Lecture Notes of the Unione Matematica Italiana

Ricardo Castano-Bernard
Fabrizio Catanese
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Homological Mirror Symmetry and Tropical Geometry

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Introduction

The workshop "Mirror Symmetry and Tropical Geometry" took place in Cetraro, Italy on July 2–8, 2011. The idea was to bring together mathematicians and physicists who worked on both topics or in related areas.

Homological Mirror Symmetry, here abbreviated as HMS, is the area of mathematics revolving around several categorical equivalences connecting symplectic and holomorphic (or algebraic) geometry. This mathematical approach to Mirror Symmetry goes back to the work of Maxim Kontsevich (1993). Further developments of Kontsevich's program was the subject of many talks at the workshop. This theme is therefore present in several papers of this volume.

Works related to Homological Mirror Symmetry include the paper on HMS for Landau–Ginzburg models by H. Ruddat, the paper of N. Sibilla on HMS for curves, the paper of Kontsevich and Y. Soibelman on complex integrable systems, and the paper by D. Favero, F. Haiden, and L. Katzarkov on the phantom categories which appear in HMS. The variety of methods ranging from homological algebra to delicate questions of symplectic topology and algebraic geometry illustrates the complexity of the subject.

The second topic of the workshop was Tropical Geometry. Roughly speaking, Tropical Geometry studies piecewise-linear objects which appear as certain "degenerations" of the corresponding algebro-geometric objects. The relationship of Tropical Geometry with Mirror Symmetry goes back to the work by Kontsevich and Y. Soibelman (2000) where methods of non-archimedean geometry (in particular, tropical curves) were used for the purposes of Homological Mirror Symmetry. Combined with the subsequent work of Mikhalkin on a "tropical" approach to Gromov–Witten theory and with the work of Gross, Siebert, and several others, Tropical Geometry has become a useful tool for people working in Mirror Symmetry.

On the other hand, "tropical" analogs of many notions of classical symplectic and algebraic geometry are interesting and nontrivial objects by themselves. The paper by G. Mikhalkin and I. Zharkov, which is devoted to the tropical analog of the intermediate Jacobian, is a good illustration of this statement. Methods of tropical geometry are also used in the paper by Kontsevich and Y. Soibelman devoted to the study of Donaldson–Thomas invariants and corresponding wall-crossing formulas.

The volume also contains several papers which are related to the main topics of the workshop in an indirect way. For example, the paper by S. Guillermou and P. Schapira is devoted to the application of the microlocal theory of sheaves developed by the second author jointly with M. Kashiwara to the displaceability problem in symplectic topology. It should be compared with attempts of several mathematicians to describe the Fukaya category (one of the main objects on the "symplectic" side of Homological Mirror Symmetry) in terms of constructible sheaves and corresponding dg-categories.

Two papers are devoted to various aspects of the moduli stacks of bundles. In the paper by O. Ben-Bassat and E. Gasparim the stack of vector bundles on a formal neighborhood of a rational curve in a surface is studied. In the paper by A. Soibelman the "very good" property introduced by Beilinson and Drinfeld in their work on the Geometric Langlands Program is generalized to the case of arbitrary parabolic bundles on a curve and then applied to the additive Deligne–Simpson problem.

A. Neitzke gives a nice review of his joint work with D. Gaiotto and G. Moore on the construction of hyperkähler metrics. Their approach is based on the thermodynamical Bethe Ansatz-type integral equation proposed by them, as well as on the "Kontsevich–Soibelman wall-crossing formulas". There are many interesting and nontrivial analogies between the paper by Neitzke and the paper by Kontsevich and Y. Soibelman in this volume.

S. Gukov and P. Sulkowski propose a way to quantize spectral curves. Then they discuss the relationship of arising "quantum spectral curves" with the topological recursion of Eynard–Orantin as well as with other topics such as A-polynomials of knots.

The paper by M. Kapranov, O. Schiffmann, and E. Vasserot is devoted to the Hall algebra of the "compactified $Spec(\mathbf{Z})$" interpreted as a curve. The "category of vector bundles" on such an object is described in Arakelov terms, as the category of metrized lattices. The (spherical) Hall algebra of this category is a shuffle algebra, similar to Hall algebras of the corresponding categories for "usual" curves. The relations in the algebra are described in terms for the (full) zeta-function.

We believe that the present volume represents a rather complete update about the state of the art in the field, and we hope that it shall become an important reference for graduate students and researchers who want to enter this exciting new field. Papers in this volume represent a tiny portion of the variety of topics discussed at the workshop. In order to give to the reader an idea about the latter we finish the Introduction with the list of talks presented at the Cetraro workshop.

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Lectures

- Mina Aganagic (Berkeley): Knot Homology from Refined Chern–Simons Theory.
- Fedor Bogomolov (NYU): On rationality of the fields of invariants of linear actions for connected nonsemisimple algebraic groups (based on my joint work with Christian Boehning and Hans-Christian Graf von Bohtmer).
- Fabrizio Catanese (Bayreuth): Special Galois coverings and the singular set of the moduli space of curves
- Alexander Efimov (Steklov Institute): Cohomological Hall algebra and Kac's conjecture
- Vladimir Fock (Strasbourg): Integrable systems, dimers and cluster varieties. (Joint work with A Marshakov)
- Kenji Fukaya (Kyoto): Homological Mirror symmetry of toric manifolds
- Alexander Goncharov (Yale): Dimers and cluster integrability
- Mark Gross (UCSD): Examples of stable log maps and tropical geometry
- Sergei Gukov (Caltech): From hyperholomorphic sheaves to quantum group invariants via Langlands duality
- Ilia Itenberg (Strasbourg): Topology of real tropical hypersurfaces
- Mikhail Kapranov (Yale): Arithmetic Hall algebras
- Ludmil Katzarkov (Miami and Vienna): Degenerations and wall crossings
- Viatcheslav Kharlamov (Strasbourg): Anti-symplectic involutions on rational symplectic 4-manifolds
- Maxim Kontsevich (IHES): Integrable systems and canonical bases
- Andrei Losev (ITEP): Homotopical beta-function of the instantonic sigma-model and bosonic string Einstein equation on schemes
- Diego Matessi (U of Milan): Conifold transitions and tropical geometry
- David Morrison (UCSB): Mirror symmetry and non-complete-intersection Calabi–Yau manifolds
- Andy Neitzke (UT Austin): A 2d-4d wall-crossing formula
- Nikita Nekrasov (IHES): Surprises with four dimensional N=2 gauge theories
- Dimitri Orlov (Steklov): Mirror symmetry, B-branes and strange Arnold duality
- Tony Pantev (UPenn): Mirror symmetry and mixed Hodge structures

- Pierre Schapira (Paris VI): Microlocal theory of sheaves and symplectic topology: results and open problems
- Bernd Siebert (Hamburg): Logarithmic Gromov–Witten invariants
- Yan Soibelman (Kansas State): Integrable systems and wall-crossing formulas
- Jake Solomon (Hebrew University): Entropy of Lagrangian submanifolds
- Piotr Sulkowski (Caltech): Quantum curves and topological recursion
- Valerio Toledano Laredo (Northeastern): Yangians, quantum loop algebras and trigonometric connections.
- Ilia Zharkov (Kansas State): Tropical Homology
- Anton Zorich (Rennes): Degeneration of flat versus hyperbolic metric on Riemann surfaces, determinant of Laplacian, and Lyapunov exponents of the Hodge bundle (in collaboration with A. Eskin and M. Kontsevich)

Other Contributions:

- Oren Ben-Bassat (University of Haifa): Deformations of Open Surfaces and their Stacks of Bundles
- Colin Diemer (U of Miami): Circuit Relations and the Secondary Stack
- David Favero (UPenn): Graded matrix factorizations, functor categories, and orbit categories
- Gabriel Kerr (U of Miami): Circuit Relation in the Symplectic Mapping Class Group
- Helge Ruddat (UC Berkeley): Mirror Symmetry partners via vanishing cycles
- Nick Sheridan (MIT): On the Homological Mirror Symmetry conjecture for pairs of pants
- Nicolo Sibilla (Northwestern): Mirror symmetry in dimension one and Fourier–Mukai transform
- Alexander Soibelman (UNC Chapel Hill): The very good property for moduli of parabolic bundles and the additive Deligne–Simpson problem

Contents

Moduli Stacks of Bundles on Local Surfaces

Oren Ben-Bassat and Elizabeth Gasparim

Abstract We give an explicit groupoid presentation of certain stacks of vector bundles on formal neighborhoods of rational curves inside algebraic surfaces. The presentation involves a Möbius type action of an automorphism group on a space of extensions.

1 Introduction

A fundamental question in algebraic geometry is to understand how rational maps on a variety X affect the moduli of vector bundles on X, that is: suppose X and Y birationally equivalent, then what is the relation between the various moduli of vector bundles on X and Y? Here we focus on the case of surfaces, in which case rational maps are obtained by blowing up (possibly singular) points. Suppose $\pi: Y \rightarrow X$ is the blow up of a point x in X, with $\ell = \pi^{-1}(x)$. Considering pullbacks, one can then study the relative situation of the moduli of vector bundles on X mapping into the moduli of vector bundles on Y. Since π is an isomorphism outside ℓ clearly the heart of the question lies in the geometry of moduli of bundles on a small neighborhood of ℓ. This question was addressed from the point of view of moduli spaces of equivalence classes of vector bundles in [15] for the case when x is a smooth point, and the geometry of the local moduli was used to prove the Atiyah–Jones conjecture for rational surfaces. In this paper we consider the

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moduli *stacks* of vector bundles in formal neighborhoods of ℓ, and give explicit groupoid presentations of such moduli stacks. The stacky point of view, besides clarifying several delicate issues about the local moduli also has the advantage that it generalises to the case of singular surfaces, where ℓ is a line with self-intersection $\ell^2 = -k < -1$. We develop the study of stacks of bundles on (completions of the) local surfaces $Z_k = \mathrm{Tot}(\mathscr{O}(-k))$ and give presentations of certain stacks of rank 2 bundles over these surfaces. The most interesting aspect of these presentations is the "Möbius" transformation (17) discussed in Sect. 2.3.

2 Local Surfaces and Vector Bundles on Them

Notation 1. *In this paper we will work with (associative, commutative, unital) $\mathbb{C}$-algebras. Therefore, affine scheme will mean the spectrum of such an algebra, and all varieties, schemes, and formal schemes are considered over $\mathbb{C}$. We will work over the site of affine schemes or $\mathbb{C}$-algebras with the faithfully flat topology. The schemes we will consider are quasi-compact and quasi-separated. For any positive integer k, we have the algebraic variety*

$$Z_k = \mathrm{Tot}(\mathscr{O}_{\mathbb{P}^1}(-k)) = Spec_{\mathbb{P}^1}\left(\bigoplus_{i=0}^{\infty} \mathscr{O}_{\mathbb{P}^1}(ik)\right)$$

and $\ell \cong \mathbb{P}^1$ its zero section, so that $\ell^2 = -k$. Let I_ℓ be the sheaf of $\mathscr{O}_{Z_k}$ ideals defining ℓ. We write $Z_k^{(n)}$ for the n^{th} infinitesimal neighborhood of ℓ and $\widehat{Z_k} = Z_k^{(\infty)}$ for the formal neighborhood of ℓ in Z_k. $\widehat{Z_k} = (\ell, lim_n \mathscr{O}_{Z_k}/I_\ell^n)$ is the formal scheme given as the formal completion of Z_k along ℓ. It is a (an inductive or direct) limit in the category of ringed spaces over $\mathbb{P}^1$. There is a presentation

$$Z_k = \left(U \bigsqcup V\right) / \sim,$$

where we will always use the charts $U = \mathbb{C}^2$ with coordinates (z, u), and $V = \mathbb{C}^2$ with coordinates (ξ, v), with $U \cap V = (\mathbb{C} - \{0\}) \times \mathbb{C}$ where the equivalence relation $\sim$ is given by the change of coordinates $(\xi, v) = (z^{-1}, z^k u)$. Note that the zero section ℓ is given in these coordinates by $u = 0$ in the U-chart and $v = 0$ in the V-chart. It is easy to see that $I_\ell \cong \mathscr{O}(k)$. In fact, I_ℓ is the line bundle associated to the divisor $-\ell$ and since $u = \xi^k v$,

$$div(u) = \ell + kf$$

where f is the fiber defined by $\xi = 0$. We similarly have

$$U^{(n)} = Spec(\mathbb{C}[z, u]/(u^{n+1}))$$

and

$$V^{(n)} = Spec(\mathbb{C}[\xi, v]/(v^{n+1})).$$

As above, we have

$$Z_k^{(n)} = \left(U^{(n)} \bigsqcup V^{(n)}\right) / \sim$$

and

$$Z_k^{(\infty)} = \widehat{Z_k} = \left(\hat{U} \bigsqcup \hat{V}\right) / \sim$$

where $\hat{U}$ and $\hat{V}$ are the formal scheme completions of U and V along ℓ.

Remark 1. Unless we explicitly state that n is finite, in each usage of the spaces $Z_k^{(n)}$ we are including the case that $n = \infty$.

These presentations are helpful for describing vector bundles. For instance by the answer to Serre's famous question (proved by Seshadri [22] and in further generality by Quillen [20] and Suslin [23]), $U = Spec(\mathbb{C}[z, u])$ has no non-trivial vector bundles; similarly this is true for $U^{(n)}$ and $\hat{U}$ by Theorem 7 of [10]. In contrast, vector bundles on $Z_k^{(n)}$ were studied on [1–3, 14]. All the schemes we have mentioned up until now are Noetherian and $\widehat{Z_k}$ is a Noetherian formal scheme. If T is an affine scheme such that $Pic(T)$ is trivial then

$$\operatorname{Pic}(\widehat{Z_k} \times T) \simeq \operatorname{Pic}(Z_k^{(n)} \times T) \simeq \operatorname{Pic}(\mathbb{P}^1 \times T) \simeq \operatorname{Pic}(\mathbb{P}^1) \simeq \mathbb{Z};$$

we will use the symbol $\mathcal{O}(j)$ for the line bundle with first Chern class j coming from $\mathbb{P}^1$ in any of these spaces. If E is a rank 2 vector bundle of first Chern class zero on $Z_k^{(n)}$ then the splitting type $j \geq 0$ of E is the integer such that the restriction of E to ℓ is isomorphic to $\mathcal{O}(j) \oplus \mathcal{O}(-j)$. For a vector bundle on $Z_k^{(n)} \times T$ we say that it has constant splitting type j if its splitting type is j over every $t \in T(\mathbb{C})$.

For our explicit presentations of stacks, we will need the following basic results about rank 2 bundles on $Z_k^{(n)}$, which we generalize from [13].

Lemma 1. *Let S be any scheme over $\mathbb{C}$ and E a rank 2 vector bundle on $Z_k^{(n)} \times S$ of constant splitting type $j \geq 0$. Then for any $s \in S(\mathbb{C})$ there is an open subscheme T of S containing s and such that the restriction of E to $Z_k^{(n)} \times T$ has the structure of an extension*

$$0 \to \mathcal{O}(-j) \to E|_{Z_k^{(n)} \times T} \to \mathcal{O}(j) \to 0.$$

Proof. By [12], Theorem 3.3 $E|_{Z_k^{(n)} \times \{s\}}$ can be written as an algebraic extension

$$0 \to \mathcal{O}(-j) \to E|_{Z_k^{(n)} \times \{s\}} \to \mathcal{O}(j) \to 0$$

where $j > 0$. Consider the leftmost injective map as a nowhere vanishing element of the space of global sections $H^0(Z_k^{(n)} \times \{s\}, E|_{Z_k^{(n)} \times \{s\}} \otimes \mathscr{O}(j))$. The pushforward $\pi_{S*}(E|_{\ell \times S} \otimes \mathscr{O}(j))$ is a vector bundle on S and we have chosen a non-zero point in the fiber over s. Choose T' open in S and containing s and an extension of the above section to an element of

$$H^0(T', (\pi_{S*}(E|_{\ell \times S} \otimes \mathscr{O}(j)))|_{T'}) = H^0(\ell \times T', E|_{\ell \times T'} \otimes \mathscr{O}(j))$$

such that this chosen global extension does not vanish on $\ell \times T'$ and hence does not vanish on $Z_k^{(n)} \times T'$, and passes through our chosen element of the fiber over s. This gives us an injective map of constant rank leading to a short exact sequence on $Z_k^{(n)} \times T'$

$$0 \to \mathscr{O}(-j) \to E_{Z_k^{(n)} \times T'} \to L \to 0$$

where L is a line bundle on $Z_k^{(n)} \times T'$ isomorphic to $\mathscr{O}(j)$ over every geometric point of T'. By the see–saw principle there is a T open in T' and containing S such that the restriction of L to $Z_k^{(n)} \times T'$ is isomorphic to $\mathscr{O}(j)$. Therefore the restriction of the above short exact sequence to $Z_k^{(n)} \times T$ gives the desired result.

Remark 2. An alternate approach to the Lemma 1 is to start with any vector bundle which has nowhere zero map of $\mathscr{O}(-j)$ to E over $\ell \times T$ for some affine scheme T and use the fact that $H^1(\ell \times T, I^m_{\ell \times T}) = 0$ for $m > 0$ to extend this map order by order to a map over $Z_k^{(n)} \times T$ which must be nowhere zero.

Lemma 2. *Let T be an affine scheme and E an algebraic extension of $\mathscr{O}_{Z_k^{(n)} \times T}$ modules*

$$0 \to \mathscr{O}(-j) \to E \to \mathscr{O}(j) \to 0,$$

over $Z_k^{(n)} \times T$ which splits over $\ell \times T$ for $j \geq 0$ then, in the chosen coordinates E can be described by a transition matrix of the form

$$\begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix}$$

on $(U^{(n)} \cap V^{(n)}) \times T$, where

$$p = \sum_{i=1}^{min(\lfloor (2j-2)/k \rfloor, n)} \sum_{l=ki-j+1}^{j-1} p_{i,l} z^l u^i. \tag{1}$$

and $p_{i.l} \in \mathscr{O}(T)$.

Proof. A Čech cohomology calculation (performed in Theorem 3.3 of [12]) shows that

$$\mathrm{Ext}^1_{Z_k^{(n)}}(\mathscr{O}(j),\mathscr{O}(-j)) = \frac{\mathbb{C}[z,z^{-1},u]/(u^{n+1})}{z^{-j}\,\mathbb{C}[z^{-1},z^k u]/((z^k u)^{n+1}) + z^j\,\mathbb{C}[z,u]/(u^{n+1})}$$

by flat base change for the diagram

$$\begin{array}{ccc} Z_k^{(n)} \times T & \xrightarrow{\pi_T} & T \\ \Big\downarrow{\scriptstyle \pi_{Z_k^{(n)}}} & & \Big\downarrow \\ Z_k^{(n)} & \longrightarrow & \{\cdot\} \end{array}$$

and the Leray spectral sequence for $\pi_{Z_k^{(n)}}$ we have

$$\begin{aligned} \mathrm{Ext}^1_{Z_k^{(n)}\times T}(\pi^*_{Z_k^{(n)}}\mathscr{O}(j),\pi^*_{Z_k^{(n)}}\mathscr{O}(-j)) &= H^1(Z_k^{(n)}\times T,\pi^*_{Z_k^{(n)}}(\mathscr{O}(-2j))) \\ &= H^0(T,R^1\pi_{T*}\pi^*_{Z_k^{(n)}}(\mathscr{O}(-2j))) \\ &= H^0(T,\mathscr{O}_T\otimes H^1(Z_k^{(n)},\mathscr{O}(-2j))) \\ &= H^0(T,\mathscr{O}_T\otimes \mathit{Ext}^1_{Z_k^{(n)}}(\mathscr{O}(j),\mathscr{O}(-j))). \quad (2) \end{aligned}$$

Remark 3. As a consequence of the above two Lemmas 2 and 1, we see that any rank 2 vector bundle on $Z_k^{(n)} \times T$ (or $\widehat{Z_k} \times T$) takes a special form locally on T and in this form it is clearly the restriction (completion) of a vector bundle on Z_k. The theorem on formal functions implies then that

$$\widehat{\mathrm{Ext}^i_{Z_k\times T}(V,W)} \cong \mathrm{Ext}^i_{\widehat{Z_k}\times T}(V,W).$$

Notation 2. *Let*

$$N_{j,k}^{(n)} = \{(i,l)|ki - j + 1 \le l \le j-1 \ \ and\ \ 1 \le i \le min(\lfloor (2j-2)/k \rfloor, n)\}.$$

Consider the algebraic variety over $\mathbb{C}$

$$W_{j,k}^{(n)} = Spec\left(\mathbb{C}[\ \ p_{i,l} \ \ | \ \ (i,l) \in N_{j,k}^{(n)}]\right). \quad (3)$$

For any fixed j,k *it remains finite dimensional even for* $n=\infty$. *If we pass to the* $\mathbb{C}$ *points then we get*

$$W_{j,k}^{(n)}(\mathbb{C}) = \{p \in \mathrm{Ext}^1_{Z_k^{(n)}}(\mathscr{O}(j),\mathscr{O}(-j)) \ \ | \ \ p|_\ell = 0\}.$$

Let

$$R_{j,k}^{(n)} = \bigoplus_{i=1}^{\lfloor (2j-2)/k \rfloor} \bigoplus_{l=ki-j+1}^{j-1} \mathbb{C} z^l u^i \subset \mathscr{O}(U^{(n)} \cap V^{(n)}). \tag{4}$$

of course $R_{j,k}^{(n)}$ is the set of $\mathbb{C}$ points of $W_{j,k}^{(n)}$ but we distinguish them because of the different notions of automorphisms of $R_{j,k}^{(n)}$ and $W_{j,k}^{(n)}$.

Remark 4. Note that in our chosen form of transition matrix from Eq. (1) we have explicitly chosen $p \in R_{j,k}^{(n)}$.

Definition 1. Consider the open cover $\{U^{(n)} \times W_{j,k}^{(n)}, V^{(n)} \times W_{j,k}^{(n)}\}$ of $Z_k^{(n)} \times W_{j,k}^{(n)}$. We define $\mathbb{E}$, sometimes called the big bundle, to be the bundle

$$\begin{array}{c} \mathbb{E} \\ \downarrow \\ Z_k^{(n)} \times W_{j,k}^{(n)} \end{array}$$

on $Z_k \times W_{j,k}^{(n)}$ defined by transition matrix

$$\begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix} \in H^0((U^{(n)} \cap V^{(n)}) \times W_{j,k}^{(n)}, \mathscr{A}ut(\mathscr{O}^{\oplus 2})).$$

Let T be an affine scheme and p a morphism from T to $W_{j,k}^{(n)}$. We denote by E_p the bundle (also described in Lemma 2) given by the pullback $(\mathrm{id}_{Z_k^{(n)}}, p)^* \mathbb{E}$ of $\mathbb{E}$ via the map

$$Z_k^{(n)} \times T \overset{(\mathrm{id}_{Z_k^{(n)}}, p)}{\to} Z_k^{(n)} \times W_{j,k}^{(n)}.$$

Lemma 3 ([4, Thm. 4.9]). *On the first formal neighborhood $Z_k^{(1)}$, two bundles E and E' with transition matrices*

$$\begin{pmatrix} z^j & p_1 \\ 0 & z^{-j} \end{pmatrix} \quad and \quad \begin{pmatrix} z^j & p_1' \\ 0 & z^{-j} \end{pmatrix}$$

respectively are isomorphic if and only if $p_1' = \lambda p_1$ for some $\lambda \in \mathbb{C}^\times$.

Remark 5. It follows from this lemma that the coarse moduli space of bundles on $Z_k^{(1)}$ coming from non-trivial extensions of $\mathscr{O}(j)$ by $\mathscr{O}(-j)$ is isomorphic to $\mathbb{P}^{2j-k-2}$.

Example 1. On higher infinitesimal neighborhoods we need to consider far more relations among extension classes then just projectivisation to obtain the moduli of bundles. The simplest of such examples occurs in the case when $k = 1$ and $j = 2$, so that our extension classes have the form

$$p = (p_{1,0} + p_{1,1}z)u + p_{2,1}zu^2.$$

The set of equivalence classes of vector bundles is then $\mathbb{C}^3/\ \sim$ where the equivalence relation is generated by

$$(p_{1,0}, p_{1,1}, p_{2,1}) \sim (\lambda p_{1,0}, \lambda p_{1,1}, \lambda p'_{2,1}) \ \ if \ \ (p_{1,0}, p_{1,1}) \neq (0,0),\ \ \lambda \neq 0,$$
$$(0,0,p_{2,1}) \sim (0,0,\lambda p_{2,1}),\ \ \lambda \neq 0.$$

Note that $p'_{2,1}$ is does not depend on p, and that the quotient topology makes the entire space the only open neighborhood of the split bundle, which is the image of the origin in $\mathbb{C}^3$.

2.1 *Stacks of Vector Bundles*

We now define the stack of bundles $\mathfrak{M}_j(Z_k^{(n)})$, the main object we seek to understand in this article.

