

and

$$\varepsilon^*(x) := \sum_{k=0}^{\infty}(\Lambda^k\varepsilon)(x) < \infty,\ x \in G, \qquad (C_2)$$

then, for every $x \in G$, the limit

$$A(x) := \lim_{n\rightarrow\infty}(\mathscr{T}^n g)(x)$$

exists and the function $A \in X^G$, defined in this way, is a fixed point of $\mathscr{T}$, with

$$d(g(x), A(x)) \leq \varepsilon^*(x),\ x \in G.$$

Moreover, if the condition

$$\lim_{n\to\infty}(\Lambda^n\varepsilon^*)(x)=0,\ \forall x\in G, \tag{C_3}$$

holds, then A is the unique fixed point of $\mathcal{T}$ with the property

$$d(g(x),A(x))\le\varepsilon^*(x),\ x\in G.$$

The proof of Theorem 3.1. We apply the above proposition by taking the mapping

$$\Lambda:\mathcal{R}_+^G\to\mathcal{R}_+^G,\ (\lambda\delta)(x):=\frac{\delta(3x)}{3},\ (\delta:G\longrightarrow\mathcal{R}_+),$$

and the operator

$$\mathcal{T}:X^G\longrightarrow X^G,\ (\mathcal{T}\psi)(x):=\frac{\psi(3x)}{3},\ (\psi:G\longrightarrow X).$$

From the definition of Λ, the relation (C_1) is obvious and (16) holds with equality.

If we take $\varepsilon(x):=\frac{\varphi(x,0,0)}{3}$, where the mapping φ is defined in Theorem 3.1, the relation (8) implies that the series

$$\varepsilon^*(x):=\sum_{k=0}^{\infty}(\Lambda^k\varepsilon)(x)=\frac{1}{3}\sum_{k=0}^{\infty}\frac{\varphi(3^kx,0,0)}{3^k}=\frac{1}{3}\Phi(x),\ \forall x\in G$$

is convergent, so (C_2) is verified.

As in the first part of the initial proof of Theorem 3.1, we have that

$$\left\|\frac{g(3x)}{3}-g(x)\right\|\le\frac{1}{3}\varphi(x,0,0),\forall x\in G,$$

where $g(x):=f(x)-f(0)$ and f satisfied the hypotheses of Theorem 3.1. This means that (17) holds. Also

$$\begin{aligned}(\Lambda^k\varepsilon^*)(x)&=\frac{(\Lambda^n\Phi)(x)}{3}=\frac{\Phi(3^nx)}{3^{n+1}}=\frac{1}{3}\sum_{k=0}^{\infty}\frac{\varphi(3^{n+k}x,0,0)}{3^{n+k}}\\&=\frac{1}{3}\sum_{p=n}^{\infty}\frac{\varphi(3^px,0,0)}{3^p},\ \forall x\in G.\end{aligned}$$

Taking on the limit in the above relation as $n\to\infty$, we obtain that (C_3) is verified.

From Proposition 4.1 it results that the limit

$$\lim_{n\to\infty}(\mathscr{T}^n g)(x) = \lim_{n\to\infty}\frac{g(3^n x)}{3^n} = \lim_{n\to\infty}\frac{f(3^n x)}{3^n}$$

exists for every $x \in G$. Moreover, the mapping $A : G \longrightarrow X$,

$$A(x) = \lim_{n\to\infty}(\mathscr{T}^n g)(x)$$

is the unique fixed point of $\mathscr{T}$, with

$$d(g(x), A(x)) \leq \varepsilon^*(x),\ \forall x \in G,$$

which implies that

$$\|f(x) - A(x) - f(0)\| \leq \frac{1}{3}\Phi(x),\ \forall x \in G.$$

To prove that the function A is a solution of the affine functional equation (3) we use (9) and the definition of A.

This completes the proof.

5 The Alternative of Fixed Point

Theorem 5.1 (The alternative of fixed point [9]). *Suppose that we are given a complete generalized metric space (Ω, d) and a strictly contractive mapping $T : \Omega \longrightarrow \Omega$ with Lipchitz constant L. Then, for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1} x) = \infty \ \textit{for all}\ n \geq 0,$$

or there exists a natural number n_0 such that

- $d(T^n x, T^{n+1} x) < \infty$ *for all* $n \geq n_0$;
- *The sequence* $(T^n x)$ *is convergent to a fixed point* y^* *of* T;
- y^* *is the unique fixed point of* T *in the set* $\Delta = \{y \in \Omega : d(T^{n_0} x, y) < \infty\}$;
- $d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)$ *for all* $y \in \Delta$.

Utilizing the above-mentioned fixed point alternative, we now obtain our result, i.e., the generalized Hyers–Ulam–Rassias stability of the functional equation (3).

From now on, let G be an abelian group and X a Banach space. Given a mapping $f : G \longrightarrow X$, we set

$$\begin{aligned} Df(x,y,z) :=\ & f(3x+y+z) + f(x+3y+z) + f(x+y+3z) + f(x) + f(y) \\ & + f(z) - 6f(x+y+z) \end{aligned}$$

for all $x, y, z \in G$. Let $\varphi : G \times G \times G \longrightarrow [0, \infty)$ be a function such that

$$\lim_{n\to\infty} \frac{\varphi(3^n x, 3^n y, 3^n z)}{3^n} = 0, \forall x, y, z \in G. \tag{18}$$

Yang-Soo Jung and Ick-Soon Jung [14] used the above result to show the stability of a cubic functional equation. In the following theorem we will use the same result to prove the properties of stability from the Theorem 3.1.

Theorem 5.2. *Suppose that a function $f : G \longrightarrow X$ satisfies the functional inequality*

$$\|Df(x, y, z)\| \le \varphi(x, y, z) \tag{19}$$

$\forall x, y, z \in G$. *If there exists $L < 1$ such that the mapping*

$$x \mapsto \psi(x) = \varphi\left(\frac{x}{3}, 0, 0\right)$$

has the property

$$\psi(x) \le 3L\psi\left(\frac{x}{3}\right) \tag{20}$$

for all $x \in G$, then there exists a unique affine function $A : G \longrightarrow X$ such that the inequality

$$\|f(x) - A(x) - f(0)\| \le \frac{L}{1-L}\psi(x) \tag{21}$$

holds for all $x \in G$.

Proof. Let us define the mapping $g : G \longrightarrow X$ such that

$$g(x) = f(x) - f(0). \tag{22}$$

It is clear that $g(0) = 0$ and $f(x) = g(x) + f(0)$.

Consider the set

$$\Omega := \{g : X \longrightarrow Y,\ g(0) = 0\}$$

and introduce the generalized metric on Ω,

$$d(g, h) = d_\psi(g, h) = \inf\{K \in (0, \infty) :\ \|g(x) - h(x)\| \le K\psi(x),\ x \in X\}.$$

It is easy to see that (Ω, d) is complete.

Now we define a function $T : \Omega \longrightarrow \Omega$ by

$Tg(x) = \frac{1}{3}g(3x)$ for all $x \in G$. Note that for all $g, h \in \Omega$,

$$\begin{aligned} d(g,h) < K \quad &\Longrightarrow \|g(x) - h(x)\| \leq K\psi(x), \quad x \in G, \\ &\Longrightarrow \left\| \frac{g(3x)}{3} - \frac{h(3x)}{3} \right\| \leq \frac{1}{3}K\psi(3x), \quad x \in G, \\ &\Longrightarrow \left\| \frac{g(3x)}{3} - \frac{h(3x)}{3} \right\| \leq LK\psi(x), \quad x \in G, \\ &\Longrightarrow d(Tg, Th) \leq LK. \end{aligned}$$

Hence we see that

$d(Tg, Th) \leq Ld(g,h)$ for all $g, h \in \Omega$, that is, T is a strictly self-mapping of Ω with the Lipschitz constant L.

If we put $y = z = 0$ in (10), we obtain

$$\|f(3x) - 3f(x) + 2f(0)\| \leq \varphi(x, 0, 0), \ \forall x \in G.$$

Using (22) in the last inequality, we have

$$\|g(3x) - 3g(x)\| \leq \varphi(x, 0, 0), \ \forall x \in G,$$

$$\left\| g(x) - \frac{g(3x)}{3} \right\| \leq \frac{1}{3}\varphi(x, 0, 0), \ \forall x \in G,$$

which is reduced to

$$\left\| g(x) - \frac{g(3x)}{3} \right\| \leq L\psi(x), \ \forall x \in G,$$

i.e., $d(g, Tg) \leq L < \infty$.

Now, from the fixed point alternative, that there exists a fixed point A of T in Ω such that

$$A(x) = \lim_{n \to \infty} \frac{g(3^n x)}{3^n} = \lim_{n \to \infty} \frac{f(3^n x)}{3^n} \tag{23}$$

for all $x \in G$ since $\lim_{n \to \infty} d(T^n f, A) = 0$

To show that the function $A : G \longrightarrow X$ is affine, let us replace x, y and z by $3^n x$, $3^n y$ and $3^n z$ in (19), respectively, and divide by 3^n.

$$\begin{aligned} &\|3^{-n} f(3^n(3x + y + z)) + 3^{-n} f(3^n(x + 3y + z)) + 3^{-n} f(3^n(x + y + 3z)) \\ &\quad + 3^{-n} f(3^n x) + 3^{-n} f(3^n y) + 3^{-n} f(3^n z) - 6.3^{-n} f(3^n(x + y + z))\| \\ &\quad \leq \frac{1}{3^n}\varphi(3^n x, 3^n y, 3^n z), \end{aligned}$$

for all $x, y, z \in G$. Taking on the limit as $n \to \infty$ in the above relation and using (18) and (23), it results

$$\|DA(x,y,z)\| = \lim_{n\to\infty} \frac{\|Df(3^n x, 3^n y, 3^n z)\|}{3^n} \leq \lim_{n\to\infty} \frac{\varphi(3^n x, 3^n y, 3^n z)}{3^n} = 0$$

for all $x, y, z \in G$, i.e., A satisfies the functional equation (3). Theorem 2.1 guarantees that A is affine.

According to the fixed point alternative, since A is the unique fixed point of T in the set $\Delta = \{h \in \Omega : d(g,h) < \infty\}$, A is unique function such that

$$\|g(x) - A(x)\| \leq K\psi(x), \ \forall x \in G$$

and some $K > 0$. Again using the fixed point alternative, we have

$$d(g, A) \leq \frac{1}{1-L} d(g, Tg)$$

and so we obtain the inequality

$$d(g, A) \leq \frac{L}{1-L}$$

which yields the inequality (21).

This completes the proof of the theorem.

From Theorem 5.2, we obtain the following corollary concerning the Hyers–Ulam–Rassias stability [27] of the functional equation (3).

Corollary 5.3. *Let G be an abelian group and X a Banach space, respectively. Let $0 \leq p < 1$ be given. Assume that $\delta \geq 0$ and $\varepsilon \geq 0$ are fixed. Suppose that a function $f : G \longrightarrow X$ satisfies the functional inequality*

$$\|Df(x,y,z)\| \leq \delta + \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p) \tag{24}$$

for all $x, y, z \in G$. Then there exists a unique affine mapping $A : G \longrightarrow X$ such that the inequality

$$\|f(x) - A(x) - f(0)\| \leq \frac{\delta}{3^{1-p} - 1} + \frac{\varepsilon}{3 - 3^p}\|x\|^p \tag{25}$$

holds for all $x \in G$.

Proof. Let $\varphi(x,y,z) := \delta + \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p), \ \forall x, y, z \in G$. Then it follows that

$$\frac{\varphi(3^n x, 3^n y, 3^n z)}{3^n} = \frac{\delta}{3^n} + 3^{n(p-1)}(\|x\|^p + \|y\|^p + \|z\|^p) \to 0$$

as $n \to \infty$, where $p < 1$, i.e., the relation (18) is true.

Since the inequality

$$\psi(x) = \delta + \frac{\varepsilon}{3^p}\|x\|^p \leq 3^p \left(\delta + \frac{\varepsilon}{3^{2p}}\|x\|^p\right)$$

i.e., $\psi(x) \leq 3.3^{p-1}\psi(\frac{x}{3})$ holds $\forall x \in G$, where $p < 1$ with $L = 3^{p-1} < 1$.

Now the inequality (21) yields the inequality (25) which completes the proof of the corollary. This completes the proof.

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On Some Integral Operators

Khalida Inayat Noor

Abstract Let $P(n,\beta)$, $\quad 0 \leq \beta < 1$, be the class of functions $p : p(z) = 1 + c_n z^n + c_{n+1}z^{n+1} + \ldots$ analytic in the unit disc E such that $Re\{p(z)\} > \beta$. The class $P_k(n,\beta)$, $\quad k \geq 2$ is defined as follows: An analytic function $p \in P_k(n,\beta), k \geq 2$, $0 \leq \beta < 1$ if and only if there exist $p_1, p_2 \in P(n,\beta)$ such that

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z).$$

In this paper, we discuss some integral operators for certain classes of analytic functions defined in E and related with the class $P_k(n,\beta)$.

Keywords Analytic functions • Integral operators • Convolution • Libera operators

1 Introduction

Let $\mathscr{A}(n)$ denote the class of functions f of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \quad (n = N = \{1, 2, 3, \ldots, \}), \tag{1}$$

analytic in the unit disc $E = \{z : |z| < 1\}$. Let $P(n,\beta)$ be the class of functions $h(z)$ of the form

$$h(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \ldots, \tag{2}$$

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which are analytic in E and satisfy $Re\{h(z)\} > \beta, 0 \leq \beta < 1, z \in$ We note that $P(1,0) \equiv P$ is the class of functions with positive real part.

Let $P_k(n,\beta), k \geq 2, 0 \leq \beta < 1$, be the class of functions p, analytic in E, such that

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z)$$

if and only if $p_1, p_2 \in P(n,\beta)$ for $z \in E$. The class $P_k(1,0) \equiv P_k$ was introduced in [6]. We note that $p \in P_k(n,\beta)$ if and only if there exists $h \in P_k(n,0)$ such that

$$p(z) = (1-\beta)h(z) + \beta,$$

Let f and g be analytic in E with $f(z)$ given by (1) and $g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k$. Then the convolution (or Hadamard product) of f and g is defined by

$$(f \star g)(z) = z + \sum_{k=n+1}^{\infty} a_k b_k z^k.$$

A function $f \in \mathscr{A}(n)$ is said to belong to the class $R_k(n,\beta), k \geq 2, 0 \leq \beta < 1$, if and only if $\frac{zf'}{f} \in P_k(n,\beta)$ for $z \in E$.

We note that $R_k(1,0) \equiv R_k$ is the class of functions with bounded radius rotation, first discussed by Tammi, see [1] and $R_2(1,0)$ consists of starlike univalent functions.

Similarly $f \in \mathscr{A}(n)$ belongs to $V_k(n,\beta)$ for $z \in E$ if and only if $\frac{(f')'}{f'} \in P_k(n,\beta)$. It is obvious that

$$f \in V_k(n,\beta) \quad \text{if and only if} \quad zf' \in R_k(n,\beta). \tag{3}$$

It may be observed that $V_2(1,0) \equiv C$, the class of convex univalent functions and $V_k(1,0) \equiv V_k$ is the class of functions with bounded boundary rotation first discussed by Paatero, see [1].

2 Preliminary Results

We need the following results in our investigation.

Lemma 2.1 ([5]). *Let $u = u_1 + iu_2$, $v = v_1 + iv_2$ and $\Psi(u,v)$ be a complex-valued function satisfying the following conditions:*

(i). $\Psi(u,v)$ *is continuous in a domain* $D \subset \mathbf{C}^2$

(ii). $(1,0) \in D$ *and* $\Psi\{(1,0)\} > 0$.

(iii). $Re\Psi(iu_2, v_1) \leq 0$ *whenever* $(iu_2, v_1) \in D$ *and* $v_1 \leq \frac{-1}{2}(1+u_2^2)$.

Let $h(z)$, *given by (2), be analytic in* E *such that* $(h(z), zh'(z)) \in D$ *and* $Re\Psi(h(z), zh'(z)) > 0$ *for all* $z \in E$, *then* $Re\{h(z)\} > 0$ *in* E.

We shall need the following result which is a modified version of Theorem 3.3e in [4, p113].

Lemma 2.2. *Let* $\beta > 0, \beta + \delta > 0$ *and* $\alpha \in [\alpha_0, 1)$, *where*

$$\alpha_0 = \max\left\{\frac{\beta - \delta - 1}{2\beta}, \frac{-\delta}{\beta}\right\}.$$

If $\left\{h(z) + \frac{zh'(z)}{\beta h(z)+\delta}\right\} \in P(1, \alpha)$ *for* $z \in E$, *then* $h \in P(1, \sigma)$ *in* E, *where*

$$\sigma(\alpha, \beta, \delta) = \left[\frac{(\beta + \delta)}{\beta\{{}_2F_1(2\beta(1-\alpha), 1, \beta + \delta + 1; \frac{r}{1+r}\}} - \frac{\delta}{\beta}\right], \tag{4}$$

where ${}_2F_1$ *denotes hypergeometric function. This result is sharp and external function is given as*

$$p_0(z) = \frac{1}{\beta g(z)} - \frac{\delta}{\beta}, \tag{5}$$

with

$$\begin{aligned} g(z) &= \int_0^1 \left(\frac{1-z}{1-tz}\right)^{2\beta(1-\alpha)} t^{(\beta+\delta-1)} dt \\ &= {}_2F_1\left(2\beta(1-\alpha), 1, \beta + \delta + 1; \frac{z}{z-1}\right). (\beta + \delta)^{-1}. \end{aligned}$$

3 Main Results

Theorem 3.1. *Let* $f \in R_k(n, \beta)$, $g \in R_k(n, \beta), \alpha, c, \delta$ *and* ν *be positively real and* $\delta = \nu = \alpha$. *Then the function* F *defined by*

$$[F(z)]^\alpha = cz^{\alpha-c}\int_0^z t^{(c-\delta-\nu)-1}(f(t))^\delta (g(t))^\nu dt \tag{6}$$

belongs to $R_k(n, \sigma)$, *where*

$$\sigma = \frac{2(2\beta c_1 + n\alpha_1)}{(n\alpha_1 - 2\beta + 2c_1) + \sqrt{(n\alpha_1 - 2\beta + 2c_1)^2 + 8(2\beta c_1 + n\alpha_1)}}, \tag{7}$$

with

$$c_1 = \frac{c-\alpha}{\alpha}, \quad \alpha_1 = \frac{1}{\alpha}.$$

Proof. First we show that there exists a function $F \in \mathscr{A}(n)$ satisfying (6). Let

$$G(z) = z^{-(\nu+\delta)} \left(f(z)\right)^{\delta} \left(g(z)\right)^{\nu} = 1 + \alpha_n z^n + \alpha_{n+1} z^{n+1} + \dots,$$

and choose the branches which equal 1 when $z = 0$. For

$$K(z) = z^{(c-\nu-\delta)-1} \left(f(z)\right)^{\delta} \left(g(z)\right)^{\nu} = z^{c-1} G(z),$$

we have

$$L(z) = \frac{c}{z^c} \int_0^z K(t)dt = 1 + \frac{c}{n+1} \alpha_n z^n + \dots,$$

where L is well defined and analytic in E. Now let

$$F(z) = \left[z^{\alpha} L(z)\right]^{\frac{1}{\alpha}} = z \left[L(z)\right]^{\frac{1}{\alpha}},$$

where we choose the branch of $[L(z)]^{\frac{1}{\alpha}}$ which equals 1 when $z = 0$. Thus $F \in \mathscr{A}(n)$ and satisfies (6).

Now, from (6), we have

$$z^{(c-\alpha-1)} \left[F(z)\right]^{\alpha} \left[(c-\alpha) + \alpha \frac{zF'(z)}{F(z)}\right] = c\left[z^{(c-\delta-\nu)-1} \left(f(z)\right)^{\delta} \left(g(z)\right)^{\nu}\right]. \tag{8}$$

We write

$$\frac{zF'(z)}{F(z)} = p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z). \tag{9}$$

Then $p(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \dots$, is analytic in E.

Logarithmic differentiation of (8) and use of (9) yields

$$(c-\alpha-1) + \alpha p(z) + \frac{\alpha z p'(z)}{(c-\alpha) + \alpha p(z)} = (c-\delta-\nu-1) + \frac{\delta z f'(z)}{f(z)} + \frac{\nu z g'(z)}{g(z)}.$$

Since $\nu + \delta = \alpha : f, g \in P_k(n, \beta)$ and it is known [2] that $P_k(n, \beta)$ is a convex set, it follows that

$$\left\{p + \frac{\frac{1}{\alpha} z p'}{p + \left(\frac{c-\alpha}{\alpha}\right)}\right\} \in P_k(n, \beta), \quad z \in E.$$

Define

$$\Phi_{\alpha,c}(z) = \frac{1}{1+c_1}\frac{z}{(1-z)^{\alpha_1+1}} + \frac{c_1}{1+c_1}\frac{z}{(1-z)^{\alpha_1+2}},$$

with $\alpha_1 = \frac{1}{\alpha}, \quad c_1 = \frac{c-\alpha}{\alpha}$.

Then, using (9), we have

$$\begin{aligned}\left(p \star \frac{\Phi_{\alpha,c}}{z}\right) &= p(z) + \frac{\alpha_1 z p'(z)}{p(z)+c_1} \\ &= \left(\frac{k}{4}+\frac{1}{2}\right)\left[p_1(z) + \frac{\alpha_1 z p_1'(z)}{p_1(z)+c_1}\right] \\ &\quad -\left(\frac{k}{4}-\frac{1}{2}\right)\left[p_2(z) + \frac{\alpha_1 z p_2'(z)}{p_2(z)+c_1}\right].\end{aligned}$$

Since $\left\{p + \frac{\alpha_1 z p'}{p+c_1}\right\} \in P_k(n,\beta)$, it follows that

$$\left\{p_i + \frac{\alpha_1 z p_i'}{p_i + c_1}\right\} \in P_k(n,\beta), \quad \text{for} \quad i = 1,2, \quad z \in E.$$

Writing $p_i(z) = (1-\sigma)H_i(z) + \sigma, \quad i = 1,2$, we have, for $z \in E$,

$$\left[(1-\sigma)H_i + \sigma + \frac{\alpha_1(1-\sigma)H_i'}{(1-\sigma)H_i + \sigma + c_1} - \beta\right] \in P(n,0).$$

We now form the functional $\Psi(u,v)$ by taking $u = H_i$ and $v = zH_i'$ and so

$$\Psi(u,v) = (\sigma-\beta) + (1-\sigma)u + \frac{\alpha_1(1-\sigma)v}{(1-\sigma)u+\sigma+c_1}.$$

It can easily be seen that:

(i) $\Psi(u,v)$ is continuous in $\mathscr{D} = \left(\mathscr{C} - \left\{\frac{\sigma+c_1}{1-\sigma}\right\}\right) \times \mathscr{C}$.
(ii) $(i,0) \in \mathscr{D}$ and $Re\{\Psi(i,0) = 1-\beta > 0$.

To verify the condition (iii) of Lemma 2.1, we proceed as follows:

For all $(iu_2, v_1) \in \mathscr{D}$ such that $\quad v_1 \le \frac{-n(1+u_2^2)}{2}$, and

$$\begin{aligned}\Re\{\Psi(iu_2, v_1)\} &= (\sigma-\beta) + \frac{\alpha_1(1-\sigma)(\sigma+c_1)v_1}{(\sigma+c_1)^2 + (1-\sigma)^2 u_2^2} \\ &\le \frac{2(\sigma-\beta)\left\{(\sigma+c_1)^2+(1-\sigma)^2u_2^2\right\} - n\alpha_1(1-\sigma)(\sigma+c_1)(1+u_2^2)}{2(\sigma+c_1)^2+(1-\sigma)^2u_2^2} \\ &= \frac{A+Bu_2^2}{2C} \qquad (10) \\ &\le 0, \quad \text{if} \quad A \le 0 \quad \text{and} \quad B \le 0,\end{aligned}$$

where

$$A = 2(\sigma - \beta)(\sigma + c_1)^2 - n\alpha_1(1-\sigma)(\sigma + c_1),$$
$$B = 2(\sigma - \beta)(1-\sigma)^2 - n\alpha_1(1-\sigma)(\sigma + c_1)$$
$$C = (\sigma + c_1)^2 + (1-\sigma)^2 u_2^2 > 0.$$

From $A = 0$, we obtain σ as given by (7) and $B \leq 0$ ensures that $0 \leq \sigma < 1$. Thus using Lemma 2.1, it follows that $H_i \in P(n, 0)$ and therefore $p_i \in P(n, \sigma)$, $i = 1, 2$. Consequently $p \in P_k(n, \sigma)$ and this completes the proof. □

Corollary 3.1. *For* $0 = c = n = 1, \beta = 0$ *and* $f = g, F \in V_k$ *implies that* $F \in R_k(\frac{1}{2})$ *and this, with* $k = 2$, *gives us a well-known result that every convex function is starlike of order* $\frac{1}{2}$ *in* E.

Corollary 3.2. *For* $n = 1$, *let* $f \in R_k(1, \sigma)$ *in Theorem 3.1. Then* $F \in R_k(1, \sigma_0)$, *where* σ_0 *is given by (2.1) with* $\beta = \alpha, \delta = (1 - \alpha)$. *This result is sharp.*

Corollary 3.3. *In (2), we take* $\nu + \delta = 1, c = 2, f = g$ *and obtain Libera's integral operator [3, 6] as:*

$$F(z) = \frac{2}{z} \int_0^z f(t)dt, \tag{11}$$

where $f \in R_k(n, \beta)$. *Then, by Theorem 3.1, it follows that* $F \in R_k(n, \sigma_1)$, *where*

$$\sigma_1 = \frac{2(2\beta + n)}{\left[(n - 2\beta + 2) + \sqrt{(n - 2\beta + 2)^2 + 8(2\beta + n)}\right]}. \tag{12}$$

For $\beta = 0$ *and* $n = 1$, *we have Libera's operator for the class* R_k *of bounded radius rotation. That is, if* $f \in R_k$ *and* F *is given by (3.6), then*

$$F \in R_k(1, \sigma_2), \quad \text{with} \quad \sigma_2 = \frac{2}{3 + \sqrt{17}}.$$

Using Theorem 3.1 and relation (3), we can prove the following.

Theorem 3.2. *Let* f *and* g *belong to* $V_k(n, \beta)$, *and let* F *be defined by (6) with* α, c, δ, ν *positively real,* $\delta + \nu = \alpha$. *Then* $F \in V_k(n, \sigma)$, *where* σ *is given by (7).*

By taking $\alpha = 1, c + \frac{1}{\lambda}, \nu + \delta = \alpha = 1$ and $f = g$ in (6), we obtain the integral operator $I_\lambda(f) = F$, defined as:

$$F(z) = \frac{1}{\lambda} \int_0^z t^{\frac{1}{\lambda} - 2} f(t)dt, \quad (\lambda > 0). \tag{13}$$

With the similar techniques, we can easily prove the following result which is stronger version than the one proved in Theorem 3.1.

Theorem 3.3. *Let* $f \in R_k(n, \gamma)$ *and let, for* $0 < \lambda \leq 1$, F *be defined by (13). Then* $F \in R_k(n, \delta^*)$, *where* δ^* *satisfies the conditions given below:*

(i) *If* $0 < \lambda \leq \frac{1}{2}$ *and* $\frac{n\lambda}{2(\lambda-1)} \leq \gamma < 1$, *then*

$$\delta^* = \delta_1 = \frac{1}{4\lambda}\left[A_1 + \sqrt{A_1^2 + 8B_1}\right] \geq 0,$$

where

$$A_1 = 2\gamma\lambda + 2\lambda - n\lambda$$
$$B_1 = \lambda\{2\gamma(1-\lambda) + n\lambda\}.$$

(ii) *If* $\frac{1}{2} < \lambda \leq 1$, $\quad \frac{n(\lambda-1)}{2\lambda} \leq \frac{n(3\lambda-\sqrt{8\lambda})}{2\lambda} \leq \gamma$, *then*

$$\delta^* = \delta_2 = \frac{1}{4\lambda}\left[A_2 + \sqrt{A_2^2 + 8B_2}\right] \geq 0,$$

where

$$A_2 = 2\lambda + 2\lambda\gamma - n\lambda$$
$$B_2 = \lambda(2\lambda\gamma + n - n\lambda).$$

(iii) *If* $\frac{1}{2} < \lambda \leq 1$, $\quad \frac{n(\lambda-1)}{2\lambda} < \frac{n(3\lambda-\sqrt{8\lambda})}{2\lambda} < \gamma < 1$, *then* $\delta_3 = \delta_1$.

Special Cases

(1). Let $\lambda = \frac{1}{2}$ in (13). Then we have Libera's operator and (i) gives us

$$\delta^* = \delta_1 = \frac{2(2\gamma + n)}{(n - 2\gamma + 2) + \sqrt{(n - 2\gamma + 2)^2 + 8(2\gamma + n)}}.$$

(2). When $\gamma = 0, \lambda = \frac{1}{2}, n = 1$, and $f \in R_k$, then $F \in R_k(1, \delta_1)$, where

$$\delta^* = \delta_1 = \frac{2}{3 + \sqrt{17}}.$$

(3). Let $\lambda = 1, \gamma = 0, n = 1$ and $f \in R_k$. Then, from (3.8), it follows that

$$F(z) = \int_0^z \frac{f(t)}{t} dt$$

and, by Theorem 3.3, $F \in R_k(\frac{1}{2})$. By using relation (3) and $k = 2$, we obtain a well-known result that every convex function is starlike of order $\frac{1}{2}$.

Theorem 3.4. *Let* $f \in R_k(n, 0), g \in R_k(n, \alpha), 0 \leq \alpha \leq 1$. *Let the function* F, *for* $b \geq 0$, *be defined as*

$$F(z) = \frac{1+b}{z^b} \int_0^z f^\alpha(t) t^{b-\alpha-1} g(t) dt. \tag{14}$$

Then $F \in R_k(n, \eta), \quad z \in E$, *where*

$$\eta = \frac{2n}{(2b+n) + \sqrt{(2b+n)^2 + 8n}}. \tag{15}$$

Proof. Set

$$\frac{zF'(z)}{F(z)} = p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z).$$

Then $p(z)$ is analytic in E and $p(0) = 1$. From (14), we have

$$\begin{aligned} p(z) + \frac{zp'(z)}{p(z)+b} &= \left[\alpha \frac{zf'(z)}{f(z)} + (1-\alpha)\right] + \frac{zg'(z)}{g(z)} - 1 \\ &= [\alpha h_1 + (1-\alpha)] + [(1-\alpha)h_2(z) + \alpha] - 1 \\ &= \alpha h_1(z) + (1-\alpha)h_2(z) = h(z), \quad h \in P_k(n, 0). \end{aligned}$$

Since $g \in P_k(n, \alpha), \quad f \in R_k(n, 0)$, it follows that $h_1, h_2 \in P_k(n, 0)$ and $P_k(n, 0)$ is a convex set. Now following the similar technique of Theorem 3.1 and using Lemma 2.1, we obtain the required result that $\frac{zF'(z)}{F(z)} = p(z) \in P_k(n, \eta)$, where η is given by (15). □

Remark 3.1. When $n = 1$, we obtain best possible value of $\eta = \sigma$ given by (2.1) with $\alpha = 0, \beta = 1, \delta = b$.

Conclusion. In this paper, we have introduced and considered a new class $P_k(n, \beta)$ of analytic function. We have discussed several special cases of this new class. We have discussed some integral operators for certain classes of analytic functions in the unit disc E and related with the new class $P_k(n, \beta)$. Results obtained in this paper can be viewed as an refinement and improvement of the previously known results in this field.

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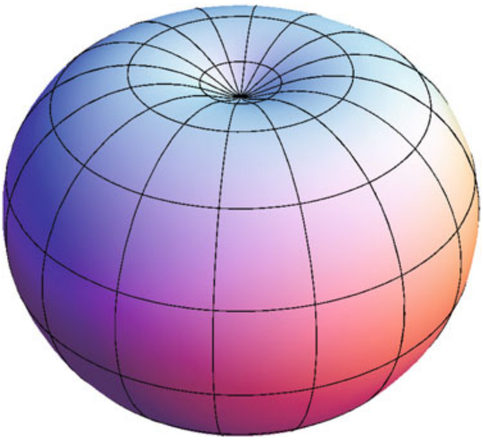
Integer Points in Large Bodies

Werner Georg Nowak

Abstract For a compact body $\mathscr{B}$ in three-dimensional Euclidean space with sufficiently smooth boundary, the number $N(\mathscr{B};t)$ of points with integer coordinates in a linearly enlarged copy $t\mathscr{B}$ is approximated in first order by the volume $\mathrm{vol}(\mathscr{B})t^3$. This article provides a survey on the state of art of research on the *lattice discrepancy* $D(\mathscr{B};t) = N(\mathscr{B};t) - \mathrm{vol}(\mathscr{B})t^3$, starting from the classic theory and emphasizing recent developments and advances.

Keywords Lattice points • Lattice discrepancy • non-convex bodies



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1 Introduction: Classic Lattice Point Theory

Let $\mathscr{B}$ denote a compact body in three-dimensional Euclidean space $\mathbb{R}^3$, with a sufficiently smooth boundary $\partial\mathscr{B}$, and t a large real parameter. It is well known that the number $N(\mathscr{B};t)$ of points with integer coordinates in a linearly dilated copy $t\mathscr{B}$ of $\mathscr{B}$ is asymptotically equal to the volume $\mathrm{vol}(\mathscr{B})t^3$. The central question of the classic *lattice point theory of large bodies*, as founded by E. Landau and others in the first decades of the twentieth century, is to estimate, from above and below, the *lattice discrepancy*

$$D(\mathscr{B};t) := N(\mathscr{B};t) - \mathrm{vol}(\mathscr{B})\,t^3\,. \tag{1}$$

Enlightening and comprehensive accounts on this theory have been given in the monographs by Fricker [8] and by Krätzel [22, 23], as well as in a more recent survey article by Ivić, Krätzel, Kühleitner, and the author [19]. The most up-to-date approach to *planar* lattice point problems ("discrete Hardy–Littlewood method") has been exposed in detail in Huxley's book [15].

1.1 The "Generic Case" of Strict Convexity and Nonzero Curvature

For the *generic case* that $\mathscr{B}$ is strictly convex and the Gaussian curvature κ of $\partial\mathscr{B}$ is everywhere bounded from above and away from zero, Hlawka [14] in 1950 proved that[1]

$$D(\mathscr{B};t) = O(t^{3/2})\,, \qquad D(\mathscr{B};t) = \Omega(t)\,, \tag{2}$$

with the constants implied possibly depending on the original body $\mathscr{B}$. (This will be the case throughout this article.) An elegant proof of this upper estimate has been given by Krätzel [23, Satz 5.15].

We remark parenthetically that in both references the problem has been treated in the more general setting of a body in k-dimensional space $\mathbb{R}^k$, $k \geq 3$. This applies to most contributions to the literature which deal with the *generic* case. The general analogues of (2) read

$$D(\mathscr{B};t) = O(t^{k-2+2/(k+1)})\,, \qquad D(\mathscr{B};t) = \Omega(t^{(k-1)/2})\,.$$

For the sake of clarity, in this article we shall restrict the discussion to dimension three.

[1] For the definitions of the order symbols O, Ω, $\ll$, $\asymp$, etc., see, e.g., Krätzel's book [22].

Hlawka's bound was improved in later decades by Krätzel and the author to $O(t^{37/25})$ [27], and $O(t^{25/17+\varepsilon})$ [28], by Müller [35] to $O(t^{63/43+\varepsilon})$, and most recently by Guo [11] to $O(t^{231/158+\varepsilon})$. Notice that $\frac{37}{25} = 1.48$, $\frac{25}{17} = 1.470\ldots$, $\frac{63}{43} = 1.4651\ldots$, and $\frac{231}{158} = 1.4620\ldots$.

Apparently the only improvement of the lower bound in (2) is due to the author [37] and reads

$$D(\mathscr{B};t) = \Omega(t\,(\log t)^{1/3})\,. \tag{3}$$

Thus, for the *generic case*, it is not known what might be the infimum of all possible exponents in this problem. The conjecture that it equals 1 may (or may not) be a hazardous guess, although it is supported not only by the lower bound (3) but also by the mean-square estimate

$$\int_1^T (D(\mathscr{B};t))^2 dt \ll T^3\,(\log T)^2\,, \tag{4}$$

which is due to Iosevich et al. [17].

1.2 Balls in $\mathbb{R}^3$

For the special case of the three-dimensional unit ball $\mathscr{B}_0$ ("sphere problem"), far better bounds are known. During several decades in the twentieth century, Vinogradov was almost the only one working on this problem, finally arriving in 1963 at [50]

$$D(\mathscr{B}_0;t) = O\left(t^{4/3}(\log t)^6\right)\,.$$

This was improved in 1995 by Chamizo and Iwaniec [6] to

$$D(\mathscr{B}_0;t) = O\left(t^{29/22+\varepsilon}\right)\,,$$

and a little later refined further by Heath-Brown [13] who obtained

$$D(\mathscr{B}_0;t) = O\left(t^{21/16+\varepsilon}\right)\,. \tag{5}$$

To understand why the results for balls are much sharper than for general bodies, it must be pointed out that here the explicit formula (see Bateman [1]) for $r_3(n)$, the number of ways to write n as a sum of three squares of integers, plays an important role. This formula involves a certain real character; to estimate the latter *on average*, results of Burgess [3,4] have been applied with gain.

The bound (5) was subsequently generalized to *rational* ellipsoids by Chamizo et al. [7]. We mention parenthetically that deep results on ellipsoids in higher dimensions have been established fairly recently by Bentkus and Götze [2] and by Götze [10].

Concerning lower estimates, Tsang [49] in 2000 proved that, somewhat sharper than (3),

$$D(\mathscr{B}_0; t) = \Omega_{\pm}\left(t(\log t)^{1/2}\right) . \tag{6}$$

The Ω_{-}-part of this result was already known to Szegö [47] in 1926.

Furthermore, there is the mean-square asymptotics

$$\int_1^T (D(\mathscr{B}_0; t))^2 dt = c_0 T^3 \log T + O(T^3) , \tag{7}$$

due to Lau [33], who improved the error term compared to much earlier work by Jarnik [21].

1.3 Bodies of Rotation (Generic Case)

Sort of "hybrids" between balls and general convex bodies are *bodies of rotation* $\mathscr{B}_{\mathrm{rot}}$, the axis of rotation being assumed throughout to coincide with one coordinate axis. In this subsection we will assume further that the Gaussian curvature κ is bounded away from zero on the sufficiently smooth boundary $\partial\mathscr{B}_{\mathrm{rot}}$ of $\mathscr{B}_{\mathrm{rot}}$. Under this condition (and some technicalities), F. Chamizo showed that [5]

$$D(\mathscr{B}_{\mathrm{rot}}; t) = O\left(t^{11/8+\varepsilon}\right) . \tag{8}$$

In the other direction, Kühleitner and the author [32] proved that

$$D(\mathscr{B}_{\mathrm{rot}}; t) = \Omega_{-}\left(t\,(\log t)^{1/3}(\log\log t)^{2(\sqrt{2}-1)/3}(\log\log\log t)^{-2/3}\right) ,$$

using an ingenious method of Soundararajan [46].

A bit special is the case of *ellipsoids of rotation*

$$\mathscr{E}_a : \qquad \frac{x_1^2 + x_2^2}{a} + a^2 x_3^2 \le 1$$

depending on one positive parameter a, and normalized such that $\mathrm{vol}(\mathscr{E}_a) = \frac{4\pi}{3}$. Krätzel and the author [29] derived an upper bound for $D(\mathscr{E}_a; t)$, uniform in a and t, with explicit numerical constants. For fixed a, this amounts to $O(t^{11/8}(\log t)^{3/8})$

which is slightly sharper than (8). Later on [41, 42], the author proved that

$$D(\mathscr{E}_a; t) = O\left(t^{3433555/2498114+\varepsilon}\right),$$

uniformly in a on every compact interval $[a_1, a_2]$, $0 < a_1 < a_2$. Note that $\frac{3433555}{2498114} = 1.37445889\ldots$, while $\frac{11}{8} = 1.375$.

2 Bodies with Boundary Points of Curvature Zero

Until now we have only discussed bodies with nonzero Gaussian curvature throughout their (smooth) boundary. In order to understand the role of boundary points of curvature zero, it may be helpful to recall the following result for the planar case which may be considered as a *main theorem of planar lattice point theory*; see Müller and the author [36].

Theorem 1. *For a compact planar domain $\mathscr{D}$ with analytic boundary $\partial\mathscr{D}$, suppose that the curvature vanishes on $\partial\mathscr{D}$ exactly in finitely many points $\mathbf{p}_j$, of orders n_j with respect to the arclength, $\partial\mathscr{D}$ possessing a rational normal vector in every $\mathbf{p}_j$. Then,*

$$\#\left(t\mathscr{D} \cap \mathbb{Z}^2\right) - \text{area}(\mathscr{D})t^2 = \sum_{\mathbf{p}_j} \sum_{k \geq 1} \mathscr{F}_{j,k}(t)\, t^{1-k/(n_j+2)} + O(t^\theta), \qquad \theta < \frac{2}{3},$$

where the functions $\mathscr{F}_{j,k}(t)$ are continuous and periodic, and can be given explicitly in terms of Fourier series.

Concerning the extension of this scenario to dimension three, the results have been rather fragmentary until very recently, and they are not really satisfactory at the time being. To start with, it is not even clear what might be "natural assumptions" about the set

$$Z = \{\mathbf{p} \in \partial\mathscr{B} : \ \kappa = 0 \ \text{ at } \ \mathbf{p}\}.$$

Should it be supposed to consist of isolated points or of whole curves on $\partial\mathscr{B}$? In the latter case it has to be expected that "almost all" normal vectors attached to points of Z are not rational.

2.1 Isolated Boundary Points of Curvature Zero

In pioneer work in 1993, which unfortunately was never published in print, Haberland [12] investigated the case that Z consists of finitely many points, each

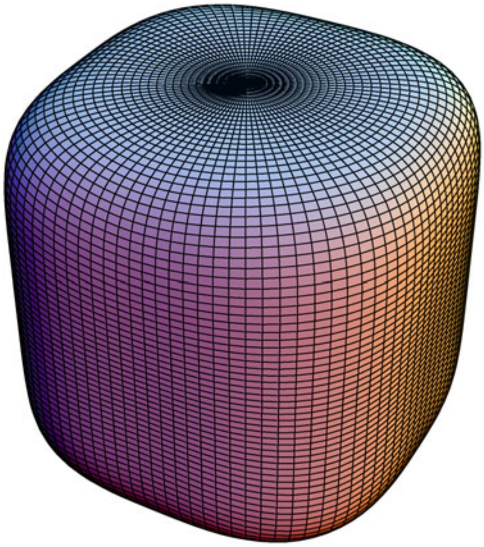


Fig. 1 Krätzel's 3D-body for $k = 6, \ell = 3$

with a rational normal vector. Assuming κ to vanish of sufficiently high order, he was able to derive an asymptotic formula for $D(\mathscr{B}; t)$.

In 2002, Peter [44] realized that an important role was played by the so-called *flat points* of $\partial\mathscr{B}$

$$F = \{\mathbf{p} \in \partial\mathscr{B} : \quad \text{both main curvatures of } \partial\mathscr{B} \text{ vanish at } \mathbf{p}\,\}\,.$$

For F finite and the normal vector of $\partial\mathscr{B}$ rational in each $\mathbf{p} \in F$, he established an asymptotics for $D(\mathscr{B}; t)$ consisting of a main term generated by the flat points and an error of $O(t^{9/5})$.

2.2 Krätzel's 3D-Body

At about the same time, Krätzel [24–26] deduced more precise estimates for certain special bodies. His research culminated in a very precise asymptotics concerning bodies (Fig. 1)

$$\mathscr{B}_{k,\ell} : \quad \left(|x_1|^\ell + |x_2|^\ell\right)^{k/\ell} + |x_3|^k \le 1\,, \tag{9}$$

where $k \ge 4$ and $\ell \ge 2$ are fixed integers, ℓ a divisor of k. As an arithmetic interpretation, counting the lattice points in $t\mathscr{B}_{k,\ell}$ yields information about the average number of solutions of a corresponding Diophantine equation.

It is straightforward to verify that $\partial\mathscr{B}_{k,\ell}$ has flat points in $(0,0,\pm 1)$, and for $\ell > 2$ also in $(0,\pm 1,0)$ and $(\pm 1,0,0)$. The normal vectors are all rational in these points. Furthermore, the Gaussian curvature identically vanishes (at least for $\ell > 2$) along the curves of intersection of $\partial\mathscr{B}_{k,\ell}$ with the coordinate planes. Krätzel's result ultimately reads

$$D(B_{k,\ell};t) = \mathscr{F}(t)t^{2-2/k} + O\left(t^{\frac{119}{73}-\frac{165}{146k}}(\log t)^{\frac{315}{146}}\right) + O(t^{3/2}(\log t)^3)\,. \qquad (10)$$

Here $\mathscr{F}(t)$ is given explicitly by an absolutely convergent Fourier series, hence a continuous periodic function; it reflects the contribution of the two flat points at $(0,0,\pm 1)$ to the lattice discrepancy. The first O-term comes from all other boundary points of curvature zero. The corresponding normal vectors are all different and *almost everywhere* irrational, thus their contribution cannot be evaluated asymptotically, but only estimated from above. To do so, Huxley's deep technique [15] has been applied.

For the special case $\ell = 2$, $\mathscr{B}_{k,2}$ is a body of rotation, generated by a Lamé curve rotating about the third coordinate axis. Here a much better error estimate can be obtained: see Krätzel and the author [30].

3 Recent Progress

3.1 General Bodies of Rotation

Further research concentrated on bodies of rotation $\mathscr{B}_{\mathrm{rot}}$ (about a coordinate axis), for at least two good reasons: The set of points on $\partial\mathscr{B}_{\mathrm{rot}}$ with Gaussian curvature zero are in a natural way and conveniently described by the *meridian curve* whose rotation generates $\partial\mathscr{B}_{\mathrm{rot}}$. Furthermore, the necessary analysis is considerably simpler than in general. Therefore, in certain interesting cases, it is possible to determine the exact order of magnitude of $D(\mathscr{B}_{\mathrm{rot}};t)$.

In this direction, pioneer work has been done by Popov [45]. The following result due to the author is a bit sharper and more general. Look at the body $\mathscr{B}_{\mathrm{rot}}$ shown in Fig. 2. Its boundary contains two *flat points* in the points of intersection of $\partial\mathscr{B}_{\mathrm{rot}}$ with the axis of rotation. Moreover, the Gaussian curvature vanishes on $\partial\mathscr{B}_{\mathrm{rot}}$ at altogether five circles: at the *equator* (borrowing this expression from geography) and in four other circles symmetric to it. (Obviously, the curvature is positive close to the equator and close to the endpoints, but negative somewhere in between.)

The situation is described in satisfactory generality in the following theorem [39].

Theorem 2. *Suppose that the boundary $\partial\mathscr{B}_{\mathrm{rot}}$ of a body of rotation $\mathscr{B}_{\mathrm{rot}}$ is smooth and generated by rotation of a meridian curve $\mathscr{C}$ about the horizontal axis. For $\rho \in C^4[0,\pi]$ positive, let*

$$\mathscr{C} = \{(x,y) = (\rho(|\theta|)\cos\theta, \rho(|\theta|)\sin\theta) :\ \theta \in [-\pi,\pi]\}\,,$$

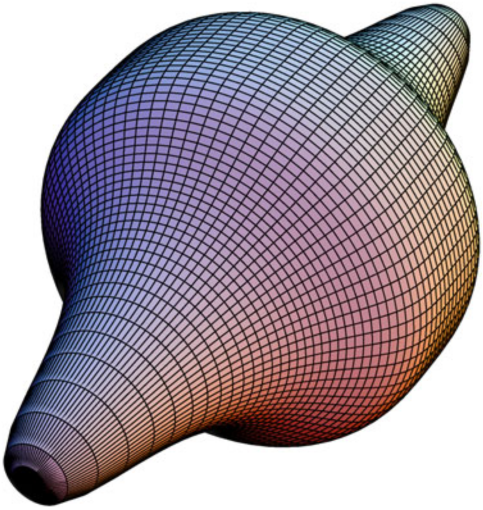


Fig. 2 A general non-convex body of rotation about a horizontal coordinate axis

with $\frac{dx}{d\theta} < 0$ *for* $0 < \theta < \pi$. *Further, the curvature of* $\mathscr{C}$ *vanishes at most at the points corresponding to finitely many* θ*-values* $0, \pi$ *and* $\theta_1, \ldots, \theta_J \in]0, \pi[$. *In these points,* $\rho(\theta)$ *is analytic. Let* N_1, N_2, *resp.,* $M_1, \ldots, M_J$ *denote the orders of the zeros of the curvature of* $\mathscr{C}$ *in these points, as a function of* θ. *Then it holds true that*

$$D(\mathscr{B}_{\text{rot}}; t) = \sum_{i=1,2} \sum_{k \geq 2} \mathscr{F}_{i,k}(t) t^{2-k/(N_i+2)} + \Delta(\mathscr{B}_{\text{rot}}; t),$$

$$\Delta(\mathscr{B}_{\text{rot}}; t) \ll \begin{cases} t^{3/2+\epsilon} & \text{for } M := \max(M_j) \leq 7, \\ t^{\frac{339M+416}{208(M+2)}+\epsilon} & \text{for } M \geq 8. \end{cases} \tag{11}$$

The functions $\mathscr{F}_i(t)$, $i = 1, 2$, *are again represented by absolutely convergent Fourier series, hence continuous and periodic.*

Remark 1. The terms $\mathscr{F}_{i,k}(t)t^{2-k/(N_i+2)}$ describe the contribution of the two flat points to $D(\mathscr{B}_{\text{rot}}; t)$. Here the normal vectors are rational, which permits an explicit evaluation. However, the leading terms $\mathscr{F}_{i,2}(t)t^{2-2/(N_i+2)}$ can supersede the error only if $N_i \geq 3$, i.e., if the curvature of the meridian curve has a zero of order at least three in this flat point. All other points of $\partial\mathscr{B}_{\text{rot}}$ of curvature zero are located on circles along which the normal vector varies, being irrational in almost all points. Hence the contribution of these points cannot be evaluated exactly, but only be estimated from above. This is done by the last bound for $\Delta(\mathscr{B}_{\text{rot}}; t)$ which has been obtained by an application of Huxley's method in its presently sharpest form [16]. However, this term is of significance only if the curvature of $\mathscr{C}$ has a zero of order ≥ 8 in those points. Otherwise, it is absorbed by the error term $O(t^{3/2+\varepsilon})$ which matches well with Hlawka's classic result (2) and which shows the limit of the method employed.

There is also a mean-square result for bodies of rotation $\mathscr{B}_{\text{rot}}$ as described in the above theorem, for which the flat points at the ends are the *only* points of $\partial\mathscr{B}_{\text{rot}}$ with curvature zero [38]: For this special case,

$$\int_1^T (\Delta(\mathscr{B}_{\text{rot}};t))^2\,dt = O\left(T^{3+\varepsilon}\right)\,,$$

where $\Delta(\mathscr{B}_{\text{rot}};t)$ is defined by (11). This matches well with the results (4) by Iosevich et al. [17] and (7) by Lau [33].

3.2 An "Exotic" Example: The Torus in $\mathbb{R}^3$

Apparently there is only one body that has been discussed in the literature in this context, whose boundary is a surface of genus exceeding zero. It is the body $\mathscr{T}$ whose boundary $\partial\mathscr{T}$ is the three-dimensional torus

$$\partial\mathscr{T}: \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (a+b\cos\alpha)\cos\beta \\ (a+b\cos\alpha)\sin\beta \\ b\sin\alpha \end{pmatrix}, \qquad 0 \le \alpha, \beta < 2\pi\,,$$

where $a > b$ are two fixed positive constants. It is worthwhile to consider this sort of "exotic" body because this will lead to a surprisingly precise result and, furthermore, provide some preparation for an important general insight to be discussed in the next subsection.

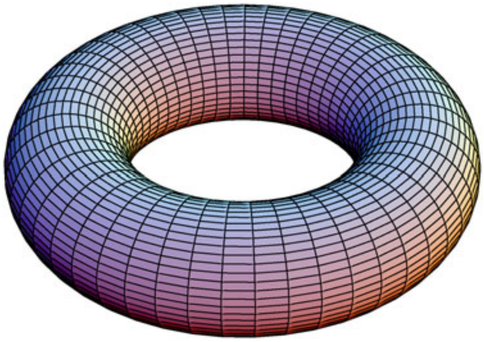
The points on $\partial\mathscr{T}$ with curvature zero are all located on the two circles

$$\mathscr{C}^{\pm}_{a,b}: \quad x^2 + y^2 = a^2\,, \quad z = \pm b\,.$$

There are no flat points at all, but the normal vectors are $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ throughout $\mathscr{C}^{\pm}_{a,b}$, i.e., they are rational and sort of "pull into the same direction". This very fact opens the way to the following quite accurate result; see[2] the author [40], as well as Garcia and the author [9] (Fig. 3).

[2]A weaker version of the first asymptotics, with error term $O(t^{3/2-1/286+\varepsilon})$, has been established by Popov [45].

Fig. 3 The torus $\partial\mathscr{T}$ for $a = 7, b = 2.5$



Theorem 3. *For any fixed reals $a > b > 0$,*

$$D(\mathscr{T};t) = \mathscr{F}_{a,b}\, t^{3/2} + O\left(t^{11/8+\varepsilon}\right)$$

with

$$\mathscr{F}_{a,b}(t) := 4a\sqrt{b}\sum_{j=1}^{\infty} j^{-3/2} \sin\left(2\pi jbt - \tfrac{\pi}{4}\right) .$$

Furthermore,

$$\int_1^T \left(D(\mathscr{T};t) - \mathscr{F}_{a,b}(t)\, t^{3/2}\right)^2 dt = O\left(T^{3+\epsilon}\right) .$$

3.3 The Effect of Destroyed Convexity: Bodies with a "Dent"[3]

Returning to bodies $\mathscr{B}$ with boundary of genus zero, we recall that it has been possible to determine the exact order of $D(\mathscr{B};t)$ only for quite special cases, namely, if $\partial\mathscr{B}$ contains flat points where the curvature vanishes of sufficiently high order. This applies to Eq. (10), to Theorem 2, as well as to the results mentioned in Sect. 2.1. In particular, for the *generic* case of strict convexity and nonvanishing curvature treated in Sect. 1.1, there is little hope to determine the optimal exponent of t when estimating $D(\mathscr{B};t)$ from above.

One might expect that the difficulty is increased if the convexity of $\mathscr{B}$ is destroyed. This is done in the most natural way by a single "dent" as shown in

[3]This subsection describes quite recent research by the author which is in course of publication elsewhere [43].

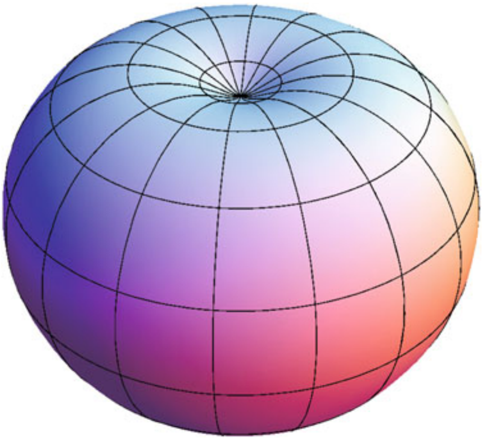


Fig. 4 Body of rotation $\mathscr{B}_{\text{rot}}$ with a "dent"

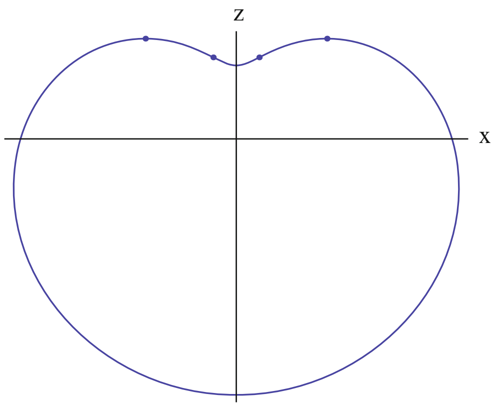


Fig. 5 The meridian curve whose rotation about the vertical axis generates $\partial\mathscr{B}_{\text{rot}}$. Its right-hand half is $\mathscr{C}^+$

Fig. 4. However, perhaps as a surprise,[4] the true order of magnitude of $D(\mathscr{B};t)$ essentially can be determined, at least for the important class of *bodies of rotation*.

Assuming w.l.o.g. the axis of rotation to be the z-axis in a Cartesian system, the body $\mathscr{B}_{\text{rot}}$ is fully described by the right-hand half $\mathscr{C}^+$ of a meridian curve in the (x,z)-plane, whose rotation about the z-axis generates $\partial\mathscr{B}_{\text{rot}}$ (Fig. 5).

We proceed to state in detail the assumptions required for the next theorem; in other words, we describe Fig. 5 exactly.

- The curve $\mathscr{C}^+$ has a natural parametrization $s \mapsto (x(s), z(s))$, with s the arclength, which is an injective map of $[0, L]$ into the (x,z)-plane $\mathbb{R}^2$. Both components are four times continuously differentiable.
- $x(0) = x(L) = 0$, and $x(s) > 0$ for $0 < s < L$. Further, $\dot{z}(0) = \dot{z}(L) = 0$, so that $\partial\mathscr{B}_{\text{rot}}$ is smooth.

[4]Or not, if one has learned the right lesson from the example of the torus.

- There exists a value $s_1 \in]0, L[$ such that $z(s)$ increases on $]0, s_1[$ and decreases on $]s_1, L[$. Denote by $z_{\max} = z(s_1)$ the global maximum of z along $\mathscr{C}^+$.
- The curvature $\kappa(s)$ of $\mathscr{C}^+$ vanishes for exactly one value $s_2 \in]0, L[$. This is a zero of first order, and $s_2 > s_1$. Hence, $\kappa(s)$ is positive on $[0, s_2[$, and negative on $]s_2, L]$. Further, $\dot{x}(s_2) < 0$.
- $x(s)$ and $z(s)$ are analytic (at least) in $s \in \{0, s_1, s_2, L\}$.

Under these assumptions, the following result can be deduced.

Theorem 4 ([43]). *For the lattice discrepancy $D(\mathscr{B}_{\mathrm{rot}}; t)$ of a body $\mathscr{B}_{\mathrm{rot}}$ whose boundary $\partial\mathscr{B}_{\mathrm{rot}}$ is generated by the rotation about the z-axis of a curve $\mathscr{C}^+$ satisfying the properties stated, it holds true that*

$$\inf\{\gamma \in \mathbb{R} : \ D(\mathscr{B}_{\mathrm{rot}}; t) = O(t^{\gamma})\} = \frac{3}{2}. \tag{12}$$

Furthermore,

$$D(\mathscr{B}_{\mathrm{rot}}; t) = c t^{3/2} \mathscr{F}_{z_{\max}}(t) + \Delta(\mathscr{B}_{\mathrm{rot}}; t), \tag{13}$$

with a constant $c \neq 0$ depending on $\mathscr{B}_{\mathrm{rot}}$,

$$\mathscr{F}_{z_{\max}}(t) := \sum_{j=1}^{\infty} j^{-3/2} \sin\left(2\pi j z_{\max} t - \frac{\pi}{4}\right),$$

and an error term $\Delta(\mathscr{B}_{\mathrm{rot}}; t)$ satisfying the mean-square estimate, for T large and every fixed $\varepsilon > 0$,

$$\int_T^{2T} (\Delta(\mathscr{B}_{\mathrm{rot}}; t))^2 \, dt \ll T^{31/9+\varepsilon}. \tag{14}$$

Remark 2. It readily follows from (13) and (14) that, for T large,

$$\int_T^{2T} (D(\mathscr{B}_{\mathrm{rot}}; t))^2 \, dt \sim \int_T^{2T} \left(c\, t^{3/2} \mathscr{F}_{z_{\max}}(t)\right)^2 dt \sim c_1 T^4,$$

with $c_1 > 0$ depending on $\mathscr{B}_{\mathrm{rot}}$. From this $D(\mathscr{B}_{\mathrm{rot}}; t) = \Omega(t^{3/2})$ is immediate. To establish (12), the upper bound $D(\mathscr{B}_{\mathrm{rot}}; t) = O(t^{3/2+\varepsilon})$ is proved by similar methods as used for Theorem 2.

To understand the result of Theorem 4 intuitively, observe that the points of $\partial\mathscr{B}_{\mathrm{rot}}$ with Gaussian curvature zero are located on two circles: the one with $z = z_{\max}$ and another one with $z = z(s_2)$. On $z = z_{\max}$, the normal vector is the same everywhere and it is rational, namely $(0, 0, 1)$. This creates the large contribution $c\, t^{3/2} \mathscr{F}_{z_{\max}}(t)$ to the lattice discrepancy. On the circle where $z = z_2$, however, the normals are all different and "most of them" are irrational. Their contribution to $D(\mathscr{B}_{\mathrm{rot}}; t)$ is proved to be small, at least "on average".

It it reasonable to conjecture that, sharper than Theorem 4, there is a *pointwise* asymptotics

$$D(\mathscr{B}_{\mathrm{rot}};t) = c\,t^{3/2}\,\mathscr{F}_{z_{\max}}(t) + O(t^{\eta}),$$

with some $\eta < \frac{3}{2}$, possibly depending on $\mathscr{B}_{\mathrm{rot}}$. However, it turns out to be very difficult to establish such a sharp and general result, without any stringent conditions on $\mathscr{C}^{+}$ which do not have an intuitive geometric interpretation.

3.4 A Look into the Toolbox

In this section we are going to give a brief account on the method used to derive the recent results described. It can be called a “cut-into-slices”-approach and is based on the identity

$$N(\mathscr{B};t) = \sum_{m_3\in\mathbb{Z}} \#\{(m_1,m_2)\in\mathbb{Z}^2:\ (m_1,m_2,m_3)\in t\mathscr{B}\}. \tag{15}$$

It works best for (but is not limited to!) bodies of rotation. We will consider an example of a body *without* rotational symmetry in the last subsection.

Step 1. In the case of a body of rotation (about the z-axis), the quantity

$$\#\{(m_1,m_2)\in\mathbb{Z}^2:\ (m_1,m_2,m_3)\in t\mathscr{B}\},$$

for m_3 fixed, enumerates the number of integer points in a circular disc. For this, one uses a truncated form of *Hardy’s identity*, in the shape

$$\sum_{0\le n\le X} r(n) - \pi X$$
$$= \frac{1}{\pi}\,X^{1/4}\sum_{1\le n\le Y}\frac{r(n)}{n^{3/4}}\cos(2\pi\sqrt{nX}-3\pi/4) + O\left(X^{1/2+\epsilon}\,Y^{-1/2}\right) + O\left(Y^{\epsilon}\right),$$

where X and Y are large real parameters, and $r(n)$ denotes the number of ways to represent $n\ge 0$ as a sum of two squares of integers.

For $r(n)$ replaced by the number-of-divisors function $d(n)$, the analogue of this formula is classic and can be found, e.g., in the book of Titchmarsh [48, p. 319]. The present assertion has been stated and applied, e.g., by Ivić [18, Eq. (1.9)]. An explicit proof has been given for a more general result by Müller [34, Lemma 3].

Step 2. After applying this identity, there remains a weighted trigonometric sum with respect to the variable m_3. To reduce its length, a device is used which is called the *Van der Corput transformation* of exponential sums. It essentially reads, writing $e(w) := e^{2\pi i w}$ as usual,

$$\sum_{m_3 \in I} G(m_3)\, e(F(m_3))$$
$$= e\left(\frac{\operatorname{sgn}(F'')}{8}\right) \sum_{m \in F'(I)} \frac{G(F'^{-1}(m))}{\sqrt{|F''(F'^{-1}(m))|}}\, e\left(F(F'^{-1}(m)) - m F'^{-1}(m)\right)$$
$$+ \text{ error terms}\,,$$

under suitable conditions on the derivatives of F and G. (E.g., $F'' \neq 0$ throughout.) For a version with very sharp error terms, see Iwaniec and Kowalski [20, Theorem 8.16 and Eq. (8.47)]. The proof of results of this kind is based on Poisson's formula and on the method of stationary phase.

Step 3. To estimate the remaining exponential sum, in many cases a result of the following shape is useful.
For $r \geq 4$ a fixed integer, and positive real parameters $M \geq 1$ and T, suppose that F is a real function on some compact interval I^* of length M, with $r+1$ continuous derivatives which satisfy throughout

$$M^{-j} T \ll F^{(j)} \ll M^{-j} T \quad \text{for} \quad j = r-2,\ r-1,\ r\,.$$

Then, for every interval $I \subseteq I^*$,

$$\sum_{m \in I} e(F(m)) \ll M^{a_r} T^{b_r} + M^{\xi_r} T^{\eta_r} + M^{\alpha_r} + M^{\gamma_r} T^{-\delta_r}\,,$$

where the nonnegative exponents $(a_r, b_r, \xi_r, \eta_r, \alpha_r, \gamma_r, \delta_r)$ are given for every $r \geq 4$ by recursive formulas.
This result is sort of a "hybrid" of the classic Van der Corput theory, as displayed, e.g., in Krätzel [22], with Huxley's [15] "*Discrete Hardy–Littlewood Method*": See the author's article [42].
If one is interested in a mean-square result instead of a point-wise one, step 3 is usually replaced by a lemma which can be found in Iwaniec and Kowalski [20, Lemma 7.1].

3.5 Krätzel's 3D-Body Revisited: A Body Without Rotational Symmetry

We conclude this article by making reference to a perhaps "exotic" example where the "cut-into-slices"-method has been successfully applied to a body which is *not* invariant under rotation: This is Krätzel's 3D-body already encountered in Sect. 2.2.

With a slight change of notation compared to (9),

$$\mathscr{B}^*_{k,m} := \{(u_1, u_2, u_3) \in \mathbb{R}^3 : \left(|u_1|^k + |u_2|^k\right)^m + |u_3|^{mk} \le 1\},$$

where $k > 2$ and $m > 1$ are fixed reals. As a variant of (15),

$$N(\mathscr{B}^*_{k,m}; t) = \sum_{|m_3| \le t} L_k\left(\left(t^{mk} - |m_3|^{mk}\right)^{1/m}\right),$$

$$L_k(W) := \sum_{|m_1|^k + |m_2|^k \le W} 1$$
$$= a_k\, W^{2/k} + 8I_k(W) - 8\Delta_k(W) + O(1).$$

Here the "slices" are Lamé discs, and $a_k = \frac{2\Gamma^2(1/k)}{k\Gamma(2/k)}$ is the area of the unit Lamé disc $|u_1|^k + |u_2|^k \le 1$. Further, with $\psi(u) := u - [u] - \frac{1}{2}$,

$$I_k(W) := \int_0^{W^{1/k}} \psi(u)\, d\left((W - u^k)^{1/k}\right),$$

$$\Delta_k(W) := \sum_{(\frac{1}{2}W)^{1/k} < n \le W^{1/k}} \psi\left((W - n^k)^{1/k}\right).$$

The integral $I_k(W)$ has been evaluated with high precision in Krätzel [22, Chap. 3.3]. The sum $\Delta_k(W)$ is treated by transition to exponential sums and the devices sketched above (Steps 2 and 3). In this way, the following result can be established.

Theorem 5 (Krätzel and the author [31]). *For fixed reals $k > 2$, $m > 1$, with $mk > \frac{7}{3}$, and large t,*

$$D(\mathscr{B}^*_{k,m}; t) = \mathscr{F}_1(t)\, t^{2-\frac{2}{mk}} + \mathscr{F}_2(t)\, t^{2-\frac{1}{mk}-\frac{1}{k}} + E_{m,k}(t),$$

with continuous periodic functions $\mathscr{F}_1(t)$, $\mathscr{F}_2(t)$. The error term is

$$E_{m,k}(t) = O\left(t^{\frac{37}{25}}\right) + O\left(t^{\frac{339}{208} - \frac{131}{104mk} + \epsilon}\right) + O\left(t^{\frac{339}{208} - \frac{235}{208k} + \epsilon}\right).$$

Here the terms $\mathscr{F}_1(t)\, t^{2-\frac{2}{mk}}$ and $\mathscr{F}_2(t)\, t^{2-\frac{1}{mk}-\frac{1}{k}}$ come from the *flat points* $(0, 0, \pm 1)$, resp., $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ of $\partial\mathscr{B}^*_{k,m}$. The two complicated O-terms represent the contribution of the curves of intersection of $\partial\mathscr{B}^*_{k,m}$ with the coordinate planes: Along these, the Gaussian curvature vanishes identically, and the normal vector varies steadily, being irrational in most points.

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On the Orderability Problem and the Interval Topology

Kyriakos Papadopoulos

Abstract The class of LOTS (linearly ordered topological spaces, i.e. spaces equipped with a topology generated by a linear order) contains many important spaces, like the set of real numbers, the set of rational numbers and the ordinals. Such spaces have rich topological properties, which are not necessarily hereditary. The Orderability Problem, a very important question on whether a topological space admits a linear order which generates a topology equal to the topology of the space, was given a general solution by van Dalen and Wattel (Gen. Topol. Appl. 3:347–354, 1973). In this article we first examine the role of the interval topology in van Dalen's and Wattel's characterization of LOTS, and we then discuss ways to extend this model to transitive relations that are not necessarily linear orders.

Keywords Orderability problem • Nest • LOTS • Interval topology

1 Introduction

> Order is a concept as old as the idea of number and much of early mathematics was devoted to constructing and studying various subsets of the real line. (Steve Purisch).

In Purisch's account of results on orderability and suborderability (see [5]), one can read the formulation and development of several orderability problems, starting from the beginning of the twentieth century and reaching our days. By an orderability problem, in topology, we mean the following. Let $(X, \mathscr{T})$ be a topological space and let X be equipped with an order relation $<$. Under what conditions will $\mathscr{T}_<$, i.e. the topology induced by the order $<$, be equal to $\mathscr{T}$? There was no general solution to this problem until the early 1970s.

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The first general solution to the characterization of LOTS (linearly ordered topological spaces, that is, spaces whose topology is generated by a linear order) was given by van Dalen and Wattel, in 1973 (see [6]). These authors succeeded to generalize and expand the properties that appear in the real line, where the natural topology equals the natural order topology. Their main tool were nests. They considered a topological space X, whose topology is generated by a subbasis $\mathscr{L} \cup \mathscr{R}$ of two nests $\mathscr{L}$ and $\mathscr{R}$ on X, whose union is T_1-separating. By considering also the order which is generated by the nest $\mathscr{L}$ on X, namely $\lhd_{\mathscr{L}}$, the authors introduced conditions such that $\mathscr{T}_{\lhd_{\mathscr{L}}}$ to be equal to $\mathscr{T}_{\mathscr{L}\cup\mathscr{R}}$. In this case, they proved the space to be LOTS, while in the case where $\mathscr{T}_{\lhd_{\mathscr{L}}} \subset \mathscr{T}_{\mathscr{L}\cup\mathscr{R}}$, the space was proved to be GO (generalized ordered, i.e. a topological subspace of a LOTS). Both in the case of GO-spaces and of LOTS the authors demanded $\mathscr{L} \cup \mathscr{R}$ to form a T_1-separating subbasis for the topology on X. The necessary and sufficient condition for both nests $\mathscr{L}$, $\mathscr{R}$ to be interlocking was added in order for the space to be LOTS.

LOTS are natural occurring topological objects and are canonical building blocks for topological examples. For example, a space which is LOTS is also monotonically normal (see, for example, [4]). On the other hand, a subspace of a LOTS is not necessarily a LOTS.

In this paper we will initially use tools from [3], where the authors revisited and simplified J. van Dalen and E. Wattel's ideas in order to construct ordinals, and we will then investigate the role of the interval topology in their solution to the orderability problem. The interval topology can be defined for any transitive order (see, for example, [1]), and we believe that it is a good candidate for replacing the topology that is generated by the T_1-separating union of two nests, $\mathscr{L}$ and $\mathscr{R}$ (the topology used by van Dalen and Wattel), in order to extend the orderability problem to nonlinearly ordered spaces.

2 Preliminaries and a Few Remarks on [6]

In this section we will introduce the machinery that will be needed in order to develop our ideas in the succeeding sections. Our main reference on standard order- and lattice-theoretic definitions will be the book [1]. A more recent account on topological properties of ordered structures is given in [2].

Definition 1. Let $(X, <)$ be a set equipped with a transitive relation $<$. We define $\uparrow A \subset X$ to be the set:

$$\uparrow A = \{x : x \in X \text{ and there exists } y \in A, \text{ such that } y < x\}.$$

We also define $\downarrow A \subset X$ to be the set:

$$\downarrow A = \{x : x \in X \text{ and there exists } y \in A, \text{ such that } x < y\}.$$

More specifically, if $A = \{y\}$, then:

$$\uparrow A = \{x : x \in X \text{ and } y < x\}$$

and

$$\downarrow A = \{x : x \in X \text{ and } x < y\}$$

From now on we will use the conventions $\uparrow x = \uparrow \{x\}$ and $\downarrow x = \downarrow \{x\}$.

We remind that the *upper topology* $\mathscr{T}_U$ is generated by the subbasis $\mathscr{S} = \{X - \downarrow x : x \in X\}$ and the *lower topology* $\mathscr{T}_l$ is generated by the subbasis $\mathscr{S} = \{X - \uparrow x : x \in X\}$. The *interval topology* $\mathscr{T}_{\text{in}}$ is defined as $\mathscr{T}_{\text{in}} = \mathscr{T}_U \vee \mathscr{T}_l$, where $\vee$ stands for supremum.

Definition 2. Let X be a set.

1. A collection $\mathscr{L}$, of subsets of X, T_0-*separates* X, if and only if for all $x, y \in X$, such that $x \neq y$, there exist $L \in \mathscr{L}$, such that $x \in L$ and $y \notin L$ or $y \in L$ and $x \notin L$.
2. Let X be a set. A collection $\mathscr{L}$, of subsets of X, T_1-*separates* X, if and only if for all $x, y \in X$, such that $x \neq y$, there exist $L, L' \in \mathscr{L}$, such that $x \in L$ and $y \notin L$ and also $y \in L'$ and $x \notin L'$.

One can easily see the link between Definition 2 and the separation axioms of topology: a topological space $(X, \mathscr{T})$ is T_0 (resp. T_1), if and only if there is a subbasis $\mathscr{S}$, for $\mathscr{T}$, which T_0-separates (resp. T_1-separates) X.

Definition 3. Let X be a set and let $\mathscr{L}$ be a family of subsets of X. $\mathscr{L}$ is a *nest* on X, if for every $M, N \in \mathscr{L}$, either $M \subset N$ or $N \subset M$.

Definition 4. Let X be a set and let $\mathscr{L}$ be a nest on X. We define an order relation on X via the nest $\mathscr{L}$, as follows:

$$x \lhd_{\mathscr{L}} y \Leftrightarrow \exists\, L \in \mathscr{L}, \text{ such that } x \in L \text{ and } y \notin L$$

It follows from Definitions 2 and 4 that if the nest $\mathscr{L}$ is T_0-separating, then the order $\lhd_{\mathscr{L}}$ is linear, provided the order is reflexive.

We note that the declaration of reflexivity in the order is very vital from a purely order-theoretic point of view: a partial order is defined to be reflexive, antisymmetric, and transitive. A linear order is a partial order plus every two distinct elements in the set can be compared to one another via the order. On the other hand, the orderability problem is in fact a topological problem of comparing two topologies, as stated in the introductory section, and J. van Dalen's and E. Wattel's proof of the general solution to this problem does not examine reflexivity or antisymmetry in the order: only transitivity and comparability of any two elements. More specifically, if x and y are distinct elements in a set X, and $\lhd_{\mathscr{L}}$ is an order relation generated by a T_0-separating nest on X, then one can easily check that it cannot happen that $x \lhd_{\mathscr{L}} y$ and simultaneously $y \lhd_{\mathscr{L}} x$. This gives us the liberty to say that the order is always antisymmetric but, still, from an order-theoretic point of view one should state explicitly whether the order is reflexive or not. In conclusion, the characterization of LOTS in [6] not only refers to linearly ordered sets but also covers cases of sets which are equipped with an order $<$ which appears to be a weaker version of the linear order.

Having this in mind, we find it important to make a distinction between $\lhd_{\mathscr{L}}$ and $\unlhd_{\mathscr{L}}$.