Definition 2.

$$\mathfrak{M}_j(Z_k^{(n)})\colon \text{Schemes} \to \text{Groupoids}$$

given by

$$T \mapsto \text{Hom}(T, \mathfrak{M}_j(Z_k^{(n)}))$$

where

$$\text{ob}(\text{Hom}(T, \mathfrak{M}_j(Z_k^{(n)})) = \{\text{rank 2 vector bundles on } Z_k^{(n)} \times T \text{ which have splitting type j and first Chern class 0 for every restriction to } Z_k^{(n)} \times \{t\}, t \in T(\mathbb{C})\} \tag{5}$$

and

$$\text{mor}(\text{Hom}(T, \mathfrak{M}_j(Z_k^{(n)}))(V_1, V_2) = \text{Isom}(V_1, V_2).$$

This is a stack [17] with respect to the faithfully flat topology on schemes ($\mathbb{C}$-algebras). Notice that there is automatically a universal bundle $\mathscr{E}$ over $Z_k^{(n)} \times \mathfrak{M}_j(Z_k^{(n)})$. We can similarly define the stack $\mathfrak{M}_j(\widehat{Z_k})$. We similarly have the stacks $\mathfrak{M}(Z_k^{(n)})$ of bundles where we drop the condition on splitting type.

There is an inverse (or projective) system of stacks of finite type over $\mathbb{C}$:

$$\cdots \to \mathfrak{M}_j(Z_k^{(3)}) \to \mathfrak{M}_j(Z_k^{(2)}) \to \mathfrak{M}_j(Z_k^{(1)}) \to \mathfrak{M}_j(Z_k^{(0)}) = \mathfrak{M}_j(\mathbb{P}^1) \qquad (6)$$

whose inverse limit in the category of algebraic stacks is $\mathfrak{M}_j(\widehat{Z_k})$. Alternatively we can consider the inverse system $\mathfrak{M}_j(Z_k^{(\bullet)})$ to be an pro-stack of pro-finite type. This type of approximation is studied in [21]. It seems difficult to compute invariants of the stacks $\mathfrak{M}_j(Z_k^{(n)})$ using only the definition above so we will find a more explicit description below.

2.2 *The Structure of Vector Bundle Isomorphisms*

Consider the bundles E_p defined in Definition 1. There is a exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}(E_p, E_{p'}) &\to \operatorname{End}(\mathscr{O}(-j) \oplus \mathscr{O}(j)) \\ &\longrightarrow \operatorname{Ext}^1(\mathscr{O}(-j) \oplus \mathscr{O}(j), \mathscr{O}(-j) \oplus \mathscr{O}(j)) \to \operatorname{Ext}^1(E_p, E_{p'}) \to 0. \qquad (7) \end{aligned}$$

We now explain the structure of isomorphisms between families of bundles coming from extensions by constructing an explicit splitting for the first non-trivial map in this sequence. If the bundles E_p and $E_{p'}$ on $Z_k^{(n)} \times T$, given by maps

$$p, p' : T \to R_{j,k}^{(n)}$$

are isomorphic (see Eq. (4)) then necessarily they have the same splitting type, and in such case we can represent them by transition matrices on

$$(U^{(n)} \cap V^{(n)}) \times T$$

by $\begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix}$ and $\begin{pmatrix} z^j & p' \\ 0 & z^{-j} \end{pmatrix}$ respectively. An isomorphism between E_p and $E_{p'}$ is given by a pair of invertible matrices

$$A = \begin{pmatrix} a_U & b_U \\ c_U & d_U \end{pmatrix}$$

regular on $U^{(n)} \times T$ and

$$B = \begin{pmatrix} a_V & b_V \\ c_V & d_V \end{pmatrix}$$

regular on $V^{(n)} \times T$, such that:

$$B \begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix} = \begin{pmatrix} z^j & p' \\ 0 & z^{-j} \end{pmatrix} A, \tag{8}$$

or equivalently

$$\begin{aligned} B &= \begin{pmatrix} z^j & p' \\ 0 & z^{-j} \end{pmatrix} A \begin{pmatrix} z^{-j} & -p \\ 0 & z^j \end{pmatrix} \\ &= \begin{pmatrix} a_U + z^{-j} p' c_U & z^{2j} b_U + z^j (p' d_U - a_U p) - p p' c_U \\ z^{-2j} c_U & d_U - z^{-j} p c_U \end{pmatrix}. \end{aligned} \tag{9}$$

Definition 3. We use the notation Y^+ to denote the terms in $Y \in \mathscr{O}((U^{(n)} \cap V^{(n)}) \times T)$ that are not regular on $V^{(n)} \times T$ and $Y^{+,\geq 2j}$ denotes the terms in Y that are not regular on $V^{(n)} \times T$ and have power of z greater than or equal to $2j$.

Lemma 4. *Suppose that $j > 0$. Then any isomorphism (A, B) between E_p and $E_{p'}$ on $Z_k^{(n)} \times T$ has the form*

$$(A, B) = (M_U, M_V) + (\Phi_U(M), \Phi_V(M)) \tag{10}$$

where

$$M = (M_U, M_V) \in \mathrm{Aut}_{Z_k^{(n)} \times T}(\mathscr{O}(j) \oplus \mathscr{O}(-j)).$$

$$M_U = \begin{pmatrix} \underline{a} & \underline{b}_U \\ \underline{c}_U & \underline{d} \end{pmatrix}$$

and

$$\Phi_U(M) = \begin{pmatrix} -(z^{-j} p' \underline{c}_U)^+ & -z^{-2j} \left(z^j (p' \underline{d} - \underline{a} p) - p p' \underline{c}_U\right)^{+,\geq 2j} \\ 0 & (z^{-j} p \underline{c}_U)^+ \end{pmatrix}$$

depends only on p, p' and M and satisfies

$$[p' \underline{d} - \underline{a} p - z^{-j} p p' \underline{c}_U] = 0 \in \mathrm{Ext}^1_{Z_k^{(n)} \times T}(\mathscr{O}(j), \mathscr{O}(-j)). \tag{11}$$

Proof. First suppose that such an isomorphism exists, between E_p and $E_{p'}$. Then we have

$$\begin{pmatrix} z^j & p' \\ 0 & z^{-j} \end{pmatrix} A - B \begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix} = 0. \tag{12}$$

The left hand side comes out to be

$$\begin{pmatrix} p'c_U + (a_U - a_V)z^j & d_U p' - a_V p + z^j b_U - z^{-j} b_V \\ c_U z^{-j} - c_V z^j & z^{-j}(d_U - d_V) - c_V p \end{pmatrix}. \tag{13}$$

The lower left corner of (13) implies first of all that c must be a section $\underline{c}$ of $\mathcal{O}(2j)$. We need to arrange for the vanishing of all terms in (13). Therefore, we need to solve the equations:

$$\begin{aligned} a_U - a_V \quad &= -z^{-j} p' \underline{c}_U \\ z^j b_U - z^{-j} b_V &= -d_U p' + a_V p \\ d_U - d_V \quad &= z^j \underline{c}_V p. \end{aligned}$$

Becasue $H^1(Z_k^{(n)} \times T, \mathcal{O})$ vanishes, the first and third equations have solutions which are unique up to global functions. Let

$$a_U = \underline{a} - (z^{-j} p' \underline{c}_U)^+$$

and

$$d_U = \underline{d} + (z^j \underline{c}_V p)^+.$$

These solve the first and third equation. If we substitute into the second equation, it reads

$$\begin{aligned} z^j b_U - z^{-j} b_V &= -(z^j \underline{c}_V p)^+ p' + (-(z^{-j} p' \underline{c}_U)^+ + z^{-j} p' \underline{c}_U) p - \underline{d} p' + \underline{a} p \\ &= -\underline{d} p' + \underline{a} p + z^{-j} p p' \underline{c}_U. \end{aligned} \tag{14}$$

This implies that

$$[p' \underline{d} - \underline{a} p - z^{-j} p p' \underline{c}_U] = 0 \in \mathrm{Ext}^1_{Z_k^{(n)} \times T}(\mathcal{O}(j), \mathcal{O}(-j)).$$

Conversely, suppose that these conditions are satisfied by some $\underline{a}$, $\underline{d}$, $\underline{c}$, p, and p', let us record the general form of an element of $\mathrm{Isom}_{Z_k^{(n)} \times T}(E_p, E_{p'})$. It remains only to determine the expression for b_U. By the assumptions we already know that

$$\left(z^j (p' \underline{d} - \underline{a} p) - p p' \underline{c}_U\right)^{+,<2j}$$

is regular on $V^{(n)} \times T$. Hence

$$b_U = \underline{b}_U - z^{-2j}\left(z^j(p'\underline{d} - \underline{a}p) - pp'\underline{c}_U\right)^{+,\geq 2j}.$$

Finally, since u divides p and p', we know that A is invertible if and only if M_U is and therefore the isomorphism (A, B) is invertible if and only if the automorphism M is invertible.

Remark 6. We conclude that the expression of the element (A, B) of $\mathrm{Hom}(E_p, E_{p'})$ under the decomposition (43)

$$\mathrm{Hom}(E_p, E_{p'}) = \mathrm{Hom}(\mathcal{O}(j), \mathcal{O}(-j)) \oplus \phi(\ker(d_1^{1,-1})) \oplus \psi(\ker(d_2^{0,0}))$$

from the appendix is satisfied if we take $\underline{b} \in \mathrm{Hom}(\mathcal{O}(j), \mathcal{O}(-j))$,

$$\psi_U(c) = \begin{pmatrix} -(z^{-j}p'\underline{c}_U)^{+} & z^{-2j}(pp'\underline{c}_U)^{+,\geq 2j} \\ \underline{c}_U & (z^{-j}p\underline{c}_U)^{+} \end{pmatrix}$$

and

$$\phi_U(\underline{a}, \underline{d}) = \begin{pmatrix} \underline{a} & -z^{-2j}\left(z^j(p'\underline{d} - \underline{a}p)\right)^{+,\geq 2j} \\ 0 & \underline{d} \end{pmatrix}.$$

2.3 Bundle Isomorphism Viewed as an Equivalence Relation

Although we have worked out the structure of the space of isomorphisms between two given bundles, this does not yet give a criterion for when two bundles are isomorphic nor does it provide any understanding of the equivalence relation on $W_{j,k}^{(n)}$ given by isomorphisms of vector bundles. We show that there are algebraic groups $G_{j,k}^{(n)}$ acting on $W_{j,k}^{(n)}$ so that the orbits of this action are identified with the equivalence classes. This action (17) takes on the familiar form of a Möbius transformation. Lange studied in [16] (see also Drézet [11]) the question of universal bundles over the projectivized space of extensions. In a specific example we study here a more difficult problem, the difference being that we do not remove the origin and we consider all vector bundle isomorphisms, not just those that correspond to scaling the extension. First we need to define the structure of a scheme on the sets $\mathrm{Aut}_{Z_k^{(n)}}(\mathcal{O}(j) \oplus \mathcal{O}(-j))$ for n finite.

Definition 4. Consider the functors from schemes to sets given by

$$T \mapsto \mathrm{Aut}_{Z_k^{(n)} \times T}(\mathcal{O}(j) \oplus \mathcal{O}(-j)).$$

These functors are $\mathbb{C}$-groups (sheaves of groups in the faithfully flat topology on schemes) and are easily seen to be representable by reduced schemes. These schemes are in fact affine, being defined inside the finite dimensional affine space

$$\mathbb{E}\mathrm{nd}_{Z_k^{(n)}}(\mathscr{O}(j) \oplus \mathscr{O}(-j))$$

defined with coordinates as in Remark 8 as the complement of the pre-image of 0 by the morphism

$$det_0 : \mathbb{E}\mathrm{nd}_{Z_k^{(n)}}(\mathscr{O}(j) \oplus \mathscr{O}(-j)) \to \mathscr{O}(Z_k^{(n)}) \to Spec(\mathbb{C}[s]).$$

sending s to the restriction of the determinant to ℓ. When we pass to $\mathbb{C}$ points we get the standard determinant followed by restriction to ℓ

$$det_0 : \mathrm{End}_{Z_k^{(n)}}(\mathscr{O}(j) \oplus \mathscr{O}(-j)) \to \mathscr{O}(Z_k^{(n)}) \to \mathscr{O}(Z_k^{(0)}) = \mathbb{C}.$$

We denote these finite dimensional algebraic groups by $G_{j,k}^{(n)}$. These form a directed system of $\mathbb{C}$-spaces (sheaf of sets for the faithfully flat topology on the category of $\mathbb{C}$-algebras) and their direct limit as a $\mathbb{C}$-space (see [9] for this yoga) is representable by an infinite dimensional algebraic variety,

$$\widetilde{G_{j,k}} = G_{j,k}^{(\infty)}$$

which has $\mathrm{Aut}_{Z_k^{(\infty)}}(\mathscr{O}(j) \oplus \mathscr{O}(-j))$ as its underlying set of $\mathbb{C}$-points. In fact, $\widetilde{G_{j,k}}$ is an infinite-dimensional algebraic group. The sequence $G_{j,k}^{(\bullet)}$

$$\cdots \to G_{j,k}^{(3)} \to G_{j,k}^{(2)} \to G_{j,k}^{(1)} \to G_{j,k}^{(0)} = \mathrm{Aut}_{\mathbb{P}^1}(\mathscr{O}(j) \oplus \mathscr{O}(-j)) \tag{15}$$

is an pro-finite-type pro-scheme. We often write elements of $\mathrm{Hom}(T, G_{j,k}^{(n)})$ as matrices.

Consider the following direct sum decomposition of the vector space of functions

$$\mathscr{O}_{Z_k^{(n)}}(U^{(n)} \cap V^{(n)}) = \mathscr{O}_{Z_k^{(n)}}(U^{(n)} \cap V^{(n)})^{\succ} \oplus \mathscr{O}_{Z_k^{(n)}}(U^{(n)} \cap V^{(n)})_{good} \oplus \mathscr{O}_{Z_k^{(n)}}(U^{(n)} \cap V^{(n)})^{\prec}$$

where the sector named "good" corresponds to the terms appearing in Lemma 1, and also

$$z^j \mathscr{O}_{Z_k^{(n)}}(U^{(n)} \cap V^{(n)})^{\prec} \subset \mathscr{O}(V^{(n)})$$

and

$$z^{-j}\mathcal{O}_{Z_k^{(n)}}(U^{(n)}\cap V^{(n)})^{\succ}\subset\mathcal{O}(U^{(n)})$$
$$q-q_{good}=q^{\succ}+q^{\prec}.$$

As in Eq. (8) we write elements of

$$\mathrm{Hom}(T,G_{j,k}^{(n)})\subset H^0(Z_k^{(n)}\times T,\mathcal{O}^{\oplus 2}\oplus\mathcal{O}(2j)\oplus\mathcal{O}(-2j))$$

in the form

$$g=\begin{pmatrix}\underline{a} & \underline{b}\\ \underline{c} & \underline{d}\end{pmatrix}. \tag{16}$$

with $\underline{b}=(\underline{b}_U,\underline{b}_V)$ and $\underline{b}_U$ holomorphic on $U^{(n)}\times T$, etc. First of all notice that the group $\mathrm{Hom}(T,G_{j,k}^{(n)})$ acts on the functions p on $U^{(n)}\cap V^{(n)}\times T$ which vanish on the zero section by the formula

$$gp=\frac{\underline{a}p-z^j\underline{b}_U}{\underline{d}-z^{-j}p\underline{c}_U}. \tag{17}$$

A special case (where $\underline{b}$ and $\underline{c}$ are taken to be zero) of this action was observed for general varieties and bundles in [11]. For n finite, such functions vanishing on ℓ belong to $u\mathbb{C}[z,z^{-1}][u]/(u^{n+1})$, in the case $n=\infty$ such functions belong to $u\mathbb{C}[z,z^{-1}][[u]]$. The action $p\mapsto gp$ does not preserve the finite dimensional space $R_{j,k}^{(n)}$ which was written in (4). This means that we need to somehow correct the morphism $(g,p)\mapsto gp$. This will happen in the next definition.

Definition 5. Define a morphism

$$G_{j,k}^{(n)}\times R_{j,k}^{(n)}\to R_{j,k}^{(n)}$$

by

$$\begin{aligned}(g,p)\mapsto g\bullet p&=\frac{\underline{a}p-z^j\underline{b}_U}{\underline{d}-z^{-j}p\underline{c}_U}-\left(\frac{\underline{a}p-z^j\underline{b}_U}{\underline{d}-z^{-j}p\underline{c}_U}\right)^{\succ}-\left(\frac{\underline{a}p-z^j\underline{b}_U}{\underline{d}-z^{-j}p\underline{c}_U}\right)^{\prec}.\\&=\left(\frac{\underline{a}p-z^j\underline{b}_U}{\underline{d}-z^{-j}p\underline{c}_U}\right)_{good}\end{aligned} \tag{18}$$

This morphism will become one of the structure maps of a groupoid (see Eq. (40)). It is not the action of a group.

Consider

$$A_g(p) = \begin{pmatrix} \underline{a} - (z^{-j} p\underline{c}_U)^+ & \underline{b}_U - z^{-2j} \left(z^j \left((g \bullet p)\underline{d} - \underline{a}p\right) - p(g \bullet p)\underline{c}_U\right)^{+,\geq 2j} \\ \underline{c}_U & \underline{d} + (z^{-j} \underline{c}_U (g \bullet p))^+ \end{pmatrix} \tag{19}$$

and

$$B_g(p) = \begin{pmatrix} \underline{a} + (z^{-j} p\underline{c}_U)^+ & \underline{b}_V + \left(z^j \left((g \bullet p)\underline{d} - \underline{a}p\right) - p(g \bullet p)\underline{c}_U\right)^{+,<2j} \\ \underline{c}_V & \underline{d} - (z^{-j} \underline{c}_U (g \bullet p))^+ \end{pmatrix}. \tag{20}$$

They are regular over $U^{(n)} \times T$ and $V^{(n)} \times T$ respectively because they satisfy $(A_g(p), B_g(p)) = (M_U, M_V)$ from Lemma 4 in the case that $p' = g \bullet p$. That is to say, they satisfy

$$B_g(p) \begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix} = \begin{pmatrix} z^j & g \bullet p \\ 0 & z^{-j} \end{pmatrix} A_g(p) \tag{21}$$

and so the pair $(A_g(p), B_g(p))$ provides an isomorphism between E_p and $E_{g \bullet p}$. We have shown the following Lemma.

Lemma 5. *There is a morphism*

$$\begin{aligned} G_{j,k}^{(n)} \times R_{j,k}^{(n)} &\to R_{j,k}^{(n)} \\ (g, p) &\mapsto g \bullet p \end{aligned} \tag{22}$$

such that for two bundles E_p and $E_{p'}$ of constant splitting type j,

$$\begin{aligned} \mathrm{Isom}_{Z_k^{(n)} \times T}(E_p, E_{p'}) &= \{g \in \mathrm{Hom}(T, G_{j,k}^{(n)}) \mid g \bullet p = p'\} \\ &= \{g \in \mathrm{Hom}(T, G_{j,k}^{(n)}) \mid \textit{11 is satisfied}\}. \end{aligned} \tag{23}$$

□

Consider the isomorphism

$$(A_{g_1}(g_2 \bullet p) A_{g_2}(p), B_{g_1}(g_2 \bullet p) B_{g_2}(p))$$

between E_p and $E_{g_1 \bullet (g_2 \bullet p)}$. In Lemma 4, we defined an element

$$g_1 \bullet_p g_2 \in G_{j,k}^{(n)}(\mathbb{C})$$

such that this isomorphism equals $(A_{g_1 \bullet_p g_2}, B_{g_1 \bullet_p g_2})$. Similarly, the isomorphism $(A_g(p)^{-1}, B_g(p)^{-1})$ between $E_{g \bullet p}$ and E_p corresponds to a an element

$$g^{(-1)_p} \in G_{j,k}^{(n)}(\mathbb{C}). \tag{24}$$

From here it is clear (since both $A_{e_{G^{(n)}_{j,k}}}(p)$ and $B_{e_{G^{(n)}_{j,k}}}(p)$ are the identity matrix) that

$$g \bullet_p g^{(-1)_p} = e_{G^{(n)}_{j,k}} = g^{(-1)_p} \bullet_p g. \tag{25}$$

Definition 6. Define $g_1 \bullet_p g_2$ and $g^{(-1)_p}$ to be the elements of $G^{(n)}_{j,k}(\mathbb{C})$ corresponding via Lemma 4 to the isomorphisms $(A_{g_1}(g_2 \bullet p)A_{g_2}(p), B_{g_1}(g_2 \bullet p)B_{g_2}(p))$ and $(A_g(p)^{-1}, B_g(p)^{-1})$ described above.

The elements $g_1 \bullet_p g_2$ vary algebraically with g_1 and g_2 and give a morphism of schemes

$$G^{(n)}_{j,k} \times G^{(n)}_{j,k} \times W^{(n)}_{j,k} \to G^{(n)}_{j,k}$$
$$(g_1, g_2, p) \mapsto g_1 \bullet_p g_2.$$

The restriction to $p = 0$ in $W^{(n)}_{j,k}$ gives us back the standard multiplication but in general this structure does depend on p.

Therefore by definition we have

$$B_{g_1}(g_2 \bullet p)B_{g_2}(p) = B_{g_1 \bullet_p g_2}(p). \tag{26}$$

(and also $A_{g_1}(g_2 \bullet p)A_{g_2}(p) = A_{g_1 \bullet_p g_2}(p)$). An immediate consequence of this together with (21) is

$$g_1 \bullet (g_2 \bullet p) = (g_1 \bullet_p g_2) \bullet p, \tag{27}$$

and we also have

$$\begin{aligned} B_{(g_1 \bullet_{(g_3 \bullet p)} g_2) \bullet_p g_3}(p) &= B_{g_1 \bullet_{(g_3 \bullet p)} g_2}(g_3 \bullet p)B_{g_3}(p) \\ &= B_{g_1}(g_2 \bullet (g_3 \bullet p))B_{g_2}(g_3 \bullet p)B_{g_3}(p) \\ &= B_{g_1}(g_2 \bullet (g_3 \bullet p))B_{g_2 \bullet_p g_3}(p) = B_{g_1 \bullet_p (g_2 \bullet_p g_3)}(p) \end{aligned} \tag{28}$$

and similarly for $A_g(p)$. Because every isomorphism (A, B) which takes one of our chosen transition matrices corresponding to a bundle E_p to another transition matrix of the same form corresponds (7) to a unique $g \in \mathrm{Hom}(T, G^{(n)}_{j,k})$ we conclude that

$$(g_1 \bullet_{(g_3 \bullet p)} g_2) \bullet_p g_3 = g_1 \bullet_p (g_2 \bullet_p g_3). \tag{29}$$

This will be used to verify the associativity of the groupoid structure. A direct inspection of (18), (19) and (20) shows that identity matrix $e_{G^{(n)}_{j,k}}$ satisfies

$$e_{G^{(n)}_{j,k}} \bullet p = p \tag{30}$$

for any p and corresponds to the identity map from E_p to itself. Therefore we of course have

$$e_{G^{(n)}_{j,k}} \bullet_p g = g = g \bullet_p e_{G^{(n)}_{j,k}} \tag{31}$$

for any p.

3 An Explicit Groupoid in Schemes

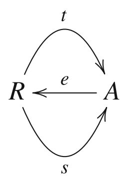
In this section we describe an explicit groupoid in schemes and show that its associated stack is isomorphic to the stack of rank 2 vector bundles of splitting type j and first Chern class 0 on $Z_k^{(n)}$.

3.1 Review of Groupoids in Schemes and Their Sheaf Theory

We begin with a review of the definition of a groupoid in schemes and the notion of a sheaf on a groupoid in schemes. Recall that a groupoid

$$\mathscr{G} = (A, R, s, t, m, e, \iota)$$

in schemes consists of schemes A (the atlas) and R (the relations), morphisms s, t, m, e, ι



$$R_t \times_{A\ s} R \xrightarrow{m} R \tag{32}$$

and

$$R \xrightarrow{\iota} R$$

which satisfy some conditions which we write below. Here

$$R_t \times_{A\ s} R = \{(r_1, r_2) \in R \times R | t(r_1) = s(r_2)\}.$$

Let p_1, p_2 be the first and second projections

$$R_t \times_{A\ s} R \overset{p_1, p_2}{\longrightarrow} R$$

and let Δ be the diagonal

$$R \times R \xleftarrow{\Delta} R.$$

The morphisms then must satisfy

$$m \circ (m, \mathrm{id}_R) = m \circ (\mathrm{id}_R, m) \tag{33}$$

on all composable elements of $R \times R \times R$,

$$t \circ m = t \circ p_2, \quad s \circ m = s \circ p_1 \tag{34}$$

on all composable elements of $R \times R$

$$m \circ (\iota, \mathrm{id}_R) \circ \Delta = e \circ s, \quad m \circ (\mathrm{id}_R, \iota) \circ \Delta = e \circ s \tag{35}$$

on R, and also

$$m \circ (\mathrm{id}_R, e \circ t) \circ \Delta = \mathrm{id}_R, \quad m \circ (e \circ s, \mathrm{id}_R) \circ \Delta = \mathrm{id}_R \tag{36}$$

on R. Notice that for any scheme S that by taking the set of morphisms of schemes from S into R and A one gets a pair of sets and these naturally form a groupoid in sets using the obvious maps. We denote this groupoid in sets by

$$\mathrm{Hom}(S, \mathscr{G}).$$

A (coherent/locally free of rank r) sheaf of modules on the groupoid consists of a (coherent/locally free of rank r) sheaf $\mathscr{S}$ of $\mathscr{O}_A$ modules on A together with an isomorphism f of sheaves of $\mathscr{O}_R$ modules over R

$$f : s^* \mathscr{S} \to t^* \mathscr{S}$$

which satisfies

$$p_2^* f \circ p_1^* f = m^* f \tag{37}$$

and

$$e^* f = \mathrm{id}. \tag{38}$$

To make sense of this equality, one must use the identities

$$s \circ p_1 = s \circ m, \quad \text{and} \quad t \circ p_2 = t \circ m.$$

3.2 Stacks from Groupoids

Let $\mathscr{G} = (A, R, s, t, m, e, \iota)$ be a groupoid in schemes.