Definition 5. Let X be a set and $x, y \in X$. We say that $x \unlhd_{\mathscr{L}} y$, if and only if either $x = y$ or there exists $L \in \mathscr{L}$, such that $x \in L$ and $y \notin L$.

From now on, whenever we write $x \lhd_{\mathscr{L}} y$, we will assume that $x \neq y$.

The order of Definition 4 was first introduced in [6] and was further examined in [3], where the authors gave the following useful (for the purposes of this paper) theorem.

Theorem 1. *Let X be a set. Suppose $\mathscr{L}$ and $\mathscr{R}$ are two nests on X. $\mathscr{L} \cup \mathscr{R}$ is T_1-separating, if and only if $\mathscr{L}$ and $\mathscr{R}$ are both T_0-separating and $\lhd_{\mathscr{L}} = \rhd_{\mathscr{R}}$.*

Definition 6 (van Dalen & Wattel). Let X be a set and let $\mathscr{L} \subset \mathscr{P}(X)$. We say that $\mathscr{L}$ is interlocking if and only if, for each $L \in \mathscr{L}$, $L = \bigcap\{N \in \mathscr{L} : L \subset N, L \neq N\}$ implies $L = \bigcup\{N \in \mathscr{L} : N \subset L, L \neq N\}$.

The following theorem, stemming from Definition 6, gives conditions so that a nest to be interlocking in linearly ordered spaces, in particular.

Theorem 2 (See [3]). *Let X be a set and let $\mathscr{L}$ be a T_0-separating nest on X. The following are equivalent:*

1. *$\mathscr{L}$ is interlocking;*
2. *for each $L \in \mathscr{L}$, if L has a $\lhd_{\mathscr{L}}$-maximal element, then $X - L$ has a $\lhd_{\mathscr{L}}$-minimal element;*
3. *for all $L \in \mathscr{L}$, either L has no $\lhd_{\mathscr{L}}$-maximal element or $X - L$ has a $\lhd_{\mathscr{L}}$-minimal element.*

Theorem 2 permits us to say that if $\mathscr{L}$ is a T_0-separating nest on X and if for every $L \in \mathscr{L}$, $X - L$ has a minimal element, then $\mathscr{L}$ is interlocking.

We remark that the subset $X = [0, 1) \cup \{2\}$, of the real line, together with its subspace topology inherited from the topology of the real line, is a non-compact GO-space, but not a LOTS: the order $\lhd_{\mathscr{L}}$, where $\mathscr{L} = X \cap \{(-\infty, a) : a \in \mathbb{R}\}$, cannot "spot" the difference between X and the space $Y = [0, 2]$, because it cannot tell whether there is a gap between [0, 1) and 2 or not. The property of interlocking (Theorem 2) is what guarantees that there are not such gaps. We will now give the version of the solution to the orderability problem that was stated by van Dalen and Wattel, as it appeared in [3].

Theorem 3 (van Dalen & Wattel). *Let $(X, \mathscr{T})$ be a topological space.*

1. *If $\mathscr{L}$ and $\mathscr{R}$ are two nests of open sets, whose union is T_1-separating, then every $\lhd_{\mathscr{L}}$-order open set is open, in X.*
2. *X is a GO space, if and only if there are two nests, $\mathscr{L}$ and $\mathscr{R}$, of open sets, whose union is T_1-separating and forms a subbasis for $\mathscr{T}$.*

3. *X is a LOTS, if and only if there are two interlocking nests $\mathscr{L}$ and $\mathscr{R}$, of open sets, whose union is T_1-separating and forms a subbasis for $\mathscr{T}$.*

In the section that follows, we will examine particular properties of the interval topology, when the order is generated by a T_0-separating nest, and we will see the key role that it plays in the characterization of LOTS. The topology $\mathscr{T}_{\mathscr{L}\cup\mathscr{R}}$ does not have any particular topological meaning when the union of $\mathscr{L}$ and $\mathscr{R}$ is not T_1-separating. The interval topology though, as being more flexible from the way that it is defined, can replace the $\mathscr{T}_{\mathscr{L}\cup\mathscr{R}}$ topology, if subjected to certain conditions. We will see this in more detail in Sect. 3 that follows.

3 Some Further Remarks on the Orderability Problem

Remark 1. Let X be a set and let $\mathscr{L}$ be a T_0-separating nest on X.

If $\trianglelefteq_{\mathscr{L}}$ is reflexive, then obviously the interval topology $\mathscr{T}_{\text{in}}^{\trianglelefteq_{\mathscr{L}}}$, via $\trianglelefteq_{\mathscr{L}}$, will be equal to the topology $\mathscr{T}_{\triangleleft_{\mathscr{L}}}$ (for a rigorous proof, one should add T_0-separation in Lemma 3 of Sect. 4). In addition, the order topology $\mathscr{T}_{\trianglelefteq_{\mathscr{L}}}$ will be equal to the discrete topology on X.

If $\triangleleft_{\mathscr{L}}$ is irreflexive, then the interval topology via $\triangleleft_{\mathscr{L}}$ will be equal to the discrete topology on X. Indeed, $\downarrow a = \{x \in X : x \triangleleft_{\mathscr{L}} a\}$ and so $X - \downarrow a = \{x \in X : a \trianglelefteq_{\mathscr{L}} x\} = (-\infty, a]$. In a similar fashion, $x - \uparrow a = [a, \infty)$ and so $(-\infty, a] \cap [a, \infty) = \{a\}$.

Given the conditions in Theorem 3 and the observations in Remark 1, we achieve the following comparisons for the topologies $\mathscr{T}_{\triangleleft_{\mathscr{L}}}$, $\mathscr{T}_{\trianglelefteq_{\mathscr{L}}}$, $\mathscr{T}_{\text{in}}^{\triangleleft_{\mathscr{L}}}$, $\mathscr{T}_{\text{in}}^{\trianglelefteq_{\mathscr{L}}}$, on a space X:

Lemma 1. *1. $\mathscr{T}_{\triangleleft_{\mathscr{L}}} = \mathscr{T}_{\mathscr{L}\cup\mathscr{R}} = \mathscr{T}_{\text{in}}^{\trianglelefteq_{\mathscr{L}}} \subseteq \mathscr{T}_{\text{in}}^{\triangleleft_{\mathscr{L}}} = \mathscr{T}_{\trianglelefteq_{\mathscr{L}}}$, provided that $\mathscr{L}$ and $\mathscr{R}$ are interlocking and $\mathscr{L} \cup \mathscr{R}$ T_1-separates X.*

2. $\mathscr{T}_{\triangleleft_{\mathscr{L}}} = \mathscr{T}_{\text{in}}^{\trianglelefteq_{\mathscr{L}}} \subseteq \mathscr{T}_{\mathscr{L}\cup\mathscr{R}} \subseteq \mathscr{T}_{\text{in}}^{\triangleleft_{\mathscr{L}}} = \mathscr{T}_{\trianglelefteq_{\mathscr{L}}}$, provided that $\mathscr{L} \cup \mathscr{R}$ T_1-separates X.

Lemma 1 permits us to restate Theorem 3, using the interval topology.

Corollary 1. *A topological space $(X, \mathscr{T})$ is:*

1. *a LOTS, iff there exists a nest $\mathscr{L}$ on X, such that $\mathscr{L}$ is T_0-separating and interlocking and also $\mathscr{T} = \mathscr{T}_{\text{in}}^{\trianglelefteq_{\mathscr{L}}}$.*
2. *a GO-space, iff there exists a nest $\mathscr{L}$ on X, such that $\mathscr{L}$ is T_0-separating and also $\mathscr{T} = \mathscr{T}_{\text{in}}^{\trianglelefteq_{\mathscr{L}}}$.*

Our proposed weaker version of the orderability problem, that will be presented in Sect. 5, will be based on observations on Lemma 1. Knowing that the interval topology can be defined via any transitive relation, a weaker version of the orderability question can be expressed as follows.

Question: Let X be a set equipped with a transitive relation $<$ and the interval topology $\mathscr{T}_{\text{in}}^{\leq}$, defined via $\leq$, where $\leq$ is $<$ plus reflexivity. Under which necessary and sufficient conditions will $\mathscr{T}_{<}$ be equal to $\mathscr{T}_{\text{in}}^{\leq}$?

In this paper we give a partial answer to this question, through Theorem 4.

4 The Order Topology and the Interval Topology in the Light of Nests

In order to give an answer to the Question of Sect. 3, we will need first to see what form do the topologies $\mathscr{T}_{\lhd_{\mathscr{L}}}$ and $\mathscr{T}_{\text{in}}^{\unlhd_{\mathscr{L}}}$ take, when the nest $\mathscr{L}$ is not necessarily T_0-separating.

Lemma 2. *Let X be a set and let $\mathscr{L}$ be a nest on X. Let also $\Delta = \{(x, x) : x \in X\}$. Then:*

1. $\lhd_{\mathscr{L}} = \bigcup_{L \in \mathscr{L}}[L \times (X - L)]$.
2. $\ntriangleleft_{\mathscr{L}} = X \times X - \lhd_{\mathscr{L}} = \bigcap_{L \in \mathscr{L}}[((X - L) \times X) \cup (X \times L)]$.
3. $\ntrianglelefteq_{\mathscr{L}} = \bigcap_{L \in \mathscr{L}}[((X - L) \times X) \cup (X \times L)] \cap (X - \Delta)$

Notation: From now on, if $U \subset X \times X$, then $U(x) = \{y \in X : (x, y) \in U\}$.

Lemma 3. *1. For each $x \in X$, $X - \uparrow x = \bigcap\{L \in \mathscr{L} : x \in L\} - \{x\}$.*
2. For each $x \in X$, $X - \downarrow x = \bigcap\{X - L : x \in X - L\} \cup \{x\}$

Proof. 1. $y \in X - \uparrow x$, if and only if $x \ntrianglelefteq_{\mathscr{L}} y$, if and only if $(x, y) \notin \unlhd_{\mathscr{L}}$, if and only if (by Lemma 2) $y \in \bigcap_{L \in \mathscr{L}}\{((X - L) \times X)(x) \cup (X \times L)(x)\}$ and $y \neq x$, if and only if $y \in \bigcap\{L \in \mathscr{L} : x \in X\} - \{x\}$.
2. In a similar fashion to 1 using Lemma 2. ■

Let us now have a closer look to the order topology. It is known that if a set X is equipped with an order $<$, then the order topology $\mathscr{T}_{<}$, on X, will be the supremum of two topologies, namely the topology $\mathscr{T}_{\leftarrow}$ and the topology $\mathscr{T}_{\rightarrow}$. In particular, $\mathscr{T}_{\leftarrow}$ is generated by a subbasis $\mathscr{S}_{\leftarrow} = \{(\leftarrow, a) : a \in X\}$, where $(\leftarrow, a) = \{x \in X : x < a\}$. Similarly, the topology $\mathscr{T}_{\rightarrow}$ is generated by a subbasis $\mathscr{S}_{\rightarrow} = \{(a, \rightarrow) : a \in X\}$, where $(a, \rightarrow) = \{x \in X : a < x\}$. We will now see how these subbasis look like when the order is generated by a not necessarily T_0-separating nest, that is $< = \lhd_{\mathscr{L}}$, where $<$ stands for any (not necessarily linear) order.

Notation: Let $\mathscr{L}$ be a nest on a set X. Then, $\mathscr{L}_a = \{L \in \mathscr{L} : a \in L\}$, for each $a \in X$.

Lemma 4. *1. For each $a \in X$, $(a, \rightarrow) = \bigcup_{L \in \mathscr{L}_a} X - L$.*
2. For each $a \in X$, $(\leftarrow, a) = \bigcup_{L \notin \mathscr{L}_a} L$.

It is easily seen that when $\mathscr{L}$ is T_0-separating, Lemmas 2 and 3 compensate to our remarks in Sect. 3.

5 A Weaker Orderability Problem

In this section we find necessary conditions, so that $\mathscr{T}_{\text{in}}^{\leq} = \mathscr{T}_{<}$, where $<$ is a transitive relation, $\leq$ is $<$ plus reflexivity and $\mathscr{T}_{\text{in}}^{\leq}$ is the interval topology that is defined via $\leq$. To achieve this, we will give conditions such that $\mathscr{T}_l = \mathscr{T}_{\leftarrow}$ and $\mathscr{T}_U = \mathscr{T}_{\rightarrow}$. The logic behind these conditions is the following. As we have seen in Theorem 1, linearity in the space is strongly related to the notion T_0-separating nest. We believe that a first step towards the generalization of the orderability problem will be to define a weaker version of T_0-separation, and we can achieve this without using the notion of nest. It seems that the use of nests was vital in the understanding and the description of the order-theoretic properties and of the topological properties of linearly ordered topological spaces. Even the fact that our approach can be stated using nests and the material that was presented in Sect. 4, we prefer to adopt a more general approach. The description of the order topology and the interval topology in Sect. 4 will permit us to see that (i) if the nest is T_0-separating we will get back to Sect. 3 and that (ii) nests might not help that much to describe sets that are not necessarily linearly ordered. This can be also seen in the topology $\mathscr{T}_{\mathscr{L}\cup\mathscr{R}}$, which loses its meaning when it does not refer to linearly ordered sets.

Let X be a set and let $<$ be a transitive relation on X.

Condition 1 Let $x, y \in X$, such that $x \nleq y$. Then, there exist $z_i \in X$, $i = 1, \cdots, n$, such that $y < z_i$ and, if $w \in X$, such that $w < z_i$, then $x \nleq w$.

Condition 2 Let $x, y \in X$, such that $y \nleq x$. Then, there exist $z_i \in X$, $i = 1, \cdots, n$, such that $z_i < y$ and, if $w \in X$, such that $z_i < w$, then $w \nleq x$.

Condition 3 Let $x, y \in X$, such that $y < x$. Then, there exist $z_i \in X$, $i = 1, \cdots, n$, such that $z_i \nleq y$ and, if $w \in X$, such that $z_i \nleq w$, then $w < x$.

Condition 4 Let $x, y \in X$, such that $x < y$. Then, there exist $z_i \in X$, $i = 1, \cdots, n$, such that $y \nleq z_i$ and, if $w \in X$, such that $w \nleq z_i$, then $x < w$.

Proposition 1. *1. $\mathscr{T}_l = \mathscr{T}_{\leftarrow}$, if and only if Conditions 1 and 3 are satisfied.*
2. $\mathscr{T}_U = \mathscr{T}_{\rightarrow}$, if and only if Conditions 2 and 4 are satisfied.

Proof. $\mathscr{T}_l \subset \mathscr{T}_{\leftarrow}$, if and only if for an arbitrary $y \in X - \uparrow x$, there exists a basic-open set B, in $\mathscr{T}_{\leftarrow}$, such that $y \in B \subset X - \uparrow x$. By Condition 1, there exist $z_i \in X$, $i = 1, \cdots, n$, such that $y \in \bigcap_{i=1}^{n} (\leftarrow, z_i) \subset X - \uparrow x$. That $\mathscr{T}_{\leftarrow} \subset \mathscr{T}_l$ follows in a similar fashion, using Condition 3 and given this, the proof of $\mathscr{T}_U = \mathscr{T}_{\rightarrow}$ is straightforward. ■

Theorem 4. *If Conditions 1, 2, 3, and 4 are satisfied, then $\mathscr{T}_{\text{in}}^{\leq} = \mathscr{T}_{<}$.*

Remark 2. If $(X, <)$ is a linearly ordered set, then Condition 1 is obviously satisfied. Indeed, if $x \nleq y$, then $y < x$. So, there exists $z = x$, such that $y < z$ and, if $w \in X$, such that $w < x$, then $x \nleq w$.

Remark 2 shows that our conditions give weaker properties than those satisfied from linear orders (and T_0-separating nests). Even the fact that we have a necessary but not yet a sufficient condition for our Question of Sect. 3, Theorem 4 permits us to conclude that the "distance" of the interval topology $\mathscr{T}_{\text{in}}^{\trianglelefteq_{\mathscr{L}}}$, from the order topology $\mathscr{T}_{<}$, depends on how weaker are our conditions from T_0-separation (-linear order).

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A Class of Functional-Integral Equations with Applications to a Bilocal Problem

Adrian Petruşel and Ioan A. Rus

Abstract Let $\alpha \leq a < b \leq \beta$ be some real numbers, $K : [\alpha, \beta] \times [\alpha, \beta] \times [\alpha, \beta] \times [\alpha, \beta] \times \mathbb{R}^m \to \mathbb{R}^m$ and $g : [\alpha, \beta] \to \mathbb{R}^m$ be continuous functions. In this work, using the Picard operator technique in a $\mathbb{R}^m_+$-metric space, we study the following functional-integral equation

$$x(t) = \int_a^b K(t, s, a, b, x(s))ds + g(t),\ t \in [\alpha, \beta].$$

As an application, the following bilocal problem

$$-x''(t) + px'(t) + qx(t) = f(t, x(t)),\ t \in [\alpha, \beta],\ \ x(a) = 0, x(b) = 0.$$

is also discussed.

Keywords $\mathbb{R}^m_+$-metric space • Fredholm integral equation • Bilocal problem • Fixed point • Picard operator • Fiber-Picard operator • Data dependence (continuity, differentiability) • Ulam-Hyers stability • Open problem

1 Introduction

Let $\alpha \leq a < b \leq \beta$ be some real numbers, $K : [\alpha, \beta] \times [\alpha, \beta] \times [\alpha, \beta] \times [\alpha, \beta] \times \mathbb{R}^m \to \mathbb{R}^m$ and $g : [\alpha, \beta] \to \mathbb{R}^m$ be continuous functions. The aim of this work is to study existence, uniqueness, dependence with respect to K and g, dependence with respect to a and b, Ulam–Hyers stability for the following functional-integral equation

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$$x(t) = \int_a^b K(t, s, a, b, x(s))ds + g(t),\ t \in [\alpha, \beta],$$

where $K \in C([\alpha, \beta]^4 \times \mathbb{R}^m, \mathbb{R}^m)$ and $g \in C[\alpha, \beta], \mathbb{R}^m$. The main tool in our approach will be the weakly Picard operator technique, see, for example, [33,36,38].

As an application, the following bilocal problem

$$-x''(t) + px'(t) + qx(t) = f(t, x(t)), t \in [\alpha, \beta],\ x(a) = 0, x(b) = 0$$

is also discussed.

The plan of our work is the following:

2 Preliminaries

2.1 Notations

The following notations will be used throughout this paper.

$$\mathbb{N} := \{0, 1, 2, \ldots\},\ \mathbb{N}^* := \{1, 2, \ldots\},\ \mathbb{R} := \text{ the set of all real numbers},$$

$$\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\},\ \mathbb{R}^m := \{x = (x_1, \ldots, x_m) \mid x_i \in \mathbb{R},\ i \in \{1, 2, \ldots m\}\},$$

$$\mathbb{R}^*_+ : = \{x \in \mathbb{R} | \ x > 0\}, \mathbb{R}^{m \times m}_+ : = \text{the set of all } m \times m \text{ matrices with elements in } \mathbb{R}_+.$$

Let X be a nonempty set and $A : X \to X$ be an operator. Then we denote:

$$\mathscr{P}(X) := \{Y \mid Y \text{ is a subset of } X\},\ P(X) := \{Y \in \mathscr{P}(X) \mid Y \text{ is nonempty}\},$$
$$I(A) := \{Y \in P(X) \mid A(Y) \subset Y\} --- \text{ the family of all nonempty invariant subsets of A.}$$

If $A : X \to X$ is an operator, then $F_A := \{x \in A \mid x = A(x)\}$ denotes the fixed point set of A, while $F_A := \{x^*\}$ means that the operator A has a unique fixed point which is denoted by x^*. The symbols $A^0 := 1_X$, $A^1 := A$, $A^2 := A \circ A, \ldots,$ $A^n := f \circ A^{n-1}$ denote the iterates of A.

2.2 Matrices Convergent to Zero

We recall first the concept of matrix convergent to zero. Throughout this paper, we denote by I the identity $m \times m$ matrix and by O the zero $m \times m$ matrix. Also, for the sake of simplicity we will make an identification between row and column vectors. in $\mathbb{R}^m$.

Definition 1. A square matrix of real numbers is said to be convergent to zero if and only if $A^n \to O$ *as* $n \to \infty$. (See, for example, [43].)

A classical result in matrix analysis is the following theorem (see, for example, [1,43]).

Theorem 1. *Let $A \in M_{mm}(\mathbb{R}_+)$. The following assertions are equivalent:*

(i) A is a matrix convergent to zero;
(ii) the spectral radius $\rho(A)$ of the matrix A is strictly less than 1, i.e., $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $\det(A - \lambda I) = 0$*;*
(iii) The matrix $(I - A)$ is nonsingular and

$$(I - A)^{-1} = I + A + \cdots + A^n + \cdots ; \tag{1}$$

(iv) The matrix $(I - A)$ is nonsingular and $(I - A)^{-1}$ has nonnegative elements;
(v) $A^n q \to O$ and $qA^n \to O$ as $n \to \infty$, for each $q \in \mathbb{R}^m$.

2.3 Cauchy Lemmas

We start by presenting the classical Cauchy's lemma, see [37].

Lemma 1. *Let $a_n, b_n \in \mathbb{R}_+$, $n \in \mathbb{N}$. We suppose that:*

(i) $\sum_{k=0}^{\infty} a_k < +\infty$;

(ii) $b_n \to 0$ *as* $n \to \infty$.
Then

$$\sum_{k=0}^{n} a_{n-k} b_k \to 0 \text{ as } n \to \infty.$$

The next result was given by Rus [31].

Lemma 2. *Let* $A_n \in R_+^{m\times m}(\mathbb{R}_+)$ *and* $B_n \in \mathbb{R}_+^m$*,* $n \in \mathbb{N}$*. We suppose that:*

(i) $\sum_{k=0}^{\infty} A_k < +\infty$;
(ii) $B_n \to 0$ *as* $n \to \infty$.

Then

$$\sum_{k=0}^{n} A_{n-k} B_k \to 0 \text{ as } n \to \infty.$$

2.4 Picard Operators in $\mathbb{R}_+^m$*-Metric Spaces*

Let (X, d) be a $\mathbb{R}_+^m$-metric space (in the sense that $d : X \times X \to \mathbb{R}_+^m$ and it satisfies the standard axioms of a metric) and let $A : X \to X$ be an operator. By definition, the operator A is said to be:

(i) an S-contraction if $S \in R_+^{m\times m}$ is a matrix convergent to zero and

$$d\left(A\left(x\right), A\left(y\right)\right) \leq Sd\left(x, y\right), \text{ for all } x, y \in X;$$

(ii) a Picard operator if $F_A = \{x^*\}$ and $A^n(x) \to x^*$ as $n \to \infty$, for all $x \in X$.
(iii) a C-Picard operator if A is a Picard operator, $C \in \mathbb{R}_+^{m\times m}$ and

$$d(x, x^*) \leq Cd(x, A(x)), \text{ for all } x \in X.$$

We recall now Perov's fixed point theorem (see Perov [23], Perov and Kibenko [24], Ortega and Rheinboldt [22], pp. 433–434).

Notice that Perov's fixed point theorem is an extension of Banach's contraction principle for single valued contractions on spaces endowed with $\mathbb{R}_+^m$-metrics.

Theorem 2 (Perov). *Let (X, d) be a complete generalized metric space and the operator $A : X \to X$ be an S-contraction then:*

(i) $F_A = \{x^*\}$;
(ii) *the sequence of successive approximations* $(x_n)_{n\in\mathbb{N}}$, $x_n := A^n(x_0)$ *is convergent to* x^*, *for all* $x_0 \in X$ *and the following estimation holds*

$$d\left(x_n, x^*\right) \leq S^n (I - S)^{-1} d(x_0, x_1); \tag{2}$$

(iii) $d(x, x^*) \leq (I - S)^{-1} d(x, A(x))$, *for all* $x \in X$;
(iv) *let* $B : X \to X$ *be an operator for which there exists* $\eta \in \mathbb{R}^m_+$ *such that*

$$d(A(x), B(x)) \leq \eta, \text{ for each } x \in X.$$

$$\text{If } F_B \neq \emptyset, \text{ then } d\left(x^*, y^*\right) \leq (I - S)^{-1} \eta, \text{ for all } y^* \in F_B.$$

In other words, from the above theorem we conclude that an S-contraction is a C-Picard operator with $C := (I - S)^{-1}$.

The following abstract data dependence lemma also holds.

Theorem 3 (Data Dependence Lemma). *Let (X, d) be an $\mathbb{R}^m_+$-metric space and $A, B : X \to X$ be two operators. We suppose that:*

(i) *A is a C-Picard operator with* $F_A = \{x_A^*\}$;
(ii) *there exists* $\eta \in \mathbb{R}^m_+$ *such that* $d(A(x), B(x)) \leq \eta$, *for all* $x \in X$;
(iii) $F_B \neq \emptyset$.

Then, $d(x, x_A^*) \leq C\eta$, *for all* $x \in F_B$.

For other considerations on this lemma, see [36, 38].

2.5 Fiber Picard Operators on $\mathbb{R}^m_+$-Metric Spaces

The following abstract problem appears when we study the differentiability of the fixed points with respect to parameters.

Problem 1. Let (X, d), (Y, ρ) be two metric spaces and

$$A : X \times Y \to X \times Y, \; (x, y) \longmapsto (B(x), C(x, y))$$

be a triangular operator. We suppose that:

(a) the operator $B : X \to X$ is Picard;
(b) the operator $C(x, \cdot) : Y \to Y$ is Picard, for each $x \in X$.

The problem is in which conditions the operator A is Picard.

In [14] Hirsch and Pugh give the following result for this problem.

Theorem 4 (Fiber Contraction Theorem). *Let* (X, d) *be a metric space and* (Y, ρ) *be a complete metric spaces. Let* $A : X \times Y \to X \times Y$, $(x, y) \longmapsto (B(x), C(x, y))$ *be a triangular operator. We suppose:*

(a) $B : X \to X$ *is a Picard operator;*
(b) there exists $\alpha \in]0, 1[$ *such that*

$$\rho(C(x, y), C(x, z)) \leq \alpha\rho(y, z), \text{ for all } x \in X \text{ and } y, z \in Y;$$

(c) $C(\cdot, y)$ *is continuous, for all* $y \in Y$.

Then A *is a Picard operator.*

The proof of this result makes use of the classical Cauchy Lemma. A more general result is the following one.

Theorem 5 (Fiber Generalized Contraction Theorem). *Let* (X, d) *be a* $\mathbb{R}^m_+$*-metric space and* (Y, ρ) *be a complete* $\mathbb{R}^m_+$*-metric spaces. Let* $A : X \times Y \to X \times Y$, $(x, y) \longmapsto (B(x), C(x, y))$ *be a triangular operator. We suppose:*

(a) $B : X \to X$ *is a Picard operator;*
(b) there exists S *a matrix convergent to zero such that*

$$\rho(C(x, y), C(x, z)) \leq S\rho(y, z), \text{ for all } x \in X \text{ and } y, z \in Y;$$

(c) $C(\cdot, y)$ *is continuous, for all* $y \in Y$.

Then A *is a Picard operator.*

Proof. Let $x_0 \in X$ and $y_0 \in Y$. Since B is Picard there exists a unique fixed point x^* of B and the sequence $x_n := B^n(x_0)$ converges to x^* as $n \to \infty$. From (b), using Perov's Theorem, we get that $C(x^*, \cdot)$ has a unique fixed point, which will be denoted by y^*. Thus (x^*, y^*) is the unique fixed point of the operator A. We will show now that

$$A^n(x_0, y_0) \to (x^*, y^*) \text{ as } n \to \infty.$$

If we denote $y_{n+1} := C(x_n, y_n)$ $(n \in \mathbb{N}^*)$, then it is easy to check that $A^n(x_0, y_0) = (x_n, y_n)$. Now we successively have:
$\rho(y_{n+1}, y^*) \leq$
$\rho(C(x_n, y_n), C(x_n, y^*)) + \rho(C(x_n, y^*), y^*) \leq S\rho(y_n, y^*) + \rho(C(x_n, y^*), y^*)$
$\leq S^2\rho(y_{n-1}, y^*) + S\rho(C(x_{n-1}, y^*), y^*) + \rho(C(x_n, y^*), y^*)$
$\leq \cdots S^{n+1}\rho(y_0, y^*) + S^n\rho(C(x_0, y^*), y^*) + \cdots + S\rho(C(x_{n-1}, y^*), y^*)$
$+\rho(C(x_n, y^*), y^*)$.

The conclusion follows now by Lemma 2.

2.6 Ulam–Hyers Stability of a Fixed Point Equations in a $\mathbb{R}_+^m$-Metric Spaces

An important stability concept is that of Ulam–Hyers stability for the fixed point equation. The following notion was given by Rus in [32].

Definition 2. Let (X, d) be a $\mathbb{R}_+^m$-metric space and $A : X \to X$ be an operator. Then, the fixed point equation

$$x = A(x) \tag{3}$$

is said to be Ulam–Hyers stable if there exists a matrix $C \in \mathbb{R}_+^{m\times m}$ such that, for any $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_m)$ with $\varepsilon_i > 0$ for $i \in \{1, \ldots, m\}$ and any ε-solution $y^* \in X$ of (3), i.e.,

$$d\left(y^*, f\left(y^*\right)\right) \le \varepsilon, \tag{4}$$

there exists a solution x^* of (3) such that

$$d\left(x^*, y^*\right) \le C\varepsilon. \tag{5}$$

We have the following abstract result (see also Rus [32]) concerning the Ulam–Hyers stability of the fixed point equation (3).

Theorem 6. *Let (X, d) be a $\mathbb{R}_+^m$-metric space and $A : X \to X$ be a C-Picard operator. Then, the fixed point equation (3) is Ulam–Hyers stable.*

For the Ulam stability of a fixed point equation, see also [6, 12, 25, 32, 34, 35].

2.7 A Special Class of Integral Equations

Let $\alpha \le a < b \le \beta$ be some real numbers, $\mathbb{B}$ be a Banach space (real or complex), $K \in C([\alpha, \beta]^2 \times \mathbb{B}, \mathbb{B})$ and $g \in C([\alpha, \beta], \mathbb{B})$. We consider the following functional-integral equation

$$x(t) = \int_a^b K(t, s, x(s))ds + g(t),\ t \in [\alpha, \beta]. \tag{6}$$

We are looking for the solution of this equation in $C([\alpha, \beta], \mathbb{B})$.

Let us now consider the following integral equation:

$$x(t) = \int_a^b K(t, s, x(s))ds + g(t),\ t \in [a, b]. \tag{7}$$

We notice that the following statements are equivalent:

(1) Equation (7) has a unique solution in $C([a,b],\mathbb{B})$;
(2) Equation (6) has a unique solution in $C([\alpha,\beta],\mathbb{B})$.

Moreover, if $u \in C([a,b],\mathbb{B})$ is the unique solution of (7), let us define $v_1 \in C([\alpha,a],\mathbb{B})$ and $v_2 \in C([b,\beta],\mathbb{B})$ be defined by:

$$v_1(t) := \int_a^b K(t,s,u(s))ds + g(t),\ t \in [\alpha,a]$$

and

$$v_2(t) := \int_a^b K(t,s,u(s))ds + g(t),\ t \in [b,\beta].$$

Then, the function $v \in C([\alpha,\beta],\mathbb{B})$ defined by

$$v(t) := \begin{cases} v_1(t),\ t \in [\alpha,a] \\ u(t),\ \ t \in [a,b] \\ v_2(t),\ t \in [b,\beta] \end{cases}$$

is the unique solution of (6). As we shall see in this paper, Eq. (6) is useful and important in the study of data dependence of the solution of the integral equation (7) with respect to a and b (see also [31]).

For the theory of Eq. (6), see also [2, 11, 13, 17, 28].

3 Existence and Uniqueness

Let $\alpha \leq a < b \leq \beta$ be some real numbers, $K \in C([\alpha,\beta]^4 \times \mathbb{R}^m, \mathbb{R}^m)$ and $g \in C([\alpha,\beta],\mathbb{R}^m)$. We consider the following functional-integral equation

$$x(t) = \int_a^b K(t,s,a,b,x(s))ds + g(t),\ t \in [\alpha,\beta]. \tag{8}$$

We have the following result.

Theorem 7. *Let us consider Eq. (8). We suppose that there exists a matrix* $Q \in \mathbb{R}_+^{m\times m}$ *such that:*

(i) for all $u = (u_1,\ldots,u_m), v = (v_1,\ldots,v_m) \in \mathbb{R}^m$ *and all* $t,s,a,b \in [\alpha,\beta]$ *we have*

$$\begin{pmatrix} |K_1(t,s,a,b,u) - K_1(t,s,a,b,v)| \\ \dots \\ |K_m(t,s,a,b,u) - K_m(t,s,a,b,v)| \end{pmatrix} \leq Q \begin{pmatrix} |u_1 - v_1| \\ \dots \\ |u_m - v_m| \end{pmatrix};$$

(ii) The matrix $S := (\beta - \alpha)Q$ is convergent to zero.

Then, we have the following conclusions:

(a) Equation (8) has in $C([\alpha,\beta],\mathbb{R}^m)$ a unique solution $x_*(\cdot,a,b)$;
(b) for all $x_0 \in C([\alpha,\beta],\mathbb{R}^m)$, the sequence $(x_n)_{n\in\mathbb{N}}$ defined by

$$x_{n+1}(t,a,b) := \int_a^b K(t,s,a,b,x_n(s,a,b))ds + g(t)$$

converges uniformly to x_* with respect to $t,a,b \in [\alpha,\beta]$. Moreover

$$\begin{pmatrix} |x_{1n}(t,a,b) - x_{1*}(t,a,b)| \\ \dots \\ |x_{mn}(t,a,b) - x_{m*}(t,a,b)| \end{pmatrix} \leq (I-S)^{-1}S^n \begin{pmatrix} |x_{10}(t,a,b) - x_{11}(t,a,b)| \\ \dots \\ |x_{m0}(t,a,b) - x_{m1}(t,a,b)| \end{pmatrix};$$

(c) the function

$$x^* : [\alpha,\beta]^3 \to \mathbb{R}^m, \ (t,a,b) \mapsto x_*(t,a,b)$$

is continuous.

Proof. We consider the operator $A : C([\alpha,\beta],\mathbb{R}^m) \to C([\alpha,\beta],\mathbb{R}^m)$ defined by

$$Ax(t) := \int_a^b K(t,s,a,b,x(s))ds + g(t), \ t \in [\alpha,\beta].$$

From (i) and (ii) it follows that A is an S-contraction on the Banach space $(C([\alpha,\beta],\mathbb{R}^m), \|\cdot\|_\infty)$, where $\|\cdot\|_\infty$ is the supremum vector-valued norm given by $\|x\|_\infty := (\|x_1\|_\infty,\dots,\|x_m\|_\infty)$. So, the proof follows from Perov's Theorem. From now on, we will denote the norm $\|\cdot\|_\infty$ by $\|\cdot\|$.

Remark 1. In the conditions of Theorem 7, the operator $A : C([a,b],\mathbb{R}^m) \to C([a,b],\mathbb{R}^m)$ (which appears in the proof of the above mentioned theorem) is C-Picard with respect to the vector-valued norm $\|\cdot\|$, with $C := (I-S)^{-1}$.

4 Data Dependence with Respect to *K* and *g*

Let us perturbe Eq. (8) as follows

$$y(t) = \int_a^b H(t, s, a, b, y(s))ds + h(t),\ t \in [\alpha, \beta], \tag{9}$$

where $H \in C([\alpha, \beta]^4 \times \mathbb{R}^m, \mathbb{R}^m)$ and $h \in C([\alpha, \beta], \mathbb{R}^m)$.

We suppose that there exist $\eta, \mu \in (\mathbb{R}_+^*)^m$ such that, for $i \in \{1, 2, \ldots, m\}$, one have:

$$|K_i(t, s, a, b, u) - H_i(t, s, a, b, u)| \le \eta_i,\ \text{for all } t, s, a, b \in [\alpha, \beta] \text{ and } u \in \mathbb{R}_+^m \tag{10}$$

and

$$|g_i(t) - h_i(t)| \le \mu_i,\ \text{for all } t \in [\alpha, \beta]. \tag{11}$$

The problem is to estimate the vectorial distance $\|\cdot\|$ in $C([\alpha, \beta], \mathbb{R}^m)$ between the unique solution of Eq. (8) and a solution of Eq. (9), if a such solution exists.

In this direction, we have:

Theorem 8. *Consider Eqs. (8) and (9). Suppose that:*

(i) the functions K and g satisfy all the assumptions of Theorem 7;
(ii) the functions K, H, g, h satisfy the assumptions (10) and (11);
(iii) Equation (9) has at least one solution.

Then, if x^* is the unique solution of Eq. (8) and y^* denotes a solution of Eq. (9), we have

$$\|x^* - y^*\| \le (I - S)^{-1}[(\beta - \alpha)\eta + \mu].$$

Proof. Let us consider the operator $A : C([\alpha, \beta], \mathbb{R}^m) \to C([\alpha, \beta], \mathbb{R}^m),\ x \mapsto Ax$ given by

$$Ax(t) := \int_a^b K(t, s, a, b, x(s))ds + g(t)$$

and the operator $B : C([\alpha, \beta], \mathbb{R}^m) \to C([\alpha, \beta], \mathbb{R}^m),\ x \mapsto Bx$ defined by

$$Bx(t) := \int_a^b H(t, s, a, b, x(s))ds + h(t).$$

By Theorem 7, we obtain that A is a $(I - S)^{-1}$-operator. On the other hand, using (10) and (11) we obtain that

$$\|Ax - Bx\| \le (\beta - \alpha)\eta + \mu.$$

Now the conclusion follows from the Data Dependence Lemma.

Remark 2. Let $K, K_n \in C([\alpha,\beta]^4 \times \mathbb{R}^m, \mathbb{R}^m)$ and $g, g_n \in C([\alpha,\beta], \mathbb{R}^m)$ be some functions, for $m \in \mathbb{N}^*$. Let us consider the following sequence of integral equations

$$y_n(t) = \int_a^b K_n(t,s,a,b,y_n(s))ds + h_n(t),\ t \in [\alpha,\beta]\ (n \in \mathbb{N}^*). \tag{12}$$

Suppose also that the sequences (K_n) and (g_n) converge, in some sense, to K and, respectively, g. The problem is to obtain sufficient conditions such that, if y_n is a solution of (12) and x^* is the unique solution of Eq. (8), the sequence (y_n) converges, in a given sense, to x^* as $n \to +\infty$.

An answer to this problem is the following theorem.

Theorem 9. *Consider Eqs. (8) and (12). Suppose that:*

(i) the functions K and g satisfy all the assumptions of Theorem 7;
(ii) the sequence (K_n) uniformly converges to K and the sequence (g_n) uniformly converges to g, as $n \to +\infty$;
(iii) for each $n \in \mathbb{N}^*$ Eq. (12) has at least one solution $y_n \in C([\alpha,\beta], \mathbb{R}^m)$.

Then, if x^* denotes the unique solution of Eq. (8), we have that (y_n) uniformly converges to x^* as $n \to +\infty$.

Proof. The conclusion follows by Theorem 8.

5 Data Dependence with Respect to *a* and *b*

Let $\alpha \le a < b \le \beta$ be some real numbers and denote $X := C([\alpha,\beta]^3, \mathbb{R}^m)$. We will consider on X the following vectorial norm:

$$\|x\| := \begin{pmatrix} \max_{t,a,b\in[\alpha,\beta]} |x_1(t,a,b)| \\ \dots \\ \max_{t,a,b\in[\alpha,\beta]} |x_m(t,a,b)| \end{pmatrix}.$$

Then $(X, \|\cdot\|)$ is a generalized complete linear normed space.

Throughout this section we suppose that $K \in C([\alpha,\beta]^4 \times \mathbb{R}^m, \mathbb{R}^m)$ and $g \in C([\alpha,\beta], \mathbb{R}^m)$ are given functions. We consider the integral equation

$$x(t,a,b) = \int_a^b K(t,s,a,b,x(s,a,b))ds + g(t),\ t \in [\alpha,\beta]. \tag{13}$$

We are looking for the solutions of this equations in X. Let us denote by $B : X \to X,\ x \mapsto Bx$ the following operator

$$Bx(t,a,b) := \int_a^b K(t,s,a,b,x(s,a,b))ds + g(t).$$

Then, Eq. (13) takes the form of a fixed point problem, as follows

$$x = Bx. \tag{14}$$

By a similar approach to the proof of Theorem 7 in Sect. 3, we can get the following existence and uniqueness result in X for Eq. (13).

Theorem 10. *Let us consider Eq. (13). We suppose that there exists a matrix $Q \in \mathbb{R}_+^{m\times m}$ such that:*

(i) for all $u = (u_1,\dots,u_m), v = (v_1,\dots,v_m) \in \mathbb{R}^m$ and all $t,s,a,b \in [\alpha,\beta]$ we have

$$\begin{pmatrix} |K_1(t,s,a,b,u) - K_1(t,s,a,b,v)| \\ \dots \\ |K_m(t,s,a,b,u) - K_m(t,s,a,b,v)| \end{pmatrix} \le Q \begin{pmatrix} |u_1 - v_1| \\ \dots \\ |u_m - v_m| \end{pmatrix};$$

(ii) The matrix $S := (\beta-\alpha)Q$ is convergent to zero.

Then, we have the following conclusions:

(a) Equation (13) has in X a unique solution $x_*(\cdot,a,b)$;
(b) for all $x^0 \in X$, the sequence $(x^n)_{n\in\mathbb{N}}$ defined by

$$x^{n+1}(t,a,b) := \int_a^b K(t,s,a,b,x^n(s,a,b))ds + g(t), n \in \mathbb{N}$$

converges uniformly on with respect to $[\alpha,\beta]^3$ to x_*.

Remark 3. Under the above hypotheses, the operator B is a C-Picard operator, with $C := (I-S)^{-1}$.

We will consider now the differentiability of the solutions with respect to a and b. In this sense, we have the following result.

Theorem 11. *Let us consider Eq. (13). We suppose that:*

(i) the functions K and g satisfy all the assumptions of Theorem 10;
(ii) $K(t,s,\cdot,\cdot,\cdot) \in C^1([\alpha,\beta]^2 \times \mathbb{R}^m, \mathbb{R}^m)$, for all $t,s \in [\alpha,\beta]$.

Then, for the unique solution $x_* \in X$ of Eq. (13) we have that

$$x_*(t,\cdot,\cdot) \in C^1([\alpha,\beta]^2, \mathbb{R}^m), \text{ for all } t \in [\alpha,\beta].$$

Proof. By Theorem 10 we get that Eq. (13) has a unique solution $x_* \in X$. Let us prove, for example, that $\frac{\partial x_*}{\partial a}$ exists and $\frac{\partial x_*}{\partial a} \in X$. To do this, we shall use the following heuristic argument (see [31,33,39,40]). We suppose that there exists $\frac{\partial x_*}{\partial a}$. Then, from Eq. (13) we get

$$\begin{aligned}\frac{\partial x_*(t,a,b)}{\partial a} &= -K(t,a,a,b,x_*(a,a,b))\\ &\quad+\int_a^b \frac{\partial K(t,s,a,b,x_*(s,a,b))}{\partial a}ds\\ &\quad+\int_a^b \left(\frac{\partial K_j(t,s,a,b,x_*(s,a,b))}{\partial x_i}\right)\frac{\partial x_*(s,a,b)}{\partial a}ds.\end{aligned}$$

This relation suggests to consider the operator $C : X \times X \to X$ defined by

$$\begin{aligned}C(x,y)(t,a,b) := -K(t,a,a,b,x(a,a,b)) + \int_a^b \frac{\partial K(t,s,a,b,x(s,a,b))}{\partial a}ds\\ +\int_a^b \left(\frac{\partial K_j(t,s,a,b,x(s,a,b))}{\partial x_i}\right)y(s,a,b)ds.\end{aligned}$$

By the assumption (i) in Theorem 10 and by assumption (ii) in Theorem 11 it follows that

$$\left(\left|\frac{\partial K_j(t,s,a,b,u)}{\partial x_i}\right|\right) \le Q, \text{ for all } t,s,a,b \in [\alpha,\beta] \text{ and } u \in \mathbb{R}_m.$$

Thus, we obtain

$$\|C(x,y^1) - C(x,y^2)\| \le S\|y^1 - y^2\|, \text{ for all } x,y^1,y^2 \in X,$$

where $S = (\beta - \alpha)Q$.

Let us consider now the operator

$$A : X \times X \to X \times X,\ A := (B,C).$$

Notice that we are now in the conditions of the Fiber Generalized Contraction Theorem. Thus, the operator A is Picard and the sequences

$$x^{n+1}(t,a,b) := \int_a^b K(t,s,a,b,x^n(s,a,b))ds + g(t), n \in \mathbb{N}$$

and

$$y^{n+1}(t,a,b) := -K(t,a,a,b,x^n(a,a,b)) + \int_a^b \frac{\partial K(t,s,a,b,x^n(s,a,b))}{\partial a} ds$$

$$+ \int_a^b \left(\frac{\partial K_j(t,s,a,b,x^n(s,a,b))}{\partial x_i} \right) y^n(s,a,b) ds$$

converge uniformly on $[\alpha,\beta]^3$ to the unique fixed point (x_*, y_*) of A. If we choose $x^0 = y^0 = 0$, then $y^1 = \frac{\partial x^1}{\partial a}$. By induction, we can prove that $y^n = \frac{\partial x^n}{\partial a}$, for $n \in \mathbb{N}^*$. In conclusion:

$$(x^n) \overset{\text{unif}}{\to} x_* \text{ and } \left(\frac{\partial x^n}{\partial a} \right) \overset{\text{unif}}{\to} y_* \text{ as } n \to +\infty.$$

These relations imply that there exists $\frac{\partial x_*}{\partial a}$ and $\frac{\partial x_*}{\partial a} = y_*$. In a similar way, one can prove that $\frac{\partial x_*}{\partial b}$ exists and $\frac{\partial x_*}{\partial b} \in X$.

Remark 4. An important particular case is when $K(t,s,a,b,u) := H(t,s,a,b) F(s,u)$, where $H \in C([\alpha,\beta]^4, \mathbb{R}^{m\times m})$ and $F \in C([\alpha,\beta] \times \mathbb{R}^m, \mathbb{R}^m)$. Thus, the corresponding equation is:

$$x(t,a,b) = \int_a^b H(t,s,a,b)F(s,x(s,a,b))ds + g(t), \; t \in [\alpha,\beta]. \tag{15}$$

For this particular case we obtain the following results.

Theorem 12. *Let us consider Eq. (15). We suppose that there exists a matrix* $Q \in \mathbb{R}_+^{m\times m}$ *such that:*

(i) for all $u = (u_1,\dots,u_m), v = (v_1,\dots,v_m) \in \mathbb{R}^m$ *and all* $s \in [\alpha,\beta]$ *we have*

$$\begin{pmatrix} |F_1(s,u) - F_1(s,v)| \\ \dots \\ |F_m(s,u) - F_m(s,v)| \end{pmatrix} \le Q \begin{pmatrix} |u_1 - v_1| \\ \dots \\ |u_m - v_m| \end{pmatrix};$$

(ii) $H \in C([\alpha,\beta]^4, \mathbb{R}^{m\times m})$, $F \in C([\alpha,\beta] \times \mathbb{R}^m, \mathbb{R}^m)$ *and* $g \in C([\alpha,\beta], \mathbb{R}^m)$*;*
(iii) The matrix $S := (\beta - \alpha)M_H Q$ *is convergent to zero, where* $M_H := \max_{[\alpha,\beta]^4} |H(t,s,a,b)|$.

Then:

(a) the operator $B : X \to X \; x \longmapsto Bx$*, given by*

$$Bx(t,a,b) := \int_a^b H(t,s,a,b)F(s,x(s,a,b))ds + g(t)$$

is C*-Picard with* $C := (I - S)^{-1}$.
(b) Equation (15) has a unique solution $x_* \in X$.

Theorem 13. *Let us consider Eq. (15). We suppose that there exists a matrix* $Q \in \mathbb{R}_+^{m\times m}$ *such that:*

(i) for all $u = (u_1, \ldots, u_m), v = (v_1, \ldots, v_m) \in \mathbb{R}^m$ *and all* $s \in [\alpha, \beta]$ *we have*

$$\begin{pmatrix} |F_1(s,u) - F_1(s,v)| \\ \cdots \\ |F_m(s,u) - F_m(s,v)| \end{pmatrix} \le Q \begin{pmatrix} |u_1 - v_1| \\ \cdots \\ |u_m - v_m| \end{pmatrix};$$

(ii) The matrix $S := (\beta - \alpha)Q$ *is convergent to zero;*
(iii) $H \in C^1([\alpha,\beta]^4, \mathbb{R}^{m\times m})$, $F \in C^1([\alpha,\beta] \times \mathbb{R}^m, \mathbb{R}^m)$, *and* $g \in C^1([\alpha,\beta], \mathbb{R}^m)$.

Then, for the unique solution $x_* \in X$ *of Eq. (15) we have* $x^*(t,\cdot,\cdot) \in C^1([\alpha,\beta]^2, \mathbb{R}^m)$.

Remark 5. Let $\alpha \le a < b \le \beta$ be some real numbers, $\varphi, \psi \in C([\alpha,\beta],[\alpha,\beta])$, $K \in C([\alpha,\beta]^4 \times \mathbb{R}^m, \mathbb{R}^m)$, and $g \in C([\alpha,\beta]^3, \mathbb{R}^m)$ be given functions. We consider the following functional-integral equation

$$x(t,a,b) = \int_{\varphi(a)}^{\psi(b)} K(t,s,a,b,x(s,a,b))ds + g(t,a,b),\ t \in [\alpha,\beta]. \tag{16}$$

We are looking for the solutions of this equations in $X := C([\alpha,\beta]^3, \mathbb{R}^n)$.

As in the case of Eq. (13) we have the following result.

Theorem 14. *Let us consider Eq. (16). We suppose that there exists a matrix* $Q \in \mathbb{R}_+^{m\times m}$ *such that:*

(i) for all $u = (u_1, \ldots, u_m), v = (v_1, \ldots, v_m) \in \mathbb{R}^m$ *and all* $t, s, a, b \in [\alpha,\beta]$ *we have*

$$\begin{pmatrix} |K_1(t,s,a,b,u) - K_1(t,s,a,b,v)| \\ \cdots \\ |K_m(t,s,a,b,u) - K_m(t,s,a,b,v)| \end{pmatrix} \le Q \begin{pmatrix} |u_1 - v_1| \\ \cdots \\ |u_m - v_m| \end{pmatrix};$$

(ii) The matrix $S := (\beta - \alpha)Q$ *is convergent to zero.*

Then, we have the following conclusions:

(a) Equation (16) has in X *a unique solution* $x_*(\cdot, a, b)$;
(b) if $K(t,s,\cdot,\cdot,\cdot) \in C^1([\alpha,\beta]^2 \times \mathbb{R}^m, \mathbb{R}^m)$, *for all* $t, s \in [\alpha,\beta]$. $g(t,\cdot,\cdot) \in C^1([\alpha,\beta]^2, \mathbb{R}^m)$, *for all* $t \in [\alpha,\beta]$, *and if* $\varphi, \psi \in C^1([\alpha,\beta],[\alpha,\beta])$, *then*

$$x_*(t,\cdot,\cdot) \in C^1([\alpha,\beta]^2, \mathbb{R}^m),\ \textit{for all } t \in [\alpha,\beta].$$

Remark 6. The particular cases:

(i) $\varphi(a) = \alpha, \psi(b) = \beta$

and

(ii) $\varphi(a) = a, \psi(b) = b$

are the most relevant for the applications.

6 Ulam–Hyers Stability

In this section we will consider Eq. (13). With respect to the Ulam–Hyers stability of this equation we can prove the following result.

Theorem 15. *Consider Eq. (13). Suppose that all the assumptions of Theorem 10 hold. Then, Eq. (13) is Ulam–Hyers stable.*

Proof. The conclusion follows by Theorem 6 and Remark 1.

Remark 7. For other considerations on Ulam stability of the fixed point equation, with emphasis on the case of integral equations, see [12, 25, 32, 34, 35].

7 Applications to Bilocal Problems

7.1 The Dirichlet Problem in the Linear Case

Let $p, q, f \in C[a,b]$ and γ_1, γ_2 be some real numbers. We will denote

$$L_0(x)(t) := -x''(t) + p(t)x(t).$$

We consider the following Dirichlet problem (also called Picard problem):

$$\begin{cases} L(x)(t) := L_0(x)(t) + q(t)x(t) = f(t),\ t \in]a,b[\\ x(a) = \gamma_1,\ x(b) = \gamma_2 \end{cases} \tag{17}$$

For this problem we have the following well-known result (see [4, 18, 19, 27, 29, 30]).

Theorem 16. *Consider the Dirichlet problem (17). Suppose that one has uniqueness for the Dirichlet problem (17). Then, the following statements are equivalent:*

(i) there exists a function v such that $L(v) \geq 0$ in $]a,b[$ and $v > 0$ on $[a,b]$;
(ii) Given any continuous function $q_1 \geq q$, one has uniqueness for the Dirichlet problem written for the operator $L_1 := L_0 + q_1$ on the interval $[a,b]$;

(iii) *for every interval* $[c, d] \subset [a, b]$ *one has uniqueness for the Dirichlet problem written for the operator* L_0 *and the interval* $[c, d]$*;*

(iv) *the Green function* $G(t, s)$ *corresponding to the operator* L *and* $[a, b]$ *satisfies the condition*

$$G(t, s) \geq 0, \text{ for all } t, s \in [a, b].$$

From the above theorem we have the following notion, see [30].

Definition 3. Concerning the Dirichlet problem (17), the interval $[a, b[$ is, by definition, a maximum uniqueness interval for L if:

(1) we have uniqueness for the Dirichlet problem corresponding to L and for all intervals $[c, d] \subset [a, b[$;
(2) we have no uniqueness for the Dirichlet problem (17) corresponding to L and $[a, b]$.

7.2 Bilocal Problems

Let $\alpha \leq a < b \leq \beta, \gamma_1, \gamma_2$ be some real numbers and let $q, f \in C[\alpha, \beta]$. We consider the following differential equation

$$L(x)(t) := -x''(t) + q(t)x(t) = f(t), \ t \in [\alpha, \beta]. \tag{18}$$

The problem is to find the solutions $x \in C^2[\alpha, \beta]$ of the above equation which also satisfy the conditions:

$$x(a) = \gamma_1, \ x(b) = \gamma_2. \tag{19}$$

In other words, the bilocal problem for Eq. (18) is to find the solutions $x \in C^2[\alpha, \beta]$ which are solutions for the Dirichlet problem corresponding to the operator L and for the interval $[a, b] \subset [\alpha, \beta]$.

In the rest of this section, we will take (for simplicity) $\gamma_1 = \gamma_2 = 0$ and we will suppose that the interval $[\alpha, \beta]$ is a subinterval of a maximum uniqueness interval corresponding to the operator L.

Let $G(t, s; a, b)$ be the Green function corresponding to L and $[a, b]$. Then, the function $x_0 \in C^2[a, b]$ defined by

$$x_0(t) := \int_a^b G(t, s; a, b) f(s) ds$$

is a solution of the Dirichlet problem

$$L(x) = f, \ x(a) = 0, x(b) = 0.$$

Let us consider now the following Cauchy problems:

$$\begin{cases} L(x)(t) = f(t), \ t \in [\alpha, a] \\ x(a) = 0, \ x'(a) = \int_a^b \frac{\partial G(a,s;a,b)}{\partial t} f(s)ds \end{cases} \tag{20}$$

and

$$\begin{cases} L(x)(t) = f(t), \ t \in [b, \beta] \\ x(b) = 0, \ x'(b) = \int_a^b \frac{\partial G(b,s;a,b)}{\partial t} f(s)ds \end{cases} \tag{21}$$

Let $x_1 \in C^2[\alpha, a]$ be the unique solution of (20) and $x_2 \in C^2[b, \beta]$ be the unique solution of (21). Then, it is easy to see that the function $x_* \in C^2[\alpha, \beta]$ defined by

$$x_*(t) := \begin{cases} x_1(t), \ t \in [\alpha, a] \\ x_0(t), \ t \in [a, b] \\ x_2(t), \ t \in [b, \beta] \end{cases}$$

is the unique solution of the following bilocal problem

$$\begin{cases} L(x)(t) = f(t), \ t \in [\alpha, \beta] \\ x(a) = 0, \ x(b) = 0. \end{cases} \tag{22}$$

For the case $a = b$, notice that $x_*(t)$ is the unique solution on $[\alpha, \beta]$ of the Cauchy problem

$$\begin{cases} L(x)(t) = f(t), \ t \in [\alpha, \beta] \\ x(a) = 0, \ x'(a) = 0. \end{cases} \tag{23}$$

By the Cauchy's formula and using Green's formula, we have

$$x_*(t) = \int_\alpha^\beta \tilde{G}(t, s; a, b) f(s)ds + h(t, a, b), \ \text{for all } t \in [\alpha, \beta], a, b \in [\alpha, \beta], a \leq b, \tag{24}$$

with suitable functions $\tilde{G} \in C([\alpha, \beta]^2 \times D)$ and $h \in C^1([\alpha, \beta] \times D)$ (where $D := \{(a, b) \in [\alpha, \beta] \times [\alpha, \beta] : \ a \leq b\}$. Moreover, the function $\tilde{G}$ has some regularity properties as in the case of Green functions.

7.3 Nonlinear Bilocal Problems

Let $\alpha \leq a < b \leq \beta$ be some real numbers, let $p, q \in C[\alpha, \beta]$ and define $L(x) := -x'' + px' + qx, \ t \in [\alpha, \beta]$.

We suppose that $[\alpha, \beta]$ is included in a maximum uniqueness interval corresponding to the operator L. Let $f \in C([\alpha, \beta] \times \mathbb{R}, \mathbb{R})$. We consider the following nonlinear bilocal problem:

$$\begin{cases} L(x)(t) = f(t, x), & t \in [\alpha, \beta] \\ x(a) = 0, \ x(b) = 0, \end{cases} \tag{25}$$

where $x \in C^2[\alpha, \beta]$.

From (24) we get that the problem (25) is equivalent to the following functional integral equation

$$x(t) = \int_{\alpha}^{\beta} \tilde{G}(t, s, a, b) f(s, x(s)) ds + h(t, a, b), \ t \in [\alpha, \beta], \tag{26}$$

where we are looking for the unknown function $x \in C[\alpha, \beta]$.

From Theorem 14 (with $m = 1$) we obtain the following result for the functional integral equation:

$$x(t, a, b) = \int_{\alpha}^{\beta} \tilde{G}(t, s, a, b) f(s, x(s, a, b)) ds + h(t, a, b), \ t \in [\alpha, \beta], \tag{27}$$

Theorem 17. *Let us consider Eq. (27), where* $f \in C([\alpha, \beta] \times \mathbb{R}, \mathbb{R})$ *and* $h \in C([\alpha, \beta]^3, \mathbb{R})$*. We suppose that there exists* $r \in \mathbb{R}_+$ *such that:*

(i) for all $u, v \in \mathbb{R}$ *and all* $t, s, a, b \in [\alpha, \beta]$ *we have*

$$|f(t, u) - f(t, v)| \leq r|u - v|;$$

(ii) $(\beta - \alpha) r < 1$.

Then, Eq. (27) has a unique solution $x_*(\cdot, a, b)$.

Proof. For the theory of bilocal problem

$$\begin{cases} L(x)(t) = f(t, x), & t \in [a, b] \\ x(a) = 0, \ x(b) = 0, \end{cases} \tag{28}$$

see [4, 5, 7, 9–11, 20, 27, 29, 41, 42].

8 Some Further Research Directions

8.1 Integral Equations in Banach Spaces

1. Let $\mathbb{B}$ a Banach space, $\alpha \leq a < b \leq \beta$ be some real numbers, $K \in [\alpha,\beta]^4 \times \mathbb{B} \to \mathbb{B}$ and $g : [\alpha,\beta] \to \mathbb{B}$ be continuous functions. The problem is to study, in $C([\alpha,\beta]^3,\mathbb{B})$, the following functional-integral equation:

$$x(t,a,b) = \int_a^b K(t,s,a,b,x(s,a,b))ds + g(t),\ t \in [\alpha,\beta], \tag{29}$$

   In this paper, we studied the case $\mathbb{B} := \mathbb{R}^m$.
2. Let $\mathbb{B}$ a Banach space, $\alpha \leq a < b \leq \beta$ be some real numbers, $\varphi,\psi \in C([\alpha,\beta],[\alpha,\beta])$, $K \in C([\alpha,\beta]^4 \times \mathbb{B},\mathbb{B})$, and $g \in C([\alpha,\beta]^3,\mathbb{B})$. The problem is to study, in $C([\alpha,\beta]^3,\mathbb{B})$, the following functional-integral equation:

$$x(t,a,b) = \int_{\varphi(a)}^{\psi(b)} K(t,s,a,b,x(s,a,b))ds + g(t,a,b),\ t \in [\alpha,\beta]. \tag{30}$$

   In this paper, we studied the case $\mathbb{B} := \mathbb{R}^m$.
3. Let $\alpha \leq a < b \leq \beta$ be some real numbers, $q \in C[\alpha,\beta]$ and $f \in C([\alpha,\beta] \times \mathbb{R}^m,\mathbb{R}^m)$. We consider, in $C^2([\alpha,\beta],\mathbb{R}^m)$, the following bilocal problem:

$$\begin{cases} -x''(t) + q(t)x(t) = f(t,x(t)),\ t \in [\alpha,\beta] \\ x(a) = 0,\ x(b) = 0, \end{cases} \tag{31}$$

   If this problem has a unique solution in $C^2([\alpha,\beta],\mathbb{R}^m)$, then the question is if the data dependence phenomenon with respect to a and b takes place, for this unique solution.

   Notice that the case $m = 1$ is treated in this paper.
4. Let $\alpha \leq t_1 < \cdots < t_k \leq \beta$ be some real numbers, $p_i \in C[\alpha,\beta]$ and $f \in C([\alpha,\beta] \times \mathbb{R},\mathbb{R})$. We consider, in $C^k([\alpha,\beta],\mathbb{R}^m)$, the following polilocal problem:

$$\begin{cases} -x^{(k)}(t) + p_1(t)x^{(k-1)}(t) + \cdots + p_k(t)x(t) = f(t,x(t)),\ t \in [\alpha,\beta] \\ x(t_1) = 0,\ldots,x(t_k) = 0, \end{cases} \tag{32}$$

   If this problem has a unique solution in $C^K([\alpha,\beta],\mathbb{R}^m)$, then the question is again to study the data dependence of this solution with respect to $t_1,\ldots,t_k$. See [18, 19, 21, 27, 44–46].

8.2 Integral Inclusions and Multivalued Bilocal Problems

1. Let $\alpha \leq a < b \leq \beta$ be some real numbers, $g : [\alpha, \beta] \to \mathbb{R}^m$ be a continuous function and let $K : [\alpha, \beta]^4 \times \mathbb{R}^m \multimap \mathbb{R}^m$ be a multivalued operator such that $K(t, \cdot, a, b, x(\cdot, a, b)) : [\alpha, \beta] \multimap \mathbb{R}^m$ is Aumann integrable, for every $x \in C(\alpha, \beta]^3, \mathbb{R}^m)$. The problem is to study, in $C([\alpha, \beta]^3, \mathbb{R}^m)$, the following functional-integral inclusion:

$$x(t, a, b) \in \int_a^b K(t, s, a, b, x(s, a, b))ds + g(t),\ t \in [\alpha, \beta], \tag{33}$$

2. Let $\alpha \leq a < b \leq \beta$ be some real numbers, $q \in C[\alpha, \beta]$ and let $F : [\alpha, \beta] \multimap \mathbb{R}$ be a continuous (in some sense) multivalued operator with compact and convex values. The problem is to study, in $C^2([\alpha, \beta], \mathbb{R})$, the following bilocal problem:

$$\begin{cases} -x''(t) + q(t)x(t) \in F(t),\ t \in [\alpha, \beta] \\ x(a) = 0,\ x(b) = 0, \end{cases} \tag{34}$$

For the above problems, see [3, 8, 15, 16, 26] and the references therein.

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Hyperbolic Wavelets

F. Schipp

Abstract In the last two decades a number of different types of wavelets transforms have been introduced in various areas of mathematics, natural sciences, and technology. These transforms can be generated by means of a uniform principle based on the machinery of harmonic analysis. In this way we pass from the affine group to the wavelet transforms, from the Heisenberg group to the Gábor transform. Taking the congruences of the hyperbolic geometry and using the same method we introduced the concept of hyperbolic wavelet transforms (HWT). These congruences can be expressed by Blaschke functions, which play an eminent role not only in complex analysis but also in control theory. Therefore we hope that the HWT will become an adequate tool in signal and system theories. In this paper we give an overview on some results and applications concerning HWT.

Keywords Wavelets • Hyperbolic geometry • Rational systems • Blaschke functions • System identification • Signal processing

1 Introduction

The evolution of the theory of Fourier-series is strongly connected with important practical applications. *Fourier* himself developed his method for solving a physical problem on heat conduction. The born and progress of many areas of mathematics are related to the same problems that inspired that application of Fourier series [1, 52]. At the beginning of the last century several function systems, that seemed rather exotic at that time, have been introduced. Their theoretical and practical importance became evident only much later. A very special one of them is the Haar orthonormal

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system defined by Alfréd Haar [30] in 1909, which looks quite artificial for the first look. It contains step functions originated from the basic function $h(x) := 1\ (x \in [0, 1/2))$, $h(x) := -1\ (x \in [1/2, 1))$, $h(x) = 0\ (x \in [1, \infty))$ by means of simple transformations, namely by translation and dilation: $h_0(x) = 1$, $h_m(x) := 2^{n/2} h(2^n x - k)\ (x \in [0, 1))$, $(m = 2^n + k, n, k \in \mathbb{N})$. The Haar-system is orthogonal in the Hilbert space $L^2 := L^2([0, 1))$ with respect to the usual scalar product, and the Haar–Fourier series of a function $f \in L^1([0, 1))$ converges to the function in both norm and almost everywhere. In particular, if the function is continuous, then the convergence is uniform. In this respect it is essentially different from the trigonometric system [58].

The Haar system was the starting point in several investigations in martingale theory, in wavelet theory, and in functional analysis [10, 58, 59, 64]. The fact that the members of the system are not continuous makes them inappropriate for approximating smooth functions. By multiple integration and orthogonalization of the Haar functions Z. Ciesielski constructed orthogonal bases of smooth functions having good approximation properties [9, 11]. Taking the Haar system and following a different path Y. Meyer, I. Daubechies among others started to construct orthonormed systems, so-called *wavelets*, of the form

$$\varphi_{n,k}(x) = 2^{n/2} \varphi(2^n x - k)\ \ (x \in \mathbb{R}, \varphi \in L^2(\mathbb{R}), \|\varphi\|_2 = 1).$$

Except from the Haar system the construction of such systems is a hard task [13, 40, 41]. Then the Fourier transform $\hat{\varphi}$ instead of the mother wavelet φ itself turned to be a good starting point. Despite the fact that φ cannot be given in an explicit form generally the wavelet Fourier series enjoy nice convergence and approximation properties. The kernel functions of the partial sums can be well estimated and the wavelet Fourier coefficients can be calculated by a fast algorithm.

In applications not only the L^p spaces but also the set of analytic functions $\mathscr{A}$ on the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and the Banach spaces related to it play important roles. Taking the integral means

$$\|f_r\|_p := \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right)^{1/p} \quad (0 < p < \infty)$$

of a function $f \in \mathscr{A}$ Frigyes Riesz [53] introduced the class of functions in $\mathscr{A}$ for which $\sup_{0<r<1} \|f_r\|_p < \infty$. Referring to a paper of Hardy [31] from 1915, in which Hardy showed that $\|f_r\|_p$ is monotonic with respect to r, he named the function class after Hardy and denoted it by $\mathscr{H}^p$. The quantity $\|f\|_{H^p} := \sup_{0<r<1} \|f_r\|_p$ defines a norm $(1 \leq p \leq \infty)$ or a quasinorm $(0 < p < 1)$ on $\mathscr{H}^p(\mathbb{D}) := \mathscr{H}^p$, and it becomes complete with respect to them. It is known that the boundary function $f(e^{it}) := \lim_{r \to 1} f(re^{it})$ exists a.e. for every $f \in \mathscr{H}^p$ $(p > 0)$, and f belongs to $L^p(\mathbb{T})$ on $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Moreover $\|f\|_{H^p} = \|f\|_{L^p(\mathbb{T})}$. The space $\mathscr{H}^\infty(\mathbb{D})$ is the collection of functions $f \in \mathscr{A}$ for which $\|f\|_{H^\infty} := \sup_{z \in \mathbb{D}} |f(z| < \infty$. The *disc algebra* $\mathfrak{A}$, i.e. the set of functions analytic on $\mathbb{D}$ and continuous on its closure is a closed subspace of $\mathscr{H}^\infty(\mathbb{D})$ [14, 28, 66].

The Hardy spaces are applied intensively not only in the theories of complex functions and Fourier series but as it turned out in the 1960s they, in particular $\mathscr{H}^2(\mathbb{D})$, and $\mathscr{H}^\infty(\mathbb{D})$, are the proper Banach spaces for mathematical modeling of problems in control and operator theories [34, 54, 62].

The simplest discrete systems in control theory can be described by linear operators of type $T : \ell^2 \to \ell^2$:

$$y = T(x) \;\; (x = (x_n, n \in \mathbb{N}), y = (y_n, n \in \mathbb{N}) \in \ell^2).$$

The sequences $x, y \in \ell^2$ are called (input, output) signals, and the norm $\|x\|_{\ell_2} := (\sum_{n\in\mathbb{N}} |x_n|^2)^{1/2}$ is the energy of x. The usual interpretation of the index n in $(x_n, n \in \mathbb{N})$ is discrete time. The *discrete linear causal and time invariant* (LTI) systems can be given by convolution operators:

$$y = T_a x := a * x, \;\; y_n = (a * x)_n := x_n a_0 + x_{n-1} a_1 + \cdots + x_0 a_n \; (n \in \mathbb{N}).$$

The map

$$x \to X, \;\; X(z) := \sum_{n\in\mathbb{N}} x_n z^n \; (z \in \mathbb{D})$$

is isometric isomorphy between ℓ^2 and the Hardy space $\mathscr{H}^2(\mathbb{D})$. The function

$$A(z) := \sum_{n\in\mathbb{N}} a_n z^n \; (z \in \mathbb{D})$$

generated by the sequence a is called the *transfer function* of the system. Taking the isometry between ℓ^2 and $\mathscr{H}^2(\mathbb{D})$ the operator $x \to T_a x$ corresponds to the operator of multiplication by the transfer function $X \to AX$. Since this is a bounded operator on $\mathscr{H}^2(\mathbb{D})$ if and only if $A \in \mathscr{H}^\infty(\mathbb{D})$ and its norm is $\|A\|_{H^\infty}$ we have that

$$\|T_\mathfrak{a}\|_{\ell^2\to\ell^2} = \|A\|_{H^\infty}.$$

This implies that $T_a \to A$ is isomorphism between the LTI systems and $\mathscr{H}^\infty(\mathbb{D})$. This makes clear the importance of Hardy spaces in mathematical modeling of system theory.

$T_a : \ell^2 \to \ell^2$ is a unitary operator, i.e. it satisfies the principle of energy conservations, if $\|T_a x\|_{\ell^2} = \|x\|_{\ell^2}$ $(x \in \ell^2)$. For the transfer function it means that $\|X\|_{L^2(\mathbb{T})} = \|AX\|_{L^2(\mathbb{T})}$ holds for every $X \in \mathscr{H}^2(\mathbb{D})$. Hence we have that the operator T_a is unitary if and only if:

$$A \in \mathscr{H}^\infty(\mathbb{D}), \;\; |A(e^{it})| = 1 \;\; (\text{a.e. } \; t \in [0, 2\pi)).$$

The functions satisfying this condition are called *inner functions*. Consequently, the LTI systems for which the energy conservation law holds can be given by the inner functions in $\mathscr{H}^\infty(D)$ [34].

It is easy to show that the functions

$$B_{\mathfrak{b}}(z) := \epsilon B_b(z),\quad B_b(z) := \frac{z-b}{1-\overline{b}z}\ \ (z \in \mathbb{C}, \mathfrak{b} = (b,\epsilon) \in \mathbb{B} := \mathbb{D} \times \mathbb{T})$$

with parameters $b \in \mathbb{D}$ and $\epsilon \in \mathbb{T}$ are bijections of the form $\mathbb{D} \to \mathbb{D}$ and $\mathbb{T} \to \mathbb{T}$, so they are inner functions. Clearly, their finite products are inner functions as well. Blaschke [2] proved that if the sequence $b_n \in \mathbb{D}$ $(n \in \mathbb{N})$ satisfies the condition

$$\sum_{n\in\mathbb{N}}(1-|b_n|) < \infty,$$

which was named after him, then taking $\epsilon_n := -\overline{b}_n/|b_n|$ $(b_n \neq 0)$, $\epsilon_n = 1$ $(b_n = 0)$ the infinite product $B(z) := \prod_{n=0}^{\infty} B_{\mathfrak{b}_n}(z)$ $(z \in \mathbb{D})$ is uniformly convergent on every compact subset of $\mathbb{D}$, and $B \in \mathscr{H}^{\infty}$ is an inner function. $B_{\mathfrak{b}}$ is called Blaschke-function, and B is called Blaschke-product.

$b \in \mathbb{D}$ is the zero of the function B_b, and $b^* := 1/\overline{b}$ which is the inverse of b with respect to the unit circle is the pole of $B_{\mathfrak{b}}$. Therefore the zeros of the Blaschke-product B are b_n and their multiplicities correspond to their occurrences in $(b_k, k \in \mathbb{N})$. Conversely, it is known [60] that the zeros of any $f \in \mathscr{H}^p(\mathbb{D})$ $(p > 0)$ satisfy the Blaschke-condition. The Blaschke-product defined by them contains the zeros of f and f can be written in the form $f = Bg$, where $g \in \mathscr{H}^p(\mathbb{D})$ does not vanish on $\mathbb{D}$ and $\|f\|_{H^p} = \|g\|_{H^p}$ [53]. This decomposition is unique except for a factor with absolute value 1.

Based on the Blaschke functions Malmquist [39] and Takenaka [63] in 1925 independently introduced a wide class of orthogonal systems of rational functions in $\mathscr{H}^2(\mathbb{D})$. These systems are now called *Malmquist–Takenaka* (MT) systems. They can be generated by an arbitrary sequence $b_n \in \mathbb{D}$ $(n \in \mathbb{N})$ and can be given in an explicit form as follows

$$\Phi_n(z) := \frac{\sqrt{1-|b_n|^2}}{1-\overline{b}_n z}\prod_{k=0}^{n-1} B_{b_k}(z)\ \ (z \in \overline{\mathbb{D}} := \mathbb{D} \cup \mathbb{T},\ b_k \in \mathbb{D}, k \in \mathbb{N}).$$

It is known that the opposite of the Blaschke-condition is necessary and sufficient for the MT-systems to be closed in the Hardy spaces $\mathscr{H}^p(\mathbb{D})$ $(1 \leq p < \infty)$ and in the disc algebra. We note that the power functions $e_n(z) := z^n$ $(z \in \mathbb{C}, n \in \mathbb{N})$ (complex trigonometric functions on $\mathbb{T}$) can be obtained from the MT-systems by choosing $b_n = 0$ $(n \in \mathbb{N})$. While the classical orthogonal systems (e.g., Jacobi-, Csebisev-, Laguerre-systems) used in physical applications and the wavelets applied in signal theory have only one or two parameters, there are infinitely many parameters that can be chosen freely in the construction of MT-systems. This makes possible to take parameters according to the problem that are optimal in a sense. Following this idea we considered the decomposition of ECG signals in rational bases. It resulted in good approximation and efficient compression of the signal.

Starting from the 1960s researchers have realized the advantage of using MT-systems with one or two parameters over the trigonometric one in applications in system theory [34]. The MT-system

$$L_n^b(z) := \frac{\sqrt{1-|b|^2}}{1-\overline{b}z} B_b^n(z) \ (z \in \overline{\mathbb{D}}, n \in \mathbb{N})$$

corresponding to the constant sequence $b_n := b, n \in \mathbb{N}$ is called *discrete Laguerre-system.* Taking the Fourier transform of it we receive the Laguerre-functions, that are especially important in theoretical physics, which justifies the name. The special MT-system, when $b_{2n} = b$, $b_{2n+1} = \overline{b}$ $(n \in \mathbb{N})$ was introduced by *Kautz* for system representation. MT-systems generated by periodic sequences $(b_n, n \in \mathbb{N})$ are called *periodic MT-systems.* A thorough summary on the application of MT-systems in control theory can be found in the book [34]. The problem of discretization of MT expansions and their applications, in particular simple representations of ECG signals, will be addressed in Sect. 5.

The Blaschke functions play important role not only in system identification or factorization of function belonging to Hardy spaces. For instance, they can be used to represent the congruences in the Poincaré model of the Bolyai–Lobachevsky geometry. Based on this property we can take them in the construction of wavelets instead of the affine transforms in $\mathbb{R}$. As a result the so-called hyperbolic wavelets are introduced. The definition and properties of them will be given in Sect. 4.

2 The Blaschke-Group

The Blaschke-function are very useful in modeling the hyperbolic geometry.

2.1 Möbius Transforms

Let the group of Möbius transforms (linear rational functions) be denoted by $\mathfrak{M}$, and the complex two dimensional linear group by $SL(2) := \{A \in \mathbb{C}^{2\times 2} : \det A = 1\}$. The map

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \to r_A(z) := \frac{a_{11}z + a_{12}}{a_{21}z + a_{22}} \quad (z \in \overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\})$$

is an $SL(2) \to \mathfrak{M}$ group homomorphism: $r_{A_1A_2} = r_{A_1} \circ r_{A_2}$, where $\circ$ stands for composition of functions. The class of unitary matrices $SU(2)$ forms a subgroup of $SL(2)$. Every matrix $A \in SU(2)$ is of the form

$$A = \begin{pmatrix} p & -q \\ \overline{q} & \overline{p} \end{pmatrix}, \quad \det A = |p|^2 + |q|^2 = 1 \ \ (p, q \in \mathbb{C}).$$

Moreover the (positive definite) quadratic form $Q_E(x) := |x_1|^2 + |x_2|^2$ $(x = (x_1, x_2) \in \mathbb{C}^2)$ is invariant with respect to the transformations in $SU(2)$: $Q_E(Ax) = Q_E(x)$ $(x \in \mathbb{C}^2, A \in SU(2))$.

The subgroup of $SL(2)$ that contains the matrices

$$B = \begin{pmatrix} p & -q \\ -\overline{q} & \overline{p} \end{pmatrix}, \quad \det B = |p|^2 - |q|^2 = 1 \ \ (p, q \in \mathbb{C})$$

will play an important role. The hyperbolic quadratic form $Q_H(x) := |x_1|^2 - |x_2|^2$ $(x = (x_1, x_2) \in \mathbb{C}^2)$ is invariant with respect to these transformations: $Q_H(Bx) = Q_H(x)$. This justifies the usual notation $SH(2)$ (or $SU(1,1)$) for this subgroup [65]. The homomorphism $B \to r_B$ will take the elements of $SH(2)$ to Blaschke-functions:

$$r_B(z) := \frac{pz - q}{\overline{q}z + \overline{p}} = \frac{p}{\overline{p}} \frac{z - q/p}{1 - z\overline{q}/\overline{p}} = \epsilon \frac{z - b}{1 - \overline{b}z} =: B_{\mathfrak{b}}(z) \ (z \in \mathbb{C})$$

$$\left(b := q/p \in \mathbb{D}, \epsilon := \frac{p}{\overline{p}} \in \mathbb{T} \right).$$

Consequently, the set of Blaschke-functions $\mathfrak{B}$ is a subgroup of the Möbius-transforms $\mathfrak{M}$. The group $(\mathfrak{B}, \circ)$ is called *Blaschke-group*.

2.2 The Parameters of the Blaschke-Group

It follows from the identity (see [32])

$$1 - |B_{\mathfrak{b}}(z)|^2 = \frac{(1 - |z|^2)(1 - |b|^2)}{|1 - \overline{b}z|^2} \ \ (z \in \overline{\mathbb{D}}, \mathfrak{b} = (b, \epsilon) \in \mathbb{B} := \mathbb{D} \times \mathbb{T})$$

that if $\mathfrak{b} \in \mathbb{B}$ then $B_{\mathfrak{b}} : \mathbb{D} \to \mathbb{D}$, and $B_{\mathfrak{b}} : \mathbb{T} \to \mathbb{T}$ is one-to-one. Moreover, the unit element of $\mathfrak{B}$ is $B_{\mathfrak{e}}$ $(\mathfrak{e} := (0, 1))$ and $B_{\mathfrak{b}^{-1}}$ $(\mathfrak{b}^{-1} := (-b\epsilon, \overline{\epsilon}))$ is the inverse of $B_{\mathfrak{b}}$. The sets

$$\mathfrak{B}_{\mathbb{I}} := \{B_{\mathfrak{b}} : \mathfrak{b} = (s, 1), s \in (-1, 1)\}, \ \ \mathfrak{B}_{\mathbb{T}} := \{B_{\mathfrak{b}} : \mathfrak{b} = (0, \epsilon), \epsilon \in \mathbb{T}\}$$

are one parameter subgroups of the Blaschke-group, which by means of

$$B_{\mathfrak{b}} = B_{(0, e^{i(\varphi + \theta)})} \circ B_{(r,1)} \circ B_{(0, e^{-i\varphi})} \ \ (\mathfrak{b} := (re^{i\varphi}, e^{i\theta}) \in \mathbb{B})$$

generate the Blaschke-group: $\mathfrak{B} = \mathfrak{B}_{\mathbb{T}} \circ \mathfrak{B}_{\mathbb{I}} \circ \mathfrak{B}_{\mathbb{T}}$. The functions in $\mathfrak{B}_{\mathbb{I}}$ map the set $\mathbb{I} := \{z \in \mathbb{D} : -1 < \Re z < 1, \Im z = 0\}$ onto itself, and $1, -1$ are fix points of them.

The disc $\mathbb{D}$ with the *pseudohyperbolic metric*

$$\rho(z_1, z_2) := \frac{|z_1 - z_2|}{|1 - z_1\overline{z}_2|} = |B_{z_2}(z_1)| \ \ (z_1, z_2 \in \mathbb{D})$$

is a complete metric space. This metric is invariant with respect to Blaschke-functions:

$$\rho(B_{\mathfrak{b}}(z_1), B_{\mathfrak{b}}(z_2)) = \rho(z_1, z_2) \ \ (z_1, z_2 \in \mathbb{D}, \mathfrak{b} \in \mathbb{B}). \tag{1}$$

This is the consequence of the identity

$$\frac{B_{\mathfrak{b}}(z_1) - B_{\mathfrak{b}}(z_2)}{1 - B_{\mathfrak{b}}(z_1)\overline{B}_{\mathfrak{b}}(z_2)} = \frac{z_1 - z_2}{1 - z_1\overline{z}_2}\frac{1 - b\overline{z}_2}{1 - \overline{b}z_2} \ \ (z_1, z_2 \in \mathbb{D}, \mathfrak{b} = (b, \epsilon) \in \mathbb{B}).$$

The property in (1) characterizes the Blaschke-functions. Namely, for every $f \in \mathscr{H}^\infty(\mathbb{D})$, $\|f\|_\infty \le 1$ we have $\rho(f(z_1), f(z_2)) \le \rho(z_1, z_2)$, and equality holds in a point $z \in \mathbb{D}$ if and only if f is a Blaschke-function [15].

The map $\mathfrak{b} \to B_{\mathfrak{b}}$ induces a group structure in the parameter set $\mathbb{B}$. Moreover

$$\mathfrak{z}\circ\mathfrak{b} = (B_{\mathfrak{b}^{-1}}(z), \zeta\eta_{\mathfrak{b}^{-1}}(z)) \ \left(\eta_{\mathfrak{b}}(z) := \overline{\epsilon}\frac{1 - z\overline{b}}{1 - \overline{z}b}, \ \mathfrak{b} = (b, \epsilon), \mathfrak{z} = (z, \zeta) \in \mathbb{B}\right). \tag{2}$$

Hence it is clear that the group operation $(\mathfrak{z}, \mathfrak{b}) \to \mathfrak{z} \circ \mathfrak{b}^{-1}$ is continuous with respect to the (euclidian) metric $\varrho(\mathfrak{b}_1, \mathfrak{b}_2) := |b_1 - b_2| + |1 - \epsilon_1\overline{\epsilon}_2|$ $(\mathfrak{b}_j = (b_j, \epsilon_j) \in \mathbb{B})$ of the space $\mathbb{B}$. Consequently, $(\mathbb{B}, \circ)$ is a locally compact continuous group.

The bijection $B_b : \mathbb{T} \to \mathbb{T}$ $(b := re^{i\varphi} \in \mathbb{D})$ can be written in the following form on the boundary $\mathbb{T}$

$$B_b(e^{it}) = e^{i\beta_b(t)}, \beta_b(t) := \varphi + \gamma_r(t - \varphi) \qquad (t \in \mathbb{R}), \tag{3}$$

where the derivative of γ_r is the Poisson-kernel function:

$$\gamma_r'(t) = P_r(t) = \frac{1 - r^2}{1 - 2r\cos t + r^2} \ (t \in \mathbb{R}), \ \gamma_r(0) = 0. \tag{4}$$

Since $P_r(t) > 0$ $(t \in \mathbb{R})$ we have that γ_r is strictly increasing, and

$$\gamma_r(t) = \int_0^t P_r(\tau)\,d\tau = 2\arctan(c(r)\tan t/2)$$
$$(c(r) := (1 + r)/(1 - r), r \in [0, 1), t \in \mathbb{R}).$$

2.3 Hyperbolic Geometry

In the disc model (PD) of Poincaré for the Bolyai–Lobachevsky geometry the Blaschke-group can be identified with the group of congruences. The lines in the model are the images of the interval $\mathbb{I}$ by Blaschke-functions: $\{l_{\mathfrak{b}} := B_{\mathfrak{b}}(\mathbb{I}) : \mathfrak{b} \in \mathbb{B}\}$. They coincide with the circular arcs and line segments within $\mathbb{D}$ that cross $\mathbb{T}$ perpendicularly. The points $B_{\mathfrak{b}}(1), B_{\mathfrak{b}}(-1) \in \mathbb{T}$ are called the points at infinity of the line $l_{\mathfrak{b}}$, and the images $B_{\mathfrak{b}}(I)$ of the intervals $I := [s_1, s_2] \subset \mathbb{I}$ are called (hyperbolic) line segments. It is easy to show that $l_{\mathfrak{b}_1} = l_{\mathfrak{b}_2}$ if and only if there exists a function $B \in \mathfrak{B}_{\mathbb{I}}$ with $B_{\mathfrak{b}_1} = B_{\mathfrak{b}_2} \circ B$. Therefore the collection of the lines in PD can be identified with the right cosets $\mathfrak{G}/\mathfrak{G}_0$. An important metric, other than ρ is the *hyperbolic metric*:

$$\rho^*(z_1, z_2) := \operatorname{arth} \rho(z_1, z_2) = \frac{1}{2} \ln\left(\frac{1 + \rho(z_1, z_2)}{1 - \rho(z_1, z_2)}\right) \qquad (z_1, z_2 \in \mathbb{D}).$$

It can be shown that taking the triangle inequality with respect to ρ^* the equality $\rho^*(z_1, z_2) = \rho^*(z_1, z_3) + \rho^*(z_3, z_2)$ holds if and only if z_3 lies on the hyperbolic segment z_1, z_2.

There exist several equivalent models for the Bolyai–Lobachevsky geometry [12]. The *half plane model of Poincareé* (HP) can be generated from the disc model by the Cayley-transform

$$\Upsilon(z) := \frac{i - z}{i + z} \qquad (z \in \overline{\mathbb{C}}).$$

This bijection maps the half plane $\mathbb{C}_+ := \{z \in \mathbb{C} : \Im z > 0\}$ onto $\mathbb{D}$, the real line $\mathbb{R}$ onto $\mathbb{T} \setminus \{-i\}$, and $\Upsilon(t) = e^{2i \arctan t} \in \mathbb{T}$ $(t \in \mathbb{R})$. The functions

$$B_{\mathfrak{b}}^{\diamond}(z) := B_{\mathfrak{b}}(\Upsilon(z)) = \epsilon^{\diamond} \frac{z - b^{\diamond}}{z - \overline{b^{\diamond}}} \qquad \left(z \in \overline{\mathbb{C}}, b^{\diamond} := \Upsilon^{-1}(b), \epsilon^{\diamond} = -\epsilon \frac{1 + b}{1 + \overline{b}}\right)$$

on the half plane correspond to the Blaschke functions. The lines in HP are the sets $\{\Upsilon^{-1}(l_{\mathfrak{b}}) : b \in \mathbb{B}\}$, the circular arcs, and half lines in $\mathbb{C}_+$ that cross the real line perpendicularly. The congruences can be given as $\Upsilon^{-1} \circ B_{\mathfrak{b}} \circ \Upsilon$ $(\mathfrak{b} \in \mathbb{B})$.

3 Wavelet, Gábor, and Voice Transforms

In order to define the continuous version of the wavelet transform let us start from a basic function $\psi \in L^2(\mathbb{R})$, called mother wavelet, and use dilation and translation to obtain the collection of functions

$$\psi_{pq}(x) = \frac{\psi((x - q)/p)}{\sqrt{p}} \qquad (x \in \mathbb{R}, (p, q) \in \mathbb{L} := (0, \infty) \times \mathbb{R}).$$

By means of this kernel function we can construct an integral operator

$$(\mathscr{W}_{\psi} f)(p,q):=\frac{1}{\sqrt{p}}\int_{\mathbb{R}} f(x)\overline{\psi}((x-q)/p)\,dx = \langle f, \psi_{pq}\rangle \; ((p,q)\in\mathbb{L},\, f\in L^2(\mathbb{R}))$$

called *wavelet transform.* It is known that under general conditions made on ψ the function f can be reconstructed from its wavelet transform and the analogue of the Plancherel formula, in other words the energy conservation principle holds for it [13, 33, 41].

Similarly to the Fourier transform there can be given a group theoretical interpretation for W_{ψ} by means of the collection $\mathfrak{L}$ of affine maps

$$\ell_{\mathfrak{a}}(x) := px + q \;(x\in\mathbb{R}, \mathfrak{a}=(p,q)\in\mathbb{L}).$$

The function set $\mathfrak{L}$ is closed for composition $\circ$, it contains the identical map $\ell_{\mathfrak{e}}$ that corresponds to $\mathfrak{e} := (1,0)$. Moreover the function in $\mathfrak{L}$ corresponding to $\mathfrak{a}^{-1} := (p^{-1}, -qp^{-1}) \in \mathbb{L}$ is the inverse function of $\ell_{\mathfrak{a}} : \ell_{\mathfrak{a}^{-1}} = \ell_{\mathfrak{a}}^{-1}$. The group $(\mathfrak{L}, \circ)$ is called *affine group.* Introducing the group operation

$$\mathfrak{a}_1 \circ \mathfrak{a}_2 := (p_1 p_2, q_1 + p_1 q_2) \;\; (\mathfrak{a}_j := (p_j, q_j) \in \mathbb{L}, j=1,2)$$

on $\mathbb{L}$ we obtain the group $(\mathbb{L}, \circ)$ which is isomorphic with the affine group, and $\ell_{\mathfrak{a}} = \ell_{\mathfrak{a}_1} \circ \ell_{\mathfrak{a}_2}$. The group operations are continuous with respect to the usual topology in $\mathbb{L}$, therefore $(\mathbb{L}, \circ)$ is a (noncommutative, locally compact) topological group.

The wavelet transform can be described by the family of operators

$$U_{\mathfrak{a}}\psi := \frac{1}{\sqrt{p}}\psi \circ \ell_{\mathfrak{a}}^{-1} \qquad (\mathfrak{a}=(p,q)\in\mathbb{L}, \psi\in L^2(\mathbb{R}))$$

as

$$(\mathscr{W}_{\psi} f)(\mathfrak{a}) = \langle f, U_{\mathfrak{a}}\psi\rangle \qquad (\mathfrak{a}=(p,q)\in\mathbb{L}, f,\psi\in L^2(\mathbb{R})). \tag{5}$$

It is easy to show that the operators $U_{\mathfrak{a}} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ $(\mathfrak{a}\in\mathbb{L})$ are *unitary representations* of $(\mathbb{L}, \circ)$ on the space $L^2(\mathbb{R})$, i.e.

$$(i) \quad \|U_{\mathfrak{a}}\psi\| = \|\psi\|, \quad (ii) \quad U_{\mathfrak{a}_1}(U_{\mathfrak{a}_1}\psi) = U_{\mathfrak{a}_1\circ\mathfrak{a}_2}\psi \quad (\mathfrak{a}, \mathfrak{a}_1, \mathfrak{a}_2 \in \mathbb{L}, \psi\in L^2(\mathbb{R})).$$

Moreover the representation is continuous in the following sense: For every function $\psi \in L^2(\mathbb{R})$ we have

$$(iii) \quad \|T_{\mathfrak{a}_n}\psi - T_{\mathfrak{a}}\psi\| \to 0, \;\text{ if }\; \mathfrak{a}_n \to \mathfrak{a} \qquad (n\to\infty).$$

Taking the discrete subgroup $(\mathbb{L}_0, \circ), \mathbb{L}_0 := \{(2^{-n}, k2^{-n}) : k, n \in \mathbb{Z}\}$ instead of $(\mathbb{L}, \circ)$ we obtain, as a generalization of Haar–Fourier coefficients, the discrete version of the wavelet transform

$$(\mathscr{W}_\psi f)(2^{-n}, k2^{-n}) = \sqrt{2^n} \int_{\mathbb{R}} f(x)\overline{\psi}(2^n x - k)\, dx \qquad (k, n \in \mathbb{Z}).$$

Referring to the relation with the affine group the map $\mathscr{W}_\psi$ is usually called *affine wavelet transform.*

This model can serve as an example for the construction of useful function transformations. Instead of the affine group one may take a locally compact topological group $(\mathbb{G}, \cdot)$ and a unitary representation $V_g : H \to H$ $(g \in \mathbb{G})$ of it. Then similarly to (5)

$$(\mathscr{V}_\psi f)(g) := \langle f, V_g \psi \rangle \ \ (g \in \mathbb{G}, f, \psi \in H)$$

will be a bounded linear operator from the Hilbert space H to the space of bounded continuous functions $C(\mathbb{G})$ defined on $\mathbb{G}$. According to *Feichtinger* and *Gröchenig* the map $\mathscr{V}_\psi$ is called *voice-transform* generated by the representation $(V_g, g \in \mathbb{G})$ [20, 21]. We say that the representation is *irreducible* if it has no proper closed invariant subspace, i.e. $V_g \psi$ $(g \in \mathbb{G})$ is a closed system in H for any $\psi \in H, \psi \neq \theta$. It can be shown that if the representation is irreducible then the voice transform is injective. Let the left invariant Haar measure on the group $\mathbb{G}$ be denoted by m, and the Hilbert space generated by the measure m on $\mathbb{G}$ by $L^2_m(\mathbb{G})$. The elements $\psi \in H$ for which $\mathscr{V}_\psi(H) \subset L^2_m(\mathbb{G})$ are called *admissible elements.*

The set H_0 of admissible elements is dense in H. Moreover, $\psi \in H_0, \psi \neq \theta$ if and only if $\mathscr{V}_\psi \psi \in L^2_m(\mathbb{G})$. One can show [29, 33] that there is positive definite quadratic form $C : H_0 \to \mathbb{R}_+$ for which

$$\langle \mathscr{V}_{\psi_1} f_1, \mathscr{V}_{\psi_2} f_2 \rangle_{L^2_m(\mathbb{G})} = C(\psi_1, \psi_2)\langle f_1, f_2 \rangle_H \qquad (f_1, f_2 \in H, \psi_1, \psi_2 \in H_0).$$

This can be considered as the analogue of the Plancherel-theorem for voice transforms. In particular, if the group $\mathbb{G}$ is unimodular, i.e. every left invariant measure is right invariant as well, then there is an absolute constant C_0 for which the equality

$$\|\mathscr{V}_\psi f\|_{L^2_m(\mathbb{G})} = C_0 \|\psi\|_H \|f\|_H \qquad (f \in H, \psi \in H_0)$$

holds. Consequently, with $\|\psi\|_H = 1/C_0$ the voice transform becomes unitary. This means not only that the analogue of the Plancherel theorem holds true under a very general condition but also it explains the special form of the formula in the particular cases by enlightening the role of the group $\mathbb{G}$.

The transform introduced by *Dénes Gábor* in 1946 can be understood as a special voice transform by taking a special representation of the *Heisenberg group.* This explains that the *Gábor-transform* is also called as *Weyl–Heisenberg wavelet* *transform.* Taking the Haar measures for the affine and Heisenberg groups one can characterize the admissible functions and the analogues of the Plancherel formula can be written in explicit forms [33, 55].

4 Hyperbolic Wavelets

In view of the geometric representation of the Blaschke-functions, and their role in control and complex function theories it seemed natural to introduce the voice transform based on the Blasche group. The basic properties of the transform are proved in [42, 49–51] and applications concerning system identification are given in the papers [6–8]. The potential applications in numerical mathematics and in the theory of complex functions are presented in [22, 23, 56, 57]. The Blaschke group was shown to be identical with the group of congruences in the PD model of hyperbolic geometry, which makes it logical to call the voice transform in question as *hyperbolic wavelet transform*

4.1 Hardy and Bergman Spaces

We will introduce a one parameter collection of hyperbolic wavelet transform. To this order we take the group $(\mathfrak{B}, \circ)$, or the isomorphic $(\mathbb{B}, \circ)$ Blaschke group, a family of Hilbert spaces $\mathfrak{H}_s$ $(s \geq -1)$ and the unitary representations $T_{\mathfrak{b}}^{[s]} : \mathfrak{H}_s \to \mathfrak{H}_s$ $(\mathfrak{b} \in \mathbb{B}, s \geq -1)$ of $(\mathbb{B}, \circ)$. If $s = -1$, then the Hilbert space $\mathfrak{H}_s$ will be the Hardy space $\mathscr{H}^2(\mathbb{D})$ (or the isomorphic $\mathscr{H}^2(\mathbb{T})$ space). In case $s > -1$ we will consider the weighted Bergman space $\mathscr{B}_s^2(\mathbb{D})$. For the definition of Bergman spaces (see [32]) let us introduce the weight function $\sigma_s(z) := (1 - |z|^2)^s$ $(z \in \mathbb{D}, s > -1)$ and the area measure $d\varsigma_s(z) := (s+1)\sigma_s(z)\, dxdy/\pi$ $(z = x + iy \in \mathbb{D})$ on the disc generated by it. If $s = -1$, then $d\varsigma_s(e^{it}) = dt/2\pi$ is the Lebesgue-measure on the torus $\mathbb{T}$. Set $\mathbb{F}_s := \mathbb{D}$ if $s > -1$, and $\mathbb{F}_s := \mathbb{T}$ if $s = -1$. Let

$$\|f\|_{p,s} := \left(\int_{\mathbb{F}_s} |f(z)|^p \, d\varsigma_s(z)\right)^{1/p} \quad (0 < p < \infty)$$

denote the norm of Lebesgue space $L_s^p(\mathbb{F}_s) := L_{\varsigma_s}^p(\mathbb{F}_s)$. The closed subspace $\mathscr{B}_s^p(\mathbb{F}_s) := L_s^p(\mathbb{D}) \cap \mathscr{A}, s > -1$ of $L_s^p(\mathbb{F}_s)$ is called weighted Bergman space. In the special case $s = -1$ let $\mathscr{B}_s^p(\mathbb{F}_s)$ be the Hardy space $\mathscr{H}^p(\mathbb{T})$. Then for every value of the parameter $s \geq -1$ we have that $\mathfrak{H}_s := \mathscr{B}_s^2(\mathbb{F}_s)$ is a Hilbert space with the scalar product

$$\langle f, g\rangle_s := \int_{\mathbb{F}_s} f(z)\overline{g}(z)\, d\varsigma_s(z) \qquad (f, g \in \mathfrak{H}_s),$$

and the subspace of complex algebraic polynomials is dense in $\mathscr{B}_s^p(\mathbb{F}_s)$ $(0 < p < \infty)$. If $s > -1$, then for the $\|\cdot\|_s$ norm of the function $A(z) := \sum_{n\in\mathbb{N}} a_n z^n$ $(z \in \mathbb{D})$ we have

$$\|A\|_s^2 = (s+1)\sum_{n=0}^{\infty} |a_n|^2 \int_0^1 r^{2n+1}(1-|r|^2)^s \, dr = \sum_{n=0}^{\infty} |a_n|^2 \lambda_n^{[s]} < \infty,$$

$$\lambda_n^{[s]} := (s+1)\int_0^1 r^{2n+1}(1-|r|^2)^s = \frac{n!\Gamma(2+s)}{\Gamma(2+s+n)} \ (s \geq -1, n \in \mathbb{N}).$$

Let $\mathfrak{h}_s$ $(s \geq -1)$ denote the set of complex sequences $a = (a_n, n \in \mathbb{N})$ for which

$$\|a\|_s := \left(\sum_{n=0}^{\infty} |a_n|^2 \lambda_n^{[s]}\right)^{1/2} < \infty.$$

Then $\mathfrak{h}_s$ is a Hilbert space with the scalar product

$$[a,b]_s := \sum_{n=0}^{\infty} a_n \overline{b}_n \lambda_n^{[s]}$$

and the map $a \to A$ is an isomorphism between $\mathfrak{h}_s$ and $\mathfrak{H}_s$, provided $s > -1$. Since $\lambda_n^{[-1]} = 1$ $(n \in \mathbb{N})$ and $\mathfrak{h}_{-1} = \ell^2, \mathfrak{H}_{-1} = \mathscr{H}^2(\mathbb{T})$ we have that the isomorphism

$$\|A\|_s = \|a\|_s \ \ (a \in \mathfrak{h}_s)$$

holds for $s = -1$ as well. Moreover, $\mathfrak{H}_{-1} \subset \mathfrak{H}_{s_1} \subset \mathfrak{H}_{s_2}$ $(-1 < s_1 < s_2)$. The power functions

$$e_n^{[s]}(z) := z^n/\sqrt{\lambda_n^{[s]}} \ \ (z \in \mathbb{F}_s, n \in \mathbb{N}, s \geq -1)$$

form an orthogonal basis in $\mathfrak{H}_s$ if $s \geq -1$ [15, 32].

4.2 Invariant Integrals, Haar Measure

The space $L^1_{-2}(\mathbb{D})$ enjoy the following invariance properties:

$$\int_{\mathbb{D}} f(w)\sigma_{-2}(w)\, du dv = \int_{\mathbb{D}} f(B_{\mathfrak{b}}(z))\sigma_{-2}(z)\, dx dy \qquad (\mathfrak{b} \in \mathbb{B}, f \in L^1_{-2}(\mathbb{D})). \tag{6}$$

Indeed, for the Jacobi determinant of the bijection $B_{\mathfrak{b}} : \mathbb{D} \to \mathbb{D}$ we have (see [32])

$$|B_{\mathfrak{b}}'(z)|^2 = \frac{(1-|b|^2)^2}{|1-\overline{b}z|^4} = \frac{(1-|B_{\mathfrak{b}}(z)|^2)^2}{(1-|z|^2)^2} = \frac{\sigma_{-2}(z)}{\sigma_{-2}(B_{\mathfrak{b}}(z))} \qquad (z \in \mathbb{D})$$

therefore (6) follows by applying the integral transform $w = B_{\mathfrak{b}}(z)$.

Similar reasoning yields that $L^1_{-1}(\mathbb{I})$ is invariant with respect to the transforms in $\mathfrak{B}_0$:

$$\int_{\mathbb{I}} f(t)\sigma_{-1}(t)\,dt = \int_{\mathbb{I}} f(B_b(t))\sigma_{-1}(t)\,dt \qquad (B_{\mathfrak{b}} := B_{(b,1)} \in \mathfrak{B}_0). \tag{7}$$

(6) implies that the integral

$$\int_{\mathbb{B}} f(\mathfrak{z})\,dm(\mathfrak{z}) := \int_0^{2\pi}\int_{\mathbb{D}} f(z,e^{it})\sigma_{-2}(z)\,dxdydt$$

defined on the group $\mathbb{B}$ is invariant with respect to the transform $\mathfrak{z} \to \mathfrak{z} \circ \mathfrak{b}$. Consequently, m is a *right invariant Haar measure* of $(\mathbb{B}, \circ)$. Indeed, by (2) we have

$$\mathfrak{z} \circ \mathfrak{b} = (B_{\mathfrak{b}^{-1}}(z), e^{it}\eta_{\mathfrak{b}^{-1}}(z)),\ \eta_{\mathfrak{b}^{-1}}(z) \in \mathbb{T} \qquad (\mathfrak{z} = (z, e^{it}),\ \mathfrak{b} \in \mathbb{B}).$$

Then by Fubini's theorem and by (7) we obtain

$$\begin{aligned}
\int_{\mathbb{B}} f(\mathfrak{z}\circ\mathfrak{a})\,dm(\mathfrak{z}) &= \int_0^{2\pi}\int_{\mathbb{D}} f(B_{\mathfrak{b}^{-1}}(z), e^{it}\eta_{\mathfrak{b}^{-1}}(z))\sigma_{-2}(z)\,dxdydt \\
&= \int_0^{2\pi}\int_{\mathbb{D}} f(B_{\mathfrak{b}^{-1}}(z), e^{it})\sigma_{-2}(z)\,dxdydt \\
&= \int_0^{2\pi}\int_{\mathbb{D}} f(z, e^{it})\sigma_{-2}(z)\,dxdydt = \int_{\mathbb{B}} f(\mathfrak{z})\,dm(\mathfrak{z}).
\end{aligned}$$

Similarly, it follows from $\mathfrak{z}^{-1} = (-e^{it}z, e^{-it})$ $(\mathfrak{z} = (z, e^{it}) \in \mathbb{B})$ that the measure m is invariant with respect to the transform $\mathfrak{z} \to \mathfrak{z}^{-1}$, and so *the Haar measure m is* unimodular.

4.3 Unitary Representations

In order to define unitary representations that generate hyperbolic wavelets we start from the linear space of Lebesgue measurable functions $\mathscr{M}(\mathbb{F}_s)$ defined on the torus $\mathbb{F}_s := \mathbb{T}$ if $s = -1$, and on the disc $\mathbb{F}_s := \mathbb{D}$ if $s > -1$, and take the maps of the form

$$T_{\mathfrak{b}} f := f \circ B_{\mathfrak{b}}^{-1} \qquad (f \in \mathscr{M}(\mathbb{F}_s),\ \mathfrak{b} \in \mathbb{B}).$$

Since

$$T_{\mathfrak{b}_1\circ\mathfrak{b}_2}f = f\circ B^{-1}_{\mathfrak{b}_1\circ\mathfrak{b}_2} = f\circ(B^{-1}_{\mathfrak{b}_2}\circ B^{-1}_{\mathfrak{b}_1}) = T_{\mathfrak{b}_1}(f\circ B^{-1}_{\mathfrak{b}_2}) = T_{\mathfrak{b}_1}(T_{\mathfrak{b}_2}f)\ (f\in\mathscr{M}(\mathbb{F}_s)),$$

we have that $(T_{\mathfrak{b}}, \mathfrak{b}\in\mathbb{B})$ is the so-called *right regular representation* of the group $(\mathfrak{B},\circ)$.

For the construction of unitary representations we will apply *multiplier representations* [42, 49–51, 65]. Let $M_{\mathfrak{b}}\in\mathscr{M}(\mathbb{F}_s)$ $(\mathfrak{b}\in\mathbb{B})$ be a collection of functions. It is easy to show that the family of operators

$$R_{\mathfrak{b}}f := M_{\mathfrak{b}^{-1}}\cdot f\circ B_{\mathfrak{b}^{-1}}\ \ (\mathfrak{b}\in\mathbb{B}, f\in\mathscr{M}(\mathbb{F}_s))$$

is a homomorphism if and only if

$$M_{\mathfrak{e}} = 1,\ M_{\mathfrak{b}_1\circ\mathfrak{b}_2} = M_{\mathfrak{b}_2}\cdot M_{\mathfrak{b}_1}\circ B_{\mathfrak{b}_2}\qquad(\mathfrak{b}_1,\mathfrak{b}_2\in\mathbb{B}).\tag{8}$$

The set of functions $(M_{\mathfrak{b}}, \mathfrak{b}\in\mathbb{B})$ that satisfy condition (8) is called *multiplier class*, and the homomorphism $(R_{\mathfrak{b}}, b\in\mathbb{B})$ generated by them is called *multiplier representation.*

Simple calculation yields that the collection of functions

$$M_{\mathfrak{b}}(z) := \epsilon M_b(z),\ M_b(z) := \frac{1-|b|^2}{(1-\bar{b}z)^2}\qquad(\mathfrak{b}=(b,\epsilon)\in\mathbb{B}, z\in\mathbb{F}_s)$$

is a multiplier class. Since

$$M_{\mathfrak{b}}(z) = B'_{\mathfrak{b}}(z)\ (z\in\mathbb{D}),\ ie^{it}M_{\mathfrak{b}}(e^{it}) = \frac{d}{dt}B_{\mathfrak{b}}(e^{it})\ (t\in\mathbb{R}),\tag{9}$$

we have by the law of differentiation of composite functions that (8) holds on $\mathbb{F}_s$:

$$M_{\mathfrak{b}_1\circ\mathfrak{b}_2} = B'_{\mathfrak{b}_1\circ\mathfrak{b}_2} = B'_{\mathfrak{b}_2}\cdot B'_{\mathfrak{b}_1}\circ B_{\mathfrak{b}_2} = M_{\mathfrak{b}_2}\cdot M_{\mathfrak{b}_1}\circ B_{\mathfrak{b}_2}.\tag{10}$$

Along with the functions $M_{\mathfrak{b}}$ $(\mathfrak{b}\in\mathbb{B})$ also their powers satisfy condition (8). Thus

$$T^{[s]}_{\mathfrak{b}}f := M^{s/2+1}_{\mathfrak{b}^{-1}}\cdot f\circ B_{\mathfrak{b}^{-1}}\ \ (s\geq -1, \mathfrak{b}\in\mathbb{B})$$

is a multiplier representation of the group $(\mathbb{B},\circ)$ on the space $\mathscr{M}(\mathbb{F}_s)$. We will show that $T^{[s]}_{\mathfrak{b}}$ is unitary on the Hilbert space $L^2_s(\mathbb{F}_s)$. Let us first take the case $s>-1$ and use the substitution $w = B_{\mathfrak{b}}(z)$. By $|B'_{\mathfrak{b}}(z)| = |M_{\mathfrak{b}}(z)|$ and (9), (10) we have

$$\frac{\sigma_s\circ B_{\mathfrak{b}}}{\sigma_s} = |M_{\mathfrak{b}}|^s,\ M_{b^{-1}}(B_{\mathfrak{b}}(z))M_{\mathfrak{b}}(z) = M_{\mathfrak{b}^{-1}\circ\mathfrak{b}} = 1.$$

Then

$$\begin{aligned}\|T_{\mathfrak{b}} f\|_{2,s}^2 &= \int_{\mathbb{D}} |M_{\mathfrak{b}^{-1}}(w)|^{s+2} |f(B_{\mathfrak{b}}^{-1}(w))|^2 \sigma_s(w)\, du dv \\ &= \int_{\mathbb{D}} |M_{\mathfrak{b}^{-1}}(B_{\mathfrak{b}}(z))|^{s+2} |f(z)|^2 \sigma_s(B_{\mathfrak{b}}(z)) |B_{\mathfrak{b}}'(z)|^2\, dy dy \\ &= \int_{\mathbb{D}} |M_{\mathfrak{b}^{-1}}(B_{\mathfrak{b}}(z))|^{s+2} |M_{\mathfrak{b}}(z)|^{s+2} |f(z)|^2 \sigma_s(z)\, dy dy = \|f\|_{2,s}^2.\end{aligned}$$

In case $s = -1$ we have that

$$\left(T_{\mathfrak{b}^{-1}}^{[-1]} f\right)(z) := M_{\mathfrak{b}}^{1/2}(z) f(B_{\mathfrak{b}}(z)) \ \ (z \in \mathbb{T},\, f \in L^2(\mathbb{T}))$$

is a unitary representation of $(\mathbb{B}, \circ)$ on the Hilbert space $L^2(\mathbb{T})$. Indeed, it follows from (3) and (4) that if $b = re^{i\varphi}$ then

$$|M_{\mathfrak{b}}(e^{it})| = \frac{1 - r^2}{1 - 2r\cos(t - \varphi) + r^2} = \beta_b'(t).$$

Therefore

$$\begin{aligned}\int_0^{2\pi} |(T_{\mathfrak{b}^{-1}}^{[-1]} f)(e^{it})|^2\, dt &= \int_0^{2\pi} |M_{\mathfrak{b}}(e^{it})| |f(e^{i\beta_{\mathfrak{b}}(t)})|^2\, dt \\ &= \int_0^{2\pi} \beta_{\mathfrak{b}}'(t) |f(e^{i\beta_{\mathfrak{b}}(t)})|^2\, dt = \int_0^{2\pi} |f(e^{is})|^2\, ds.\end{aligned}$$

It is known that if $s \geq -1$ then the space $\mathfrak{H}_s$ is a closed subspace of the Hilbert space $L_s^2(\mathbb{F}_s)$ (see [32]). Consequently, the representation $(T_{\mathfrak{b}}^{[s]}, b \in \mathbb{B})$ is unitary also on this space.

By

$$\|T_{\mathfrak{b}}^{[s]} f - T_{\mathfrak{a}}^{[s]} f\|_{2,s} := \|T_{\mathfrak{a}^{-1} \circ \mathfrak{b}}^{[s]} f - f\|_{2,s} \qquad (f \in \mathfrak{H}_s)$$

we have that continuity is equivalent to the condition that the convergence

$$\|T_{\mathfrak{c}}^{[s]} e_n^{[s]} - e_n^{[s]}\|_{2,s} \to 0 \ (n \in \mathbb{N}, \mathfrak{c} \to \mathfrak{e}).$$

holds for the orthonormed basis of the space $\mathfrak{H}_s$ if $\mathfrak{c} \to \mathfrak{e}$. This convergence is easy to show.

4.4 The Hyperbolic Wavelet Transform

The voice transform

$$(\mathscr{T}_{\psi,s} f)(\mathfrak{b}) := \langle f, T_{\mathfrak{b}}^{[s]} \psi \rangle_s \qquad (\mathfrak{b} \in \mathbb{B},\, f, \psi \in \mathfrak{H}_s)$$

generated by the unitary representations $T_{\mathfrak{b}}^{[s]} : \mathfrak{H}_s \to \mathfrak{H}_s$ $(s \geq -1)$ of the Blaschke group $(\mathbb{B}, \circ)$ is called *hyperbolic wavelet transform* (HWT), and the functions

$$T_{\mathfrak{b}}^{[s]}\psi \quad (\psi \in \mathfrak{H}_s, \mathfrak{b} \in \mathbb{B}, s \geq -1)$$

themselves are called *hyperbolic wavelets* (HW).

Since the unitary operator $T_{\mathfrak{b}}^{[s]}$ is a bijection of $\mathfrak{H}_s$ onto itself we have that it takes any orthonormal basis in $\mathfrak{H}_s$ into an orthonormal basis. In particular the systems

$$L_n^{[b,s]} := T_{\mathfrak{b}^{-1}}^{[s]} e_n^{[s]} = M_{\mathfrak{b}}^{s/2+1} \cdot e_n^{[s]} \circ B_{\mathfrak{b}} \qquad (\mathfrak{b} = (b, 1) \in \mathbb{B})$$

form an orthonormal basis of the Hilbert space $\mathfrak{H}_s$ for any value of the parameter $b \in \mathbb{D}$. Hence in case $s = -1$ we obtain the discrete Laguerre orthonormal system, used widely in control theory, on $H^2(\mathbb{T}) : L_n^{[b,-1]} = L_n^b$ $(b \in \mathbb{D})$.

In the papers [42, 50, 51] we presented the matrix of the representation $T_{\mathfrak{b}}^{[s]}$: $\mathfrak{H}_s \to \mathfrak{H}_s$ in the basis $e_n^{[s]}$ $(n \in \mathbb{N})$:

$$t_{mn}^{[s]}(\mathfrak{b}) := \langle T_{\mathfrak{b}}^{[s]} e_n^{[s]}, e_m^{[s]} \rangle_s = \langle e_n^{[s]}, T_{\mathfrak{b}^{-1}}^{[s]} e_m^{[s]} \rangle_s \qquad (m, n \in \mathbb{N}, s \geq -1).$$

The elements of this matrix can be expressed by the Jacobi polynomials, in special case by the Zernike polynomials. Using this relation new formulas can be derived for these polynomials. This way, for instance, we showed the *addition formulas for the Zernike functions*, which play fundamental role in optics.

The matrix of this representation is diagonal on the subgroup $\mathbb{B}_{\mathbb{T}} := \{(0, \epsilon) : \epsilon \in \mathbb{T}\}$. Indeed, since $T_{(0,\epsilon)}^{[s]} e_n^{[s]} = \epsilon^{-(s/2+n+1)} e_n^{[s]}$ we have

$$t_{mn}^{[s]}((0, \epsilon)) = \delta_{mn} \epsilon^{-(s/2+n+1)} \qquad (m, n \in \mathbb{N}).$$

Using the decomposition $\mathfrak{b} := (re^{i\varphi}, e^{i\theta}) = (0, e^{i(\varphi+\theta)}) \circ (r, 1) \circ (0, e^{-i\varphi}) =: (0, \epsilon_1) \circ (r, 1) \circ (0, \epsilon_2)$ the representation in question can be written in the form $T_{\mathfrak{b}}^{[s]} = T_{(0,\epsilon_1)}^{[s]} \circ T_{(r,1)}^{[s]} \circ T_{(0,\epsilon_2)}^{[s]}$, and the elements $t_{mn}^{[s]}(r, 1)$ can be expressed by the Jacobi polynomials. If $s = -1$, then $t_{mn}(re^{i\varphi}) := t_{mn}^{[-1]}(re^{i\varphi}, 1)$ can be expressed by the Zernike functions $Y_n^{\ell}(r, \varphi)$:

$$t_{mn}(re^{i\varphi}) = \frac{(-1)^m \sqrt{1 - r^2}}{\sqrt{n + m + 1}} Y_{\min\{m,n\}}^{|m-n|}(r, \varphi).$$

The property $T^{[s]}(\mathfrak{b}_1 \circ \mathfrak{b}_2) = T^{[s]}(\mathfrak{b}_1) T^{[s]}(\mathfrak{b}_2)$ is equivalent to the identity

$$t_{mn}^{[s]}(\mathfrak{b}_1 \circ \mathfrak{b}_2) = \sum_{k \in \mathbb{N}} t_{mk}^{[s]}(\mathfrak{b}_1) t_{kn}^{[s]}(\mathfrak{b}_2) \qquad (m, n \in \mathbb{N}).$$

From this identity we can derive the so-called *addition formulas for the Jacobi polynomials and for the Zernike functions.*

It can be shown that the representation $T^{[s]}$ is irreducible on the space $\mathfrak{H}_s$ $(s \geq -1)$. Consequently, the hyperbolic wavelet transform $\mathscr{T}_{\psi,s}$ generated by it is injective.

In the papers [42, 50, 51] we investigated the admissible elements of $\mathfrak{H}_s$ and gave the analogues of the Plancherel formula for hyperbolic wavelet transform. For instance, for the case $s = 0$, i.e. for the Bergman space $\mathfrak{B}_0(\mathbb{D}) = \mathscr{B}^2(\mathbb{D})$ we proved that every $\psi \in \mathscr{B}^2(\mathbb{D})$ is admissible and

$$\|\mathscr{T}_\psi^{[0]} f\|_{L^2_m(\mathbb{B})} = c\|\psi\|_{\mathscr{B}^2(\mathbb{D})}\|f\|_{\mathscr{B}^2(\mathbb{D})},$$

where c is an absolute constant, and m is the Haar measure of the group $(\mathbb{B}, \circ)$.

In the papers [20, 21] atoms and atomic decompositions are constructed for a wide class of Banach spaces by discretizing the voice transform of general locally compact groups induced by square integrable and integrable representations. Applying these general results atomic decompositions are constructed for a wide class of Bergman spaces by means of HWT in [23, 44]. In [22, 43, 45] the problem of discretization of HWT is addressed based on the discrete subgroups of $\mathbb{B}$. Among others MRA decompositions and construction of orthogonal bases are considered.

4.5 Poles, Eigenvalues, Identification

The hyperbolic wavelet transform can be applied for determining the poles of rational functions, the eigenvalues of matrices and for system identification [6–8, 56, 57]. To this order let us start from the hyperbolic wavelet transform (discrete Laguerre coefficients) $\mathscr{T}_n := \mathscr{T}_{e_n}^{[-1]}$ generated by the basis functions $e_n(z) := e_n^{[-1]}(z) := z^n$ $(n \in \mathbb{N}, z \in \mathbb{C})$ of the Hardy space $H^2(\mathbb{T})$:

$$(\mathscr{T}_n f)(b) := \langle f, T_{(b,1)}^{[-1]} e_n\rangle = \langle f, L_n^b\rangle \ (n \in \mathbb{N}, b \in \mathbb{D}, \langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{-1}).$$

We will define a map, that will be useful in several respects, which takes the elements of $\mathscr{H}^2(\mathbb{D})$ to sequences of functions $\mathscr{T} f := (\mathscr{T}_n f, n \in \mathbb{N})$ analytic on $\mathbb{D}$. For example, the ℓ^2norm of the sequence is a constant function on the disc: $\|(\mathscr{T} f)(b)\|_{\ell^2} = \|f\|_{H^2(\mathbb{D})}$ $(f \in \mathscr{H}^2(\mathbb{D}), b \in \mathbb{D})$. We showed in [56] that the nonlinear sequence of functionals

$$(Q_n f)(b) := \frac{(\mathscr{T}_{n+1} f)(b)}{(\mathscr{T}_n f)(b)} \ \ (n \in \mathbb{N}).$$

can be used for calculating the poles of rational functions. To the description of the set of convergence we will use the PD model and the pseudohyperbolic metric ρ. Namely, let f be a proper rational function whose poles lie outside the closed disc

$\overline{\mathbb{D}}$. Let $a_i \in \mathbb{D}$ $(i = 1, 2, \ldots, N)$ denote the inverse poles of f, i.e. $a_i^* := 1/\overline{a}_i$ $(i = 1, \ldots, N)$ are the poles of f. It can be shown that except for the points of the perpendicular bisector of the hyperbolic segment

$$D_{ij} := \{b \in \mathbb{D} : \rho(b, a_i) = \rho(b, a_j)\},$$

the sequence $((Q_n f)(b), n \in \mathbb{N})$ converges for every $b \in \mathbb{D}$ and the limits form a subset of $\mathbb{D}$ with at most N elements. Indeed, if

$$D_0 := \bigcup_{1 \leq i < j \leq N} D_{ij}\ , \quad D_i := \{b \in \mathbb{D} : \rho(b, a_i) > \max_{1 \leq j \leq n, j \neq i} \rho(b, a_j)\},$$

then the sets D_i $(1 \leq i \leq N)$ are pairwise disjoint and $\mathbb{D} = \cup_{0 \leq i \leq N} D_i$. It can be proved that for every $b \in D_i$ the sequence $((Q_n f)(b), n \in \mathbb{N})$ converges to the same point in $b_i \in \mathbb{D}$ and so a_i is easy to reconstruct:

$$b_i = (Qf)(b) := \lim_{n \to \infty} (Q_n f)(b), \;\; B_b^{-1}(\overline{b}_i) = a_i \qquad (b \in D_i, 1 \leq i \leq N).$$

For poles with multiplicity one the numbers

$$q_i(b) := \max_{1 \leq j \leq N, j \neq i} \rho(b, a_j)/\rho(b, a_i) \qquad (b \in D_i, 1 \leq i \leq N)$$

can be used also for measuring the rate of convergence:

$$|b_i - (Q_n f)(b)| = O(q_i^n(b)) \qquad (b \in D_i, n \to \infty).$$

In case of poles with multiplicity higher than one, the rate of convergence is $|b_i - (Q_n f)(b)| = O(1/n)$.

We note that depending on the locations of the inverse poles it may happen that D_i is empty for some i. In that case the pole a_i^* is called a *hidden pole*. By means of the nonlinear operator

$$(Sf)(b) := (B_b^{-1}(\overline{Q} f))(b) \qquad (b \in \mathbb{B} \setminus D_0)$$

we can obtain all of the poles, except for the hidden ones, of a rational function f. Separating these poles and repeating the process we can get every pole.

We construct a similar algorithm for calculating the eigenvalues of matrices [57]. Let us suppose that the eigenvalues $\lambda_1, \ldots, \lambda_N$ of the matrix $A \in \mathbb{C}^{N \times N}$ lie in $\mathbb{D}$. Starting from an arbitrary vector $x_0 \in \mathbb{C}^N$ and using the so-called Mises iteration we calculate the sequence

$$x_{n+1} = Ax_n \in \mathbb{C}^N \qquad (n \in \mathbb{N}).$$

This recursion can also be considered as special discrete time invariant system. The algorithm will be presented in this special case, noting that the method can be extended to any discrete time invariant system. Let

$$F(z) := \sum_{n=0}^{\infty} x_n z^n \ (z \in \mathbb{D})$$

stand for the transfer function of the system. The function $F : \mathbb{D} \to \mathbb{C}^N$ is analytic and

$$F(z) - x_0 = \sum_{n=0}^{\infty} x_{n+1} z^n = A \left(\sum_{n=0}^{\infty} x_n z^n \right) = AF(z).$$

Hence we have that the transfer function can be written as

$$F(z) = (I - zA)^{-1} x_0 \qquad (z \in \mathbb{D}).$$

F is an analytic rational function on the closed disc $\overline{\mathbb{D}}$, which can be expressed by the minimal polynomial

$$p_A(z) := \prod_{j=1}^{m} (z - \lambda_j)^{\nu_j} \qquad (z \in \mathbb{C}, m \le N, \nu_1 + \cdots + \nu_m \le N)$$

of the matrix A :

$$F(z) = \sum_{j=1}^{m} \sum_{k=0}^{\nu_j - 1} \frac{z^k}{(1 - \lambda_j z)^{k+1}} h_{ij}(A) x_0.$$

Here the h_{ij}'s are the Hermite type base polynomials generated by the zeros of the minimal polynomial p_A. Using the method given above for coordinate functions F_j of F we obtain the non-hidden inverse poles $a_i = \lambda_i$, or eigenvalues.

5 Discrete Orthonormal Systems

In practical applications of orthogonal expansion we always use a discrete version of the system. This means that instead of the original continuous system $\varphi_n : I \to \mathbb{C}$ $(n \in \mathbb{N})$ we take the restriction of the first N members onto a subset $I_N \subset I$ with N elements. The discretization process produces a suitable discrete system if for a proper discrete scalar product of the form

$$[f,g]_N := \sum_{t\in I_N} f(t)\overline{g}(t)\nu_N(t) \qquad (\nu_N(t)>0), \tag{11}$$

the discrete system will be orthonormal, i.e.

$$[\varphi_n,\varphi_m]_N = \delta_{mn} \qquad (0\le m,n<N). \tag{12}$$

Such process is called *orthogonal discretization.* In this case the Nth partial sum of the Fourier expansion will interpolate the function in the pint of I_N. For the trigonometric case the discrete system is generated by equidistant partitions.

In the orthogonalization of the Malmquist–Takenaka systems we will make use the fact that they can be originated from the trigonometric system by means of an argument transform:

$$B_b(e^{it}) = e^{i\beta_b(t)}, \beta_b(t) = \varphi + \gamma_r(t-\varphi)\ (t\in\mathbb{R}, b = re^{i\varphi}\in\mathbb{D}),$$

where γ_r is the integral function of the Poisson kernel (see (4)). Then it follows that the Blascke products in the MT functions can be written in the form

$$\prod_{k=0}^{N-1} B_{b_k}(e^{it}) = e^{iN\theta_N(t)},\ \theta_N(t) = \frac{1}{N}\sum_{k=0}^{N-1}\beta_{b_k}(t),$$

where $\theta_N : \mathbb{R}\to\mathbb{R}$ is a strictly increasing function for which $\theta_N(t+2\pi) = \theta_N(t)+2\pi$ $(t\in\mathbb{R})$. The Dirichlet kernel functions of the MT systems, similarly to the trigonometric case, can be given in a closed form [43]:

$$\sum_{j=0}^{N-1}\Phi_j(z)\overline{\Phi}_j(\zeta) = \frac{\prod_{k=0}^{N-1} B_{b_k}(z)\overline{B}_{b_k}(\zeta)-1}{z\overline{\zeta}-1}\ \ (z,\zeta\in\overline{\mathbb{D}}).$$

Hence

$$\sum_{j=0}^{N-1}\Phi_j(z_k)\overline{\Phi}_j(z_\ell) = \delta_{k\ell}\Lambda_N(z_k),\ \Lambda_N(z) = \sum_{j=0}^{N-1}\frac{1-|b_j|^2}{|1-\overline{b}_j z|^2} \qquad (0\le k,\ell<N)$$

in the points of the set

$$\mathbb{T}_N := \{z_k := e^{i\tau_k} : \tau_k := \theta_N^{-1}(t_0+2k\pi/N), 0\le k<N\}.$$

Equivalently, $\alpha_{jk} := \Phi_j(z_k)/\sqrt{\Lambda_N(z_k)}$ $(0\le j,k<N)$ is orthogonal. Consequently,

$$\sum_{k=0}^{N-1} \alpha_{rk}\overline{\alpha}_{sk} = \sum_{k=0}^{N-1} \Phi_r(z_k)\overline{\Phi}_s(z_k)/\Lambda_N(z_k) = \delta_{rs}.$$

Thus we obtain an orthogonal discretization of the MT systems of the form (11) on the set $I_N := \mathbb{T}_N$ and with the weight function $\nu_N = 1/\Lambda_N$.

We note that by means of the Christoffel–Darboux formula the orthogonal discretization of the polynomial system P_n^ν $(n \in \mathbb{N})$ with respect to the weight function $\nu : I \to (0,\infty)$ is obtained in a similar way. Namely, taking the set of the zeros of P_N^ν as I_N the functions $\varphi_n = P_n^\nu$ $(0 \le n < N)$ satisfy the discrete orthogonality (12), with $\nu_N(t)$ $(t \in I_N)$ being the Christoffel–Darboux numbers in this case [61]. We used these ideas for the orthogonal discretization of the Zernike functions, and for other discrete orthogonal systems, and also for the construction of approximation, interpolation processes [16, 47, 48]. We found that these results can be effectively applied for the mathematical representation of the human cornea and for the approximation of functions defined on the surface of a ball [17–19, 48].

The MT systems share a number of properties with the orthogonal polynomials. The roots of the classical orthogonal polynomials are related to electrostatic equilibrium [61]. Similar interpretation can be given for the discretization set $\mathbb{T}_N$ [46]. Namely, setting

$$\omega_1(z) := \prod_{j=0}^{N-1}(z-b_j),\ \omega_2(z) := \prod_{j=0}^{N-1}(1-\overline{b}_j z),$$
$$\omega(z) := \omega_1'(z)\omega_2(z) - \omega_2'(z)\omega_1(z) \qquad (z \in \mathbb{C})$$

we have that if $\lambda \in \mathbb{C}$ is a root of the polynomial ω of order $2(N-1)$ then this holds also for $\lambda^* := 1/\overline{\lambda}$, the mirror image of λ with respect to $\mathbb{T}$, with the same multiplicity. Let $\lambda_k \in \mathbb{D}$ $(k = 1,\ldots,s)$ denote the pairwise distinct roots of ω in the set $\mathbb{D}$, and let m_k be the multiplicity of λ_k. Then the following equilibrium condition holds true:

$$\sum_{k=1,k\neq n}^{N} \frac{1}{z_n - z_k} = \frac{1}{2}\sum_{j=1}^{s}\left(\frac{m_j}{z_n-\lambda_j} + \frac{m_j}{z_n-\lambda_j^*}\right) \qquad (n = 1,2,\ldots,N).$$

In particular, if $b_0 = \cdots = b_{N-1} = b$, then b and b^* are zeros of ω with multiplicity $(N-1)$, and the equilibrium equation is:

$$\sum_{k=1,k\neq n}^{N} \frac{1}{z_n - z_k} = \frac{N-1}{2}\left(\frac{1}{z_n-b} + \frac{1}{z_n-b^*}\right) \qquad (n = 1,2,\ldots,N). \quad (13)$$

The last two equations can be understood as electrostatic equilibrium conditions. The two-dimensional vector

$$F_{nk} = \frac{1}{\overline{z}_n - \overline{z}_k} = \frac{1}{|z_n - z_k|} \frac{z_n - z_k}{|z_n - z_k|}$$

is the force between two unit charges with same polarity, where the Coulomb force is reciprocally proportional with the distance. For the interpretation of the second equation let us place N unit charges along the unit circle that can freely move and fix two charges of value $(N-1)/2$ at the points b and b^*. The forces generated by the fixed charges are called outer forces while those generated by the moving charges are called inner forces. Equation (13) shows that for any particle z_n the sum of the outer forces acting on it is equal to the sum of the inner forces.

We have used the discrete version of the MT systems in system identification, and in approximation and compression of ECG signals. In connection with system identification we mention two results which are related to summation of expansions with respect to periodic MT systems (especially Laguerre, and Kautz series). It may occur that these expansions, similarly to the situation in the trigonometric case, do not converge even for continuous functions. In order to fix this problem we showed that for a wide collection of summation processes, so-called θ summations, the θ means of continuous functions with respect to periodic MT systems converge uniformly. We proved similar results for discrete periodic MT systems as well [3–5].

In case $b_0 = 0$ the MT system, similarly to the trigonometric system, can be extended to an orthonormal system on $L^2(\mathbb{T})$ by adding the functions $\Phi_{-n}(z) = \overline{\Phi_n}(z)$ $(z \in \mathbb{T}, n \in \mathbb{N})$. Then both the real $\Re\Phi_n$ and the imaginary $\Im\Phi_n$ $(n \in \mathbb{N})$ systems are orthogonal in $L^2(\mathbb{T})$. We used 2π periodic functions to model the ECG curves. The typical segments, like the QRS complex, of it are of similar shape as the linear combinations of real and imaginary parts of the basic functions

$$r_{b,j}(z) := \frac{1}{(1 - \overline{b}z)^j} \qquad (b \in \mathbb{D}, j = 1, 2, \ldots, z \in \mathbb{T})$$

that generate the MT systems. This observation was the motive behind the representation of the ECG signals in MT bases rather than in trigonometric or wavelet ones [24–27]. We have worked out several algorithms for finding the optimal parameters $(b_j, j \in \mathbb{N})$ for the discrete orthogonal real MT systems. Based on our experiments we found it advantageous to use three parameters, i.e. inverse poles $b_1, b_2, b_3 \in \mathbb{D}$ and repeating them periodically. Then the approximation of an ECG signal is performed in two steps: First the distance

$$\mathscr{L}_s := \text{span}\{\Re r_{b_j,k}, \Im r_{b_j,k} : 1 \le j \le 3, 1 \le k \le s\}$$

of f from the subspace

$$\text{dist}(b_1, b_2, b_3) := \min_{g \in \mathscr{L}_r} \|f - g\|$$

is calculated, then it is minimalized with respect to the parameters b_1, b_2, b_3 [35–38].

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One Hundred Years Uniform Distribution Modulo One and Recent Applications to Riemann's Zeta-Function

Jörn Steuding

Dedicated to the Memory of Professor Wolfgang Schwarz

Abstract We start with a brief account of the theory of uniform distribution modulo one founded by Weyl and others around 100 years ago (which is neither supposed to be complete nor historically depleting the topic). We present a few classical implications to diophantine approximation. However, our main focus is on applications to the Riemann zeta-function. Following Rademacher and Hlawka, we show that the ordinates of the nontrivial zeros of the zeta-function $\zeta(s)$ are uniformly distributed modulo one. We conclude with recent investigations concerning the distribution of the roots of the equation $\zeta(s) = a$, where a is any complex number, and further questions about such uniformly distributed sequences.

Keywords Riemann zeta-function • Zeros • a-points • Uniform distribution modulo one

1 Dense Sequences and Classical Diophantine Approximation

There are several opportunities to motivate uniform distribution modulo one. We start with a remarkable observation from the Middle Ages due to the French mathematician Nicole Oresme. In his work *De proportionibus proportionum* from around 1360 he wrote that

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> *it is probable that two proposed unknown ratios are incommensurable because if many unknown ratios are proposed it is most probable that any [one] would be incommensurable to any [other].*[1]

In another work entitled *Tractatus de commensurabilitate vel incommensurabilitate motuum cell* from this time Oresme considered two bodies moving on a circle with uniform but incommensurable velocities; here, he claimed that

> *no sector of a circle is so small that two such bodies could not conjunct in it at some future time, and could not have conjuncted in it sometime [in the past].*[2]

These sentences form part of Oresme's refutation of astrology. He considered the future as essentially unpredictable, a modern viewpoint which was pretty controversial to the standards of his contemporaries. The above quotations indicate a deep understanding of irrationality and circle rotations. In modern mathematical language Oresme's observation is that rational numbers form a negligible set (of Lebesgue measure zero) and that the multiples of an irrational number lie dense in the unit interval; with this statement Oresme was more than half a millennium ahead of his time although his reasoning had gaps; we refer to [92] for a detailed analysis of his thinking. Oresme is also well known for his opposition to Aristotle's astronomy; indeed, he thought about rotation of the Earth about two centuries before Copernicus. Moreover, Oresme wrote an interesting treatise on the speed of light and he invented a kind of coordinate geometry before Descartes, to mention just a few of his ingenious ideas.

We continue with an interesting phenomenon about irregularities in the distribution of digits in statistical data: in 1881, Simon Newcomb noticed that in books consisting tabulars with values for the logarithm those pages starting with digit 1 were looking more used than others. In 1938, this phenomenon was rediscovered and popularized by the physicist Frank Benford [3] who gave further examples from statistics about American towns. According to this distribution a set of numbers is said to be Benford *distributed* if the leading digit equals $k \in \{1, 2, \ldots, 9\}$ for $\log_{10}(1 + \frac{1}{k})$ percent instances. Thus, slightly more than 30 % of the numbers in a Benford distributed data set have leading digit 1, and only about 6 % start with digit 7. Benford's law is supposed to hold for quite many sequences as constants in physics and stock market values. An example for which Benford's law is known to hold is the sequence of Fibonacci numbers $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$, however, the sequence of primes is not Benford distributed as was proved by Jolissaint [50] and Diaconis [15]. Recent investigations show that certain stochastic processes, e.g., the geometric Brownian motion or the $3X + 1$-iteration due to Collatz satisfy Benford's law as shown by Kontorovich and Miller [55].

Here is an illustrating example of a deterministic sequence which follows Benford's law. Considering the powers of two, we notice that among the first of those powers,

$$1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8092, \ldots ,$$

[1] The English translation is taken from [92].

[2] The English translation is taken from [92].

there are indeed more integers starting with digit 1 than with digit 3. Obviously, a power of 2 with a decimal expansion of $m + 1$ digits has leading digit k if, and only if,

$$10^m k \leq 2^n < 10^m (k+1) \qquad \text{for} \quad k \in \{0, 1, \ldots, 9\};$$

taking the logarithm gives

$$m + \log_{10} k \leq n \log_{10} 2 < m + \log_{10}(k+1).$$

For a real number x we introduce the decomposition in its integral and fractional parts by writing $x = \lfloor x \rfloor + \{x\}$ with $\lfloor x \rfloor$ being the largest integer less than or equal to x and $\{x\} \in [0, 1)$ the fractional part. Consequently, the latter inequalities transform into

$$\log_{10} k \leq \{n \log_{10} 2\} < \log_{10}(k+1).$$

Since the logarithm is concave, the interval $[\log_{10} k, \log_{10}(k+1))$ is larger for small k, so, heuristically, the chance is larger that $n \log_{10} 2$ has fractional part in such an interval as n ranges through the set of positive integers. In the next section we shall show that the sequence of numbers $\log_{10} x_n = n \log_{10} 2$ is *uniformly distributed modulo* 1 which implies that indeed the proportion of 2^n with leading digit $k \in \{1, 2, 3, \ldots, 9\}$ equals the length of the interval $[\log_{10} k, \log_{10}(k+1))$, that is

$$\log_{10}(k+1) - \log_{10} k = \log_{10}(1 + \tfrac{1}{k}).$$

In particular, $\log_{10} 2 \approx 30.1\,\%$ of the powers of 2 have a decimal expansion with leading digit 1 whereas the leading digit equals 7 for only approximately 5.8 %. On the contrary, powers of 10 have always leading digit 1 in the decimal system. This shows that the arithmetical nature of $\log_{10} 2$ is relevant for the proportion with which leading digits appear.

For the first we shall generalize this problem to powers of some positive integer a with respect to expansions to an arbitrary base b. Here both, a and b are positive integers at least two. We shall use classical inhomogeneous Diophantine approximation in order to show that any possible digit will appear as leading digit of a power of a if, and only if, $\log_b a$ is irrational. A theorem of Leopold Kronecker [57] from 1884 states that *given* $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ *and* $\beta \in \mathbb{R}$, *for any* $N \in \mathbb{N}$ *and any* $\varepsilon > 0$, *there exist integers* $n > N$ *and* m *such that*

$$|n\alpha - m - \beta| < \varepsilon.$$

There are many proofs of this theorem; see [41] for a collection of such proofs. An elementary proof could start with the observation that, if $\alpha \notin \mathbb{Q}$, then there exist integers k, ℓ such that $|k\alpha - \ell| < \varepsilon$. Hence, the sequence $\{k\alpha\}, \{2k\alpha\}, \ldots$ provides a chain of points across $[0, 1)$ where the distance between consecutive points is less than ε. Applying Kronecker's approximation theorem, we find that for any fixed $k \in \{1, \dots, b-1\}$, for any $\beta \in (\log_b k, \log_b(k+1))$, and any $\varepsilon > 0$, there do exist integers m, n such that

$$|n \log_b a - m - \beta| < \varepsilon$$

provided $\log_b a$ is irrational. Thus, for sufficiently small ε, the number a^n has leading digit k with respect to its expansion in base b. Otherwise, if $\log_b a$ is rational, the sequence $\{n \log_b a\}$ is periodic and the distribution of leading digits of a^n differs from Benford's law. We may interpret Kronecker's approximation theorem as follows: *Given an irrational* α*, the sequence* $\{n\alpha\}$ *lies dense in the unit interval* $[0, 1)$ *as* n *ranges through* $\mathbb{N}$. This is nothing but Oresme's statement from the beginning! In the next section we shall strengthen this approximation theorem significantly.

2 Uniform Distribution Modulo One

Given a dense sequence in the unit interval, e.g., the fractional parts of the numbers $n\alpha$ with some explicit irrational real number α, it is natural to ask how this sequence is distributed: *are there subintervals that contain* only a few *elements of this sequence? How soon does a sequence meet a given subinterval?* The elaborated study of such dense sequences was started around 1909 by three mathematicians independently.

The Latvian mathematician Piers Bohl [7] was the first to succeed with a quantitative improvement of Kronecker's denseness theorem. He came across the following Diophantine result:

> *We consider a onesided unlimited yarn with an infinite number of knots in such a way that the first knot is at the end of the yarn and the other knots follow equally spaced with subsequent distance* $r > 0$. *(...) On a circle of circumference* 1 *we take a line segment* AB *of length* s $(0 < s < 1)$ *and wrap the yarn starting from some arbitrary point on the circumference. The number of the first* n *knots which after wrapping the yarn fall into the line segment* AB *is denoted by* $\varphi(n)$. *(...) If* r *is an irrational number, then (...)* $\lim_{n\to\infty} \frac{\varphi(n)}{n} = s$.[3]

[3]This is the author's free translation of the original German text: "Wir denken uns nun einen einseitig unbegrenzten Faden und nehmen auf demselben eine unbegrenzte Zahl von Knoten in der Weise an, daß der erste Knoten mit dem Fadenende zusammenfällt, während die übrigen im Abstände $r > 0$ der Reihe nach aufeinander folgen. (...) Auf einem Kreise vom Umfang 1 nehmen wir (...) eine Strecke AB von der Länge s $(0 < s < 1)$ an und wickeln den Faden von irgendeinem Punkte ausgehend auf die Peripherie auf. Die Anzahl derjenigen unter den n ersten Knoten, welche bei der Aufwickelung auf der Strecke AB liegen, bezeichnen wir mit $\varphi(n)$. (...) Ist r eine Irrationalzahl, so folgt (...) $\lim_{n\to\infty} \frac{\varphi(n)}{n} = s$".

Bohl's reasoning was of geometrical nature and rather complicated; his motivation originated from astronomical questions.

In order to formulate his result in modern language we begin with a crucial definition: a sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is said to be *uniformly distributed modulo one* (resp. *equidistributed*) if for all α, β with $0 \leq \alpha < \beta \leq 1$ the proportion of the fractional parts of the x_n in the interval $[\alpha, \beta)$ corresponds to its length in the following sense:

$$\lim_{N\to\infty} \frac{1}{N} \sharp\{1 \leq n \leq N \,:\, \{x_n\} \in [\alpha, \beta)\} = \beta - \alpha.$$

Obviously, it suffices to consider only intervals of the form $[0, \beta)$ with arbitrary $\beta \in (0, 1)$.

Theorem 1. *Given a real number* α*, the sequence* $(n\alpha)_{n\in\mathbb{N}}$ *is uniformly distributed modulo one if, and only if,* α *is irrational.*

This theorem is due to Bohl [7] and it provides an immediate solution of the problem concerning the powers of two from the previous section. Since $\log_{10} 2$ is irrational, an application of Theorem 1 shows that the proportion of positive integers n for which the inequalities $\log_{10} k \leq \{n \log_{10} 2\} < \log_{10}(k + 1)$ hold equals the length of the interval, that is $\log_{10}(1 + \frac{1}{k})$, as predicted by Benford's law. We shall give an elegant and short proof of Bohl's theorem below.

Around the same time Wacław Sierpińksi [80, 81] gave an independent proof of this result; his motivation was of pure arithmetical nature. Finally, there is to mention Hermann Weyl [95, 96] who at the same time was investigating Gibb's phenomenon in Fourier analysis; he was faced with essentially the same arithmetical question as Bohl and Sierpinski. In view of these rather different motivations uniform distribution was indeed a *hot topic* around 1909/1910. A little later, Felix Bernstein [4] observed the similarities in the papers of Bohl, Sierpinksi, and Weyl. Interestingly, his approach is based on Lebesgue theory, a modern tool in that time which turned out to be not appropriate with respect to uniform distribution modulo one (as follows from Theorem 2 below). The paper of Bernstein was rather influential; it stands at the beginning of further investigations of illustrious mathematicians.

Once Harald Bohr said "To illustrate to what extent Hardy and Littlewood in the course of the years came to be considered as the leaders of recent English mathematical research, I may report what an excellent colleague once jokingly said: 'Nowadays, there are only three really great English mathematicians: Hardy, Littlewood, and Hardy-Littlewood"'. (cf. [66]). In 1912, the third of these three English mathematicians, Hardy and Littlewood [38, 39] started his (their) research on uniform distribution and succeeded to prove

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \exp(\pi i n^k \alpha) = 0 \tag{1}$$

for fixed $k \neq 0$ and irrational α. Their aim were applications to the Riemann zeta-function, however, of different nature than our later applications. The same can be said about the work of Bohr and Courant [8]. We refer to Binder and Hlawka [5] for a detailed historical account of their work and the very beginnings of uniform distribution theory. We remark that besides Cambridge, where Hardy and Littlewood were doing their work, Göttingen was the place of location giving the impetus on uniform distribution theory. Here Bernstein was working as a professor, Weyl as young docent, and Bohr was visiting Courant.

Another impact of Bernstein's paper [4] was the new awakening of Weyl's old interest in questions on rational approximation. The following quote is from a late work of Weyl [102]:

> *When the problem and Bohl's paper were pointed out to me by Felix Bernstein in 1913, it started me on my investigations on Diophantine approximations…*

Indeed, Weyl starts his pathbreaking article [100] with almost the same words as the above quotation of Bohl's theorem. We continue with presenting the main results from Weyl's papers [98–100].

Theorem 2. *A sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is uniformly distributed modulo one if, and only if, for any Riemann integrable function $f : [0,1] \to \mathbb{C}$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{x_n\}) = \int_0^1 f(x)\,\mathrm{d}x. \tag{2}$$

By this criterion uniform distribution modulo one can be characterized by a certain property in some class of functions. This point of view is completely different from previous approaches and might be seen as starting point of any deeper study of uniform distribution. Moreover, Weyl's first theorem may be interpreted as a forerunner of the celebrated Birkhoff pointwise ergodic theorem [6]; nowadays, any treatise on ergodic theory with applications in number theory as, for instance [18], includes uniform distribution modulo one and, in particular, Weyl's theorem as motivation for the concept of ergodicity. And indeed, in 1913/1914, Rosenthal [75,76] showed the impossibility of the *strong* ergodicity hypothesis from statistical mechanics and how a *weak* ergodicity hypothesis can be used as substitute; for the latter purpose he used ideas similar to those of Bohr and Weyl.

Proof. Given $\alpha, \beta \in [0,1]$, denote by $\chi_{[\alpha,\beta)}$ the indicator function of the interval $[\alpha,\beta)$, i.e.,

$$\chi_{[\alpha,\beta)}(x) = \begin{cases} 1 & \text{if } \alpha \leq x < \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously,

$$\int_0^1 \chi_{[\alpha,\beta)}(x)\,\mathrm{d}x = \beta - \alpha.$$

Therefore, the sequence (x_n) is uniformly distributed modulo 1 if, and only if, for any pair $\alpha, \beta \in [0, 1)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{[\alpha,\beta)}(\{x_n\}) = \int_0^1 \chi_{[\alpha,\beta)}(x)\,\mathrm{d}x.$$

Assuming the asymptotic formula (2) for any Riemann integrable function f, it follows that (x_n) is indeed uniformly distributed modulo one.

In order to show the converse implication we suppose that (x_n) is uniformly distributed modulo 1. Then (2) holds for $f = \chi_{[\alpha,\beta)}$ and, consequently, for any linear combination of such indicator functions. In particular, we may deduce that (2) is true for any step function. For any real-valued Riemann integrable function f and any $\varepsilon > 0$, we can find step functions t_-, t_+ such that

$$t_-(x) \le f(x) \le t_+(x) \qquad \text{for all} \quad x \in [0, 1],$$

and

$$\int_0^1 (t_+(x) - t_-(x))\,\mathrm{d}x < \varepsilon.$$

Hence,

$$\int_0^1 f(x)\,\mathrm{d}x \ge \int_0^1 t_-(x)\,\mathrm{d}x > \int_0^1 t_+(x)\,\mathrm{d}x - \varepsilon,$$

and

$$\frac{1}{N} \sum_{n=1}^{N} f(\{x_n\}) - \int_0^1 f(x)\,\mathrm{d}x \le \frac{1}{N} \sum_{n=1}^{N} t_+(\{x_n\}) - \int_0^1 t_+(x)\,\mathrm{d}x + \varepsilon,$$

which is less than 2ε for all sufficiently large N. Analogously, we obtain a similar lower bound. Consequently, (2) holds for all real-valued Riemann integrable functions f. The case of complex-valued Riemann integrable functions can be deduced from the real case by treating the real and imaginary part of f separately. □

We shall illustrate Theorem 2 with an example of a sequence which is not uniformly distributed modulo one. For this purpose we consider the fractional parts of the numbers $x_n = \log n$ and the function defined by $f(u) = \exp(2\pi i u)$. An easy computation shows

$$\sum_{n=1}^{N} f(\log n) = \sum_{n=1}^{N} n^{2\pi i} = \sum_{n=1}^{N} \left(\frac{n}{N}\right)^{2\pi i} N^{2\pi i} \sim N^{1+2\pi i} \int_0^1 u^{2\pi i}\,\mathrm{d}u = \frac{N^{1+2\pi i}}{1+2\pi i}$$

which is not $o(N)$. Hence, the sequence $(\log n)_n$ is not uniformly distributed modulo 1. Actually, this is the reason why we have been surprised by Benford's law: if (x_n) is uniformly distributed modulo one, then $(\log x_n)$ is Benford distributed. As a matter of fact, the Benford distribution is nothing else than the probability law of the mantissa with respect to the basis.