We associate to it a stack $[\mathscr{G}]$ defined as the stack on the fppf site associated to the prestack pre-$[\mathscr{G}]$ which associates to any test scheme T the groupoid in sets

$$\text{pre-}[\mathscr{G}](T) = \text{Hom}(T, \mathscr{G}).$$

Notice that such a morphism consists of a map from maps from T to A, and T to R which satisfy the obvious compatibilities.

Remark 7. In the case that $R = G \times A$ and the groupoid structure is just given by a group action of G on A, we may denote the associated quotient stack by $[A/G]$, leaving the structure implicit.

There is an equivalence [17] of Abelian categories of coherent sheaves which takes vector bundles to vector bundles

$$\text{Coh}(\mathscr{G}) \xrightarrow{\cong} \text{Coh}([\mathscr{G}]). \tag{39}$$

Definition 7. We denote by $[\mathscr{S}]$ the sheaf on $[\mathscr{G}]$ corresponding to a sheaf $\mathscr{S}$ on $\mathscr{G}$ under the equivalence (39) given above.

3.3 Groupoid Presentations for Stacks of Rank 2 Bundles

We define a groupoid in schemes to be called $\mathscr{G}_{j,k}^{(n)}$. The atlas of $\mathscr{G}_{j,k}^{(n)}$ is $W_{j,k}^{(n)}$ and the relations are $G_{j,k}^{(n)} \times W_{j,k}^{(n)}$.

The arrow s is given by the projection

$$G_{j,k}^{(n)} \times W_{j,k}^{(n)} \xrightarrow{s} W_{j,k}^{(n)}.$$

defined by

$$(g, p) \mapsto p.$$

The arrow t is given by the map

$$G_{j,k}^{(n)} \times W_{j,k}^{(n)} \xrightarrow{t} W_{j,k}^{(n)}. \tag{40}$$

defined by

$$(g, p) \mapsto g \bullet p.$$

where $g \bullet p$ is defined in Definition 5. The multiplication

$$m : (G^{(n)}_{j,k} \times W^{(n)}_{j,k})_s \times_{W^{(n)}_{j,k}} {}_t(G^{(n)}_{j,k} \times W^{(n)}_{j,k}) \to G^{(n)}_{j,k} \times W^{(n)}_{j,k}$$

is given by

$$m\left((g_1, g_2 \bullet p), (g_2, p)\right) = (g_1 \bullet_p g_2, p)$$

where $g_1 \bullet_p g_2$ is defined in Definition 6.

The identity section is defined by

$$e(p) = (\mathrm{id}, p)$$

and the inverse is defined by

$$\iota(g, p) = (g^{(-1)_p}, g \bullet p)$$

where $g^{(-1)_p}$ was defined in Definition 5. The associativity condition (33) follows from (29). The conditions (34), (36) and (35) follow from (27), (31), and (25).

We get an inverse system $\mathscr{G}^{(\bullet)}_{j,k}$ in the category of groupoids in schemes:

$$\cdots \to \mathscr{G}^{(3)}_{j,k} \to \mathscr{G}^{(2)}_{j,k} \to \mathscr{G}^{(1)}_{j,k} \to \mathscr{G}^{(0)}_{j,k}. \tag{41}$$

and the inverse limit is $\widetilde{\mathscr{G}_{j,k}} = \mathscr{G}^{(\infty)}_{j,k}$.

3.4 *The Morphism Defined via the Big Bundle* $\mathbb{E}$

The big bundle $\mathbb{E}$ defines a morphism of stacks from $W^{(n)}_{j,k}$ to $\mathfrak{M}_j(Z^{(n)}_k)$ as follows. Given an affine scheme T, we have a map

$$\varphi_T \colon \mathrm{Hom}(T, W^{(n)}_{j,k}) \to \mathrm{Hom}(T, \mathfrak{M}_j(Z^{(n)}_k))$$
$$f \mapsto (\mathrm{id}, f)^*\mathbb{E}$$

given by sending f to the pullback of $\mathbb{E}$ via the map

$$(\mathrm{id}, f) : Z^{(n)}_k \times T \to Z^{(n)}_k \times W^{(n)}_{j,k}.$$

Lemma 6. *For each $j \geq 0$ the substacks*

$$\mathfrak{M}_{\leq j}(Z_k^{(n)}) = \bigcup_{0 \leq i \leq j} \mathfrak{M}_i(Z_k^{(n)})$$

of $\mathfrak{M}(Z_k^{(n)})$ are given by

$$T \mapsto \Big\{ E \in \mathfrak{M}(Z_k^{(n)})(T) | \pi_{T*}(E \otimes \mathscr{O}(j)) \text{ is generated by global sections and} R^1\pi_{T*}(E \otimes \mathscr{O}(j)) = 0 \Big\}.$$

Proof. By Serre's theorem, $\mathfrak{M}(Z_k^{(n)})$ is covered by the open substacks defined

$$T \mapsto \Big\{ E \in \mathfrak{M}(Z_k^{(n)})(T) | \pi_{T*}(E \otimes \mathscr{O}(j)) \text{ is generated by global sections and} R^1\pi_{T*}(E \otimes \mathscr{O}(j)) = 0 \Big\}.$$

In order to show the Lemma we can work locally in the site, and show the equivalence using the prestacks pre-$[\mathscr{G}_{j,k}^{(n)}]$. First suppose that E has constant (in T) splitting type less than or equal to j. Using Lemma 1, we can assume (after shrinking T) that is an extension of $\mathscr{O}(i)$ by $\mathscr{O}(-i)$ for $0 \leq i \leq j$. Then $E \otimes \mathscr{O}(i)$ is an extension of $\mathscr{O}(2i)$ by $\mathscr{O}$. Due to the fact that $H^1(Z_k^{(n)} \times T, \mathscr{O}) = 0$, the resulting sequence on global sections is exact. Both of the line bundles $\mathscr{O}(2i)$ and $\mathscr{O}$ are generated by their global sections, and the fact that $\pi_{T*}(E \otimes \mathscr{O}(i))$ and therefore $\pi_{T*}(E \otimes \mathscr{O}(j))$ is generated by its global sections follows. However, $H^1(Z_k^{(n)}, \mathscr{O}(a))$ vanishes for $a \geq 0$ and therefore $R^1\pi_{T*}(E \otimes \mathscr{O}(j))$ vanishes. Conversely, suppose that $\pi_{T*}(E \otimes \mathscr{O}(j))$ is generated by global sections and $R^1\pi_{T*}(E \otimes \mathscr{O}(j)) = 0$. The second condition implies (see Remark 8) that for every geometric point t of T, the splitting type of the restriction of E to $Z_k^{(n)} \times \{t\}$ is less than or equal to j. Therefore, E belongs to $\mathfrak{M}_{\leq j}(Z_k^{(n)})(T)$.

3.5 The Universal Bundle $\tilde{\mathscr{E}}$

We now construct the universal bundle on the groupoid

$$Z_k^{(n)} \times \mathscr{G}_{j,k}^{(n)}.$$

The groupoid in question has atlas $Z_k^{(n)} \times W_{j,k}^{(n)}$ and relations $Z_k^{(n)} \times G_{j,k}^{(n)} \times W_{j,k}^{(n)}$. We use the description of sheaves on groupoids in schemes given in Sect. 3.1. We start with the big bundle $\mathbb{E}$ on $Z_k^{(n)} \times W_{j,k}^{(n)}$ which was defined in Definition 1. Consider the map in

$$\mathrm{Isom}_{Z_k^{(n)} \times G_{j,k}^{(n)} \times W_{j,k}^{(n)}}((\mathrm{id}_{Z_k^{(n)}}, t)^*\mathbb{E}, (\mathrm{id}_{Z_k^{(n)}}, s)^*\mathbb{E})$$

given by the pair

$$(A_g(p), B_g(p)) \in \mathrm{Aut}\left(U^{(n)} \times G_{j,k}^{(n)} \times W_{j,k}^{(n)}, \mathscr{O}^{\oplus 2}\right) \\ \times \mathrm{Aut}\left(V^{(n)} \times G_{j,k}^{(n)} \times W_{j,k}^{(n)}, \mathscr{O}^{\oplus 2}\right)$$

which was defined in Eqs. (19) and (20). We need to consider the pullbacks of the isomorphism to

$$Z_k^{(n)} \times (G_{j,k}^{(n)} \times W_{j,k}^{(n)})_s \times_{W_{j,k}^{(n)}} {}_t(G_{j,k}^{(n)} \times W_{j,k}^{(n)})$$

via the maps

$$(\mathrm{id}_{Z_k^{(n)}}, m), (\mathrm{id}_{Z_k^{(n)}}, p_1), (\mathrm{id}_{Z_k^{(n)}}, p_2)$$

where m, p_1, p_2 are the maps

$$(G_{j,k}^{(n)} \times W_{j,k}^{(n)})_s \times_{W_{j,k}^{(n)}} {}_t(G_{j,k}^{(n)} \times W_{j,k}^{(n)}) \to G_{j,k}^{(n)} \times W_{j,k}^{(n)}$$

given by

$$m\,((g_1, g_2 \bullet p), (g_2, p)) = (g_1 \bullet_p g_2, p)$$
$$p_1\,((g_1, g_2 \bullet p), (g_2, p)) = (g_1, g_2 \bullet p)$$

and

$$p_2\,((g_1, g_2 \bullet p), (g_2, p)) = (g_2, p).$$

These pullbacks are described by the pairs of elements of

$$\mathrm{Aut}\left(U^{(n)} \times (G_{j,k}^{(n)} \times W_{j,k}^{(n)})_s \times_{W_{j,k}^{(n)}} {}_t(G_{j,k}^{(n)} \times W_{j,k}^{(n)}), \mathscr{O}^{\oplus 2}\right)$$

and

$$\mathrm{Aut}\left(V^{(n)} \times (G_{j,k}^{(n)} \times W_{j,k}^{(n)})_s \times_{W_{j,k}^{(n)}} {}_t(G_{j,k}^{(n)} \times W_{j,k}^{(n)}), \mathscr{O}^{\oplus 2}\right)$$

given by

$$(A_{g_1 \bullet_p g_2}(p), B_{g_1 \bullet_p g_2}(p)),$$

$$(A_{g_1}(g_2 \bullet p), B_{g_1}(g_2 \bullet p)),$$

and

$$(A_{g_2}(p), B_{g_2}(p))$$

respectively. Therefore identity (37) follows from (26) while (38) follows from (30) and consequently we have defined a vector bundle on the groupoid in accordance with the description in Sect. 3.1.

3.6 The Equivalence of Stacks

Let us first mention groupoid presentations in the case of line bundles.

The stack of line bundles on the $Z_k^{(n)}$ is equivalent to

$$\mathbb{Z} \times [\bullet/\mathscr{O}(Z_k^{(n)})^\times].$$

For example when $k = 1, n = \infty$ this stack is equivalent to

$$\mathbb{Z} \times [\bullet/\mathbb{C}[[x, y]]^\times].$$

In Sect. 3 we defined a groupoid in schemes

$$\mathscr{G}_{j,k}^{(n)} = (G_{j,k}^{(n)} \times W_{j,k}^{(n)}, W_{j,k}^{(n)}, m, e, \iota),$$

the associated pre-stack pre-$[\mathscr{G}_{j,k}^{(n)}]$ and the associated stack $[\mathscr{G}_{j,k}^{(n)}]$ on the fppf site.

Theorem 3. *The natural map $W_{j,k}^{(n)} \to \mathfrak{M}_j(Z_k^{(n)})$ given by the big bundle $\mathbb{E}$ which was defined in Definition 1 induces an isomorphism of stacks*

$$[\mathscr{G}_{j,k}^{(n)}] \cong \mathfrak{M}_j(Z_k^{(n)}).$$

Furthermore, there is a vector bundle

$$\begin{array}{c}[\tilde{\mathscr{E}}]\\ \downarrow\\ Z_k^{(n)} \times [\mathscr{G}_{j,k}^{(n)}]\end{array}$$

whose pullback to $Z_k^{(n)} \times W_{j,k}^{(n)}$ *is the big bundle* $\mathbb{E}$*, and is identified via the above isomorphism with the universal bundle* $\mathscr{E}$ *on* $Z_k^{(n)} \times \mathfrak{M}_j(Z_k^{(n)})$.

Here, $[\mathscr{G}_{j,k}^{(n)}]$ is the stack associated to the groupoid $\mathscr{G}_{j,k}^{(n)}$. This association is reviewed in Sect. 3.2.

Proof. We will prove this theorem by first defining a morphism of stacks over the fppf site and then show that it is locally in the site an equivalence of categories. Consider the morphism of pre-stacks

$$\text{pre-}F : \text{pre-}[\mathscr{G}_{j,k}^{(n)}] \to \mathfrak{M}_j(Z_k^{(n)})$$

by

$$\text{pre-}F_T(f) = (\mathrm{id}_{Z_k^{(n)}}, f)^* \tilde{\mathscr{E}}$$

where f is a morphism of groupoids from T to $\mathscr{G}_{j,k}^{(n)}$. Because $\mathfrak{M}_j(Z_k^{(n)})$ is already a stack over the fppf site, we get for free a morphism of the associated stacks over the fppf site

$$F : [\mathscr{G}_{j,k}^{(n)}] \to \mathfrak{M}_j(Z_k^{(n)}).$$

In order to show that this is an equivalence we need only to show that it is locally an isomorphism. Consider a vector bundle E on $Z_k^{(n)} \times T$ for an affine $\mathbb{C}$-scheme T and write it somehow (it does not matter how) as an extension of $\mathscr{O}(j)$ by $\mathscr{O}(-j)$ possibly after renaming T. Using Eq. (2) we have

$$\begin{aligned}
&\mathrm{Ext}^1_{Z_k^{(n)} \times T}(\pi^*_{Z_k^{(n)}} \mathscr{O}(j), \pi^*_{Z_k^{(n)}} \mathscr{O}(-j)) \\
&\quad = H^0(T, \mathscr{O}_T \otimes \mathit{Ext}^1_{Z_k^{(n)}}(\mathscr{O}(j), \mathscr{O}(-j))) = \mathrm{Hom}(T, W_{j,k}^{(n)}).
\end{aligned}$$

We can conclude that choosing (locally in the test schemes) the structure of an extension gives maps from T to the atlas of $\mathscr{G}_{j,k}^{(n)}$. It remains to show that the ambiguity in such choices is given by maps from T to the relations of $\mathscr{G}_{j,k}^{(n)}$. Suppose we have two maps p and p' from T to $W_{j,k}^{(n)}$. We need to show that

$$\mathrm{Isom}_{[\mathscr{G}_{j,k}^{(n)}](T)}(p, p') \cong \mathrm{Isom}_{Z_k^{(n)} \times T}((\mathrm{id}_{Z_k^{(n)}}, p)^* \mathbb{E}, (\mathrm{id}_{Z_k^{(n)}}, p')^* \mathbb{E}).$$

We have already naturally identified these two sets in Lemma 4.

We can use some easy observations about the explicit presentation we have established to give some properties of the stacks $\mathfrak{M}_j(Z_k^{(n)})$. First of all $G_{j,k}^{(n)}$ and

$W_{j,k}^{(n)}$ are reduced, irreducible, affine algebraic varieties. Notice that s is a projection and the map t factors as a Zariski open embedding followed by a projection

$$\begin{array}{ccc} R = G_{j,k}^{(n)} \times W_{j,k}^{(n)} & \longrightarrow & \mathbb{E}\mathrm{nd}_{Z_k^{(n)}}(\mathcal{O}(j) \oplus \mathcal{O}(-j)) \times W_{j,k}^{(n)} \\ & \overset{t}{\searrow} & \downarrow \\ & & A = W_{j,k}^{(n)}. \end{array}$$

where the horizontal map is

$$(g, p) \mapsto (g, g \bullet p).$$

The following could be concluded from the general construction of these stacks of vector bundles using Quot schemes due to Laumon and Moret-Bailly but we can give here a direct proof.

Corollary 1. *For every finite n, the stack $\mathfrak{M}_j(Z_k^{(n)})$ is an Artin stack.*

Proof. When n is finite then $G_{j,k}^{(n)}$ and $W_{j,k}^{(n)}$ are smooth affine varieties of finite type. By [17], Cor. 4.7, in order to conclude that it is an Artin stack, we need to show that s and t are flat and that the morphism

$$(s, t) : R \to A \times A$$

is separated and quasi-compact. Since n is finite, s and t are in fact smooth and therefore certainly flat. Quasi-compactness is obvious since R is quasi-compact. To see that (s, t) is separated we need to see that the induced diagonal

$$R \to R_{(s,t)} \times_{A \times A} {}_{(s,t)} R \tag{42}$$

is closed. Notice that $R_{(s,t)} \times_{A\times A} {}_{(s,t)} R$ is a closed subvariety of

$$G_{j,k}^{(n)} \times G_{j,k}^{(n)} \times W_{j,k}^{(n)}$$

defined by the equation

$$g_1 \bullet p = g_2 \bullet p.$$

The image of the diagonal (42) is therefore closed, being just the intersection inside

$$G_{j,k}^{(n)} \times G_{j,k}^{(n)} \times W_{j,k}^{(n)}$$

of

$$R_{(s,t)} \times_{A \times A} {}_{(s,t)}R$$

with the closed subvariety

$$\Delta_{G_{j,k}^{(n)}} \times W_{j,k}^{(n)}.$$

where

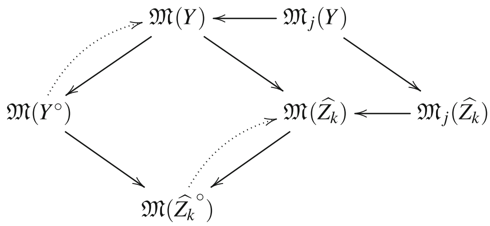
$$\Delta_{G_{j,k}^{(n)}} \subset G_{j,k}^{(n)} \times G_{j,k}^{(n)}$$

is the diagonal.

4 Applications

In a forthcoming article [5] we will use these groupoid presentations to calculate the space of deformations of the moduli stacks $\mathfrak{M}_j(Z_k^{(n)})$. To do this one must calculate the cohomology of the tangent complex (thought of as a complex of coherent sheaves) on these stacks. We then consider deformations of the $Z_k^{(n)}$. These include both classical and non-commutative deformations of the type considered in [8, 24]. By considering stacks of vector bundles over universal families of these deformations we get natural deformations of the stacks $\mathfrak{M}_j(Z_k^{(n)})$. We investigate the corresponding map from deformations of $Z_k^{(n)}$ to deformations of $\mathfrak{M}_j(Z_k^{(n)})$. This map is neither injective nor surjective. Such maps are well understood for the case of curves (see for example [19]); whereas for surfaces such maps are only understood in a few special cases, such as Mukai's [18] description for the case of $K3$ surfaces. In general such maps are quite mysterious for the case of surfaces. Thus, it is interesting to look at the question in the intermediate case of formal neighborhoods of curves inside surfaces.

Consider a proper algebraic surface X over $\mathbb{C}$. By attaching the stacks $\mathfrak{M}_j(\widehat{Z_k})$ to $\mathfrak{M}(X)$ in the correct way one gets certain substacks $\mathfrak{M}_j(Y)$ of the stack of vector bundles on the blow up of X at some point. Consider the punctured space $Z_k^\circ = Z_k - \ell$ and the punctured formal neighborhood $\widehat{Z_k}^\circ$ which is defined in [7] using Berkovich's analytic geometry. Now let Y is any algebraic surface containing a rational curve ℓ with $\ell^2 = -k$, $k > 0$ then let $Y^\circ = Y - \ell$. Let $\mathfrak{M}(Y)$ be the stack of all vector bundles of rank 2 whose restriction to ℓ has first Chern class zero, while $\mathfrak{M}(Y^\circ)$ and $\mathfrak{M}(\widehat{Z_k}^\circ)$ are the stacks of all vector bundles of rank 2 on Y° and $\widehat{Z_k}^\circ$ respectively. By taking stacks of vector bundles and using the main theorem of [7], we get a fiber product diagram of stacks along with the substacks of splitting type j,



consisting of the above diagram with the solid arrows only. The dotted curved arrows going up here exist only in the case that $k = 1$ and when the image of ℓ is a smooth point under the contraction of ℓ. Suppose we are in this case and $\pi : Y \to X$ is the contraction of ℓ. Then the dotted arrows are sections of the arrows in the opposite direction and are given by extending a bundle from $Y^\circ = Y - \ell \cong X - \{x\}$ to a bundle in $\mathfrak{M}(X)$ by taking the double dual of its pushforward and then pulling back the bundle via π to Y (and similarly on the other side). This diagram is an algebraic version of the holomorphic patching construction used in [15] and can be used to get information about the relationship of $\mathfrak{M}_j(Y)$ and $\mathfrak{M}(Y - \ell)$ from the relationship of $\mathfrak{M}_j(\widehat{Z_k})$ and $\mathfrak{M}(\widehat{Z_k}^\circ)$. This version of patching using stacks is a much more powerful construction, in particular avoiding all-together the use of framings, hence eliminating the unnecessarily complicated issues of infinite dimensionality of the space of reframings of each individual bundle. In this article we have focused on a description of $\mathfrak{M}_j(\widehat{Z_k})$. The application to topological information will appear in a forthcoming article [6] where we use the groupoid presentation to compute homology, cohomology and homotopy groups of the stacks of bundles.

Another reason why using stacks of bundles is preferable for gluing purposes over the construction via framings is that framings (in the sense of trivialising sections) simply do not exist in general. For the case of a surface with a -1 line it turns out to be possible to add framings to all holomorphic bundles, that is, every bundle on $\hat{Z}_1$ is trivial on $\hat{Z}_1^\circ$, so one can consider pairs of bundles together with framings, and glue by identifying framings. However, for elements of $\mathfrak{M}_j(\hat{Z}_k)$ only those satisfying $j = 0 \mod k$ are trivial on $\widehat{Z_k}^\circ$. This argument becomes even more relevant if one considers curves inside threefolds. For instance over completion $\widehat{W_1}$ of the resolved conifold $W_1 = \text{Tot}(\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ we can consider also rank 2 bundles with splitting $(j, -j)$ and define stacks $\mathfrak{M}_j(\widehat{W_1})$ but here only the trivial bundle is frameable in the sense of [15].

Appendix A: Some Cohomology Groups

The ring of global functions on $\widehat{Z_k}$ is

$$\mathcal{O}(\widehat{Z_k}) = \mathbb{C}[[x_0, x_1, \ldots, x_k]] \Big/ \sum_{i=0}^{k-2} \sum_{j=i+2}^{k} \left(x_i x_j - x_{i+1} x_{j-1} \right),$$

and for $Z_k^{(n)}$ one gets $\mathscr{O}(Z_k^{(n)}) = \mathscr{O}(Z_k)/m^{n+1}$ where m is the ideal $(x_0, \ldots, x_k)$. Note that here $x_i = z^i u$ in terms of the original coordinates on U and $U^{(n)}$. The zeroth cohomology is the torsion-free $\mathscr{O}(\widehat{Z_k})$ module

$$H^0(\widehat{Z_k}, \mathscr{O}(s)) = \bigoplus_{ki+s-l\geq 0, l\geq 0, i\geq 0} \mathbb{C}z^l u^i \subset \mathscr{O}(\hat{U}).$$

Similarly, we have the $\mathscr{O}(Z_k^{(n)})$ module

$$H^0(Z_k^{(n)}, \mathscr{O}(s)) = \bigoplus_{ki+s-l\geq 0, l\geq 0, n\geq i\geq 0} \mathbb{C}z^l u^i \subset \mathscr{O}(U^{(n)}).$$

Remark 8. The set $H^0(Z_k^{(n)}, \mathscr{O}(s))$ is the $\mathbb{C}$ points of the spectrum of the polynomial algebra freely generated over $\mathbb{C}$ by variables indexed by pairs (l, i) such that $ki + s - l \geq 0, l \geq 0, n \geq i \geq 0$. It is also easy to see that $H^1(Z_k^{(n)}, \mathscr{O}(s))$ vanishes for $s \geq 0$.