The converse of Weyl's Theorem was found by de Bruijn and Post [12]: *given a function* $f : [0,1) \to \mathbb{C}$ *with the property that for any uniformly distributed sequence* (x_n) *the limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{x_n\})$$

exists, then f *is Riemann integrable.* It is interesting that here the Riemann integral is superior to the Lebesgue integral (different from ergodic theory where it is vice versa). In fact, Theorem 2 does not hold for Lebesgue integrable functions f in general since f might vanish at each point $\{x_n\}$ but have a non-vanishing integral. This subtle difference is related to a rather important application of uniformly distributed sequences, namely so-called Monte-Carlo methods and their use in numerical integration: if N points are *uniformly distributed* in the square $[-1,1]^2$ in the Euclidean plane and the number M counts those points which lie inside the unit circle centred at the origin, then the quotient M/N is a good guess for the area π of the unit disk. In view of this idea uniformly distributed sequences can be used to evaluate numerically certain integrals for which there is no elementary method, e.g. the Gaussian integral $\int \exp(-x^2)\,\mathrm{d}x$. More on this topic can be found in [45].

Our next aim is another characterization of uniform distribution modulo one, also due to Weyl. For abbreviation, we write $\mathrm{e}(\xi) = \exp(2\pi i \xi)$ for $\xi \in \mathbb{R}$ which translates the $2\pi i$-periodicity of the exponential function to 1-periodicity: $\mathrm{e}(\xi) = \mathrm{e}(\xi + \mathbb{Z})$.

Theorem 3. *A sequence* $(x_n)_{n\in\mathbb{N}}$ *of real numbers is uniformly distributed modulo one if, and only if, for any integer* $m \neq 0$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathrm{e}(m x_n) = 0. \tag{3}$$

Proof. Suppose the sequence (x_n) is uniformly distributed modulo one, then Theorem 2 applied with $f(x) = \mathrm{e}(mx)$ shows

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathrm{e}(m x_n) = \int_0^1 \mathrm{e}(mx)\,\mathrm{d}x.$$

For any integer $m \neq 0$ the right-hand side equals zero which gives (3).

For the converse implication suppose (3) for all integers $m \neq 0$. Starting with a trigonometric polynomial

$$P(x) = \sum_{m=-M}^{+M} a_m \,\mathrm{e}(mx) \qquad \text{with} \quad a_m \in \mathbb{C},$$

we compute

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} P(\{x_n\}) = \sum_{m=-M}^{+M} a_m \cdot \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathrm{e}(mx_n) = a_0 = \int_0^1 P(x)\,\mathrm{d}x. \quad (4)$$

Recall Weierstraß' approximation theorem which claims that, for any continuous 1-periodic function f and any $\varepsilon > 0$, there exists a trigonometric polynomial P such that

$$|f(x) - P(x)| < \varepsilon \qquad \text{for} \quad 0 \leq x < 1 \quad (5)$$

(this can be proved with Fourier analysis). Using this approximating polynomial, we deduce

$$\begin{aligned} &\left| \frac{1}{N} \sum_{n=1}^{N} f(\{x_n\}) - \int_0^1 f(x)\,\mathrm{d}x \right| \\ &\quad \leq \left| \frac{1}{N} \sum_{n=1}^{N} (f(\{x_n\}) - P(\{x_n\})) \right| + \left| \frac{1}{N} \sum_{n=1}^{N} P(\{x_n\}) - \int_0^1 P(x)\,\mathrm{d}x \right| \\ &\qquad + \left| \int_0^1 (P(x) - f(x))\,\mathrm{d}x \right|. \end{aligned}$$

The first and the third terms on the right are less than ε thanks to (5); the second term is small by (4). Hence, formula (2) holds for all continuous, 1-periodic functions f. Denoting by $\chi_{[\alpha,\beta)}$ the indicator function of the interval $[\alpha, \beta)$ (as in the proof of the previous theorem), for any $\varepsilon > 0$, there exist continuous 1-periodic functions f_-, f_+ satisfying

$$f_-(x) \leq \chi_{[\alpha,\beta)}(x) \leq f_+(x) \qquad \text{for all} \quad 0 \leq x < 1,$$

and

$$\int_0^1 (f_+(x) - f_-(x))\,\mathrm{d}x < \varepsilon.$$

This leads to

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{[\alpha,\beta)}(\{x_n\}) = \int_0^1 \chi_{[\alpha,\beta)}(x)\,\mathrm{d}x.$$

Hence, the sequence (x_n) is uniformly distributed modulo one. □

A probabilistic proof can be found in Elliott's monography [20].

As an easy application of the latter criterion we shall deduce Bohl's Theorem 1: if α is irrational, then $\mathrm{e}(m\alpha) \neq 1$ for any $0 \neq m \in \mathbb{Z}$ and the formula for the finite geometric series yields

$$\sum_{n=1}^{N} \mathrm{e}(mn\alpha) = \mathrm{e}(m\alpha)\frac{1-\mathrm{e}(mN\alpha)}{1-\mathrm{e}(m\alpha)}$$

for all integers $m \neq 0$. Since this quantity is bounded independently of N, it follows that (3) holds with $x_n = n\alpha$. Otherwise, $\alpha = \frac{a}{b}$ for some integers a, b with $b \neq 0$; in this case, the limit is different from zero for all integer multiples m of b and Theorem 3 implies the assertion. For an elementary proof see Miklavc [68].

Weyl [100] gave the following polynomial generalization of Bohl's Theorem 1 extending the result on the uniform distribution of αn^k implied from (1) significantly: *If $P = a_d X^d + \cdots + a_1 X + a_0$ is a polynomial with real coefficients, where at least one coefficient a_j with $j \neq 0$ is irrational, then the values $P(n)$ are uniformly distributed modulo one as n ranges through $\mathbb{N}$.*

In the brief introduction to uniform distribution modulo one above we have closely followed Weyl [100]. It should be noticed that Theorem 3 was already known to Weyl as early as Summer 1913 previous to Theorem 2; in [100] he wrote about Bohl's theorem:

> *The claim, that this sequence is everywhere dense, is the content of a famous approximation theorem due to Kronecker. The present stronger theorem has been presented first by myself in a talk at the Göttingen Mathematical Society in Summer 1913 and it had been proved in a similar way as here.*[4]

However, after its presentation at the meeting of the *Göttingen Mathematical Society* in July 1913 Weyl did not intend to publish this criterion immediately since at that time he was much impressed by Bohr's approach to related problems (see [98]). The first publication of Weyl's elegant characterization of uniform distribution is [99]. In view of Hardy and Littlewood's estimate (1) it follows from Theorem 3 that for an arbitrary positive integer k the sequence $n^k\alpha$ is uniformly distributed modulo one if α is irrational. It might have been that Weyl was inspired by (1) to consider

[4]This is the author's free translation of the original German text: "Die Behauptung, daß diese Punktfolge überall dicht liegt, ist der Inhalt eines berühmten Approximationssatzes von Kronecker. Das vorliegende viel schärfere Theorem ist zuerst im Sommer 1913 von mir in einem Vortrag in der Göttinger Mathematischen Gesellschaft aufgestellt und ähnliche Weise wie hier bewiesen worden".

exponential sums with respect to uniform distribution and his extension to more general polynomials changed his reservation to publish his results.

The year 1913 must have been a very important year for Hermann Weyl for various reasons. Not only that he gave birth to the theory of uniform distribution modulo one, in the same year Weyl married Helene Joseph, a student of the philosopher Husserl, he left Göttingen for Zurich where he became full professor at the Polytechnic Zurich,[5] and he published his famous treatise on Riemann surfaces [97]. At that time Weyl was 27 years old. His later works to mathematics include his important contributions to the theory of group representations, mathematical physics, and philosophy of mathematics. In 1919, Weyl adopted Brouwer's ideas about intuitionism and in particular Weyl's approach to uniform distribution modulo one was based on non-constructive mathematics. This is neither curious nor tragic since Weyl was discussing this type of questions with a certain gingerliness. As Taschner [84] showed, Weyl's reasoning can be made constructive.

It might be interesting to notice that in the mathematical diaries of Adolf Hurwitz [46] one can find an entry from April 1914[6] dealing with Sierpiński's theorem [80, 81] claiming that, for any $a \in [0, 1]$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m\leq n} \{m\alpha + a\} = \tfrac{1}{2}$$

if, and only if, α is irrational. At that time young Weyl and the established Hurwitz were colleagues at the ETH Zurich but it seems that the elder did not know about the younger's work on this topic beyond the papers [95, 96] from 1909/1910. Actually, Sierpiński used a result of Hurwitz in his second paper [81] and we may guess that this started Hurwitz's interest on this topic. In his diary Hurwitz proves a generalization of Sierpiński's theorem which is very close to Weyl's Theorem 2, namely: *if f is* Riemann *integrable on* $[0, 1]$ *and α irrational, then*

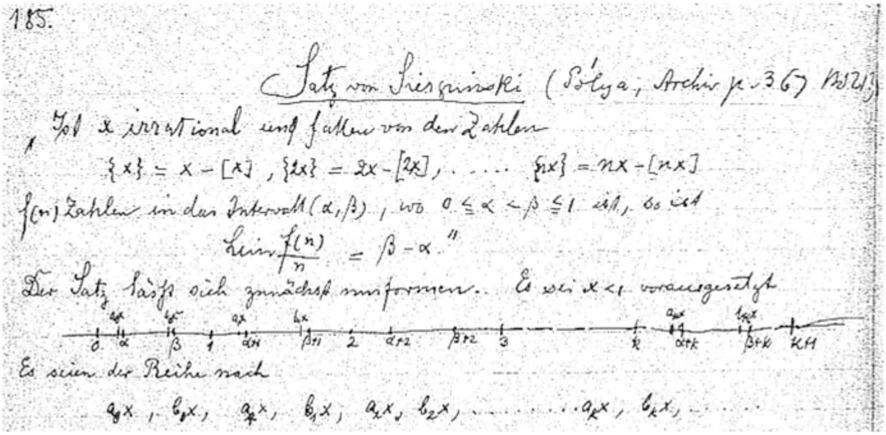
$$\lim_{n\to\infty} \frac{1}{n}(f(\{\alpha\}) + f(\{2\alpha\}) + \cdots + f(\{n\alpha\})) = \int_0^1 f(x)\,\mathrm{d}x$$

(by slight modification of his notation from the diary). It is also mentioned that this holds with replacing the left-hand side by $\lim_{n\to\infty} \frac{1}{n}(f(\{c+\alpha\}) + f(\{c+2\alpha\}) + \cdots + f(\{c+n\alpha\}))$, where c is an arbitrary real number (Fig. 1 and Fig. 2).

The strong criterion, Weyl's Theorem 3, can be applied and extended in various ways giving generalizations beyond Bohl's theorem. He himself strengthened results of Hardy and Littlewood on sequences of the form $n^k\alpha$ (as already mentioned above), mathematical billiards, and the three-body problem (see [100]). One of the most spectacular results is due to Vinogradov [87] being the main ingredient in his

[5] *Eidgenössische Hochschule Zürich* (ETH).

[6] A precise date is impossible because this entry is without date, however, comparing with other date entries one may deduce that Hurwitz wrote this in between April 2 and April 30.


185.

Satz von Sierpiński (Pólya, Archiv p. 367 1912)

„Ist x irrational und fallen von den Zahlen

$\{x\} = x - [x],\ \{2x\} = 2x - [2x],\ \ldots\ \{nx\} = nx - [nx]$

f(n) Zahlen in das Intervall (α, β), wo $0 \leq \alpha < \beta \leq 1$ ist, so ist

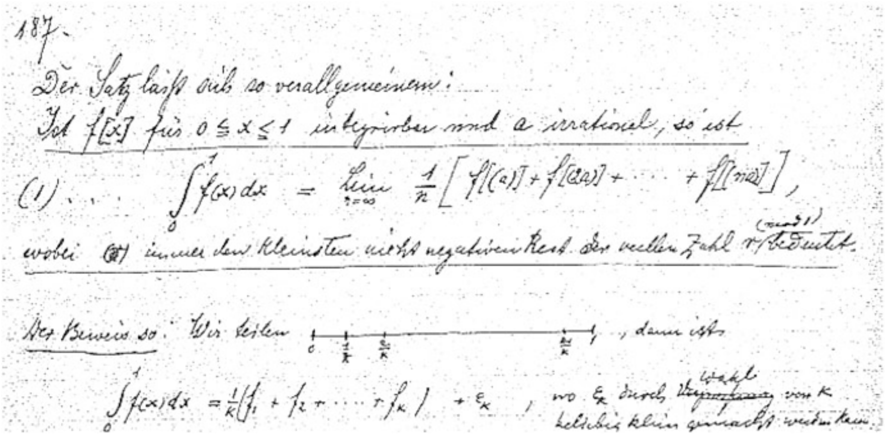
$\lim \frac{f(n)}{n} = \beta - \alpha$."

Der Satz lässt sich zunächst umformen. Es sei x < 1 vorausgesetzt.

Es seien der Reihe nach

$a_0x,\ b_0x,\ a_1x,\ b_1x,\ a_2x,\ b_2x,\ \ldots\ a_kx,\ b_kx,\ \ldots$

Fig. 1 Adolf Hurwitz, Mathematische Tagebücher, no. 26, p. 185: Sierpiński's theorem. As follows from a handwritten note [72] Hurwitz's protégé Pólya gave the inspiration


187.

Der Satz lässt sich so verallgemeinern:

Ist f[x] für $0 \leq x \leq 1$ integrierbar und a irrational, so ist

(1) ... $\int_0^1 f(x)dx = \lim_{n=\infty} \frac{1}{n}\left[f[(a)] + f[(2a)] + \cdots + f[(na)]\right]$,

wobei (x) immer den kleinsten nicht negativen Rest der reellen Zahl x (mod 1) bedeutet.

Der Beweis so: Wir teilen $0, \frac{1}{k}, \frac{2}{k}, \ldots, \frac{k}{k}$, dann ist

$\int_0^1 f(x)dx = \frac{1}{k}(f_1 + f_2 + \cdots + f_k) + \varepsilon_k$, wo ε_k durch Wahl von k beliebig klein gemacht werden kann.

Fig. 2 Adolf Hurwitz, Mathematische Tagebücher, no. 26, p. 187: Hurwitz's generalization

proof of the *ternary* Goldbach conjecture that any sufficiently large odd integer can be represented as a sum of three primes. For this purpose he found a nontrivial estimate for the exponential sum $\sum_{p_n \leq N} e(p_n\alpha)$, where p_n denotes the nth prime (in ascending order). Vinogradov proved that for irrational α the sequence $(p_n\alpha)$ is uniformly distributed modulo 1. In order to get an impression on the depth of this result one may notice that in case of rational α this question is intimately related to the distribution of primes in arithmetic progressions.[7] The *binary* Goldbach

[7]Recently, H. Helfgott published an article "Major arcs for Goldbach's theorem" (see arXiv:1305.2897) and another article "Numerical Verification of the Ternary Goldbach Conjecture

conjecture that any even integer larger than two is representable as sum of two primes is wide open.

We conclude this section with another open question. It is not known whether the sequence of powers $(\frac{3}{2})^n$ or the numbers $\exp(n)$ are uniformly distributed modulo one. Koksma [53] showed that almost all sequences (α^n) with $\alpha > 1$ are uniformly distributed, however, there is no single α with this property explicitly known. On the contrary, if α is a Salem number, i.e., all algebraic conjugates of α (except α) have absolute value less than one, then the sequence (α^n) is not uniformly distributed. An excellent reading on the beautiful theory of uniform distribution modulo one are the monographs [10] and [58] by Bugeaud and Kuipers and Niederreiter, respectively.

3 Basic Theory of the Riemann Zeta-Function

Prime numbers are the fascinating multiplicative atoms from which the integers are built. It was the young Gauss who was the first to conjecture the true order of growth for the number $\pi(x)$ of primes $p \leq x$. In a letter to Encke from Christmas 1849 Gauss wrote

> *You have reminded me of my own pursuit of the same subect, whose first beginnings occurred a very long time ago, in 1792 or 1793, when I had procured for myself Lambert's supplement to the table of logarithms. Before I had occupied myself with the finer investigations of higher arithmetic, one of my first projects was to direct my attention to the decreasing frequency of prime numbers, to which end I counted them up in several chiliads and recorded the results on one of the affixed white sheets. I soon recognized, that under all variations of this frequency, on average, it is nearly inversely proportional to the logarithm, so that the number of all prime numbers under a given boundary n were nearly expressed through the integral*
>
> $$\int \frac{\mathrm{d}n}{\log n},$$
>
> *if the integral is understood hyperbolic.* (see [86]).[8]

---

up to 8.875e30" (see arXiv:1305.3062) which is joint work with D.J. Platt; both pieces together imply the full ternary Goldbach conjecture provided that there is no serious gap in their reasoning.

[8] "Die gütige Mittheilung Ihrer Bemerkungen über die Frequenz der Primzahlen ist mir in mehr als einer Beziehung interessant gewesen. Sie haben mir meine eigenen Beschäftigungen mit demselben Gegenstande in Erinnerung gebracht, deren erste Anfänge in eine sehr entfernte Zeit fallen, ins Jahr 1792 oder 1793, wo ich mir die Lambertschen Supplemente zu den Logarithmentafeln angeschafft hatte. Es war noch ehe ich mit feineren Untersuchungend er höheren Arithmetik mich befasst hatte eines meiner ersten Geschäfte, meine Aufmerksamkeit auf die abnehmende Frequenz der Primzahlen zu richten, zu welchem Zweck ich die einzelnen Chiliaden abzählte, und die Resultate auf einem der angehefteten weissen Blätter verzeichnete. Ich erkannte bald, dass unter allen Schwankungen diese Frequenz durchschnittlich nahe dem Logarithmen verkehrt proportional sei, so dass die Anzahl aller Primzahlen unter einer gegebenen Grenze n nahe durch das Integral $\int \frac{\mathrm{d}n}{\log n}$ ausgedrückt werde, wenn der hyperbolische Logarithm. verstanden werde".

The appearing integral is the logarithmic integral; if the upper limit equals x, then it is asymptotically equal to $\frac{x}{\log x}$, hence Gauss' conjecture can be made precise in writing

$$\pi(x) \sim \mathrm{li}\,(x) := \int_2^x \frac{\mathrm{d}u}{\log u}\,, \qquad \text{as} \quad x \to \infty, \tag{6}$$

using a modified logarithmic integral.

In 1859, Riemann [74] figured out how the distribution of prime numbers can be studied by means of analysis; in contrast to previous work of Euler on the zeta-function Riemann had the stronger tools of complex analysis at hand. In the following we shall briefly survey his remarkable insights in the close relation between primes and the zeta-function.

For $\mathrm{Re}\, s > 1$, the Riemann *zeta-function* is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}; \tag{7}$$

here the product is taken over all prime numbers p. The identity between the series and the product is an analytic version of the unique prime factorization of the integers as becomes obvious by expanding each factor of the product into a geometric series. This type of series is called a Dirichlet series and a product over primes as above is referred to as Euler product. It is not difficult to show that both, the series and the product in (7) converge absolutely for all complex numbers s with $\mathrm{Re}\, s > 1$. We need an analytic continuation of the zeta-function to the left of this half-plane of absolute convergence. Following Riemann [74] we substitute $u = \pi n^2 x$ in Euler's representation of the Gamma-function,

$$\Gamma(u) = \int_0^\infty u^{z-1} \exp(-u)\, \mathrm{d}u,$$

and obtain

$$\Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \frac{1}{n^s} = \int_0^\infty x^{\frac{s}{2}-1} \exp(-\pi n^2 x)\, \mathrm{d}x. \tag{8}$$

Summing up over all $n \in \mathbb{N}$ yields

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \int_0^\infty x^{\frac{s}{2}-1} \exp(-\pi n^2 x)\, \mathrm{d}x.$$

On the left-hand side we find the Dirichlet series defining $\zeta(s)$; in view of its convergence, the latter formula is valid for $\mathrm{Re}\, s > 1$. On the right-hand side we may interchange summation and integration, justified by absolute convergence. Thus we obtain

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\int_0^\infty x^{\frac{s}{2}-1}\sum_{n=1}^{\infty}\exp(-\pi n^2x)\,\mathrm{d}x.$$

We split the integral at $x=1$ and get

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\left\{\int_0^1+\int_1^\infty\right\}x^{\frac{s}{2}-1}\omega(x)\,\mathrm{d}x, \tag{9}$$

where the series $\omega(x)$ is given in terms of the theta-function of Jacobi:

$$\omega(x):=\tfrac{1}{2}\left(\vartheta(x)-1\right)\qquad\text{with}\quad\vartheta(x):=\sum_{n=-\infty}^{+\infty}\exp(-\pi n^2x).$$

In view of the functional equation for the theta-function, we have

$$\omega\left(\frac{1}{x}\right)=\frac{1}{2}\left(\theta\left(\frac{1}{x}\right)-1\right)=\sqrt{x}\omega(x)+\frac{1}{2}(\sqrt{x}-1),$$

which can be deduced from Poisson's summation formula. By the substitution $x\mapsto\frac{1}{x}$ it turns out that the first integral in (9) equals

$$\int_1^\infty x^{-\frac{s}{2}-1}\omega\left(\frac{1}{x}\right)\,\mathrm{d}x=\int_1^\infty x^{-\frac{s+1}{2}}\omega(x)\,\mathrm{d}x+\frac{1}{s-1}-\frac{1}{s}.$$

Using this in (9) yields

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\frac{1}{s(s-1)}+\int_1^\infty\left(x^{-\frac{s+1}{2}}+x^{\frac{s}{2}-1}\right)\omega(x)\,\mathrm{d}x. \tag{10}$$

Since $\omega(x)\ll\exp(-\pi x)$, the integral converges for all values of s, and thus (10) holds, by analytic continuation, throughout the complex plane. The right-hand side remains unchanged by $s\mapsto 1-s$. This proves

Theorem 4. *The zeta-function $\zeta(s)$ has an analytic continuation to $\mathbb{C}\setminus\{1\}$ with a simple pole at $s=1$ and satisfies*

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{11}$$

Riemann's functional equation (11) in combination with the Euler product disclose important information about the analytic behaviour of the zeta-function. In view of the Euler product (7) it is easily seen that $\zeta(s)$ has no zeros in the half-plane $\operatorname{Re}s>1$. It follows from the functional equation (11) and from basic properties of the Gamma-function that $\zeta(s)$ vanishes in $\operatorname{Re} s < 0$ exactly at the so-called *trivial zeros* $s = -2n$ with $n \in \mathbb{N}$. All other zeros of $\zeta(s)$ are said to be *nontrivial*, and we denote them by $\rho = \beta + i\gamma$. Obviously, they lie inside the so-called *critical strip* $0 \le \operatorname{Re} s \le 1$, and they are non-real. The functional equation (11) and the identity $\zeta(\overline{s}) = \overline{\zeta(s)}$ show some symmetries of $\zeta(s)$. In particular, the nontrivial zeros of $\zeta(s)$ are distributed symmetrically with respect to the real axis and to the vertical line $\operatorname{Re} s = \frac{1}{2}$. It was Riemann's ingenious contribution to number theory to point out how the distribution of these nontrivial zeros is linked to the distribution of prime numbers. Riemann conjectured the asymptotics for the number $N(T)$ of nontrivial zeros $\rho = \beta + i\gamma$ with $0 < \gamma < T$ (counted according to multiplicities). This conjecture was proved in 1895 by von Mangoldt [90,91] who found more precisely, as $T \to \infty$,

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T). \tag{12}$$

Here and elsewhere the nontrivial zeros are counted according to their multiplicity. There is no multiple zero known, however, one cannot exclude their existence so far; it follows from (12) that the multiplicity of a nontrivial zero $\rho = \beta + i\gamma$ is bounded by $O(\log |\gamma|)$. Since there are no zeros on the real line except the trivial ones, and nontrivial zeros are symmetrically distributed with respect to the real axis, it suffices to study the distribution of zeros in the upper half-plane.

Riemann worked with the function $t \mapsto \zeta(\frac{1}{2} + it)$ and wrote that *very likely all roots t are real,*[9] i.e., all nontrivial zeros lie on the so-called *critical line* $\operatorname{Re} s = \frac{1}{2}$. This is the famous, yet unproved Riemann hypothesis which we rewrite equivalently as

Riemann's hypothesis. $\zeta(s) \neq 0$ *for* $\operatorname{Re} s > \frac{1}{2}$.

In support of his conjecture, Riemann calculated some zeros; the first one with positive imaginary part is $\rho = \frac{1}{2} + i14.134\ldots$. Furthermore, he conjectured that there exist constants A and B such that

$$\tfrac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \exp(A + Bs)\prod_{\rho}\left(1 - \frac{s}{\rho}\right)\exp\left(\frac{s}{\rho}\right), \tag{13}$$

where the product on the right is taken over all nontrivial zeros (the trivial zeta zeros are cancelled by the poles of the Gamma-factor). This latter conjecture was shown by Hadamard [35] in 1893 (on behalf of his theory of product representations of entire functions). Finally, Riemann conjectured the so-called *explicit formula* which states that

[9] The original German text is: "und es ist wahrscheinlich, daß alle Wurzeln den Realteil 1/2 haben: Hiervon wäre allerdings ein strenger Beweis zu wünschen; ich habe indess die Aufsuchung desselben nach einigen flüchtigen vergeblichen Versuchen vorläufig bei Seite gelassen, da er für den nächsten Zweck meiner Untersuchung entbehrlich schien".

$$\pi(x) + \sum_{n=2}^{\infty} \frac{\pi\left(x^{\frac{1}{n}}\right)}{n} = \mathrm{li}(x) - \sum_{\substack{\rho=\beta+i\gamma \\ \gamma>0}} \left(\mathrm{li}(x^{\rho}) + \mathrm{li}(x^{1-\rho})\right)$$
$$+ \int_{x}^{\infty} \frac{\mathrm{d}u}{u(u^2-1)\log u} - \log 2 \qquad (14)$$

for any $x \geq 2$ not being a prime power (otherwise a term $\frac{1}{2k}$ has to be added on the left-hand side where k stems from $x = p^k$). The appearing *modified* integral logarithm is defined by

$$\mathrm{li}(x^{\beta+i\gamma}) = \int_{(-\infty+i\gamma)\log x}^{(\beta+i\gamma)\log x} \frac{\exp(z)}{z + \delta i\gamma}\,\mathrm{d}z,$$

where $\delta = +1$ if $\gamma > 0$ and $\delta = -1$ otherwise. The explicit formula was proved by von Mangoldt [90] in 1895 as a consequence of both product representations for $\zeta(s)$, the Euler product (7) and the Hadamard product (13). Building on these ideas, Hadamard [36] and de la Vallée-Poussin [13] gave (independently) in 1896 the first proof of Gauss' conjecture (6), the celebrated prime number theorem. For technical reasons it is of advantage to work with the logarithmic derivative of $\zeta(s)$ which is for $\mathrm{Re}\, s > 1$ given by

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s},$$

where the von Mangoldt Λ-function is defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ with } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases} \qquad (15)$$

Important information concerning the prime counting function $\pi(x)$ can be recovered from information about

$$\psi(x) := \sum_{n \leq x} \Lambda(n) = \sum_{p \leq x} \log p + O\left(x^{\frac{1}{2}} \log x\right).$$

Partial summation gives $\pi(x) \sim \frac{\psi(x)}{\log x}$. First of all, we shall express $\psi(x)$ in terms of the zeta-function. If c is a positive constant, then

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s}\,\mathrm{d}s = \begin{cases} 1 & \text{if } x > 1, \\ 0 & \text{if } 0 < x < 1. \end{cases}$$

This yields the so-called Perron formula: for $x \notin \mathbb{Z}$ and $c > 1$,

$$\psi(x) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,\mathrm{d}s. \tag{16}$$

Moving the path of integration to the left, we find that the latter expression is equal to the corresponding sum of residues, that are the residues of the integrand at the pole of $\zeta(s)$ at $s = 1$, at the zeros of $\zeta(s)$, and at the additional pole of the integrand at $s = 0$. The main term turns out to be

$$\mathrm{Res}_{s=1}\left\{-\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\right\} = \lim_{s\to 1}(s-1)\left(\frac{1}{s-1} + O(1)\right)\frac{x^s}{s} = x,$$

whereas each nontrivial zero ρ gives the contribution

$$\mathrm{Res}_{s=\rho}\left\{-\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\right\} = -\frac{x^\rho}{\rho}.$$

By the same reasoning, the trivial zeros altogether contribute

$$\sum_{n=1}^{\infty}\frac{x^{-2n}}{2n} = -\tfrac{1}{2}\log\left(1-\frac{1}{x^2}\right).$$

Incorporating the residue at $s = 0$, this leads to the *exact* explicit formula

$$\psi(x) = x - \sum_{\rho}\frac{x^\rho}{\rho} - \tfrac{1}{2}\log\left(1-\frac{1}{x^2}\right) - \log(2\pi),$$

which is equivalent to Riemann's formula (14). This formula is valid for any $x \notin \mathbb{Z}$. Notice that the right-hand side of this formula is not absolutely convergent. If $\zeta(s)$ would have only finitely many nontrivial zeros, the right-hand side would be a continuous function of x, contradicting the jumps of $\psi(x)$ for prime powers x. Going on it is more convenient to cut the integral in (16) at $t = \pm T$ which leads to the truncated version

$$\psi(x) = x - \sum_{|\gamma|<T}\frac{x^\rho}{\rho} + O\left(\frac{x}{T}(\log(xT))^2\right), \tag{17}$$

valid for all values of x. Next we need information on the distribution of the nontrivial zeros. Already the non-vanishing of $\zeta(s)$ on the line $\mathrm{Re}\, s = 1$ yields the asymptotic relations $\psi(x) \sim x$, resp. $\pi(x) \sim \mathrm{li}\,(x)$, which is Gauss' conjecture (6) and sufficient for many applications. However, more precise asymptotics with a remainder term can be obtained by a zero-free region inside the critical strip. The largest known zero-free region for $\zeta(s)$ was found by Vinogradov [88] and Korobov [56] (independently) in 1958 who proved

$$\zeta(s) \neq 0 \qquad \text{in} \quad \operatorname{Re} s \geq 1 - c(\log(|t|+3))^{-\frac{1}{3}}(\log\log(|t|+3))^{-\frac{2}{3}},$$

where c is some positive absolute constant. In combination with the Riemann–von Mangoldt formula (12) we can estimate the sum over the nontrivial zeros in (17). Balancing T and x, we obtain the prime number theorem with the sharpest known remainder term: *there exists an absolute positive constant C such that for sufficiently large x*

$$\pi(x) = \operatorname{li}(x) + O\left(x \exp\left(-C \frac{(\log x)^{\frac{3}{5}}}{(\log\log x)^{\frac{1}{5}}}\right)\right).$$

By the explicit formula (17) the impact of the Riemann hypothesis on the prime number distribution becomes visible. In 1900, von Koch [89] showed that for fixed $\theta \in [\frac{1}{2}, 1)$

$$\pi(x) - \operatorname{li}(x) \ll x^{\theta+\varepsilon} \qquad \Longleftrightarrow \qquad \zeta(s) \neq 0 \quad \text{for} \quad \operatorname{Re} s > \theta\,; \tag{18}$$

equivalently, one can replace the left-hand side by $\psi(x) - x$; here ε stands for an arbitrary small positive constant. In view of the existence of zeros on the critical line an error term with $\theta < \frac{1}{2}$ is impossible. Hardy [37] proved that infinitely many zeros lie on the critical line. Refining a method of Atle Selberg [78], Levinson [63] localized more than one third of the nontrivial zeros of the zeta-function on the critical line, and as Heath-Brown [42] and Selberg (unpublished) discovered, those zeros are all simple. The current record is due to Bui et al. [11] who showed, by extending Levinson's method, that more than 41 % of the zeros are on the critical line and more than 40.5 % are simple and on the critical line. For further reading on the theory of the Riemann zeta-function we refer to the classical monograph [85] by Titchmarsh and the current book [48] by Ivić; many historical details about prime numbers can be found in Schwarz's survey [77] and Narkiewicz's monograph [70].

4 The Ordinates of Zeta Zeros are Uniformly Distributed Modulo One

Obviously, the trivial zeros are not uniformly distributed modulo one. In 1956 Rademacher [73] proved on the contrary the remarkable result that the ordinates of the nontrivial zeros of the zeta-function are uniformly distributed modulo one provided that the Riemann hypothesis is true; later Elliott [19] remarked that the latter condition can be removed, and (independently) Hlawka [44] obtained the following unconditional

Theorem 5. *For any real number $\alpha \neq 0$ the sequence $\alpha\gamma$, where γ ranges through the set of positive ordinates of the nontrivial zeros of $\zeta(s)$ in ascending order, is uniformly distributed modulo one. In particular, the ordinates of the nontrivial zeros of the zeta-function are uniformly distributed modulo one.*

Proof. We need some deeper results from zeta-function theory. We start with a theorem of Landau [59] who proved, for $x > 1$,

$$\sum_{0<\gamma<T} x^{\rho} = -\Lambda(x)\frac{T}{2\pi} + O(\log T), \tag{19}$$

where the summation is over all nontrivial zeros $\rho = \beta + i\gamma$ and $\Lambda(x)$ is the von Mangoldt Λ-function, defined by (15); if $x \in (0,1)$ one has to replace $\Lambda(x)$ by $x\Lambda(\frac{1}{x})$ because of the symmetrical distribution of nontrivial zeros. (We shall give a proof of Landau's formula in the following section!) Let $x > 1$. In view of (19) and the Riemann–von Mangoldt-formula (12) it follows that

$$\frac{1}{N(T)}\sum_{0<\gamma<T} x^{\rho} \ll \frac{\log x}{\log T}. \tag{20}$$

To avoid the assumption of the Riemann hypothesis we observe that

$$\begin{aligned} |x^{\frac{1}{2}+i\gamma} - x^{\beta+i\gamma}| &\leq \max\{x^{\beta}, x^{\frac{1}{2}}\}|\exp\left(\left(\tfrac{1}{2}-\beta\right)\log x\right) - 1| \\ &\leq \max\{x^{\beta}, x^{\frac{1}{2}}\}\log x\,|\beta - \tfrac{1}{2}|. \end{aligned}$$

Thus,

$$\frac{1}{N(T)}\sum_{0<\gamma<T}|x^{\frac{1}{2}+i\gamma} - x^{\beta+i\gamma}| \leq \frac{\max\left\{x, x^{\frac{1}{2}}\right\}|\log x|}{N(T)}\sum_{0<\gamma<T}|\beta - \tfrac{1}{2}|. \tag{21}$$

In the sequel the implicit constants may depend on x. Next we shall use a result of Littlewood [65], namely

$$\sum_{0<\gamma\leq T}|\beta - \tfrac{1}{2}| \ll T\log\log T. \tag{22}$$

It should be noted that Selberg [78] improved upon this result in replacing the right-hand side by T by integration of his density theorem

$$N(\sigma, T) \ll T^{1-\frac{1}{4}(\sigma-\frac{1}{2})}\log T \tag{23}$$

for the number $N(\sigma, T)$ of hypothetical zeros $\rho = \beta + i\gamma$ with $0 < \gamma \le T$ and $\beta > \sigma$, i.e.,

$$\sum_{0<\gamma\le T} |\beta - \tfrac{1}{2}| = \int_{\frac{1}{2}}^{1} N(\sigma, T)\, \mathrm{d}\sigma \ll T.$$

Both estimates indicate that most of the zeta zeros are clustered around the critical line.[10]

Inserting this in (21) and using (12) leads to

$$\frac{1}{N(T)} \sum_{0<\gamma\le T} \left(x^{\frac{1}{2}+i\gamma} - x^{\beta+i\gamma} \right) \ll \frac{\log\log T}{\log T}.$$

Thus, it follows from (20) that also

$$\frac{1}{N(T)} \sum_{0<\gamma\le T} x^{\frac{1}{2}+i\gamma} \ll \frac{\log\log T}{\log T}.$$

Letting $x = z^m$ with some real number $z > 1$ and $m \in \mathbb{Z}$, we deduce

$$\frac{1}{N(T)} \sum_{0<\gamma\le T} \exp(im\gamma\log z) \ll \frac{\log\log T}{\log T},$$

which tends to zero as $T \to \infty$. Hence, it follows from Weyl's criterion, Theorem 3, that the sequence of numbers $\alpha\gamma$ with $\alpha = \frac{\log z}{2\pi}$ is uniformly distributed modulo one. □

Elliott's paper [19] is from 1972; it is a transcript of a talk he had given at a meeting on number theory at Oberwolfach in 1968. The main focus of his work, however, was on the frequency of negative values of the Legendre symbol. Elliott's approach is based on the following formula (in slightly different notation)

$$\begin{aligned} \sum_{|\gamma_n|<T} \exp(i\omega\gamma_n) &= 2\mathrm{Re} \int_0^{T-} \exp(i\omega y)\, \mathrm{d}N(y) \\ &= 2\mathrm{Re} \left(i\omega \int_0^{T-} S(y) \exp(i\omega y)\, \mathrm{d}y \right) + O(\log T), \end{aligned}$$

[10] A different approach to results around the sum of terms $|\beta - \frac{1}{2}|$ is due to Kondratyuk [54] based on a variant of the Carleman–Nevanlinna theorem.

where $S(t) := \frac{1}{\pi} \arg \zeta(\frac{1}{2} + it)$ is the argument of the zeta-function on the critical line[11] and $\omega \neq 0$ is a fixed real number. Using a conditional asymptotic formula for the second moment of $S(t)$ (under assumption of the Riemann hypothesis) due to Selberg (unpublished), Rademacher's theorem follows. This is followed by an added note saying

> *In a lecture given at the same meeting in Oberwolfach, Professor Selberg indicated that he had improved his result concerning* $\arg \zeta(\frac{1}{2} + it)$ *to give unconditional information concerning the distribution of the values of* $\zeta(s)$ *in regions centred on the line* $\sigma = \frac{1}{2}$*. In particular it is possible with a suitable interpretation to give an unconditional form of Theorem 2.*

In this quotation Theorem 2 is exactly the same statement as in Theorem 5 above. The paper [45] by Hlawka is from 1975 and does not include a reference to Elliott's paper. Hlawka's approach is slightly different; his proof is more or less identical to our reasoning above. A last word about the reception of these papers. It seems that Elliott [19] was unaware of Rademacher's work [73] since he does not cite his paper. On the contrary, Hlawka [45] quotes Rademacher's paper but not the one by Elliott. In his *Zentralblatt* review Hlawka wrote that he learned about Elliott's previous result by his colleague Bundschuh.[12] It should be mentioned that Hlawka [45] also gave a multidimensional analogue of the above theorem. Another proof of the uniform distribution modulo one of the ordinates was given by Fujii [24]; this paper contains as well further related results.

It is a long-standing conjecture that the ordinates of the nontrivial zeros are linearly independent over the rationales. So $\gamma + \gamma'$ should never equal another ordinate of a zeta zero. Of course, one should not expect any algebraic relation for the zeta zeros, hence it is reasonable to expect the converse. Ingham [47] observed an interesting impact on the distribution of values of the Möbius μ-function. Let $M(x) = \sum_{n \leq x} \mu(n)$, where $\mu(n)$ is defined by

$$\zeta(s)^{-1} = \prod_p \left(1 - \frac{1}{p^s}\right) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},$$

the latter identity being valid for $\mathrm{Re}\, s > 1$. It is not difficult to deduce that $\mu(n) = (-1)^r$ if n is squarefree and r denotes the number of prime divisors of n; otherwise, $\mu(n) = 0$. Ingham showed that, if the ordinates of the nontrivial zeros are indeed linearly independent over the rationals, then $\limsup_{x\to\infty} M(x)x^{-\frac{1}{2}} = +\infty$ and $\liminf_{x\to\infty} M(x)x^{-\frac{1}{2}} = -\infty$ which should be compared with (24) below.

Since $\mu(n) \in \{0, \pm 1\}$ one may interpret $M(x)$ as the realization of a one-dimensional symmetric random walk starting at zero. It was Denjoy [14] who argued

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[11] Defined by continuous variation from the principal branch of the logarithm on the real axis.

[12] "Der Referent [Hlawka] wurde von Herrn Bundschuh aufmerksam gemacht, daß auch P.D.T.A. Elliott (...) diese Tatsache bemerkt hat".

as follows. Assume that $\{X_n\}$ is a sequence of random variables with distribution $\mathbf{P}(X_n = +1) = \mathbf{P}(X_n = -1) = \frac{1}{2}$. Define

$$S_0 = 0 \qquad \text{and} \qquad S_n = \sum_{j=1}^{n} X_j,$$

then $\{S_n\}$ is a symmetrical random walk in $\mathbb{Z}$ with starting point at 0. By the theorem of Moivre-Laplace this can be made more precise. It follows that

$$\lim_{n\to\infty} \mathbf{P}\left\{|S_n| < cn^{\frac{1}{2}}\right\} = \frac{1}{\sqrt{2\pi}} \int_{-c}^{c} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x.$$

Since the right-hand side above tends to 1 as $c \to \infty$, we obtain

$$\lim_{n\to\infty} \mathbf{P}\left\{|S_n| \ll n^{\frac{1}{2}+\varepsilon}\right\} = 1$$

for every $\varepsilon > 0$. We observe that this might be regarded as a model for the value-distribution of Möbius μ-function. The law of the iterated logarithm in order to get the strong estimate for a symmetric random walk

$$\lim_{n\to\infty} \mathbf{P}\left\{|S_n| \ll (n \log\log n)^{\frac{1}{2}}\right\} = 1.$$

This suggests for $M(x)$ the upper bound $(x \log\log x)^{\frac{1}{2}}$ which is pretty close to the so-called weak Mertens hypothesis stating

$$\int_1^X \left(\frac{M(x)}{x}\right)^2 \mathrm{d}x \ll \log X.$$

The latter bound implies the Riemann hypothesis and that all zeros are simple. On the contrary, Odlyzko and te Riele [71] disproved the original Mertens hypothesis [67], i.e., $|M(x)| < x^{\frac{1}{2}}$, by showing

$$\liminf_{x\to\infty} \frac{M(x)}{x^{\frac{1}{2}}} < -1.009 \qquad \text{and} \qquad \limsup_{x\to\infty} \frac{M(x)}{x^{\frac{1}{2}}} > 1.06; \tag{24}$$

for more details see the notes to Sect. 14 in Titchmarsh [85].

Landau's formula and variations have been used, in particular by Fujii in a series of papers, in order to evaluate discrete moments of the zeta-function or its derivative near or at its zeros, e.g. $\sum_{0<\gamma<T} \zeta'(\frac{1}{2} + i\gamma)$ with very precise error terms under assumption of the Riemann hypothesis (see [27]). Furthermore, Fujii [28] investigated the sequence $\gamma + \gamma'$ where both γ and γ' range through the set of positive ordinates of zeta zeros (in ascending order). Assuming the Riemann hypothesis, he obtained an asymptotic formula for

$$\sum_{\substack{0<\gamma,\gamma'<T\\ \gamma+\gamma'<T}} x^{i(\gamma+\gamma')} \sim \frac{\Lambda(x)^2}{x}\frac{T^2}{8\pi^2} + \frac{x^{iT}}{\log x}\frac{T(\log T)^2}{4\pi^2 i}$$

with an explicit error term. Since the number of terms $\gamma + \gamma' < T$ is asymptotically equal to $\frac{T^2(\log T)^2}{8\pi^2}$, it follows that the sequence of $\gamma + \gamma'$ is uniformly distributed modulo one. This has been used by Egami and Matsumoto [17] to motivate a related conjecture on distances between different pairs of zero ordinates in order to show that a certain multiple zeta-function has a natural boundary.

5 Questions Around the Distribution of Values of $\zeta(s)$

Bounds for the Riemann zeta-function rely heavily on estimates for certain exponential sums. In order to see that consider the Dirichlet polynomial obtained from the defining series for $\zeta(s)$, i.e.,

$$\sum_{n\le x}\frac{1}{n^s} = 1 + \sum_{1<n\le x}\frac{1}{n^\sigma}\exp(-it\log n),$$

where we have written $s = \sigma + it$. One may hope to estimate $\zeta(\sigma + it)$ by finding a good bound for this Dirichlet polynomial; of course, we exclude here any treatment of the tail of the series expansion. By partial summation it suffices to consider the latter sum in case of $\sigma = 0$. Replacing the logarithm $\log n$ by an appropriate polynomial P of sufficiently high degree, the problem is reduced to an estimation of the exponential sum

$$\sum_{1<n\le x}\frac{1}{n^\sigma}\exp(-itP(n)).$$

This type of quantity was already treated by Weyl [100] when he was generalizing Bohl's theorem from the case of linear polynomials to arbitrary polynomials. The works of Hardy and Littlewood [38, 39] had been following this line of investigation (see [85], Chap. V); exponential sums have found further applications in their approach to the Waring problem by introducing the circle method. In 1921, Weyl [101] pushed his method further to deal with exponential sums associated with the zeta-function and obtained stronger bounds for $\zeta(1 + it)$. Later Vinogradov gave another, in the case of the zeta-function more powerful method to bound exponential sums which led him to the still best known zero-free region so far (see [85], Chap. V for more details).

Studies on the general distribution of values of the zeta-function started with the research of Bohr and his school. In fact Bohr and his contemporaries were using Diophantine approximation in order to prove that the zeta-function assumes *large* and *small* values. For the sake of simplicity, let us consider the truncated Euler product

$$\prod_{p \le x} \left(1 - \frac{1}{p^s}\right)^{-1}; \tag{25}$$

by observing $p^s = p^\sigma \exp(it \log p)$ one may use a multi-dimensional version of Kronecker's approximation theorem in order to find some real number τ such that the values $\frac{1}{2\pi} t \log p$ are close to $\frac{1}{2}$ modulo one for all primes $p \le x$ which in turn implies that p^{it} is close to -1. This leads to a small value for the truncated Euler product and thereby proves that $\inf |\zeta(s)| = 0$ in the half-plane of absolute convergence although $\zeta(s)$ does not vanish (see [85], Chap. VIII for details). Inside the critical strip the situation is more subtle since (25) does not converge any longer to $\zeta(s)$. Nevertheless, by mean-square approximation this idea can be transported in some way to deduce similar results for $\zeta(s)$ on and around vertical lines $\sigma + i\mathbb{R}$ inside the critical strip. A central role is played by the following extension of Kronecker's approximation theorem due to Weyl [100]: *Let $a_1, \ldots, a_N$ be real numbers, linearly independent over $\mathbb{Q}$, and let γ be a subregion of the N-dimensional unit cube with Jordan content Γ.*[13] *Then*

$$\lim_{T \to \infty} \frac{1}{T} \operatorname{meas} \{\tau \in (0, T) \, : \, (\tau a_1, \ldots, \tau a_N) \in \gamma \bmod 1\} = \Gamma.$$

Moreover, suppose that the curve

$$\{\omega(\tau)\} := (\{\omega_1(\tau)\}, \ldots, \{\omega_N(\tau)\}),$$

is uniformly distributed mod 1 *in* $\mathbb{R}^N$ (extending the discrete and one-dimensional definition from Sect. 2 in a natural way). *Let $\mathscr{D}$ be a closed and Jordan measurable subregion of the unit cube in $\mathbb{R}^N$ and let Ω be a family of complex-valued continuous functions defined on $\mathscr{D}$. If Ω is uniformly bounded and equicontinuous, then*

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(\{\omega(\tau)\}) \mathbf{1}_{\mathscr{D}}(\tau) \, \mathrm{d}\tau = \int_{\mathscr{D}} f(x_1, \ldots, x_N) \, \mathrm{d}x_1 \ldots \mathrm{d}x_N$$

uniformly with respect to $f \in \Omega$, where $\mathbf{1}_{\mathscr{D}}(\tau)$ is equal to 1 *if $\omega(\tau) \in \mathscr{D}$* mod 1, *and* 0 *otherwise.* A proof of Weyl's theorem can be found in his paper [98] as well as in Karatsuba and Voronin [52]. In [41], Sect. 23.6, Hardy and Wright state a multi-dimensional analogue of Kronecker's theorem and comment on this result as *one of those mathematical theorems which assert (…) that what is not impossible will happen sometimes however improbable it may be*. Outside mathematics this is

[13]Note that the notion of Jordan content is more restrictive than the notion of Lebesgue measure. But, if the Jordan content exists, then it is also defined in the sense of Lebesgue and equal to it.

known as " Murphy's law". The unique prime factorization of integers implies the linear independence of the logarithms of the prime numbers over the field of rational numbers. Thus, in some sense, the logarithms of prime numbers behave like random variables and everything that can happen will happen!

Exploiting this idea, and sometimes further methods, namely addition of convex sets, Bohr and his school obtained plenty of remarkable and beautiful results on the value-distribution of the zeta-function, e.g. that the set of values of $\zeta(\sigma + it)$ is dense in $\mathbb{C}$ as t ranges through $\mathbb{R}$ for any fixed $\frac{1}{2} < \sigma \leq 1$ (see [85], Chap. XI). Notice that the problem whether the values taken by the zeta-function on the critical line lie dense in the complex plane is still unsolved.

However, the most spectacular statement in the value-distribution theory of the Riemann zeta-function was found by Voronin [93] in 1975 who discovered the following remarkable approximation property of the zeta-function: *Let* $0 < r < \frac{1}{4}$ *and* $g(s)$ *be a non-vanishing continuous function defined on the disk* $|s| \leq r$*, which is analytic in the interior of the disk. Then, for any* $\varepsilon > 0$*, there exists a real number* $\tau > 0$ *such that*

$$\max_{|s| \leq r} \left| \zeta\left(s + \tfrac{3}{4} + i\tau\right) - g(s) \right| < \varepsilon;$$

moreover, the set of all $\tau \in [0, T]$ *with this property has positive lower density with respect to the Lebesgue measure.* This is the so-called universality theorem since it allows the approximation of a huge class of target functions by a single function, namely the Riemann zeta-function. Also here a key role in Voronoin's proof is played by Weyl's refinement of Kronecker's approximation theorem. Besides Voronin's original proof there is a probabilistic approach to universality due to Bagchi, Reich, Laurinčikas and further developed by many others (see [62, 82]). In this method the pointwise ergodic theorem due to Birkhoff replaces the use of Weyl's uniform distribution theorem in Voronin's approach.

It was Edmund Landau [60] who started in his invited talk at the occasion of the fifth International Mathematical Congress held at Cambridge in 1912 a new direction in the value distribution theory of the zeta-function. He announces this line of investigation as follows:

> *Now let me discuss some different investigations about* $\zeta(s)$*. Given an analytic function, the points for which this function is* 0 *are very important; however, of equal interest are those points where the function assumes a given value* a*. It is easy to prove that* $\zeta(s)$ *takes any value* a*. But where do the roots of* $\zeta(s) = a$ *lie?*[14]

Notice that for Landau the distribution of the roots of

$$\zeta(s) = a$$

[14] "Ich komme jetzt zu einigen anderen Untersuchungen über $\zeta(s)$. Es sind bei einer analytischen Funktion die Punkte, an denen sie 0 ist, zwar sehr wichtig; ebenso interessant sind aber die Punkte, an denen sie einen bestimmten Wert a annimmt. Zu beweisen, dass $\zeta(s)$ jeden Wert a annimmt, ist ein leichtes. Wo liegen aber die Wurzeln von $\zeta(s) = a$?"

is for each complex value a equally important. These roots are called a-points and will be denoted by $\rho_a = \beta_a + i\gamma_a$. In the following year Landau [9] proved that there is an a-point near any trivial zero $s = -2n$ for any sufficiently large positive integer n, which we shall call trivial. One can show that the trivial a-points are not uniformly distributed modulo one (since they lie too close to the trivial zeros $-2n$, see [64]). All other a-points are said to be nontrivial. For any fixed a, there exist left and right half-planes free of nontrivial a-points (see formula (28), resp. [9]). Moreover, Landau [9] obtained an asymptotic formula for the number $N_a(T)$ of nontrivial a-values with imaginary part γ_a satisfying $0 < \gamma_a \leq T$, namely,

$$N_a(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e c_a} + O(\log T), \tag{26}$$

as $T \to \infty$, where $c_a = 1$ if $a \neq 1$, and $c_1 = 2$. Here and in the sequel the a-points are counted according to multiplicities and the multiplicity of an a-point $\rho_a = \beta_a + i\gamma_a$ is therefore bounded by $O(\log(3 + |\gamma_a|))$. The paper [9] of Bohr et al. is of special interest in our context: it was published exactly 100 years ago, when in Göttingen Weyl was proving his powerful criterion, Theorem 3, and Landau was at the same time professor at Göttingen. This very paper consists of three independent chapters, the first belonging essentially to Bohr, the second to Landau, and the third to Littlewood; it had been submitted in November 1913, the same year as Weyl proved his powerful criterion for uniform distribution modulo one.[15] The asymptotic formula (26) extends the Riemann–von Mangoldt-formula for the number $N(T)$ of nontrivial zeros to arbitrary a-points and shows that the main term is independent of a. Finally, Landau [9] proved that almost all a-points are clustered around the critical line provided the Riemann hypothesis is true. The latter assumption was removed by Levinson [64] who showed that *all but* $O(N_a(T)/\log\log T)$ *of the* a*-points* $\rho_a = \beta_a + i\gamma_a$ *with imaginary part in* $\gamma_a \in (T, 2T)$ *satisfy*

$$|\beta_a - \tfrac{1}{2}| < \frac{(\log\log T)^2}{\log T}. \tag{27}$$

His reasoning is based on the identity

$$2\pi \sum_{\substack{T<\gamma_a\leq T+U \\ \beta_a>-b}} (\beta_a + b) = \int_T^{T+U} \log|\zeta(-b+it) - a|\,\mathrm{d}t$$
$$-U\log|1-a| + O(\log T)$$

with some real constant b (as follows from Littlewood's lemma).

[15] And we shall make use of both results later on!

In some literature, density theorems showing that *most* of the zeros of $\zeta(s)$ lie close to the critical line were interpreted as an indicator for the truth of the Riemann hypothesis, however, this is only correct if the quantitative difference to the clustering of arbitrary a-points is taken into account (as, for example, (23) vs. Levinson's theorem above). We illustrate this observation with a quotation from Levinson [64]:

> *In his recent book (...) Edwards states that the clustering of the zeros of $\zeta(s)$ near $\sigma = 1/2$, first proved by Bohr and Landau (...), is the best existing evidence for the Riemann Hypothesis. Titchmarsh (...) also emphasizes with italics the clustering phenomenon of the zeros of $\zeta(s)$. It will be shown here that for any complex a the roots of $\zeta(s) = a$ cluster at $\sigma = 1/2$ and so, in this sense, the case $a = 0$ is not special. However, (...) it is clear that the clustering for the case $a = 0$ is more pronounced than for $a \neq 0$...*

The books in question are [16] and [85] by Edwards and Titchmarsh, respectively. It seems that Landau's conditional results on the distribution of a-points have been forgotten. As Levinson pointed out, a general clustering of a-points around the critical line is true, not only for zeros. However, in the case of zeros the quantity of those zeros which do not lie inside this cluster set are smaller than for any other a-points.

6 Generalizing Landau's Theorem and Applications

Landau's formula, resp. theorem (19) has been extended and generalized in different ways. For instance, Kaczorowski et al. [51] introduced weights in order to obtain an error term of more flexible shape. We aim at an application to the distribution of a-points, hence our generalization is completely different: Following [83] we start with

Theorem 6. *Let x be a positive real number $\neq 1$. Then, as $T \to \infty$,*

$$\sum_{0<\gamma_a<T} x^{\rho_a} = (\alpha(x) - x\Lambda(\tfrac{1}{x}))\frac{T}{2\pi} + O\big(T^{\frac{1}{2}+\varepsilon}\big),$$

where $\alpha(x)$ and $\Lambda(x)$ equal the Dirichlet series coefficients in (29) and (30), respectively, if $x = n$ or $x = 1/n$ for some integer $n \geq 2$, and zero otherwise.

The implicit constant in the error term may depend here and elsewhere on x. The theorem gives an explicit formula with a-points in place of zeros. The case $a = 0$ was first treated by Landau [59]; later improvements, resp. generalizations are due to Gonek [33], Fujii [25–27] (with an improved uniform error estimate), and Murty and Murty [69] (for L-functions from the Selberg class). For the special case $a = 2$ Hille [43] proved that the coefficients $f(n)$ of the Dirichlet series for $(\zeta(s) - 2)^{-1}$ count the number of representations of n as a product of integers strictly greater than one; this allows a simple computation of the arithmetical function $n \mapsto \alpha(n)$ via the convolution $-\log = \alpha * f$. For other values of a the author is not aware of any number-theoretical meaning of α.

In order to generalize Landau's formula (19) we shall use some ideas from Garunkštis and Steuding [30].

Proof. First we assume $a \neq 1$. Since $\zeta(s) - a$ has a convergent Dirichlet series representation for sufficiently large $\mathrm{Re}\, s$, it follows that there exists a half-plane $\mathrm{Re}\, s > B$ which is free of a-points. In order to compute such an abscissa B explicitly, we assume $\sigma := \mathrm{Re}\, s > 1$ and estimate

$$|\zeta(s) - 1| \leq \sum_{n\geq 2} n^{-\sigma} < \int_1^\infty u^{-\sigma}\,\mathrm{d}u = \frac{1}{\sigma - 1}.$$

Thus,

$$\zeta(s) - a \neq 0 \qquad \text{for} \quad \sigma > 1 + \frac{1}{|a-1|}. \tag{28}$$

Consequently, as shown by Landau [61], the inverse $(\zeta(s) - a)^{-1}$ has a convergent Dirichlet series expansion in the same half-plane. After multiplying with the convergent Dirichlet series for $\zeta'(s)$, we end up with

$$\frac{\zeta'(s)}{\zeta(s) - a} = \sum_{n\geq 2} \frac{\alpha(n)}{n^s}; \tag{29}$$

In case of $a = 0$ this equals the logarithmic derivative of the zeta-function

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n\geq 2} \frac{\Lambda(n)}{n^s} \tag{30}$$

which plays a central role in Landau's proof of Theorem 6 in the special case $a = 0$ as well as in proofs of other explicit formulae in prime number theory. Notice that $\alpha(n) = -\Lambda(n)$ if $a = 0$. Moreover, we observe that both series have no constant term since the series for $\zeta'(s)$ has not. By partial summation it follows that the abscissa of convergence and the abscissa of absolute convergence of an ordinary Dirichlet series differ by at most one. Hence, the abscissa of absolute convergence of the Dirichlet series (29) is less than or equal to $B := 2 + |a-1|^{-1}$ (see [40]). In view of (26) for any positive T_0 we can find some $T \in [T_0, T_0 + 1)$ such that the distance between T to the nearest ordinate γ_a of the a-points is bounded by $(\log T)^{-1}$. Moreover, let $b := 1 + (\log T)^{-1}$. Then only finitely many a-points lie to the left of the vertical line $\mathrm{Re}\, s = 1 - b$. Finally, note that the logarithmic derivative of $\zeta(s) - a$ has simple poles at each a-point with residue equal to the order. Hence

$$\sum_{0<\gamma_a<T} x^{\rho_a} = \frac{1}{2\pi i} \int_{\mathsf{R}} x^s \frac{\zeta'(s)}{\zeta(s) - a}\,\mathrm{d}s,$$

where R denotes the counterclockwise oriented rectangle with vertices $B+i, B+iT, 1-b+iT, 1-b+i$ and the error term arises from possible contributions of a-points outside R. We rewrite

$$\int_{\mathsf{R}} x^s \frac{\zeta'(s)}{\zeta(s)-a}\,\mathrm{d}s = \left\{\int_{B+i}^{B+iT} + \int_{B+iT}^{1-b+iT} + \int_{1-b+iT}^{1-b+i} + \int_{1-b+i}^{B+i}\right\} x^s \frac{\zeta'(s)}{\zeta(s)-a}\,\mathrm{d}s = \sum_{j=1}^{4} I_j,$$

say. We start with the vertical integral on the right-hand side. Interchanging summation and integration we find

$$I_1 = \sum_{n\geq 2}\alpha(n)\int_{B+i}^{B+iT}\left(\frac{x}{n}\right)^s \mathrm{d}s = i\alpha(x)T + O(1),$$

where $\alpha(x)$ equals the coefficient $\alpha(n)$ in the Dirichlet series expansion (29) if $x = n$ or $x = 1/n$, and $\alpha(x) = 0$ otherwise (i.e., $x \neq n, 1/n$ for all $2 \leq n \in \mathbb{N}$).

Next we consider the horizontal integrals. Recall the functional equation,

$$\zeta(s) = \Delta(s)\zeta(1-s)\,, \quad \text{where} \quad \Delta(s) = 2(2\pi)^{s-1}\sin\tfrac{\pi s}{2}\Gamma(s).$$

By Stirling's formula, we find

$$\zeta(\sigma \pm it) \asymp |t|^{\frac{1}{2}-\sigma}|\zeta(1-\sigma \mp it)|,$$

uniformly in σ, as $|t| \to \infty$. This in combination with the Phragmén-Lindelöf principle yields the bound

$$\zeta(\sigma+it) \ll t^{\mu(\sigma)+\varepsilon},$$

where

$$\mu(\sigma) \ll \begin{cases} 0 & \text{if} \quad \sigma > 1, \\ \frac{1}{2}(1-\sigma) & \text{if} \quad 0 \leq \sigma \leq 1, \\ \frac{1}{2}-\sigma & \text{if} \quad \sigma < 0. \end{cases}$$

Using the partial fraction decomposition for the logarithmic derivative (as in [30]), we get

$$I_2 = -\left\{\int_{-(\log T)^{-1}}^{1} + \int_1^B\right\} x^{\sigma+iT}\frac{\zeta'(\sigma+iT)}{\zeta(\sigma+iT)-a}\mathrm{d}\sigma \ll xT^{\frac{1}{2}+\varepsilon} + x^B T^{\varepsilon}.$$

Next we evaluate the vertical integral on the left-hand side $\mathrm{Re}\, s = 1-b$ of the contour. It is not difficult to show

$$\zeta(\sigma + it) \gg \frac{t^{\frac{1}{2}-\sigma}}{\log t} \tag{31}$$

(otherwise consult Lemma 4 in [30]). Hence, the left-hand side has absolute value larger than $1/2$ for $t \geq t_0$. For such values of t we may expand the logarithmic derivative into a geometric series

$$\frac{\zeta'(s)}{\zeta(s) - a} = \frac{\zeta'}{\zeta}(s)\frac{1}{1 - a/\zeta(s)} = \frac{\zeta'}{\zeta}(s)\left(1 + \sum_{k\geq 1}\left(\frac{a}{\zeta(s)}\right)^k\right).$$

This gives

$$I_3 = O(1) - \int_{1-b+it_0}^{1-b+iT} x^s \frac{\zeta'}{\zeta}(s)\left(1 + \sum_{k\geq 1}\left(\frac{a}{\zeta(s)}\right)^k\right) \mathrm{d}s.$$

In view of (31) we find

$$\begin{aligned}\int_{1-b+it_0}^{1-b+iT} x^s \frac{\zeta'}{\zeta}(s)\sum_{k\geq 1}\left(\frac{a}{\zeta(s)}\right)^k \mathrm{d}s &\ll x^{1-b}T(\log T)\sum_{k\geq 1}\left(\frac{\log T}{T^{\frac{1}{2}}}\right)^k \\ &\ll x^{1-b}T^{\frac{1}{2}}(\log T)^2.\end{aligned}$$

Using the functional equation, we get in view of (30)

$$\begin{aligned}-\int_{1-b+it_0}^{1-b+iT} x^s \frac{\zeta'}{\zeta}(s)\,\mathrm{d}s &= \int_{1-b+it_0}^{1-b+iT} x^s\left(\frac{\zeta'}{\zeta}(1-s) - \frac{\Delta'}{\Delta}(s)\right)\mathrm{d}s \\ &= -ix^{1-b}\sum_{n\geq 2}\Lambda(n)n^{-b}\int_{t_0}^{T}(xn)^{it}\,\mathrm{d}t + \\ &\quad + ix^{1-b}\int_{t_0}^{T} x^{it}(\log\tfrac{t}{2\pi} + O(t^{-1}))\,\mathrm{d}t.\end{aligned}$$

The first term on the right-hand side equals $ix\Lambda(\frac{1}{x})T + O(1)$ whereas the second term can be bounded by $\log T/|\log x|$. Finally, the remaining horizontal integral is independent of T, hence $I_4 \ll 1 + x^B$. Thus we arrive at

$$\sum_{0<\gamma_a<T} x^{\rho_a} = \left\{\alpha(x) - x\Lambda(\tfrac{1}{x})\right\}\frac{T}{2\pi} + O_x\left(T^{\frac{1}{2}+\varepsilon}\right).$$

In order to have the asymptotic formula uniform for all T we add an error of size $O(\log T)$. This proves the theorem for $a \neq 1$. If $a = 1$, we consider the function

$$\ell(s) = 2^s(\zeta(s) - 1) = 1 + \sum_{n\geq 3} \left(\frac{2}{n}\right)^s$$

and its logarithmic derivative

$$\frac{\ell'}{\ell}(s) = \log 2 + \frac{\zeta'(s)}{\zeta(s) - 1}.$$

Applying contour integration to this logarithmic derivative yields the asymptotic formula for $a = 1$. □

The above reasoning with $x = 1$ and a more careful treatment of the error term leads to the asymptotic formula (26) for $N_a(T)$.

Next, using our new explicit formula, we shall generalize the result on the uniform distribution modulo one from the zeros to a-points:

Theorem 7. *For any complex number a and any real $\alpha \neq 0$, the sequence of numbers $\alpha\gamma_a$ (with γ_a denoting the ordinates of the a-points) are uniformly distributed modulo one.*

In view of our generalization of Landau's theorem the proof of the latter result is straightforward.

Proof. Recall Levinson's theorem [64] from the previous section that all but $O(N_a(T)/\log\log T)$ of the a-points $\rho_a = \beta_a + i\gamma_a$ with imaginary part in $\gamma_a \in (T, 2T)$ satisfy (27). More precisely, let $\delta(T) = (\log\log T)^2/\log T$; then Levinson showed that the number of a-points $\rho_a = \beta_a + i\gamma_a$ for which $|\beta_a - 1/2| > \delta$ and $T < \gamma_a < 2T$ is bounded by $T\log T/\log\log T$. This yields

$$\sum_{T<\gamma_a\leq 2T} |\beta_a - \tfrac{1}{2}| = \left\{ \sum_{\substack{T<\gamma_a\leq 2T \\ \beta_a - 1/2 > \delta}} + \sum_{\substack{T<\gamma_a\leq 2T \\ |\beta_a - 1/2| \leq \delta}} + \sum_{\substack{T<\gamma_a\leq 2T \\ \beta_a - 1/2 < -\delta}} \right\} |\beta_a - \tfrac{1}{2}|$$

$$\ll \frac{T\log T}{\log\log T} + T(\log\log T)^2.$$

Using this with $2^{-k}T$ in place of T and adding the corresponding estimates over all $k \in \mathbb{N}$, we deduce

$$\sum_{0<\gamma_a\leq T} |\beta_a - \tfrac{1}{2}| \ll \frac{T\log T}{\log\log T} = o(N_a(T)). \tag{32}$$

Since

$$\exp(y) - 1 = \int_0^y \exp(t)\,\mathrm{d}t \ll |y| \max\{1, \exp(y)\},$$

we find, for $x \neq 1$,

$$|x^{\frac{1}{2}+i\gamma_a} - x^{\beta_a+i\gamma_a}| \leq x^{\beta_a}|\exp((\tfrac{1}{2}-\beta_a)\log x)-1| \leq |\beta_a - \tfrac{1}{2}|\max\{x^{\beta_a}, x^{\frac{1}{2}}\}|\log x|.$$

Hence,

$$\frac{1}{N_a(T)} \sum_{0<\gamma_a\leq T} |x^{\frac{1}{2}+i\gamma_a} - x^{\beta_a+i\gamma_a}| \leq \frac{X}{N_a(T)} \sum_{0<\gamma_a\leq T} |\beta_a - \tfrac{1}{2}|,$$

where $X = \max\{x^B, 1\}|\log x|$ and B is the upper bound for the real parts of the a-points. In view of (32) we have

$$\frac{1}{N_a(T)} \sum_{0<\gamma_a\leq T} \left(x^{\frac{1}{2}+i\gamma_a} - x^{\beta_a+i\gamma_a}\right) \ll \frac{X}{\log\log T}.$$

Recall Theorem 6,

$$\sum_{0<\gamma_a<T} x^{\beta_a+i\gamma_a} \ll T;$$

here and in the sequel we drop the dependency on x since only the limit as $T \to \infty$ is relevant. Hence, we obtain

$$\frac{1}{N_a(T)} \sum_{0<\gamma_a\leq T} x^{\frac{1}{2}+i\gamma_a} \ll \frac{1}{\log\log T}.$$

Let $x = z^m$ with some positive real number $z \neq 1$ and $m \in \mathbb{N}$. Then, after dividing the previous formula by $x^{\frac{1}{2}}$, we may deduce

$$\lim_{T\to\infty} \frac{1}{N_a(T)} \sum_{0<\gamma_a\leq T} \exp(im\gamma_a \log z) = 0$$

Now Weyl's criterion, Theorem 3, implies that the sequence of numbers $\frac{1}{2\pi}\gamma_a \log z$ is uniformly distributed modulo one. □

We shall give a new application of Theorem 7. Given sequences of monotonically increasing positive real numbers $\mathfrak{a} = (a_n)_n$ and $\mathfrak{b} = (b_k)_k$, both being uniformly distributed modulo one, Akbary and Murty [1] showed that then also the union of both, namely the sequence $\mathfrak{a} \cup \mathfrak{b} := (a_n, b_k)_{n,k}$ is uniformly distributed modulo one. Here the sequence $(a_n, b_k)_{n,k}$ is ordered according to the absolute value of its elements. The easy proof is as follows. Denote by $N_{\mathfrak{a}}(x)$ the number of elements a_n from $\mathfrak{a}$ satisfying $a_n \leq x$. Then,

$$\frac{1}{N_{\mathfrak{a}\cup\mathfrak{b}}(x)}\left|\sum_{a_n,b_k\le x}\mathrm{e}(m(a_n,b_k))\right| = \frac{1}{N_{\mathfrak{a}}(x)+N_{\mathfrak{b}}(x)}\left|\sum_{a_n\le x}\mathrm{e}(ma_n)+\sum_{b_k\le x}\mathrm{e}(mb_k)\right|$$
$$\le \frac{1}{N_{\mathfrak{a}}(x)}\left|\sum_{a_n\le x}\mathrm{e}(ma_n)\right| + \frac{1}{N_{\mathfrak{b}}(x)}\left|\sum_{b_k\le x}\mathrm{e}(mb_k)\right|.$$

Hence, Weyl's criterion, Theorem 3, implies the assertion. As a consequence, we may deduce from Theorem 7 the following

Corollary 1. *Let $M \in \mathbb{N}$ and $\alpha_1,\ldots,\alpha_M$ be arbitrary positive real numbers and $a_1,\ldots,a_M$ be arbitrary complex numbers. Then the sequence*

$$\cup_{1\le m\le M}(\alpha_m\gamma_{a_m}) = \{\alpha_1\gamma_{a_1},\ldots,\alpha_M\gamma_{a_M}\}$$

is uniformly distributed modulo one. In particular, the ordinates of the zeros of $P(\zeta(s))$ are uniformly distributed modulo one, where P is any non-constant polynomial with complex coefficients.

The application to $P(\zeta(s))$ follows from the factorization $P(\zeta) = \prod_j(\zeta - a_j)$ with certain complex numbers a_j by the fundamental theorem of algebra and an application of the uniform distribution modulo one of the union of the imaginary parts of the a_j-points.

7 Discrepancy and Further Concluding Remarks

We conclude with a few further problems related to applications of uniform distribution modulo one in the context of the Riemann zeta-function. Already Weyl noticed that the appearing limits are uniform which has been studied ever since under the notion of *discrepancy*. This topic has important applications, for instance, in billiards where we may ask how soon an aperiodic ray of light will visit a given domain? First results for effective billiards are due to Weyl [99], interesting and surprising results on square billiards have recently been discovered by Beck [2] showing that the typical billiard path is extremely uniform far beyond what one might expect. Also important in this setting are effective versions of the inhomogeneous Kronecker approximation theorem from the introductory section as, for example, [94]. In the case of the zeros of the Riemann zeta-function first estimates for the discrepancy were already given by Hlawka [45]; using the Erdös-Turán inequality he proved

$$\sup_{0\le\alpha\le1}\frac{1}{N(T)}\left|\sharp\left\{0<\gamma<T\,:\,\{\gamma\tfrac{1}{2\pi}\log X\}-\alpha N(T)\right\}\right| \ll \frac{\log X}{\log\log T},$$

valid for $X > 1$ and all sufficiently large T, where C is an explicit positive constant; the right-hand side can be replaced by $\frac{\log X}{\log T}$ if the Riemann hypothesis is assumed. Further results in this direction are due to Fujii in a series of papers [23, 24, 29], Akbary and Murty [1], and Ford, Soundararajan, respectively. In Zaharescu [21, 22] connections to Montgomery's pair correlation conjecture and the distribution of primes in short intervals are established. In all these investigations the dependency of the error term in Landau's explicit formula on x is relevant. Improvements of the explicit formula with an error terms that is uniform in x were given in particular by Fujii [25, 26] and Gonek [32, 33].

The most natural question seems to be *whether the uniform distribution modulo one is a common feature for all arithmetical L-functions.* There is no precise definition of an L-function. M.N. Huxley said "What is a zeta-function (or an L-function)? We know one when we see one". Therefore it seems natural to consider classes of L-functions. The Selberg class provides a rather general axiomatic setting for L-functions (see [79] and [82] for its definition); the most simple examples are Dirichlet L-functions to residue class characters (or L-series as in the above quotation). These L-functions share many patterns with the Riemann zeta-function and for those it was already noticed by Hlawka [44] that the results for the zeta-function carry over without any difficulty. However, the situation for L-functions associated with modular forms is more delicate; Hlawka finishes his investigations with the following words:

> *The investigations can be generalized to L-series as well as to Dedekind zeta-functions. (...) It would be interesting to extend this to other zeta-functions as, for example, $\sum \frac{\tau(n)}{n^s}$, where τ is the well-known Ramanujan function.*[16]

The Ramanujan function $n \mapsto \tau(n)$ provides the Fourier coefficients of the modular discriminant; the associated L-function is the prototype of a degree two element of the Selberg class arising from a modular form. Akbary and Murty [1] proved conditionally uniform distribution modulo one for the non-trivial zeros of L-functions $L(s)$ from a certain class containing the Selberg class; however, their condition is a conjecture on power moments, resp. an analogue of Levinson's bound, namely the so-called *average density hypothesis* claiming that

$$\sum_{\substack{0 \leq \gamma < T \\ \beta > \frac{1}{2}}} \left(\beta - \tfrac{1}{2}\right) = o(N_L(T)), \tag{33}$$

where the nontrivial zeros of $L(s)$ are denoted by $\beta + i\gamma$ and their number up to height T is counted by $N_L(T)$. Such an estimate implies that the zeros are clustered around the critical line. In the case of L-functions to modular forms Akbary and Murty [1] succeeded to prove (33), however, different to Hlwaka's statement, for

[16] "Die Überlegungen lassen sich auf L-Reihen wie auf die Dedekindsche Zetafunktion übertragen. (...) Interessant wäre es, dies auf andere Zetafunktionen, wie z.B. auf die $\sum \frac{\tau(n)}{n^s}$ auszudehnen, wo τ die bekannte Ramanujansche Funktion ist."

Dedekind zeta-function the uniform distribution modulo one of the zeros has been proved only in the case of Abelian number fields $\mathbb{K}/\mathbb{Q}$ (since in this case the Dedekind zeta-function splits into a product of more simple L-functions). Obvious question are whether Condition (33) can be proved in general and what can be done with respect to a-points. A certain progress here is due to Jakhlouti et al. [49] who considered an extension of Theorem 7 to L-functions with polynomial Euler products, and Garunkštis et al. [31] obtained an analogue for certain Selberg zeta-functions. The distribution of a-points of an L-function from the Selberg class has been started already with Selberg's influential paper [79]; further results in this direction are in particular due to Gonek et al. [34].

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On the Energy of Graphs

Irene Triantafillou

Abstract The energy of a graph, $E(G)$, is the sum of the absolute values of its eigenvalues. The energy concept has received a high interest over the last decade, at first due to its various applications in chemistry and then in its own right. This paper focuses on some of the most important results on the bounds for the energy of general graphs and the energy of bipartite graphs. Some known bounds for the change in the energy of a graph after deleting a vertex or an edge are also considered.

Keywords Energy of graphs • Graph energy change • Bipartite graphs

1 Introduction and Preliminaries

Let $A(G)$ be the *adjacency matrix* of a simple, finite, undirected graph, G, with vertex set $V(G)$ and edge set $E(G)$. The set of the eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of $A(G)$ is the *spectrum* of G. As the adjacency matrix is symmetric, its eigenvalues are real and have a sum equal to zero.

According to the interlacing theorem [22]: If G is a graph with spectrum $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and the spectrum of the graph obtained upon deleting a vertex u_1, $G - u_1$, is $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1}$, then the spectrum of $G - u_1$ is "interlaced" with the spectrum of G, and

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n. \tag{1}$$

Another important property of the eigenvalues of a graph G is: The number of closed walks of length k in G equals $\sum_i \lambda_i^k$, the kth spectral moment of $A(G)$. In particular, the trace of A^2 is:

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$$\sum_i \lambda_i^2 = 2m, \tag{2}$$

and since $\sum_i \lambda_i = 0$,

$$\sum_{i<j} \lambda_i \lambda_j = -m. \tag{3}$$

A graph, G, is *singular* if the adjacency matrix, $A(G)$, is a singular matrix. The *nullity*, $\eta(G)$, of a singular graph G is the algebraic multiplicity of the eigenvalue zero in the graph's spectrum.

A *strongly regular graph* with parameters (v, k, λ, μ) is a graph with v vertices, such that each vertex has precisely k neighbors, every pair of its adjacent vertices has λ common neighbors, and every pair of non-adjacent vertices has μ common neighbors.

A $2-(v, k, \lambda)$-*design* is a family of k blocks of a set of v points, such that each 2-set of points lies in exactly λ blocks.

A *semiregular bipartite graph* is a bipartite graph whose vertices in the same class of bipartition have the same degree.

The concept of graph energy was first defined by Gutman in [9] and it emerged from the idea of Hückel energy in theoretical chemistry. Coulson [4] provided the following integral formula for the energy of a graph, $E(G)$:

$$E(G) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left(n - \frac{\mathrm{i}x\phi'(\mathrm{i}x)}{\phi(\mathrm{i}x)} \right) dx, \tag{4}$$

where $\phi(x)$ is the characteristic polynomial of graph G, and $\phi'(x)$ its derivative.

The energy of a graph, G, is defined as the sum of the absolute values of its eigenvalues:

$$E(G) = \sum_{i=1}^{n} |\lambda_i|. \tag{5}$$

In Sect. 2 of this paper, we focus on some of the most important bounds for the energy of a general graph and of a bipartite graph, in terms of a graph's vertices, edges, degree sequence, and spectral moments. In Sect. 3, we cite several known bounds for the change in energy upon deleting a vertex or edge. We conclude this paper with some additional bounds for the energy of bipartite graphs.

We denote the *complete graph* of order n by K_n and the *complete r-partite graph* by $K_{t,t,\ldots,t}$. The *path* and *cycle* with n vertices are denoted by P_n and C_n, respectively. By T_n we denote the *tree* and by S_n the *star graph* of order n.

2 Graph Energy Bounds

The calculation of the energy of certain graphs, such as the path, the cycle, or the complete graph, is straightforward as their spectrum is known. In this section, we focus on some of the most important bounds for the energy of general graphs and the energy of bipartite graphs.

McClelland considers the n vertices and m edges of a graph G for the following energy bounds:

Theorem 1 ([19]). *For an* (n, m)*-graph* G*,*

$$\sqrt{2m + n(n-1)\left|detA\right|^{2/n}} \leq E(G) \leq \sqrt{2mn}. \tag{6}$$

The upper bound is obtained by the use of the Cauchy–Schwartz inequality to $(1, 1, \dots, 1)$ and $(|\lambda_1|, |\lambda_2|, \dots, |\lambda_n|)$, so that

$$E(G) \leq \sqrt{n}\sqrt{\sum_i \lambda_i^2}. \tag{7}$$

Since $\sum_i \lambda_i^2 = 2m$, we get the desired result.

For the lower bound, Eq. (2) and the arithmetic-geometric means inequality is used in

$$E^2(G) = \left(\sum_i |\lambda_i|\right)^2 = \sum_i |\lambda_i|^2 + 2\sum_{i<j} \left|\lambda_i \lambda_j\right|. \tag{8}$$

The upper and lower bound of Ineq. (6) can be improved for singular graphs, by taking into account only the nonzero eigenvalues:

Proposition 1 ([9]). *Let* G *be a graph on* n *vertices, and nullity* $\eta(G)$*. Then,*

$$E(G) \leq \sqrt{2(n - \eta(G))m}. \tag{9}$$

Proposition 2 ([2]). *Let* G *be a graph on* n *vertices, and nullity* $\eta(G)$*. Then,*

$$E(G) \geq n - \eta(G). \tag{10}$$

In regard to the edges of a graph, a lower and upper bound for the energy of a graph was given:

Theorem 2 ([3]). *Let* G *be a graph with* m *edges. Then,*

$$2\sqrt{m} \leq E(G) \leq 2m. \tag{11}$$

On the left side equality holds if and only if G is a complete bipartite graph and equality on the right side if and only if G consists of m copies of the complete graph K_2.

The lower bound of the above inequality can be easily derived by using Eqs. (2) and (3) in $E^2(G) = \sum_{i=1}^{n} \lambda_i^2 + 2\left|\sum_{i<j} \lambda_i \lambda_j\right|$.

The upper bound takes into consideration that the maximum number of vertices of a graph with m edges is $2m$ (this implies m copies of the complete graph K_2).

If G is a graph with no isolated vertices, then a lower bound with reference to its vertices is given by the following theorem.

Theorem 3 ([10]). *Let G be a n-vertex graph, with no isolated vertices. Then,*

$$E(G) \geq 2\sqrt{n-1}. \tag{12}$$

If G is connected, then $m \geq n-1$ and by using the left side of Ineq. (11), $E(G) \geq 2\sqrt{m} \geq 2\sqrt{n-1}$. The proof is analogous if G is disconnected.

From Ineq. (12), it is obvious that the star graph has minimum energy among all n-vertex graphs with no isolated vertices.

An upper bound for a graph with n vertices was given by Koolen and Moolton:

Theorem 4 ([12]). *Let G be a graph with n vertices. Then,*

$$E(G) \leq \frac{1}{2}n\left(\sqrt{n}+1\right), \tag{13}$$

with equality if and only if G is a strongly regular graph with parameters $\left(n, \frac{n+\sqrt{n}}{2}, \frac{n+2\sqrt{n}}{4}, \frac{n+2\sqrt{n}}{4}\right)$.

and later an upper bound for a bipartite graph with n vertices was provided by the same authors:

Theorem 5 ([13]). *Let G be a n-vertex bipartite graph. Then,*

$$E(G) \leq \frac{1}{\sqrt{8}}n\left(\sqrt{n}+\sqrt{2}\right). \tag{14}$$

Equality holds if and only if $n = 2v$ *and G is the incidence graph of a* $2-\left(v, \frac{v+\sqrt{v}}{2}, \frac{v+2\sqrt{v}}{4}\right)$*-design.*

In order to prove Ineq. (14), the following bound was taken into account:

Theorem 6 ([13]). *Let G be a bipartite graph with* $n > 2$ *vertices, and* $m \geq \frac{n}{2}$ *edges. Then,*

$$E(G) \leq 2\left(\frac{2m}{n}\right) + \sqrt{(n-2)\left[2m - 2\left(\frac{2m}{n}\right)^2\right]}. \tag{15}$$

Moreover, equality holds in (15) if and only if at least one of the following statements is true:

1. $n = 2m$ *and* $G = mK_2$.
2. $n = 2t$, $m = t^2$ *and* $G = K_{t,t}$.
3. $n = 2u$, $2\sqrt{m} < n < 2m$, *and* G *is the incidence graph of a symmetric* $2-(u,k,\lambda)$*-design with* $k = \frac{2m}{n}$ *and* $\lambda = \frac{k(k-1)}{u-1}$.

In the above theorem, Eq. (2) and the symmetry of the spectrum of the bipartite graph G are considered, in order to get:

$$\sum_{i=2}^{n-1} \lambda_i^2 = 2m - 2\lambda_1^2. \tag{16}$$

By the use of the Cauchy–Schwartz inequality to $(1, 1, \ldots, 1)$, $(|\lambda_2|, \ldots, |\lambda_{n-1}|)$, we get:

$$\sum_{i=2}^{n-1} |\lambda_i| \le \sqrt{(n-2)(2m - 2\lambda_1^2)}. \tag{17}$$

It follows that

$$E(G) \le 2\lambda_1 + \sqrt{(n-2)(2m - 2\lambda_1^2)}. \tag{18}$$

Since the function $F(x) := 2x + \sqrt{(n-2)(2m-2x^2)}$ decreases on the interval $\sqrt{\frac{2m}{n}} < x \le \sqrt{m}$ and $2m \ge n$, the result is obtained.

In order to get Ineq. (14), we only need to consider that Ineq. (15) is maximized when $m = \frac{n^2 + n\sqrt{2n}}{8}$. For a general graph Ineq. (15) is written:

Theorem 7 ([12]). *Let G be a graph with n vertices and m edges. If* $2m \ge n$, *then*

$$E(G) \le \left(\frac{2m}{n}\right) + \sqrt{(n-1)\left[2m - \left(\frac{2m}{n}\right)^2\right]}. \tag{19}$$

Moreover, equality holds in (19) if and only if $G \cong \frac{n}{2}K_2$, *or* $G \cong K_n$, *or G is a noncomplete connected strongly regular graph with two nontrivial eigenvalues both having absolute values equal to* $\sqrt{\frac{2m - \left(\frac{2m}{n}\right)^2}{n-1}}$.

An upper bound for the energy of a graph in terms of its vertex degree sequence is:

Theorem 8 ([28]). *If G is a graph with n vertices, m edges, and vertex degree sequence* $d_1, d_2, \ldots, d_n$ *then*

$$E(G) \leq \sqrt{\frac{\sum_{i=1}^{n} d_i^2}{n}} + \sqrt{(n-1)\left(2m - \frac{\sum_{i=1}^{n} d_i^2}{n}\right)}. \tag{20}$$

Equality in (20) holds if and only if G is either $\frac{n}{2}K_2$ *(if* $n = 2m$*),* K_n *(if* $m = n(n-1)/2$*), or a noncomplete connected strongly regular graph with two nontrivial eigenvalues both with absolute value* $\sqrt{\left(2m - \left(\frac{2m}{n}\right)^2\right)/\left(n-1\right)}$ *or* nK_1 *(if* $m = 0$*).*

For the proof of Ineq. (20), as in the proof of Ineq. (15), the Cauchy–Schwartz inequality is applied to $(1, 1, \ldots, 1)$, $(|\lambda_2|, \ldots, |\lambda_n|)$, to get

$$E(G) \leq \lambda_1 + \sqrt{(n-1)(2m - \lambda_1^2)} \tag{21}$$

and the inequation $\lambda_1 \geq \sqrt{\frac{\sum_{i=1}^{n} d_i^2}{n}}$ [27] is taken into account. In a similar way, the following bound for bipartite graphs was proved by also considering the symmetry of the spectrum of bipartite graphs.

Theorem 9 ([28]). *If G is a bipartite graph with* $n > 2$ *vertices, m edges, and vertex degree sequence* $d_1, d_2, \ldots, d_n$*, then*

$$E(G) \leq 2\sqrt{\frac{\sum_{i=1}^{n} d_i^2}{n}} + \sqrt{(n-2)\left(2m - \frac{2\sum_{i=1}^{n} d_i^2}{n}\right)}. \tag{22}$$

Equality in (22) holds if and only if G is either $\frac{n}{2}K_2$*, a complete bipartite graph, or the incidence graph of a symmetric* $2-(v,k,\lambda)$*-design with* $k = \frac{2m}{n}$ *and* $\lambda = \frac{k(k-1)}{v-1}$*,* $(n = 2v)$*, or* nK_1*.*

The 2-degree sequence, t_i, of a vertex $u_i \in V(G)$ is the sum of degrees of the vertices adjacent to u_i.

An upper bound in terms of a graph's degree sequence and 2-degree sequence is:

Theorem 10 ([26]). *Let G be a nonempty graph with n vertices, m edges, degree sequence* $d_1, d_2, \ldots, d_n$ *and 2-degree sequence* $t_1, t_2, \ldots, t_n$*. Then,*

$$E(G) \leq \sqrt{\frac{\sum_{i=1}^{n} t_i^2}{\sum_{i=1}^{n} d_i^2}} + \sqrt{(n-1)\left(2m - \frac{\sum_{i=1}^{n} t_i^2}{\sum_{i=1}^{n} d_i^2}\right)}. \tag{23}$$

Equality holds if and only if one of the following statements holds:

1. $G \cong \frac{n}{2}K_2$.
2. $G \cong K_n$.
3. *G is a nonbipartite connected p-pseudo-regular graph with three distinct eigenvalues* $\left(p, \sqrt{\frac{2m-p^2}{n-1}}, -\sqrt{\frac{2m-p^2}{n-1}}\right)$ *where* $p > \sqrt{\frac{2m}{n}}$*.*

For the proof of Ineq. (23), the Cauchy–Schwartz inequality is applied to get the function $F(x) := x + \sqrt{(n-1)(2m - x^2)}$ and since $\lambda_1 \geq \sqrt{\frac{\sum_{i=1}^n t_i^2}{\sum_{i=1}^n d_i^2}}$ the result is obtained.

For bipartite graphs Ineq. (23) is written:

Theorem 11 ([26]). *Let $G = (X, Y)$ be a nonempty bipartite graph with $n > 2$ vertices, m edges, degree sequence $d_1, d_2, \ldots, d_n$, and 2-degree sequence $t_1, t_2, \ldots, t_n$. Then,*

$$E(G) \leq 2\sqrt{\frac{\sum_{i=1}^n t_i^2}{\sum_{i=1}^n d_i^2}} + \sqrt{(n-2)\left(2m - 2\frac{\sum_{i=1}^n t_i^2}{\sum_{i=1}^n d_i^2}\right)}. \tag{24}$$

Equality holds if and only if one of the following statements holds:

1. $G \cong \frac{n}{2}K_2$.
2. $G \cong K_{r_1,r_2} \cup (n - r_1 - r_2)K_1$, *where* $r_1 r_2 = m$.
3. G *is a connected* (p_x, p_y)*-pseudo-semiregular bipartite graph with four distinct eigenvalues* $\left(\sqrt{p_x p_y}, \sqrt{\frac{2m-2p_x p_y}{n-2}}, -\sqrt{\frac{2m-2p_x p_y}{n-2}}, -\sqrt{p_x p_y}\right)$, *where* $\sqrt{p_x p_y} > \sqrt{\frac{2m}{n}}$.

Theorem 12 ([18]). *Let G be a nonempty simple graph with n vertices and m edges and let σ_i be the sum of the 2-degrees of vertices adjacent to vertex v_i. Then,*

$$E(G) \leq \sqrt{\frac{\sum_{i=1}^n \sigma_i^2}{\sum_{i=1}^n t_i^2}} + \sqrt{(n-1)\left(2m - \frac{\sum_{i=1}^n \sigma_i^2}{\sum_{i=1}^n t_i^2}\right)}. \tag{25}$$

Equality holds if and only if one of the following statements holds:

1. $G \cong \frac{n}{2}K_2$.
2. $G \cong K_n$.
3. G *is a non-bipartite connected graph satisfying* $\frac{\sigma_1}{t_1} = \cdots = \frac{\sigma_n}{t_n}$ *and has three distinct eigenvalues* $\left(p, \sqrt{\frac{2m-p^2}{n-1}}, -\sqrt{\frac{2m-p^2}{n-1}}\right)$ *where* $p = \frac{\sigma_1}{t_1} = \cdots = \frac{\sigma_n}{t_n} > \sqrt{\frac{2m}{n}}$.

For Ineq. (25), $\lambda_1 \geq \sqrt{\frac{\sum_{i=1}^n \sigma_i^2}{\sum_{i=1}^n t_i^2}}$ is taken under consideration.

For bipartite graphs the above theorem can be written:

Theorem 13 ([18]). *Let $G = (X, Y)$ be a nonempty bipartite graph with $n > 2$ vertices and m edges. Then,*

$$E(G) \leq 2\sqrt{\frac{\sum_{i=1}^n \sigma_i^2}{\sum_{i=1}^n t_i^2}} + \sqrt{(n-2)\left(2m - 2\frac{\sum_{i=1}^n \sigma_i^2}{\sum_{i=1}^n t_i^2}\right)}. \tag{26}$$

Equality holds if and only if one of the following statements holds:

1. $G \cong \frac{n}{2}K_2$.
2. $G \cong K_{r_1,r_2} \cup (n - r_1 - r_2)K_1$, *where* $r_1 r_2 = m$.
3. G *is a connected bipartite graph with* $V = \{v_1, v_2, \dots, v_s\} \cup \{v_{s+1}, v_{s+2}, \dots, v_n\}$ *such that* $\frac{\sigma_1}{t_1} = \cdots = \frac{\sigma_s}{t_s}$ *and* $\frac{\sigma_{s+1}}{t_{s+1}} = \cdots = \frac{\sigma_n}{t_n}$, *and has four distinct eigenvalues* $\left(\sqrt{p_x p_y}, \sqrt{\frac{2m-2p_x p_y}{n-2}}, -\sqrt{\frac{2m-2p_x p_y}{n-2}}, -\sqrt{p_x p_y}\right)$, *where* $p_x = \frac{\sigma_1}{t_1} = \cdots = \frac{\sigma_s}{t_s}$, $p_y = \frac{\sigma_{s+1}}{t_{s+1}} = \cdots = \frac{\sigma_n}{t_n}$ *and* $\sqrt{p_x p_y} > \sqrt{\frac{2m}{n}}$.

An upper bound that involves a graph's spectral moments is:

Theorem 14 ([15]). *Let* G *be a nonempty graph on* n *vertices. If* $\sqrt{\frac{\sum_{i=1}^n \sigma_i^2}{\sum_{i=1}^n t_i^2}} \geq \left(\frac{M_{2k}}{n}\right)^{\frac{1}{2k}}$, *where* k *is a positive integer, then the inequality*

$$E(G) \leq \sqrt{\frac{\sum_{i=1}^n \sigma_i^2}{\sum_{i=1}^n t_i^2}} + (n-1)^{\frac{2k-1}{2k}} \left(M_{2k} - \left(\frac{\sum_{i=1}^n \sigma_i^2}{\sum_{i=1}^n t_i^2}\right)^k\right)^{\frac{1}{2k}} \tag{27}$$

holds. Moreover, equality in 27 holds if and only if G *is either* $\frac{n}{2}K_2$, K_n, *or a non-bipartite connected graph satisfying* $\frac{\sigma_1}{t_1} = \cdots = \frac{\sigma_n}{t_n}$ *and has three distinct eigenvalues* $\left(p, \left(\frac{M_{2k}-p^{2k}}{n-1}\right)^{\frac{1}{2k}}, -\left(\frac{M_{2k}-p^{2k}}{n-1}\right)^{\frac{1}{2k}}\right)$, *where* $p = \frac{\sigma_1}{t_1} = \cdots = \frac{\sigma_n}{t_n} > \left(\frac{M_{2k}}{n}\right)^{\frac{1}{2k}}$.

If G is a bipartite graph, Ineq. (27) is written:

Theorem 15 ([15]). *Let* $G = (X, Y)$ *be a nonempty bipartite graph with* $n > 2$ *vertices and* m *edges. If* $\sqrt{\frac{\sum_{i=1}^n \sigma_i^2}{\sum_{i=1}^n t_i^2}} \geq \left(\frac{M_{2k}}{n}\right)^{\frac{1}{2k}}$, *where* k *is a positive integer, then the inequality*

$$E(G) \leq 2\sqrt{\frac{\sum_{i=1}^n \sigma_i^2}{\sum_{i=1}^n t_i^2}} + (n-2)^{\frac{2k-1}{2k}} \left(M_{2k} - 2\left(\frac{\sum_{i=1}^n \sigma_i^2}{\sum_{i=1}^n t_i^2}\right)^k\right)^{\frac{1}{2k}} \tag{28}$$

holds. Moreover, equality in (28) holds if and only if G *is either* $\frac{n}{2}K_2$, $K_{r_1,r_2} \cup (n - r_1 - r_2)K_1$, *where* $r_1 r_2 = m$, *or a connected bipartite graph with* $V = \{v_1, v_2, \dots, v_s\} \cup \{v_{s+1}, v_{s+2}, \dots, v_n\}$ *such that* $\frac{\sigma_1}{t_1} = \cdots = \frac{\sigma_s}{t_s}$ *and* $\frac{\sigma_{s+1}}{t_{s+1}} = \cdots = \frac{\sigma_n}{t_n}$, *and has four distinct eigenvalues*

$\left(\sqrt{p_x p_y}, \left(\frac{M_{2k}-2(p_x p_y)^k}{n-2}\right)^{\frac{1}{2k}}, -\left(\frac{M_{2k}-2(p_x p_y)^k}{n-2}\right)^{\frac{1}{2k}}, -\sqrt{p_x p_y}\right)$, *where* $p_x = \frac{\sigma_1}{t_1} = \cdots = \frac{\sigma_s}{t_s}$, $p_y = \frac{\sigma_{s+1}}{t_{s+1}} = \cdots = \frac{\sigma_n}{t_n}$ *and* $\sqrt{p_x p_y} > \left(\frac{M_{2k}}{n}\right)^{\frac{1}{2k}}$.

In regard to the k-degree $d_k(v)$ of a vertex $v \in G$, which is defined as the number of walks of length k of G starting at v, the next upper bound was given:

Theorem 16 ([11]). *Let G be a connected graph with n $(n \geq 2)$ vertices and m edges. Then,*

$$E(G) \leq \sqrt{\frac{\sum_{v\in V(G)} d_{k+1}^2(v)}{\sum_{v\in V(G)} d_k^2(v)}} + \sqrt{(n-1)\left(2m - \frac{\sum_{v\in V(G)} d_{k+1}^2(v)}{\sum_{v\in V(G)} d_k^2(v)}\right)}. \quad (29)$$

Equality holds if and only if G is the complete graph K_n, or G is a strongly regular graph with two nontrivial eigenvalues both with absolute value $\sqrt{\frac{2m-\left(\frac{2m}{n}\right)^2}{n-1}}$.

If G is a bipartite graph, the above inequality is:

Theorem 17 ([11]). *Let G be a connected bipartite graph with n $(n \geq 2)$ vertices and m edges. Then,*

$$E(G) \leq 2\sqrt{\frac{\sum_{v\in V(G)} d_{k+1}^2(v)}{\sum_{v\in V(G)} d_k^2(v)}} + \sqrt{(n-2)\left(2m - 2\frac{\sum_{v\in V(G)} d_{k+1}^2(v)}{\sum_{v\in V(G)} d_k^2(v)}\right)}. \quad (30)$$

Equality holds if and only if G is the complete bipartite graph or G is the incidence graph of a symmetric $2-(v,k,\lambda)$-design with $k = \frac{2m}{n}$, $n = 2v$ and $\lambda = \frac{k(k-1)}{v-1}$.

3 Graph Energy Change

In this section, we consider the change in the energy of a graph G when a vertex or an edge is deleted. Whereas when we remove a vertex u, the energy of the induced subgraph $G-u$ always decreases, for the subgraph $G-e$, which is obtained upon deleting an edge e, it has been proved that the energy may increase, decrease, or stay the same [5].

By the interlacing theorem it can be easily shown that:

$$E(G-u) \leq E(G). \quad (31)$$

For singular graphs, Ineq. (31) is improved if the null spread of a vertex u, $\eta_u(G) = \eta(G) - \eta(G-u)$ [7], is taken into consideration:

Theorem 18 ([24]). *Let $G = (V, E)$ be a graph and $u \in V$. If $n_u(G) = -1$, then*

$$E(G - u) \leq E(G) - (|\lambda_l| + |\lambda_m|), \tag{32}$$

where λ_l and λ_m are the smallest nonnegative and the largest nonpositive eigenvalue, respectively.

In the case where G is a connected graph of nullity $\eta(G) = n - 2$, the equality holds if and only if G is a star graph and u is the center vertex of the graph.

Theorem 19 ([24]). *Let $G = (V, E)$ be a graph and $u \in V$. If $n_u(G) = 0$, then*

$$E(G - u) \leq E(G) - |\lambda_i| \tag{33}$$

where λ_i is either the smallest nonnegative or the largest nonpositive eigenvalue of G.

In a similar way, Ineq. (31) can be improved to also include non-singular graphs. For example:

Let $G = (V, E)$ be a graph and $u \in V$. Let $m(G)$ be the multiplicity of a nonnegative eigenvalue λ for G, $m(G - u)$ be the multiplicity of λ for $G - u$, and $m_u(G) = m(G) - m(G - u)$ be the vertex spread of λ. Then by the interlacing inequalities,

Theorem 20. *Let $G = (V, E)$ be a graph and $u \in V$. If $m_u(G) = 0$, then*

$$E(G - u) \leq \max\{E(G) - \lambda_l, E(G) - |\lambda_m|\} \tag{34}$$

where λ_l (resp. λ_m) is the smallest positive (resp. largest negative) eigenvalue of G.

Since the energy of a graph decreases with the removal of a vertex, it is clear that if H is an induced subgraph of graph G, then

$$E(H) \leq E(G). \tag{35}$$

If we examine the subgraph $G - e$, obtained by deleting edge e from a graph G, we find that the change in energy does not always decrease. In fact, it may increase or stay the same. For example, let us consider the complete bipartite graph $K_{2,2}$ with nonzero eigenvalues $-2, 2$ in its spectrum. Then if we delete an edge e, the subgraph $K_{2,2} - e$ is the path P_3, with nonzero eigenvalues $-\sqrt{2}$ and $\sqrt{2}$ in its spectrum. Thus, the energy decreases upon deleting an edge. However, for the complete bipartite graph $K_{2,3}$ with known nonzero eigenvalues $\sqrt{6}, -\sqrt{6}$, the energy increases if we remove an edge as we find the spectrum of the subgraph $K_{2,3} - e$ to be $\{2.136, 0.662, 0, -0.662, -2.136\}$.

It has been shown that:

Proposition 3 ([24]). *Let $K_{p,q}$ be a complete bipartite graph, with $p+q>4$. Then, if we remove an edge e:*

$$E(K_{p,q}-e)=2\sqrt{pq-1+2\sqrt{(p-1)(q-1)}}. \tag{36}$$

By Ineq. (11),

$$E(K_{p,q}-e)-E(K_{p,q})\geq 2\left(\sqrt{pq-1+2\sqrt{(p-1)(q-1)}}-\sqrt{pq}\right), \tag{37}$$

and the energy of the complete bipartite graph increases after removing an edge.

To obtain Ineq. (36) the symmetry of the graph's spectrum is considered for its four nonzero eigenvalues $\mu_1\geq\mu_2\geq\mu_3\geq\mu_4$, so that the graph's characteristic polynomial is written:

$$x^{p+q-4}(x^4-(\mu_1^2+\mu_2^2)x^2+\mu_1^2\mu_2^2). \tag{38}$$

By Eq. (2) and the trace of A^4, we obtain the desired conclusion.

For complete multipartite graphs:

Theorem 21 ([1]). *Let $G=K_{t_1,t_2,\ldots,t_k}$ be a complete k-partite graph, with $k\geq 2$, $t_i\geq 2$, for $i=1,\ldots,k$. Then for every edge e,*

$$E(K_{t_1,t_2,\ldots,t_k}-e)\geq E(K_{t_1,t_2,\ldots,t_k}). \tag{39}$$

Another example of a graph that increases its energy after an edge is removed is the singular hypercube Q_n (the hypercube with even vertices).

Theorem 22 ([24]). *Let Q_{2k} be a singular hypercube. If $Q_{2k}-e$ is its subgraph after removing edge e, then:*

$$E(Q_{2k}-e)\geq E(Q_{2k}). \tag{40}$$

For Ineq. (40), the following lemma is considered for the adjacency matrix of the hypercube: $A(Q_n)=\begin{bmatrix} A(Q_{n-1}) & I_{2^{n-1}} \\ I_{2^{n-1}} & A(Q_{n-1}) \end{bmatrix}$, where $I_{2^{n-1}}$ denotes the identity matrix:

Lemma 1 ([23]). *For a partitioned matrix $C=\begin{bmatrix} A & X \\ Y & B \end{bmatrix}$, where both A and B are square matrices, we have:*

$$\sum_j s_j(A)+\sum_j s_j(B)\leq\sum_j s_j(C), \tag{41}$$

where $s_j(\cdot)$ denote the singular values of a matrix.

Let $G - \{m\}$ denote the graph obtained from G by deleting all m edges of a subgraph H but keeping all vertices of H. If G_1 and G_2 are two graphs without common vertices, let $G_1 \oplus G_2$ denote the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. Hence, $A(G_1 \oplus G_2) = A(G_1) + A(G_2)$.

Theorem 23 ([5]). *If F is a cut set of a simple graph G, then*

$$E(G - F) \leq E(G). \tag{42}$$

To obtain Ineq. (42), Lemma 1 is applied to $A(G) = \begin{bmatrix} A(H) & X \\ X^T & A(K) \end{bmatrix}$, where H and K are two complementary induced subgraphs of G, such that $G - F = H \oplus K$.

Theorem 24 ([8]). *Let A and B be two $n \times n$ complex matrices. Then*

$$\sum_{i=1}^{n} s_i(A + B) + \leq \sum_{i=1}^{n} s_i(A) + \sum_{i=1}^{n} s_i(B), \tag{43}$$

where $s_j(\cdot)$ denote the singular values of a matrix.

Moreover equality holds if and only if there exists a unitary matrix P such that PA and PB are both positive semi-definite.

Theorem 25 ([6]). *Let H be an induced subgraph of a graph G. Then,*

$$E(G) - E(H) \leq E(G - \{m\}) \leq E(G) + E(H). \tag{44}$$

Moreover,

1. *if H is nonsingular, then the left equality holds if and only if $G = H \oplus (G - H)$.*
2. *the right equality holds if and only if $m = 0$.*

For the left side of Ineq. (44) Theorem 24 is applied to

$$A(G) = \begin{bmatrix} A(H) & X^T \\ X & A(G-H) \end{bmatrix} = \begin{bmatrix} A(H) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & X^T \\ X & A(G-H) \end{bmatrix}, \tag{45}$$

where X represents edges connecting H and $G - H$.

For the right side of Ineq. (44) the same theorem is applied to

$$A(G - \{m\}) = A(G) + \begin{bmatrix} -A(H) & 0 \\ 0 & 0 \end{bmatrix}. \tag{46}$$

It has been shown that:

Theorem 26 ([5]). *For any simple graph G with at least one edge,*

$$E(G) \geq 2. \tag{47}$$

Since the complete graph K_2 is an induced subgraph of G, by Ineq. (35) the proof of the above bound is trivial.

Corollary 1 ([6]). *Let e be an edge of a graph G. Then the subgraph with the edge set $\{e\}$ is induced and nonsingular, hence*

$$E(G) - 2 \leq E(G - \{e\}) \leq E(G) + 2. \tag{48}$$

Moreover,

1. *the left equality holds if and only if e is an isolated edge of G.*
2. *the right equality never holds.*

From Ineq. (48), it is clear that:

1. if e is an edge of a connected graph G such that $E(G) = E(G - \{e\}) + 2$, then $G = K_2$.
2. there are no graphs G such that $E(G - \{e\}) = E(G) + 2$.

4 Energy of Bipartite Graphs

We conclude this paper with some additional bounds for the energy of bipartite graphs.

Theorem 27 ([2]). *If G is a connected bipartite graph of rank r, then*

$$E(G) \geq \sqrt{(r+1)^2 - 5}. \tag{49}$$

Theorem 28 ([2]). *Let G be a bipartite graph with at least four vertices. If G is not full rank, then*

$$E(G) \geq 1 + rank(G). \tag{50}$$

Theorem 29 ([21]). *Let G be a bipartite graph with 2N vertices. Then,*

$$E(G) \geq 2m\sqrt{\frac{m}{q}}, \tag{51}$$

where $q = \sum_{i=1}^{N} \lambda_i^4$. The equality holds if and only if $G = NK_2$ or G is the direct sum of isolated vertices and complete bipartite graphs $K_{r_1,s_1}, \ldots, K_{r_j,s_j}$ such that $r_1 s_1 = \cdots = r_j s_j$.

It has been shown [21] that Ineq. (51) remains also true if G is a bipartite graph with $2N + 1$ vertices.

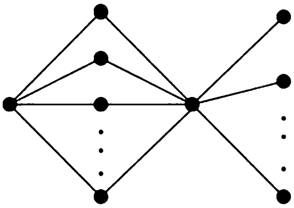

Fig. 1 The graph $B_{n,m}$

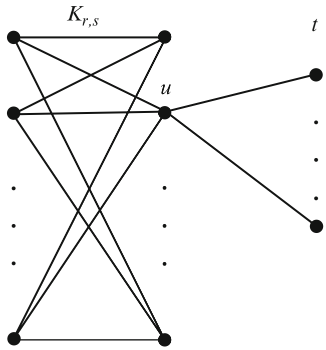

Fig. 2 The graph G_{rst}

Let $B_{n,m}$ be the bipartite (n, m)-graph with two vertices on one side, one of which is connected to all vertices on the other side, as illustrated in Fig. 1.

Theorem 30 ([17]). *$B_{n,m}$ $(n \leq m \leq 2(n-2))$ is the unique graph with minimal energy in all bipartite connected (n, m)-graphs.*

An n-vertex graph G is said to be hypoenergetic if $E(G) < n$. It has been shown [20] that almost all graphs are hyperenergetic ($E(G) > 2(n-1)$), which implies that there are but a few hypoenergetic graphs.

Theorem 31 ([25]). *Let $G \cong K_{n_1,n_2}$, $n_1 \neq n_2$. Then G is hypoenergetic.*

Since the spectrum of K_{n_1,n_2} is known, the proof of the above theorem is trivial.

Theorem 32 ([16]). *The complete bipartite graph $K_{2,3}$ is the only hypoenergetic connected cycle-containing (or cyclic) graph with maximum degree $\Delta \leq 3$.*

Let G_{rst} be a graph of order n constructed as shown in Fig. 2, by identifying the center of a star $K_{1,t}$ with a vertex of a complete bipartite graph $K_{r,s}$, where $r + s + t = n$.

Theorem 33 ([25]). *Among the complete bipartite graphs with pendent vertices attached, some are hypoenergetic.*

It is well known [14] that the graph $G \cong G_{rst}$ has a nullity of $\eta(G) = n - 4$. By Ineq. (9) and after some calculations, it can be easily shown that the graph G_{rst} is hypoenergetic for $t \geq r > s \geq 5$.

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Implicit Contractive Maps in Ordered Metric Spaces

Mihai Turinici

Abstract In Part 1, we show that most of the implicit contractions introduced by Wardowski [Fixed Point Theory Appl., 2012, 2012:94] are Matkowski type contractions. In Part 2, some limit type extensions are obtained for the fixed point result (involving implicit contractions) due to Altun and Simsek [Fixed Point Theory Appl., Volume 2010, Article ID 621469]. Moreover, the connections with a lot of related statements in the area due to Agarwal, El-Gebeily, and O'Regan [Appl. Anal. 87:109–116, 2008] are also discussed. Finally, in Part 3, a non-limit counterpart of these results is given, under the same general context.

Keywords Metric space • Convergent and Cauchy sequence • Fixed point • Picard operator • Wardowski function • Limit and non-limit implicit contraction • Boyd-Wong and Matkowski admissible function

1 Wardowski Type Contractions

1.1 Introduction

Let in the following X be a nonempty set. Call the subset Y of X, *almost singleton* (abbreviated: *asingleton*), if $[y_1, y_2 \in Y \Longrightarrow y_1 = y_2]$; and *singleton*, if, in addition, Y is nonempty; note that, in this case, $Y = \{y\}$, for some (uniquely determined) $y \in X$. By a *sequence* in X we mean any map $n \mapsto x(n) := x_n$ from $N = \{0, 1, \ldots\}$ to X; also written as $(x_n; n \geq 0)$; or, simply, (x_n). Further, let us say that $(y_i; i \geq 0)$ is a *subsequence* of $(x_n; n \geq 0)$ if $(y_i = x_{n(i)}; i \geq 0)$, where $(n(i); i \geq 0)$ is a sequence in N with $n(i) \to \infty$ as $i \to \infty$.

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By a *metric* over X, we mean any map $d : X \times X \to R_+ := [0, \infty[$, supposed to be *symmetric* $[d(x,y) = d(y,x), \forall x, y \in X]$, *triangular* $[d(x,z) \leq d(x,y) + d(y,z), \forall x, y, z \in X]$, and *reflexive-sufficient* ($x = y$ iff $d(x,y) = 0$); in this case, (X, d) is called a *metric space*.

(A) Having this pair, we introduce a d-convergence and a d-Cauchy structure on X as follows. Given the sequence (x_n) in X and the point $x \in X$, we say that (x_n), *d-converges* to x (written as: $x_n \xrightarrow{d} x$) provided $d(x_n, x) \to 0$ as $n \to \infty$; i.e.,

$$\forall \varepsilon > 0, \exists i = i(\varepsilon): \ n \geq i \Longrightarrow d(x_n, x) < \varepsilon;$$

or, equivalently:

$$\forall \varepsilon > 0, \exists i = i(\varepsilon): \ n \geq i \Longrightarrow d(x_n, x) \leq \varepsilon;$$

The set of all such points x will be denoted $\lim_n(x_n)$; it is an asingleton, by the properties of $d(.,.)$. If $\lim_n(x_n)$ is nonempty, then (x_n) is called *d-convergent*.

By this very definition, we have the hereditary property:

$$\begin{aligned} &x_n \xrightarrow{d} x \text{ implies } y_i \xrightarrow{d} x, \\ &\text{for each subsequence } (y_i; i \geq 0) \text{ of } (x_n; n \geq 0). \end{aligned} \tag{1}$$

Moreover, it is clear that

$$[x_n = u, \ \forall n \geq 0] \text{ implies } x_n \xrightarrow{d} u. \tag{2}$$

As a consequence, the convergence structure $(\xrightarrow{d})$ has all regularity properties required in Kasahara [13].

Further, call the sequence (x_n), *d-Cauchy* when $d(x_m, x_n) \to 0$ as $m, n \to \infty$ with $m < n$; i.e.,

$$\forall \varepsilon > 0, \exists j = j(\varepsilon): \ j \leq m < n \Longrightarrow d(x_m, x_n) < \varepsilon;$$

or, equivalently,

$$\forall \varepsilon > 0, \exists j = j(\varepsilon): \ j \leq m < n \Longrightarrow d(x_m, x_n) \leq \varepsilon.$$

As before, we have the hereditary property

$$\begin{aligned} &(x_n) \text{ is } d\text{-Cauchy implies } (y_i) \text{ is } d\text{-Cauchy}, \\ &\text{for each subsequence } (y_i; i \geq 0) \text{ of } (x_n; n \geq 0). \end{aligned} \tag{3}$$

Finally, call $(x_n; n \geq 0)$, *d-semi-Cauchy*, when $d(x_n, x_{n+1}) \to 0$; and *strong d-semi-Cauchy*, provided $(d(x_n, x_{n+i}) \to 0$, as $n \to \infty$, for each $i \geq 1)$. Clearly, the metrical properties of d give

$$(\forall \text{ sequence}): d\text{-Cauchy} \Longrightarrow \text{strong } d\text{-semi-Cauchy} \Longleftrightarrow \ d\text{-semi-Cauchy}.$$

In addition, each d-convergent sequence is d-Cauchy, as it can be directly seen.

(B) Let again (X, d) be a metric space. In the following, a useful property is described for the d-semi-Cauchy sequences in X which are not d-Cauchy. Let us say that the subset Θ of $R^0_+ :=]0, \infty[$ is $(>)$-*cofinal* in R^0_+, when: for each $\varepsilon \in R^0_+$, there exists $\theta \in \Theta$ with $\varepsilon > \theta$. Further, given the sequence $(r_n; n \geq 0)$ in R_+ and the point $r \in R_+$, let us write

$r_n \to r+$ (respectively, $r_n \to r++$), if $r_n \to r$ and
$r_n \geq r$ (respectively, $r_n > r$), for all $n \geq 0$ large enough.

Proposition 1. *Suppose that* $(x_n; n \geq 0)$ *is a sequence in* X *with*

(a01) $r_n := d(x_n, x_{n+1}) > 0$, *for all* $n \geq 0$
(a02) $(x_n; n \geq 0)$ *is* d*-semi-Cauchy but not* d*-Cauchy.*

Further, let Θ *be a* $(>)$*-cofinal part of* R^0_+*. There exist then a number* $b \in \Theta$*, such that: for each* $\eta \in]0, b/3[$*, a rank* $j(\eta) \geq 0$*, and a couple of rank-sequences* $(m(j); j \geq 0)$*,* $(n(j); j \geq 0)$*, may be found, with*

$$j \leq m(j) < n(j),\ d(x_{m(j)}, x_{n(j)}) > b,\ \forall j \geq 0 \tag{4}$$

$$j \geq j(\eta) \Longrightarrow r_{m(j)}, r_{n(j)-1}, r_{n(j)} < \eta < b/3 \tag{5}$$

$$n(j) - m(j) \geq 2,\ d(x_{m(j)}, x_{n(j)-1}) \leq b,\ \forall j \geq j(\eta) \tag{6}$$

$$\begin{array}{l}(u_j(0,0) := d(x_{m(j)}, x_{n(j)}); j \geq 0) \textit{ is a sequence in } R^0_+ \\ \textit{with } u_j(0,0) \to b++ \textit{ as } j \to \infty\end{array} \tag{7}$$

$$\begin{array}{l}(u_j(p,q) := d(x_{m(j)+p}, x_{n(j)+q}); j \geq j(\eta)) \textit{ is a sequence in } R^0_+ \\ \textit{with } u_j(p,q) \to b \textit{ as } j \to \infty,\ \forall p,q \in \{0,1\}.\end{array} \tag{8}$$

Proof. By definition, the d-Cauchy property of our sequence writes:

$\forall \varepsilon \in R^0_+, \exists k = k(\varepsilon)\colon k \leq m < n \Longrightarrow d(x_m, x_n) \leq \varepsilon.$

As Θ is a $(>)$-cofinal part in R^0_+, this property may be also written as

$\forall \theta \in \Theta, \exists k = k(\theta)\colon k \leq m < n \Longrightarrow d(x_m, x_n) \leq \theta.$

The negation of this property means: there exists $b \in \Theta$ such that, $\forall j \geq 0$:

$$A(j) := \{(m,n) \in N \times N; j \leq m < n, d(x_m, x_n) > b\} \neq \emptyset.$$

Having this precise, denote, for each $j \geq 0$,

$m(j) = \min \mathrm{Dom}(A(j)),\ n(j) = \min A(m(j)).$

The couple of rank-sequences $(m(j); j \geq 0)$, $(n(j); j \geq 0)$ fulfills (4); hence, the first half of (7). On the other hand, letting $j(\eta) \geq 0$ be such that

$$r_i := d(x_i, x_{i+1}) < \eta < b/3, \text{ for all } i \geq j(\eta),$$

it is clear that (by (4), the first part), that (5) and (6) hold too. This yields (by the triangular inequality), for all $j \geq j(\eta)$;

$$b < d(x_{m(j)}, x_{n(j)}) \leq d(x_{m(j)}, x_{n(j)-1}) + r_{n(j)-1} \leq b + r_{n(j)-1};$$

so, passing to limit as $j \to \infty$ gives the second half of (7). Finally, $\forall j \geq j(b)$,

$$d(x_{m(j)}, x_{n(j)+1}) \leq d(x_{m(j)}, x_{n(j)}) + r_{n(j)},$$
$$d(x_{m(j)}, x_{n(j)+1}) \geq d(x_{m(j)}, x_{n(j)}) - r_{n(j)} > 2b/3.$$

This gives the case $(p = 0, q = 1)$ of (8). The remaining alternatives (modulo (p, q)) of this relation are obtained in a similar way.

(C) Let X be a nonempty set. Further, take some $T \in \mathscr{F}(X)$. [Here, given the nonempty sets A and B, $\mathscr{F}(A, B)$ stands for the class of all functions $f : A \to B$; when $A = B$, we write $\mathscr{F}(A, A)$ as $\mathscr{F}(A)$.] Denote $\text{Fix}(T) = \{z \in X; z = Tz\}$; each element of this set is called *fixed* under T. For many practical and theoretical reasons, it is useful to determine whether $\text{Fix}(T)$ is nonempty; and, if this holds, to establish whether T is *fix-asingleton* (i.e.: $\text{Fix}(T)$ is asingleton); or, equivalently: T is *fix-singleton* (in the sense: $\text{Fix}(T)$ is singleton); A similar problem is to be formulated with respect to the iterates T^k, where $k \geq 1$.

Concerning the posed problem, the following concepts establish the directions under which the investigation be conducted (cf. Rus [29, Chap. 2, Sect. 2.2]):

- **(1a)** We say that T is a *Picard operator* (modulo d) if, for each $x \in X$, $(T^n x; n \geq 0)$ is d-convergent (hence, d-Cauchy); so that, $\lim_n(T^n x)$ is a singleton
- **(1b)** We say that T is a *strong Picard operator* (modulo d) if, for each $x \in X$, $(T^n x; n \geq 0)$ is d-convergent (hence, d-Cauchy); and $z := \lim_n(T^n x)$ is an element of $\text{Fix}(T)$
- **(1c)** We say that T is a *globally strong Picard operator* (modulo d) when it is a strong Picard operator (modulo d) and T is fix-asingleton (hence, fix-singleton).

  The general sufficient conditions for such properties are being founded on *orbital* properties (in short: o-properties). Call the sequence $(z_n; n \geq 0)$ in X, T*-orbital*, when it is a subsequence of $(T^n x; n \geq 0)$, for some $x \in X$.
- **(1d)** We say that (X, d) is *o-complete*, provided (for each o-sequence): d-Cauchy implies d-convergent
- **(1e)** Call T, (o, d)*-continuous* if: whenever (z_n) is o-sequence and $z_n \xrightarrow{d} z$ then $Tz_n \xrightarrow{d} Tz$.

If the orbital properties are ignored, then each o-convention becomes an ordinary one. For example, we say that (X, d) is *complete*, when each d-Cauchy sequence is d-convergent. Likewise, T is termed d*-continuous*, provided $z_n \xrightarrow{d} z$ implies $Tz_n \xrightarrow{d} Tz$.

(D) Finally, the specific conditions for such properties are represented by "functional" metric contractions. Call T, (d,φ)-*contractive* (for some $\varphi \in \mathscr{F}(R_+)$), when

(a03) $\quad d(Tx,Ty) \le \varphi(d(x,y)), \forall x,y \in X, x \ne y.$

The functions to be considered here are to be taken according to the lines below.

Call $\varphi \in \mathscr{F}(R_+)$, *regressive* if $[\varphi(0) = 0$ and $\varphi(t) < t$, $\forall t > 0]$; the class of all these will be denoted as $\mathscr{F}(re)(R_+)$, For any $\varphi \in \mathscr{F}(re)(R_+)$ and any $s \in R_+^0$, put

(a04) $\quad \Lambda_+\varphi(s) = \inf_{\varepsilon>0} \Phi(s+)(\varepsilon)$; where $\Phi(s+)(\varepsilon) = \sup \varphi(]s, s+\varepsilon[)$;
(a05) $\quad \Lambda^+\varphi(s) = \sup\{\varphi(s), \Lambda_+\varphi(s)\}$.

By this very definition, we have the representation (for all $s \in R_+^0$)

$$\Lambda^+\varphi(s) = \inf_{\varepsilon>0} \Phi[s+](\varepsilon);\ \text{where}\ \Phi[s+](\varepsilon) = \sup\{\varphi([s, s+\varepsilon[). \tag{9}$$

From the regressive property of φ, these limit quantities are finite; precisely,

$$0 \le \varphi(s) \le \Lambda^+\varphi(s) \le s, \quad \forall s \in R_+^0. \tag{10}$$

The following additional properties will be useful for us:

Lemma 1. *Let $\varphi \in \mathscr{F}(re)(R_+)$ and $s \in R_+^0$ be arbitrary fixed. Then,*

i) $\limsup_n(\varphi(t_n)) \le \Lambda^+\varphi(s)$, *for each sequence (t_n) in R_+ with $t_n \to s+$; hence, in particular, for each sequence (t_n) in R_+ with $t_n \to s++$*
ii) *there exists a sequence (r_n) in R_+^0 with $r_n \to s+$ and $\varphi(r_n) \to \Lambda^+\varphi(s)$.*

Proof. i) Given $\varepsilon > 0$, there exists a rank $p(\varepsilon) \ge 0$ such that $s \le t_n < s + \varepsilon$, for all $n \ge p(\varepsilon)$; hence

$$\limsup_n(\varphi(t_n)) \le \sup\{\varphi(t_n); n \ge p(\varepsilon)\} \le \Phi[s+](\varepsilon).$$

It suffices taking the infimum over $\varepsilon > 0$ in this relation to get the desired fact.

ii) When $\Lambda^+\varphi(s) = 0$, the written conclusion is clear, with $(r_n = s; n \ge 0)$; for, in this case, $\varphi(s) = 0$. Suppose now that $\Lambda^+\varphi(s) > 0$. By definition,

$$\forall \varepsilon \in]0, \Lambda^+\varphi(s)[,\ \exists \delta \in]0, \varepsilon[:\ \Lambda^+\varphi(s) - \varepsilon < \Lambda^+\varphi(s) \le \Phi[s+](\delta) < \Lambda^+\varphi(s) + \varepsilon.$$

This tells us that there must be some r in $[s, s+\delta[$ with

$$\Lambda^+\varphi(s) - \varepsilon < \varphi(r) < \Lambda^+\varphi(s) + \varepsilon.$$

Taking a sequence (ε_n) in $]0, \Lambda^+\varphi(s)[$ with $\varepsilon_n \to 0$, there exists a corresponding sequence (r_n) in R_+^0 with $r_n \to s+$ and $\varphi(r_n) \to \Lambda^+\varphi(s)$.

We say that $\varphi \in \mathscr{F}(re)(R_+)$ is *Boyd–Wong admissible at* $s \in R^0_+$, provided $\Lambda^+\varphi(s) < s$. Denote by $BW(\varphi)$ the set of all such points; i.e., $BW(\varphi) = \{s \in R^0_+; \Lambda^+\varphi(s) < s\}$. We say that $\varphi \in \mathscr{F}(re)(R_+)$ is *almost Boyd–Wong admissible* (in short: *a-BW-adm*), when

$BW(\varphi)$ is (>)-cofinal in R^0_+: for each $\varepsilon > 0$ there exists $\theta \in BW(\varphi)$ with $\varepsilon > \theta$.

Further, let us say that $\varphi \in \mathscr{F}(re)(R_+)$ is *compatible*, provided

(a06) for each sequence $(r_n; n \geq 0)$ in R^0_+, with $r_{n+1} \leq \varphi(r_n)$, $\forall n \geq 0$, we must have $r_n \to 0$.

If both these properties hold, we say that $\varphi \in \mathscr{F}(re)(R_+)$ is *compatible almost Boyd–Wong admissible*; in short, c-a-BW-adm. Two basic examples of such objects are given below.

I) We say that $\varphi \in \mathscr{F}(re)(R_+)$ is *Boyd–Wong admissible*, when $BW(\varphi) = R^0_+$; i.e., by the adopted conventions,

(a07) $\Lambda^+\varphi(s) < s$ (or, equivalently: $\Lambda_+\varphi(s) < s$), for all $s > 0$.

(This concept is related to the developments in Boyd and Wong [8]; we do not give details.) In particular, $\varphi \in \mathscr{F}(re)(R_+)$ is Boyd–Wong admissible provided it is upper semicontinuous at the right on R^0_+:

$$\Lambda^+\varphi(s) = \varphi(s), \text{ (or, equivalently: } \Lambda_+\varphi(s) \leq \varphi(s)), \ \forall s \in R^0_+.$$

Note that this is fulfilled when φ is continuous at the right on R^0_+; for, in such a case, $\Lambda_+\varphi(s) = \varphi(s)$, $\forall s \in R^0_+$.

Now, if $\varphi \in \mathscr{F}(re)(R_+)$ is Boyd–Wong admissible, then it is trivially shown to be almost Boyd–Wong admissible. Concerning the compatibility question, we have

Lemma 2. *Let $\varphi \in \mathscr{F}(re)(R_+)$ be Boyd–Wong admissible. Then, φ is compatible; hence, a fortiori, compatible almost Boyd–Wong admissible.*

Proof. Let $(r_n; n \geq 0)$ be a sequence in R^0_+ with $r_{n+1} \leq \varphi(r_n)$, $\forall n \geq 0$. As $\varphi \in \mathscr{F}(re)(R_+)$, (r_n) is strictly descending in R_+; hence, $r := \lim_n(r_n)$ exists in R_+ and $[r_n > r, \forall n]$. We have (again via $\varphi \in \mathscr{F}(re)(R_+)$) $r_{n+1} \leq \varphi(r_n) < r_n$, $\forall n \geq 0$. This, along with $r_n \to r$ as $n \to \infty$, yields $\lim_n \varphi(r_n) = r$; wherefrom $\Lambda^+\varphi(r) = r$; contradiction. Hence, $r = 0$, as desired.

II) Call $\varphi \in \mathscr{F}(re)(R_+)$, *Matkowski admissible*, provided

(a08) φ is increasing and Matkowski ($\varphi^n(t) \to 0$ as $n \to \infty$, $\forall t > 0$).

Here, φ^n stands for the n-th iterate of φ, $\forall n$. (The proposed convention is related to the developments in Matkowski [16].) Note that, for each $\varphi \in \mathscr{F}(re)(R_+)$, this property is quite distinct from the preceding one (Boyd–Wong admissible), as simple examples show. However, as a completion of the above statement, we do have

Lemma 3. *Let $\varphi \in \mathscr{F}(re)(R_+)$ be Matkowski admissible. Then, necessarily, φ is compatible and almost Boyd–Wong admissible.*

Proof. i) Let $(r_n; n \geq 0)$ be a sequence in R_+^0 with $r_{n+1} \leq \varphi(r_n)$, $\forall n \geq 0$. As $\varphi \in \mathscr{F}(re)(R_+)$, (r_n) is strictly descending in R_+; hence, $r := \lim_n r_n$ exists in R_+ and $[r_n > r, \forall n]$. Further, as φ is increasing, this yields $r < r_n \leq \varphi^n(r_0)$, $\forall n$; so, by the Matkowski property, we get $r = 0$; hence, φ is compatible.

ii) Let $\Gamma := \Gamma(\varphi)$ stand for the subset of all $r > 0$ where φ is discontinuous at r. By a well-known property of such functions (cf. Natanson, [18, Chap. 8, Sect. 1]), Γ is at most denumerable. Denote, for simplicity, $\Theta = R_+^0 \setminus \Gamma$. Each $r \in \Theta$ is a (bilateral) continuity point of φ; and then, $\Lambda^+\varphi(r) = \varphi(r) < r$; so that, φ is Boyd–Wong admissible at r. Moreover, as Θ is dense (hence, all the more, $(>)$-cofinal) in R_+^0, φ is almost Boyd–Wong admissible.

(E) The usefulness of these concepts is to be judged from the following fixed point statement in the area, referred to as: BWM-caa theorem. Let (X, d) be a metric space; and $T \in \mathscr{F}(X)$ be a selfmap of X.

Theorem 1. *Supposed that T is (d, φ)-contractive, for some compatible almost Boyd–Wong admissible function $\varphi \in \mathscr{F}(re)(R_+)$. In addition, suppose that (X, d) is complete. Then, T is a global strong Picard operator (modulo d).*

A proof of this result will be given a bit further. For the moment (see the preceding developments), two particular cases are of interest for us:

Case 1. Each Boyd–Wong admissible function $\varphi \in \mathscr{F}(re)(R_+)$ is compatible almost Boyd–Wong admissible. Then, BWM-caa theorem is just the 1969 one in Boyd and Wong [8]; referred to as: Boyd–Wong theorem. For example, such a property holds when φ is *linear* (i.e.: $\varphi(t) = \alpha t$, $t \in R_+$, for some $\alpha \in [0, 1[$); in this case, the above result is nothing else than the 1922 Banach's contraction mapping principle [5].

Case 2. Any Matkowski admissible function $\varphi \in \mathscr{F}(re)(R_+)$ is compatible almost Boyd–Wong admissible. Then, BWM-caa theorem is just the 1975 one in Matkowski [16]; referred to as: Matkowski theorem.

(F) Let $\Gamma(X, d)$ stand for the class of all globally strong Picard (modulo d) operators. An interesting local type problem concerning these data is that of determining the subclass of all $T \in \Gamma(X, d)$, fulfilling an evaluation like

$d(x, \mathrm{Fix}(T)) \leq \Phi(d(x, Tx))$, for all $x \in X$;

where the function $\Phi \in \mathscr{F}(R_+)$ depends on T. This is a *Hyers–Ulam stability* question related to the considered class. A solution to this problem is obtainable for a limited family of contractions described by means of the list in Rhoades [28]. Some related facts may be found in the 1998 monograph by Hyers et al. [11]; see also the 2010 volume edited by Pardalos et al. [21, Part I].

Now, these particular cases of the BWM-caa theorem found some interesting applications in operator equations theory; so, they were the subject of various extensions. For example, a natural way of doing this consists in taking implicit "functional" contractive conditions like

(a09) $F(d(Tx, Ty), d(x, y)) \leq 0, \forall x, y \in X, x \neq y;$

where $F : R_+^2 \to R \cup \{-\infty, \infty\}$ is an appropriate function. Some partial details about the possible choices of F may be found in the 1976 paper by Turinici [32]. Recently, an interesting contractive condition of this type was introduced in Wardowski [41]. It is our aim in the following to show that, concerning the fixed point question, a reduction to Matkowski theorem is possible for most of these contractions. For the remaining ones, we provide a result where some specific requirements posed by Wardowski are shown to be superfluous.

1.2 Left-Continuous Wardowski Functions

Call $F : R_+ \to R \cup \{-\infty\}$, a *semi-Wardowski* function, provided

(b01) F is reflexive-sufficient: $F(t) = -\infty$ if and only if $t = 0$
(b02) F is strictly increasing: $t < s \Longrightarrow F(t) < F(s)$.

As a consequence of these facts, the lateral limits

$F(t-0) := \lim_{s \to t-} F(s)$, $F(t+0) := \lim_{s \to t+} F(s)$

exist, for each $t > 0$; in addition,

$$-\infty < F(t-0) \le F(t) \le F(t+0) < \infty, \ \forall t > 0. \tag{11}$$

Note that, in general, F is not continuous. However, by the specific properties of the monotone functions, we have (cf. Natanson [18, Chap. 8, Sect. 1]):

Proposition 2. *Suppose that $F : R_+ \to R \cup \{-\infty\}$ is a semi-Wardowski function. Then, there exists a denumerable subset $\Delta := \Delta(F) \subset R_+^0$, such that*

$$F(t-0) = F(t) = F(t+0), \textit{ for each } t \in R_+^0 \setminus \Delta. \tag{12}$$

In addition, the following property of such objects is useful for us.

Lemma 4. *Let $F : R_+ \to R \cup \{-\infty\}$ be a semi-Wardowski function. Then,*

$$\forall t, s \in R_+ : \ F(t) < F(s) \Longrightarrow t < s. \tag{13}$$

Proof. Take the numbers $t, s \in R_+$ according to the premise of this relation. Clearly, $s > 0$; and, from this, the case $t = 0$ is proved; hence, we may assume that $t > 0$. The alternative $t = s$ gives $F(t) = F(s)$; in contradiction with $F(t) < F(s)$. Likewise, the alternative $t > s$ yields (as F is strictly increasing) $F(t) > F(s)$; again in contradiction with $F(t) < F(s)$. Hence, $t < s$; and the conclusion follows.

(A) Now, let us add one more condition (upon such functions)

(b03) F is coercive: $F(t) \to -\infty$, when $t \to 0+$; i.e.: $F(0+) = -\infty$.

When F satisfies (b01)–(b03), it will be referred to as a *Wardowski* function.

Lemma 5. *Suppose that $F : R_+ \to R \cup \{-\infty\}$ is a Wardowski function. Then, for each sequence (t_n) in R_+^0,*

$$F(t_n) \to -\infty \text{ implies } t_n \to 0. \tag{14}$$

Proof. Suppose that this is not true: there must be some $\varepsilon > 0$ such that

for each n, there exists $m > n$, such that: $t_m \geq \varepsilon$.

We get therefore a subsequence $(s_n := t_{i(n)})$ of (t_n) such that $s_n \geq \varepsilon$ (hence, $F(s_n) \geq F(\varepsilon)$), $\forall n$. This, however, contradicts the property $F(s_n) \to -\infty$ (obtained via $F(t_n) \to -\infty$); hence, the conclusion.

(B) Suppose that $F : R_+ \to R \cup \{-\infty\}$ is a Wardowski function; and let $a > 0$ be fixed in the sequel. Denote, for each $t \geq 0$,

(b04) $M(a, F)(t) = \{s \geq 0; a + F(s) \leq F(t)\}$, $\varphi(t) = \sup M(a, F)(t)$.

Note that, from (b01), $M(a, F)(t)$ is nonempty, for each $t \geq 0$; and $M(a, F)(0) = \{0\}$ [hence $\varphi(0) = 0$]. In addition, for each $t > 0$ one has, by a preceding fact,

$$s \in M(a, F)(t) \Longrightarrow F(s) < F(t) \Longrightarrow s < t;$$

wherefrom, again for all $t > 0$,

$$M(a, F)(t) \subseteq [0, t[; \text{ whence, } \varphi(t) \leq t. \tag{15}$$

Moreover, as F is (strictly) increasing,

$$0 \leq t_1 \leq t_2 \Longrightarrow M(a, F)(t_1) \subseteq M(a, F)(t_2) \Longrightarrow \varphi(t_1) \leq \varphi(t_2); \tag{16}$$

so that, φ is increasing on R_+.

A basic problem to be posed is that of determining sufficient conditions under F such that the (increasing) function φ be regressive Matkowski. The following answer is available.

Proposition 3. *Suppose that the Wardowski function $F : R_+ \to R \cup \{-\infty\}$ fulfills*

(b05) F is left-continuous: $F(t - 0) = F(t)$, $\forall t > 0$.

Then, the associated (increasing) function φ is regressive and Matkowski.

Proof. There are two steps to be passed.

Step 1. Let $t > 0$ be arbitrary fixed; and put $u := \varphi(t)$. If $u = 0$, we are done; so, without loss, we may assume that $u > 0$. By the very definition of this number,

$$a + F(s) \leq F(t), \text{ for all } s \in [0, u[.$$

So, passing to limit as ($s \to u-$), one gets (by the left continuity of F)

$$a + F(u) \le F(t); \text{ i.e.: } a + F(\varphi(t)) \le F(t). \tag{17}$$

This in turn yields $F(\varphi(t)) < F(t)$; whence $\varphi(t) < t$. As $t > 0$ was arbitrarily chosen, it follows that φ is regressive.

Step 2. Fix some $t > 0$; and let the iterative sequence (t_n) be given as [$t_0 = t$, $t_{n+1} = \varphi(t_n)$; $n \ge 0$]. If $t_n = 0$, for some index $n \ge 1$, we are done; so, without loss, one may assume that [$t_n > 0$, $\forall n \ge 1$]. By the relation above,

$$a + F(t_{n+1}) \le F(t_n), \text{ [hence, } a \le F(t_n) - F(t_{n+1})\text{]}, \forall n.$$

Adding the first n inequalities, we derive

$$na \le F(t_0) - F(t_n) \text{ [hence: } F(t_n) \le F(t_0) - na\text{]}, \forall n;$$

so that, passing to limit as $n \to \infty$, one gets $F(t_n) \to -\infty$. This, by a preceding auxiliary statement, gives $t_n \to 0$. As $t > 0$ was arbitrarily chosen, it results that φ has the Matkowski property, as claimed.

(C) Let (X, d) be a metric space; and $T \in \mathscr{F}(X)$, a selfmap of X. Given the real number $a > 0$ and the Wardowski function $F : R_+ \to R \cup \{-\infty\}$, let us say that T is (a, F)-*contractive*, if

(b06) $\quad a + F(d(Tx, Ty)) \le F(d(x, y)), \forall x, y \in X, x \ne y.$

Our first main result is

Theorem 2. *Suppose that T is (a, F)-contractive, for some $a > 0$ and some left-continuous Wardowski function $F : R_+ \to R \cup \{-\infty\}$. In addition, let (X, d) be complete. Then, T is a globally strong Picard map (modulo d).*

Proof. By a preceding statement, the associated increasing function $\varphi \in \mathscr{F}(R_+)$ is regressive Matkowski. On the other hand, the imposed contractive hypothesis tells us that T is (d, φ)-contractive. This, along with Matkowski theorem, gives us all conclusions we need.

(D) Given $k > 0$, let us say that $F : R_+ \to R \cup \{-\infty\}$ is k-*regular*, provided

(b07) $\quad t^k F(t) \to 0$, as $t \to 0+$.

As a direct consequence of the above result, we have, formally,

Theorem 3. *Suppose that T is (a, F)-contractive, for some $a > 0$ and some left-continuous Wardowski function $F : R_+ \to R \cup \{-\infty\}$; supposed to be k-regular, for some $k \in]0, 1[$. In addition, let (X, d) be complete. Then, T is a globally strong Picard map (modulo d).*

For a different proof, we refer to the developments below. Note that all examples in Wardowski [41] are illustrations of this last result; such as

(b08) $F(0) = -\infty$; $F(t) = \log(\alpha t^2 + \beta t) + \gamma t$, $t > 0$,
(b09) $F(0) = -\infty$; $F(t) = (-1)(t^{-\delta})$, $t > 0$;

here, log is the natural logarithm and $\alpha, \beta, \gamma > 0$, $\delta \in]0, 1[$ are constants. The case $\delta = 1$ of the last example is not reducible to this particular statement. However, it is possible to handle it with the aid of the first main result; we do not give details.

1.3 Discontinuous Wardowski Functions

Let us now return to the general setting above. As the preceding facts show, the left continuity requirement upon F is essential for the first main result to be deduced from Matkowski theorem. In the absence of this, the reduction argument above does not work; and then, the question arises as to what extent is our first main result retainable in such an extended setting. Strange enough, a positive answer to this is still available, via standard metrical procedures.

Let (X, d) be a metric space; and $T \in \mathscr{F}(X)$ be a selfmap of X. Our second main result is (cf. Turinici [39]):

Theorem 4. *Suppose that T is (a, F)-contractive, for some $a > 0$ and some Wardowski function $F : R_+ \to R \cup \{-\infty\}$. Then, T is globally strong Picard (modulo d).*

Proof. From the (a, F)-contractive condition, it results that

$$T \text{ is a strict contraction: } d(Tx, Ty) < d(x, y),\ \forall x, y \in X, x \neq y. \tag{18}$$

This firstly gives us that $\text{Fix}(T)$ is a singleton. As a second consequence,

$$T \text{ is nonexpansive: } d(Tx, Ty) \leq d(x, y),\ \forall x, y \in X; \tag{19}$$

hence, in particular, T is d-continuous on X. So, it remains only to prove that T is strong Picard (modulo d). Let x_0 be arbitrary fixed in X; and put $(x_n := T^n x_0;$ $n \geq 0)$. If $x_k = x_{k+1}$ for some $k \geq 0$, we are done; so, without loss, one may assume that

(c01) $\rho_n := d(x_n, x_{n+1}) > 0$, for all $n \geq 0$.

Part 1. By the contractive condition, we have

$$a + F(\rho_{n+1}) \leq F(\rho_n), \text{ [hence, } a \leq F(\rho_n) - F(\rho_{n+1})], \forall n.$$

Adding the first n inequalities, one gets

$$na \leq F(\rho_0) - F(\rho_n) \text{ [hence: } F(\rho_n) \leq F(\rho_0) - na], \forall n;$$

so that, passing to limit as $n \to \infty$, one derives $F(\rho_n) \to -\infty$. This, along with a preceding auxiliary fact, gives $\rho_n \to 0$; hence, $(x_n; n \geq 0)$ is d-semi-Cauchy.

Part 2. Let $\Delta := \Delta(F)$ stand for the subset of all $s \in R^0_+$ where F is discontinuous at s; note that, by a preceding auxiliary fact, it is (at most) denumerable. This tells us that $\Theta := R^0_+ \setminus \Delta$ is dense in R^0_+; hence, all the more, $(>)$-cofinal in R^0_+. Assume by contradiction that (x_n) is not d-Cauchy. By an auxiliary statement about d-semi-Cauchy sequences, there exists then a number $b \in \Theta$, such that: for each $\eta \in]0, b/3[$, a rank $j(\eta) \geq 0$, and a couple of rank-sequences $(m(j); j \geq 0)$, $(n(j); j \geq 0)$, may be found with the properties (4)–(8). By the contractive condition, we have

$$a + F(d(x_{m(j)+1}, x_{n(j)+1})) \leq F(d(x_{m(j)}, x_{n(j)})), \ \forall j.$$

Passing to limit as $j \to \infty$ one gets, from the choice of b,

$$a + F(b) \leq F(b); \text{ hence, } a \leq 0.$$

The obtained contradiction tells us that (x_n) is d-Cauchy.

Part 3. By the completeness assumption, $x_n \xrightarrow{d} z$ as $n \to \infty$, for some (uniquely determined) $z \in X$. As T is d-continuous (see above), $y_n := Tx_n \xrightarrow{d} Tz$. On the other hand, $(y_n = x_{n+1}; n \geq 0)$ is a subsequence of (x_n); and this yields (as d= metric) $z = Tz$. Hence, T is a strong Picard operator (modulo d); and the proof is complete.

Let again (X, d) be a metric space; and $T \in \mathscr{F}(X)$ be a selfmap of X. From the second main result, we have formally

Theorem 5. *Suppose that T is (a, F)-contractive, for some $a > 0$ and some Wardowski function $F(.)$; supposed to be k-regular, for some $k \in]0, 1[$. In addition, let (X, d) be complete. Then, T is global strong Picard (modulo d).*

This result is, essentially, the one in Wardowski [41]. For completeness reasons, we shall provide its proof, with some modifications.

Proof. **(Theorem 5)** As in the second main result, we only have to establish that T is strong Picard (modulo d). Let x_0 be arbitrary fixed in X; and put $(x_n := T^n x_0; n \geq 0)$. If $x_k = x_{k+1}$ for some $k \geq 0$, we are done; so, without loss, one may assume that $(\rho_n := d(x_n, x_{n+1}); n \geq 0)$ is a sequence in R^0_+. By the contractive condition,

$$a + F(\rho_{n+1}) \leq F(\rho_n), \text{ [hence, } a \leq F(\rho_n) - F(\rho_{n+1})], \forall n.$$

Adding the first n inequalities, one gets

$$na \le F(\rho_0) - F(\rho_n) \text{ [hence: } F(\rho_n) \le F(\rho_0) - na], \forall n. \tag{20}$$

This firstly gives $F(\rho_n) \to -\infty$; wherefrom, by a preceding auxiliary fact, $\rho_n \to 0$; hence, $(x_n; n \ge 0)$ is d-semi-Cauchy. Secondly, the same relation yields

$$na\rho_n^k \le [F(\rho_0) - F(\rho_n)]\rho_n^k, \ \forall n. \tag{21}$$

By the convergence property of $(\rho_n; n \ge 0)$ and the k-regularity of F, the limit of the right-hand side is zero; so, given $\beta > 0$, there must be some rank $i = i(\beta)$ with

$$[F(\rho_0) - F(\rho_n)]\rho_n^k \le \beta, \ \forall n \ge i.$$

Combining with the same relation above yields (after transformations)

$$\rho_n \le (\beta/an)^{1/k}, \text{ for all } n \ge i.$$

This, along with convergence of the series $\sum_{n\ge 1} n^{-1/k}$ tells us that so is the series $\sum_n \rho_n$; wherefrom, $(x_n; n \ge 0)$ is d-Cauchy. The last part is identical with the one of our second main result; and conclusion follows.

Note, finally, that all these conclusions are extendable to the framework of quasi-ordered metric spaces under the lines in Agarwal et al. [1]; see also Turinici [36].

2 Limit Implicit Contractions

2.1 Introduction

Let X be a nonempty set, and $d(.,.)$ be a (standard) metric over it. Call the relation $(\le)$ on X, *quasi-order*, provided it is *reflexive* ($x \le x$, for all $x \in X$) and *transitive* ($x \le y$ and $y \le z$ imply $x \le z$); the structure $(X, d, \le)$ will be referred to as a *quasi-ordered metric space*. We say that the subset Y of X is $(\le)$-*asingleton*, if $[y_1, y_2 \in Y, \ y_1 \le y_2]$ imply $y_1 = y_2$; and $(\le)$-*singleton*, if, in addition, Y is nonempty. Clearly, in the *amorphous* case $(\le) = X \times X$, $(\le)$-asingleton (resp., $(\le)$-singleton) is identical with asingleton (resp., singleton); but, in general, this cannot be true.

Now, take some $T \in \mathscr{F}(X)$. Assume in the following that

(a01) T is semi-progressive: $X(T, \le) := \{x \in X; x \le Tx\} \ne \emptyset$
(a02) T is increasing: $x \le y$ implies $Tx \le Ty$.

We have to determine circumstances under which $\mathrm{Fix}(T)$ be nonempty; and, if this holds, to establish whether T is *fix-*$(\le)$*-asingleton* (i.e.: $\mathrm{Fix}(T)$ is $(\le)$-asingleton);

or, equivalently: T is *fix-*$(\leq)$*-singleton* (in the sense: $\text{Fix}(T)$ is $(\leq)$-singleton); a similar problem is to be formulated with respect to the iterates T^k, where $k \geq 1$. Note that the introduction of a quasi-order structure over X yields rather deep modifications of the working context. This is shown from the list of basic concepts to be considered (cf. Turinici [38]):

(1a) We say that T is a *Picard operator* (modulo $(d, \leq)$) if, for each $x \in X(T, \leq)$, $(T^n x; n \geq 0)$ is d-convergent (hence, d-Cauchy); so that, $\lim_n(T^n x)$ is a singleton

(1b) We say that T is a *strong Picard operator* (modulo $(d, \leq)$) when, for each $x \in X(T, \leq)$, $(T^n x; n \geq 0)$ is d-convergent (hence, d-Cauchy); and $z := \lim_n(T^n x)$ is an element of $\text{Fix}(T)$

(1c) We say that T is a *Bellman Picard operator* (modulo $(d, \leq)$) when, for each $x \in X(T, \leq)$, $(T^n x; n \geq 0)$ is d-convergent (hence, d-Cauchy); and $z := \lim_n(T^n x)$ is an element of $\text{Fix}(T)$, with $T^n x \leq z$, for all $n \geq 0$

(1d) We say that T is a *globally strong (resp., Bellman) Picard operator* (modulo $(d, \leq)$) when it is a strong (resp., Bellman) Picard operator (modulo $(d, \leq)$) and T is fix-$(\leq)$-asingleton (hence, fix-$(\leq)$-singleton).

In particular, when $(\leq) = X \times X$, the list of such notions is identical with the one we already encountered; because, in this case, $X(T, \leq) = X$.

The sufficient (regularity) conditions for such properties are being founded on *ascending orbital* concepts (in short: a-o-concepts). Namely, call the sequence $(z_n; n \geq 0)$ in X, *ascending*, if $z_i \leq z_j$ for $i \leq j$; and T*-orbital*, when it is a subsequence of $(T^n x; n \geq 0)$, for some $x \in X$; the intersection of these notions is just the precise one.

(1e) Call (X, d), *a-o-complete*, provided (for each a-o-sequence) d-Cauchy $\Longrightarrow$ d-convergent

(1f) We say that T is *$(a - o, d)$-continuous*, if ((z_n)=a-o-sequence and $z_n \xrightarrow{d} z$) imply $Tz_n \xrightarrow{d} Tz$

(1g) Call $(\leq)$, *$(a - o, d)$-self-closed*, when the d-limit of each d-convergent a-o-sequence is an upper bound of it.

When the orbital properties are ignored, these conventions may be written in the usual way; we do not give details.