Appendix B: The Cohomology Spectral Sequence of $\mathscr{H}om(E, F)$

Consider a scheme Z covered by just two affine open sets U_1 and U_2 and two rank 2 vector bundles E and F on Z which trivialize on the U_i. Assume also that $H^1(Z, \mathscr{O}) = 0$. The Čech complex for computing the cohomology of $\mathscr{H}om(E, F)$ on Z looks like

$$\mathrm{Hom}_{U_1}(E|_{U_1}, F|_{U_1}) \oplus \mathrm{Hom}_{U_2}(E|_{U_2}, F|_{U_2}) \to \mathrm{Hom}_{U_1\cap U_2}(E|_{U_1\cap U_2}, F|_{U_1\cap U_2}).$$

If we choose local trivializations for $E|_{U_1}, E|_{U_2}$ and $F|_{U_1}, F|_{U_2}$ then the complex becomes

$$\mathrm{Hom}_{U_1}(\mathscr{O}^{\oplus 2}, \mathscr{O}^{\oplus 2}) \oplus \mathrm{Hom}_{U_2}(\mathscr{O}^{\oplus 2}, \mathscr{O}^{\oplus 2}) \to \mathrm{Hom}_{U_1\cap U_2}(\mathscr{O}^{\oplus 2}, \mathscr{O}^{\oplus 2})$$

with differential

$$(A, B) \mapsto G_E A - B G_F$$

where G_E, G_F are the transition matrices of E and F. On the other hand suppose we know that E and F can be written on Z as extensions of line bundles L_2 by L_1. By choosing local splittings the Čech complex becomes

$$\mathrm{End}_{U_1}(\mathscr{O}^{\oplus 2}) \oplus \mathrm{End}_{U_2}(\mathscr{O}^{\oplus 2}) \xrightarrow{D_1} \mathrm{End}_{U_1\cap U_2}(\mathscr{O}^{\oplus 2})$$

$$D_1(N_1, N_2) = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} N_1 - N_2 \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix},$$

$$D_2(M_1, M_2) = \begin{pmatrix} g_1 & p_E \\ 0 & g_2 \end{pmatrix} M_1 - M_2 \begin{pmatrix} g_1 & p_F \\ 0 & g_2 \end{pmatrix}.$$

$$\ker(D_1) \xrightarrow{\overline{D_2}} \operatorname{coker}(D_1)$$

Let us compute the cohomology groups

$$\ker(\overline{D_2}) = \operatorname{Hom}(E, F) \cong H^0(X, \mathscr{H}om(E, F))$$

and

$$\operatorname{coker}(\overline{D_2}) = \operatorname{Ext}^1(E, F) \cong H^1(X, \mathscr{H}om(E, F))$$

in terms of the extension and cohomology groups of the L_i. The filtration on $\mathscr{H}om(E, F)$ reads

$$0 \subset \mathscr{H}om(L_2, L_1) \subset \mathscr{H}om(E, L_1) + \mathscr{H}om(L_2, F) \subset \mathscr{H}om(E, F)$$

with associated graded pieces $\mathscr{H}om(L_2, L_1)$, $\mathscr{E}nd(L_1) \oplus \mathscr{E}nd(L_2)$, and $\mathscr{H}om(L_1, L_2)$. The associated spectral sequence computing the cohomology $\mathscr{H}om(E, F)$ has an E_1 term which looks like

$q=2$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$q=1$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$q=0$	$\operatorname{Hom}(L_1, L_2)$	0	$\vdots$	0
$q=-1$		$\operatorname{End}(L_1) \oplus \operatorname{End}(L_2)$	$\operatorname{Ext}^1(L_2, L_1)$	0
$q=-2$			$\operatorname{Hom}(L_2, L_1)$	0
$q=-3$				0
	$p=0$	$p=1$	$p=2$	$p=3$

The E_2 term looks like

$q=2$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$q=1$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$q=0$	$\mathrm{Hom}(L_1,L_2)$	0	$\vdots$	0
$q=-1$		$\ker(d_1^{1,-1})$	$\mathrm{coker}(d_1^{1,-1})$	0
$q=-2$			$\mathrm{Hom}(L_2,L_1)$	0
$q=-3$				0
	$p=0$	$p=1$	$p=2$	$p=3$

The E_3 term looks like

$q=2$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$q=1$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$q=0$	$\ker(d_2^{0,0})$	0	$\vdots$	0
$q=-1$		$\ker(d_1^{1,-1})$	$\mathrm{coker}(d_1^{1,-1})/\mathrm{im}(d_2^{0,0})$	0
$q=-2$			$\mathrm{Hom}(L_2,L_1)$	0
$q=-3$				0
	$p=0$	$p=1$	$p=2$	$p=3$

The first differential we consider is

$$H^0(X,\mathscr{O})^{\oplus 2} = \mathrm{End}(L_1) \oplus \mathrm{End}(L_2) \overset{d_1^{1,-1}}{\to} \mathrm{Ext}^1(L_2, L_1).$$

It is the connecting map for the cohomology of the short exact sequence

$$\begin{aligned} 0 &\to \mathscr{H}om(L_2, L_1) \to \mathscr{H}om(L_2, F) + \mathscr{H}om(E, L_1) \\ &\to \mathscr{E}nd(L_1) \oplus \mathscr{E}nd(L_2) \to 0 \end{aligned}$$

Consider the induced filtration on $\mathrm{Hom}(E, F)$ given by

$$0 \subset \mathrm{Hom}(L_2, L_1) \subset \mathrm{Hom}(E, L_1) + \mathrm{Hom}(L_2, F) \subset \mathrm{Hom}(E, F).$$

One has

$$\frac{\mathrm{Hom}(E, F)}{\mathrm{Hom}(E, L_1) + \mathrm{Hom}(L_2, F)} \cong \ker(d_2^{0,0}) \subset \mathrm{Hom}(L_1, L_2),$$

and

$$\frac{\mathrm{Hom}(E, L_1) + \mathrm{Hom}(L_2, F)}{\mathrm{Hom}(L_2, L_1)} \cong \ker(d_1^{1,-1}) \subset H^0(X, \mathscr{O})^{\oplus 2}.$$

For any choices of splittings

$$\mathrm{Hom}(E, F) \xleftarrow{\psi} \ker(d_2^{0,0}) \subset \mathrm{Hom}(L_1, L_2)$$

and

$$\mathrm{Hom}(E, L_1) + \mathrm{Hom}(L_2, F) \xleftarrow{\phi} \ker(d_1^{1,-1}) \subset H^0(X, \mathscr{O})^{\oplus 2}$$

we get a decomposition

$$\mathrm{Hom}(E, F) = \mathrm{Hom}(L_2, L_1) \oplus \phi(\ker(d_1^{1,-1})) \oplus \psi(\ker(d_2^{0,0})). \tag{43}$$

We record formulas for $d_1^{1,-1}$ and $d_2^{0,0}$ in the case that $X = Z_k^{(n)} \times T$ for some affine scheme T, $L_1 = \mathscr{O}(-j)$, $L_2 = \mathscr{O}(j)$, $E = E_p$, $F = E_{p'}$.

$$d_1^{1,-1} : H^0(X, (L_1 \otimes L_1^\vee) \oplus (L_2 \otimes L_2^\vee)) \to \mathrm{Ext}^1(L_2, L_1)$$

We compute

$$\begin{pmatrix} z^j & p' \\ 0 & z^{-j} \end{pmatrix} \begin{pmatrix} \underline{a} & 0 \\ 0 & \underline{d} \end{pmatrix} - \begin{pmatrix} \underline{a} & 0 \\ 0 & \underline{d} \end{pmatrix} \begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix} = \begin{pmatrix} 0 & \underline{d}\,p' - \underline{a}\,p \\ 0 & 0 \end{pmatrix}.$$

Therefore the element of $\mathrm{Ext}^1(L_2, L_1)$ to which the pair $(\underline{a}, \underline{d})$ maps is represented by $(\underline{d}\,p' - \underline{a}\,p)|_{(U^{(n)} \cap V^{(n)}) \times T}$. The differential

$$\begin{aligned} d_1^{1,-1} : H^0(X, \mathscr{O}^{\oplus 2}) &\to \mathrm{Ext}^1(\mathscr{O}(j), \mathscr{O}(-j)) \\ (\underline{a}, \underline{d}) &\mapsto \underline{d}\,p' - \underline{a}\,p. \end{aligned}$$

In order to write down the next differential

$$d_2^{0,0} : \mathrm{Hom}(\mathcal{O}(-j), \mathcal{O}(j)) \to \mathrm{Ext}^1(\mathcal{O}(j), \mathcal{O}(-j))/\mathrm{image}(d_1^{1,-1}),$$

we choose regular functions α_U, δ_U on U and α_V, δ_V on V such that

$$-z^{-j} p' \underline{c}_U = \alpha_U - \alpha_V$$
$$z^j p \underline{c}_U = \delta_U - \delta_V$$

so

$$d_2^{0,0}(\underline{c}) = \delta_U p' - \alpha_V p.$$

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An Orbit Construction of Phantoms, Orlov Spectra, and Knörrer Periodicity

David Favero, Fabian Haiden, and Ludmil Katzarkov

1 Introduction

The notion of a phantom category is a recently coined term, referring to a nontrivial triangulated or dg-category with vanishing Grothendieck group and/or Hochschild homology. In this note, we refer to categories with vanishing Grothendieck group as K-phantoms and categories with vanishing Hochschild homology and HH-phantoms. When both of these invariants vanish, we follow [13], and refer to these categories simply as phantoms.

The existence of phantom Fukaya–Seidel categories was conjectured in [10]. This conjecture was based on a combination of the seminal works of Donaldson, Kotschick, and Okonek and van de Ven [11, 18, 20], who distinguished smooth structures on Barlow surfaces and Del Pezzo surfaces of degree one using the moduli space of instanton bundles. It was also inspired by the study of the behavior of D-branes under phase transition following Witten and Aspinwall [4, 26].

By the homological mirror symmetry conjecture, it stands to reason that phantom categories also appear as triangulated categories coming from algebraic geometry. Indeed, the first example of such a category was revealed in [9] where the authors construct an admissible subcategory of the derived category of coherent sheaves on a Godeaux surface with vanishing Hochschild homology (an HH-phantom). A different construction of an HH-phantom was provided in [3], and strengthened in [13] which provides the first example of a phantom admissible subcategory of a bounded derived category of coherent sheaves on a smooth projective variety. In fact, in this example it is shown that both the Hochschild homology and all of the algebraic K-groups of the admissible subcategory vanish. A different example of a

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geometric phantom is constructed in [8]. In that case the phantom is an admissible subcategory of the derived category of a determinantal Barlow surface.

In this note we show that geometric phantom categories are not accidental and can naturally appear in an infinite sequence of components of the moduli of dg categories. We provide a basic but non-trivial set of examples of K-phantom categories: matrix factorizations for odd-dimensional A_{2m}-singularities. An important property of our example is that its Orlov spectrum is not a consecutive sequence of integers, a phenomenon connected to birational geometry in [7]. We discuss how this example fits with results from string theory appearing in [2, 4] and outline a description of geometric transformations which might produce phantoms which we plan to explore in future work.

The content of the paper is arranged as follows. In Sect. 3 we discuss the properties of the category $\mathrm{MF}(k[[x,y]], x^{n+1}+y^2)$, and show that it has vanishing K_0 group. In Sect. 4, we demonstrate that the Orlov Spectrum of this example is not a consecutive sequence of integers for $n > 7$. In Sect. 5 we discuss the geometric significance of phantom categories in a conjectural framework.

2 Adding Quadratic Terms to the Potential

Suppose $(R, \mathfrak{m})$ is a regular local ring and $f \in \mathfrak{m}$. Following Eisenbud [12] we consider *matrix factorizations*

$$P_0 \xrightarrow{p_0} P_1 \xrightarrow{p_1} P_0, \qquad p_0 p_1 = f \cdot \mathbf{1}, \quad p_1 p_0 = f \cdot \mathbf{1} \tag{1}$$

where P_0 and P_1 are free R-modules of finite rank. Matrix factorizations form a differential $\mathbb{Z}/(2)$-graded category $\mathrm{MF}(R, f)$, c.f. Orlov [21], and the homotopy category $H(\mathrm{MF}(R, f))$ is triangulated. We will denote a quadruple as in (1) by (P_0, P_1, p_0, p_1).

The aim of this section is to discuss the relation between the categories

$$\mathscr{D} = \mathrm{MF}(R, f) \qquad \text{and} \qquad \mathscr{C} = \mathrm{MF}(R[[y]], f + y^2). \tag{2}$$

We begin by describing an adjunction

$$F : \mathscr{D} \to \mathscr{C}, \quad G : \mathscr{C} \to \mathscr{D} \tag{3}$$

with F left adjoint to G. The functor F assigns to an object (P_0, P_1, p_0, p_1) of $\mathscr{D}$ the object

$$\left((P_0 \oplus P_1) \otimes_R R[[y]], (P_0 \oplus P_1) \otimes_R R[[y]], \begin{bmatrix} y & p_1 \\ p_0 & -y \end{bmatrix}, \begin{bmatrix} y & p_1 \\ p_0 & -y \end{bmatrix}\right) \tag{4}$$

of $\mathscr{C}$, to a morphism (f_0, f_1) in degree 0 the morphism

$$\left(\begin{bmatrix} f_0 \otimes 1 & 0 \\ 0 & f_1 \otimes 1 \end{bmatrix}, \begin{bmatrix} f_0 \otimes 1 & 0 \\ 0 & f_1 \otimes 1 \end{bmatrix}\right) \tag{5}$$

and to a morphism (f_0, f_1) in degree 1 the morphism

$$\left(\begin{bmatrix} 0 & f_1 \otimes 1 \\ f_0 \otimes 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & f_1 \otimes 1 \\ f_0 \otimes 1 & 0 \end{bmatrix}\right). \tag{6}$$

The functor G maps an object (P_0, P_1, p_0, p_1) of $\mathscr{C}$ to the object

$$(P_0/yP_0, P_1/yP_1, \bar{p}_0, \bar{p}_1) \tag{7}$$

of $\mathscr{D}$, where $\bar{p}_i$ denote the induced maps, and to a morphism (f_0, f_1), of either degree, the induced morphism $(\bar{f}_0, \bar{f}_1)$. On an object (P_0, P_1, p_0, p_1) of $\mathscr{C}$ the counit ε is defined by

$$((1, -\widetilde{p_1}), (\widetilde{p_0}, 1)) \in \mathrm{Hom}(P_0 \oplus P_1, P_0) \oplus \mathrm{Hom}(P_0 \oplus P_1, P_1) \tag{8}$$

where

$$p_i = \bar{p}_i \otimes 1 + \widetilde{p_i}\, y. \tag{9}$$

On an object (P_0, P_1, p_0, p_1) of $\mathscr{D}$ the unit η is defined by

$$\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \in \mathrm{Hom}(P_0, P_0 \oplus P_1) \oplus \mathrm{Hom}(P_1, P_0 \oplus P_1). \tag{10}$$

We note that

$$G(F(X)) \cong X \oplus X[1] \tag{11}$$

and thus for objects X, Y of $\mathscr{D}$

$$\mathrm{Hom}_{\mathscr{C}}(FX, FY) \cong \mathrm{Hom}_{\mathscr{D}}(X, GFY) \tag{12}$$

$$\cong \bigoplus_{i \in \{0,1\}} \mathrm{Hom}_D(X, Y[i]) \tag{13}$$

which shows that the full subcategory of $\mathscr{C}$ generated by objects in the image of F is the category of orbits under the action of $\mathbb{Z}/(2)$ on $\mathscr{D}$ via the shift functor, c.f. [15].

We specialize to the case of a power series ring $R = k[[x_0, \ldots, x_m]]$ over an algebraically closed field k of characteristic zero. Under these assumptions, Knörrer [17] shows that for an indecomposable object $X \in H(\mathscr{D})$ with $X \ncong X[1]$ the object

$F(X)$ is indecomposable as well, while for an indecomposable X with $X \cong X[1]$ we have that $F(X) \cong Y \oplus Y[1]$ for some indecomposable $Y \in H(\mathscr{C})$. The same property also holds for the functor G. Moreover, an object $Y \in H(\mathscr{C})$ is in the essential image of F if and only if $Y \cong Y[1]$.

3 The A_{2m} Singularity

Let k be an algebraically closed field of characteristic zero and fix an even integer $n = 2m$. Consider the category of matrix factorizations

$$\mathscr{C} = \mathrm{MF}(k[[x, y]], x^{n+1} + y^2) \tag{14}$$

associated with the A_n curve singularity.

For $0 \le i \le n + 1$ define matrix factorizations

$$W_i : \quad R^2 \xrightarrow{\varphi_i} R^2 \xrightarrow{\varphi_i} R^2 \tag{15}$$

where $R = k[[x, y]]$ and

$$\varphi_i := \begin{bmatrix} y & x^i \\ x^{n+1-i} & -y \end{bmatrix}. \tag{16}$$

By [25] these are all the indecomposable objects and

$$W_0 \cong 0, \qquad W_i \cong W_{n+1-i} \tag{17}$$

in the triangulated homotopy category $H(\mathscr{C})$. Moreover, for $0 < i < n + 1$ there are triangles

$$W_i \longrightarrow W_{i-1} \oplus W_{i+1} \longrightarrow W_i \longrightarrow W_i[1] \tag{18}$$

as follows from [25].

We claim that

$$K_0(\mathscr{C}) = 0. \tag{19}$$

Indeed, writing $[X] \in K_0(\mathscr{C})$ for the K-theory class of an object $X \in \mathscr{C}$, we see from (17) and (18) that $[W_0] = 0$, $[W_i] = i[W_1]$ for $0 < i < n + 1$, so $[W_1]$ is a generator. But $(n - 1)[W_1] = 0$, and since $W_i[1] \cong W_i$ we also have $2[W_1] = 0$, so $[W_1] = 0$, since n was assumed to be even.

We note that, by Knörrer periodicity (see [17]), $\mathscr{C}$ is equivalent to any of the categories

$$\mathrm{MF}(k[[x_0, \ldots, x_m]], x_0^{n+1} + x_1^2 + \ldots + x_m^2) \tag{20}$$

with m odd. The relation to the corresponding category for even m is a special instance of the discussion in the previous section with

$$\mathscr{D} = \mathrm{MF}(k[[x]], x^{n+1}), \qquad \mathscr{C} = \mathrm{MF}(k[[x, y]], x^{n+1} + y^2). \tag{21}$$

Indecomposable objects of $H(\mathscr{D})$ are given by

$$V_i : \quad k[[x]] \xrightarrow{x^{n+1-i}} k[[x]] \xrightarrow{x^i} k[[x]] \tag{22}$$

for $1 \leq i \leq n$, see for example [21]. We have

$$F(V_i) = F(V_{n+1-i}) = W_i, \qquad G(W_i) \cong V_i \oplus V_{n+1-i} = V_i \oplus V_i[1] \tag{23}$$

so in this case the essential image of F is all of $\mathscr{C}$, hence $\mathscr{C}$ is the orbit category of $\mathscr{D}$ under the action of $\mathbb{Z}/(2)$ via the shift functor.

Remark 1. Let $\mathscr{A} = MF_{gr}(k[[X]], x^{n+1})$ denote the dg-category of graded matrix factorizations of the A_n-singularity. The group $\mathbb{Z}$ acts on $\mathscr{A}$ via grading shift, and the corresponding orbit dg-category, in the sense of [15], is equivalent to $MF(k[[X]], x^{n+1})$ by a general result from [16]. The category $\mathscr{A}$ itself coincides with the bounded derived category of finite-dimensional representations of an A_n quiver, as was show in [22].

4 Comparison of Orlov Spectra

Let us recall the following definitions. For a more complete treatment see, [7, 23]. Let $\mathscr{T}$ be a triangulated category. For a full subcategory, $\mathscr{I}$, of $\mathscr{T}$ we denote by $\langle \mathscr{I} \rangle$ the full subcategory of $\mathscr{T}$ whose objects are isomorphic to summands of finite coproducts of shifts of objects in $\mathscr{I}$. In other words, $\langle \mathscr{I} \rangle$ is the smallest full subcategory containing $\mathscr{I}$ which is closed under isomorphisms, shifting, and taking finite coproducts and summands. For two full subcategories, $\mathscr{I}_1$ and $\mathscr{I}_2$, we denote by $\mathscr{I}_1 * \mathscr{I}_2$ the full subcategory of objects, T, such that there is a distinguished triangle,

$$I_1 \to T \to I_2 \to I_1[1],$$

with $I_i \in \mathscr{I}_i$. Set

$$\mathscr{I}_1 \diamond \mathscr{I}_2 := \langle \mathscr{I}_1 * \mathscr{I}_2 \rangle,$$
$$\langle \mathscr{I} \rangle_0 := \langle \mathscr{I} \rangle,$$

and, for $n \geq 1$, inductively define,

$$\langle \mathscr{I} \rangle_n := \langle \mathscr{I} \rangle_{n-1} \diamond \langle \mathscr{I} \rangle.$$

Similarly we define

$$\langle \mathscr{I} \rangle_\infty := \bigcup_{n \geq 0} \langle \mathscr{I} \rangle_n.$$

For an object, $X \in \mathscr{T}$, we notationally identify X with the full subcategory consisting of E in writing, $\langle X \rangle_n$. The reader is warned that, in some of the previous literature, $\langle \mathscr{I} \rangle_0 := 0$ and $\langle \mathscr{I} \rangle_1 := \langle \mathscr{I} \rangle$. We follow the notation in [5, 7]. With our convention, the index equals the number of cones allowed.

Definition 1. Let X be an object $\mathscr{T}$. If there is an n with $\langle X \rangle_n = \mathscr{T}$, we set

$$\circleddash(X) := \min \{n \geq 0 \mid \langle X \rangle_n = \mathscr{T}\}.$$

Otherwise, we set $\circleddash(X) := \infty$. We call $\circleddash(X)$ the **generation time** of X.

Definition 2. Let X be an object of a triangulated category, $\mathscr{T}$. The **Orlov spectrum** of $\mathscr{T}$, denoted OSpec $\mathscr{T}$, is the set,

$$\text{OSpec}\, \mathscr{T} := \{\circleddash(X) \mid X \in \mathscr{T},\ \circleddash(X) < \infty\} \subseteq \mathbb{Z}_{\geq 0}.$$

Let $F : \mathscr{T} \to \mathscr{R}$ be an exact functor between triangulated categories. If every object in $\mathscr{R}$ is isomorphic to a direct summand of an object in the image of F, we say that F is **dense**.

Lemma 1. *If $F : \mathscr{T} \to \mathscr{R}$ is dense and X is a strong generator, then,*

$$\circleddash(F(X)) \leq \circleddash(X).$$

Proof. If X is a generator of $\mathscr{T}$ with minimal generation time t, then $\mathscr{T} = \langle X \rangle_t$. Now as F is an exact functor,

$$F(\mathscr{T}) \subset \langle F(X) \rangle_t.$$

Since every object of $\mathscr{R}$ is a summand of an object $F(\mathscr{T})$, we see that $\mathscr{R} = \langle F(X) \rangle_t$ and the formula follows.

Proposition 1. *Let R be a complete regular ring. For any $f \in R$ one has:*

$$\text{OSpec}(\text{MF}(R, f)) = \text{OSpec}(\text{MF}(R[[y]], f + y^2))$$

Proof. We demonstrate that,

$$\circleddash(X) = \circleddash(F(X)).$$

Notice that F is dense as, by Proposition 2.6 of [17], for any object, $A \in \mathrm{MF}(R[[y]], f + y^2)$, $A \oplus A[1]$ is in the image of F. Meanwhile, G is dense by (11). Therefore,

$$\Clocklogo(X) = \Clocklogo(G(F(X)) \leq \Clocklogo(F(X)) \leq \Clocklogo(X),$$

where the first equality follows from the (11) and the two inequalities are applications of Lemma 1. Therefore the map,

$$\begin{aligned} \mathrm{OSpec}(\mathrm{MF}(R, f)) &\to \mathrm{OSpec}(\mathrm{MF}(R[[y]], f + y^2)) \\ \Clocklogo(X) &\mapsto \Clocklogo(F(X)), \end{aligned}$$

provides an inclusion of Orlov spectra.

By the symmetry of the situation which follows from Knörrer Periodicity (Theorem 3.1 of [17]), we have the other inclusion as well.

Remark 2. Proposition 1 can also be obtained as a consequence of viewing $\mathrm{MF}(R, f)$ as a $\mathbb{Z}_2$-orbit category of $\mathrm{MF}(R[[y]], f + y^2)$ and applying Proposition 9.8 of [6].

Corollary 1. *The category,* $\mathrm{MF}(k[[x, y]], x^{2m+1})$ *has the following properties:*

1. $\mathrm{MF}(k[[x, y]], x^{2m+1})$ *is a* K*-phantom i.e.* $\mathrm{K}_0(\mathrm{MF}(k[[x, y]], x^{2m+1}) = 0$
2. $\mathrm{OSpec}(\mathrm{MF}(k[[x, y]], x^{2m+1}) = \{0, 1, \ldots, \lceil \frac{m}{s} \rceil - 1, \ldots, \lceil \frac{m}{2} \rceil - 1, m - 1\}$

Proof. 1) was explained in the paragraph following (19). 2) Follows from Proposition 1 and Theorem 4.14 of [7]

5 Conjectures

In this section, we propose that the following conjectual procedures could lead to the creation of phantom categories:

1. Rational blow downs and smoothings of surfaces.
2. Degenerations and smoothings of threefolds.

These ideas were inspired by [9], which provided the first example of an HH-phantom as a subcategory of the derived category of coherent sheaves on the classical Godeaux surface. This led the third author to immediately conjecture that the derived category of coherent sheaves on a Barlow surface would contain a phantom subcategory in the stronger sense. This intuition was brought to fruition in [8] for generic determinantal Barlow surfaces and was based on the following rationale.

After blowing-up, a determinantal Barlow surface degenerates to a two sheeted covering of $\mathbb{P}^1 \times \mathbb{P}^1$ with a singular ramification curve. These degenerations and blow-ups replace a cohomology class from $h^{2,0}$ with one in $h^{1,1}$. The new $h^{1,1}$ class is spanned by a single exceptional object. In the mirror this procedure creates a deep singularity and the phantom subcategory, $\mathscr{A} \subseteq \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)$, can be viewed through Homological Mirror Symmetry, as the Fukaya–Seidel category of this singularity.

In general, from the perspective of [14], the Hochschild homology of the Fukaya–Seidel category of a hypersurface singularity is computed as the hypercohomology of a sheaf of vanishing cycles on the singular locus associated to the potential. Therefore, the fact that the Fukaya–Seidel category contains an HH-phantom is equivalent to the existence of a connected component of the singular locus whose associated sheaf of vanishing cycles has trivial cohomology. Hence, we can look for symplectic procedures which create such singular fibers.

Indeed, the required geometric procedures in the case of the Barlow surface, appear naturally and robustly in physics where they are known as conifold and extremal exoflop transitions [4, 26]. Pushing the envelope, these procedures suggest a general framework for producing phantom categories.

Let us consider an example appearing in [24]: a conic bundle over $\mathbb{P}^2$ with a discriminant curve of degree 12. There is a degeneration consisting of a sequence of exoflops [4], which reduces the intermediate Jacobian of this conic bundle to zero. This is the analog of the transition which on the Barlow surface modifies an $h^{2,0}$ class into an $h^{1,1}$ class. While the intermediate Jacobian is eliminated, the singular fiber in the mirror has degenerated but not disappeared. However, all the cohomology has Hodge degree (p, p), hence the presence of an exceptional collection forces the existence of an *HH*-phantom.

Conjecture 1. Let X be the threefold obtained from the conic bundle after degeneration and smoothing described above. There is a semi-orthogonal decomposition,

$$\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X) = \langle \mathscr{A}, E_1, \dots, E_{112} \rangle,$$

where $\mathscr{A}$ is a phantom category and E_i are exceptional objects.

Semi-orthogonal components have also been described as a categorical analog of the Griffiths–Clemens component by Kuznetsov, see for example [19]. The conic bundle above is not rational by [24]. Therefore, the example above indicates that phantom categories, at least in some cases, can be a finer invariant than the classical Griffith–Clemens component. This suggests a method to understanding birational geometry for three dimensional conic bundles if the following conjecture holds:

Conjecture 2. For a generic three dimensional conic bundle over a rational surface, there exists a deformation of the complex structure and resolution of singularities, X, such that $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)$ contains an admissible phantom subcategory.

This conjecture is based on ideas from [1, 2]. Indeed, [1] provides a mirror symmetry construction for conic bundles which can be degenerated as in [2] so that the mirror still contains a deep singular fiber but whose Hodge cycles are all of type (p, p). In the example above the phantom subcategory corresponds to a component of the singular locus of this mirror given by an elliptic curve whose associated sheaf of vanishing cycles F has trivial hypercohomology.

Conversely, the following conjecture follows from the Jordan–Hölder property for semi-orthogonal decompositions conjectured by Kuznetsov see for example, [19].

Conjecture 3. For any rational threefold, X, $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$ does not contain an admissible phantom subcategory.

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Microlocal Theory of Sheaves and Tamarkin's Non Displaceability Theorem

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Abstract This paper is an attempt to better understand Tamarkin's approach of classical non-displaceability theorems of symplectic geometry, based on the microlocal theory of sheaves, a theory whose main features we recall here. If the main theorems are due to Tamarkin, our proofs may be rather different and in the course of the paper we introduce some new notions and obtain new results which may be of interest.

1 Introduction

In [12], D. Tamarkin gives a totally new approach for treating classical problems of non-displaceability in symplectic geometry. His approach is based on the microlocal theory of sheaves, introduced and systematically developed in [3–5]. (Note however that the use of the microlocal theory of sheaves also appeared in a related context in [7–9].)

The aim of this paper was initially to better understand Tamarkin's ideas and to give more accessible proofs by making full use of the tools of [5] and of the recent paper [2]. But when working on this subject, we found some new results which may be of interest. In particular, we make here a systematic study of the category of torsion objects.

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Let us first briefly recall the main facts of the microlocal theory of sheaves. Consider a real manifold M of class C^∞ and a commutative unital ring $\mathbf{k}$ of finite global dimension. Denote by $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ the bounded derived category of sheaves of $\mathbf{k}$-modules on M. In [5], the authors attach to an object F of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ its singular support, or microsupport, $\mathrm{SS}(F)$, a closed subset of T^*M, the cotangent bundle to M. The microsupport is conic for the action of $\mathbb{R}^+$ on T^*M and is involutive (i.e., co-isotropic). The microsupport allows one to localize the triangulated category $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$, and in particular to define the category $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M; U)$ for an open subset $U \subset T^*M$. This theory is "conic", that is, it is invariant by the $\mathbb{R}^+$-action and is related to the homogeneous symplectic structure rather than the symplectic structure.

In order to get rid of the homogeneity, a classical trick is to add a variable which replaces it. This trick appears for example in the complex case in [10] where a deformation quantization ring (with an $\hbar$-parameter) is constructed on the cotangent bundle T^*X to a complex manifold X by using the ring of microdifferential operators of [11] on $T^*(X \times \mathbb{C})$. Coming back to the real setting, denote by t a coordinate on $\mathbb{R}$, by $(t;\tau)$ the associated coordinates on $T^*\mathbb{R}$, by $T^*_{\{\tau>0\}}(M \times \mathbb{R})$ the open subset $\{\tau > 0\}$ of $T^*(M \times \mathbb{R})$ and consider the map

$$\rho\colon T^*_{\{\tau>0\}}(M \times \mathbb{R}) \to T^*M, \quad (x,t;\xi,\tau) \mapsto (x;\xi/\tau).$$

Tamarkin's idea is to work in the localized category $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times\mathbb{R}}; \{\tau > 0\})$, the localization of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times\mathbb{R}})$ by the triangulated subcategory $\mathsf{D}^{\mathrm{b}}_{\{\tau\le 0\}}(\mathbf{k}_{M\times\mathbb{R}})$ consisting of sheaves with microsupport contained in the set $\{\tau \le 0\}$. He first proves the useful result which asserts that this localized category is equivalent to the left orthogonal to $\mathsf{D}^{\mathrm{b}}_{\{\tau\le 0\}}(\mathbf{k}_{M\times\mathbb{R}})$ and that the convolution by the sheaf $\mathbf{k}_{\{t\ge 0\}}$ is a projector on this left orthogonal.

Let us introduce the notation $\mathsf{D}^{\mathrm{b}}(\mathbf{k}^\gamma_M) := \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times\mathbb{R}}; \{\tau > 0\})$ and, for a closed subset $A \subset T^*M$, let us denote by $\mathsf{D}^{\mathrm{b}}_A(\mathbf{k}^\gamma_M)$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}^\gamma_M)$ consisting of objects with microsupport contained in $\rho^{-1}A$.

The first result of Tamarkin is a separability theorem. If A and B are two compact subsets of T^*M, $F \in \mathsf{D}^{\mathrm{b}}_A(\mathbf{k}^\gamma_M)$, $G \in \mathsf{D}^{\mathrm{b}}_B(\mathbf{k}^\gamma_M)$, and if $A \cap B = \emptyset$, then $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}^\gamma_M)}(F,G) \simeq 0$.

The second result of Tamarkin is a Hamiltonian isotopy invariance theorem, up to torsion, that is, after killing what he calls the torsion objects. An object $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}^\gamma_M)$ is torsion if there exists $c \ge 0$ such that the natural map $F \to T_{c*}(F)$ is zero, $T_{c*}(F)$ denoting the image of F by the translation $t \mapsto t + c$ in the t-variable. Let I be an open interval of $\mathbb{R}$ containing $[0,1]$ and let $\Phi = \{\varphi_s\}_{s\in I}$ be a Hamiltonian isotopy (with $\varphi_0 = \mathrm{id}$) such that there exists a compact set $C \subset T^*M$ satisfying $\varphi_s|_{T^*M\setminus C} = \mathrm{id}_{T^*M\setminus C}$ for all $s \in I$. Tamarkin constructs a functor $\Psi\colon \mathsf{D}^{\mathrm{b}}_A(\mathbf{k}^\gamma_M) \to \mathsf{D}^{\mathrm{b}}_{\varphi_1(A)}(\mathbf{k}^\gamma_M)$ such that $\Psi(F)$ is isomorphic to F modulo torsion, for any $F \in \mathsf{D}^{\mathrm{b}}_A(\mathbf{k}^\gamma_M)$.

From these two results he easily deduces that if $A, B \subset T^*M$ are compact sets and if there exist $F \in \mathsf{D}^{\mathrm{b}}_A(\mathbf{k}^\gamma_M)$, $G \in \mathsf{D}^{\mathrm{b}}_B(\mathbf{k}^\gamma_M)$ such that the map $\mathrm{RHom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}^\gamma_M)}(F, G) \to \mathrm{RHom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}^\gamma_M)}(F, T_c(G))$ is not zero for all $c \geq 0$, then the sets A and B are mutually non displaceable, that is, for any Hamiltonian isotopy Φ as above and any $s \in I$, $A \cap \varphi_s(B) \neq \emptyset$.

Let us describe the contents of this paper.

In Sect. 2 we recall some constructions and results of [5] on the microlocal theory of sheaves.

In Sect. 3 we recall the main theorem of [2] which allows one to quantize homogeneous Hamiltonian isotopies and we also give some geometrical tools linking homogeneous and non homogeneous symplectic geometry.

In Sect. 4 we study convolution of sheaves on a trivial vector bundle $E = M \times V$ over M as well as the category $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_E; U_\gamma)$, the localization of the category $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)$ on $U_\gamma = E \times V \times \mathrm{Int}(\gamma_0^\circ)$ where $\mathrm{Int}(\gamma_0^\circ)$ is the interior of the polar cone to a closed convex proper cone γ_0 in V. We prove in particular a separability theorem in this category.

In Sect. 5 we introduce the Tamarkin category $\mathsf{D}^{\mathrm{b}}(\mathbf{k}^\gamma_M)$, that is, the category $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_E; U_\gamma)$ for $E = M \times \mathbb{R}$ and $\gamma_0 = \{t \geq 0\}$.

In Sect. 6 we make a systematic study of the category $\mathscr{N}_{\mathrm{tor}}$ of torsion objects, proving that this category is triangulated and also proving that, under some hypothesis on the microsupport, an object is torsion if and only if its restriction to one point is torsion (Theorem 6.12).

Finally, in Sect. 7 we give a proof of the Hamiltonian isotopy invariance theorem of Tamarkin. The existence of the functor Ψ mentioned above is now an easy consequence on the results of [2], and one checks that this functor induces a functor isomorphic to the identity functor modulo torsion. As already mentioned, Tamarkin's non displaceability theorem is an easy corollary of the preceding results.

Note that, for the purposes we have in mind, we do not need to consider the unbounded derived category $\mathsf{D}(\mathbf{k}_M)$, as did Tamarkin, but only its full triangulated category $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)$ consisting of locally bounded objects. Also note that our notations, as well as our proofs, may seriously differ from Tamarkin's ones.

In future work, motivated by the papers of Fukaya–Seidel–Smith [1] and Nadler [8], we plan to use the tools developed here to study sheaves associated with smooth Lagrangian manifolds.

2 Microlocal Theory of Sheaves

In this section, we recall some definitions and results from [5], following its notations with the exception of slight modifications. We consider a real manifold M of class C^∞.

Some Geometrical Notions ([5, § 4.2, § 6.2])

For a locally closed subset A of M, one denotes by $\mathrm{Int}(A)$ its interior and by $\overline{A}$ its closure. One denotes by Δ_M or simply Δ the diagonal of $M \times M$.

One denotes by $\tau: TM \to M$ and $\pi: T^*M \to M$ the tangent and cotangent bundles to M. If $L \subset M$ is a (smooth) submanifold, we denote by $T_L M$ its normal bundle and $T^*_L M$ its conormal bundle. They are defined by the exact sequences

$$0 \to TL \to L \times_M TM \to T_L M \to 0,$$
$$0 \to T^*_L M \to L \times_M T^*M \to T^*L \to 0.$$

One identifies M to $T^*_M M$, the zero-section of T^*M. One sets $\dot{T}^*M := T^*M \setminus T^*_M M$ and one denotes by $\dot{\pi}_M: \dot{T}^*M \to M$ the projection.

Let $f: M \to N$ be a morphism of real manifolds. To f are associated the tangent morphisms

$$\begin{array}{ccccc} TM & \xrightarrow{f'} & M \times_N TN & \xrightarrow{f_\tau} & TN \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} \\ M & = & M & \xrightarrow{f} & N. \end{array} \tag{1}$$

By duality, we deduce the diagram:

$$\begin{array}{ccccc} T^*M & \xleftarrow{f_d} & M \times_N T^*N & \xrightarrow{f_\pi} & T^*N \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ M & = & M & \xrightarrow{f} & N. \end{array} \tag{2}$$

One sets

$$T^*_M N := \mathrm{Ker}\, f_d = f_d^{-1}(T^*_M M).$$

Note that, denoting by Γ_f the graph of f in $M \times N$, the projection $T^*(M \times N) \to M \times T^*N$ identifies $T^*_{\Gamma_f}(M \times N)$ and $M \times_N T^*N$.

For two subsets $S_1, S_2 \subset M$, their Whitney normal cone, denoted $C(S_1, S_2)$, is the closed cone of TM defined as follows. Let (x) be a local coordinate system and let $(x; v)$ denote the associated coordinate system on TM. Then

$$\begin{cases} (x_0; v_0) \in C(S_1, S_2) \subset TM \text{ if and only if there exists a sequence} \\ \{(x_n, y_n, c_n)\}_n \subset S_1 \times S_2 \times \mathbb{R}^+ \text{ such that } x_n \xrightarrow{n} x_0,\ y_n \xrightarrow{n} x_0 \text{ and} \\ c_n(x_n - y_n) \xrightarrow{n} v_0. \end{cases}$$

For a subset S of M and a smooth closed submanifold L of M, the Whitney normal cone of S along L, denoted $C_L(S)$, is the image in $T_L M$ of $C(L,S)$. If $L = \{p\}$, we write $C_p(S)$ instead of $C_{\{p\}}(S)$.

Now consider the homogeneous symplectic manifold T^*M: it is endowed with the Liouville 1-form given in a local homogeneous symplectic coordinate system $(x;\xi)$ on T^*M by

$$\alpha_M = \langle \xi, dx \rangle.$$

The antipodal map a_M is defined by:

$$a_M : T^*M \to T^*M, \quad (x;\xi) \mapsto (x;-\xi). \tag{3}$$

If A is a subset of T^*M, we denote by A^a instead of $a_M(A)$ its image by the antipodal map.

We shall use the Hamiltonian isomorphism $H: T^*(T^*M) \xrightarrow{\sim} T(T^*M)$ given in a local symplectic coordinate system $(x;\xi)$ by

$$H(\langle \lambda, dx \rangle + \langle \mu, d\xi \rangle) = -\langle \lambda, \partial_\xi \rangle + \langle \mu, \partial_x \rangle.$$

Definition 2.1 (See [5, Def. 6.5.1]). A subset S of T^*M is co-isotropic (one also says involutive) at $p \in T^*M$ if for any $\theta \in T^*_p T^*M$ such that the Whitney normal cone $C_p(S,S)$ is contained in the hyperplane $\{v \in TT^*M; \langle v, \theta \rangle = 0\}$, one has $-H(\theta) \in C_p(S)$. A set S is co-isotropic if it is so at each $p \in S$.

When S is smooth, one recovers the usual notion.

Microsupport

We consider a commutative unital ring $\mathbf{k}$ of finite global dimension (e.g. $\mathbf{k} = \mathbb{Z}$). We denote by $\mathsf{D}(\mathbf{k}_M)$ (resp. $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$) the derived category (resp. bounded derived category) of sheaves of $\mathbf{k}$-modules on M.

Recall the definition of the microsupport (or singular support) $\mathrm{SS}(F)$ of a sheaf F.

Definition 2.2 (See [5, Def. 5.1.2]). Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ and let $p \in T^*M$. One says that $p \notin \mathrm{SS}(F)$ if there exists an open neighborhood U of p such that for any $x_0 \in M$ and any real C^1-function φ on M defined in a neighborhood of x_0 satisfying $d\varphi(x_0) \in U$ and $\varphi(x_0) = 0$, one has $(\mathrm{R}\Gamma_{\{x;\varphi(x)\geq 0\}}(F))_{x_0} \simeq 0$.

In other words, $p \notin \mathrm{SS}(F)$ if the sheaf F has no cohomology supported by "half-spaces" whose conormals are contained in a neighborhood of p.

- By its construction, the microsupport is closed and is $\mathbb{R}^+$-conic, that is, invariant by the action of $\mathbb{R}^+$ on T^*M.

- $SS(F) \cap T^*_M M = \pi_M(SS(F)) = \mathrm{Supp}(F)$.
- The microsupport satisfies the triangular inequality: if $F_1 \to F_2 \to F_3 \xrightarrow{+1}$ is a distinguished triangle in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$, then $SS(F_i) \subset SS(F_j) \cup SS(F_k)$ for all $i, j, k \in \{1, 2, 3\}$ with $j \neq k$.

Theorem 2.3 (See [5, Th. 6.5.4]). *Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$. Then its microsupport* $SS(F)$ *is co-isotropic.*

In the sequel, for a locally closed subset Z in M, we denote by $\mathbf{k}_Z$ the constant sheaf with stalk $\mathbf{k}$ on Z, extended by 0 on $M \setminus Z$.

Example 2.4. (i) If F is a non-zero local system on a connected manifold M, then $SS(F) = T^*_M M$, the zero-section.
(ii) If N is a smooth closed submanifold of M and $F = \mathbf{k}_N$, then $SS(F) = T^*_N M$, the conormal bundle to N in M.
(iii) Let φ be C^1-function with $d\varphi(x) \neq 0$ when $\varphi(x) = 0$. Let $U = \{x \in M; \varphi(x) > 0\}$ and let $Z = \{x \in M; \varphi(x) \geq 0\}$. Then

$$SS(\mathbf{k}_U) = U \times_M T^*_M M \cup \{(x; \lambda d\varphi(x)); \varphi(x) = 0, \lambda \leq 0\},$$
$$SS(\mathbf{k}_Z) = Z \times_M T^*_M M \cup \{(x; \lambda d\varphi(x)); \varphi(x) = 0, \lambda \geq 0\}.$$

(iv) Let $(X, \mathscr{O}_X)$ be a complex manifold and let $\mathscr{M}$ be a coherent module over the ring $\mathscr{D}_X$ of holomorphic differential operators. (Hence, $\mathscr{M}$ represents a system of linear partial differential equations on X.) Denote by $F = \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$ the complex of holomorphic solutions of $\mathscr{M}$. Then $SS(F) = \mathrm{char}(\mathscr{M})$, the characteristic variety of $\mathscr{M}$.

Functorial Operations (Proper and Non-characteristic Cases)

Let M and N be two real manifolds. We denote by q_i $(i = 1, 2)$ the i-th projection defined on $M \times N$ and by p_i $(i = 1, 2)$ the i-th projection defined on $T^*(M \times N) \simeq T^*M \times T^*N$.

Definition 2.5. Let $f: M \to N$ be a morphism of manifolds and let $\Lambda \subset T^*N$ be a closed $\mathbb{R}^+$-conic subset. One says that f is non-characteristic for Λ (or else, Λ is non-characteristic for f, or f and Λ are transversal) if

$$f_\pi^{-1}(\Lambda) \cap T^*_M N \subset M \times_N T^*_N N.$$

A morphism $f: M \to N$ is non-characteristic for a closed $\mathbb{R}^+$-conic subset Λ of T^*N if and only if $f_d: M \times_N T^*N \to T^*M$ is proper on $f_\pi^{-1}(\Lambda)$ and in this case $f_d f_\pi^{-1}(\Lambda)$ is closed and $\mathbb{R}^+$-conic in T^*M.

We denote by ω_M the dualizing complex on M. Recall that ω_M is isomorphic to the orientation sheaf shifted by the dimension. We also use the notation $\omega_{M/N}$ for the relative dualizing complex $\omega_M \otimes f^{-1}\omega_N^{\otimes -1}$. We have the duality functors

$$\mathrm{D}_M(\bullet) = \mathrm{R}\mathscr{H}om(\bullet, \omega_M), \tag{4}$$

$$\mathrm{D}'_M(\bullet) = \mathrm{R}\mathscr{H}om(\bullet, \mathbf{k}_M). \tag{5}$$

Theorem 2.6 (See [5, § 5.4]). *Let $f: M \to N$ be a morphism of manifolds, let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ and let $G \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_N)$.*

(i) *One has*

$$\mathrm{SS}(F \overset{\mathrm{L}}{\boxtimes} G) \subset \mathrm{SS}(F) \times \mathrm{SS}(G),$$

$$\mathrm{SS}(\mathrm{R}\mathscr{H}om(q_1^{-1}F, q_2^{-1}G)) \subset \mathrm{SS}(F)^a \times \mathrm{SS}(G).$$

(ii) *Assume that f is proper on* $\mathrm{Supp}(F)$. *Then* $\mathrm{SS}(\mathrm{R}f_!F) \subset f_\pi f_d^{-1}\mathrm{SS}(F)$.
(iii) *Assume that f is non-characteristic with respect to* $\mathrm{SS}(G)$. *Then the natural morphism $f^{-1}G \otimes \omega_{M/N} \to f^!(G)$ is an isomorphism. Moreover* $\mathrm{SS}(f^{-1}G) \cup \mathrm{SS}(f^!G) \subset f_d f_\pi^{-1}\mathrm{SS}(G)$.
(iv) *Assume that f is smooth (that is, submersive). Then $\mathrm{SS}(F) \subset M \times_N T^*N$ if and only if, for any $j \in \mathbb{Z}$, the sheaves $H^j(F)$ are locally constant on the fibers of f.*

For the notion of a cohomologically constructible sheaf we refer to [5, § 3.4].

Corollary 2.7. *Let $F_1, F_2 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$.*

(i) *Assume that* $\mathrm{SS}(F_1) \cap \mathrm{SS}(F_2)^a \subset T^*_M M$. *Then*

$$\mathrm{SS}(F_1 \overset{\mathrm{L}}{\otimes} F_2) \subset \mathrm{SS}(F_1) + \mathrm{SS}(F_2).$$

(ii) *Assume that* $\mathrm{SS}(F_1) \cap \mathrm{SS}(F_2) \subset T^*_M M$. *Then*

$$\mathrm{SS}(\mathrm{R}\mathscr{H}om(F_1, F_2)) \subset \mathrm{SS}(F_1)^a + \mathrm{SS}(F_2).$$

Moreover, assuming that F_1 is cohomologically constructible, the natural morphism $\mathrm{D}' F_1 \overset{\mathrm{L}}{\otimes} F_2 \to \mathrm{R}\mathscr{H}om(F_1, F_2)$ *is an isomorphism.*

The next result follows immediately from Theorem 2.6(ii). It is a particular case of the microlocal Morse lemma (see [5, Cor. 5.4.19]), the classical theory corresponding to the constant sheaf $F = \mathbf{k}_M$.