Concerning these concepts, the following simple fact is useful for us:

Proposition 4. *Suppose that T is globally Bellman Picard (modulo $(d, \leq)$). Then, $(X(T, \leq), \leq)$ is a Zorn quasi-ordered structure, in the sense:*

i) *each $x \in X(T, \leq)$ is majorized by an element $z \in \text{Fix}(T) \subseteq X(T, \leq)$*
ii) *any $w \in \text{Fix}(T)$ is $(\leq)$-maximal in $X(T, \leq)$: $w \leq x \in X(T, \leq) \Longrightarrow x \leq w$.*

Proof. i) Evident, by definition.

ii) Let $w \in \text{Fix}(T)$ and $x \in X(T, \leq)$ be such that $w \leq x$. By the preceding fact, we have $x \leq z$, for some $z \in \text{Fix}(T)$. This yields $w \leq z$; and then, as T is fix-$(\leq)$-asingleton, $w = z$; whence (combining with the conclusion above) $x \leq w$.

This auxiliary statement shows the important role of global Bellman Picard operators within the above operator classes. Note that the introduced concept is related to the well-known Bellman integral inequality; cf Turinici [37].

The final lot of (regularity) conditions for our problem are of metrical-contractive type. Denote, for $x, y \in X$:

$$H(x, y) = \max\{d(x, Tx), d(y, Ty)\},\ L(x, y) = (1/2)[d(x, Ty) + d(Tx, y)],$$
$$G_1(x, y) = d(x, y),\ G_2(x, y) = \max\{G_1(x, y), H(x, y)\},$$
$$G_3(x, y) = \max\{G_2(x, y), L(x, y)\} = \max\{G_1(x, y), H(x, y), L(x, y)\}.$$

Given $G \in \{G_1, G_2, G_3\}$, $\varphi \in \mathscr{F}(R_+)$, we say that T is $(d, \leq; G, \varphi)$-*contractive* if

(a03) $\quad d(Tx, Ty) \leq \varphi(G(x, y)), \forall x, y \in X, x \leq y.$

The regularity conditions imposed upon φ are of the type discussed in the preliminary part. Namely, call $\varphi \in \mathscr{F}(re)(R_+)$, *Boyd–Wong admissible*, provided $\Lambda^+\varphi(s) < s$, $\forall s \in R_+^0$; and *Matkowski admissible*, if [φ is increasing and $\varphi^n(t) \to 0$ as $n \to \infty$, for all $t > 0$]. The union of these properties will be referred to as: $\varphi \in \mathscr{F}(re)(R_+)$ is *Boyd–Wong–Matkowski admissible*; in short: *BWM-admissible*.

The following *Extended BWM theorem* is our starting point.

Theorem 6. *Suppose that T is $(d, \leq; G, \varphi)$-contractive, for some mapping $G \in \{G_1, G_2, G_3\}$ and some BWM-admissible function $\varphi \in \mathscr{F}(re)(R_+)$. In addition, let (X, d) be a-o-complete. Then,*

i) *If T is $(a - o, d)$-continuous, then it is globally strong Picard (modulo $(d, \leq)$)*
ii) *If $(\leq)$ is $(a - o, d)$-self-closed, then T is a globally Bellman Picard operator (modulo $(d, \leq)$).*

As before, a proof of this result will be given a bit further. For the moment, some particular cases will be discussed.

Case 1. Suppose that φ is Boyd–Wong admissible. Then, if $G = G_1$, Extended BWM theorem is a quasi-order extension of the Boyd–Wong theorem [8]. In particular, when φ is *linear* ($\varphi(t) = \alpha t$, $t \in R_+$, for some $\alpha \in [0, 1[$), this version of Extended BWM theorem gives the statement in Ran and Reurings [26]; see also Nieto and Rodriguez-Lopez [19].

Case 2. Suppose that φ is Matkowski admissible. If $G = G_3$, then Extended BWM theorem is a particular case of the one in Turinici [38], (expressed in terms of convergence structures); cf. O'Regan and Petruşel [20]. On the other hand when the orbital properties are ignored, then, **i)** if $G = G_3$ and $(\leq)$ is an order, Extended BWM theorem is just the 2008 statement in Agarwal et al. [1], **ii)** if $G = G_1$, Extended BWM theorem is the 1986 result in Turinici [37]; which, in turn, extends, when $(\leq) = X \times X$, the Matkowski theorem [16].

The obtained variants of Extended BWM theorem found some basic applications to existence results for linear and nonlinear operator equations; see the quoted papers for details. As a consequence, the question of extending this result is useful from both a theoretical and a practical perspective. A statement of this type was given (in the ordered framework) by Altun and Simsek [4], via "implicit" type contractive conditions like

(a04) $F(d(Tx,Ty),d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(Tx,y)) \leq 0$, for all $x,y \in X$ with $x \leq y$;

where $F : R^6_+ \to R$ is a function. However, it does not include in a complete manner the already quoted 2008 "explicit" statement above; so, we may ask whether this is possible in some way. It is our aim in the following to show that a positive answer to the posed question is available. Precisely, a "limit" type extension of the Altun–Simsek result is formulated, so as to include the results above. Further aspects occasioned by these developments are also discussed.

2.2 Main Result

Let $(X,d,\leq)$ be a quasi-ordered metric space; and $T \in \mathscr{F}(X)$ be a selfmap of X; supposed to be semi-progressive and increasing.

(A) Denote, for each $x,y \in X$

(b01) $M_1(x,y) = d(Tx,Ty)$, $M_2(x,y) = d(x,y)$, $M_3(x,y) = d(x,Tx)$,
$M_4(x,y) = d(y,Ty)$, $M_5(x,y) = d(x,Ty)$, $M_6(x,y) = d(Tx,y)$,
$\mathscr{M}(x,y) = (M_1(x,y), M_2(x,y), M_3(x,y), M_4(x,y), M_5(x,y), M_6(x,y))$,
$\mathscr{M}^1(x,y) = (M_2(x,y), M_3(x,y), M_4(x,y), M_5(x,y), M_6(x,y))$.

Given $F \in \mathscr{F}(R^6_+, R)$, let us say that T is $(d,\leq;\mathscr{M};F)$*-contractive*, provided

(b02) $F(\mathscr{M}(x,y)) \leq 0$, for all $x,y \in X$ with $x \leq y$, $x \neq y$.

The class of functions F appearing here is described under the lines below.

(2a) Call $F \in \mathscr{F}(R^6_+, R)$, *compatible*, provided:

(b03) for each couple of sequences $(r_n; n \geq 0)$ in R^0_+ and $(s_n; n \geq 0)$ in R_+ with
$F(r_n, r_{n-1}, r_{n-1}, r_n, s_{n-1}, 0) \leq 0$, $\forall n \geq 1$, and $|s_{n-1} - r_{n-1}| \leq r_n$, $\forall n \geq 1$,
we must have $r_n \to 0$ (hence, $s_n \to 0$)

(2b) Let us say that $F \in \mathscr{F}(R^6_+, R)$ is *(3,4)-normal*, in case

(b04) $F(r,r,0,0,r,r) > 0$, for all $r > 0$.

The next properties will necessitate some conventions. Take some point (in R^6_+) $W = (w_1,w_2,w_3,w_4,w_5,w_6)$; as well as a rank $j \in \{1,2,3,4,5,6\}$. We say that the sequence $(t^n := (t^n_1, t^n_2, t^n_3, t^n_4, t^n_5, t^n_6); n \geq 0)$ in R^6_+ is

I) *j-right* at W, if $t^n_i \to w_i$, $i \neq j$, $t^n_j \to w_j ++$
II) *j-point* at W, if $t^n_i \to w_i$, $i \neq j$, $t^n_j = w_j$, $\forall n$.

(2c) Given $b > 0$, call the function F, *2-right-lim-positive* at b, if $\limsup_n F(t^n) > 0$, for each 2-right at $(b,b,0,0,b,b)$, sequence $(t^n; n \geq 0)$ in $(R^0_+)^6$. The class of all these $b > 0$ will be denoted as $\mathrm{Pos}(2 - right - lim; F)$. In this case, we say that F is *almost 2-right-lim-positive*, if $\Theta := \mathrm{Pos}(2 - right - lim; F)$ is $(>)$-cofinal in R^0_+ [for each $\varepsilon \in R^0_+$ there exists $\theta \in \Theta$ with $\varepsilon > \theta$]; and *2-right-lim-positive*, if $\Theta = R^0_+$.

(2d) Given $b > 0$, call the function F, *4-point-lim-positive* at b, if $\limsup_n F(t^n) > 0$, for each 4-point at $(b, 0, 0, b, b, 0)$ sequence $(t^n; n \geq 0)$ in $(R^0_+)^6$. The class of all these $b > 0$ will be denoted as $\mathrm{Pos}(4 - point - lim; F)$. In this case, we say that F is *4-point-lim-positive*, if this last set is identical with R^0_+.

(B) Having these precise, we may now pass to our first main result:

Theorem 7. *Assume that T is $(d, \leq; \mathscr{M}; F)$-contractive, for some function $F \in \mathscr{F}(R^6_+, R)$, endowed with the properties: compatible, (3,4)-normal, almost 2-right-lim-positive, and 4-point-lim-positive. In addition, let (X, d) be a-o-complete. Then,*

- **i)** *If T is $(a - o, d)$-continuous, then it is globally strong Picard (modulo $(d, \leq)$)*
- **ii)** *If $(\leq)$ is $(a - o, d)$-self-closed, then T is a global Bellman Picard operator (modulo $(d, \leq)$).*

Proof. We first check the fix $(\leq)$-asingleton property of T. Let $z_1, z_2 \in \mathrm{Fix}(T)$ be such that $z_1 \leq z_2$ and $z_1 \neq z_2$. By the contractive condition,

$$F(\rho, \rho, 0, 0, \rho, \rho) \leq 0, \text{ where } \rho := d(z_1, z_2) > 0.$$

This, however, is in contradiction with F being (3,4)-normal; and our claim follows. It remains now to establish that T is a strong/Bellman Picard operator (modulo $(d, \leq)$). Let $x_0 \in X(T, \leq)$ be arbitrary fixed; and put $(x_n = T^n x_0, n \geq 0)$; clearly, $(x_n; n \geq 0)$ is ascending-orbital (by the general conditions imposed upon T). If $x_n = x_{n+1}$ for some $n \geq 0$, we are done; so, without loss, one may assume that

$x_n \neq x_{n+1}$ (hence, $r_n := d(x_n, x_{n+1}) > 0$), $\forall n$.

Step 1. Denote for simplicity $(s_n := d(x_n, x_{n+2}); n \geq 0)$; it is a sequence in R_+. By the contractive condition attached to (x_{n-1}, x_n) we have

$$F(r_n, r_{n-1}, r_{n-1}, r_n, s_{n-1}, 0) \leq 0, \quad \forall n \geq 1. \tag{22}$$

Combining with the evaluation $(\forall n \geq 1)$

$$|s_{n-1} - r_{n-1}| = |d(x_{n-1}, x_{n+1}) - d(x_{n-1}, x_n)| \leq d(x_n, x_{n+1}) = r_n, \tag{23}$$

one gets (via F=compatible) that

$$(d(x_n, x_{n+1}) > 0, \forall n, \text{ and) } d(x_n, x_{n+1}) \to 0 \text{ as } n \to \infty; \tag{24}$$

hence (by definition), $(x_n; n \geq 0)$ is d-semi-Cauchy.

Step 2. As F is almost 2-right-lim-positive, $\Theta := \text{Pos}(2-right-lim; F)$ appears as ($>$)-cofinal in R^0_+. We show that $(x_n; n \geq 0)$ is d-Cauchy. Suppose not; then, by a preliminary statement, there exists a number $b \in \Theta$, such that: for each $\eta \in]0, b/3[$ (fixed in the sequel), a rank $j(\eta) \geq 0$, and a couple of rank-sequences $(m(j); j \geq 0)$, $(n(j); j \geq 0)$, may be found with the properties (4)–(8). By the very definition of Θ, F is 2-right-lim-positive at b. On the other hand, by the working hypothesis above,

$$\begin{aligned}&(t_3^j := r_{m(j)}; j \geq 0) \text{ and } (t_4^j := r_{n(j)}; j \geq 0)\\ &\text{are sequences in } R^0_+ \text{ with } t_3^j, t_4^j \to 0 \text{ as } j \to \infty.\end{aligned} \tag{25}$$

Moreover, taking (7) into account, yields

$$\begin{aligned}&(t_2^j := d(x_{m(j)}, x_{n(j)}); j \geq 0)\\ &\text{is a sequence in } R^0_+ \text{ with } t_2^j \to b++ \text{ as } j \to \infty.\end{aligned} \tag{26}$$

Finally, by the relation (8), one gets

$$\begin{aligned}&(t_1^j := d(x_{m(j)+1}, x_{n(j)+1}); j \geq j(\eta)), \text{ and}\\ &(t_5^j := d(x_{m(j)}, x_{n(j)+1}); j \geq j(\eta)),\ (t_6^j := d(x_{m(j)+1}, x_{n(j)}); j \geq j(\eta))\\ &\text{are sequences in } R^0_+ \text{ with } t_1^j, t_5^j, t_6^j \to b \text{ as } j \to \infty.\end{aligned} \tag{27}$$

Now, by (7) again, the contractive condition applies to $(x_{m(j)}, x_{n(j)})$, for all $j \geq 0$; and yields:

$$F(t_1^j, t_2^j, t_3^j, t_4^j, t_5^j, t_6^j) \leq 0, \ \forall j \geq 0.$$

This gives $\limsup_j F(t_1^j, t_2^j, t_3^j, t_4^j, t_5^j, t_6^j) \leq 0$; hence, F is not 2-right-lim-positive at b; contradiction.

Step 3. As $(x_n; n \geq 0)$ is an ascending-orbital d-Cauchy sequence, there exists, by the a-o-completeness property of (X, d), some point $z \in X$ with $x_n \xrightarrow{d} z$ as $n \to \infty$. So, if T is $(a-o, d)$-continuous, $y_n := Tx_n \xrightarrow{d} Tz$ as $n \to \infty$. In addition, as $(y_n = x_{n+1}; n \geq 0)$ is a subsequence of $(x_n; n \geq 0)$, we have $y_n \xrightarrow{d} z$ as $n \to \infty$; hence, (as d=metric), $z = Tz$. Suppose now that $(\leq)$ is $(a-o, d)$-self-closed; note that, as a consequence, $x_n \leq z$, $\forall n$. Two cases may occur.

Case 3-1. There exists a sequence of ranks $(k(i); i \geq 0)$ with $k(i) \to \infty$ as $i \to \infty$ in such a way that $x_{k(i)} = z$ (hence $x_{k(i)+1} = Tz$), for all i. This, and $(x_{k(i)+1}; i \geq 0)$ being a subsequence of $(x_n; n \geq 0)$, gives $z \in \text{Fix}(T)$.

Case 3-2. There exists some rank $h \geq 0$ such that

(b05) $\quad n \geq h \Longrightarrow (x_n \leq z \text{ and}) \ x_n \neq z.$

Suppose by contradiction that $z \neq Tz$; i.e.: $b := d(z, Tz) > 0$. From the imposed assumptions, F is 4-point-lim-positive at b. On the other hand, the semi d-Cauchy and convergence properties above give (for all $n \geq h$)

$$\begin{aligned}&(t_2^n := d(x_n, z); n \geq h), \text{ and}\\ &(t_3^n := d(x_n, x_{n+1}); n \geq h), (t_6^n := d(x_{n+1}, z); n \geq h)\\ &\text{are sequences in } R_+^0 \text{ with } t_2^n, t_3^n, t_6^n \to 0 \text{ as } n \to \infty.\end{aligned} \tag{28}$$

Further, the same convergence relation assures us that

$$\exists n(b) \geq h: \ 0 < d(x_n, z) < b/2, \ \ \forall n \geq n(b).$$

Combining with the evaluation

$$|d(x_n, Tz) - b| \leq d(x_n, z) < b/2, \ \forall n \geq n(b),$$

we get

$$\begin{aligned}&(t_1^n := d(x_{n+1}, Tz); n \geq n(b)), (t_5^n := d(x_n, Tz); n \geq n(b))\\ &\text{are sequences in } R_+^0 \text{ with } t_1^n \to b, t_5^n \to b \text{ if } n \to \infty.\end{aligned} \tag{29}$$

The contractive condition applies to (x_n, z) (for all $n \geq h$); and yields

$$F(t_1^n, t_2^n, t_3^n, t_4^n, t_5^n, t_6^n) \leq 0, \ \forall n \geq n(b);$$

where $(t_4^n = b; n \geq 0)$. This gives $\limsup_n F(t_1^n, t_2^n, t_3^n, t_4^n, t_5^n, t_6^n) \leq 0$; hence, F is not 4-point-lim-positive at b; contradiction. So, necessarily, $z \in \mathrm{Fix}(T)$; and the proof is thereby complete.

2.3 Explicit Versions

In the following, we show that certain explicit counterparts of the main result include the (already exposed) BWM-caa theorem and Extended BWM theorem; this, among others, provides the promised proof of both these.

Let us say that $A \in \mathscr{F}(R_+^5, R_+)$ is *5-semi-altering*, if it is increasing and continuous in all variables. In the following, three such functions will be used:

(c01) $G_1(t_1, t_2, t_3, t_4, t_5) = t_1, \ G_2(t_1, t_2, t_3, t_4, t_5) = \max\{t_1, t_2, t_3\}$,
$G_3(t_1, t_2, t_3, t_4, t_5) = \max\{t_1, t_2, t_3, (1/2)(t_4 + t_5)\}$, $(t_1, t_2, t_3, t_4, t_5) \in R_+^5$.

Let $(X, d, \leq)$ be a quasi-ordered metric space; and $T \in \mathscr{F}(X)$ be a selfmap of X; supposed to be semi-progressive and increasing. Denote, for each $x, y \in X$,

$M_1(x, y) = d(Tx, Ty)$, $M_2(x, y) = d(x, y)$, $M_3(x, y) = d(x, Tx)$,

$$M_4(x,y) = d(y,Ty),\, M_5(x,y) = d(x,Ty),\, M_6(x,y) = d(Tx,y),$$
$$\mathscr{M}(x,y) = (M_1(x,y), M_2(x,y), M_3(x,y), M_4(x,y), M_5(x,y), M_6(x,y)),$$
$$\mathscr{M}^1(x,y) = (M_2(x,y), M_3(x,y), M_4(x,y), M_5(x,y), M_6(x,y)).$$

Given $G \in \{G_1, G_2, G_3\}$ and $\psi \in \mathscr{F}(R_+)$, let us say that T is $(d, \le; G, \mathscr{M}^1; \psi)$-*contractive* if

(c02) $\quad d(Tx,Ty) \le \psi(G(\mathscr{M}^1(x,y))),\ \forall x,y \in X, x \le y, x \ne y.$

Note that the introduced convention amounts to saying that T is $(d, \le; \mathscr{M}; F)$-contractive, where the *associated to* (G,ψ) function $F \in \mathscr{F}(R_+^6, R)$ is taken as

(c03) $\quad F(t_1,\dots,t_6) = t_1 - \psi(G(t_2,\dots,t_6)),\ (t_1,t_2,\dots,t_6) \in R_+^6.$

We want to determine under which conditions about ψ is our main result applicable to $(X,d,\le)$ and the function F.

(A) Remember that $\psi \in \mathscr{F}(re)(R_+)$ is called *compatible*, when

(c04) $\quad$ for each sequence $(r_n; n \ge 0)$ in R_+^0, with $r_n \le \psi(r_{n-1})$, $\forall n \ge 1$, we must have $r_n \to 0$.

Also, $\psi \in \mathscr{F}(re)(R_+)$ is called *almost Boyd–Wong admissible*, when: for each $\varepsilon > 0$ there exists $b \in]0,\varepsilon[$ such that $\Lambda^+\psi(b) < b$.

Sufficient conditions for such properties were already given. Here, we just note that, either of the functions below fulfills both these:

(3a) $\psi \in \mathscr{F}(re)(R_+)$ is Boyd–Wong admissible: $\Lambda^+\psi(s) < s$, for each $s > 0$
(3b) $\psi \in \mathscr{F}(re)(R_+)$ is Matkowski admissible:
ψ is increasing and $[\psi^n(t) \to 0$ as $n \to \infty$, for each $t > 0]$.

Proposition 5. *Let the function $\psi \in \mathscr{F}(re)(R_+)$ be compatible and almost Boyd–Wong admissible. Then, the associated to (G,ψ) function F is compatible, (3,4)-normal, almost 2-right-lim-positive, and 4-point-lim positive.*

Proof. The argument will be divided into several steps.

Part 1 (F is compatible). Let $(r_n) \subset R_+^0$, $(s_n) \subset R_+$ be sequences fulfilling

(c08) $\quad F(r_n, r_{n-1}, r_{n-1}, r_n, s_{n-1}, 0) \le 0$ and $|s_{n-1} - r_{n-1}| \le r_n$, $\forall n \ge 1$.

From $s_{n-1} \le r_{n-1} + r_n \le 2\max\{r_{n-1}, r_n\}$, $\forall n \ge 1$, we have, for all $n \ge 1$,

$$G_1(r_{n-1}, r_{n-1}, r_n, s_{n-1}, 0) = r_{n-1},$$
$$G_k(r_{n-1}, r_{n-1}, r_n, s_{n-1}, 0) = \max\{r_{n-1}, r_n\},\ k \in \{2,3\}.$$

This, along with $\psi \in \mathscr{F}(re)(R_+)$, yields (for any choice of G), $r_n \le \psi(r_{n-1}), \forall n \ge 1$; wherefrom (as ψ is compatible) $r_n \to 0$ as $n \to \infty$; and the claim follows.

Part 2 (F is (3,4)-normal). Let $r > 0$ be arbitrary fixed. By definition,

$$G_k(r,0,0,r,r) = r,\ k \in \{1,2,3\};$$

so that

$$F(r, r, 0, 0, r, r) = r - \psi(G(r, 0, 0, r, r)) = r - \psi(r) > 0;$$

and, from this, we are done.

Part 3 (F is almost 2-right-lim-positive). As ψ is almost Boyd–Wong admissible, for each $\varepsilon > 0$ there exists $r \in]0, \varepsilon[$ with $\Lambda^+\psi(r) < r$. We show that the function F defined above is 2-right-lim-positive at r. Let $(t_i^n; n \geq 0)$, $i \in \{1, 2, 3, 4, 5, 6\}$, be sequences in R_+^0 with (as $n \to \infty$)

$t_i^n \to r, i \in \{1, 5, 6\}; t_i^n \to 0, i \in \{3, 4\}; t_2^n \to r++.$

By definition, there exists some rank $n(r)$ such that, for all $n \geq n(r)$,

$$G_1(t_2^n, t_3^n, t_4^n, t_5^n, t_6^n) = t_2^n > r,\ G_k(t_2^n, t_3^n, t_4^n, t_5^n, t_6^n) \geq t_2^n > r,\ k \in \{2, 3\};$$

and, as G is continuous in its variables,

$$G(t_2^n, t_3^n, t_4^n, t_5^n, t_6^n) \to G(r, 0, 0, r, r) = r;$$

hence, summing up,

$$G(t_2^n, t_3^n, t_4^n, t_5^n, t_6^n) \to r++, \text{ as } n \to \infty.$$

As a consequence,

$$\limsup_n F(t_1^n, t_2^n, t_3^n, t_4^n, t_5^n, t_6^n) \geq$$
$$r - \liminf_n \psi(G(t_2^n, t_3^n, t_4^n, t_5^n, t_6^n)) \geq r - \Lambda^+\psi(r) > 0;$$

and this proves our assertion.

Part 4 (F is 4-point-lim-positive). Let $r > 0$ be arbitrary fixed. We have to show that F is 4-point-lim-positive at r. Let $(t_i^n; n \geq 0)$, $i \in \{1, 2, 3, 4, 5, 6\}$, be sequences in R_+^0 with $(t_4^n = r; n \geq 0)$; and (as $n \to \infty$)

$t_i^n \to r, i \in \{1, 5\}; t_i^n \to 0, i \in \{2, 3, 6\}.$

There exists some rank $n(r)$ in such a way that

$$(\forall n \geq n(r)) :\ t_i^n < 3r/2,\ i \in \{1, 5\};\ t_i^n < r/2,\ i \in \{2, 3, 6\}.$$

Combining with the choice of $(t_4^n; n \geq 0)$ yields, for all $n \geq n(r)$,

$$G_1(t_2^n, t_3^n, t_4^n, t_5^n, t_6^n) = t_2^n < r/2;$$
$$\text{hence } \psi(G_1(t_2^n, t_3^n, t_4^n, t_5^n, t_6^n)) = \psi(t_2^n) < t_2^n < r/2;$$

$$G_k(t_2^n, t_3^n, t_4^n, t_5^n, t_6^n) = r,\ k \in \{2, 3\}.$$

This yields, in case $G = G_1$,

$$\limsup_n F(t_1^n, t_2^n, t_3^n, t_4^n, t_5^n, t_6^n) \geq r - r/2 = r/2 > 0;$$

and, in case $G \in \{G_2, G_3\}$,

$$\limsup_n F(t_1^n, t_2^n, t_3^n, t_4^n, t_5^n, t_6^n) = r - \psi(r) > 0.$$

As a consequence of these, the claim follows; and the argument is complete.

Now, by simply combining the obtained fact with our main result above, one gets the so-called Comp-almost Boyd–Wong theorem:

Theorem 8. *Suppose that T is $(d, \leq; G, \mathscr{M}^1; \psi)$-contractive, for some semi-altering $G \in \{G_1, G_2, G_3\}$ and some compatible almost Boyd–Wong admissible function $\psi \in \mathscr{F}(re)(R_+)$. In addition, assume that (X, d) is a-o-complete. Then,*

i) *If T is $(a-o, d)$-continuous, then it is globally strong Picard (modulo $(d, \leq)$)*
ii) *If $(\leq)$ is $(a-o, d)$-self-closed, then T is a globally Bellman Picard operator (modulo $(d, \leq)$).*

(B) The following particular cases are of interest:

Case 1. Assume that $(\leq) = X \times X$. Then, if $G = G_1$, Comp-almost Boyd–Wong theorem is just BWM-caa theorem; and, if $G = G_3$, Comp-almost Boyd–Wong theorem yields a direct extension of the statement in Hardy and Rogers [9].

Case 2. Passing to the general case (modulo $(\leq)$), note that any BWM-admissible function is compatible and almost Boyd–Wong admissible. Then, Comp-almost Boyd–Wong theorem is just Extended BWM theorem.

2.4 Global Aspects

In the following, a certain "global" version of the main result is given. As before, $(X, d, \leq)$ is a quasi-ordered metric space; and $T \in \mathscr{F}(X)$ is a selfmap of X; assumed to be semi-progressive, increasing.

Let $F \in \mathscr{F}(R_+^6, R)$ be compatible [see above]. For an application of the main result, it will suffice that F be (in addition) (3,4)-normal, almost 2-right-lim-positive, and 4-point-lim-positive. We shall try to assure this under the global condition

(d01) F is lower semicontinuous (in short: lsc) on R_+^6: $\liminf_n F(t_1^n, \ldots, t_6^n) \geq F(a_1, \ldots, a_6)$, whenever $t_i^n \to a_i$, $i \in \{1, \ldots, 6\}$.

Note that, in such a case, the 2-right-lim-positive property is obtainable from

(d02) (F is (3,4)-normal): $F(r, r, 0, 0, r, r) > 0$, $\forall r > 0$.

On the other hand, the 4-point-lim-positive property is holding, as long as

(d03) (F is (2,3,6)-normal): $F(r, 0, 0, r, r, 0) > 0$, $\forall r > 0$.

An application of our main result yields the following practical statement:

Theorem 9. *Assume that T is $(d, \leq, F)$-contractive, for some compatible lsc $F \in \mathscr{F}(R^6_+, R)$ which is both (3,4)-normal and (2,3,6)-normal. In addition, assume that (X, d) is a-o-complete. Then, conclusions of the main result are retainable.*

The following particular case is of interest. Assume that (in addition to the lsc property) the global condition holds

(d04) F is $(2, \ldots, 6)$-decreasing: $F(t_1, .)$ is decreasing in each argument, $\forall t_1 \in R_+$.

Then, the compatibility condition upon F is deductible from:

(d05) F is almost-compatible: for each sequence (r_n) in R^0_+, with $F(r_n, r_{n-1}, r_{n-1}, r_n, r_n + r_{n-1}, 0) \leq 0$, $\forall n \geq 1$, we must have $r_n \to 0$.

In particular this last condition is deductible from

(d06) F is ψ-compatible ($F(u, v, v, u, u + v, 0) \leq 0 \Longrightarrow u \leq \psi(v)$) for some compatible function $\psi \in \mathscr{F}(re)(R_+)$.

This is just the main result in Altun and Simsek [4]; obtained (under a different approach) with the lsc condition upon F being substituted by a continuity assumption upon the same.

Now, technically speaking, this last condition was introduced so as to be applicable for functions F attached to couples (G, ψ) (see above); where the compatible $\psi \in \mathscr{F}(re)(R_+)$ is either increasing or continuous. In the former case, F is $(2, \ldots, 6)$-decreasing; but, not in general lsc. In the latter case, F is neither lsc nor $(2, \ldots, 6)$-decreasing. As a consequence of this, no variant of the Comp-almost Boyd–Wong theorem involving such functions is deductible from the above statement. Further aspects may be found in Popa and Mocanu [25]; see also Vetro and Vetro [40].

Finally, note that these techniques are applicable as well to coincidence point results. In this case, it is possible to include the statement in Akkouchi [2]; we do not give further details.

3 Non-limit Approach

3.1 Introduction

In the following, a non-limit approach for the results above is being performed, so as to give a complementary perspective about these.

Let (X, d) be a metric space; and T be a selfmap of X. As before, we have to determine whether Fix(T) is nonempty; and, if this holds, to establish whether T is *fix-asingleton*; or, equivalently: T is *fix-singleton*. A similar problem is to be formulated with respect to the iterates T^k, where $k \geq 1$.

The specific directions under which this problem is to be solved were already listed in the previous context involving (amorphous) metrical structures. Sufficient general conditions for getting such properties are being founded on the orbital concepts (in short: o-concepts) we just introduced. The specific conditions for the same involve contractive properties, expressed as follows. Letting $C \in \mathscr{F}(R^3_+, R_+)$ be a function, we say that T is *(d, C)-contractive*, if

(a01) $d(Tx, Ty) \leq C(d(x, y), d(x, Tx), d(y, Ty)), \forall x, y \in X.$

The class of such functions $C(.)$ to be considered may be described along the following lines. Let Θ be a $(>)$-cofinal part of R^0_+. We say that $C \in \mathscr{F}(R^3_+, R_+)$ is *Q3-normal (modulo Θ)*, if it satisfies

(1a) the global conditions:

(a02) $C(w, 0, 0) < w$, for all $w > 0$
(a03) $u, v > 0$ and $u \leq C(v, v, u)$ imply $u \leq v$;

(1b) the local conditions: $\forall r > 0, \exists e(r) > 0, \exists h(r) \in]0, r[$, such that:

(a04) $u, v \in [r, r + e(r)[, u \leq v \Longrightarrow C(v, v, u) \leq h(r)$
(a05) $u, v \in]0, e(r)[\Longrightarrow C(u, v, r) \leq h(r)$;

(1c) the Θ-local condition: $\forall r \in \Theta, \exists e(r) > 0, \exists h(r) \in]0, r[$, such that:

(a06) $u \in]r, r + e(r)[, v, w \in]0, e(r)[\Longrightarrow C(u, v, w) \leq h(r).$

When the $(>)$-cofinal part Θ of R^0_+ is generically taken, the resulting property will be referred to as: $C \in \mathscr{F}(R^3_+, R_+)$ is *almost Q3-normal*. And, if $\Theta = R^0_+$, we say that $C \in \mathscr{F}(R^3_+, R_+)$ is *Q3-normal*.

The following answer (referred to as: Reich theorem) to the posed problem is now available:

Theorem 10. *Suppose that T is (d, C)-contractive, for some Q3-normal function $C \in \mathscr{F}(R^3_+, R_+)$. In addition, assume that (X, d) is o-complete. Then, T is a globally strong Picard operator (modulo d).*

In particular, when the orbital concepts are ignored, this result is just the one in Turinici [33]. As precise there, the quoted statement was constructed by taking as model the 1972 Reich's contribution [27]; this is a strong motivation for us to introduce our convention. Further aspects may be found in Turinici [31].

Now, a natural way of extending these explicit results is by considering implicit functional contractive conditions like

(a07) $F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0, \forall x, y \in X, x \neq y;$

where $F : R^6_+ \to R$ is an appropriate function. Some concrete examples may be found in Akkouchi and Popa [3], or Berinde and Vetro [6]; see also Nashine et al. [17]). Concerning this aspect, note that in almost all papers based on implicit techniques—including the ones we just quoted—it is asserted that the starting point in the area is represented by the contributions due to Popa [22–24]. Unfortunately, these affirmations are not true; see in this direction the "old" implicit approaches in Turinici [32, 34]. It is our aim in the following to give a (non-limit type) extension of these results to the realm of quasi-ordered metric spaces; which, in addition, includes the explicit statements above. As a matter of fact, further structural extensions of these developments are possible; we do not give details.

3.2 Main Result

Let X be a nonempty set. Take a metric $d(.,.)$ on X as well as a quasi-order $(\leq)$ over the same; the structure $(X, d, \leq)$ will be then called a *quasi-ordered metric space*. Further, let $T \in \mathscr{F}(X)$ be a selfmap of X; supposed to be semi-progressive and increasing. The specific directions under which the fixed point problem is to be solved were already listed in the previous context involving quasi-ordered metrical structures. Sufficient general conditions for getting such properties are being founded on the ascending-orbital concepts. (in short: a-o-concepts) we just introduced. The specific conditions for the same involve contractive properties, expressed as follows. Given $F \in \mathscr{F}(R^6_+, R)$, we say that T is $(d, \leq; F)$ *contractive*, provided

(b01) $F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(Tx, y)) \leq 0$, for all $x, y \in X$ with $x \leq y, x \neq y$.

The class of such functions is to be described as follows. Let Θ be a $(>)$-cofinal part of R^0_+ (for each $\varepsilon > 0$, there exists $\theta \in \Theta$ with $\varepsilon > \theta$). We say that $F \in \mathscr{F}(R^6_+, R)$ is *P6-normal (modulo Θ)*, provided it satisfies

(2a) the global conditions

(b02) $w > 0 \Rightarrow F(w, w, 0, 0, w, w) > 0$
(b03) $u, v > 0,\ p \leq u + v,\ F(u, v, v, u, p, 0) \leq 0 \Rightarrow u \leq v$;

(2b) the local conditions: $\forall r > 0, \exists a(r) \in]0, r[$ such that

(b04) $u, v \in [r, r + a(r)[,\ u \leq v,\ p \leq u + v \Rightarrow F(u, v, v, u, p, 0) > 0$
(b05) $t, p \in]r - a(r), r + a(r)[,\ u, v, q \in]0, a(r)[\Rightarrow F(t, u, v, r, p, q) > 0$;

(2c) the Θ-local condition: $\forall r \in \Theta, \exists a(r) \in]0, r[$ such that

(b06) $t, p, q \in]r - a(r), r + a(r)[,\ u \in]r, r + a(r)[,\ v, w \in]0, a(r)[\Rightarrow F(t, u, v, w, p, q) > 0$.

When the $(>)$-cofinal part Θ of R^0_+ is generically taken, the resulting property will be referred to as: $F \in \mathscr{F}(R^6_+, R)$ is *almost P6-normal*. And, if $\Theta = R^0_+$, we say that $F \in \mathscr{F}(R^6_+, R)$ is *P6-normal.*

Our first main result can be stated as follows.

Theorem 11. *Suppose that T is $(d, \leq, F)$-contractive, where $F \in \mathscr{F}(R^6_+, R_+)$ is almost P6-normal. In addition, assume that (X, d) is a-o-complete. Then,*

i) *If T is $(a - o, d)$-continuous, then it is globally strong Picard (modulo $(d, \leq)$)*
ii) *If $(\leq)$ is $(a - o, d)$-self-closed, then T is a globally Bellman Picard operator (modulo $(d, \leq)$).*

Proof. First, we prove the fix-$(\leq)$-asingleton property for T. Let $z_1, z_2 \in \text{Fix}(T)$ be such that $z_1 \leq z_2$ and $z_1 \neq z_2$; hence, $r := d(z_1, z_2) > 0$. From the contractive condition, we get $F(r, r, 0, 0, r, r) \leq 0$; in contradiction with the first global property of F; hence, the claim. It remains to establish that T is a strong/Bellman Picard operator (modulo $(d, \leq)$). Take some $x_0 \in X(T, \leq)$ and consider the sequence $(x_n := T^n x_0; n \geq 0)$; it is ascending and orbital. If $x_n = x_{n+1}$ for some $n \geq 0$, the conclusion follows. So, without loss, one may assume that

$x_n \neq x_{n+1}$ (hence, $r_n := d(x_n, x_{n+1}) > 0$), for all $n \geq 0$.

I) Denote for simplicity $(s_n := d(x_n, x_{n+2}); n \geq 0)$; it is a sequence in R_+. Again by the contractive condition,

$$F(r_{n+1}, r_n, r_n, r_{n+1}, s_n, 0) \leq 0, \ \forall n \geq 0. \tag{30}$$

On the other hand, the triangle inequality gives $(\forall n \geq 0)$

$$s_n = d(x_n, x_{n+2}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) = r_n + r_{n+1}. \tag{31}$$

Combining with the second global property of F, one gets $r_{n+1} \leq r_n$, $\forall n \geq 0$. The sequence $(r_n; n \geq 0)$ (in R^0_+) is then decreasing; hence $r := \lim_n(r_n)$ exists and $r_n \geq r$, for all $n \geq 0$. Suppose by contradiction that $r > 0$; and let the quantity $a(r) \in]0, r[$ be the one given by the first local property of F. By definition, there may find some rank $n(r) \geq 0$, such that

$$n \geq n(r) \Rightarrow r_n, r_{n+1} \in [r, r + a(r)[. \tag{32}$$

By the first local property in question, we thus have

$$F(r_{n+1}, r_n, r_n, r_{n+1}, s_n, 0) > 0, \ \forall n \geq n(r); \tag{33}$$

which contradicts, for $n \geq n(r)$, a previous relation involving these data. Hence, $r = 0$; i.e.,

$$r_n := d(x_n, x_{n+1}) \to 0, \text{ as } n \to \infty; \tag{34}$$

or, in other words: $(x_n; n \geq 0)$ is d-semi-Cauchy.

II) We now claim that $(x_n; n \geq 0)$ is d-Cauchy. Suppose that this is not true. From the imposed hypothesis upon F, there exists a $(>)$-cofinal subset Θ of R_+^0, such that F is P6-normal (modulo Θ). This, by a preliminary statement involving d-semi-Cauchy sequences, assures us that there exists a number $\theta \in \Theta$, such that: for each $\eta \in]0, \theta/3[$, there may be found a rank $j(\eta) \geq 0$, and a couple of rank-sequences $(m(j); j \geq 0)$, $(n(j); j \geq 0)$, with the properties (4)–(8). Let $a(\theta)$ be the number given by the Θ-local condition upon F at θ; and take $\eta \in [0, \theta/3[$ according to $0 < \eta < a(\theta)/3$. From the contractive condition, we have

$$F(d(x_{m(j)+1}, x_{n(j)+1}), d(x_{m(j)}, x_{n(j)}), r_{m(j)}, r_{n(j)},$$
$$d(x_{m(j)}, x_{n(j)+1}), d(x_{m(j)+1}, x_{n(j)}) \leq 0, \ \forall j \geq j(\eta).$$

On the other hand, by the relations in the quoted preliminary statement,

$$0 < r_{n(j)-1} \leq r_{m(j)} < \eta < (1/3)a(\varepsilon) < a(\varepsilon), \ \forall j \geq j(\eta). \tag{35}$$

This yields (for the same ranks j)

$$\begin{aligned} &d(x_{m(j)}, x_{n(j)}) - r_{m(j)} - r_{n(j)} \leq \\ &d(x_{m(j)+1}, x_{n(j)+1}) \leq d(x_{m(j)}, x_{n(j)}) + r_{m(j)} + r_{n(j)}, \end{aligned} \tag{36}$$

$$\theta < d(x_{m(j)}, x_{n(j)}) \leq d(x_{m(j)}, x_{n(j)-1}) + r_{n(j)-1} \leq \theta + r_{n(j)-1}, \tag{37}$$

$$d(x_{m(j)}, x_{n(j)}) - r_{n(j)} \leq d(x_{m(j)}, x_{n(j)+1}) \leq d(x_{m(j)}, x_{n(j)}) + r_{n(j)}, \tag{38}$$

$$d(x_{m(j)}, x_{n(j)}) - r_{m(j)} \leq d(x_{m(j)+1}, x_{n(j)}) \leq d(x_{m(j)}, x_{n(j)}) + r_{m(j)}. \tag{39}$$

From these facts, it easily follows, $\forall j \geq j(\eta)$

$$\begin{aligned} &d(x_{m(j)+1}, x_{n(j)+1}), d(x_{m(j)}, x_{n(j)+1}), d(x_{m(j)+1}, x_{n(j)}) \in \\ &]\theta - a(\theta), \theta + a(\theta)[, \ \ d(x_{m(j)}, x_{n(j)}) \in]\theta, \theta + a(\theta)[. \end{aligned} \tag{40}$$

Taking into account the Θ-local property for F, one has

$$F(d(x_{m(j)+1}, x_{n(j)+1}), d(x_{m(j)}, x_{n(j)}), r_{m(j)}, r_{n(j)},$$
$$d(x_{m(j)}, x_{n(j)+1}), d(x_{m(j)+1}, x_{n(j)}) > 0, \ \forall j \geq j(\eta).$$

This, however, is in contradiction with a previous relation involving these quantities; so that $(x_n; n \geq 0)$ is d-Cauchy, as claimed.

III) Since (X, d) is a-o-complete, $x_n \xrightarrow{d} z$, for some $z \in X$. If T is $(a - o, d)$-continuous, $y_n := Tx_n \xrightarrow{d} Tz$ as $n \to \infty$. In addition, as $(y_n = x_{n+1}; n \geq 0)$ is a subsequence of $(x_n; n \geq 0)$, we have $y_n \xrightarrow{d} z$ as $n \to \infty$; hence (as d=metric), $z = Tz$. Suppose now that $(\leq)$ is $(a - o, d)$-self-closed; note that, as a consequence, $x_n \leq z$, $\forall n$. Two cases may occur.

Case 1. There exists a sequence of ranks $(k(i); i \geq 0)$ with $k(i) \to \infty$ as $i \to \infty$ in such a way that $x_{k(i)} = z$ (hence $x_{k(i)+1} = Tz$), for all i. This, and $(x_{k(i)+1}; i \geq 0)$ being a subsequence of $(x_n; n \geq 0)$, gives $z \in \mathrm{Fix}(T)$.

Case 2. There exists some rank $h \geq 0$ such that

(b07) $n \geq h \Longrightarrow x_n \neq z.$

Suppose by contradiction that $z \neq Tz$; hence, $r := d(z, Tz) > 0$. From the contractive property, we must have

$$F(d(x_{n+1}, Tz), d(x_n, z), d(x_n, x_{n+1}), r, d(x_n, Tz), d(x_{n+1}, z)) \leq 0, \ \forall n \geq h.$$

On the other hand, let $a(r) \in]0, r[$ be the quantity given by the second local property of F. From the d-semi-Cauchy property and the convergence relation involving $(x_n; n \geq 0)$, there exists some rank $n(r) \geq 0$, such that

$$0 < d(x_n, x_{n+1}), d(x_n, z), d(x_{n+1}, z) < (1/3)a(r) < a(r), \ \forall n \geq n(r). \quad (41)$$

Moreover, from the triangle inequality, we have

$$r - d(x_n, z) \leq d(x_n, Tz) \leq r + d(x_n, z), \ \forall n \geq 0;$$

so that (combining with the preceding relation)

$$d(x_n, Tz), d(x_{n+1}, Tz) \in]r - a(r), r + a(r)[, \forall n \geq n(r). \quad (42)$$

These, along with the quoted local property of F, give

$$F(d(x_{n+1}, Tz), d(x_n, z), d(x_n, x_{n+1}), r, d(x_n, Tz), d(x_{n+1}, z)) > 0, \ \forall n \geq n(r);$$

in contradiction, for $n \geq n(r)$, with a preceding relation involving these data. Therefore, $z = Tz$; and this completes the argument.

The developments above allow a direct extension to the class of generalized metric spaces. Precisely, let X be a nonempty set. By a *generalized metric* on X, we mean, any map $d : X \times X \to R_+ \cup \{\infty\}$; supposed to be *symmetric* [$d(x, y) = d(y, x)$, $\forall x, y \in X$], *triangular* [$d(x, z) \leq d(x, y) + d(y, z)$, $\forall x, y, z \in X$], and *reflexive-sufficient* [$d(x, y) = 0$ iff $x = y$]. In other words, $d(., .)$ has all the properties of a metric; but its values may be infinite; in this case, the structure (X, d) will be called a *generalized metric space*. Some basic examples are to be found in Luxemburg [15] and Jung [12]; and an interesting application of these to approximate multipliers/centralizers may be found in Bodaghi et al. [7]. Another basic example is represented by the so-called *Thompson's metric* [30], constructed over convex cones in a normed space. A general fixed point theory over such structures may be found in the 1997 book by Hyers et al. [10, Chap. 5]; further extensions and some applications to projective Volterra integral equations may be found in Turinici [35].

Now, let $(\leq)$ be a quasi-order on X; the resulting structure will be called a *quasi-ordered generalized metric space*. Further, let $T \in \mathscr{F}(X)$ be increasing and

(b08) T is finitely semi-progressive: $X(T, \leq; \infty) := \{x \in X; x \leq Tx, d(x, Tx) < \infty\}$ is nonempty.

An extension of the main result above to such a framework is now possible, by the described argument; we do not give details.

3.3 Particular Case

Let $(X, d, \leq)$ be a quasi-ordered metric space. Further, let $T \in \mathscr{F}(X)$ be a selfmap of X; supposed to be semi-progressive and increasing.

(A) Concerning the main result above, a basic particular case is the one characterized as the associated function F does not depend on its last two variables. The obtained problem can be stated under the lines below. Given $E \in \mathscr{F}(R_+^4, R)$, we say that T is $(d, \leq; E)$ *contractive*, provided

(c01) $E(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty)) \leq 0$ for all $x, y \in X$ with $x \leq y, x \neq y$.

The class of such functions is introduced as follows. Let Θ be a $(>)$-cofinal part of R_+^0. We say that $E \in \mathscr{F}(R_+^4, R)$ is *P4-normal (modulo Θ)*, provided it satisfies

(3a) the global conditions

(c02) $w > 0 \Longrightarrow E(w, w, 0, 0) > 0$
(c03) $u, v > 0$, $E(u, v, v, u) \leq 0 \Longrightarrow u \leq v$;

(3b) the local conditions: $\forall r > 0, \exists a(r) \in]0, r[$ such that

(c04) $u, v \in [r, r + a(r)[, u \leq v \Longrightarrow E(u, v, v, u) > 0$
(c05) $t \in]r - a(r), r + a(r)[, u, v \in]0, a(r)[\Longrightarrow E(t, u, v, r) > 0$;

(3c) the Θ-local condition: $\forall r \in \Theta, \exists a(r) \in]0, r[$ such that

(c06) $t \in]r - a(r), r + a(r)[$, $u \in]r, r + a(r)[$, $v, w \in]0, a(r)[\Rightarrow E(t, u, v, w) > 0$.

When the $(>)$-cofinal part Θ of R_+^0 is generically taken, the resulting property will be referred to as: $E \in \mathscr{F}(R_+^4, R)$ is *almost P4-normal*. And, if $\Theta = R_+^0$, we say that $E \in \mathscr{F}(R_+^4, R)$ is *P4-normal*.

By the preceding developments, we directly get our second main result:

Theorem 12. *Suppose that T is $(d, \leq, E)$-contractive, where $E \in \mathscr{F}(R_+^4, R)$ is almost P4-normal. In addition, assume that (X, d) is a-o-complete. Then, conclusions of the first main result are retainable.*

(B) This setting is the most appropriate one to make a comparison with Reich theorem. The following intermediary statement is useful for our purpose.

Theorem 13. *Suppose that T is $(d, \leq; C)$-contractive, for some almost Q3-normal function $C \in \mathscr{F}(R_+^3, R_+)$. In addition, assume that (X, d) is a-o-complete. Then, conclusions of the first main result are retainable.*

Proof. By definition, there exists a $(>)$-cofinal part Θ of R_+^0 such that C is Q3-normal (modulo Θ). Let $E \in \mathscr{F}(R_+^4, R)$ be defined as

$$E(t, u, v, w) = t - C(u, v, w), \ (t, u, v, w) \in R_+^4;$$

it will be referred to as the *associated* to C function. We show that E is P4-normal (modulo Θ); and, from this, all is clear (by the preceding statement).

i) Let $w > 0$ be arbitrary fixed. By the first global property of C, one has $E(w, w, 0, 0) = w - C(w, 0, 0) > 0$.

ii) Let $u, v > 0$ be such that $E(u, v, v, u) \leq 0$. By definition, this yields $u \leq C(v, v, u)$; and then, the second global property of C gives $u \leq v$.

iii) Let $r > 0$ be given. By the first local property of C, there exist $e(r) > 0$, $h(r) \in]0, r[$, with: $u, v \in [r, r + e(r)[$ and $u \leq v$ imply $C(v, v, u) \leq h(r)$. This yields

$$E(u, v, v, u) = u - C(v, v, u) \geq r - h(r) > 0;$$

so, E fulfills the first local property, with $a(r) = e(r)$.

iv) Let $r > 0$ be arbitrary fixed. By the second local property of C, there exist $e(r) > 0$, $h(r) \in]0, r[$, such that: $u, v \in]0, e(t)[\implies C(u, v, r) \leq h(t)$. Denote

$$a(r) = (1/2) \min\{e(r), r - h(r)\}; \text{ (hence, } a(r) < r - h(r)).$$

For each $t \in]r - a(r), r + a(r)[$, $u, v \in]0, a(r)[$, we then get

$$E(t, u, v, r) = t - C(u, v, r) > r - a(r) - h(r) > 0;$$

whence, E fulfills the second global property.

v) Let $r \in \Theta$ be arbitrary fixed. By hypothesis, there exist $e(r) > 0$, $h(r) \in]0, r[$, such that, the relation below holds

$$u \in]r, r + e(r)[\text{ and } v, w \in]0, e(t)[\text{ imply } C(u, v, w) \leq h(t).$$

Denote (as before) $a(r) = (1/2) \min\{e(r), r - h(r)\}$. Then, for each (t, u, v, w) with $t \in]r - a(r), r + a(r)[$, $u \in]r, r + a(r)[$, $v, w \in]0, a(r)[$,

$$E(t, u, v, w) = t - C(u, v, w) > r - a(r) - h(r) > 0;$$

so that, E fulfills the Θ-local property. This ends the argument.

Note, finally, that this result includes both the Boyd–Wong theorem and the Matkowski theorem for the class of quasi-ordered metric spaces. On the other hand, by an appropriate choice of our data, the obtained theorem includes the“altering" fixed point statement in Khan et al. [14]. Further aspects may be found in the paper by Turinici [34].

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Higher Dimensional Continuous Wavelet Transform in Wiener Amalgam Spaces

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Abstract Norm convergence and convergence at Lebesgue points of the inverse wavelet transform are obtained for functions from the L_p and Wiener amalgam spaces.

Keywords Continuous wavelet transform • Wiener amalgam spaces • θ-summability • Inversion formula

1 Introduction

A general method of summation, the so-called θ-summation method, which is generated by a single function θ and which includes all well-known summability methods, is studied intensively in the literature (see, e.g., Butzer and Nessel [2], Trigub and Belinsky [22], Gát [8], Goginava [9], Simon [19] and Weisz [26, 27]). The means generated by the θ-summation are defined by

$$\sigma_T^\theta f(x) := \int_{\mathbb{R}^d} \theta\left(\frac{-t_1}{T_1}, \dots, \frac{-t_d}{T_d}\right) \hat{f}(t) e^{2\pi \imath x \cdot t}\, dt.$$

In Feichtinger and Weisz [6, 7, 25] we have proved that under some conditions of the Fourier transform of θ, the θ-means $\sigma_T^\theta f$ converge to f almost everywhere or at Lebesgue points and in norm as $T \to \infty$, whenever f is in the $L_p(\mathbb{R}^d)$ space or in a Wiener amalgam space.

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There is a close connection between summability of Fourier transforms and the inverse of the continuous wavelet transform. Using the summability results we can obtain convergence results for the inverse continuous wavelet transform. The continuous wavelet transform of f with respect to a wavelet g is defined by

$$W_g f(x,s) = \langle f, T_x D_s g \rangle \qquad (x \in \mathbb{R}^d, s \in \mathbb{R}, s \neq 0),$$

where D_s is the dilation operator and T_x the translation operator. Under some conditions on g and γ the inversion formula holds for all $f \in L_2(\mathbb{R}^d)$:

$$\int_0^\infty \int_{\mathbb{R}^d} W_g f(x,s) T_x D_s \gamma \, \frac{dx\,ds}{s^{d+1}} = C_{g,\gamma} f,$$

where the equality is understood in a vector-valued weak sense (see Daubechies [5] and Gröchenig [11]). The convergence of this integral is an important problem. In fact, there are several results on the convergence of the inverse continuous or discrete wavelet transform (see, e.g., [1,3,4,13–16,30–32]). In this paper we summarize the results about the convergence of

$$\lim_{S\to 0, T\to\infty} \int_S^T \int_{\mathbb{R}^d} W_g f(x,s) T_x D_s \gamma \, \frac{dx\,ds}{s^{d+1}}$$

including the case when $T = \infty$.

2 Wiener Amalgam Spaces

Let us fix $d \geq 1$, $d \in \mathbb{N}$. For a set $\mathbb{Y} \neq \emptyset$, let $\mathbb{Y}^d$ be its Cartesian product $\mathbb{Y} \times \cdots \times \mathbb{Y}$ taken with itself d-times. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$ set $u \cdot x := \sum_{k=1}^d u_k x_k$ and $|x| = \|x\|_2$. We briefly write $L_p(\mathbb{R}^d)$ or $L_p(\mathbb{R}^d, dx)$ instead of $L_p(\mathbb{R}^d, \lambda)$ space equipped with the norm (or quasi-norm)

$$\|f\|_p := \left(\int_{\mathbb{R}^d} |f|^p \, d\lambda \right)^{1/p} \qquad (0 < p \leq \infty)$$

with the usual modification for $p = \infty$, where λ is the Lebesgue measure. The $\ell_p(\mathbb{Z}^d)$ $(1 \leq p \leq \infty)$ spaces containing sequences are defined in the usual way. The set of sequences $(a_k, k \in \mathbb{Z}^d)$ with the property $\lim_{|k|\to\infty} a_k = 0$ is denoted by $c_0(\mathbb{Z}^d)$ and it is equipped with the $\ell_\infty(\mathbb{Z}^d)$ norm.

Set $\log^+ u = \max(0, \log u)$. A function f is in the set $L_p(\log L)^k(\mathbb{R}^d)$ $(1 \leq p < \infty)$ if

$$\|f\|_{L_p(\log L)^k} := \left(\int_{\mathbb{R}^d} |f|^p (\log^+ |f|)^k \, d\lambda \right)^{1/p} < \infty.$$

Of course, if $k = 0$, then $L_p(\log L)^k(\mathbb{R}^d) = L_p(\mathbb{R}^d)$.

The space of continuous functions with the supremum norm is denoted by $C(\mathbb{R}^d)$ and we will use $C_0(\mathbb{R}^d)$ for the space of continuous functions vanishing at infinity. $C_c(\mathbb{R}^d)$ denotes the space of continuous functions having compact support.

A measurable function f belongs to the *Wiener amalgam space* $W(L_p, \ell_q)(\mathbb{R}^d)$ $(1 \le p, q \le \infty)$ if

$$\|f\|_{W(L_p,\ell_q)} := \left(\sum_{k \in \mathbb{Z}^d} \|f(\cdot + k)\|_{L_p[0,1)^d}^q \right)^{1/q} < \infty,$$

with the obvious modification for $q = \infty$. If we replace here the space $L_p[0,1)^d$ by $L_p(\log L)^k[0,1)^d$, then we get the definition of $W(L_p(\log L)^k, \ell_q)(\mathbb{R}^d)$. $W(L_p, c_0)(\mathbb{R}^d)$ is defined analogously $(1 \le p \le \infty)$. The space $W(L_\infty, \ell_1)(\mathbb{R}^d)$ is called the *Wiener algebra*.

Let us introduce another type of Wiener amalgam spaces. Let $(i_1, \ldots, i_d)$ be a permutation of $(1, \ldots, d)$. A function f belongs to the space $W_K(L_p, \ell_\infty)(\mathbb{R}^d)$ $(1 \le p \le \infty)$ if

$$\|f\|_{W_K(L_p,\ell_\infty)} := \sup_{(i_1,\ldots,i_d)} \left(\sup_{n_{i_1} \in \mathbb{Z}} \int_{n_{i_1}}^{n_{i_1}+1} \cdots \sup_{n_{i_d} \in \mathbb{Z}} \int_{n_{i_d}}^{n_{i_d}+1} |f(x)|^p \, dx_{i_d} \cdots dx_{i_1} \right)^{1/p} < \infty.$$

If we replace $|f(x)|^p$ by $|f(x)|^p (\log^+ |f(x)|)^k$ in the previous integral, then we get the definition of $W_K(L_p(\log L)^k, \ell_\infty)(\mathbb{R}^d)$. In the one-dimensional case the W_K spaces are the same as the usual W spaces. A function $f \in W_K(L_p, \ell_\infty)(\mathbb{R}^d)$ belongs to the space $W_K(L_p, c_0)(\mathbb{R}^d)$ $(1 \le p \le \infty)$ if for all $\epsilon > 0$ there exists $K \in \mathbb{N}$ such that

$$\left\| f \, \mathbf{1}_{([-K,K]^d)^c} \right\|_{W_K(L_p,\ell_\infty)} < \epsilon.$$

The space $W_K(L_p(\log L)^k, c_0)(\mathbb{R}^d)$ is defined analogously. We say that a function $f \in W_K(L_p, \ell_\infty)(\mathbb{R}^d)$ is in the space $W_K(L_p, c_1)(\mathbb{R}^d)$ $(1 \le p \le \infty)$ if for all $\epsilon > 0$ there exists $K \in \mathbb{N}$ such that for all fixed $x_{i_2}, \ldots, x_{i_d}$

$$\left\| f(x) \, \mathbf{1}_{([-K,K])^c}(x_{i_1}) \right\|_{W(L_p,\ell_\infty)(\mathbb{R},dx_{i_1})} < \epsilon,$$

where $(i_1, \ldots, i_d)$ is an arbitrary permutation of $(1, \ldots, d)$. Obviously, $C_c(\mathbb{R}^d) \subset W_K(L_p, c_1)(\mathbb{R}^d)$.

It is easy to see that $W(L_p, \ell_p)(\mathbb{R}^d) = L_p(\mathbb{R}^d)$,

$$W_K(L_{p_1}, \ell_\infty)(\mathbb{R}^d) \supset W_K(L_{p_2}, \ell_\infty)(\mathbb{R}^d) \qquad (p_1 \le p_2),$$

$$W(L_{p_1}, \ell_q)(\mathbb{R}^d) \supset W(L_{p_2}, \ell_q)(\mathbb{R}^d) \qquad (p_1 \le p_2),$$

$$W(L_p, \ell_{q_1})(\mathbb{R}^d) \subset W(L_p, \ell_{q_2})(\mathbb{R}^d) \qquad (q_1 \le q_2)$$

$(1 \le p_1, p_2, q_1, q_2 \le \infty)$. Thus

$$W(L_\infty, \ell_1)(\mathbb{R}^d) \subset L_p(\mathbb{R}^d) \subset W(L_1, \ell_\infty)(\mathbb{R}^d) \qquad (1 \le p \le \infty).$$

For all $1 \le p \le \infty$,

$$W(L_p, \ell_\infty)(\mathbb{R}^d) \supset W_K(L_p, \ell_\infty)(\mathbb{R}^d)$$
$$W(L_p(\log L)^k, \ell_\infty)(\mathbb{R}^d) \supset W_K(L_p(\log L)^k, \ell_\infty)(\mathbb{R}^d).$$

Moreover, for $1 \le p < r \le \infty$,

$$W_K(L_p(\log L)^{d-1}, \ell_\infty)(\mathbb{R}^d) \supset C_0(\mathbb{R}^d),$$
$$W_K(L_p(\log L)^{d-1}, \ell_\infty)(\mathbb{R}^d) \supset W_K(L_r, \ell_\infty)(\mathbb{R}^d) \supset L_r(\mathbb{R}^d),$$
$$W_K(L_p, \ell_\infty)(\mathbb{R}^d) \supset W_K(L_p(\log L)^{d-1}, \ell_\infty)(\mathbb{R}^d) \supset L_p(\log L)^{d-1}(\mathbb{R}^d),$$
$$W_K(L_p, \ell_\infty)(\mathbb{R}^d) \supset L_p(\mathbb{R}^d).$$

3 Convergence of the Inverse Wavelet Transform as $S \to 0, T \to \infty, S, T \in \mathbb{R}_+$

3.1 θ-Summability of Fourier Transforms

The *Fourier transform* of $f \in L_1(\mathbb{R}^d)$ is given by

$$\hat{f}(x) := \int_{\mathbb{R}^d} f(t) e^{-2\pi \imath x \cdot t} \, dt \qquad (x \in \mathbb{R}^d),$$

where $\imath = \sqrt{-1}$. Suppose that $f \in L_p(\mathbb{R}^d)$ for some $1 \le p \le 2$. The *Dirichlet integrals* $\tau_T f$ and $\tau_{S,T} f$ are introduced by

$$\tau_T f(x) := \int_{|v| \le T} \hat{f}(v) e^{2\pi \imath x \cdot v} \, dv = \int_{\mathbb{R}^d} f(x-u) D_T(u) \, du \qquad (T > 0)$$

and

$$\tau_{S,T} f(x) := \int_{S < |v| \le T} \hat{f}(v) e^{2\pi \imath x \cdot v} \, dv \qquad (S, T > 0).$$

It is known (see, e.g., Grafakos [10]) that for $f \in L_2(\mathbb{R}^d)$

$$\lim_{S \to 0, T \to \infty} \tau_{S,T} f = f \qquad \text{in the } L_2\text{-norm}. \tag{1}$$

In the one-dimensional case this convergence holds almost everywhere and in the L_p-norm for all $f \in L_p(\mathbb{R})$ with $1 < p < \infty$.

We have shown in [23] that the *Dirichlet kernel* D_t satisfies $|D_t| \le Ct^d$ and

$$D_t(u) := \int_{\{|v|\le t\}} e^{2\pi\imath u\cdot v}\, dv = |u|^{-d/2} t^{d/2} J_{d/2}(2\pi|u|t),$$

where

$$J_\nu(t) = \frac{(t/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu+1/2)} \int_{-1}^{1} e^{\imath ts}(1-s^2)^{\nu-1/2}\, ds \qquad (\nu > -1/2, t > 0)$$

are the *Bessel functions* (see Stein and Weiss [20] or Weisz [27]).

Using a summability method, we obtain more general convergence results. To define a general summability method, the so-called θ-summability, let θ_0 be even and continuous on $\mathbb{R}$, differentiable on $(0,\infty)$ and $\theta(u) := \theta_0(|u|)$. We suppose that $\theta \in L_1(\mathbb{R}^d)$ and

$$\int_0^\infty (r \vee 1)^d |\theta_0'(r)|\, dr < \infty.$$

For $T > 0$ the *θ-means* of a function $f \in L_p(\mathbb{R}^d)$ $(1 \le p \le 2)$ are defined by

$$\sigma_T^\theta f(x) := \int_{\mathbb{R}^d} \theta_0\left(\frac{|v|}{T}\right) \hat{f}(v) e^{2\pi\imath x\cdot v}\, dv.$$

It is easy to see that

$$\sigma_T^\theta f(x) = \int_{\mathbb{R}^d} f(x-u) K_T^\theta(u)\, du,$$

where

$$K_T^\theta(u) = \int_{\mathbb{R}^d} \theta_0\left(\frac{|v|}{T}\right) e^{2\pi\imath u\cdot v}\, dv = T^d \hat{\theta}(Tu).$$

On the other hand,

$$K_T^\theta(u) = \frac{-1}{T} \int_{\mathbb{R}^d} \int_{|v|}^\infty \theta_0'\left(\frac{t}{T}\right) dt\, e^{2\pi\imath u\cdot v}\, dv = \frac{-1}{T} \int_0^\infty \theta_0'\left(\frac{t}{T}\right) D_t(u)\, dt.$$

If $\hat{\theta} \in L_1(\mathbb{R}^d)$, then we can extend the θ-means by

$$\sigma_T^\theta f := f * K_T^\theta \qquad (T > 0)$$

for all $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$, where $*$ denotes the convolution. This summation contains all well-known summability methods, such as the Fejér, Riesz, Weierstrass, Abel, Picard, Bessel, etc. summability methods.

3.2 Norm Convergence of the θ-Summability

The following two results about the norm convergence of $\sigma_T^\theta f$ were proved by Feichtinger and Weisz [6, 23].

Theorem 1 ([6]). *If* $1 \le p, q < \infty$, $\theta \in L_1(\mathbb{R}^d)$, $\hat{\theta} \in L_1(\mathbb{R}^d)$ *and* $f \in W(L_p, \ell_q)(\mathbb{R}^d)$, *then*

$$\lim_{T\to\infty} \sigma_T^\theta f = \theta(0) f \qquad \text{in the } W(L_p, \ell_q)\text{-norm.}$$

The same holds for $C_0(\mathbb{R}^d)$, $W(C, \ell_q)(\mathbb{R}^d)$ *and for* $W(L_p, c_0)(\mathbb{R}^d)$.

Theorem 2 ([23]). *If* $1 \le p < \infty$, $1 < q < \infty$, $\theta \in L_1(\mathbb{R}^d)$ *and*

$$\left|\hat{\theta}(x)\right| \le C |x|^{-d-\epsilon} \qquad (x \neq 0)$$

for some $\epsilon > 0$, *then for all* $f \in W(L_p, \ell_q)(\mathbb{R}^d)$,

$$\lim_{T\to 0} \sigma_T^\theta f = 0 \qquad \text{in the } W(L_p, \ell_q)\text{-norm.}$$

The same holds for $C_0(\mathbb{R}^d)$, $W(C, \ell_q)(\mathbb{R}^d)$ *and for* $W(L_p, c_0)(\mathbb{R}^d)$.

Of course, the conditions of Theorem 2 imply $\hat{\theta} \in L_1(\mathbb{R}^d)$. The theorem is not true for the $L_1(\mathbb{R}^d)$ space, because if $f, \hat{\theta} \ge 0$, then

$$\int_{\mathbb{R}^d} \left|\sigma_T^\theta f(x)\right| dx = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(t) T^d \hat{\theta}\left(T(x-t)\right) dt \right| dx = \|f\|_1 \left\|\hat{\theta}\right\|_1$$

and this does not tend to 0.

3.3 Almost Everywhere Convergence of the θ-Summability

First we introduce the concept of Lebesgue points. A point $x \in \mathbb{R}^d$ is called a *Lebesgue point* of f if

$$\lim_{h\to 0} \frac{1}{(2h)^d} \int_{-h}^{h} \cdots \int_{-h}^{h} |f(x+u) - f(x)|\, du = 0.$$

It is known that almost every point $x \in \mathbb{R}^d$ is a Lebesgue point of all functions f from the space $W(L_1, \ell_\infty)(\mathbb{R}^d)$ (see Feichtinger and Weisz [7]).

Theorem 3 ([7]). *Suppose that* $\theta \in L_1(\mathbb{R})$ *and* $\left|\hat{\theta}(x)\right| \leq C|x|^{-d-\epsilon}$ $(x \neq 0, \epsilon > 0)$. *Then*

$$\lim_{T\to\infty} \sigma_T^\theta f(x) = \theta(0) f(x)$$

for all Lebesgue points of $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$.

The limit of $\sigma_T^\theta f(x)$ as $T \to 0$ is equal to 0 at each point.

Theorem 4 ([23]). *Suppose that* $\theta \in L_1(\mathbb{R})$, *and* $\left|\hat{\theta}(x)\right| \leq C|x|^{-d-\epsilon}$ $(x \neq 0, \epsilon > 0)$. *Then*

$$\lim_{T\to 0} \sigma_T^\theta f(x) = 0$$

for all $x \in \mathbb{R}^d$ *and* $f \in W(L_1, c_0)(\mathbb{R}^d)$.

One can show easily that for all $1 \leq p, q < \infty$

$$W(L_1, c_0)(\mathbb{R}) \supset W(L_p, c_0)(\mathbb{R}), W(L_p, \ell_q)(\mathbb{R}), L_p(\mathbb{R}).$$

3.4 Multi-Dimensional Continuous Wavelet Transform

Translation and *dilation* of a function f are defined, respectively, by

$$T_x f(t) := f(t - x) \qquad \text{and} \qquad D_s f(t) := |s|^{-d/2} f(s^{-1}t),$$

where $t, x, \omega \in \mathbb{R}^d, s \in \mathbb{R}, s \neq 0$.

The continuous wavelet transform of f with respect to a wavelet g is defined by

$$W_g f(x, s) := |s|^{-d/2} \int_{\mathbb{R}^d} f(t)\overline{g(s^{-1}(t - x))}\, dt = \langle f, T_x D_s g\rangle,$$

$(x \in \mathbb{R}^d, s \in \mathbb{R}, s \neq 0)$ when the integral does exist. A function f is radial if there exists a one-variable function η such that $f(x) = \eta(|x|)$. We suppose that $g, \gamma \in L_2(\mathbb{R}^d)$ are radial functions. Then $\hat{g}$ and $\hat{\gamma}$ are also radial functions. Set $\mu(|x|) := \hat{g}(x)$ and $\nu(|x|) := \hat{\gamma}(x)$. Suppose that

$$\int_0^\infty \left|\overline{\hat{g}}(s\omega)\hat{\gamma}(s\omega)\right| \frac{ds}{s} = \int_0^\infty \left|\overline{\mu(s)}\nu(s)\right| \frac{ds}{s} < \infty \tag{2}$$

for some fixed $|\omega| = 1$ and let

$$C_{g,\gamma} := \int_0^\infty \overline{\mu(s)}\nu(s)\, \frac{ds}{s}.$$

We define also

$$C_g := \int_0^\infty |\mu(s)|^2 \, \frac{ds}{s} \qquad \text{and} \qquad C_\gamma := \int_0^\infty |\nu(s)|^2 \, \frac{ds}{s}.$$

By Hölder's inequality if C_g and C_γ are both finite, then so is $C_{g,\gamma}$. Plancherel's theorem is well known for Fourier transforms: if $f, g \in L_2(\mathbb{R})$, then

$$\langle f, g \rangle = \left\langle \hat{f}, \hat{g} \right\rangle \qquad \text{and} \qquad \|f\|_2 = \left\| \hat{f} \right\|_2.$$

The analogues of these results remain true for continuous wavelet transforms, too (see, e.g., Daubechies [5] and Gröchenig [11]).

Theorem 5. *Suppose that $g \in L_2(\mathbb{R}^d)$ is a radial function such that $C_g < \infty$. If $f \in L_2(\mathbb{R}^d)$, then*

$$\int_0^\infty \int_{\mathbb{R}^d} \left| W_g f(x,s) \right|^2 \frac{dx\, ds}{s^{d+1}} = C_g \|f\|_2^2.$$

Theorem 6. *Suppose that $g, \gamma \in L_2(\mathbb{R}^d)$ are radial functions such that $C_g < \infty$ and $C_\gamma < \infty$. If $f_1, f_2 \in L_2(\mathbb{R}^d)$, then*

$$\int_0^\infty \int_{\mathbb{R}^d} W_g f_1(x,s) \overline{W_\gamma f_2(x,s)} \, \frac{dx\, ds}{s^{d+1}} = C_{g,\gamma} \langle f_1, f_2 \rangle .$$

Here in the last theorem we have to suppose that C_g and C_γ are both finite, it is not enough that $C_{g,\gamma}$ is finite (see [29]). However, if $g, \gamma \in L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$, then it is enough to suppose the weaker condition $C_{g,\gamma} \in \mathbb{C}$.

Under the above conditions, i.e., $g, \gamma \in L_2(\mathbb{R}^d)$ are radial functions such that C_g and C_γ are finite, the inversion formula holds for all $f \in L_2(\mathbb{R}^d)$:

$$\int_0^\infty \int_{\mathbb{R}^d} W_g f(x,s) T_x D_s \gamma \, \frac{dx ds}{s^{d+1}} = C_{g,\gamma} f,$$

where the equality is understood in a vector-valued weak sense (see, e.g., Christensen [3], Chui [4], Daubechies [5] or Gröchenig [11]). The convergence of the integral on the left-hand side is very important, so we will investigate the convergence of

$$\rho_{S,T} f := \int_S^T \int_{\mathbb{R}^d} W_g f(x,s) T_x D_s \gamma \, \frac{dx ds}{s^{d+1}}$$

and

$$\rho_S f := \int_S^\infty \int_{\mathbb{R}^d} W_g f(x,s) T_x D_s \gamma \, \frac{dx ds}{s^{d+1}},$$

where $0 < S < T < \infty$ and $S \to 0$ and $T \to \infty$. In the next sections we investigate the almost everywhere and norm convergence of $\rho_{S,T} f$ and $\rho_S f$.

3.5 Convergence of $\rho_{S,T} f$

The one-dimensional version of the next theorem is due to Rao et al. [16] and the higher dimensional version to the author [28].

Theorem 7 ([16, 28]). *Assume that $g, \gamma \in L_2(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$ are radial functions such that (2) holds. If $f \in L_2(\mathbb{R}^d)$, then*

$$\lim_{S\to 0, T\to\infty} \rho_{S,T} f = C_{g,\gamma} f \qquad \text{in the } L_2\text{-norm.}$$

In the one-dimensional case this convergence holds almost everywhere and in the L_p-norm for all $f \in L_p(\mathbb{R})$ with $1 < p < \infty$.

In the proof of this theorem we have shown that

$$\rho_{S,T} f(t) = \int_0^\infty \overline{\mu(s)} \nu(s) \tau_{s/T, s/S} f(t) \, \frac{ds}{s} \qquad (f \in L_2(\mathbb{R}^d)).$$

Then the theorem follows from (1) (see [28]). Under much more stronger conditions, if $g, \gamma \in C^\infty(\mathbb{R}^d)$, another version of the norm convergence was proved by Wilson [30, 31] for all $f \in L_p(\mathbb{R}^d)$ $(1 < p < \infty)$.

Supposing more conditions about g and γ, Li and Sun [15] extended this result as follows. We call h a radial log-majorant function if h is radial, positive, decreasing as a function on $(0, \infty)$ and $h(\cdot) \ln(2 + |\cdot|) \in L_1(\mathbb{R}^d)$. It is easy to see that in this case $h \in L_2(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$ and even $h \in W(L_\infty, \ell_1)(\mathbb{R}^d)$. Let

$$C'_{g,\gamma} := -\int_{\mathbb{R}^d} (\overline{g} * \gamma)(x) \ln|x| \, dx.$$

Theorem 8 ([15]). *Suppose that the radial functions g and γ have radial log-majorants, $\int_{\mathbb{R}^d} (\overline{g} * \gamma)(x) \, dx = 0$ and $C'_{g,\gamma}$ is finite.*

1. If $1 < p < \infty$ and $f \in L_p(\mathbb{R}^d)$, then

$$\lim_{S\to 0, T\to\infty} \rho_{S,T} f = C'_{g,\gamma} f \qquad \text{in the } L_p\text{-norm.}$$

2. *If* $1 \le p < \infty$ *and* $f \in L_p(\mathbb{R}^d)$*, then*

$$\lim_{S\to 0, T\to\infty} \rho_{S,T} f(x) = C'_{g,\gamma} f(x)$$

for all Lebesgue-points of f*.*

Of course, under some conditions, $C_{g,\gamma}$ coincides with $C'_{g,\gamma}$ (see Rubin and Shamir [17] and Saeki [18]). In special case (amongst others if f is bounded and continuous), 2 was proved by Holschneider and Tchamitchain [12]. In the next theorem we do not suppose that $g, \gamma \in W(L_\infty, \ell_1)(\mathbb{R}^d)$, instead we suppose more smoothness and that $g, \gamma \in L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$, and we show similar results for functions from the Wiener amalgam spaces. Let $d_0 = \lceil d/2 \rceil$, the upper integer part of $d/2$.