Corollary 2.8. *Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$, let $\varphi: M \to \mathbb{R}$ be a function of class C^1 and assume that φ is proper on* $\mathrm{supp}(F)$. *Let $a < b$ in $\mathbb{R}$ and assume that $d\varphi(x) \notin \mathrm{SS}(F)$ for $a \le \varphi(x) < b$. Then the natural morphism*
$\mathrm{R}\Gamma(\varphi^{-1}(]-\infty, b[); F) \to \mathrm{R}\Gamma(\varphi^{-1}(]-\infty, a[); F)$ *is an isomorphism.*

Corollary 2.9. *Let I be a contractible manifold and let $p\colon M \times I \to M$ be the projection. If $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times I})$ satisfies $\mathrm{SS}(F) \subset T^*M \times T^*_I I$, then $F \simeq p^{-1}\mathrm{R}p_*F$.*

Proof. It follows from Theorem 2.6(iv) that the restriction $F|_{\{x\}\times I}$ is locally constant for any $x \in M$. Then the result follows from [5, Prop. 2.7.8]. □

Corollary 2.10. *Let I be an open interval of $\mathbb{R}$ and let $q\colon M \times I \to I$ be the projection. Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times I})$ such that $\mathrm{SS}(F) \cap (T^*_M M \times T^*I) \subset T^*_{M\times I}(M \times I)$ and q is proper on $\mathrm{Supp}(F)$. Then we have isomorphisms $\mathrm{R}\Gamma(M; F_s) \simeq \mathrm{R}\Gamma(M; F_t)$ for any $s, t \in I$.*

Proof. It follows from Theorem 2.6 that $\mathrm{SS}(\mathrm{R}q_*(F)) \subset T^*_I I$. Hence, there exists $V \in \mathsf{D}^{\mathrm{b}}(\mathbf{k})$ and an isomorphism $\mathrm{R}q_*(F) \simeq V_I$. (Recall that $V_I = a_I^{-1}V$, where $a_I \to \mathrm{pt}$ is the projection and V is identified to a sheaf on pt.) Since we have $\mathrm{R}\Gamma(M; F_s) \simeq (\mathrm{R}q_*(F))_s$ the result follows. □

Kernels ([5, § 3.6])

Notation 2.11. *Let M_i ($i = 1,2,3$) be manifolds. For short, we write $M_{ij} := M_i \times M_j$ ($1 \le i, j \le 3$) and $M_{123} = M_1 \times M_2 \times M_3$. We denote by q_i the projection $M_{ij} \to M_i$ or the projection $M_{123} \to M_i$ and by q_{ij} the projection $M_{123} \to M_{ij}$. Similarly, we denote by p_i the projection $T^*M_{ij} \to T^*M_i$ or the projection $T^*M_{123} \to T^*M_i$ and by p_{ij} the projection $T^*M_{123} \to T^*M_{ij}$. We also need to introduce the map p_{12^a}, the composition of p_{12} and the antipodal map on T^*M_2.*

Let $A \subset T^*M_{12}$ and $B \subset T^*M_{23}$. We set

$$\begin{aligned} A \times_{T^*M_{2^a}} B &= p_{12}^{-1}(A) \cap p_{2^a 3}^{-1}(B) \\ A \overset{a}{\circ} B &= p_{13}(A \times_{T^*M_{2^a}} B) \\ &= \{(x_1, x_3; \xi_1, \xi_3) \in T^*M_{13};\ \text{there exists } (x_2;\xi_2) \in T^*M_2, \\ &\qquad (x_1, x_2; \xi_1, \xi_2) \in A, (x_2, x_3; -\xi_2, \xi_3) \in B\}. \end{aligned} \tag{6}$$

We consider the operation of composition of kernels:

$$\begin{aligned} \circ\colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{12}}) \times \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{23}}) &\to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{13}}) \\ (K_1, K_2) &\mapsto K_1 \circ K_2 := \mathrm{R}q_{13!}(q_{12}^{-1}K_1 \overset{\mathrm{L}}{\otimes} q_{23}^{-1}K_2). \end{aligned} \tag{7}$$

Let $A_i = \mathrm{SS}(K_i) \subset T^*M_{i,i+1}$ and assume that

$$\begin{cases} \text{(i) } q_{13} \text{ is proper on } q_{12}^{-1}\,\mathrm{supp}(K_1) \cap q_{23}^{-1}\,\mathrm{supp}(K_2), \\ \text{(ii) } p_{12}^{-1}A_1 \cap p_{2^a3}^{-1}A_2 \cap (T^*_{M_1}M_1 \times T^*M_2 \times T^*_{M_3}M_3) \\ \qquad\qquad \subset T^*_{M_1\times M_2\times M_3}(M_1 \times M_2 \times M_3). \end{cases} \tag{8}$$

It follows from Theorem 2.6 that under the assumption (8) we have:

$$\mathrm{SS}(K_1 \circ K_2) \subset A_1 \overset{a}{\circ} A_2. \tag{9}$$

Characteristic Inverse Images

Theorem 2.6 treats the easy cases of external tensor product or external Hom, non-characteristic inverse images or proper direct image. In order to treat more general cases we introduce some additional geometrical notions.

Let Λ be a smooth Lagrangian submanifold of T^*M. The Hamiltonian isomorphism defines an isomorphism

$$T^*\Lambda \simeq T_\Lambda T^*M.$$

Let $j\colon L \hookrightarrow M$ be the embedding of a smooth submanifold L of M. The Liouville form defines an embedding

$$T^*L \hookrightarrow T^*T^*_L M \simeq T_{T^*_L M} T^*M.$$

Now consider a morphism of manifolds $f\colon M \to N$ and let us identify M to the graph of f in $M \times N$. For a subset $B \subset T^*N$ one sets:

$$f^\sharp(B) = T^*M \cap C_{T^*_M(M\times N)}(T^*_M M \times B). \tag{10}$$

In local symplectic coordinate systems $(x;\xi)$ on M and $(y;\eta)$ on N one has

$$\begin{cases} (x_0;\xi_0) \in f^\sharp(B) \text{ if and only if there exist sequences } \{x_n\}_n \subset \\ M \text{ and } \{(y_n;\eta_n)\}_n \subset B \text{ such that} \\ x_n \to x_0,\ {}^t f'(x_n)\cdot\eta_n \xrightarrow{n} \xi_0 \text{ and } |y_n - f(x_n)|\cdot|\eta_n| \xrightarrow{n} 0. \end{cases} \tag{11}$$

For two closed $\mathbb{R}^+$-conic subsets A and B of T^*M one sets

$$A \mathbin{\hat{+}} B = T^*M \cap C(A, B^a). \tag{12}$$

Here, $C(A, B^a)$ is considered as a subset of T^*T^*M via the Hamiltonian isomorphism and T^*M is embedded into T^*T^*M via the Liouville form α_M. In a local coordinate system, one has

$$\begin{cases} (z_0;\zeta_0) \in A \mathbin{\hat{+}} B \text{ if and only if there exist sequences} \\ \{(x_n;\xi_n)\}_n \text{ in } A \text{ and } \{(y_n;\eta_n)\}_n \text{ in } B \text{ such that } x_n \xrightarrow{n} z_0, \\ y_n \xrightarrow{n} z_0,\ \xi_n + \eta_n \xrightarrow{n} \zeta_0 \text{ and } |x_n - y_n|\cdot|\xi_n| \xrightarrow{n} 0. \end{cases} \tag{13}$$

Theorem 2.12 (See [5, Cor. 6.4.4, 6.4.5]). *Let* $F_1, F_2 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ *and let* $G \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_N)$. *Then*

$$\begin{aligned}
&\mathrm{SS}(F_1 \overset{\mathrm{L}}{\otimes} F_2) \subset \mathrm{SS}(F_1) \hat{+} \mathrm{SS}(F_2),\\
&\mathrm{SS}(\mathrm{R}\mathscr{H}om(F_1, F_2)) \subset \mathrm{SS}(F_2) \hat{+} \mathrm{SS}(F_1)^a,\\
&\mathrm{SS}(f^{-1}G) \cup \mathrm{SS}(f^!G) \subset f^\sharp(\mathrm{SS}(G)).
\end{aligned}$$

Non Proper Direct Images

We shall also need a direct image theorem in a non proper case.

Consider a *constant* linear map u of *trivial* vector bundles over M, that is, we assume that $E_i = M \times V_i$ $(i = 1, 2)$ and $u\colon V_1 \to V_2$ is a linear map. The map u defines the maps described by the diagram

$$\begin{array}{ccccc}
 & & T^*M \times V_1 \times V_2^* & & \\
 & \swarrow^{u_d} & & \searrow^{u_\pi} & \\
T^*M \times V_1 \times V_1^* & & & & T^*M \times V_2 \times V_2^*. \\
 & \searrow_{v_\pi} & & \swarrow_{v_d} & \\
 & & T^*M \times V_2 \times V_1^* & &
\end{array}$$

Note that for a subset A of T^*E_1 we have

$$u_\pi(u_d^{-1}(A)) = v_d^{-1}(v_\pi(A)). \tag{14}$$

Notation 2.13. *Let* $u\colon E_1 \to E_2$ *be a constant linear map of trivial vector bundles over* M *and let* $A \subset T^*E_1$ *be a closed subset. We set*

$$u_\sharp(A) = v_d^{-1}(\overline{v_\pi(A)}). \tag{15}$$

In Lemmas 2.14 and 2.15 below we use the notations $\bigoplus_n G_n$ and $\prod_n G_n$ for a family $\{G_n\}_{n\in\mathbb{N}}$ in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$. We define it as follows. Let $p\colon M \times \mathbb{N} \to M$ be the projection. Then we have a unique $G \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times\mathbb{N}})$ such that $G|_{M\times\{n\}} \simeq G_n$, for all n, and we set $\bigoplus_n G_n := \mathrm{R}p_!G$ and $\prod_n G_n := \mathrm{R}p_*G$.

Lemma 2.14. *Let* M *be a manifolds and let* $\{U_n\}_{n\in\mathbb{N}}$ *be an increasing sequence of open subsets of* M *such that* $M = \bigcup_n U_n$. *Then, for any* $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$, *we have the distinguished triangles*

$$\bigoplus_n F_{U_n} \xrightarrow{\mathrm{id}-s_1} \bigoplus_n F_{U_n} \to F \xrightarrow{+1}, \quad F \to \prod_n \mathrm{R}\Gamma_{U_n}(F) \xrightarrow{\mathrm{id}-s_2} \prod_n \mathrm{R}\Gamma_{U_n}(F) \xrightarrow{+1},$$

where s_1 is the sum of the natural morphisms $F_{U_n} \to F_{U_{n+1}}$ and s_2 the product of the natural morphisms $\mathrm{R}\Gamma_{U_{n+1}}(F) \to \mathrm{R}\Gamma_{U_n}(F)$ for $n \geq 0$ and the zero morphism for $n = -1$.

Proof. These triangles arise from similar exact sequences of sheaves when F is a flabby sheaf. The exactness can be checked easily on the stalks in the first case and on sections over any open subset in the second case. □

Lemma 2.15. *Let $f\colon M \to N$ be a morphism of manifolds and let $\{U_n\}_{n\in\mathbb{N}}$ be an increasing sequence of open subsets of M such that $M = \bigcup_n U_n$. Then, for any $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$, we have*

$$\mathrm{SS}(\mathrm{R}f_!F) \subset \overline{\bigcup_n \mathrm{SS}(\mathrm{R}f_!(F_{U_n}))}, \qquad \mathrm{SS}(\mathrm{R}f_*F) \subset \overline{\bigcup_n \mathrm{SS}(\mathrm{R}f_*\mathrm{R}\Gamma_{U_n}(F))}.$$

Proof. We can check, similarly as in [5, Exe. V.7], that for any family $\{G_n\}_{n\in\mathbb{N}}$ in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_N)$ we have $\mathrm{SS}(\bigoplus_n G_n) \cup \mathrm{SS}(\prod_n G_n) \subset \overline{\bigcup_n \mathrm{SS}(G_n)}$. Then the result follows from Lemma 2.14 and the fact that $\mathrm{R}f_!$ commutes with $\oplus$ and $\mathrm{R}f_*$ with $\prod$. □

The following result is due to Tamarkin [12, Lem. 3.3] but our proof is completely different.

Theorem 2.16. *Let $u\colon E_1 \to E_2$ be a constant linear map of trivial vector bundles over M and let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{E_1})$. Then $\mathrm{SS}(\mathrm{R}u_!F) \subset u_\sharp(\mathrm{SS}(F))$. The same estimate holds with $\mathrm{R}u_!F$ replaced with $\mathrm{R}u_*F$.*

Proof. (i) By decomposing u by its graph, one is reduced to prove the result for an immersion and for a projection. Since the case of an immersion is obvious, we restrict ourselves to the case where $E = M \times V$ and $u\colon E \to M$ is the projection. Moreover the result is local on M and we may assume that M is an open subset in a vector space W.

(ii) We consider $(x_0;\xi_0) \in T^*M \simeq M \times W^*$ such that $(x_0;\xi_0) \notin u_\sharp(\mathrm{SS}(F))$. We will prove that $(x_0;\xi_0) \notin \mathrm{SS}(\mathrm{R}u_!F) \cup \mathrm{SS}(\mathrm{R}u_*F)$. If $\xi_0 = 0$, then $F|_{U\times V} \simeq 0$ for some neighborhood U of x_0 and the result follows easily. Hence we assume that $\xi_0 \neq 0$. Up to shrinking M we may find an open cone $C \subset W^* \times V^*$ such that $(\xi_0, 0) \in C$ and $\mathrm{SS}(F) \cap ((M \times V) \times C) = \emptyset$.

(iii) We choose an open convex cone $\gamma \subset W \times V$ such that $\overline{\gamma} \cap (\{0\} \times V) = \{(0,0)\}$ and $\gamma^\circ \subset C$. We also choose two sequences of points $\{z_n\}_{n\in\mathbb{N}}$, resp. $\{z'_n\}_{n\in\mathbb{N}}$, of $W \times V$ such that $W \times V$ is the increasing union of the cones $\gamma_n = z_n - \gamma$, resp. $\gamma'_n = z'_n + \gamma$. By Lemma 2.15 it is enough to show

$$(\mathrm{SS}(\mathrm{R}u_*\mathrm{R}\Gamma_{\gamma_n}F) \cup \mathrm{SS}(\mathrm{R}u_!(F_{\gamma'_n}))) \cap (M \times (C \cap (W^* \times \{0\})) = \emptyset.$$

(iv) By Lemma 4.16 below $\mathrm{SS}(\mathbf{k}_{\overline{\gamma_n}}) \subset (W \times V) \times (-C)$. Using $\mathrm{D}'_M(\mathbf{k}_{\overline{\gamma_n}}) \simeq \mathbf{k}_{\gamma_n}$ we deduce $\mathrm{SS}(\mathbf{k}_{\gamma_n}) \subset (W \times V) \times C$. Similarly $\mathrm{SS}(\mathbf{k}_{\gamma'_n}) \subset (W \times V) \times (-C)$. Since $\mathrm{SS}(F) \cap ((M \times V) \times C) = \emptyset$, Corollary 2.7 gives

$$(\mathrm{SS}(\mathrm{R}\Gamma_{\gamma_n}F) \cup \mathrm{SS}((F_{\gamma'_n}))) \cap ((M \times V) \times C) = \emptyset.$$

Since $\overline{\gamma} \cap (\{0\} \times V) = \{(0,0)\}$ the map $u: M \times V \to M$ is proper on all $\overline{\gamma_n}$ and $\overline{\gamma_n'}$ and the result follows from Theorem 2.6(ii).

□

For a trivial vector bundle $E = M \times V$ we denote by

$$\hat{\pi}_E: T^*E \to T^*M \times V^*, \tag{16}$$

or $\hat{\pi}$ if there is no risk of confusion, the natural projection. We say that a subset of $T^*M \times V^*$ is a cone if it is stable by the multiplicative action of $\mathbb{R}^+$ given by

$$\lambda \cdot (x; \xi, v) = (x; \lambda\xi, \lambda v). \tag{17}$$

We will be mainly concerned with the case where $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)$ has a microsupport bounded by $\hat{\pi}_E^{-1}(A)$ for some closed cone $A \subset T^*M \times V^*$.

Let $u: E_1 = M \times V_1 \to E_2 = M \times V_2$ be a constant linear map of trivial vector bundles over M and denote by

$$\tilde{u}_d: T^*M \times V_2^* \to T^*M \times V_1^* \tag{18}$$

the map associated with u.

Corollary 2.17. *Let $u: E_1 \to E_2$ be a constant linear map of trivial vector bundles over M and let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{E_1})$. Assume that $\mathrm{SS}(F) \subset \hat{\pi}_{E_1}^{-1}(A_1)$ for a closed cone $A_1 \subset T^*M \times V_1^*$. Then $\mathrm{SS}(\mathrm{R}u_!F) \subset \hat{\pi}_{E_2}^{-1}\tilde{u}_d^{-1}(A_1)$. The same estimate holds with $\mathrm{R}u_!F$ replaced with $\mathrm{R}u_*F$.*

Proof. We have $v_\pi(\hat{\pi}_{E_1}^{-1}(A_1)) = A_1 \times V_2$ and this set is closed. We thus have

$$\begin{aligned} u_\sharp(\hat{\pi}_{E_1}^{-1}(A_1)) &= v_d^{-1}(v_\pi(\hat{\pi}_{E_1}^{-1}(A_1))) = u_\pi(u_d^{-1}(A_1 \times V_1)) \\ &= \tilde{u}_d^{-1}(A_1) \times V_2 = \hat{\pi}_{E_2}^{-1}\tilde{u}_d^{-1}(A_1). \end{aligned}$$

□

Localization

Let $\mathscr{T}$ be a triangulated category. Recall that a null system $\mathscr{N}$ is the set of objects of a strictly full triangulated subcategory (where *strictly full* means full and with the property that if one has an isomorphism $F \simeq G$ in $\mathscr{T}$ with $F \in \mathscr{N}$, then $G \in \mathscr{N}$). The localization $\mathscr{T}/\mathscr{N}$ is a well defined triangulated category (we skip the problem of universes). Its objects are those of $\mathscr{T}$ and a morphism $u: F_1 \to F_2$ in $\mathscr{T}$ becomes an isomorphism in $\mathscr{T}/\mathscr{N}$ if, after embedding this morphism in a distinguished triangle $F_1 \to F_2 \to F_3 \xrightarrow{+1}$, one has $F_3 \in \mathscr{N}$.

Recall that the left orthogonal $\mathscr{N}^{\perp,l}$ of $\mathscr{N}$ is the full triangulated subcategory of $\mathscr{T}$ defined by:

$$\mathscr{N}^{\perp,l} = \{F \in \mathscr{T}; \mathrm{Hom}_{\mathscr{T}}(F, G) \simeq 0 \text{ for all } G \in \mathscr{N}\}.$$

By classical results (see e.g., [6, Exe. 10.15]), if the embedding $\mathscr{N}^{\perp,l} \hookrightarrow \mathscr{T}$ admits a left adjoint, or equivalently, if for any $F \in \mathscr{T}$, there exists a distinguished triangle $F' \to F \to F'' \xrightarrow{+1}$ with $F' \in \mathscr{N}^{\perp,l}$ and $F'' \in \mathscr{N}$, then there is an equivalence $\mathscr{N}^{\perp,l} \simeq \mathscr{T}/\mathscr{N}$.

Of course, there are similar results with the right orthogonal $\mathscr{N}^{\perp,r}$.

Now let U be a subset of T^*M and set $Z = T^*M \setminus U$. The full subcategory $\mathsf{D}^{\mathrm{b}}_Z(\mathbf{k}_M)$ of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ consisting of sheaves F such that $\mathrm{SS}(F) \subset Z$ is a strictly full triangulated subcategory. One sets

$$\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M; U) := \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)/\mathsf{D}^{\mathrm{b}}_Z(\mathbf{k}_M),$$

the localization of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ by $\mathsf{D}^{\mathrm{b}}_Z(\mathbf{k}_M)$. Hence, the objects of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M; U)$ are those of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ but a morphism $u\colon F_1 \to F_2$ in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ becomes an isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M; U)$ if, after embedding this morphism in a distinguished triangle $F_1 \to F_2 \to F_3 \xrightarrow{+1}$, one has $\mathrm{SS}(F_3) \cap U = \emptyset$.

For a closed subset A of U, $\mathsf{D}^{\mathrm{b}}_A(\mathbf{k}_M; U)$ denotes the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M; U)$ consisting of objects whose microsupports have an intersection with U contained in A.

Quantized Symplectic Isomorphisms ([5, §7.2])

Consider two manifolds M and N, two conic open subsets $U \subset T^*M$ and $V \subset T^*N$ and a homogeneous symplectic isomorphism χ:

$$T^*N \supset V \xrightarrow[\chi]{\sim} U \subset T^*M. \tag{19}$$

Denote by V^a the image of V by the antipodal map a_N on T^*N and by Λ the image of the graph of φ by $\mathrm{id}_U \times a_N$. Hence Λ is a conic Lagrangian submanifold of $U \times V^a$. A quantized contact transformation (a QCT, for short) above χ is a kernel $K \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times N})$ such that $\mathrm{SS}(K) \cap (U \times V^a) \subset \Lambda$ and satisfying some technical properties that we do not recall here, so that the kernel K induces an equivalence of categories

$$K \circ \bullet\colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}_N; V) \xrightarrow{\sim} \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M; U). \tag{20}$$

Given χ and $q \in V$, $p = \chi(q) \in U$, there exists such a QCT after replacing U and V by sufficiently small neighborhoods of p and q.

Simple Sheaves ([5, §7.5])

Let $\Lambda \subset \dot{T}^*M$ be a locally closed conic Lagrangian submanifold and let $p \in \Lambda$. Simple sheaves along Λ at p are defined in [5, Def. 7.5.4].

When Λ is the conormal bundle to a submanifold $N \subset M$, that is, when the projection $\pi_M|_\Lambda \colon \Lambda \to M$ has constant rank, then an object $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ is simple along Λ at p if $F \simeq \mathbf{k}_N\,[d]$ in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M; p)$ for some shift $d \in \mathbb{Z}$.

If $\mathrm{SS}(F)$ is contained in Λ on a neighborhood of Λ, Λ is connected and F is simple at some point of Λ, then F is simple at every point of Λ.

The Functor *μhom* ([5, §4.4, §7.2])

The functor of microlocalization along a submanifold has been introduced by Mikio Sato in the 70's and has been at the origin of what is now called "microlocal analysis". A variant of this functor, the bifunctor

$$\mu hom \colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)^{\mathrm{op}} \times \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{T^*M}) \tag{21}$$

has been constructed in [5]. Let us only recall the properties of this functor that we shall use. For $F, G \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$, with F cohomologically constructible, we have

$$\mathrm{R}\pi_{M*}\mu hom(F,G) \simeq \mathrm{R}\mathscr{H}om\,(F,G),$$
$$\mathrm{R}\pi_{M!}\mu hom(F,G) \simeq \mathrm{D}'_M(F)\overset{\mathrm{L}}{\otimes}G$$

and we deduce the distinguished triangle

$$\mathrm{D}'_M(F)\overset{\mathrm{L}}{\otimes}G \to \mathrm{R}\mathscr{H}om\,(F,G) \to \mathrm{R}\dot{\pi}_{M*}(\mu hom(F,G)|_{\dot{T}^*M}) \xrightarrow{+1}. \tag{22}$$

Let $\Lambda \subset \dot{T}^*M$ be a locally closed smooth conic Lagrangian submanifold and let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ be simple along Λ. Then

$$\mu hom(F,F)|_\Lambda \simeq \mathbf{k}_\Lambda. \tag{23}$$

3 Quantization of Hamiltonian Isotopies

In this section, we recall the main theorem of [2].

We first recall some notions of symplectic geometry. Let $\mathfrak{X}$ be a symplectic manifold with symplectic form ω. We denote by $\mathfrak{X}^a$ the same manifold endowed with the symplectic form $-\omega$. The symplectic structure induces the Hamiltonian isomorphism $\mathbf{h} \colon T\mathfrak{X} \xrightarrow{\sim} T^*\mathfrak{X}$ by $\mathbf{h}(v) = \iota_v(\omega)$, where ι_v denotes the contraction with v (in case $\mathfrak{X}$ is a cotangent bundle we have $\mathbf{h} = -H^{-1}$, where H is used in Definition 2.1). To a vector field v on $\mathfrak{X}$ we associate in this way a 1-form $\mathbf{h}(v)$ on $\mathfrak{X}$. For a C^∞-function $f\colon \mathfrak{X} \to \mathbb{R}$, the Hamiltonian vector field of f is by definition $H_f := -\mathbf{h}^{-1}(df)$.

A vector field v is called symplectic if its flow preserves ω. This is equivalent to $\mathscr{L}_v(\omega) = 0$ where $\mathscr{L}_v$ denotes the Lie derivative of v. By Cartan's formula ($\mathscr{L}_v = d\,\iota_v + \iota_v\, d$) this is again equivalent to $d(\mathbf{h}(v)) = 0$ (recall that $d\omega = 0$). The vector field v is called Hamiltonian if $\mathbf{h}(v)$ is exact, or equivalently $v = H_f$ for some function f on $\mathfrak{X}$.

Let I be an open interval of $\mathbb{R}$ containing the origin and let $\Phi\colon \mathfrak{X} \times I \to \mathfrak{X}$ be a map such that $\varphi_s := \Phi(\cdot, s)\colon \mathfrak{X} \to \mathfrak{X}$ is a symplectic isomorphism for each $s \in I$ and is the identity for $s = 0$. The map Φ induces a time dependent vector field on $\mathfrak{X}$

$$v_\Phi := \frac{\partial \Phi}{\partial s}\colon \mathfrak{X} \times I \to T\mathfrak{X}. \tag{24}$$

The "time dependent" 1-form $\beta = \mathbf{h}(v_\Phi)\colon \mathfrak{X} \times I \to T^*\mathfrak{X}$ satisfies $d(\beta_s) = 0$ for any $s \in I$. The map Φ is called a Hamiltonian isotopy if $v_{\Phi,s}$ is Hamiltonian, that is, if β_s is exact, for any s. In this case we can write $\beta_s = -d(f_s)$ for some C^∞-function $f\colon \mathfrak{X} \times I \to \mathbb{R}$. Hence we have

$$\frac{\partial \Phi}{\partial s} = H_{f_s}.$$

The fact that the isotopy Φ is Hamiltonian can be interpreted as a geometric property of its graph as follows. For a given $s \in I$ we let Λ_s be the graph of φ_s^{-1} and we let Λ' be the family of Λ_s's:

$$\Lambda_s = \{(\varphi_s(v), v)\,;\, v \in \mathfrak{X}^a\} \subset \mathfrak{X} \times \mathfrak{X}^a,$$
$$\Lambda' = \{(\varphi_s(v), v, s)\,;\, v \in \mathfrak{X}^a,\ s \in I\} \subset \mathfrak{X} \times \mathfrak{X}^a \times I.$$

Thus Λ_s is a Lagrangian submanifold of $\mathfrak{X} \times \mathfrak{X}^a$. Now we can see that Φ is a Hamiltonian isotopy if and only if there exists a Lagrangian submanifold $\Lambda \subset \mathfrak{X} \times \mathfrak{X}^a \times T^*I$ such that, for any $s \in I$,

$$\Lambda_s = \Lambda \circ T_s^* I. \tag{25}$$

(Here, the notation $\bullet \circ \bullet$ is a slight generalization of (6) to the case where the symplectic manifolds are no more cotangent bundles.) In this case Λ is written

$$\Lambda = \left\{\big(\Phi(v,s), v, s, -f(\Phi(v,s), s)\big)\,;\, v \in \mathfrak{X}, s \in I\right\}, \tag{26}$$

where the function $f\colon \mathfrak{X} \times I \to \mathbb{R}$ is defined up to addition of a function depending on s by $v_{\Phi,s} = H_{f_s}$.

Homogeneous Case

Let us come back to the case $\mathfrak{X} = \dot{T}^*M$ and consider $\Phi\colon \dot{T}^*M \times I \to \dot{T}^*M$ such that

$$\begin{cases} \varphi_s \text{ is a homogeneous symplectic isomorphism for each } s \in I, \\ \varphi_0 = \mathrm{id}_{\dot{T}^*M}\,. \end{cases} \tag{27}$$

In this case Φ is a Hamiltonian isotopy and there exists a unique homogeneous function f such that $v_{\Phi,s} = H_{f_s}$. It is given by

$$f = \langle \alpha, v_\Phi \rangle \colon \dot{T}^*M \times I \to \mathbb{R}. \tag{28}$$

Since f is homogeneous of degree 1 in the fibers of $\dot{T}^*M$, the Lagrangian submanifold Λ of $\dot{T}^*M \times \dot{T}^*M \times T^*I$ associated to f in (26) is $\mathbb{R}^+$-conic.

We say that $F \in \mathsf{D}(\mathbf{k}_M)$ is locally bounded if for any relatively compact open subset $U \subset M$ we have $F|_U \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_U)$. We denote by $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)$ the full subcategory of $\mathsf{D}(\mathbf{k}_M)$ consisting of locally bounded objects.

Theorem 3.1 ([2, Th 4.3]). *Consider a homogeneous Hamiltonian isotopy Φ satisfying the hypotheses* (27)*. Let us consider the following conditions on $K \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M\times M\times I})$:*

(a) $\mathrm{SS}(K) \subset \Lambda \cup T^*_{M\times M\times I}(M \times M \times I)$*,*
(b) $K_0 \simeq \mathbf{k}_\Delta$*,*
(c) *both projections* $\mathrm{Supp}(K) \rightrightarrows M \times I$ *are proper,*
(d) $K_s \circ K_s^{-1} \simeq K_s^{-1} \circ K_s \simeq \mathbf{k}_\Delta$*, where* $K_s^{-1} = v^{-1}\mathrm{R}\mathscr{H}om(K_s, \omega_M \boxtimes \mathbf{k}_M)$ *and* $v(x, y) = (y, x)$*.*

Then we have

(i) *The conditions (a) and (b) imply the other two conditions (c) and (d).*
(ii) *There exists K satisfying (a)–(d).*
(iii) *Moreover such a K satisfying the conditions (a)–(d) is unique up to a unique isomorphism.*

We shall call K the *quantization* of Φ on I, or the quantization of the family $\{\varphi_s\}_{s\in I}$.

Non Homogeneous Case

Theorem 3.1 is concerned with homogeneous Hamiltonian isotopies. The next result will allow us to adapt it to non homogeneous cases. Let $\Phi\colon T^*M \times I \to T^*M$ be a Hamiltonian isotopy and assume

$$\begin{cases} \text{there exists a compact set } C \subset T^*M \text{ such that } \varphi_s|_{T^*M \setminus C} \text{ is the} \\ \text{identity for all } s \in I. \end{cases} \tag{29}$$

We denote by $T^*_{\{\tau>0\}}(M \times \mathbb{R})$ the open subset $\{\tau > 0\}$ of $T^*(M \times \mathbb{R})$ and we define the map

$$\rho\colon T^*_{\{\tau>0\}}(M \times \mathbb{R}) \to T^*M, \quad (x,t;\xi,\tau) \mapsto (x;\xi/\tau). \tag{30}$$

We let $\pi_0(\dot{T}^*M)$ be the set of connected components of $\dot{T}^*M$. Hence $\pi_0(\dot{T}^*M)$ consists of two points if $\dim M = 1$ and one point if $\dim M > 1$. See Remark 3.3 below.

Proposition 3.2 ([2, Prop. A.6]). *There exist a homogeneous Hamiltonian isotopy* $\tilde{\Phi}\colon \dot{T}^*(M \times \mathbb{R}) \times I \to \dot{T}^*(M \times \mathbb{R})$ *and* C^∞*-functions* $u\colon T^*M \times I \to \mathbb{R}$ *and* $v\colon I \times \pi_0(\dot{T}^*M) \to \mathbb{R}$ *such that the following diagram commutes:*

$$\begin{array}{ccc} T^*_{\{\tau>0\}}(M \times \mathbb{R}) \times I & \xrightarrow{\ \widetilde{\Phi}\ } & T^*_{\{\tau>0\}}(M \times \mathbb{R}) \\ \downarrow{\scriptstyle \rho \times \mathrm{id}_I} & & \downarrow{\scriptstyle \rho} \\ T^*M \times I & \xrightarrow{\ \Phi\ } & T^*M \end{array}$$

and

$$\tilde{\Phi}((x;\xi),(t;\tau),s) = ((x';\xi'),(t + u(x;\xi/\tau,s);\tau)), \quad \textit{for } \tau > 0, \tag{31}$$

$$\tilde{\Phi}((x;\xi),(t;0),s) = ((x;\xi),(t + v(s,[(x;\xi)]);0)), \tag{32}$$

where $(x';\xi'/\tau) = \varphi_s(x;\xi/\tau)$*. Moreover we have* $u(x;\xi/\tau,s) = v(s,[(x;\xi/\tau)])$ *for* $(x;\xi/\tau) \notin C$*.*

Remark 3.3. If $\dim M = 1$, $T^*M \setminus M$ has two connected components, and one has to consider two functions v_- and v_+, one for each connected component. Hence, as mentioned to us by Damien Callaque, Proposition A.6 of [2] should be corrected accordingly. This has no consequence for the rest of the paper.

4 Convolution and Localization

Most of the ideas of this section are due to Tamarkin [12]. The reader will be aware that our notations do not follow Tamarkin's ones. We also give some proofs which may be rather different from Tamarkin's original ones.

In all this section, we consider a trivial vector bundle

$$q\colon E = M \times V \to M \tag{33}$$

and a trivial cone $\gamma = M \times \gamma_0 \subset E$ such that

$$\gamma_0 \text{ is a closed convex proper cone of } V \text{ containing } 0 \text{ and } \gamma_0 \neq \{0\}. \tag{34}$$

The polar cone $\gamma_0^\circ \subset V^*$ is the closed convex cone given by

$$\gamma_0^\circ = \{\theta \in V^*; \langle \theta, v\rangle \geq 0\} \text{ for all } v \in \gamma_0.$$

Many results could be generalized to general vector bundles and general proper convex cones, but in practice we shall use these results with $V = \mathbb{R}$ and $\gamma_0 = \{t \in \mathbb{R}; t \geq 0\}$. Recall that a subset in $T^*M \times V^*$ is a cone if it is invariant by the diagonal action of $\mathbb{R}^+$ (see (17)).

Definition 4.1. A closed cone $A \subset T^*M \times V^*$ is called a strict γ-cone if $A \subset (T^*M \times \mathrm{Int}\gamma_0^\circ) \cup T^*_M M \times \{0\}$.

Example 4.2. Assume $V = \mathbb{R}$ and M is open in $\mathbb{R}^n$. Denote by $(t;\tau)$ the coordinates on $T^*\mathbb{R}$ and by $(x;\xi)$ the coordinates on T^*M. Let $\gamma_0 = \{t \in \mathbb{R}; t \geq 0\}$. Then a closed cone $A \subset T^*M \times V^*$ is a strict γ-cone if, for any compact subset $C \subset M$, there exists $a \in \mathbb{R}, a > 0$ such that $\tau \geq a|\xi|$ for all $(x;\xi,\tau) \in A \cap (\pi_M^{-1}(C) \times V^*)$.

Remark 4.3. If $f: N \to M$ is a morphism of manifolds and $A \subset T^*M \times V^*$ is a strict γ-cone, then $f \times \mathrm{id}_V: N \times V \to M \times V$ is non-characteristic for $\hat{\pi}_E^{-1}(A)$ (where $\hat{\pi}_E^{-1}$ is defined in (16)).

In the sequel, we consider the maps

$$\begin{gathered} q_1, q_2, s: V \times V \to V, \\ q_1(v_1,v_2) = v_1, \quad q_2(v_1,v_2) = v_2, \quad s(v_1,v_2) = v_1 + v_2. \end{gathered} \tag{35}$$

If there is no risk of confusion, we still denote by q_1, q_2, s the associated maps $M \times V \times V \to M \times V$.

We denote by δ_M the diagonal embedding

$$\delta_M: M \hookrightarrow M \times M \tag{36}$$

and if there is no risk of confusion, we still denote by δ_M the associated map $M \times V \times V \hookrightarrow M \times M \times V \times V$, that is, the map $E \times_M E \hookrightarrow E \times E$.

The maps s and δ_M give rise to the maps:

$$\begin{aligned} &T^*(E \times_M E) \xleftarrow{(\delta_M)_d} M \times_{M \times M} T^*(E \times_M E) \xrightarrow{(\delta_M)_\pi} T^*(E \times E), \\ &T^*(E \times_M E) \xleftarrow{s_d} V \times_{V \times V} T^*(E \times_M E) \xrightarrow{s_\pi} T^*E. \end{aligned}$$

On T^*E we have the antipodal map a, but there is another involution associated with a and the involution $(x,y) \mapsto (x,-y)$ on E. We denote by α the involution of T^*E

$$\alpha\colon (x, y; \xi, \eta) \mapsto (x, -y; -\xi, \eta) \tag{37}$$

and for a subset $A \subset T^*E$ we denote by A^α its image by this involution. We also denote by α the involution of $T^*M \times V^*$ defined by $(x; \xi, \eta) \mapsto (x; -\xi, \eta)$. Hence for $A \subset T^*M \times V^*$ we have, using the notation (16), $\hat{\pi}_E^{-1}(A^\alpha) = \hat{\pi}_E^{-1}(A)^\alpha$.

Convolution

Recall the notations (10) and (15).

Notation 4.4. *For two closed subsets A and B in T^*E, we set*

$$A \mathbin{\hat{\star}} B := s_\sharp \delta_M^\sharp (A \times B). \tag{38}$$

In general, the calculation of $A \mathbin{\hat{\star}} B$ is difficult. In Lemmas 4.5 and 4.7 below we consider special situations in which this calculation is easy.

Lemma 4.5. *Let A' and B' be two closed cones in V^*. Set $A = T^*M \times V \times A'$ and $B = T^*M \times V \times B'$. Then*

$$A \mathbin{\hat{\star}} B = A \cap B. \tag{39}$$

Proof. Using the hypothesis on A and B, it follows from (11) that

$$\delta_M^\sharp (A \times B) = T^*M \times V \times V \times A' \times B'.$$

Then the result follows from Corollary 2.17. □

Notation 4.6. *Let A and B be two closed cones in $T^*M \times V^*$. We set*

$$\begin{aligned} A \underset{M}{+} B = \{(x; \xi, \eta) \in T^*M \times V^*; &\text{ there exist } \xi_1, \xi_2 \in T_x^*M \text{ such} \\ &\text{that } (x; \xi_1, \eta) \in A,\ (x; \xi_2, \eta) \in B \text{ and } \xi = \xi_1 + \xi_2\}. \end{aligned} \tag{40}$$

Lemma 4.7. *Consider two closed strict γ-cones A and B in $T^*M \times V^*$. Then $A \underset{M}{+} B$ is also a strict γ-cone and $\hat{\pi}_E^{-1}(A) \mathbin{\hat{\star}} \hat{\pi}_E^{-1}(B) = \hat{\pi}_E^{-1}(A \underset{M}{+} B)$.*

In particular, if $A \cap B \subset T_M^*M \times \{0\}$, then

$$(\hat{\pi}_E^{-1}(A) \mathbin{\hat{\star}} (\hat{\pi}_E^{-1}(B))^\alpha) \cap (T_M^*M \times T^*V) \subset T_E^*E.$$

Proof. The fact that $A \underset{M}{+} B$ is a strict γ-cone follows easily from the definition.

By Remark 4.3, $\hat{\pi}_E^{-1}(A) \times \hat{\pi}_E^{-1}(B)$ is non-characteristic for the inclusion $\delta_M\colon M \times V \times V \to M \times M \times V \times V$ and we may replace $\delta_M^\sharp$ by $\delta_{M,d}\delta_{M,\pi}^{-1}$ in (38). We find $\delta_M^\sharp(\hat{\pi}_E^{-1}(A) \times (\hat{\pi}_E^{-1}(B))) = \hat{\pi}_{M\times V\times V}^{-1}(C_1)$, where

$$C_1 = \{(x;\xi,\eta_1,\eta_2) \in T^*M \times V^* \times V^*; \text{there exist } \xi_1,\xi_2 \in T^*_xM \text{ such that } (x;\xi_1,\eta_1) \in A,\ (x;\xi_2,\eta_2) \in B \text{ and } \xi = \xi_1+\xi_2\}$$

and the result follows. □

Using the notations (35), the convolution of sheaves is defined by:

Definition 4.8. For $F, G \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)$, we set

$$F \star G := \mathrm{R}s_!(q_1^{-1}F \overset{\mathrm{L}}{\otimes} q_2^{-1}G) \simeq \mathrm{R}s_!\delta_M^{-1}(F \overset{\mathrm{L}}{\boxtimes} G), \tag{41}$$

$$F \star_{np} G := \mathrm{R}s_*(q_1^{-1}F \overset{\mathrm{L}}{\otimes} q_2^{-1}G) \simeq \mathrm{R}s_*\delta_M^{-1}(F \overset{\mathrm{L}}{\boxtimes} G). \tag{42}$$

The morphism $\mathbf{k}_\gamma \to \mathbf{k}_{M\times\{0\}}$ gives the morphism

$$F \star_{np} \mathbf{k}_\gamma \to F. \tag{43}$$

Recall the following result:

Proposition 4.9 (Microlocal Cut-Off Lemma [5, Prop. 5.2.3, 3.5.4]). *Let* $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)$*. Then* $\mathrm{SS}(F) \subset T^*M \times V \times \gamma_0^\circ$ *if and only if the morphism* (43) *is an isomorphism.*

If γ_0 has a non-empty interior we have $\mathbf{k}_{\gamma_0} \simeq \mathrm{D}'_V(\mathbf{k}_{\mathrm{Int}\gamma_0})$ and we deduce from Corollary 2.7(ii) that

$$F \star_{np} \mathbf{k}_\gamma \simeq \mathrm{R}s_*\mathrm{R}\Gamma_{M\times V\times \mathrm{Int}\gamma_0}(q_1^{-1}F). \tag{44}$$

Following Tamarkin [12], we introduce a right adjoint to the convolution functor by setting for $F, G \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)$

$$\mathscr{H}om^\star(G,F) := \mathrm{R}q_{1*}\mathrm{R}\mathscr{H}om(q_2^{-1}G, s^!F). \tag{45}$$

Hence for $F_1, F_2, F_3 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)$, we have

$$\mathrm{RHom}(F_1 \star F_2, F_3) \simeq \mathrm{RHom}(F_1, \mathscr{H}om^\star(F_2,F_3)). \tag{46}$$

We use the notation:

$$i\colon E \to E \text{ denotes the involution } (x,y) \mapsto (x,-y). \tag{47}$$

Lemma 4.10. *For* $F, G \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)$ *we have*

$$\mathscr{H}om^\star(G,F) \simeq \mathrm{R}s_*\mathrm{R}\mathscr{H}om(q_2^{-1}i^{-1}G, q_1^!F),$$

$$F \star G \simeq \mathrm{R}q_{1!}(s^{-1}F \overset{\mathrm{L}}{\otimes} q_2^{-1}i^{-1}G).$$

Proof. We only prove the first isomorphism, the second one being similar. We set $f := (s, -q_2)\colon E \times_M E \to E \times_M E$, $(x, v_1, v_2) \mapsto (x, v_1 + v_2, -v_2)$. We find $f \circ f = \mathrm{id}$, $s = q_1 \circ f$, $q_2 \circ f = i \circ q_2$. Since f is an isomorphism $\mathrm{R}\mathscr{H}om$ commutes with $f^{-1} \simeq f^!$. Since $f \circ f = \mathrm{id}$ we have $f^{-1} = f_*$. We deduce the isomorphisms:

$$\begin{aligned}\mathscr{H}om^*(G, F) &\simeq \mathrm{R}q_{1*}\mathrm{R}\mathscr{H}om(q_2^{-1}G, s^!F)\\ &\simeq \mathrm{R}q_{1*}\mathrm{R}\mathscr{H}om(f^{-1}q_2^{-1}i^{-1}G, f^!q_1^!F)\\ &\simeq \mathrm{R}q_{1*}f^{-1}\mathrm{R}\mathscr{H}om(q_2^{-1}i^{-1}G, q_1^!F)\\ &\simeq \mathrm{R}s_*\mathrm{R}\mathscr{H}om(q_2^{-1}i^{-1}G, q_1^!F).\end{aligned}$$

□

Proposition 4.11. *For $F_1, F_2, F_3 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)$ we have*

$$\begin{aligned}(F_1 \star F_2) \star F_3 &\simeq F_1 \star (F_2 \star F_3),\\ \mathscr{H}om^*(F_1 \star F_2, F_3) &\simeq \mathscr{H}om^*(F_1, \mathscr{H}om^*(F_2, F_3)).\end{aligned} \tag{48}$$

Proof. (i) The first isomorphism is proved in the same way as the associativity of the composition of kernels: we check easily that both sides are isomorphic to $\mathrm{R}\sigma_!(q_1^{-1}(F_1)\overset{\mathrm{L}}{\otimes}q_2^{-1}(F_2)\overset{\mathrm{L}}{\otimes}q_3^{-1}(F_3))$ where $\sigma\colon M \times V^3 \to M \times V$ is given by $\sigma(x, v_1, v_2, v_3) = (x, v_1 + v_2 + v_3)$ and $q_i\colon M \times V^3 \to M \times V$ is the projection on the ith factor V.

(ii) We use the Yoneda embedding to prove the second isomorphism. We apply the functor $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)}(H, \bullet)$ for any $H \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)$ to each term of this formula. One gets an isomorphism in view of the adjunction isomorphism (46) and the associativity of $\star$ proved in (i).

□

Proposition 4.12. *Let $q\colon E \to M$ and $q'\colon M \times V \times V \to M$ be the projections. For $F, G, H \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)$ we have*

$$\mathrm{R}q_*(\mathrm{R}\mathscr{H}om(F, \mathscr{H}om^*(G, H))) \simeq \mathrm{R}q_*(\mathrm{R}\mathscr{H}om(F \star G, H)), \tag{49}$$

$$\begin{aligned}\mathrm{R}q_!((F \star G)\overset{\mathrm{L}}{\otimes}H) &\simeq \mathrm{R}q'_!(q_1^{-1}F\overset{\mathrm{L}}{\otimes}q_2^{-1}G\overset{\mathrm{L}}{\otimes}s^{-1}H)\\ &\simeq \mathrm{R}q_!(F\overset{\mathrm{L}}{\otimes}(i^{-1}G \star H)).\end{aligned} \tag{50}$$

Proof. The first isomorphism follows by adjunction from (41) and (45), using $q \circ q_1 = q \circ s$. The second and third ones follow from the projection formula, the identities $q \circ q_1 = q' = q \circ s$ and Lemma 4.10. □

Recall that the involution $(\bullet)^\alpha$ is defined in (37).

Proposition 4.13. *For $F, G \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)$ we have*

$$\begin{aligned} &\mathrm{SS}(F \star G) \subset \mathrm{SS}(F) \hat{\star}\, \mathrm{SS}(G), \\ &\mathrm{SS}(\mathscr{H}om^*(G, F)) \subset \mathrm{SS}(F) \hat{\star}\, \mathrm{SS}(G)^{\alpha}. \end{aligned} \tag{51}$$

Proof. Both inclusions in (51) follow from (41), (38) and Theorems 2.12 and 2.16. For the second one we also use Lemma 4.10 and $\mathrm{SS}(i^{-1}G)^a = \mathrm{SS}(G)^{\alpha}$. □

Using (51) and (39), we get:

Corollary 4.14. *Let $F, G \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)$ and assume that there exist closed cones $A', B' \subset V^*$ such that $\mathrm{SS}(F) \subset T^*M \times V \times A'$ and $\mathrm{SS}(G) \subset T^*M \times V \times B'$. Then*

$$\begin{aligned} &\mathrm{SS}(F \star G) \subset T^*M \times V \times (A' \cap B'), \\ &\mathrm{SS}(\mathscr{H}om^*(G, F)) \subset T^*M \times V \times (A' \cap B'). \end{aligned} \tag{52}$$

Corollary 4.15. *Let $F, G \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)$ and assume that there exist closed strict γ-cones A and B in $T^*M \times V^*$ such that $\mathrm{SS}(F) \subset \hat{\pi}_E^{-1}(A)$ and $\mathrm{SS}(G) \subset \hat{\pi}_E^{-1}(B)$. Let N be a submanifold of M and $j\colon N \times V \to M \times V$ the inclusion. Then*

$$j^{-1}\mathscr{H}om^*(F, G) \simeq \mathscr{H}om^*(j^{-1}F, j^{-1}G).$$

Proof. By Proposition 4.13 and Lemma 4.7, $\mathrm{SS}(\mathscr{H}om^*(F, G)) \subset \hat{\pi}_E^{-1}(A \underset{M}{+} B)$ and $A \underset{M}{+} B$ is a strict γ-cone. By Remark 4.3, we deduce $j^!H \simeq j^{-1}H \otimes \omega_{N\times V|M\times V}$ for $H = F, G$ or $\mathscr{H}om^*(F, G)$. This gives the first and last steps in the sequence of isomorphisms, where we set $j' = j \times \mathrm{id}_V$:

$$\begin{aligned} j^{-1}\mathscr{H}om^*(F, G) &\simeq j^!\mathrm{R}q_{1*}\mathrm{R}\mathscr{H}om\,(q_2^{-1}F, s^!G) \otimes \omega_{N\times V|M\times V}^{\otimes -1} \\ &\simeq \mathrm{R}q_{1*}j'^!\mathrm{R}\mathscr{H}om\,(q_2^{-1}F, s^!G) \otimes \omega_{N\times V|M\times V}^{\otimes -1} \\ &\simeq \mathrm{R}q_{1*}\mathrm{R}\mathscr{H}om\,(j'^{-1}q_2^{-1}F, j'^!s^!G) \otimes \omega_{N\times V|M\times V}^{\otimes -1} \\ &\simeq \mathrm{R}q_{1*}\mathrm{R}\mathscr{H}om\,(q_2^{-1}j^{-1}F, s^!j^!G) \otimes \omega_{N\times V|M\times V}^{\otimes -1} \\ &\simeq \mathscr{H}om^*(j^{-1}F, j^{-1}G). \end{aligned}$$

□

Kernels Associated with Cones

Recall that we consider a trivial vector bundle $E = M \times V$ and a trivial cone $\gamma = M \times \gamma_0$ satisfying (34). For another proper closed convex cone $\lambda_0 \subset V$ such that $\lambda_0 \subset \gamma_0$, setting $\lambda = M \times \lambda_0$, we shall use the exact sequence of sheaves:

$$0 \to \mathbf{k}_{\gamma\setminus\lambda} \to \mathbf{k}_\gamma \to \mathbf{k}_\lambda \to 0. \tag{53}$$

Lemma 4.16. *Let $\lambda_0 \subset \gamma_0$ be closed convex proper cones. Then*

$$\begin{aligned} &\mathrm{SS}(\mathbf{k}_\gamma) \subset T^*_M M \times V \times \gamma_0^\circ, \\ &\mathrm{SS}(\mathbf{k}_{\gamma\setminus\lambda}) \subset T^*_M M \times V \times (\lambda_0^\circ \setminus \mathrm{Int}(\gamma_0^\circ)). \end{aligned}$$

Proof. Since our sheaves are inverse images of sheaves on V we may as well assume that M is a point. Since our sheaves are conic in the sense of [5, §5.5] their microsupports are biconic. Now, a closed biconic subset A of $V \times V^*$ satisfies $A \subset V \times (A \cap \{0\} \times V^*)$. Hence we only have to check the inclusions at the origin.