Theorem 9 ([23]). *Assume that* $g, \gamma \in L_1(\mathbb{R}^d)$ *are radial functions,* μ *and* ν *are* d_0*-times differentiable,* $\mu^{(j)}$ *and* $\nu^{(j)}$ $(j = 0, \dots, d_0)$ *are bounded,*

$$\left|\mu^{(j)}(r)\right|, \left|\nu^{(j)}(r)\right| \le C r^\alpha \qquad (0 < r \le 1, j = 0, \dots, d_0/2) \tag{3}$$

for some $\alpha > 0$*,*

$$\int_0^\infty \left|\mu^{(j)}(r)\right|^2 r^{d-1}\, dr < \infty, \quad \int_0^\infty \left|\nu^{(j)}(r)\right|^2 r^{d-1}\, dr < \infty \quad (j = 0, \dots, d_0/2)$$

$$\lim_{u\to\infty} \mu^{(j)}(r) r^{d/2-3/2} = 0, \quad \lim_{u\to\infty} \nu^{(j)}(r) r^{d/2-3/2} = 0 \quad (j = 0, \dots, (d_0-1)/2).$$

1. *If* $1 \le p < \infty$*,* $1 < q < \infty$ *and* $f \in W(L_p, \ell_q)(\mathbb{R}^d)$*, then*

$$\lim_{S\to 0, T\to\infty} \rho_{S,T} f = C_{g,\gamma} f \qquad \text{in the } W(L_p, \ell_q)\text{-norm.}$$

The same holds for $C_0(\mathbb{R}^d)$*,* $W(C, \ell_q)(\mathbb{R}^d)$ *and for* $W(L_p, c_0)(\mathbb{R}^d)$*.*

2. *If* $f \in W(L_1, c_0)(\mathbb{R}^d)$*, then*

$$\lim_{S\to 0, T\to\infty} \rho_{S,T} f(x) = C_{g,\gamma} f(x)$$

for all Lebesgue-points of f*.*

The first parts of Theorems 8 and 9 do not hold for the $L_1(\mathbb{R}^d)$ space (i.e., $p = q = 1$), even in the one-dimensional case. The key point of the proof is that we can lead back the problem to the summability of Fourier transforms:

$$\rho_{S,T} f = \sigma^\theta_{1/S} f - \sigma^\theta_{1/T} f$$

for all $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$, where

$$\theta(\xi) := \theta_0(|\xi|) := \int_{|\xi|\le s} \overline{\mu(s)}\nu(s)\, \frac{ds}{s},$$

Moreover, we have verified in [23] that $\theta, \hat{\theta} \in L_1(\mathbb{R}^d)$ and

$$\left|\hat{\theta}(x)\right| \le C\,|x|^{-d-\alpha/4} \qquad (x \ne 0)\,.$$

The theorem follows from the results of Sects. 3.2 and 3.3.

Note that $\theta(0) = C_{g,\gamma}$ is finite, because

$$\left|C_{g,\gamma}\right| \le \int_0^\infty |\mu(r)|\,|\nu(r)|\,\frac{dr}{r} = \int_{0<r\le 1} |\mu(r)|\,|\nu(r)|\,\frac{dr}{r} + \int_{r>1} |\mu(r)|\,|\nu(r)|\,r^{d-1}\,dr < \infty.$$

Obviously, the conditions of Theorem 9 imply that $g, \gamma \in L_2(\mathbb{R}^d)$, thus the assumptions of Theorem 7 are satisfied in this case, too.

If $g \in L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$, $\hat{g}(0) = 0$ and

$$\int_{\mathbb{R}^d} |x|^\alpha\,|g(x)|\;dx < \infty$$

for some $0 < \alpha \le 1$, then

$$\begin{aligned}
|\mu(|\omega|)| = |\hat{g}(\omega)| &= \left|\int_{\mathbb{R}^d} g(x)\left(e^{-2\pi\imath x\cdot\omega} - 1\right)\,dx\right| \\
&\le 2\pi \int_{\{x:|x\cdot\omega|\le 1\}} |g(x)|\,|x\cdot\omega|\;dx + 2\int_{\{x:|x\cdot\omega|\ge 1\}} |g(x)|\;dx \\
&\le 2\pi \int_{\{x:|x\cdot\omega|\le 1\}} |g(x)|\,|x\cdot\omega|^\alpha\;dx + 2\int_{\{x:|x\cdot\omega|\ge 1\}} |g(x)|\,|x\cdot\omega|^\alpha\;dx \\
&\le 2\pi\,|\omega|^\alpha \int_{\mathbb{R}^d} |g(x)|\,|x|^\alpha\;dx.
\end{aligned}$$

Thus (3) is true for $j = 0$. Some inequalities with respect to $\rho_{S,T} f$ can be found in [23].

3.6 Convergence of $\rho_S f$

The convergence of $\rho_S f$ is different from that of $\rho_{S,T} f$. In this section we will prove more general results for $\rho_S f$. In [28] we have also shown that

$$\rho_S f(t) = \int_0^\infty \overline{\mu(s)} \nu(s) \tau_{s/S} f(t) \frac{ds}{s} \qquad (f \in L_2(\mathbb{R}^d)).$$

This implies

Theorem 10 ([28]). *Assume that $g, \gamma \in L_2(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$ and $\hat{g}$ and $\hat{\gamma}$ are radial functions such that (2) holds. If $f \in L_2(\mathbb{R}^d)$, then*

$$\lim_{S \to 0} \rho_S f = C_{g,\gamma} f \qquad \textit{in the } L_2\textit{-norm.}$$

In the one-dimensional case this convergence holds almost everywhere and in the L_p-norm for all $f \in L_p(\mathbb{R})$ with $1 < p < \infty$.

Under the same conditions as in Theorem 9,

$$\rho_S f = \sigma_{1/S}^\theta f \tag{4}$$

for all $f \in L_1(\mathbb{R}^d)$. Opposite to Theorems 8 and 9, now we obtain convergence of $\rho_S f$ in the L_1-norm.

Theorem 11 ([15]). *Suppose that the radial functions g and γ have radial log-majorants, $\int_{\mathbb{R}^d} (\overline{g} * \gamma)(x)\, dx = 0$ and $C'_{g,\gamma}$ is finite.*

1. If $1 \le p < \infty$ and $f \in L_p(\mathbb{R}^d)$, then

$$\lim_{S \to 0} \rho_S f = C'_{g,\gamma} f \qquad \textit{in the } L_p\textit{-norm.}$$

2. If $1 \le p < \infty$ and $f \in L_p(\mathbb{R}^d)$, then

$$\lim_{S \to 0} \rho_S f(x) = C'_{g,\gamma} f(x)$$

for all Lebesgue-points of f.

This result was also shown in Saeki [18] for $1 < p < \infty$.

Theorem 12 ([23]). *Assume the same conditions as in Theorem 9.*

1. If $f \in L_1(\mathbb{R}^d)$, then

$$\lim_{S \to 0} \rho_S f = C_{g,\gamma} f \qquad \textit{in the } L_1\textit{-norm.}$$

2. If $f \in L_1(\mathbb{R}^d)$, then

$$\lim_{S\to 0} \rho_S f(x) = C_{g,\gamma} f(x)$$

for all Lebesgue-points of f.

We can generalize these theorems as follows. If we suppose a little bit more about γ, then we obtain sharper results. Equation (4) holds for all $f \in L_p(\mathbb{R}^d)$, as $\gamma \in L_1(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)$, and for all $f \in W(L_p, \ell_q)(\mathbb{R}^d)$, as $\gamma \in W(L_\infty, \ell_1)(\mathbb{R}^d)$ $(1 \le p, q < \infty)$. The next theorem follows from this.

Theorem 13 ([23]). *Besides the conditions of Theorem 9 assume that $\gamma \in L_\infty(\mathbb{R}^d)$ and $1 \le p, q < \infty$.*

1. If $f \in L_p(\mathbb{R}^d)$, then

$$\lim_{S\to 0} \rho_S f = C_{g,\gamma} f \qquad \text{in the } L_p\text{-norm.}$$

If in addition $\gamma \in W(L_\infty, \ell_1)(\mathbb{R}^d)$, then the convergence holds in the $W(L_p, \ell_q)$-norm for all $f \in W(L_p, \ell_q)(\mathbb{R}^d)$.

2. If $f \in L_p(\mathbb{R}^d)$, then

$$\lim_{S\to 0} \rho_S f(x) = C_{g,\gamma} f(x)$$

for all Lebesgue-points of f. If in addition $\gamma \in W(L_\infty, \ell_1)(\mathbb{R}^d)$, then the convergence holds at each Lebesgue-point of $f \in W(L_1, \ell_q)(\mathbb{R}^d)$.

4 Convergence of the Inverse Wavelet Transform in Pringsheim's Sense, as $S \to 0, T \to \infty, S, T \in \mathbb{R}_+^d$

4.1 θ-Summability of Fourier Transforms in Pringsheim's Sense

In this section we investigate the convergence in Pringsheim's sense, i.e., if $S, T \in \mathbb{R}_+^d$ and $S \to 0, T \to \infty$, in other words, $S_j \to 0, T_j \to \infty$ for all $j = 1, \ldots, d$. Now we define the Dirichlet integral by

$$\tau_T f(x) := \int_{-T_1}^{T_1} \cdots \int_{-T_d}^{T_d} \hat{f}(u) e^{2\pi \imath x \cdot u} \, du$$

$(T = (T_1, \dots, T_d) \in \mathbb{R}_+^d)$. Then

$$\lim_{T\to\infty} \tau_T f = f \qquad \text{in the } L_p(\mathbb{R}^d)\text{-norm}$$

for all $f \in L_p(\mathbb{R}^d)$, $1 < p < \infty$ (see, e.g., Grafakos [10] or [27]), but there is no almost everywhere convergence for $\tau_T f$ $(T \in \mathbb{R}_+^d)$. Using a summability method, we can generalize these results again. Let $\theta = \theta_1 \times \cdots \times \theta_d$ and $\theta_j \in L_1(\mathbb{R}) \cap C_0(\mathbb{R})$ for all $j = 1, \dots, d$. The θ-*means* of $f \in L_p(\mathbb{R}^d)$ $(1 \le p \le 2)$ are defined by

$$\sigma_T^\theta f(x) := \int_{\mathbb{R}^d} \left(\prod_{j=1}^d \theta_j \left(\frac{-t_j}{T_j} \right) \right) \hat{f}(t) e^{2\pi \imath x \cdot t} \, dt.$$

Then

$$\sigma_T^\theta f(x) = \int_{\mathbb{R}^d} f(x-t) K_T^\theta(t) \, dt = f * K_T^\theta(x) \qquad (x \in \mathbb{R}^d, T \in \mathbb{R}_+^d)$$

and

$$K_T^\theta(x) = \int_{\mathbb{R}^d} \left(\prod_{j=1}^d \theta_j \left(\frac{-t_j}{T_j} \right) \right) e^{2\pi \imath x \cdot t} \, dt = \left(\prod_{j=1}^d T_j \hat{\theta}_j \left(T_j x_j \right) \right)$$

$(x \in \mathbb{R}^d)$. If $\hat{\theta} \in L_1(\mathbb{R}^d)$, then we can extend this definition in the following way:

$$\sigma_T^\theta f := f * K_T^\theta \qquad (T \in \mathbb{R}_+^d)$$

for all $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$. Note that $\theta \in L_1(\mathbb{R}^d)$ and $\hat{\theta} \in L_1(\mathbb{R}^d)$ imply $\theta \in C_0(\mathbb{R}^d)$.

4.2 Norm Convergence of the θ-Summability in Pringsheim's Sense

The analogues of Theorems 1 and 2 can be formulated as follows.

Theorem 14 ([6]). *If $1 \le p, q < \infty$, $\theta_j \in L_1(\mathbb{R})$ and $\hat{\theta}_j \in L_1(\mathbb{R})$ for all $j = 1, \dots, d$, then*

$$\lim_{T\to\infty} \sigma_T^\theta f = \theta(0) f \qquad \textit{in the } W(L_p, \ell_q)\textit{-norm}$$

for all $f \in W(L_p, \ell_q)(\mathbb{R}^d)$. The same holds for $C_0(\mathbb{R}^d)$, $W(C, \ell_q)(\mathbb{R}^d)$ and for $W(L_p, c_0)(\mathbb{R}^d)$.

If at least one coordinate of T, say T_i, tends to zero, then $\sigma_T^\theta f \to 0$ in the $L_p(\mathbb{R}^d)$-norm.

Theorem 15 ([24]). *If* $1 \le p < \infty$, $1 < q < \infty$, $\theta_j \in L_1(\mathbb{R})$ *and*

$$\left|\widehat{\theta}_j(x)\right| \le C|x|^{-2} \qquad (x \ne 0)$$

for all $j = 1, \ldots, d$, *then*

$$\lim_{\substack{T_i \to 0 \\ T_j \to 0,\infty, j=1,\ldots,d, j\ne i}} \sigma_T^\theta f = 0 \qquad \text{in the } W(L_p, \ell_q)\text{-norm}$$

for all $f \in W(L_p, \ell_q)(\mathbb{R}^d)$. *The same holds for* $C_0(\mathbb{R}^d)$, $W(C, \ell_q)(\mathbb{R}^d)$ *and for* $W(L_p, c_0)(\mathbb{R}^d)$.

Similarly to Theorem 2, this theorem does not hold for the $L_1(\mathbb{R})$ space, either.

4.3 Almost Everywhere Convergence of the θ-Summability in Pringsheim's Sense

The *strong maximal function* is defined by

$$M_s f(x) := \sup_{x\in I} \frac{1}{|I|} \int_I |f| \, d\lambda \qquad \left(x \in \mathbb{R}^d\right),$$

where $f \in L_1^{\text{loc}}(\mathbb{R}^d)$ and the supremum is taken over all rectangles $I \subset \mathbb{R}^d$ with sides parallel to the axes. A point $x \in \mathbb{R}^d$ is called a *strong Lebesgue point* of f if $M_s f(x)$ is *finite* and

$$\lim_{h\to 0} \frac{1}{\prod_{j=1}^d (2h_j)} \int_{-h_1}^{h_1} \ldots \int_{-h_d}^{h_d} |f(x+u) - f(x)| \, du = 0.$$

It is known that almost every point $x \in \mathbb{R}^d$ is a Lebesgue point of all functions f from the space $W_K(L_1(\log L)^{d-1}, \ell_\infty)(\mathbb{R}^d) \supset L_1(\log L)^{d-1}(\mathbb{R}^d)$, $L_p(\mathbb{R}^d)$, $1 < p \le \infty$ (see Feichtinger and Weisz [7, 25]).

Theorem 16 ([25]). *Suppose that* $\theta_j \in L_1(\mathbb{R})$ *and* $\left|\widehat{\theta}_j(x)\right| \le C/|x|^2$ $(x \ne 0)$ *for all* $j = 1, \ldots, d$. *Then*

$$\lim_{T\to\infty} \sigma_T^\theta f(x) = \theta(0) f(x)$$

for all strong Lebesgue points of $f \in W_K(L_1(\log L)^{d-1}, \ell_\infty)(\mathbb{R}^d)$.

For the almost everywhere convergence of $\sigma_T^\theta f$ as $T \to 0$, we obtained the following results in [24].

Theorem 17 ([24]). *Suppose that* $\theta_j \in L_1(\mathbb{R})$ *and* $\left|\widehat{\theta_j}(x)\right| \leq C/|x|^2$ $(x \neq 0)$ *for all* $j = 1, \ldots, d$. *Then*

$$\lim_{\substack{T_i \to 0 \\ T_j \to 0,\infty, j=1,\ldots,d, j\neq i}} \sigma_T^\theta f = 0 \qquad a.e.$$

for all $f \in W_K(L_1(\log L)^{d-1}, c_0)(\mathbb{R}^d)$.

One can show easily that

$$\begin{aligned} &W_K(L_1(\log L)^{d-1}, c_0)(\mathbb{R}^d) \supset W(L_p, c_0)(\mathbb{R}^d), W(L_p, \ell_q)(\mathbb{R}^d), \\ &\quad L_1(\log L)^{d-1}(\mathbb{R}^d), L_p(\mathbb{R}^d) \end{aligned} \tag{5}$$

for $1 < p < \infty$, $1 \leq q < \infty$. We can show everywhere convergence of the θ-means as well.

Theorem 18. *Suppose that* $\theta_j \in L_1(\mathbb{R})$, $\left|\widehat{\theta_j}(x)\right| \leq C/|x|^2$ $(x \neq 0)$ *for all* $j = 1, \ldots, d$ *and* $M_s f(x)$ *is finite. Then*

$$\lim_{\substack{T_i \to 0 \\ T_j \to 0,\infty, j=1,\ldots,d, j\neq i}} \sigma_T^\theta f(x) = 0$$

for all $x \in \mathbb{R}^d$ *and* $f \in W_K(L_1, c_1)(\mathbb{R}^d)$.

4.4 Continuous Wavelet Transform in Pringsheim's Sense

In the one-dimensional case we suppose either that g and γ are even functions or

$$\int_{-\infty}^{\infty} \left|\overline{\hat{g}}(s)\hat{\gamma}(s)\right| \frac{ds}{|s|} < \infty. \tag{6}$$

Here we will consider rather the second version. The continuous wavelet transform of f with respect to a wavelet g is defined by

$$W_g f(x, s) := |s|^{-1/2} \int_{\mathbb{R}} f(t)\overline{g(s^{-1}(t-x))}\, dt = \langle f, T_x D_s g \rangle,$$

$(x \in \mathbb{R}, s \in \mathbb{R}, s \neq 0)$ when the integral does exist.

In this section instead of the functions g and γ, we consider the Kronecker product of the one-dimensional functions g_j and γ_j, $j = 1, \dots, d$. The continuous wavelet transform of f with respect to the wavelets g_j is defined now by

$$W_g f(x,s) := \left(\prod_{j=1}^{d} |s_j|^{-1/2} \right) \int_{\mathbb{R}^d} f(t) \left(\prod_{j=1}^{d} \overline{g_j(s_j^{-1}(t_j - x_j))} \right)$$

$$dt = \left\langle f, \prod_{j=1}^{d} T_{x_j} D_{s_j} g_j \right\rangle,$$

where $x \in \mathbb{R}^d, s = (s_1, \dots, s_d) \in \mathbb{R}^d, s_j \neq 0$, $D_{s_j} f(t) := |s_j|^{-1/2} f(s_j^{-1} t)$. Furthermore, let us define

$$C_{g,\gamma} := \prod_{j=1}^{d} \int_{-\infty}^{\infty} \overline{\hat{g}_j}(s_j) \hat{\gamma}_j(s_j) \frac{ds_j}{|s_j|} = \prod_{j=1}^{d} C_{g_j,\gamma_j},$$

$$C_g := \prod_{j=1}^{d} \int_{-\infty}^{\infty} \left| \hat{g}_j(s_j) \right|^2 \frac{ds_j}{|s_j|} = \prod_{j=1}^{d} C_{g_j}$$

and

$$C_\gamma := \prod_{j=1}^{d} \int_{-\infty}^{\infty} \left| \hat{\gamma}_j(s_j) \right|^2 \frac{ds_j}{|s_j|} = \prod_{j=1}^{d} C_{\gamma_j}.$$

The analogues of the Plancherel's theorem read as follows.

Theorem 19. *Suppose that $g_j \in L_2(\mathbb{R}^d)$ and $C_{g_j} < \infty$ for all $j = 1, \dots, d$. If $f \in L_2(\mathbb{R}^d)$, then*

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| W_g f(x,s) \right|^2 \frac{dx\, ds}{\prod_{j=1}^{d} s_j^2} = C_g \|f\|_2^2.$$

Theorem 20. *Suppose that $g_j, \gamma_j \in L_2(\mathbb{R}^d)$, $C_{g_j} < \infty$ and $C_{\gamma_j} < \infty$ for all $j = 1, \dots, d$. If $f_1, f_2 \in L_2(\mathbb{R}^d)$, then*

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W_g f_1(x,s) \overline{W_\gamma f_2(x,s)} \frac{dx\, ds}{\prod_{j=1}^{d} s_j^2} = C_{g,\gamma} \langle f_1, f_2 \rangle.$$

In case $g_j, \gamma_j \in L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$ it is enough to suppose that C_{g_j,γ_j} is finite for all $j = 1, \dots, d$ (see [29]). Under the same conditions as in Theorem 20, the inversion formula holds again in vector-valued weak sense, for all $f \in L_2(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W_g f(x,s)\left(\prod_{j=1}^{d}\frac{T_{x_j}D_{s_j}\gamma_j}{s_j^2}\right)dxds = C_{g,\gamma}f.$$

For the convergence of the inverse continuous wavelet transform we will consider

$$\rho_{S,T}f := \int_{S_1\le|s_1|\le T_1}\cdots\int_{S_d\le|s_d|\le T_d}\int_{\mathbb{R}^d} W_g f(x,s)\left(\prod_{j=1}^{d}\frac{T_{x_j}D_{s_j}\gamma_j}{s_j^2}\right)dxds$$

and

$$\rho_S f := \int_{S_1\le|s_1|}\cdots\int_{S_d\le|s_d|}\int_{\mathbb{R}^d} W_g f(x,s)\left(\prod_{j=1}^{d}\frac{T_{x_j}D_{s_j}\gamma_j}{s_j^2}\right)dxds,$$

where $S=(S_1,\dots,S_d), T=(T_1,\dots,T_d)\in\mathbb{R}_+^d$, $0<S_j<T_j<\infty$ and $S\to 0$ and $T\to\infty$ in the Pringsheim's sense, i.e. each $S_j\to 0$ and $T_j\to\infty$.

4.5 Convergence of $\rho_{S,T}f$ in Pringsheim's Sense

The next result can be proved similarly to Theorem 7.

Theorem 21 ([28]). *Assume that $g_j,\gamma_j \in L_2(\mathbb{R})\cap L_1(\mathbb{R})$ satisfy (6) for all $j=1,\dots,d$. If $f\in L_p(\mathbb{R}^d)$ with $1<p<\infty$, then*

$$\lim_{S\to 0, T\to\infty}\rho_{S,T}f = C_{g,\gamma}f \qquad \text{in the } L_p\text{-norm.}$$

The conditions of the theorem imply

$$\rho_{S,T}f(t) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\prod_{j=1}^{d}\frac{\overline{\hat{g}_j}(s_j)\widehat{\gamma_j}(s_j)}{s_j}\right)\rho_{s_1/T_1,\dots,s_d/T_d;s_1/S_1,\dots,s_d/S_d}f(t)\,ds$$

for all $f\in L_p(\mathbb{R}^d)$, $1<p<\infty$.

To formulate the corresponding version of Theorem 8 let us introduce

$$C'_{g,\gamma} := (-2)^d\prod_{j=1}^{d}\int_{-\infty}^{\infty}(\overline{g_j}*\gamma_j)(s)\ln|s|\,ds = \prod_{j=1}^{d}C'_{g_j,\gamma_j}.$$

Theorem 22 ([21]). *Suppose that g_j and γ_j have log-majorants, $\int_{-\infty}^{\infty}(\overline{g_j}*\gamma_j)(s)\,ds=0$ and C'_{g_j,γ_j} is finite for all $j=1,\dots,d$.*

1. *If* $1 < p < \infty$ *and* $f \in L_p(\mathbb{R}^d)$*, then*

$$\lim_{S\to 0, T\to\infty} \rho_{S,T} f = C'_{g,\gamma} f \qquad \textit{in the } L_p\textit{-norm.}$$

2. *If* $f \in L_p(\mathbb{R}^d)$ *with* $1 < p < \infty$ *or* $f \in L_1(\log L)^{d-1}(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$*, then*

$$\lim_{S\to 0, T\to\infty} \rho_{S,T} f = C'_{g,\gamma} f \qquad a.e.$$

Now we define

$$\theta_j(\xi_j) = \int_{|\xi_j|\le|s_j|} \overline{\widehat{g_j}(s_j)}\, \widehat{\gamma_j}(s_j) \frac{ds_j}{|s_j|}$$

for all $j = 1, \ldots, d$ and let $\theta := \theta_1 \times \cdots \times \theta_d$. The corresponding θ-means are denoted by σ_T^θ $(T \in \mathbb{R}_+^d)$. Under the conditions of the next theorem, we have shown in [24] that $\theta_j, \widehat{\theta_j} \in L_1(\mathbb{R})$,

$$\left|\widehat{\theta_j}(x)\right| \le C\, |x|^{-2} \qquad (x \ne 0)$$

and

$$\rho_{S^{(1)},S^{(2)}} f = \sum_{\epsilon_i\in\{0,1\}} (-1)^{\epsilon_1+\cdots+\epsilon_d+d}\, \sigma^\theta_{1/S_1^{(2-\epsilon_1)},\ldots,1/S_d^{(2-\epsilon_d)}} f$$

for all $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$.

Theorem 23 ([24]). *Assume that* $g_j, \gamma_j \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$*,* $\widehat{g_j}$ *and* $\widehat{\gamma_j}$ *are differentiable,* $\widehat{g_j}'$ *and* $\widehat{\gamma_j}'$ *are bounded,*

$$\left|\widehat{g_j}(x)\right|, \left|\widehat{\gamma_j}(x)\right| \le C\, |x|^\alpha \qquad (0 < |x| \le 1)$$

for some $\alpha > 1/2$ *and for all* $j = 1, \ldots, d$*.*

1. *If* $1 \le p < \infty$*,* $1 < q < \infty$ *and* $f \in W(L_p, \ell_q)(\mathbb{R}^d)$*, then*

$$\lim_{S\to 0, T\to\infty} \rho_{S,T} f = C_{g,\gamma} f \qquad \textit{in the } W(L_p,\ell_q)\textit{-norm.}$$

The same holds for $C_0(\mathbb{R}^d)$*,* $W(C, \ell_q)(\mathbb{R}^d)$ *and for* $W(L_p, c_0)(\mathbb{R}^d)$*.*

2. *If* $f \in W_K(L_1(\log L)^{d-1}, c_0)(\mathbb{R}^d)$*, then*

$$\lim_{S\to 0, T\to\infty} \rho_{S,T} f = C_{g,\gamma} f \qquad a.e.$$

3. *If* $f \in W_K(L_1(\log L)^{d-1}, \ell_\infty)(\mathbb{R}^d) \cap W_K(L_1, c_1)(\mathbb{R}^d)$*, then*

$$\lim_{S\to 0, T\to\infty} \rho_{S,T} f(x) = C_{g,\gamma} f(x)$$

for all strong Lebesgue-points of f.

For some spaces contained in $W_K(L_1(\log L)^{d-1}, c_0)(\mathbb{R}^d)$ we refer to (5). As we mentioned after Theorem 9, the function g_j satisfies the conditions of Theorem 23 if $g_j \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, $\widehat{g_j}(0) = 0$ and

$$\int_{\mathbb{R}} (1+|x|) \left|g_j(x)\right| dx < \infty.$$

4.6 Convergence of $\rho_S f$ in Pringsheim's Sense

Theorem 21 holds in this case, too.

Theorem 24 ([28]). *Assume that* $g_j, \gamma_j \in L_2(\mathbb{R}) \cap L_1(\mathbb{R})$ *satisfy (6) for all* $j = 1, \ldots, d$*. If* $f \in L_p(\mathbb{R}^d)$ *with* $1 < p < \infty$*, then*

$$\lim_{S\to 0} \rho_S f = C_{g,\gamma} f \qquad \text{in the } L_p\text{-norm.}$$

Now the operator ρ_S can be expressed by

$$\rho_S f(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\prod_{j=1}^{d} \frac{\overline{\hat{g}_j}(s_j) \widehat{\gamma_j}(s_j)}{s_j} \right) \rho_{s_1/S_1, \ldots, s_d/S_d} f(t)\, ds$$

for all $f \in L_p(\mathbb{R}^d)$, $1 < p < \infty$.

Theorem 25 ([21]). *Suppose that* g_j *and* γ_j *have log-majorants,* $\int_{-\infty}^{\infty} (\overline{g_j} * \gamma_j)(s)\, ds = 0$ *and* C'_{g_j,γ_j} *is finite for all* $j = 1, \ldots, d$.

1. *If* $1 < p < \infty$ *and* $f \in L_p(\mathbb{R}^d)$*, then*

$$\lim_{S\to 0} \rho_S f = C'_{g,\gamma} f \qquad \text{in the } L_p\text{-norm.}$$

2. *If* $f \in L_1(\log L)^{d-1}(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$*, then the convergence holds in the* L_1*-norm*

$$\lim_{S\to 0} \rho_S f = C'_{g,\gamma} f \qquad \text{a.e.}$$

Under the conditions of Theorem 23, ρ_S can be characterized as

$$\rho_S f = \sigma^{\theta}_{1/S_1, \ldots, 1/S_d} f \tag{7}$$

for all $f \in L_1(\mathbb{R}^d)$.

Theorem 26 ([24]). *Assume the same conditions as in Theorem 23.*

1. *If* $f \in L_1(\mathbb{R}^d)$*, then*

$$\lim_{S \to 0} \rho_S f = C_{g,\gamma} f \qquad \text{in the } L_1\text{-norm.}$$

2. *If* $f \in W_K(L_1(\log L)^{d-1}, \ell_\infty)(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$*, then*

$$\lim_{S \to 0} \rho_S f(x) = C_{g,\gamma} f(x)$$

for all strong Lebesgue-points of f*.*

Note that

$$W_K(L_1(\log L)^{d-1}, \ell_\infty)(\mathbb{R}^d) \supset L_1(\log L)^{d-1}(\mathbb{R}^d), L_p(\mathbb{R}^d) \qquad (1 < p \leq \infty).$$

If $\gamma \in L_1(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)$, then (7) holds for all $f \in L_p(\mathbb{R}^d)$, if $\gamma \in W(L_\infty, \ell_1)(\mathbb{R}^d)$, then for all $f \in W(L_p, \ell_q)(\mathbb{R}^d)$ $(1 \leq p, q < \infty)$. The next theorem follows from this.

Theorem 27 ([23]). *Besides the conditions of Theorem 23 assume that* $1 \leq p, q < \infty$ *and* $\gamma_j \in L_\infty(\mathbb{R}^d)$ *for all* $j = 1, \ldots, d$*.*

1. *If* $f \in L_p(\mathbb{R}^d)$*, then*

$$\lim_{S \to 0} \rho_S f = C_{g,\gamma} f \qquad \text{in the } L_p(\mathbb{R}^d)\text{-norm.}$$

If in addition $\gamma_j \in W(L_\infty, \ell_1)(\mathbb{R})$*, then the convergence holds in the* $W(L_p, \ell_q)(\mathbb{R}^d)$*-norm for all* $f \in W(L_p, \ell_q)(\mathbb{R}^d)$*.*

2. *If* $f \in L_1(\log L)^{d-1}(\mathbb{R}^d)$*, then*

$$\lim_{S \to 0} \rho_S f(x) = C_{g,\gamma} f(x)$$

for all strong Lebesgue-points of f*. If in addition* $\gamma_j \in W(L_\infty, \ell_1)(\mathbb{R})$*, then the convergence holds at each strong Lebesgue-point of* $f \in W_K(L_1(\log L)^{d-1}, \ell_\infty)(\mathbb{R}^d) \cap W(L_1, \ell_q)(\mathbb{R}^d)$*.*

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Multidimensional Hilbert-Type Integral Inequalities and Their Operators Expressions

Bicheng Yang

Abstract In this chapter, by the use of the methods of weight functions and techniques of Real Analysis, we provide a general multidimensional Hilbert-type integral inequality with a non-homogeneous kernel and a best possible constant factor. The equivalent forms, the reverses and some Hardy-type inequalities are obtained. Furthermore, we consider the operator expressions with the norm, some particular inequalities with the homogeneous kernel and a large number of particular examples.

Keywords Multidimensional Hilbert-type integral inequality • Weight function • Equivalent form • Hilbert-type integral operator

Mathematics Subject Classification 26D15, 31A10, 47A07

1 Introduction

Suppose that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(x), g(y) \geq 0, f \in L^p(\mathbf{R}_+), g \in L^q(\mathbf{R}_+)$,

$$||f||_p = \left\{ \int_0^\infty f^p(x)dx \right\}^{\frac{1}{p}} > 0,$$

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$||g||_q > 0$. We have the following Hardy–Hilbert's integral inequality (cf. [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} ||f||_p ||g||_q, \tag{1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. If $a_m, b_n \geq 0, a = \{a_m\}_{m=1}^\infty \in l^p, b = \{b_n\}_{n=1}^\infty \in l^q$,

$$||a||_p = \left\{ \sum_{m=1}^\infty a_m^p \right\}^{\frac{1}{p}} > 0,$$

$||b||_q > 0$, then we have the following discrete Hardy–Hilbert's inequality with the same best constant $\frac{\pi}{\sin(\pi/p)}$:

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} ||a||_p ||b||_q. \tag{2}$$

Inequalities (1) and (2) are important in Analysis and its applications (cf. [1–6]).

In 1998, by introducing an independent parameter $\lambda \in (0, 1]$, Yang [7] gave an extension of (1) for $p = q = 2$. In 2009 and 2011, Yang [3,4] gave some extensions of (1) and (2) as follows: If $\lambda_1, \lambda_2 \in \mathbf{R} = (-\infty, \infty), \lambda_1 + \lambda_2 = \lambda, k_\lambda(x, y)$ is a nonnegative homogeneous function of degree $-\lambda$ in $\mathbf{R}_+^2$, with

$$k(\lambda_1) = \int_0^\infty k_\lambda(t, 1) t^{\lambda_1 - 1} dt \in \mathbf{R}_+ = (0, \infty),$$
$$\phi(x) = x^{p(1-\lambda_1)-1}, \psi(y) = y^{q(1-\lambda_2)-1} (x, y \in \mathbf{R}_+),$$

$f(x), g(y) \geq 0$, satisfying

$$f \in L_{p,\phi}(\mathbf{R}_+) = \left\{ f; ||f||_{p,\phi} := \left\{ \int_0^\infty \phi(x) |f(x)|^p dx \right\}^{\frac{1}{p}} < \infty \right\},$$

$g \in L_{q,\psi}(\mathbf{R}_+), ||f||_{p,\phi}, ||g||_{q,\psi} > 0$, then we have

$$\int_0^\infty \int_0^\infty k_\lambda(x, y) f(x) g(y) dxdy < k(\lambda_1) ||f||_{p,\phi} ||g||_{q,\psi}, \tag{3}$$

where the constant factor $k(\lambda_1)$ is the best possible. Moreover, if $k_\lambda(x, y)$ is finite and $k_\lambda(x, y) x^{\lambda_1 - 1} (k_\lambda(x, y) y^{\lambda_2 - 1})$ is strict decreasing with respect to $x > 0 (y > 0)$, then for $a_m, b_n \geq 0$,

$$a = \{a_m\}_{m=1}^{\infty} \in l_{p,\phi} = \left\{a; ||a||_{p,\phi} := \left\{\sum_{n=1}^{\infty} \phi(n)|a_n|^p\right\}^{\frac{1}{p}} < \infty\right\},$$

$b = \{b_n\}_{n=1}^{\infty} \in l_{q,\psi}$, $||a||_{p,\phi}, ||b||_{q,\psi} > 0$, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} k_\lambda(m,n)a_m b_n < k(\lambda_1)||a||_{p,\phi}||b||_{q,\psi}, \tag{4}$$

where the constant factor $k(\lambda_1)$ is still the best possible.

Clearly, for $\lambda = 1, k_1(x,y) = \frac{1}{x+y}, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$, (3) reduces to (1), while (4) reduces to (2). Some other results including multidimensional Hilbert-type integral inequalities are provided by Yang et al. [8], Krnić and Pečarić [9], Yang and Rassias [10, 11], Azar [12], Arpad and Choonghong [13], Kuang and Debnath [14], Zhong [15], Hong [16], Zhong and Yang [17], Yang and Krnić [18], and Li and He [19].

In this chapter, by the use of the methods of weight functions and techniques of real analysis, we give a general multidimensional Hilbert-type integral inequality with a nonhomogeneous kernel and a best possible constant factor. The equivalent forms, the reverses and some Hardy-type inequalities are obtained. Furthermore, we consider the operator expressions with the norm, some particular inequalities with the homogeneous kernel and a large number of particular examples.

2 Some Lemmas

If $i_0, j_0 \in \mathbf{N}$($\mathbf{N}$ is the set of positive integers), $\alpha, \beta > 0$, we put

$$||x||_\alpha := \left(\sum_{k=1}^{i_0} |x_k|^\alpha\right)^{\frac{1}{\alpha}} \quad (x = (x_1, \ldots, x_{i_0}) \in \mathbf{R}^{i_0}),$$

$$||y||_\beta := \left(\sum_{k=1}^{j_0} |y_k|^\beta\right)^{\frac{1}{\beta}} \quad (y = (y_1, \ldots, y_{j_0}) \in \mathbf{R}^{j_0}).$$

Lemma 1. *If* $s \in \mathbf{N}, \gamma, M > 0$, $\Psi(u)$ *is a nonnegative measurable function in* $(0, 1]$, *and*

$$D_M := \left\{x \in \mathbf{R}_+^s; 0 < u = \sum_{i=1}^{s}\left(\frac{x_i}{M}\right)^\gamma \le 1\right\},$$

then we have the following expression (cf. [6]):

$$\int \cdots \int_{D_M} \Psi\left(\sum_{i=1}^{s}\left(\frac{x_i}{M}\right)^{\gamma}\right) dx_1 \cdots dx_s \\ = \frac{M^s \Gamma^s\left(\frac{1}{\gamma}\right)}{\gamma^s \Gamma\left(\frac{s}{\gamma}\right)} \int_0^1 \Psi(u) u^{\frac{s}{\gamma}-1} du. \tag{5}$$

In view of (5) and the conditions, it follows that

(i) for

$$\mathbf{R}_+^s = \left\{x \in \mathbf{R}_+^s; 0 < u = \sum_{i=1}^{s}\left(\frac{x_i}{M}\right)^{\gamma} \le 1 (M \to \infty)\right\},$$

we have

$$\int \cdots \int_{\mathbf{R}_+^s} \Psi\left(\sum_{i=1}^{s}\left(\frac{x_i}{M}\right)^{\gamma}\right) dx_1 \cdots dx_s \\ = \lim_{M\to\infty} \frac{M^s \Gamma^s\left(\frac{1}{\gamma}\right)}{\gamma^s \Gamma\left(\frac{s}{\gamma}\right)} \int_0^1 \Psi(u) u^{\frac{s}{\gamma}-1} du; \tag{6}$$

(ii) for

$$\{x \in \mathbf{R}_+^s; ||x||_\gamma \ge 1\} \\ = \left\{x \in \mathbf{R}_+^s; \frac{1}{M^\gamma} < u = \sum_{i=1}^{s}\left(\frac{x_i}{M}\right)^{\gamma} \le 1 (M \to \infty)\right\},$$

setting $\Psi(u) = 0\left(u \in \left(0, \frac{1}{M^\gamma}\right)\right)$, we have

$$\int \cdots \int_{\{x\in\mathbf{R}_+^s; ||x||_\gamma \ge 1\}} \Psi\left(\sum_{i=1}^{s}\left(\frac{x_i}{M}\right)^{\gamma}\right) dx_1 \cdots dx_s \\ = \lim_{M\to\infty} \frac{M^s \Gamma^s\left(\frac{1}{\gamma}\right)}{\gamma^s \Gamma\left(\frac{s}{\gamma}\right)} \int_{\frac{1}{M^\gamma}}^1 \Psi(u) u^{\frac{s}{\gamma}-1} du; \tag{7}$$

(iii) for

$$\{x \in \mathbf{R}_+^s; ||x||_\gamma \le 1\} \\ = \left\{x \in \mathbf{R}_+^s; 0 < u = \sum_{i=1}^{s}\left(\frac{x_i}{M}\right)^{\gamma} \le \frac{1}{M^\gamma}\right\},$$

setting $\Psi(u) = 0\big(u \in \big(\frac{1}{M^\gamma}, \infty\big)\big)$, we have

$$\int \cdots \int_{\{x \in \mathbf{R}_+^s; ||x||_\gamma \leq 1\}} \Psi\left(\sum_{i=1}^{s}\left(\frac{x_i}{M}\right)^\gamma\right) dx_1 \cdots dx_s = \frac{M^s \Gamma^s\left(\frac{1}{\gamma}\right)}{\gamma^s \Gamma\left(\frac{s}{\gamma}\right)} \int_0^{\frac{1}{M^\gamma}} \Psi(u) u^{\frac{s}{\gamma}-1} du. \tag{8}$$

Lemma 2. *For* $s \in \mathbf{N}, \gamma > 0$, $\varepsilon > 0$, *we have*

$$\int \cdots \int_{\{x \in \mathbf{R}_+^s; ||x||_\gamma \geq 1\}} ||x||_\gamma^{-s-\varepsilon} dx_1 \cdots dx_s = \frac{\Gamma^s\left(\frac{1}{\gamma}\right)}{\varepsilon \gamma^{s-1} \Gamma\left(\frac{s}{\gamma}\right)}, \tag{9}$$

$$\int \cdots \int_{\{x \in \mathbf{R}_+^s; ||x||_\gamma \leq 1\}} ||x||_\gamma^{-s+\varepsilon} dx_1 \cdots dx_s = \frac{\Gamma^s\left(\frac{1}{\gamma}\right)}{\varepsilon \gamma^{s-1} \Gamma\left(\frac{s}{\gamma}\right)}. \tag{10}$$

Proof. By (7), it follows

$$\begin{aligned}
&\int \cdots \int_{\{x \in \mathbf{R}_+^s; ||x||_\gamma \geq 1\}} ||x||_\gamma^{-s-\varepsilon} dx_1 \cdots dx_s \\
&= \int \cdots \int_{\{x \in \mathbf{R}_+^s; ||x||_\gamma \geq 1\}} \left\{ M \left[\sum_{i=1}^{s}\left(\frac{x_i}{M}\right)^\gamma\right]^{\frac{1}{\gamma}} \right\}^{-s-\varepsilon} dx_1 \cdots dx_s \\
&= \lim_{M \to \infty} \frac{M^s \Gamma^s\left(\frac{1}{\gamma}\right)}{\gamma^s \Gamma\left(\frac{s}{\gamma}\right)} \int_{\frac{1}{M^\gamma}}^{1} (M u^{1/\gamma})^{-s-\varepsilon} u^{\frac{s}{\gamma}-1} du \\
&= \frac{\Gamma^s\left(\frac{1}{\gamma}\right)}{\varepsilon \gamma^{s-1} \Gamma\left(\frac{s}{\gamma}\right)}.
\end{aligned}$$

By (8), we find

$$\begin{aligned}
&\int \cdots \int_{\{x \in \mathbf{R}_+^s; ||x||_\gamma \leq 1\}} ||x||_\gamma^{-s+\varepsilon} dx_1 \cdots dx_s \\
&= \int \cdots \int_{\{x \in \mathbf{R}_+^s; ||x||_\gamma \leq 1\}} \left\{ M \left[\sum_{i=1}^{s}\left(\frac{x_i}{M}\right)^\gamma\right]^{\frac{1}{\gamma}} \right\}^{-s+\varepsilon} dx_1 \cdots dx_s \\
&= \frac{M^s \Gamma^s\left(\frac{1}{\gamma}\right)}{\gamma^s \Gamma\left(\frac{s}{\gamma}\right)} \int_0^{\frac{1}{M^\gamma}} (M u^{1/\gamma})^{-s+\varepsilon} u^{\frac{s}{\gamma}-1} du = \frac{\Gamma^s\left(\frac{1}{\gamma}\right)}{\varepsilon \gamma^{s-1} \Gamma\left(\frac{s}{\gamma}\right)}.
\end{aligned}$$

Hence, we have (9) and (10). The lemma is proved.

Note. By (9) and (10), for $\delta = \pm 1$, we have the following unified expression:

$$\int \cdots \int_{\{x \in \mathbf{R}_+^s; ||x||_\gamma^\delta \geq 1\}} ||x||_\gamma^{-s-\delta\varepsilon} dx_1 \cdots dx_s = \frac{\Gamma^s\left(\frac{1}{\gamma}\right)}{\varepsilon \gamma^{s-1} \Gamma\left(\frac{s}{\gamma}\right)}. \tag{11}$$

Definition 1. If $x = (x_1, \ldots, x_{i_0}) \in \mathbf{R}_+^{i_0}$, $y = (y_1, \ldots, y_{j_0}) \in \mathbf{R}_+^{j_0}$, $h(u)$ is a nonnegative measurable function in $\mathbf{R}_+$, $\sigma \in \mathbf{R}$, $\delta \in \{-1, 1\}$, then we define two weight functions $\omega_\delta(\sigma, y)$ and $\varpi_\delta(\sigma, x)$ as follows:

$$\omega_\delta(\sigma, y) := ||y||_\beta^\sigma \int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) \frac{dx}{||x||_\alpha^{i_0 - \delta\sigma}}, \tag{12}$$

$$\varpi_\delta(\sigma, x) := ||x||_\alpha^{\delta\sigma} \int_{\mathbf{R}_+^{j_0}} h(||x||_\alpha^\delta ||y||_\beta) \frac{dy}{||y||_\beta^{j_0 - \sigma}}. \tag{13}$$

By (6), we find

$$\begin{aligned}
\omega_\delta(\sigma, y) &= ||y||_\beta^\sigma \int_{\mathbf{R}_+^{i_0}} \frac{h\left(M^\delta \left[\sum_{i=1}^{i_0} \left(\frac{x_i}{M}\right)^\alpha\right]^{\frac{\delta}{\alpha}} ||y||_\beta\right)}{M^{i_0 - \delta\sigma} \left[\sum_{i=1}^{i_0} \left(\frac{x_i}{M}\right)^\alpha\right]^{\frac{i_0 - \delta\sigma}{\alpha}}} dx \\
&= ||y||_\beta^\sigma \lim_{M \to \infty} \frac{M^{i_0} \Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0} \Gamma\left(\frac{i_0}{\alpha}\right)} \int_0^1 \frac{h(M^\delta u^{\frac{\delta}{\alpha}} ||y||_\beta)}{M^{i_0 - \delta\sigma} u^{\frac{i_0 - \delta\sigma}{\alpha}}} u^{\frac{i_0}{\alpha} - 1} du \\
&= ||y||_\beta^\sigma \lim_{M \to \infty} \frac{M^{\delta\sigma} \Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0} \Gamma\left(\frac{i_0}{\alpha}\right)} \int_0^1 h(M^\delta u^{\frac{\delta}{\alpha}} ||y||_\beta) u^{\frac{\delta\sigma}{\alpha} - 1} du.
\end{aligned}$$

Setting $v = M^\delta u^{\frac{\delta}{\alpha}} ||y||_\beta$ in the above integral, in view of $\delta = \pm 1$, we obtain

$$\omega_\delta(\sigma, y) = K_2(\sigma) := \frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0 - 1} \Gamma\left(\frac{i_0}{\alpha}\right)} k(\sigma), \tag{14}$$

where $k(\sigma) = \int_0^\infty h(v) v^{\sigma - 1} dv$.

By (6), setting $v = M ||x||_\alpha^\delta u^{\frac{1}{\beta}}$, we find

$$\varpi_\delta(\sigma, x) = ||x||_\alpha^{\delta\sigma} \int_{\mathbf{R}_+^{j_0}} \frac{h\left(M ||x||_\alpha^\delta \left[\sum_{j=1}^{j_0} \left(\frac{y_j}{M}\right)^\beta\right]^{\frac{1}{\beta}}\right)}{M^{j_0 - \sigma} \left[\sum_{j=1}^{j_0} \left(\frac{y_j}{M}\right)^\beta\right]^{\frac{j_0 - \sigma}{\beta}}} dy$$

$$= ||x||_\alpha^{\delta\sigma} \lim_{M\to\infty} \frac{M^{j_0}\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0}\Gamma\left(\frac{j_0}{\beta}\right)} \int_0^1 \frac{h\left(M||x||_\alpha^\delta u^{\frac{1}{\beta}}\right)}{M^{j_0-\sigma}u^{\frac{j_0-\sigma}{\beta}}} u^{\frac{j_0}{\beta}-1} du$$

$$= ||x||_\alpha^{\delta\sigma} \lim_{M\to\infty} \frac{M^{\sigma}\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0}\Gamma\left(\frac{j_0}{\beta}\right)} \int_0^1 h\left(M||x||_\alpha^\delta u^{\frac{1}{\beta}}\right) u^{\frac{\sigma}{\beta}-1} du$$

$$= K_1(\sigma) := \frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)} k(\sigma). \tag{15}$$

Lemma 3. *As the assumptions of Definition 1, for* $k(\sigma) \in \mathbf{R}_+$, $p \in \mathbf{R}\backslash\{0,1\}$, $\frac{1}{p} + \frac{1}{q} = 1$, *setting*

$$\tilde{I} := \int_{\left\{x\in\mathbf{R}_+^{i_0};||x||_\alpha^\delta \geq 1\right\}} ||x||_\alpha^{\delta\sigma-\frac{\delta\varepsilon}{p}-i_0} \left[\int_{\left\{y\in\mathbf{R}_+^{j_0};||y||_\beta \leq 1\right\}} h\left(||x||_\alpha^\delta ||y||_\beta\right) ||y||_\beta^{\sigma+\frac{\varepsilon}{q}-j_0} dy\right] dx, \tag{16}$$

then we have

$$\varepsilon\tilde{I} \geq \tilde{K}(\sigma) + o(1)(\varepsilon \to 0^+), \tag{17}$$

where $\tilde{K}(\sigma) := L(\alpha,\beta)k(\sigma)$,

$$L(\alpha,\beta) := \frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)} \frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}. \tag{18}$$

Moreover, if there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, $k(\tilde{\sigma}) \in \mathbf{R}$, then we have

$$\varepsilon\tilde{I} = \tilde{K}(\sigma) + o(1)(\varepsilon \to 0^+). \tag{19}$$

Proof. For $\varepsilon > 0$, setting $\tilde{\sigma} = \sigma + \frac{\varepsilon}{q}$ and

$$H(||x||_\alpha^\delta) := ||x||_\alpha^{\delta\tilde{\sigma}} \int_{\left\{y\in\mathbf{R}_+^{j_0};||y||_\beta \leq 1\right\}} h\left(||x||_\alpha^\delta ||y||_\beta\right) ||y||_\beta^{\tilde{\sigma}-j_0} dy,$$

in view of (16), it follows

$$\tilde{I} = \int_{\left\{x\in\mathbf{R}_+^{i_0};||x||_\alpha^\delta \geq 1\right\}} ||x||_\alpha^{-\delta\varepsilon-i_0} H\left(||x||_\alpha^\delta\right) dx.$$

Putting

$$\Psi(u) = h(||x||_\alpha^\delta M u^{\frac{1}{\beta}}) M^{\tilde{\sigma}-j_0} u^{\frac{1}{\beta}(\tilde{\sigma}-j_0)},$$

by (8), we find

$$\begin{aligned} H(||x||_\alpha^\delta) &= ||x||_\beta^{\delta\tilde{\sigma}} \frac{M^{j_0}\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0}\Gamma\left(\frac{j_0}{\beta}\right)} \\ &\quad \times \int_0^{\frac{1}{M^\beta}} h(||x||_\alpha^\delta M u^{\frac{1}{\beta}}) M^{\tilde{\sigma}-j_0} u^{\frac{1}{\beta}(\tilde{\sigma}-j_0)} u^{\frac{j_0}{\beta}-1} du \\ &= ||x||_\alpha^{\delta\tilde{\sigma}} \frac{M^{\tilde{\sigma}}\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0}\Gamma\left(\frac{j_0}{\beta}\right)} \int_0^{\frac{1}{M^\beta}} h(||x||_\alpha^\delta M u^{\frac{1}{\beta}}) u^{\frac{\tilde{\sigma}}{\beta}-1} du. \end{aligned}$$

Setting $v = ||x||_\alpha^\delta M u^{\frac{1}{\beta}}$ in the above, it follows

$$L(||x||_\alpha^\delta) = \frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)} \int_0^{||x||_\alpha^\delta} h(v) v^{\tilde{\sigma}-1} dv.$$

Putting $\Psi(u) = M^{-\delta\varepsilon-i_0} u^{\frac{1}{\alpha}(-\delta\varepsilon-i_0)} H(M^\delta u^{\frac{\delta}{\alpha}})$, for $\delta = 1$, by (7), we obtain

$$\begin{aligned} \tilde{I} &= \lim_{M\to\infty} \frac{M^{i_0}\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0}\Gamma\left(\frac{i_0}{\alpha}\right)} \int_{\frac{1}{M^\alpha}}^1 M^{-\varepsilon-i_0} u^{\frac{1}{\alpha}(-\varepsilon-i_0)} H(M u^{\frac{1}{\alpha}}) u^{\frac{i_0}{\alpha}-1} du \\ &= \lim_{M\to\infty} \frac{M^{-\varepsilon}\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0}\Gamma\left(\frac{i_0}{\alpha}\right)} \int_{\frac{1}{M^\alpha}}^1 H(M u^{\frac{1}{\alpha}}) u^{\frac{-\varepsilon}{\alpha}-1} du \\ &= \frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)} \int_1^\infty H(t) t^{-\varepsilon-1} dt (t = M u^{\frac{1}{\alpha}}); \end{aligned}$$

for $\delta = -1$, by (8), we still find that

$$\begin{aligned} \tilde{I} &= \frac{M^{i_0}\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0}\Gamma\left(\frac{i_0}{\alpha}\right)} \int_0^{\frac{1}{M^\alpha}} M^{\varepsilon-i_0} u^{\frac{1}{\alpha}(\varepsilon-i_0)} H(M^{-1} u^{\frac{-1}{\alpha}}) u^{\frac{i_0}{\alpha}-1} du \\ &= \frac{M^{\varepsilon}\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0}\Gamma\left(\frac{i_0}{\alpha}\right)} \int_0^{\frac{1}{M^\alpha}} H(M^{-1} u^{\frac{-1}{\alpha}}) u^{\frac{\varepsilon}{\alpha}-1} du \\ &= \frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)} \int_1^\infty H(t) t^{-\varepsilon-1} dt (t = M^{-1} u^{\frac{-1}{\alpha}}). \end{aligned}$$

Hence, we find

$$\begin{aligned}
\varepsilon\tilde{I} &= \varepsilon L(\alpha,\beta)\int_1^\infty t^{-\varepsilon-1}\int_0^t h(v)v^{\tilde{\sigma}-1}dvdt \\
&= \varepsilon L(\alpha,\beta)\left[\int_1^\infty t^{-\varepsilon-1}\int_0^1 h(v)v^{\tilde{\sigma}-1}dvdt\right. \\
&\quad \left. + \int_1^\infty t^{-\varepsilon-1}\int_1^t h(v)v^{\tilde{\sigma}-1}dvdt\right]
\end{aligned}$$

$$= \varepsilon L(\alpha,\beta)\left[\frac{1}{\varepsilon}\int_0^1 h(v)v^{\tilde{\sigma}-1}dvdt + \int_1^\infty\left(\int_v^\infty t^{-\varepsilon-1}dt\right)h(v)v^{\tilde{\sigma}-1}dv\right]$$

$$= L(\alpha,\beta)\left[\int_0^1 h(v)v^{\tilde{\sigma}-1}dvdt + \int_1^\infty h(v)v^{\left(\sigma-\frac{\varepsilon}{p}\right)-1}dv\right]. \tag{20}$$

By Fatou lemma (cf. [20]), it follows

$$\begin{aligned}
\underline{\lim}_{\varepsilon\to 0^+}\varepsilon\tilde{I} &= L(\alpha,\beta)\,\underline{\lim}_{\varepsilon\to 0^+}\left[\int_0^1 h(v)v^{\tilde{\sigma}-1}dvdt + \int_1^\infty h(v)v^{\left(\sigma-\frac{\varepsilon}{p}\right)-1}dv\right] \\
&\geq L(\alpha,\beta)\left[\int_0^1 \underline{\lim}_{\varepsilon\to 0^+} h(v)v^{\tilde{\sigma}-1}dvdt\right. \\
&\quad \left. + \int_1^\infty \underline{\lim}_{\varepsilon\to 0^+} h(v)v^{\left(\sigma-\frac{\varepsilon}{p}\right)-1}dv\right] = L(\alpha,\beta)k(\sigma),
\end{aligned}$$

and then (17) follows.

Moreover, for $0<\varepsilon<\delta_0\min\{|p|,|q|\}, \tilde{\sigma}\in\left(\sigma-\frac{1}{2}\delta_0,\sigma+\frac{1}{2}\delta_0\right)$, since

$$\begin{aligned}
h(v)v^{\tilde{\sigma}-1} &\leq h(v)v^{(\sigma-\frac{1}{2}\delta_0)-1}(v\in(0,1]), \\
&\quad 0\leq\int_0^1 h(v)v^{(\sigma-\frac{1}{2}\delta_0)-1}\leq k\left(\sigma-\frac{1}{2}\delta_0\right)<\infty, \\
h(v)v^{\tilde{\sigma}-1} &\leq h(v)v^{(\sigma+\frac{1}{2}\delta_0)-1}(v\in[1,\infty)), \\
&\quad 0\leq\int_1^\infty h(v)v^{(\sigma+\frac{1}{2}\delta_0)-1}\leq k\left(\sigma+\frac{1}{2}\delta_0\right)<\infty,
\end{aligned}$$

by Lebesgue control convergence theorem (cf. [20]), it follows that

$$\int_0^1 h(v)v^{\tilde{\sigma}-1}dv = \int_0^1 h(v)v^{\sigma-1}dv + o_1(1)(\varepsilon\to 0^+),$$

$$\int_1^\infty h(v)v^{(\sigma-\frac{\varepsilon}{p})-1}dv = \int_1^\infty h(v)v^{\sigma-1}dv + o_2(1)(\varepsilon\to 0^+).$$

Then by (20), (19) follows. The lemma is proved.

Lemma 4. *As the assumptions of Definition 1, if* $p \in \mathbf{R}\backslash\{0,1\}$, $\frac{1}{p}+\frac{1}{q}=1$, $f(x) = f(x_1,\ldots,x_{i_0}) \ge 0$, $g(y) = g(y_1,\ldots,y_{j_0}) \ge 0$, *then*

(i) for $p>1$, *we have the following inequality:*

$$J_1 := \left\{\int_{\mathbf{R}_+^{j_0}} \frac{||y||_\beta^{p\sigma-j_0}}{[\omega_\delta(\sigma,y)]^{p-1}} \left(\int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) f(x)dx\right)^p dy\right\}^{\frac{1}{p}}$$
$$\le \left\{\int_{\mathbf{R}_+^{i_0}} \varpi_\delta(\sigma,x)||x||_\alpha^{p(i_0-\delta\sigma)-i_0} f^p(x)dx\right\}^{\frac{1}{p}}; \tag{21}$$

(ii) for $0<p<1$, *or* $p<0$, *we have the reverse of (21).*

Proof. (i) For $p>1$, by Hölder's inequality with weight (cf. [21]), it follows

$$\int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) f(x)dx$$
$$= \int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) \left[\frac{||x||_\alpha^{(i_0-\delta\sigma)/q} f(x)}{||y||_\beta^{(j_0-\sigma)/p}}\right] \left[\frac{||y||_\beta^{(j_0-\sigma)/p}}{||x||_\alpha^{(i_0-\delta\sigma)/q}}\right] dx$$
$$\le \left\{\int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) \frac{||x||_\alpha^{(i_0-\delta\sigma)(p-1)}}{||y||_\beta^{j_0-\sigma}} f^p(x)dx\right\}^{\frac{1}{p}}$$
$$\times \left\{\int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) \frac{||y||_\beta^{(j_0-\sigma)(q-1)}}{||x||_\alpha^{i_0-\delta\sigma}} dx\right\}^{\frac{1}{q}}$$
$$= [\omega_\delta(\sigma,y)]^{\frac{1}{q}} ||y||_\beta^{\frac{j_0}{p}-\sigma}$$
$$\times \left\{\int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) \frac{||x||_\alpha^{(i_0-\delta\sigma)(p-1)}}{||y||_\beta^{j_0-\sigma}} f^p(x)dx\right\}^{\frac{1}{p}}. \tag{22}$$

Then by Fubini theorem (cf. [20]), we have

$$J_1 \le \left\{\int_{\mathbf{R}_+^{j_0}} \left[\int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) \frac{||x||_\alpha^{(i_0-\delta\sigma)(p-1)}}{||y||_\beta^{j_0-\sigma}} f^p(x)dx\right] dy\right\}^{\frac{1}{p}}$$
$$= \left\{\int_{\mathbf{R}_+^{i_0}} \left[\int_{\mathbf{R}_+^{j_0}} h(||x||_\alpha ||y||_\beta) \frac{||x||_\alpha^{(i_0-\delta\sigma)(p-1)}}{||y||_\beta^{j_0-\sigma}} dy\right] f^p(x)dx\right\}^{\frac{1}{p}}$$

$$= \left\{ \int_{\mathbf{R}_+^{i_0}} \varpi_\delta(\sigma, x) ||x||_\alpha^{p(i_0-\delta\sigma)-i_0} f^p(x) dx \right\}^{\frac{1}{p}}. \tag{23}$$

Hence, (21) follows.

(ii) For $0 < p < 1$, or $p < 0$, by the reverse Hölder's inequality with weight (cf. [21]), we obtain the reverse of (22). Then by Fubini theorem, we still can obtain the reverse of (21). The lemma is proved.

Lemma 5. *As the assumptions of Lemma 4, then*

(i) for $p > 1$*, we have the following inequality equivalent to (21):*

$$I := \int_{\mathbf{R}_+^{j_0}} \int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) f(x) g(y) dx dy$$
$$\leq \left\{ \int_{\mathbf{R}_+^{i_0}} \varpi_\delta(\sigma, x) ||x||_\alpha^{p(i_0-\delta\sigma)-i_0} f^p(x) dx \right\}^{\frac{1}{p}}$$
$$\times \left\{ \int_{\mathbf{R}_+^{j_0}} \omega_\delta(\sigma, y) ||y||_\beta^{q(j_0-\sigma)-j_0} g^q(y) dy \right\}^{\frac{1}{q}}; \tag{24}$$

(ii) for $0 < p < 1$*, or* $p < 0$*, we have the reverse of (24) equivalent to the reverse of (21).*

Proof. (i) For $p > 1$, by Hölder's inequality (cf. [21]), it follows

$$I = \int_{\mathbf{R}_+^{j_0}} \frac{||y||_\beta^{\frac{j_0}{q}-(j_0-\sigma)}}{[\omega_\delta(\sigma, y)]^{\frac{1}{q}}} \left[\int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) f(x) dx \right]$$
$$\times \left[[\omega_\delta(\sigma, y)]^{\frac{1}{q}} ||y||_\beta^{(j_0-\sigma)-\frac{j_0}{q}} g(y) \right] dy$$

$$\leq J_1 \left\{ \int_{\mathbf{R}_+^{j_0}} \omega_\delta(\sigma, y) ||y||_\beta^{q(j_0-\sigma)-j_0} g^q(y) dy \right\}^{\frac{1}{q}}. \tag{25}$$

Then by (21), we have (24).

On the other hand, assuming that (24) is valid, we set

$$g(y) := \frac{||y||_\beta^{p\sigma-j_0}}{[\omega_\delta(\sigma, y)]^{p-1}} \left(\int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) f(x) dx \right)^{p-1}, \ y \in \mathbf{R}_+^{j_0}.$$

Then it follows

$$J_1^p = \int_{\mathbf{R}_+^{j_0}} \omega_\delta(\sigma, y)||y||_\beta^{q(j_0-\sigma)-j_0} g^q(y)dy.$$

If $J_1 = 0$, then (21) is trivially valid; if $J_1 = \infty$, then by (23), (21) keeps the form of equality $(= \infty)$. Suppose that $0 < J_1 < \infty$. By (24), we have

$$\begin{aligned} 0 < \int_{\mathbf{R}_+^{j_0}} \omega_\delta(\sigma, y)||y||_\beta^{q(j_0-\sigma)-j_0} g^q(y)dy = J_1^p = I \\ \leq \left\{ \int_{\mathbf{R}_+^{i_0}} \varpi_\delta(\sigma, x)||x||_\alpha^{p(i_0-\delta\sigma)-i_0} f^p(x)dx \right\}^{\frac{1}{p}} \\ \times \left\{ \int_{\mathbf{R}_+^{j_0}} \omega_\delta(\sigma, y)||y||_\beta^{q(j_0-\sigma)-j_0} g^q(y)dy \right\}^{\frac{1}{q}} < \infty. \end{aligned}$$

It follows

$$\begin{aligned} J_1 = \left\{ \int_{\mathbf{R}_+^{j_0}} \omega_\delta(\sigma, y)||y||_\beta^{q(j_0-\sigma)-j_0} g^q(y)dy \right\}^{\frac{1}{p}} \\ \leq \left\{ \int_{\mathbf{R}_+^{i_0}} \varpi_\delta(\sigma, x)||x||_\alpha^{p(i_0-\delta\sigma)-i_0} f^p(x)dx \right\}^{\frac{1}{p}}, \end{aligned}$$

and then (21) follows. Hence, (21) and (24) are equivalent.

(ii) For $0 < p < 1$, or $p < 0$, by the same way, we can obtain the reverse of (24) equivalent to the reverse of (21). The lemma is proved.

3 Main Results and Operator Expressions

Setting

$$\Phi_\delta(x) := ||x||_\alpha^{p(i_0-\delta\sigma)-i_0},$$
$$\Psi(y) := ||y||_\beta^{q(j_0-\sigma)-j_0} (x \in \mathbf{R}_+^{i_0}, y \in \mathbf{R}_+^{j_0}),$$

by Lemmas 3–5, it follows

Theorem 1. *Suppose that* $\alpha, \beta > 0,\ \sigma \in \mathbf{R}, h(v) \geq 0,$

$$k(\sigma) = \int_0^\infty h(v)v^{\sigma-1}dv \in \mathbf{R}_+,$$

$$K(\sigma) := \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{q}} k(\sigma),$$

$\delta \in \{-1, 1\}, p \in \mathbf{R}\backslash\{0, 1\}, \frac{1}{p} + \frac{1}{q} = 1, \ f(x) = f(x_1, \ldots, x_{i_0}) \geq 0, \ g(y) = g(y_1, \ldots, y_{j_0}) \geq 0,$

$$0 < ||f||_{p,\Phi_\delta} = \left\{\int_{\mathbf{R}_+^{i_0}} \Phi_\delta(x) f^p(x)dx\right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||g||_{q,\Psi} = \left\{\int_{\mathbf{R}_+^{j_0}} \Psi(y)g^q(y)dy\right\}^{\frac{1}{q}} < \infty.$$

(i) If $p > 1$*, then we have the following equivalent inequalities with the best possible constant factor* $K(\sigma)$ *:*

$$I = \int_{\mathbf{R}_+^{j_0}} \int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) f(x)g(y)dxdy < K(\sigma)||f||_{p,\Phi_\delta}||g||_{q,\Psi}, \quad (26)$$

$$J := \left\{\int_{\mathbf{R}_+^{j_0}} ||y||_\beta^{p\sigma-j_0} \left(\int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) f(x)dx\right)^p dy\right\}^{\frac{1}{p}} < K(\sigma)||f||_{p,\Phi_\delta}; \quad (27)$$

(ii) if $0 < p < 1$*, or* $p < 0$*, there exists a constant* $\delta_0 > 0$*, such that for any* $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$*,* $k(\tilde{\sigma}) \in \mathbf{R}$*, then we still have the equivalent reverses of (26) and (27) with the same best constant factor* $K(\sigma)$*.*

Proof. (i) For $p > 1$, by the conditions, we can prove that (22) takes the form of strict inequality for a.e. $y \in \mathbf{R}_+^{j_0}$. Otherwise, if (22) takes the form of equality for a $y \in \mathbf{R}_+^{j_0}$, then there exist constants A and B, which are not all zero, such that

$$A\frac{||x||_\alpha^{(i_0-\delta\sigma)(p-1)}}{||y||_\beta^{j_0-\sigma}} f^p(x) = B\frac{||y||_\beta^{(j_0-\sigma)(q-1)}}{||x||_\alpha^{i_0-\delta\sigma}} \text{ a.e. in } x \in \mathbf{R}_+^{i_0}. \quad (28)$$

If $A = 0$, then $B = 0$, which is impossible; if $A \neq 0$, then (28) reduces to

$$||x||_\alpha^{p(i_0-\delta\sigma)-i_0} f^p(x) = \frac{B||y||_\beta^{q(j_0-\sigma)}}{A||x||_\alpha^{i_0}} \text{ a.e. in } x \in \mathbf{R}_+^{i_0},$$

which contradicts the fact that $0 < ||f||_{p,\Phi_\delta} < \infty$. In fact, by (9) (for $\varepsilon \to 0^+$), it follows

$$\int_{\mathbf{R}_+^{i_0}} ||x||_\alpha^{-i_0} dx \geq \int_{\left\{x \in \mathbf{R}_+^{i_0}; ||x||_\alpha \geq 1\right\}} ||x||_\alpha^{-i_0} dx = \infty.$$

Hence (22) still takes the form of strict inequality. By (14) and (15), we obtain (27).

Similarly to (25), we still have

$$I \leq J \left\{ \int_{\mathbf{R}_+^{j_0}} ||y||_\beta^{q(j_0-\sigma)-j_0} g^q(y) dy \right\}^{\frac{1}{q}}. \tag{29}$$

Then by (29) and (27), we have (26). It is evident that by Lemma 5 and the assumptions, inequalities (27) and (26) are also equivalent.

For $\varepsilon > 0$, we set $\tilde{f}(x), \tilde{g}(y)$ as follows:

$$\tilde{f}(x) := \begin{cases} 0, & 0 < ||x||_\alpha^\delta < 1, \\ ||x||_\alpha^{\delta\left(\sigma - \frac{\varepsilon}{p}\right) - i_0}, & ||x||_\alpha^\delta \geq 1, \end{cases}$$

$$\tilde{g}(y) := \begin{cases} ||y||_\beta^{\sigma + \frac{\varepsilon}{q} - j_0}, & 0 < ||y||_\beta \leq 1, \\ 0, & ||y||_\beta \geq 1. \end{cases}$$

In view of (11) and (10), it follows

$$\begin{aligned} &||\tilde{f}||_{p,\Phi_\delta} ||\tilde{g}||_{q,\Psi} \\ &= \left\{ \int_{\left\{x \in \mathbf{R}_+^{i_0}; ||x||_\alpha^\delta \geq 1\right\}} ||x||_\alpha^{-i_0-\delta\varepsilon} dx \right\}^{\frac{1}{p}} \left\{ \int_{\left\{y \in \mathbf{R}_+^{j_0}; ||y||_\beta \leq 1\right\}} ||y||_\beta^{-j_0+\varepsilon} dy \right\}^{\frac{1}{q}} \\ &= \frac{1}{\varepsilon} \left\{ \frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)} \right\}^{\frac{1}{p}} \left\{ \frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)} \right\}^{\frac{1}{q}}. \end{aligned}$$

If there exists a constant $K \leq K(\sigma)$, such that (26) is valid when replacing $K(\sigma)$ by K, then in particular, by (16) and (17), we have

$$\begin{aligned}
\tilde{K}(\sigma) + o(1) &\leq \varepsilon \tilde{I} \\
&= \varepsilon \int_{\mathbf{R}_+^{j_0}} \int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) \tilde{f}(x)\tilde{g}(y)dxdy \\
&< \varepsilon K ||\tilde{f}||_{p,\Phi_\delta} ||\tilde{g}||_{q,\Psi} \\
&= K \left\{ \frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)} \right\}^{\frac{1}{p}} \left\{ \frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)} \right\}^{\frac{1}{q}},
\end{aligned}$$

and then we find $K(\sigma) \leq K(\varepsilon \to 0^+)$. Hence $K = K(\sigma)$ is the best possible constant factor of (26).

By the equivalency, we can prove that the constant factor $K(\sigma)$ in (27) is the best possible. Otherwise, we would reach a contradiction by (29) that the constant factor $K(\sigma)$ in (26) is not the best possible.

(ii) For $0 < p < 1$, or $p < 0$, by the same way, we still can obtain the equivalent reverses of (26) and (27). For $\varepsilon > 0$, we set $\tilde{f}(x), \tilde{g}(y)$ as the case of $p > 1$. If there exists a constant $K \geq K(\sigma)$, such that the reverse of (26) is valid when replacing $K(\sigma)$ by K, then in particular, by (16) and (19), we have

$$\begin{aligned}
\tilde{K}(\sigma) + o(1) &= \varepsilon \tilde{I} \\
&= \varepsilon \int_{\mathbf{R}_+^{j_0}} \int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) \tilde{f}(x)\tilde{g}(y)dxdy \\
&> \varepsilon K ||\tilde{f}||_{p,\Phi_\delta} ||\tilde{g}||_{q,\Psi} \\
&= K \left\{ \frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)} \right\}^{\frac{1}{p}} \left\{ \frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)} \right\}^{\frac{1}{q}},
\end{aligned}$$

and then we find $K(\sigma) \geq K(\varepsilon \to 0^+)$. Hence $K = K(\sigma)$ is the best possible constant factor of the reverse of (26). By the equivalency, we can prove that the constant factor $K(\sigma)$ in the reverse of (27) is the best possible. Otherwise, we would reach a contradiction by the reverse of (29) that the constant factor $K(\sigma)$ in the reverse of (26) is not the best possible. The theorem is proved.

In particular, for $\delta = 1$ in Theorem 1, we have

Corollary 1. *Suppose that* $\alpha, \beta > 0,\ \sigma \in \mathbf{R}, h(v) \geq 0,$

$$k(\sigma) = \int_0^\infty h(v)v^{\sigma-1}dv \in \mathbf{R}_+,$$

$$K(\sigma) = \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)} \right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)} \right]^{\frac{1}{q}} k(\sigma),$$

$p \in \mathbf{R}\backslash\{0,1\}$, $\frac{1}{p}+\frac{1}{q}=1$, $f(x)=f(x_1,\ldots,x_{i_0})\geq 0$, $g(y)=g(y_1,\ldots,y_{j_0})\geq 0$,

$$0<||f||_{p,\Phi_1}=\left\{\int_{\mathbf{R}_+^{i_0}}\Phi_1(x)f^p(x)dx\right\}^{\frac{1}{p}}<\infty,$$

$$0<||g||_{q,\Psi}=\left\{\int_{\mathbf{R}_+^{j_0}}\Psi(y)g^q(y)dy\right\}^{\frac{1}{q}}<\infty.$$

(i) If $p>1$, then we have the following equivalent inequalities with the best possible constant factor $K(\sigma)$:

$$I=\int_{\mathbf{R}_+^{j_0}}\int_{\mathbf{R}_+^{i_0}}h(||x||_\alpha||y||_\beta)f(x)g(y)dxdy<K(\sigma)||f||_{p,\Phi_1}||g||_{q,\Psi},\quad (30)$$

$$J:=\left\{\int_{\mathbf{R}_+^{j_0}}||y||_\beta^{p\sigma-j_0}\left(\int_{\mathbf{R}_+^{i_0}}h(||x||_\alpha||y||_\beta)f(x)dx\right)^p dy\right\}^{\frac{1}{p}}$$
$$<K(\sigma)||f||_{p,\Phi_1};\quad (31)$$

(ii) if $0<p<1$, or $p<0$, there exists a constant $\delta_0>0$, such that for any $\tilde{\sigma}\in(\sigma-\delta_0,\sigma+\delta_0)$, $k(\tilde{\sigma})\in\mathbf{R}$, then we still have the equivalent reverses of (30) and (31) with the same best constant factor $K(\sigma)$.

For $i_0=j_0=\alpha=\beta=1$ in Corollary 1, we have

Corollary 2. *Assuming that $\sigma\in\mathbf{R}, k(\sigma)\in\mathbf{R}_+, p\in\mathbf{R}\backslash\{0,1\}, \frac{1}{p}+\frac{1}{q}=1$, we set*

$$\varphi(x):=x^{p(1-\sigma)-1},\psi(y):=y^{q(1-\sigma)-1}(x,y>0).$$

If $f(x)\geq 0$, $g(y)\geq 0$,

$$0<||f||_{p,\varphi}=\left\{\int_0^\infty\varphi(x)f^p(x)dx\right\}^{\frac{1}{p}}<\infty,$$

$$0<||g||_{q,\psi}=\left\{\int_0^\infty\psi(y)g^q(y)dy\right\}^{\frac{1}{q}}<\infty,$$

then (i) for $p>1$, we have the following equivalent inequalities with the best possible constant factor $k(\sigma)$:

$$\int_0^\infty\int_0^\infty h(xy)f(x)g(y)dxdy<k(\sigma)||f||_{p,\varphi}||g||_{q,\psi},\quad (32)$$

$$\left\{\int_0^\infty y^{p\sigma-1}\left[\int_0^\infty h(xy)f(x)dx\right]^p dy\right\}^{\frac{1}{p}} < k(\sigma)||f||_{p,\varphi}; \tag{33}$$

(ii) for $0 < p < 1$*, or* $p < 0$*, there exists a constant* $\delta_0 > 0$*, such that for any* $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$*,* $k(\tilde{\sigma}) \in \mathbf{R}$*, we have the equivalent reverses of (32) and (33) with the same best constant factor.*

As the assumptions of Theorem 1, for $p > 1$, in view of $J < K(\sigma)||f||_{\Phi_\delta}$, we can give the following definition:

Definition 2. Define a multidimensional Hilbert-type integral operator

$$T : \mathbf{L}_{p,\Phi_\delta}(\mathbf{R}_+^{i_0}) \to \mathbf{L}_{p,\Psi^{1-p}}(\mathbf{R}_+^{j_0}) \tag{34}$$

as follows: For $f \in \mathbf{L}_{p,\Phi_\delta}(\mathbf{R}_+^{i_0})$, there exists a unique representation

$$Tf \in \mathbf{L}_{p,\Psi^{1-p}}(\mathbf{R}_+^{j_0}),$$

satisfying

$$(Tf)(y) := \int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) f(x) dx (y \in \mathbf{R}_+^{j_0}). \tag{35}$$

For $g \in \mathbf{L}_{q,\Psi}(\mathbf{R}_+^{j_0})$, we define the following formal inner product of Tf and g as follows:

$$(Tf, g) := \int_{\mathbf{R}_+^{j_0}} \int_{\mathbf{R}_+^{i_0}} h(||x||_\alpha^\delta ||y||_\beta) f(x) g(y) dx dy. \tag{36}$$

Then by Theorem 1, for $p > 1, 0 < ||f||_{p,\Phi_\delta}, ||g||_{q,\Psi} < \infty$, we have the following equivalent inequalities:

$$(Tf, g) < K(\sigma)||f||_{p,\Phi_\delta}||g||_{q,\Psi}, \tag{37}$$

$$||Tf||_{p,\Psi^{1-p}} < K(\sigma)||f||_{p,\Phi_\delta}. \tag{38}$$

It follows that T is bounded with

$$||T|| := \sup_{f(\neq\theta)\in\mathbf{L}_{p,\Phi_\delta}(\mathbf{R}_+^{i_0})} \frac{||Tf||_{p,\Psi^{1-p}}}{||f||_{p,\Phi_\delta}} \le K(\sigma).$$

Since the constant factor $K(\sigma)$ in (38) is the best possible, we have

$$||T|| = K(\sigma) = \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)} \right]^{\frac{1}{p}} \times \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)} \right]^{\frac{1}{q}} k(\sigma). \tag{39}$$

4 A Corollary for $\delta = -1$

Corollary 3. *Suppose that* $\alpha, \beta > 0,\ \mu, \sigma \in \mathbf{R}, \mu + \sigma = \lambda, k_\lambda(x, y) \geq 0$ *is a homogeneous function of degree* $-\lambda$*,*

$$k_\lambda(\sigma) := \int_0^\infty k_\lambda(1, v) v^{\sigma-1} dv \in \mathbf{R}_+,$$

$$K_\lambda(\sigma) := \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)} \right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)} \right]^{\frac{1}{q}} k_\lambda(\sigma),$$

$p \in \mathbf{R}\backslash\{0, 1\},\ \frac{1}{p} + \frac{1}{q} = 1, \Phi(x) := x^{p(i_0-\mu)-i_0}, F(x) = F(x_1, \ldots, x_{i_0}) \geq 0,$ $g(y) = g(y_1, \ldots, y_{j_0}) \geq 0,$

$$0 < ||F||_{p,\Phi} = \left\{ \int_{\mathbf{R}_+^{i_0}} \Phi(x) F^p(x) dx \right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||g||_{q,\Psi} = \left\{ \int_{\mathbf{R}_+^{j_0}} \Psi(y) g^q(y) dy \right\}^{\frac{1}{q}} < \infty.$$

(i) If $p > 1$*, then we have the following equivalent inequalities with the best possible constant factor* $K(\sigma)$:

$$\int_{\mathbf{R}_+^{j_0}} \int_{\mathbf{R}_+^{i_0}} k_\lambda(||x||_\alpha, ||y||_\beta) F(x) g(y) dx dy < K_\lambda(\sigma) ||F||_{p,\Phi} ||g||_{q,\Psi}, \tag{40}$$

$$\left\{ \int_{\mathbf{R}_+^{j_0}} ||y||_\beta^{p\sigma-j_0} \left(\int_{\mathbf{R}_+^{i_0}} k_\lambda(||x||_\alpha, ||y||_\beta) F(x) dx \right)^p dy \right\}^{\frac{1}{p}} < K_\lambda(\sigma) ||F||_{p,\Phi}; \tag{41}$$

(ii) if $0 < p < 1$, *or* $p < 0$, *there exists a constant* $\delta_0 > 0$, *such that for any* $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, $k_\lambda(\tilde{\sigma}) \in \mathbf{R}$, *then we still have the equivalent reverses of (40) and (41) with the same best constant factor* $K_\lambda(\sigma)$.

In particular, for $i_0 = j_0 = \alpha = \beta = 1, \varphi_1(x) := x^{p(1-\mu)-1}$, if $F(x) \geq 0$, $g(y) \geq 0$,

$$0 < ||F||_{p,\varphi_1} = \left\{ \int_0^\infty \varphi_1(x) F^p(x) dx \right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||g||_{q,\psi} = \left\{ \int_0^\infty \psi(y) g^q(y) dy \right\}^{\frac{1}{q}} < \infty,$$

then (i) for $p > 1$, we have the following equivalent inequalities with the best possible constant factor $k_\lambda(\sigma)$:

$$\int_0^\infty \int_0^\infty k_\lambda(x,y) F(x) g(y) dx dy < k_\lambda(\sigma) ||F||_{p,\varphi_1} ||g||_{q,\psi}, \tag{42}$$

$$\left\{ \int_0^\infty y^{p\sigma-1} \left[\int_0^\infty k_\lambda(x,y) F(x) dx \right]^p dy \right\}^{\frac{1}{p}} < k_\lambda(\sigma) ||F||_{p,\varphi_1}; \tag{43}$$

(ii) for $0 < p < 1$, or $p < 0$, there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, $k_\lambda(\tilde{\sigma}) \in \mathbf{R}$, we have the equivalent reverses of (42) and (43) with the same best constant factor $k_\lambda(\sigma)$.

Proof. For $\delta = -1$ in Theorem 1, setting $h(u) = k_\lambda(1,u)$ and $||x||_\alpha^\lambda f(x) = F(x)$, since $\mu = \lambda - \sigma$, by simplifications, we can obtain (40) and (41) (for $p > 1$). It is evident that (40) and (41) are equivalent with the same best constant factor $K_\lambda(\sigma)$. By the same way, we can show the cases in $0 < p < 1$ or $p < 0$. The corollary is proved.

Remark 1. Inequality (42), (43) is equivalent to (32), (33). In fact, Setting $x = \frac{1}{X}, h(u) = k_\lambda(1,u)$ in (32), (33), replacing $X^\lambda f(\frac{1}{X})$ by $F(X)$, by simplification, we obtain (42), (43). On the other hand, by (42), (43), we can deduce (32), (33).

5 Two Classes of Hardy-Type Inequalities

If $h(v) = 0 (v > 1)$, then

$$h(||x||_\alpha^\delta ||y||_\beta) = 0(||x||_\alpha^\delta > ||y||_\beta^{-1}),$$

by Theorem 1, we have the following first class of Hardy-type inequalities:

Corollary 4. *Suppose that* $\alpha, \beta > 0,\ \sigma \in \mathbf{R}, h(v) \geq 0,$

$$k_1(\sigma) := \int_0^1 h(v) v^{\sigma-1} dv \in \mathbf{R}_+,$$

$$H_1(\sigma) := \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)} \right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)} \right]^{\frac{1}{q}} k_1(\sigma),$$

$\delta \in \{-1, 1\}, p \in \mathbf{R}\backslash\{0, 1\},\ \frac{1}{p} + \frac{1}{q} = 1,\ f(x) = f(x_1, \ldots, x_{i_0}) \geq 0,\ g(y) = g(y_1, \ldots, y_{j_0}) \geq 0,$

$$0 < ||f||_{p,\Phi_\delta} = \left\{ \int_{\mathbf{R}_+^{i_0}} \Phi_\delta(x) f^p(x) dx \right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||g||_{q,\Psi} = \left\{ \int_{\mathbf{R}_+^{j_0}} \Psi(y) g^q(y) dy \right\}^{\frac{1}{q}} < \infty.$$

(i) If $p > 1$*, then we have the following equivalent inequalities with the best possible constant factor* $H_1(\sigma)$*:*

$$\int_{\mathbf{R}_+^{j_0}} \left[\int_{\left\{x \in \mathbf{R}_+^{i_0}; ||x||_\alpha^\delta \leq ||y||_\beta^{-1}\right\}} h(||x||_\alpha^\delta ||y||_\beta) f(x) dx \right] g(y) dy < H_1(\sigma) ||f||_{p,\Phi_\delta} ||g||_{q,\Psi}, \tag{44}$$

$$\left\{ \int_{\mathbf{R}_+^{j_0}} ||y||_\beta^{p\sigma - j_0} \left(\int_{\left\{x \in \mathbf{R}_+^{i_0}; ||x||_\alpha^\delta \leq ||y||_\beta^{-1}\right\}} h\left(||x||_\alpha^\delta ||y||_\beta\right) f(x) dx \right)^p dy \right\}^{\frac{1}{p}} < H_1(\sigma) ||f||_{p,\Phi_\delta}; \tag{45}$$

(ii) If $0 < p < 1$*, or* $p < 0$*, there exists a constant* $\delta_0 > 0$*, such that for any* $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$*,* $k_1(\tilde{\sigma}) \in \mathbf{R}$*, then we still have the equivalent reverses of (44) and (45) with the same best constant factor* $H_1(\sigma)$*.*

For $i_0 = j_0 = \alpha = \beta = 1, \delta = 1$ in Corollary 4, we have

Corollary 5. *Assuming that* $\sigma \in \mathbf{R}, k_1(\sigma) \in \mathbf{R}_+, p \in \mathbf{R}\backslash\{0, 1\}, \frac{1}{p} + \frac{1}{q} = 1$*, we set*

$$\varphi(x) := x^{p(1-\sigma)-1}, \psi(y) := y^{q(1-\sigma)-1} (x, y > 0).$$

If $f(x) \geq 0,\ g(y) \geq 0,$

$$0 < ||f||_{p,\varphi} = \left\{ \int_0^\infty \varphi(x) f^p(x) dx \right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||g||_{q,\psi} = \left\{ \int_0^\infty \psi(y) g^q(y) dy \right\}^{\frac{1}{q}} < \infty,$$

then (i) for $p > 1$, we have the following equivalent inequalities with the best possible constant factor $k_1(\sigma)$:

$$\int_0^\infty \left(\int_0^{\frac{1}{y}} h(xy) f(x) dx \right) g(y) dy < k_1(\sigma) ||f||_{p,\varphi} ||g||_{q,\psi}, \tag{46}$$

$$\left\{ \int_0^\infty y^{p\sigma-1} \left[\int_0^{\frac{1}{y}} h(xy) f(x) dx \right]^p dy \right\}^{\frac{1}{p}} < k_1(\sigma) ||f||_{p,\varphi}; \tag{47}$$

(ii) for $0 < p < 1$, or $p < 0$, there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, $k_1(\tilde{\sigma}) \in \mathbf{R}$, we have the equivalent reverses of (46) and (47) with the same best constant factor $k_1(\sigma)$.

If $k_\lambda(x, y) = 0 (x < y)$, by (42) and (43), we have

Corollary 6. *Assuming that $\mu, \sigma \in \mathbf{R}$, $\mu + \sigma = \lambda$,*

$$k_\lambda^{(1)}(\sigma) := \int_0^1 k_\lambda(1, v) v^{\sigma-1} dv \in \mathbf{R}_+,$$

$p \in \mathbf{R} \backslash \{0, 1\}$, $\frac{1}{p} + \frac{1}{q} = 1$, $\varphi_1(x) := x^{p(1-\mu)-1}$, *if* $F(x) \geq 0$, $g(y) \geq 0$,

$$0 < ||F||_{p,\varphi_1} = \left\{ \int_0^\infty \varphi_1(x) F^p(x) dx \right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||g||_{q,\psi} = \left\{ \int_0^\infty \psi(y) g^q(y) dy \right\}^{\frac{1}{q}} < \infty,$$

then (i) for $p > 1$, we have the following equivalent inequalities with the best possible constant factor $k_\lambda^{(1)}(\sigma)$:

$$\int_0^\infty \left[\int_y^\infty k_\lambda(x, y) F(x) dx \right] g(y) dy < k_\lambda^{(1)}(\sigma) ||F||_{p,\varphi_1} ||g||_{q,\psi}, \tag{48}$$

$$\left\{ \int_0^\infty y^{p\sigma-1} \left[\int_y^\infty k_\lambda(x, y) F(x) dx \right]^p dy \right\}^{\frac{1}{p}} < k_\lambda^{(1)}(\sigma) ||F||_{p,\varphi_1}; \tag{49}$$

(ii) for $0 < p < 1$, or $p < 0$, there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, $k_\lambda^{(1)}(\tilde{\sigma}) \in \mathbf{R}$, we have the equivalent reverses of (48) and (49) with the same best constant factor $k_\lambda^{(1)}(\sigma)$.

If $h(v) = 0 (0 < v < 1)$, then

$$h(||x||_\alpha^\delta ||y||_\beta) = 0(||x||_\alpha^\delta < ||y||_\beta^{-1}),$$

by Theorem 1, we have the following second class of Hardy-type inequalities:

Corollary 7. *Suppose that $\alpha, \beta > 0,\ \sigma \in \mathbf{R}, h(v) \geq 0,$*

$$k_2(\sigma) := \int_1^\infty h(v) v^{\sigma-1} dv \in \mathbf{R}_+,$$

$$H_2(\sigma) := \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{q}} k_2(\sigma),$$

$\delta \in \{-1, 1\}, p \in \mathbf{R}\backslash\{0, 1\}$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x) = f(x_1, \ldots, x_{i_0}) \geq 0$, $g(y) = g(y_1, \ldots, y_{j_0}) \geq 0$,

$$0 < ||f||_{p,\Phi_\delta} = \left\{\int_{\mathbf{R}_+^{i_0}} \Phi_\delta(x) f^p(x) dx\right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||g||_{q,\Psi} = \left\{\int_{\mathbf{R}_+^{j_0}} \Psi(y) g^q(y) dy\right\}^{\frac{1}{q}} < \infty.$$

(i) If $p > 1$, then we have the following equivalent inequalities with the best possible constant factor $H_2(\sigma)$:

$$\int_{\mathbf{R}_+^{j_0}} \left[\int_{\left\{x \in \mathbf{R}_+^{i_0}; ||x||_\alpha^\delta \geq ||y||_\beta^{-1}\right\}} h\left(||x||_\alpha^\delta ||y||_\beta\right) f(x) dx\right] g(y) dy \\ < H_2(\sigma) ||f||_{p,\Phi_\delta} ||g||_{q,\Psi}, \tag{50}$$

$$\left\{\int_{\mathbf{R}_+^{j_0}} ||y||_\beta^{p\sigma - j_0} \left(\int_{\left\{x \in \mathbf{R}_+^{i_0}; ||x||_\alpha^\delta \geq ||y||_\beta^{-1}\right\}} h(||x||_\alpha^\delta ||y||_\beta) f(x) dx\right)^p dy\right\}^{\frac{1}{p}} \\ < H_2(\sigma) ||f||_{p,\Phi_\delta}; \tag{51}$$

(ii) if $0 < p < 1$, *or* $p < 0$, *there exists a constant* $\delta_0 > 0$, *such that for any* $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, $k_2(\tilde{\sigma}) \in \mathbf{R}$, *then we still have the equivalent reverses of (50) and (51) with the same best constant factor* $H_2(\sigma)$.

For $i_0 = j_0 = \alpha = \beta = 1, \delta = 1$ in Corollary 7, we have

Corollary 8. *Assuming that* $\sigma \in \mathbf{R}, k_2(\sigma) \in \mathbf{R}_+, p \in \mathbf{R}\backslash\{0, 1\}, \frac{1}{p} + \frac{1}{q} = 1$, *we set*

$$\varphi(x) = x^{p(1-\sigma)-1}, \psi(y) = y^{q(1-\sigma)-1}(x, y > 0).$$

If $f(x) \geq 0$, $g(y) \geq 0$,

$$0 < ||f||_{p,\varphi} = \left\{\int_0^\infty \varphi(x) f^p(x) dx\right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||g||_{q,\psi} = \left\{\int_0^\infty \psi(y) g^q(y) dy\right\}^{\frac{1}{q}} < \infty.$$

then (i) for $p > 1$, *we have the following equivalent inequalities with the best possible constant factor* $k_2(\sigma)$:

$$\int_0^\infty \left(\int_{\frac{1}{y}}^\infty h(xy) f(x) dx\right) g(y) dy < k_2(\sigma)||f||_{p,\varphi}||g||_{q,\psi}, \tag{52}$$

$$\left\{\int_0^\infty y^{p\sigma-1}\left[\int_{\frac{1}{y}}^\infty h(xy) f(x) dx\right]^p dy\right\}^{\frac{1}{p}} < k_2(\sigma)||f||_{p,\varphi}; \tag{53}$$

(ii) for $0 < p < 1$, *or* $p < 0$, *there exists a constant* $\delta_0 > 0$, *such that for any* $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, $k_2(\tilde{\sigma}) \in \mathbf{R}$, *we have the equivalent reverses of (52) and (53) with the same best constant factor* $k_2(\sigma)$.

If $k_\lambda(x, y) = 0(x > y)$, by (42) and (43), we have

Corollary 9. *Assuming that* $\mu, \sigma \in \mathbf{R}, \mu + \sigma = \lambda$,

$$k_\lambda^{(2)}(\sigma) := \int_1^\infty k_\lambda(1, v) v^{\sigma-1} dv \in \mathbf{R}_+,$$

$p \in \mathbf{R}\backslash\{0, 1\}, \frac{1}{p} + \frac{1}{q} = 1, \varphi_1(x) := x^{p(1-\mu)-1}$, *if* $F(x) \geq 0$, $g(y) \geq 0$,

$$0 < ||F||_{p,\varphi_1} = \left\{\int_0^\infty \varphi_1(x) F^p(x) dx\right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||g||_{q,\psi} = \left\{\int_0^\infty \psi(y) g^q(y) dy\right\}^{\frac{1}{q}} < \infty,$$

then (i) for $p > 1$, we have the following equivalent inequalities with the best possible constant factor $k_\lambda^{(2)}(\sigma)$:

$$\int_0^\infty \left[\int_0^y k_\lambda(x,y)F(x)dx\right] g(y)dy < k_\lambda^{(2)}(\sigma)||F||_{p,\varphi_1}||g||_{q,\psi}, \tag{54}$$

$$\left\{\int_0^\infty y^{p\sigma-1}\left[\int_0^y k_\lambda(x,y)F(x)dx\right]^p dy\right\}^{\frac{1}{p}} < k_\lambda^{(2)}(\sigma)||F||_{p,\varphi_1}; \tag{55}$$

(ii) for $0 < p < 1$, or $p < 0$, there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, $k_\lambda^{(2)}(\tilde{\sigma}) \in \mathbf{R}$, we have the equivalent reverses of (54) and (55) with the same best constant factor $k_\lambda^{(2)}(\sigma)$.

6 Multidimensional Hilbert-Type Inequalities with Two Variables

Suppose that $u_i(s_i), u_i'(s_i) > 0, u_i(a_i^+) = 0, u_i(b_i^-) = \infty(-\infty \le a_i < b_i \le \infty, i = 1,\ldots,i_0)$, $u(s) = (u_1(s_1),\ldots,u_{i_0}(s_{i_0})), v_j(t_j), v_j'(t_j) > 0, v_j(c_j^+) = 0, v_j(d_j^-) = \infty \quad (-\infty \le c_j < d_j \le \infty, j = 1,\ldots,j_0)$,
$v(t) = (v_1(t_1),\ldots,v_{j_0}(t_{j_0}))$,

$$\tilde{\Phi}_\delta(s) := \frac{||u(s)||_\alpha^{p(i_0-\delta\sigma)-i_0}}{\left[\Pi_{i=1}^{i_0} u_i'(s_i)\right]^{p-1}}, \tilde{\Psi}(t) := \frac{||v(t)||_\alpha^{q(j_0-\sigma)-j_0}}{\left[\Pi_{j=1}^{j_0} v_j'(t_j)\right]^{q-1}}.$$

Setting $x = u(s), y = v(t)$ in Theorem 1, for

$$F(s) := \Pi_{i=1}^{i_0} u_i'(s_i) f(u(s)), G(t) := \Pi_{j=1}^{j_0} v_j'(t_j) g(v(t)),$$

we have

Theorem 2. *Suppose that $\alpha, \beta > 0$, $\sigma \in \mathbf{R}, h(v) \ge 0$,*

$$k(\sigma) = \int_0^\infty h(v)v^{\sigma-1}dv \in \mathbf{R}_+,$$

$$K(\sigma) = \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{q}} k(\sigma),$$

$\delta \in \{-1,1\}, p \in \mathbf{R}\backslash\{0,1\}$, $\frac{1}{p} + \frac{1}{q} = 1$, $F(s) = F(s_1,\ldots,s_{i_0}) \ge 0$, $G(t) = G(t_1,\ldots,t_{j_0}) \ge 0$,

$$0 < ||F||_{p,\tilde{\Phi}_\delta} = \left\{ \int_{\{s\in\mathbf{R}^{i_0};a_i<s_i<b_i\}} \tilde{\Phi}_\delta(s) F^p(s) ds \right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||G||_{q,\tilde{\Psi}} = \left\{ \int_{\{t\in\mathbf{R}^{j_0};c_j<t_j<d_j\}} \tilde{\Psi}(t) G^q(t) dt \right\}^{\frac{1}{q}} < \infty.$$

(i) *If $p > 1$, then we have the following equivalent inequalities with the best possible constant factor $K(\sigma)$:*

$$\int_{\{t\in\mathbf{R}^{j_0};c_j<t_j<d_j\}} \int_{\{s\in\mathbf{R}^{i_0};a_i<s_i<b_i\}} h(||u(s)||_\alpha^\delta ||v(t)||_\beta) F(s) G(t) ds dt < K(\sigma) ||F||_{p,\tilde{\Phi}_\delta} ||g||_{q,\tilde{\Psi}}, \tag{56}$$

$$\left\{ \int_{\{t\in\mathbf{R}^{j_0};c_j<t_j<d_j\}} ||v(t)||_\beta^{p\sigma-j_0} \Pi_{j=1}^{j_0} v_j'(t_j) \left(\int_{\{s\in\mathbf{R}^{i_0};a_i<s_i<b_i\}} h(||u(s)||_\alpha^\delta ||v(t)||_\beta) F(s) ds \right)^p dt \right\}^{\frac{1}{p}} < K(\sigma) ||F||_{p,\tilde{\Phi}_\delta}; \tag{57}$$

(ii) *if $0 < p < 1$, or $p < 0$, there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, $k(\tilde{\sigma}) \in \mathbf{R}$, then we still have the equivalent reverses of (56) and (57) with the same best constant factor $K(\sigma)$.*

In particular, for $i_0 = j_0 = \alpha = \beta = 1$,

$$\tilde{\phi}_\delta(s) := \frac{(u(s))^{p(1-\delta\sigma)-1}}{[u'(s)]^{p-1}}, \tilde{\psi}(t) := \frac{(v(t))^{q(1-\sigma)-1}}{[v'(t)]^{q-1}},$$

$$0 < ||F||_{p,\tilde{\phi}_\delta} = \left\{ \int_a^b \tilde{\phi}_\delta(s) F^p(s) ds \right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||G||_{q,\tilde{\psi}} = \left\{ \int_c^d \tilde{\Psi}(t) G^q(t) dt \right\}^{\frac{1}{q}} < \infty,$$

(i) if $p > 1$, then we have the following equivalent inequalities with the best possible constant factor $k(\sigma)$:

$$\int_c^d \int_a^b h(u^\delta(s)v(t)) F(s) G(t) ds dt < k(\sigma) ||F||_{p,\tilde{\phi}_\delta} ||G||_{q,\tilde{\psi}}, \tag{58}$$

$$\left\{\int_c^d (v(t))^{p\sigma-1}v'(t)\left(\int_a^b h(u^\delta(s)v(t))F(s)ds\right)^p dt\right\}^{\frac{1}{p}} < k(\sigma)||F||_{p,\tilde{\Phi}_\delta}; \tag{59}$$

(ii) if $0 < p < 1$, or $p < 0$, there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, $k(\tilde{\sigma}) \in \mathbf{R}$, then we still have the equivalent reverses of (58) and (59) with the same best constant factor $k(\sigma)$.

In particular, for $\gamma, \eta > 0, u_i(s_i) = s_i^\gamma, u_i'(s_i) = \gamma s_i^{\gamma-1}, u_i(0^+) = 0, u_i(\infty) = \infty (a_i = 0, b_i = \infty, i = 1, \ldots, i_0)$, $\hat{u}(s) = (s_1^\gamma, \ldots, s_{i_0}^\gamma), v_j(t_j) = t_j^\eta, v_j'(t_j) = \eta t_j^{\eta-1}, v_j(0^+) = 0, v_j(\infty) = \infty (c_j = 0, d_j = \infty, j = 1, \ldots, j_0)$, $\hat{v}(t) = (t_1^\eta, \ldots, t_{j_0}^\eta)$, and

$$\tilde{\Phi}_\delta(s) = \frac{1}{\gamma^{i_0(p-1)}}\hat{\Phi}_\delta(s), \hat{\Phi}_\delta(s) := \frac{||\hat{u}(s)||_\alpha^{p(i_0-\delta\sigma)-i_0}}{\left(\Pi_{i=1}^{i_0} s_i^{\gamma-1}\right)^{p-1}},$$

$$\tilde{\Psi}(t) = \frac{1}{\eta^{j_0(q-1)}}\hat{\Psi}(t), \hat{\Psi}(t) := \frac{||\hat{v}(t)||_\alpha^{q(j_0-\sigma)-j_0}}{\left(\Pi_{j=1}^{j_0} t_j^{\eta-1}\right)^{q-1}}$$

in Theorem 2, we have

Corollary 10. *Suppose that* $\alpha, \beta, \gamma, \eta > 0,\ \sigma \in \mathbf{R}, h(v) \geq 0$,

$$k(\sigma) = \int_0^\infty h(v)v^{\sigma-1}dv \in \mathbf{R}_+,$$

$$K(\sigma) = \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}}\left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{q}} k(\sigma),$$

$\delta \in \{-1, 1\}, p \in \mathbf{R}\backslash\{0, 1\},\ \frac{1}{p} + \frac{1}{q} = 1,\ F(s) = F(s_1, \ldots, s_{i_0}) \geq 0,\ G(t) = G(t_1, \ldots, t_{j_0}) \geq 0,$

$$0 < ||F||_{p,\hat{\Phi}_\delta} = \left\{\int_{\mathbf{R}_+^{i_0}} \hat{\Phi}_\delta(s)F^p(s)ds\right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||G||_{q,\hat{\Psi}} = \left\{\int_{\mathbf{R}_+^{j_0}} \hat{\Psi}(t)G^q(t)dt\right\}^{\frac{1}{q}} < \infty.$$

(i) If $p > 1$, then we have the following equivalent inequalities with the best possible constant factor $\frac{1}{\gamma^{i_0/q}\eta^{j_0/p}}K(\sigma)$:

$$\int_{\mathbf{R}_+^{j_0}}\int_{\mathbf{R}_+^{i_0}} h(||\hat{u}(s)||_\alpha^\delta||\hat{v}(t)||_\beta)F(s)G(t)dsdt < \frac{1}{\gamma^{i_0/q}\eta^{j_0/p}}K(\sigma)||F||_{p,\hat{\Phi}_\delta}||G||_{q,\hat{\Psi}}, \tag{60}$$

$$\left\{\int_{\mathbf{R}_+^{j_0}}||\hat{v}(t)||_\beta^{p\sigma-j_0}\Pi_{j=1}^{j_0}t_j^{\eta-1}\left(\int_{\mathbf{R}_+^{i_0}}h(||\hat{u}(s)||_\alpha^\delta||\hat{v}(t)||_\beta) F(s)ds\right)^p dt\right\}^{\frac{1}{p}} < \frac{1}{\gamma^{i_0/q}\eta^{j_0/p}}K(\sigma)||F||_{p,\hat{\Phi}_\delta}; \tag{61}$$

(ii) if $0 < p < 1$, or $p < 0$, there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, $k(\tilde{\sigma}) \in \mathbf{R}$, then we still have the equivalent reverses of (60) and (61) with the same best constant factor $\frac{1}{\gamma^{i_0/q}\eta^{j_0/p}}K(\sigma)$.

In particular, for $i_0 = j_0 = \alpha = \beta = 1$,

$$\hat{\phi}_\delta(s) := s^{p(1-\delta\gamma\sigma)-1}, \hat{\psi}(t) := t^{q(1-\eta\sigma)-1},$$

$$0 < ||F||_{p,\hat{\phi}_\delta} = \left\{\int_0^\infty \hat{\phi}_\delta(s)F^p(s)ds\right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||G||_{q,\hat{\psi}} = \left\{\int_0^\infty \hat{\psi}(t)G^q(t)dt\right\}^{\frac{1}{q}} < \infty,$$

(i) if $p > 1$, then we have the following equivalent inequalities with the best possible constant factor $\frac{1}{\gamma^{1/q}\eta^{1/p}}k(\sigma)$:

$$\int_0^\infty\int_0^\infty h(s^{\gamma\delta}t^\eta)F(s)G(t)dsdt < \frac{1}{\gamma^{1/q}\eta^{1/p}}k(\sigma)||F||_{p,\hat{\phi}_\delta}||G||_{q,\hat{\psi}}, \tag{62}$$

$$\left\{\int_0^\infty t^{p\eta\sigma-1}\left(\int_0^\infty h(s^{\gamma\delta}t^\eta)F(s)ds\right)^p dt\right\}^{\frac{1}{p}} < \frac{1}{\gamma^{1/q}\eta^{1/p}}k(\sigma)||F||_{p,\hat{\phi}_\delta}; \tag{63}$$

(ii) if $0 < p < 1$, or $p < 0$, there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, $k(\tilde{\sigma}) \in \mathbf{R}$, then we still have the equivalent reverses of (62) and (63) with the same best constant factor $\frac{1}{\gamma^{1/q}\eta^{1/p}}k(\sigma)$.

For $\delta = -1, h(u) = k_\lambda(1,u), ||u(s)||_\alpha^\lambda F(s) = f(s), \mu = \lambda - \sigma$ and

$$\tilde{\Phi}(s) := \frac{||u(s)||_\alpha^{p(i_0-\mu)-i_0}}{\left[\Pi_{i=1}^{i_0} u_i'(s_i)\right]^{p-1}}$$

in Theorem 2, by simplifications, we have

Corollary 11. *Suppose that* $\alpha, \beta > 0,\ \lambda, \mu, \sigma \in \mathbf{R}, \mu + \sigma = \lambda, k_\lambda(x,y)(\geq 0)$ *is a homogeneous function of degree* $-\lambda$ *in* $\mathbf{R}_+^2$*, with*

$$k_\lambda(\sigma) = \int_0^\infty k_\lambda(1,v) v^{\sigma-1} dv \in \mathbf{R}_+,$$

$$K_\lambda(\sigma) = \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{q}} k_\lambda(\sigma),$$

$p \in \mathbf{R}\backslash\{0,1\},\ \frac{1}{p} + \frac{1}{q} = 1,\ f(s) = f(s_1, \ldots, s_{i_0}) \geq 0,\ G(t) = G(t_1, \ldots, t_{j_0}) \geq 0,$

$$0 < ||f||_{p,\tilde{\Phi}} = \left\{\int_{\left\{s\in\mathbf{R}^{i_0};a_i<s_i<b_i\right\}} \tilde{\Phi}(s) f^p(s) ds\right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||G||_{q,\tilde{\Psi}} = \left\{\int_{\left\{t\in\mathbf{R}^{j_0};c_j<t_j<d_j\right\}} \tilde{\Psi}(t) G^q(t) dt\right\}^{\frac{1}{q}} < \infty.$$

(i) If $p > 1$*, then we have the following equivalent inequalities with the best possible constant factor* $K(\sigma)$ *:*

$$\int_{\left\{t\in\mathbf{R}^{j_0};c_j<t_j<d_j\right\}} \int_{\left\{s\in\mathbf{R}^{i_0};a_i<s_i<b_i\right\}} k_\lambda(||u(s)||_\alpha, ||v(t)||_\beta) f(s) G(t) ds dt < K_\lambda(\sigma) ||f||_{p,\tilde{\Phi}} ||G||_{q,\tilde{\Psi}}, \tag{64}$$

$$\left\{\int_{\left\{t\in\mathbf{R}^{j_0};c_j<t_j<d_j\right\}} ||v(t)||_\beta^{p\sigma-j_0} \Pi_{j=1}^{j_0} v_j'(t_j) \left(\int_{\left\{s\in\mathbf{R}^{i_0};a_i<s_i<b_i\right\}} k_\lambda(||u(s)||_\alpha, ||v(t)||_\beta) \times f(s) ds\right)^p dt\right\}^{\frac{1}{p}} < K_\lambda(\sigma) ||f||_{p,\tilde{\Phi}}; \tag{65}$$

(ii) if $0 < p < 1$*, or* $p < 0$*, there exists a constant* $\delta_0 > 0$*, such that for any* $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$*,* $k_\lambda(\tilde{\sigma}) \in \mathbf{R}$*, then we still have the equivalent reverses of (64) and (65) with the same best constant factor* $K_\lambda(\sigma)$*.*

In particular, for $i_0 = j_0 = \alpha = \beta = 1$,

$$\tilde{\phi}(s) := \frac{(u(s))^{p(1-\mu)-1}}{[u'(s)]^{p-1}}, \tilde{\psi}(t) = \frac{(v(t))^{q(1-\sigma)-1}}{[v'(t)]^{q-1}},$$

$$0 < ||f||_{p,\tilde{\phi}} = \left\{\int_a^b \tilde{\phi}(s) f^p(s) ds\right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||G||_{q,\tilde{\psi}} = \left\{\int_c^d \tilde{\Psi}(t) G^q(t) dt\right\}^{\frac{1}{q}} < \infty,$$

(i) if $p > 1$, then we have the following equivalent inequalities with the best possible constant factor $k_\lambda(\sigma)$:

$$\int_c^d \int_a^b k_\lambda(u(s), v(t)) f(s) G(t) ds dt < k_\lambda(\sigma) ||f||_{p,\tilde{\phi}} ||G||_{q,\tilde{\psi}}, \quad (66)$$

$$\left\{\int_c^d (v(t))^{p\sigma-1} v'(t) \left(\int_a^b k_\lambda(u(s), v(t)) f(s) ds\right)^p dt\right\}^{\frac{1}{p}} < k_\lambda(\sigma) ||f||_{p,\tilde{\phi}}; \quad (67)$$

(ii) if $0 < p < 1$, or $p < 0$, there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, $k_\lambda(\tilde{\sigma}) \in \mathbf{R}$, then we still have the equivalent reverses of (66) and (67) with the same best constant factor $k_\lambda(\sigma)$.

In particular, for $u_i(s_i) = \ln s_i, u_i'(s_i) = s_i^{-1}, u_i(1^+) = 0, u_i(\infty) = \infty (a_i = 1, b_i = \infty, i = 1, \ldots, i_0)$, $U(s) = (\ln s_1, \ldots, \ln s_{i_0})$, $v_j(t_j) = \ln t_j, v_j'(t_j) = t_j^{-1}, v_j(1^+) = 0, v_j(\infty) = \infty (c_j = 1, d_j = \infty, j = 1, \ldots, j_0)$, $V(t) = (\ln t_1, \ldots, \ln t_{j_0})$, and

$$\tilde{\Phi}(s) = \hat{\Phi}(s) := \frac{||U(s)||_\alpha^{p(i_0-\mu)-i_0}}{\left(\Pi_{i=1}^{i_0} s_i\right)^{1-p}},$$

$$\tilde{\Psi}(t) = \hat{\Psi}(t) := \frac{||V(t)||_\alpha^{q(j_0-\sigma)-j_0}}{\left(\Pi_{j=1}^{j_0} t_j\right)^{1-q}}$$

in Corollary 10, we have

Corollary 12. *Suppose that* $\alpha, \beta > 0,\ \lambda, \mu, \sigma \in \mathbf{R}, \mu + \sigma = \lambda, k_\lambda(x, y) (\geq 0)$ *is a homogeneous function of degree* $-\lambda$ *in* $\mathbf{R}_+^2$*, with*

$$k_\lambda(\sigma) = \int_0^\infty k_\lambda(1, v) v^{\sigma-1} dv \in \mathbf{R}_+,$$

$$K_\lambda(\sigma) = \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{q}} k_\lambda(\sigma),$$

$p \in \mathbf{R}\backslash\{0,1\}$, $\frac{1}{p}+\frac{1}{q}=1$, $f(s)=f(s_1,\ldots,s_{i_0}) \geq 0$, $G(t)=G(t_1,\ldots,t_{j_0}) \geq 0$,

$$0 < ||f||_{p,\hat{\Phi}} = \left\{\int_{\{s\in\mathbf{R}^{i_0};1<s_i<\infty\}} \hat{\Phi}(s) f^p(s) ds\right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||G||_{q,\hat{\Psi}} = \left\{\int_{\{t\in\mathbf{R}^{j_0};1<t_j<\infty\}} \hat{\Psi}(t) G^q(t) dt\right\}^{\frac{1}{q}} < \infty.$$

(i) *If $p > 1$, then we have the following equivalent inequalities with the best possible constant factor $K(\sigma)$:*

$$\int_{\{t\in\mathbf{R}^{j_0};1<t_j<\infty\}} \int_{\{s\in\mathbf{R}^{i_0};1<s_i<\infty\}} k_\lambda(||U(s)||_\alpha, ||V(t)||_\beta) f(s) G(t) ds dt < K_\lambda(\sigma) ||f||_{p,\hat{\Phi}} ||G||_{q,\hat{\Psi}}, \tag{68}$$

$$\left\{\int_{\{t\in\mathbf{R}^{j_0};1<t_j<\infty\}} ||V(t)||_\beta^{p\sigma-j_0} \Pi_{j=1}^{j_0} t_j^{-1} \left(\int_{\{s\in\mathbf{R}^{i_0};1<s_i<\infty\}} k_\lambda(||U(s)||_\alpha, ||V(t)||_\beta) \times f(s) ds\right)^p dt\right\}^{\frac{1}{p}} < K_\lambda(\sigma) ||f||_{p,\hat{\Phi}}; \tag{69}$$

(ii) *if $0 < p < 1$, or $p < 0$, there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, $k_\lambda(\tilde{\sigma}) \in \mathbf{R}$, then we still have the equivalent reverses of (68) and (69) with the same best constant factor $K_\lambda(\sigma)$.*

In particular, for $i_0 = j_0 = \alpha = \beta = 1$,

$$\tilde{\phi}(s) = \hat{\phi}(s) := \frac{(\ln s)^{p(1-\mu)-1}}{s^{1-p}}, \tilde{\psi}(t) = \hat{\psi}(t) := \frac{(\ln t)^{q(1-\sigma)-1}}{t^{1-q}},$$

$$0 < ||f||_{p,\hat{\phi}} = \left\{\int_1^\infty \hat{\phi}(s) f^p(s) ds\right\}^{\frac{1}{p}} < \infty,$$

$$0 < ||G||_{q,\hat{\psi}} = \left\{\int_1^\infty \hat{\psi}(t) G^q(t) dt\right\}^{\frac{1}{q}} < \infty,$$

(i) if $p > 1$, then we have the following equivalent inequalities with the best possible constant factor $k_\lambda(\sigma)$:

$$\int_1^\infty \int_1^\infty k_\lambda(\ln s, \ln t) f(s) G(t) ds dt < k_\lambda(\sigma) ||f||_{p,\hat{\phi}} ||G||_{q,\hat{\psi}}, \tag{70}$$

$$\left\{ \int_1^\infty (\ln t)^{p\sigma - 1} \frac{1}{t} \left(\int_1^\infty k_\lambda(\ln s, \ln t) f(s) ds \right)^p dt \right\}^{\frac{1}{p}} < k_\lambda(\sigma) ||f||_{p,\hat{\phi}}; \tag{71}$$

(ii) if $0 < p < 1$, or $p < 0$, there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, $k_\lambda(\tilde{\sigma}) \in \mathbf{R}$, then we still have the equivalent reverses of (70) and (71) with the same best constant factor $k_\lambda(\sigma)$.

7 Some Particular Examples on the Norm

Example 1. For $h(v) = \frac{|\ln v|^\gamma}{(1+v)^\lambda} (\gamma \ge 0, \mu, \sigma > 0, \mu + \sigma = \lambda)$, we have

$$k(\sigma) = k_\gamma(\sigma) := \int_0^\infty \frac{|\ln v|^\gamma}{(1+v)^\lambda} v^{\sigma - 1} dv.$$

Since $\frac{|\ln v|^\gamma}{(1+v)^{\lambda/2}} v^{\frac{\sigma}{2}} \to 0 (v \to 0^+$ or $v \to \infty)$, there exists a constant number $L > 0$, such that

$$0 < \frac{|\ln v|^\gamma}{(1+v)^{\lambda/2}} v^{\frac{\sigma}{2}} \le L (v \in \mathbf{R}_+).$$

Then it follows that

$$0 < k_\gamma(\sigma) \le L \int_0^\infty \frac{v^{(\sigma/2)-1} dv}{(1+v)^{\lambda/2}} = LB\left(\frac{\sigma}{2}, \frac{\mu}{2}\right) < \infty,$$

and $k_\gamma(\sigma) \in \mathbf{R}_+$. We find

$$k_0(\sigma) = \int_0^\infty \frac{1}{(1+v)^\lambda} v^{\sigma-1} dv = B(\sigma, \mu). \tag{72}$$

For $\gamma \ge 0$, we obtain

$$\begin{aligned} k_\gamma(\sigma) &= \int_0^1 \frac{(-\ln v)^\gamma v^{\sigma-1}}{(1+v)^\lambda} dv + \int_1^\infty \frac{(\ln v)^\gamma v^{\sigma-1}}{(1+v)^\lambda} dv \\ &= \int_0^1 \frac{(-\ln v)^\gamma}{(1+v)^\lambda} \left(v^{\sigma-1} + v^{\mu-1} \right) dv \end{aligned}$$

$$= \int_0^1 (-\ln v)^\gamma \sum_{k=0}^\infty \binom{-\lambda}{k} \left(v^{k+\sigma-1} + v^{k+\mu-1}\right) dv$$
$$= \sum_{k=0}^\infty \binom{-\lambda}{k} \int_0^1 (-\ln v)^\gamma \left(v^{k+\sigma-1} + v^{k+\mu-1}\right) dv.$$

Setting $t = -\ln v$, we find

$$\begin{aligned} k_\gamma(\sigma) &= \sum_{k=0}^\infty \binom{-\lambda}{k} \int_0^\infty t^{(\gamma+1)-1} \left[e^{-t(k+\sigma)} + e^{-t(k+\mu)}\right] dt \\ &= \Gamma(\gamma+1) \sum_{k=0}^\infty \binom{-\lambda}{k} \left[\frac{1}{(k+\sigma)^{\gamma+1}} + \frac{1}{(k+\mu)^{\gamma+1}}\right]. \end{aligned} \tag{73}$$

In view of Theorem 1 and (39), we have

$$\begin{aligned} ||T|| = K_\gamma(\sigma) &:= \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \\ &\times \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{q}} k_\gamma(\sigma). \end{aligned} \tag{74}$$

Example 2. For $h(v) = \frac{|\ln v|^\gamma}{1+v^\lambda} (\gamma \geq 0, \mu, \sigma > 0, \mu + \sigma = \lambda)$, we have

$$k(\sigma) = l_\gamma(\sigma) := \int_0^\infty \frac{|\ln v|^\gamma}{1+v^\lambda} v^{\sigma-1} dv.$$

Since $\frac{|\ln v|^\gamma}{(1+v^\lambda)^{1/2}} v^{\frac{\sigma}{2}} \to 0 (v \to 0^+ \text{ or } v \to \infty)$, there exists a constant number $L > 0$, such that

$$0 < \frac{|\ln v|^\gamma}{(1+v^\lambda)^{1/2}} v^{\frac{\sigma}{2}} \leq L (v \in \mathbf{R}_+).$$

Then it follows that

$$\begin{aligned} 0 < l_\gamma(\sigma) &\leq L \int_0^\infty \frac{v^{(\sigma/2)-1} dv}{(1+v^\lambda)^{1/2}} \\ &= \frac{L}{\lambda} \int_0^\infty \frac{u^{(\sigma/2\lambda)-1} dv}{(1+u)^{1/2}} = \frac{L}{\lambda} B\left(\frac{\sigma}{2\lambda}, \frac{\mu}{2\lambda}\right) < \infty, \end{aligned}$$

and $l_\gamma(\sigma) \in \mathbf{R}_+$. We find

$$l_0(\sigma) = \int_0^\infty \frac{1}{1+v^\lambda} v^{\sigma-1} dv = \frac{\pi}{\lambda \sin\left(\frac{\pi\sigma}{\lambda}\right)}. \tag{75}$$

For $\gamma \geq 0$, we obtain

$$\begin{aligned} l_\gamma(\sigma) &= \int_0^1 \frac{(-\ln v)^\gamma v^{\sigma-1}}{1+v^\lambda} dv + \int_1^\infty \frac{(\ln v)^\gamma v^{\sigma-1}}{1+v^\lambda} dv \\ &= \int_0^1 \frac{(-\ln v)^\gamma}{1+v^\lambda} \left(v^{\sigma-1} + v^{\mu-1}\right) dv \\ &= \int_0^1 (-\ln v)^\gamma \sum_{k=0}^\infty (-1)^k \left(v^{k\lambda+\sigma-1} + v^{k\lambda+\mu-1}\right) dv \\ &= \sum_{k=0}^\infty (-1)^k \int_0^1 (-\ln v)^\gamma \left(v^{k\lambda+\sigma-1} + v^{k\lambda+\mu-1}\right) dv. \end{aligned}$$

Setting $t = -\ln v$, we find

$$\begin{aligned} l_\gamma(\sigma) &= \sum_{k=0}^\infty (-1)^k \int_0^\infty t^{(\gamma+1)-1} \left[e^{-t(k\lambda+\sigma)} + e^{-t(k\lambda+\mu)}\right] dt \\ &= \Gamma(\gamma+1) \sum_{k=0}^\infty (-1)^k \left[\frac{1}{(k\lambda+\sigma)^{\gamma+1}} + \frac{1}{(k\lambda+\mu)^{\gamma+1}}\right]. \end{aligned} \tag{76}$$

In view of Theorem 1 and (39), we have

$$\begin{aligned} ||T|| = L_\gamma(\sigma) &:= \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \\ &\times \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]_\gamma^{\frac{1}{,q}} l_\gamma(\sigma). \end{aligned} \tag{77}$$

Example 3. For $h(v) = \frac{|\ln v|^\gamma}{(\max\{1,v\})^\lambda} (\gamma \geq 0, \mu, \sigma > 0, \mu + \sigma = \lambda)$, we have

$$\begin{aligned} k(\sigma) &= \int_0^\infty \frac{|\ln v|^\gamma}{(\max\{1,v\})^\lambda} v^{\sigma-1} dv \\ &= \int_0^1 (-\ln v)^\gamma v^{\sigma-1} dv + \int_1^\infty \frac{(\ln v)^\gamma}{v^\lambda} v^{\sigma-1} dv \\ &= \int_0^1 (-\ln v)^\gamma \left(v^{\sigma-1} + v^{\mu-1}\right) dv. \end{aligned}$$

Setting $t = -\ln v$, we find

$$k(\sigma) = \int_0^\infty t^\gamma \left[e^{-(\sigma-1)t} + e^{-(\mu-1)t}\right] e^{-t} dt$$
$$= \int_0^\infty t^{(\gamma+1)-1} \left(e^{-\sigma t} + e^{-\mu t}\right) dt$$
$$= \Gamma(\gamma+1) \left(\frac{1}{\sigma^{\gamma+1}} + \frac{1}{\mu^{\gamma+1}}\right) \in \mathbf{R}_+. \tag{78}$$

In view of Theorem 1 and (39), we have

$$||T|| = K(\sigma) = \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}}$$
$$\times \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]_\gamma^{\frac{1}{,q}} \Gamma(\gamma+1) \left(\frac{1}{\sigma^{\gamma+1}} + \frac{1}{\mu^{\gamma+1}}\right). \tag{79}$$

Example 4. For $h(v) = \frac{|\ln v|^\gamma}{|1-v|^\lambda}(\gamma \geq 0, \mu, \sigma > 0, \mu + \sigma = \lambda < 1)$, we have

$$k(\sigma) = \tilde{k}_\gamma(\sigma) := \int_0^\infty \frac{|\ln v|^\gamma}{|1-v|^\lambda} v^{\sigma-1} dv.$$

We find

$$\tilde{k}_0(\sigma) = \int_0^\infty \frac{v^{\sigma-1}}{|1-v|^\lambda} dv$$
$$= \int_0^1 (1-v)^{-\lambda} v^{\sigma-1} dv + \int_1^\infty \frac{v^{\sigma-1}}{(v-1)^\lambda} dv$$
$$= \int_0^1 (1-v)^{(1-\lambda)-1} v^{\sigma-1} dv + \int_0^1 (1-u)^{(1-\lambda)-1} u^{\mu-1} du$$
$$= B(1-\lambda, \sigma) + B(1-\lambda, \mu). \tag{80}$$

For $\gamma \geq 0$, we obtain

$$\tilde{k}_\gamma(\sigma) = \int_0^1 \frac{(-\ln v)^\gamma v^{\sigma-1}}{(1-v)^\lambda} dv + \int_1^\infty \frac{(\ln v)^\gamma v^{\sigma-1}}{(v-1)^\lambda} dv$$
$$= \int_0^1 \frac{(-\ln v)^\gamma}{(1-v)^\lambda} \left(v^{\sigma-1} + v^{\mu-1}\right) dv.$$

Setting $0 < \delta < \min\{\mu, \sigma\}$, since $(-\ln v)^{\gamma} v^{\delta} \to 0 (v \to 0^{+})$, there exists a constant $L > 0$, such that $0 < (-\ln v)^{\gamma} v^{\delta} \le L (v \in (0, 1])$, and then it follows

$$\begin{aligned} 0 < \tilde{k}_{\gamma}(\sigma) &\le L \int_0^1 \frac{v^{\sigma-\delta-1} + v^{\mu-\delta-1}}{(1-v)^{\lambda}} dv \\ &= L(B(1-\lambda, \sigma-\delta) + B(1-\lambda, \mu-\delta). \end{aligned}$$

Hence $\tilde{k}_{\gamma}(\sigma) \in \mathbf{R}_{+}$, and

$$\begin{aligned} \tilde{k}_{\gamma}(\sigma) &= \int_0^1 (-\ln v)^{\gamma} \sum_{k=0}^{\infty} (-1)^k \binom{-\lambda}{k} \left(v^{k+\sigma-1} + v^{k+\mu-1}\right) dv \\ &= \sum_{k=0}^{\infty} (-1)^k \binom{-\lambda}{k} \int_0^1 (-\ln v)^{\gamma} \left(v^{k+\sigma-1} + v^{k+\mu-1}\right) dv. \end{aligned}$$

Setting $t = -\ln v$, we find

$$\begin{aligned} \tilde{k}_{\gamma}(\sigma) &= \sum_{k=0}^{\infty} (-1)^k \binom{-\lambda}{k} \int_0^{\infty} t^{(\gamma+1)-1} \left[e^{-t(k+\sigma)} + e^{-t(k+\mu)}\right] dt \\ &= \Gamma(\gamma+1) \sum_{k=0}^{\infty} (-1)^k \binom{-\lambda}{k} \left[\frac{1}{(k+\sigma)^{\gamma+1}} + \frac{1}{(k+\mu)^{\gamma+1}}\right]. \end{aligned} \tag{81}$$

In view of Theorem 1 and (39), we have

$$\begin{aligned} ||T|| = \tilde{K}_{\gamma}(\sigma) &:= \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \\ &\times \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{q}} \tilde{k}_{\gamma}(\sigma) (\gamma \ge 0). \end{aligned} \tag{82}$$

Example 5. For $h(v) = \frac{|\ln v|^{\gamma}}{|v^{\lambda}-1|} (\gamma > 0, \mu, \sigma > 0, \mu + \sigma = \lambda)$, we have

$$k(\sigma) = \hat{k}_{\gamma}(\sigma) := \int_0^{\infty} \frac{|\ln v|^{\gamma}}{|v^{\lambda} - 1|} v^{\sigma-1} dv.$$

We find

$$\hat{k}_1(\sigma) = \int_0^\infty \frac{(\ln v)v^{\sigma-1}}{v^\lambda - 1} dv$$
$$= \frac{1}{\lambda^2} \int_0^\infty \frac{(\ln u)u^{(\sigma/\lambda)-1} du}{u-1} = [\frac{\pi}{\lambda \sin\left(\frac{\pi\sigma}{\lambda}\right)}]^2. \tag{83}$$

For $\gamma > 0$, we obtain

$$\hat{k}_\gamma(\sigma) = \int_0^1 \frac{(-\ln v)^\gamma v^{\sigma-1}}{1 - v^\lambda} dv + \int_1^\infty \frac{(\ln v)^\gamma v^{\sigma-1}}{v^\lambda - 1} dv$$
$$= \int_0^1 \frac{(-\ln v)^\gamma}{1 - v^\lambda} \left(v^{\sigma-1} + v^{\mu-1}\right) dv$$

$$= \int_0^1 (-\ln v)^\gamma \sum_{k=0}^\infty \left(v^{k\lambda+\sigma-1} + v^{k\lambda+\mu-1}\right) dv$$
$$= \sum_{k=0}^\infty \int_0^1 (-\ln v)^\gamma \left(v^{k\lambda+\sigma-1} + v^{k\lambda+\mu-1}\right) dv.$$

Setting $t = -\ln v$, we find

$$\hat{k}_\gamma(\sigma) = \sum_{k=0}^\infty \int_0^\infty t^{(\gamma+1)-1} \left[e^{-t(k\lambda+\sigma)} + e^{-t(k\lambda+\mu)}\right] dt$$
$$= \Gamma(\gamma+1) \sum_{k=0}^\infty \left[\frac{1}{(k\lambda+\sigma)^{\gamma+1}} + \frac{1}{(k\lambda+\mu)^{\gamma+1}}\right] \in \mathbf{R}_+. \tag{84}$$

In view of Theorem 1 and (39), we have

$$||T|| = \hat{K}_\gamma(\sigma) := \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}}$$
$$\times \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{,q}} \hat{k}_\gamma(\sigma). \tag{85}$$

Lemma 6. *If* $\mathbf{C}$ *is the set of complex numbers and* $\mathbf{C}_\infty = \mathbf{C} \cup \{\infty\}$, $z_k \in \mathbf{C}\backslash\{z|Rez \geq 0, Imz = 0\}(k = 1, 2, \ldots, n)$ *are different points, the function* $f(z)$ *is analytic in* $\mathbf{C}_\infty$ *except for* $z_i\,(i = 1, 2, \ldots, n)$*, and* $z = \infty$ *is a zero point of* $f(z)$ *whose order is not less than 1, then for* $\alpha \in \mathbf{R}$*, we have*

$$\int_0^{\infty} f(x)x^{\alpha-1}dx = \frac{2\pi i}{1-e^{2\pi\alpha i}}\sum_{k=1}^{n} Res[f(z)z^{\alpha-1}, z_k], \tag{86}$$

where $0 < Im \ln z = \arg z < 2\pi$. *In particular, if* $z_k (k = 1, \dots, n)$ *are all poles of order 1, setting* $\varphi_k(z) = (z - z_k) f(z)(\varphi_k(z_k) \neq 0)$, *then*

$$\int_0^{\infty} f(x)x^{\alpha-1}dx = \frac{\pi}{\sin \pi\alpha}\sum_{k=1}^{n}(-z_k)^{\alpha-1}\varphi_k(z_k). \tag{87}$$

Proof. By Pan et al. [22, p. 118], we have (86). We find

$$\begin{aligned} 1 - e^{2\pi\alpha i} &= 1 - \cos 2\pi\alpha - i \sin 2\pi\alpha \\ &= -2i \sin \pi\alpha(\cos \pi\alpha + i \sin \pi\alpha) \\ &= -2i e^{i\pi\alpha} \sin \pi\alpha. \end{aligned}$$

In particular, since $f(z)z^{\alpha-1} = \frac{1}{z-z_k}(\varphi_k(z)z^{\alpha-1})$, it is obvious that

$$Res[f(z)z^{\alpha-1}, -a_k] = z_k{}^{\alpha-1}\varphi_k(z_k) = -e^{i\pi\alpha}(-z_k)^{\alpha-1}\varphi_k(z_k).$$

Then by (86), we obtain (87). The lemma is proved.

Example 6. For $s \in \mathbf{N}$, $0 < a_1 < \cdots < a_s$, we set

$$h(v) = \frac{1}{\prod_{k=1}^{s}(v^{\lambda/s} + a_k)}(0 < \sigma < \lambda)$$

By (87), setting $u = v^{\lambda/s}$, we find

$$\begin{aligned} k(\sigma) = k_s(\sigma) &:= \int_0^{\infty} \frac{1}{\prod_{k=1}^{s}\left(v^{\lambda/s} + a_k\right)} v^{\sigma-1}dv \\ &= \frac{s}{\lambda}\int_0^{\infty} \frac{1}{\prod_{k=1}^{s}(u + a_k)} u^{\frac{s\sigma}{\lambda}-1}du \end{aligned}$$

$$= \frac{\pi s}{\lambda \sin\left(\frac{\pi s\sigma}{\lambda}\right)}\sum_{k=1}^{s} a_k^{\frac{s\sigma}{\lambda}-1}\prod_{j=1(j\neq k)}^{s}\frac{1}{a_j - a_k} \in \mathbf{R}_+. \tag{88}$$

In view of Theorem 1 and (39), we have

$$||T|| = K_s(\sigma) := \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)} \right]^{\frac{1}{p}} \times \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)} \right]^{\frac{1}{q}} k_s(\sigma). \tag{89}$$

Example 7. For $c > 0, 0 < \gamma < \pi$, We set

$$h(v) = \frac{1}{v^\lambda + \sqrt{c}v^{\lambda/2}\cos\gamma + \frac{c}{4}}(0 < \sigma < \lambda).$$

Putting $z_1 = -\frac{\sqrt{c}}{2}e^{i\gamma}, z_2 = -\frac{\sqrt{c}}{2}e^{-i\gamma}$, by (87), it follows

$$\begin{aligned}
k(\sigma) = c_\gamma(\sigma) &:= \int_0^\infty \frac{v^{\sigma-1}}{v^\lambda + \sqrt{c}v^{\lambda/2}\cos\gamma + \frac{c}{4}}dv \\
&= \frac{2}{\lambda}\int_0^\infty \frac{u^{\frac{2\sigma}{\lambda}-1}}{u^2 + \sqrt{c}u\cos\gamma + \frac{c}{4}}du \\
&= \frac{2}{\lambda}\int_0^\infty \frac{u^{\frac{2\sigma}{\lambda}-1}}{(u-z_1)(u-z_2)}du \\
&= \frac{2\pi}{\lambda\sin\left(\frac{2\pi\sigma}{\lambda}\right)}\left[\left(\frac{\sqrt{c}}{2}e^{i\gamma}\right)^{\frac{2\sigma}{\lambda}-1}\frac{\sqrt{c}}{2(e^{-i\gamma}-e^{i\gamma})}\right. \\
&\qquad \left. + \left(\frac{\sqrt{c}}{2}e^{-i\gamma}\right)^{\frac{2\sigma}{\lambda}-1}\frac{\sqrt{c}}{2(e^{i\gamma}-e^{-i\gamma})}\right] \\
&= \left(\frac{\sqrt{c}}{2}\right)^{\frac{2\sigma}{\lambda}}\frac{2\pi\sin\gamma\left(1-\frac{2\sigma}{\lambda}\right)}{\lambda\sin\gamma\sin\left(\frac{2\pi\sigma}{\lambda}\right)} \in \mathbf{R}_+.
\end{aligned} \tag{90}$$

In view of Theorem 1 and (39), we have

$$||T|| = C_\gamma(\sigma) := \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)} \right]^{\frac{1}{p}} \times \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)} \right]^{\frac{1}{q}} c_\gamma(\sigma). \tag{91}$$

Example 8. We set

$$h(v) = \frac{(\min\{v,1\})^{\eta}}{(\max\{v,1\})^{\lambda+\eta}} (\eta > -\min\{\sigma,\mu\}, \sigma+\mu=\lambda).$$

Then we find

$$\begin{aligned} k(\sigma) &= \int_0^{\infty} \frac{(\min\{v,1\})^{\eta} v^{\sigma-1}}{(\max\{v,1\})^{\lambda+\eta}} dv = \int_0^1 v^{\eta+\sigma-1} dv + \int_1^{\infty} \frac{v^{\sigma-1} dv}{v^{\lambda+\eta}} \\ &= \frac{\lambda+2\eta}{(\sigma+\eta)(\mu+\eta)} \in \mathbf{R}_+. \end{aligned} \tag{92}$$

In view of Theorem 1 and (39), we have

$$\begin{aligned} ||T|| = K_{\eta}(\sigma) &:= \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)} \right]^{\frac{1}{p}} \\ &\times \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)} \right]^{\frac{1}{,q}} \frac{\lambda+2\eta}{(\sigma+\eta)(\mu+\eta)}. \end{aligned} \tag{93}$$

Example 9. We set

$$h(v) = \ln\left(\frac{b+v^{\gamma}}{a+v^{\gamma}}\right) (0 \le a < b, 0 < \sigma < \gamma).$$

We find

$$\begin{aligned} k(\sigma) &= \int_0^{\infty} \ln\left(\frac{b+v^{\gamma}}{a+v^{\gamma}}\right) v^{\sigma-1} dv \\ &= \frac{1}{\sigma} \int_0^{\infty} \ln\left(\frac{b+v^{\gamma}}{a+v^{\gamma}}\right) dv^{\sigma} \\ &= \frac{1}{\sigma} \left[v^{\sigma} \ln\left(\frac{b+v^{\gamma}}{a+v^{\gamma}}\right) |_0^{\infty} \right. \\ &\qquad \left. + \gamma \int_0^{\infty} \left(\frac{1}{a+v^{\gamma}} - \frac{1}{b+v^{\gamma}}\right) v^{\sigma+\gamma-1} dv \right] \\ &= \frac{b-a}{\sigma} \int_0^{\infty} \frac{u^{\left(1+\frac{\sigma}{\gamma}\right)-1}}{(u+a)(u+b)} du. \end{aligned}$$

For $a > 0$, by (87), we have

$$k(\sigma) = \frac{(b-a)\pi}{\sigma \sin \pi\left(1+\frac{\sigma}{\gamma}\right)}\left(\frac{a^{\frac{\sigma}{\gamma}}}{b-a} + \frac{b^{\frac{\sigma}{\gamma}}}{-b+a}\right)$$
$$= \frac{\left(b^{\frac{\sigma}{\gamma}} - a^{\frac{\sigma}{\gamma}}\right)\pi}{\sigma \sin\left(\frac{\pi\sigma}{\gamma}\right)} \in \mathbf{R}_+. \tag{94}$$

By using the simple way, we still can obtain (94) for $a = 0$.

In view of Theorem 1 and (39), we have

$$||T|| = K(\sigma) := \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \times \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{q}} \frac{\left(b^{\frac{\sigma}{\gamma}} - a^{\frac{\sigma}{\gamma}}\right)\pi}{\sigma \sin\left(\frac{\pi\sigma}{\gamma}\right)}. \tag{95}$$

Example 10. We set

$$h(v) = e^{-\rho v^{\gamma}} (\rho, \gamma, \sigma > 0).$$

Setting $u = \rho v^{\gamma}$, we find

$$k(\sigma) = \int_0^\infty e^{-\rho v^{\gamma}} v^{\sigma-1} dv = \frac{1}{\gamma e^{\sigma/\gamma}} \int_0^\infty e^{-u} u^{\frac{\sigma}{\gamma}-1} du$$
$$= \frac{1}{\gamma \rho^{\sigma/\gamma}} \Gamma\left(\frac{\sigma}{\gamma}\right) \in \mathbf{R}_+. \tag{96}$$

In view of Theorem 1 and (39), we have

$$||T|| = K(\sigma) := \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \times \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{q}} \frac{1}{\gamma \rho^{\sigma/\gamma}} \Gamma\left(\frac{\sigma}{\gamma}\right). \tag{97}$$

Example 11. We set

$$h(v) = \arctan \rho v^{-\gamma} (\rho > 0, 0 < \sigma < \gamma).$$

We find

$$
\begin{aligned}
k(\sigma) &= \int_0^\infty v^{\sigma-1}(\arctan \rho v^{-\gamma})dv \\
&= \frac{1}{\sigma}\int_0^\infty (\arctan \rho v^{-\gamma})dv^\sigma \\
&= \frac{1}{\sigma}\left[(\arctan \rho v^{-\gamma})v^\sigma|_0^\infty + \int_0^\infty \frac{\gamma\rho v^{\sigma-\gamma-1}}{1+(\rho v^{-\gamma})^2}dv\right] \\
&= \frac{\rho^{\frac{\sigma}{\gamma}}}{2\sigma}\int_0^\infty \frac{1}{1+u}u^{\left(\frac{1}{2}-\frac{\sigma}{2\gamma}\right)-1}du \\
&= \frac{\rho^{\frac{\sigma}{\gamma}}}{2\sigma}\frac{\pi}{\sin\pi\left(\frac{1}{2}-\frac{\sigma}{2\gamma}\right)} = \frac{\rho^{\frac{\sigma}{\gamma}}\pi}{2\sigma\cos\left(\frac{\pi\sigma}{2\gamma}\right)} \in \mathbf{R}_+,
\end{aligned} \tag{98}
$$

In view of Theorem 1 and (39), we have

$$
\begin{aligned}
||T|| = K(\sigma) &:= \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \\
&\times \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{q}} \frac{\rho^{\frac{\sigma}{\gamma}}\pi}{2\sigma\cos\pi\left(\frac{\sigma}{2\gamma}\right)}.
\end{aligned} \tag{99}
$$

Example 12. We set

$$
h(v) = \csc h(\rho v^\gamma) = \frac{2}{e^{\rho v^\gamma} - e^{-\rho v^\gamma}}(\rho > 0, \sigma > \gamma > 0),
$$

where $\csc h(u) = \frac{2}{e^u - e^{-u}}$ is hyperbolic cosecant function [23]. We find

$$
\begin{aligned}
k(\sigma) = a_\gamma(\sigma) &:= \int_0^\infty v^{\sigma-1}\csc h(\rho v^\gamma)dv \\
&= \int_0^\infty \frac{2v^{\sigma-1}}{e^{\rho v^\gamma} - e^{-\rho v^\gamma}}dv \\
&= \int_0^\infty \frac{2v^{\sigma-1}e^{-\rho v^\gamma}dv}{1-e^{-2\rho v^\gamma}} = 2\int_0^\infty v^{\sigma-1}\sum_{k=0}^\infty e^{-(2k+1)\rho v^\gamma}dv \\
&= 2\sum_{k=0}^\infty \int_0^\infty v^{\sigma-1}e^{-(2k+1)\rho v^\gamma}dv.
\end{aligned}
$$

Setting $u = (2k+1)\rho v^\gamma$, we obtain

$$\begin{aligned} a_\gamma(\sigma) &= \frac{2}{\gamma\rho^{\sigma/\gamma}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{\sigma/\gamma}} \int_0^\infty u^{\frac{\sigma}{\gamma}-1} e^{-u} du \\ &= \frac{2}{\gamma\rho^{\sigma/\gamma}} \Gamma\left(\frac{\sigma}{\gamma}\right) \left[\sum_{k=1}^{\infty} \frac{1}{k^{\sigma/\gamma}} - \sum_{k=1}^{\infty} \frac{1}{(2k)^{\sigma/\gamma}}\right] \\ &= \frac{2}{\gamma\rho^{\sigma/\gamma}} \Gamma\left(\frac{\sigma}{\gamma}\right) \left(1 - \frac{1}{2^{\sigma/\gamma}}\right) \zeta\left(\frac{\sigma}{\gamma}\right) \in \mathbf{R}_+, \end{aligned} \tag{100}$$

where, $\zeta\left(\frac{\sigma}{\gamma}\right) = \sum_{k=1}^{\infty} \frac{1}{k^{\sigma/\gamma}} \left(\frac{\sigma}{\gamma} > 1\right)$ ($\zeta(\cdot)$ is the Riemann's zeta function [24]).

In view of Theorem 1 and (39), we have

$$\begin{aligned} \|T\| = A_\gamma(\sigma) &:= \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \\ &\times \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{q}} a_\gamma(\sigma). \end{aligned} \tag{101}$$

Example 13. We set

$$h(v) = \sec h(\rho v^\gamma) = \frac{2}{e^{\rho v^\gamma} + e^{-\rho v^\gamma}} (\rho, \sigma, \gamma > 0),$$

where $\sec h(u) = \frac{2}{e^u + e^{-u}}$ is hyperbolic secant function. We find

$$\begin{aligned} k(\sigma) = b_\gamma(\sigma) &:= \int_0^\infty v^{\sigma-1} \sec h(\rho v^\gamma) dv \\ &= \int_0^\infty \frac{2v^{\sigma-1} dv}{e^{\rho v^\gamma} + e^{-\rho v^\gamma}} = \int_0^\infty \frac{2v^{\sigma-1} e^{-\rho v^\gamma} dv}{1 + e^{-2\rho v^\gamma}} \\ &= 2\int_0^\infty v^{\sigma-1} \sum_{k=0}^{\infty} (-1)^k e^{-(2k+1)\rho v^\gamma} dv \\ &= 2\sum_{k=0}^{\infty} (-1)^k \int_0^\infty v^{\sigma-1} e^{-(2k+1)\rho v^\gamma} dv. \end{aligned}$$

Setting $u = (2k+1)\rho v^{\gamma}$, we obtain

$$b_{\gamma}(\sigma) = \frac{2}{\gamma\rho^{\sigma/\gamma}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{\sigma/\gamma}} \int_0^{\infty} u^{\frac{\sigma}{\gamma}-1} e^{-u} du$$
$$= \frac{1}{\gamma\rho^{\sigma/\gamma} 2^{(\sigma/\gamma)-1}} \Gamma\left(\frac{\sigma}{\gamma}\right) \varsigma\left(\frac{\sigma}{\gamma}\right) \in \mathbf{R}_+, \tag{102}$$

where

$$\varsigma\left(\frac{\sigma}{\gamma}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{\sigma/\gamma}} \left(\frac{\sigma}{\gamma} > 0\right).$$

In view of Theorem 1 and (39), we have

$$||T|| = B_{\gamma}(\sigma) := \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}}$$
$$\times \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{q}} b_{\gamma}(\sigma). \tag{103}$$

Example 14. We set

$$h(v) = \coth h(\rho v^{\gamma}) - 1 = \frac{e^{\rho v^{\gamma}} + e^{-\rho v^{\gamma}}}{e^{\rho v^{\gamma}} - e^{-\rho v^{\gamma}}} - 1$$
$$= \frac{2e^{-\rho v^{\gamma}}}{e^{\rho v^{\gamma}} - e^{-\rho v^{\gamma}}} (\rho > 0, \sigma > \gamma > 0),$$

where $\coth h(u) = \frac{e^u + e^{-u}}{e^u - e^{-u}}$ is hyperbolic cotangent function. We find

$$k(\sigma) = c_{\gamma}(\sigma) := \int_0^{\infty} v^{\sigma-1}(\coth h(\rho v^{\gamma}) - 1) dv$$
$$= \int_0^{\infty} \frac{2e^{-\rho v^{\gamma}} v^{\sigma-1}}{e^{\rho v^{\gamma}} - e^{-\rho v^{\gamma}}} dv = \int_0^{\infty} \frac{2e^{-2\rho v^{\gamma}} v^{\sigma-1}}{1 - e^{-2\rho v^{\gamma}}} dv$$
$$= 2\int_0^{\infty} v^{\sigma-1} \sum_{k=0}^{\infty} e^{-2(k+1)\rho v^{\gamma}} dv$$
$$= 2\sum_{k=1}^{\infty} \int_0^{\infty} v^{\sigma-1} e^{-2k\rho v^{\gamma}} dv.$$

Setting $u = 2k\rho v^{\gamma}$, we obtain

$$c_{\gamma}(\sigma) = \frac{2}{\gamma\rho^{\sigma/\gamma}} \sum_{k=1}^{\infty} \frac{1}{(2k)^{\sigma/\gamma}} \int_{0}^{\infty} u^{\frac{\sigma}{\gamma}-1} e^{-u} du$$
$$= \frac{1}{\gamma\rho^{\sigma/\gamma} 2^{(\sigma/\gamma)-1}} \Gamma\left(\frac{\sigma}{\gamma}\right) \zeta\left(\frac{\sigma}{\gamma}\right) \in \mathbf{R}_{+}. \tag{104}$$

In view of Theorem 1 and (39), we have

$$||T|| = C_{\gamma}(\sigma) := \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \times \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{q}} c_{\gamma}(\sigma). \tag{105}$$

Example 15. We set

$$h(v) = 1 - \tan h(\rho v^{\gamma}) = 1 - \frac{e^{\rho v^{\gamma}} - e^{-\rho v^{\gamma}}}{e^{\rho v^{\gamma}} + e^{-\rho v^{\gamma}}}$$
$$= \frac{2e^{-\rho v^{\gamma}}}{e^{\rho v^{\gamma}} + e^{-\rho v^{\gamma}}} (\rho, \sigma, \gamma > 0),$$

where $\tan h(u) = \frac{e^{u}+e^{-u}}{e^{u}-e^{-u}}$ is hyperbolic tangent function. We find

$$k(\sigma) = d_{\gamma}(\sigma) := \int_{0}^{\infty} v^{\sigma-1}(1 - \tan h(\rho v^{\gamma}))dv$$
$$= \int_{0}^{\infty} \frac{2e^{-\rho v^{\gamma}} v^{\sigma-1}}{e^{\rho v^{\gamma}} + e^{-\rho v^{\gamma}}} dv = \int_{0}^{\infty} \frac{2e^{-2\rho v^{\gamma}} v^{\sigma-1}}{1 + e^{-2\rho v^{\gamma}}} dv$$
$$= 2\int_{0}^{\infty} v^{\sigma-1} \sum_{k=0}^{\infty} (-1)^{k} e^{-2(k+1)\rho v^{\gamma}} dv$$
$$= 2\sum_{k=1}^{\infty} (-1)^{k-1} \int_{0}^{\infty} v^{\sigma-1} e^{-2k\rho v^{\gamma}} dv.$$

Setting $u = 2k\rho v^{\gamma}$, we obtain

$$d_{\gamma}(\sigma) = \frac{2}{\gamma\rho^{\sigma/\gamma}} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k)^{\sigma/\gamma}} \int_{0}^{\infty} u^{\frac{\sigma}{\gamma}-1} e^{-u} du$$
$$= \frac{1}{\gamma\rho^{\sigma/\gamma} 2^{(\sigma/\gamma)-1}} \Gamma\left(\frac{\sigma}{\gamma}\right) \xi\left(\frac{\sigma}{\gamma}\right) \in \mathbf{R}_{+}, \tag{106}$$

where, $\xi(\frac{\sigma}{\gamma}) := \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^{\sigma/\gamma}}$.

In view of Theorem 1 and (39), we have

$$||T|| = D_{\gamma}(\sigma) := \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)} \right]^{\frac{1}{p}} \times \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)} \right]^{\frac{1}{q}} d_{\gamma}(\sigma). \tag{107}$$

Note. The following references [24–31] provide an extensive theory and applications of Analytic Number Theory relating to Riemann's zeta function that will provide a source study for further research on Hilbert-type inequalities.

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