Then the first inclusion follows from [5, Prop. 5.3.1].

For the second inclusion we use the Sato–Fourier transform $(\cdot)^\wedge\colon \mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_V) \to \mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_{V^*})$ defined in [5, §3.7] ($\mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_V)$ denotes the subcategory of complexes with conic cohomology). We have $(\mathbf{k}_{\gamma_0})^\wedge \simeq \mathbf{k}_{\mathrm{Int}\gamma_0^\circ}$ and we deduce the distinguished triangle

$$(\mathbf{k}_{\gamma_0\setminus\lambda_0})^\wedge \to \mathbf{k}_{\mathrm{Int}\gamma_0^\circ} \to \mathbf{k}_{\mathrm{Int}\lambda_0^\circ} \xrightarrow{+1} .$$

Hence $(\mathbf{k}_{\gamma_0\setminus\lambda_0})^\wedge \simeq \mathbf{k}_{\mathrm{Int}\lambda_0^\circ\setminus\mathrm{Int}\gamma_0^\circ}[-1]$ and we conclude with [5, Prop. 5.5.5] which implies $\mathrm{SS}(F) \cap T^*_0 V = \mathrm{supp}(F^\wedge)$ for $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_V)$. □

We introduce the kernel:

$$L_\gamma := \mathbf{k}_\gamma \star\colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}_E) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_E). \tag{54}$$

The morphism $\mathbf{k}_\gamma \to \mathbf{k}_{\{0\}}$ induces a morphism of functors $\varepsilon\colon L_\gamma \to \mathrm{id}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)}$. By (48) we have $L_\gamma \circ L_\gamma \simeq L_\gamma$. Hence, the pair (L_γ, ε) is a projector in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)^{\mathrm{op}}$ in the sense of [6, Chap. 5]. It will be convenient to write L_γ with the language of kernels as in (7). We define $\gamma^+ \subset E \times E$ by

$$\gamma^+ = \{(x, v, x', v') \in E \times E;\ v - v' \in \gamma_0\}. \tag{55}$$

Then

$$L_\gamma \simeq \mathbf{k}_{\gamma^+} \circ \cdot. \tag{56}$$

In the sequel we set

$$\begin{aligned} U_\gamma &:= T^*M \times V \times \mathrm{Int}(\gamma_0^\circ), \\ Z_\gamma &:= T^*E \setminus U_\gamma. \end{aligned} \tag{57}$$

Proposition 4.17. *Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)$.*

(i) $\mathrm{SS}(L_\gamma F) \subset \overline{U_\gamma} = T^*M \times V \times \gamma_0^\circ$.

(ii) *Consider a distinguished triangle* $L_\gamma F \to F \to G \xrightarrow{+1}$. *Then* $\mathrm{SS}(G) \subset Z_\gamma$. *In particular,* $\mathrm{SS}(L_\gamma F) \subset (T^*M \times V \times \partial\gamma_0^\circ) \cup (\mathrm{SS}(F) \cap U_\gamma)$.

(iii) *Let* $G \in \mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\mathbf{k}_E)$. *Then* $\mathrm{R}q_*\mathrm{R}\Gamma_\gamma(G) \simeq 0$. *In particular,* $\mathrm{R}\Gamma_\gamma(E;G) \simeq 0$.

Proof. (i) follows from (52) and Lemma 4.16.

(ii) Using the exact sequence (53), we have $G \simeq \mathbf{k}_{\gamma\setminus\{0\}} \star F$. Then the result again follows from (52) and Lemma 4.16.

(iii) We set $H = \mathrm{R}\Gamma_\gamma(G) \simeq \mathrm{R}\mathscr{H}om(\mathbf{k}_\gamma, G)$. It follows from Theorem 2.12 that $\mathrm{SS}(H) \subset Z_\gamma$. Choose a vector $\xi \in \mathrm{Int}(\gamma_0^\circ)$ and consider the projection

$$\theta\colon M \times V \to M \times \mathbb{R}, \quad \theta(x,v) = (x; \langle \xi, v\rangle).$$

Since γ is a proper cone, θ is proper on $\operatorname{supp} H$ and we get by Theorem 2.6 that $\mathrm{SS}(\mathrm{R}\theta_*(H)) \subset \{\tau \le 0\}$ where $(t;\tau)$ are the coordinates on $T^*\mathbb{R}$. Moreover, $\operatorname{supp} \mathrm{R}\theta_*(H) \subset M \times \{t \ge 0\}$.

Now it is enough to prove that $\mathrm{R}\Gamma(U \times \mathbb{R}; \mathrm{R}\theta_*(H)) = 0$, for any open subset U of M. Denote by $p\colon U \times \mathbb{R} \to \mathbb{R}$ the projection and set $\tilde{H} = \mathrm{R}p_*\mathrm{R}\theta_*(H)$. Although p is not proper on $\mathrm{supp}(\tilde{H})$, one easily checks that $\mathrm{SS}(\tilde{H}) \subset \{t \ge 0, \tau \le 0\}$ and this implies $\tilde{H} \simeq 0$. (This is a special case of Corollary 2.8.) □

The next lemma follows immediately from the adjunction formula (46).

Lemma 4.18. *Let $F, G \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)$ and assume that $L_\gamma F \xrightarrow{\sim} F$. Then we have* $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)}(F,G) \simeq \mathrm{R}\Gamma_\gamma(E; \mathscr{H}om^*(F,G))$.

Proposition 4.19. (a) *Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)$. Then $F \in \mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\mathbf{k}_E)^{\perp,l}$ if and only if the natural morphism $L_\gamma F \to F$ is an isomorphism.*

(b) *Let $G \in \mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\mathbf{k}_E)$. Then $L_\gamma G \simeq 0$.*

Proof. (a)-(i) Assume $F \simeq L_\gamma F$. Let $G \in \mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\mathbf{k}_E)$ and set $H := \mathscr{H}om^*(F,G)$. Then H belongs to $\mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\mathbf{k}_E)$ by (52) and $\mathrm{R}\Gamma_\gamma(E;H) \simeq 0$ by Proposition 4.17. Since $F \simeq L_\gamma F$, we get $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)}(F,G) = 0$ by Lemma 4.18.

(a)-(ii) Assume that $F \in \mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\mathbf{k}_E)^{\perp,l}$ and consider a distinguished triangle $L_\gamma F \to F \to G \xrightarrow{+1}$. By (a)-(i) $L_\gamma F$ also belongs to $\mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\mathbf{k}_E)^{\perp,l}$. Hence so does G. On the other hand, $G \in \mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\mathbf{k}_E)$ by Proposition 4.17. Hence, $G \simeq 0$.

(b) Let $G \in \mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\mathbf{k}_E)$ and consider a distinguished triangle $L_\gamma G \to G \to H \xrightarrow{+1}$. Since both G and H belong to $\mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\mathbf{k}_E)$, so does $L_\gamma G$. Since $L_\gamma G$ belongs to $\mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\mathbf{k}_E)^{\perp,l}$, it is 0.

□

Remark 4.20. One can also consider the projector

$$R_\gamma := \mathscr{H}om^*(\mathbf{k}_\gamma, \bullet) \colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}_E) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_E). \tag{58}$$

Then we obtain similar results to Propositions 4.17, 4.19 and Lemma 4.18 with R_γ instead of L_γ. Note that the pair (L_γ, R_γ) is a pair of adjoint functors:

$$\begin{aligned}\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)}(L_\gamma F, G) &\simeq \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)}(F, R_\gamma G) \\ &\simeq \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)}(\mathbf{k}_\gamma, \mathscr{H}om^*(F, G)).\end{aligned}$$

Note that $\mathbf{k}_\gamma$ is cohomologically constructible. If we assume that $\mathrm{Int}(\gamma) \neq \emptyset$, then $\mathrm{D}'\mathbf{k}_\gamma \simeq \mathbf{k}_{\mathrm{Int}(\gamma)}$ and one deduces from Lemma 4.10 that

$$\mathscr{H}om^*(\mathbf{k}_\gamma, \mathbf{k}_\gamma) \simeq \mathbf{k}_{\mathrm{Int}(-\gamma)}[d_V], \tag{59}$$

where d_V is the dimension of V.

Projector and Localization

Recall that $E = M \times V$ is a trivial vector bundle over M, γ_0 is a cone satisfying (34) and the sets U_γ and Z_γ are defined in (57). By definition $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_E; U_\gamma)$ is a localization of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)$ and we let $Q_\gamma \colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}_E) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_E; U_\gamma)$ be the functor of localization.

Proposition 4.21. (i) *The functor L_γ defined in* (54) *takes its values in $\mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\mathbf{k}_E)^{\perp,l}$ and sends $\mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\mathbf{k}_E)$ to* 0. *It factorizes through Q_γ and induces a functor $l_\gamma \colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}_E; U_\gamma) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)$ such that $L_\gamma \simeq l_\gamma \circ Q_\gamma$.*
(ii) *The functor l_γ is left adjoint to Q_γ and induces an equivalence $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_E; U_\gamma) \simeq \mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\mathbf{k}_E)^{\perp,l}$.*

This is visualized by the diagram

$$\begin{array}{ccccc} \mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\mathbf{k}_E) & \hookrightarrow & \mathsf{D}^{\mathrm{b}}(\mathbf{k}_E) & \xrightarrow{Q_\gamma} & \mathsf{D}^{\mathrm{b}}(\mathbf{k}_E; U_\gamma) \\ & & & \searrow^{L_\gamma} & \big\downarrow{\scriptstyle l_\gamma}\,\wr \\ & & & & \mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\mathbf{k}_E)^{\perp,l}. \end{array} \tag{60}$$

Proof. This follows from Proposition 4.19 together with the classical results on the localization of triangulated categories recalled in Sect. 2 (see e.g., [6, Exe. 10.15]). □

In particular, we have for $F, G \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)$

$$\begin{aligned}\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_E; U_\gamma)}(Q_\gamma(F), Q_\gamma(G)) &\simeq \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)}(L_\gamma(F), G) \\ &\simeq \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)}(L_\gamma(F), L_\gamma(G)).\end{aligned} \tag{61}$$

There is a similar result to Proposition 4.21, replacing the functor L_γ with the functor R_γ. The functor R_γ takes its values in $\mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\mathbf{k}_E)^{\perp,r}$ and sends $\mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\mathbf{k}_E)$ to 0. It factorizes through Q_γ and induces a functor $r_\gamma\colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}_E; U_\gamma) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)$ such that $R_\gamma \simeq r_\gamma \circ Q_\gamma$.

We notice that, for $F \in \mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\mathbf{k}_E)^{\perp,l}$ or $G \in \mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\mathbf{k}_E)^{\perp,r}$, we have

$$\mathcal{H}om^*(F, G) \in \mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\mathbf{k}_E)^{\perp,r}. \tag{62}$$

By Proposition 4.17 (used with R_γ instead of L_γ) we obtain in particular

$$\mathcal{H}om^*(F, G) \in \mathsf{D}^{\mathrm{b}}_{\overline{U_\gamma}}(\mathbf{k}_M). \tag{63}$$

Notation 4.22. *Let us set for short*

$$\begin{aligned} \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M^\gamma) &:= \mathsf{D}^{\mathrm{b}}(\mathbf{k}_E; U_\gamma), \\ \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M^{\gamma,l}) &:= \mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\mathbf{k}_E)^{\perp,l}, \\ \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M^{\gamma,r}) &:= \mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\mathbf{k}_E)^{\perp,r}. \end{aligned} \tag{64}$$

When $M = \mathrm{pt}$, we set

$$\mathsf{D}^{\mathrm{b}}(\mathbf{k}^\gamma) := \mathsf{D}^{\mathrm{b}}(\mathbf{k}^\gamma_{\mathrm{pt}}) \tag{65}$$

and similarly with $\mathsf{D}^{\mathrm{b}}(\mathbf{k}^{\gamma,l})$ and $\mathsf{D}^{\mathrm{b}}(\mathbf{k}^{\gamma,r})$.

Denote by $p\colon E = M \times V \to V$ the projection and denote by Γ^γ the functor

$$\Gamma^\gamma(\bullet) = \mathrm{RHom}(\mathbf{k}_\gamma, \bullet)\colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}^\gamma) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}). \tag{66}$$

We get the diagram of categories in which the horizontal arrows are equivalences

$$\begin{array}{ccccc} \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M^{\gamma,l}) & \xleftarrow[\sim]{l_\gamma} & \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M^\gamma) & \xrightarrow[\sim]{r_\gamma} & \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M^{\gamma,r}) \\ \downarrow{\scriptstyle \mathrm{R}p_!} & & & & \downarrow{\scriptstyle \mathrm{R}p_*} \\ \mathsf{D}^{\mathrm{b}}(\mathbf{k}^{\gamma,l}) & \xleftarrow[\sim]{l_\gamma} & \mathsf{D}^{\mathrm{b}}(\mathbf{k}^\gamma) & \xrightarrow[\sim]{r_\gamma} & \mathsf{D}^{\mathrm{b}}(\mathbf{k}^{\gamma,r}) \\ & & \downarrow{\scriptstyle \Gamma^\gamma} & & \\ & & \mathsf{D}^{\mathrm{b}}(\mathbf{k}). & & \end{array} \tag{67}$$

Note that by Lemma 4.18, for $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M^{\gamma,l})$ or $G \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M^{\gamma,r})$, we have

$$\mathrm{RHom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M^\gamma)}(F, G) \simeq \Gamma^\gamma \circ \mathrm{R}p_* \mathcal{H}om^*(F, G). \tag{68}$$

Embedding the Category $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ into $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M^{\gamma})$

Recall that $q\colon E \to M$ denotes the projection and consider the functor

$$\Psi_\gamma\colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_E), \quad F \mapsto q^{-1}F \otimes \mathbf{k}_\gamma.$$

Lemma 4.23. *One has the isomorphism of functors* $L_\gamma \circ \Psi_\gamma \xrightarrow{\sim} \Psi_\gamma$.

Proof. One has

$$\begin{aligned} L_\gamma \circ \Psi_\gamma(F) &= \mathrm{R}s_!(q_1^{-1}\mathbf{k}_\gamma \otimes q_2^{-1}(q^{-1}F \otimes \mathbf{k}_\gamma)) \\ &\simeq \mathrm{R}s_!(q_1^{-1}\mathbf{k}_\gamma \otimes q_2^{-1}\mathbf{k}_\gamma \otimes q_2^{-1}(q^{-1}F)) \\ &\simeq \mathrm{R}s_!(q_1^{-1}\mathbf{k}_\gamma \otimes q_2^{-1}\mathbf{k}_\gamma \otimes s^{-1}(q^{-1}F)) \\ &\simeq \mathrm{R}s_!(q_1^{-1}\mathbf{k}_\gamma \otimes q_2^{-1}\mathbf{k}_\gamma) \otimes q^{-1}F \\ &\simeq \mathbf{k}_\gamma \otimes q^{-1}F. \end{aligned}$$

□

In the sequel, we consider Ψ_γ as a functor

$$\Psi_\gamma\colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M^{\gamma}). \tag{69}$$

Proposition 4.24. *The functor* Ψ_γ *in* (69) *is fully faithful.*

Proof. Let $F, G \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$. Then

$$\begin{aligned} &\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)}(\mathbf{k}_\gamma \otimes q^{-1}G, \mathbf{k}_\gamma \otimes q^{-1}F) \\ &\qquad \simeq \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)}(G, \mathrm{R}q_*\mathrm{R}\mathscr{H}om(\mathbf{k}_\gamma, q^{-1}F \otimes \mathbf{k}_\gamma)) \\ &\qquad \simeq \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)}(G, \mathrm{R}q_*(q^{-1}F \otimes \mathbf{k}_\gamma)). \end{aligned}$$

Hence, it is enough to check the isomorphism

$$F \xrightarrow{\sim} \mathrm{R}q_*(q^{-1}F \otimes \mathbf{k}_\gamma). \tag{70}$$

Denote by $\tilde{q}$ the projection $\gamma \to M$. The isomorphism (70) reduces to

$$F \simeq \mathrm{R}\tilde{q}_*\tilde{q}^{-1}F$$

and this last isomorphism follows from the fact that γ is a closed convex cone, hence is contractible (see for example [5, Prop. 2.7.8]). □

A Cut-Off Result

Recall that we consider a trivial vector bundle $E = M \times V$ and a trivial cone $\gamma = M \times \gamma_0$ satisfying (34). We also recall that a subset of $T^*M \times V^*$ is a cone if it is stable by the action (17). The map $\hat{\pi}$ is defined in (16) and we have set (see (57)):

$$U_\gamma = T^*M \times V \times \mathrm{Int}\gamma_0^\circ.$$

By the equivalence l_γ of Proposition 4.21, any object $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_E; U_\gamma)$ has a canonical representative in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)$ again denoted by F and we have $F \simeq L_\gamma(F)$. By Proposition 4.17(i) we have $\mathrm{SS}(F) \subset \overline{U_\gamma}$.

We first state a kind of cut-off lemma in the case where M is a point.

Lemma 4.25. *Let V be a vector space and $\gamma \subset V$ a closed convex proper cone containing* 0*. Set $U_\gamma := V \times \mathrm{Int}\gamma^\circ$ and $Z_\gamma := T^*V \setminus U_\gamma$. Let $F \in \mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\mathbf{k}_V)^{\perp,l}$. We assume that there exists a closed cone $A \subset V^*$ such that*

(i) $A \subset \mathrm{Int}\gamma^\circ \cup \{0\}$,
(ii) $\mathrm{SS}(F) \cap U_\gamma \subset V \times A$.

Then $\mathrm{SS}(F) \subset (\mathrm{SS}(F) \cap U_\gamma) \cup T^*_V V$.

Proof. (i) Up to enlarging A we may as well assume that $\mathrm{SS}(F) \cap U_\gamma \subset V \times \mathrm{Int}A$. We set $\lambda = A^\circ$. Hence λ is a closed convex proper cone of V and we have

$$\lambda^\circ \setminus \{0\} \subset \mathrm{Int}(\gamma^\circ), \tag{71}$$

$$\mathrm{SS}(F) \cap U_\gamma \subset V \times \mathrm{Int}(\lambda^\circ). \tag{72}$$

We will prove that $L_\lambda(F)$ satisfies the conclusion of the lemma as well as the isomorphism $L_\lambda(F) \xrightarrow{\sim} F$.

(ii) By (72) and Proposition 4.17(ii) we have

$$\mathrm{SS}(F) \subset V \times (\partial\gamma^\circ \cup \mathrm{Int}(\lambda^\circ)). \tag{73}$$

By (52) we deduce

$$\begin{aligned}\mathrm{SS}(L_\lambda F) &\subset V \times (\lambda^\circ \cap (\partial\gamma^\circ \cup \mathrm{Int}(\lambda^\circ))) \\ &= V \times (\mathrm{Int}(\lambda^\circ) \cup \{0\}) \\ &\subset U_\gamma \cup T^*_V V.\end{aligned}$$

(iii) It remains to see that $F \simeq L_\lambda(F)$. We consider the distinguished triangle $\mathbf{k}_{\lambda\setminus\gamma} \star F \to L_\lambda F \to L_\gamma F \xrightarrow{+1}$. We have $L_\gamma F \xrightarrow{\sim} F$. By (73), Lemma 4.16 and (52) we have

$$\mathrm{SS}(\mathbf{k}_{\lambda\setminus\gamma} \star F) \subset V \times ((\gamma^\circ \setminus \mathrm{Int}(\lambda^\circ)) \cap (\partial\gamma^\circ \cup \mathrm{Int}(\lambda^\circ))) \subset Z_\gamma,$$

which shows that $L_\lambda F \to F$ is an isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_V; U_\gamma)$. By Proposition 4.21 we obtain $F \simeq L_\gamma(L_\lambda F)$. But $\mathbf{k}_\gamma \star \mathbf{k}_\lambda \simeq \mathbf{k}_\lambda$ and we get finally $F \simeq L_\lambda F$.

□

Now we extend Lemma 4.25 to the case of an arbitrary manifold M. We consider a finite dimensional real vector space $E = E' \times E''$ with $E' = \mathbb{R}^d$. We write $x = (x', x'') \in E' \times E''$ and $x' = (x'_1, \dots, x'_d) \in \mathbb{R}^d$. We set $U =]-1,1[^d \times E''$. We choose a diffeomorphism $\varphi\colon]-1,1[\xrightarrow{\sim} \mathbb{R}$ such that $d\varphi(t) \geq 1$ for all $t \in]-1,1[$ and we define

$$\Phi\colon U \xrightarrow{\sim} E, \quad \Phi(x'_1, \dots, x'_d, x'') = (\varphi(x'_1), \dots, \varphi(x'_d), x'').$$

Lemma 4.26. *In the preceding situation, consider two closed convex proper cones $\gamma_0 \subset E''$ and $C_1 \subset E^*$ such that $C_1 \subset (E'^* \times \mathrm{Int}(\gamma_0^\circ)) \cup \{(0,0)\}$. Then there exists another closed convex proper cone $C_2 \subset E^*$ such that $C_2 \subset (E'^* \times \mathrm{Int}(\gamma_0^\circ)) \cup \{(0,0)\}$ and*

$$\Phi_\pi \Phi_d^{-1}(U \times C_1) \subset E \times C_2.$$

Proof. (i) We assume that $\mathrm{Int}(\gamma_0^\circ)$ is non empty (otherwise the lemma is trivial). Then a closed cone of E^* is contained in $(E'^* \times \mathrm{Int}(\gamma_0^\circ)) \cup \{(0,0)\}$ if and only if it is contained in $C_{a,D} := \mathbb{R}_{\geq 0} \cdot ([-a,a]^d \times D)$ for some $a > 0$ and some compact subset $D \subset \mathrm{Int}(\gamma_0^\circ)$. Hence we may assume $C_1 = C_{a,D}$.

(ii) Denote by $(x'; \xi')$ the coordinates on $\mathbb{R}^d \times (\mathbb{R}^d)^*$. We may assume that $E'' = \mathbb{R}^m$ and we denote by $(x''; \xi'')$ the coordinates on $E'' \times (E'')^*$. The change of coordinates Φ given by $y'_i = \varphi(x'_i)$ $(i = 1, \dots, d)$, $y'' = x''$ associates the coordinates $(y; \eta) = (y', y''; \eta', \eta'')$ to the coordinates $(x'_1, \dots, x'_d, x''; \xi'_1, \dots, \xi'_d, \xi'')$ with

$$y'_i = \varphi(x'_i), \quad \eta'_i = d\varphi^{-1}(x'_i) \cdot \xi'_i, \quad (i = 1, \dots, d),$$
$$y'' = x'', \quad \eta'' = \xi''.$$

Since $d\varphi(t) \geq 1$, we get that $\Phi_\pi \Phi_d^{-1}(U \times C_{a,D}) \subset E \times C_{a,D}$ and we may choose $C_2 = C_{a,D}$.

□

Theorem 4.27. *Let $F \in \mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\mathbf{k}_E)^{\perp,l}$. We assume that there exists $A \subset T^*M \times V^*$ such that*

(i) *A is a closed strict γ-cone (see Definition 4.1),*
(ii) $\mathrm{SS}(F) \cap U_\gamma \subset \hat{\pi}_E^{-1}(A)$.

Then $\mathrm{SS}(F) \subset (\mathrm{SS}(F) \cap U_\gamma) \cup T^*_E E$.

Proof. Since the statement is local on M we may assume that M is an open subset of a vector space W. Then A is a closed subset of $M \times W^* \times V^*$. For any $x \in M$, $A_x := A \cap (\{x\} \times W^* \times V^*)$ is a cone satisfying

$$A_x \subset (W^* \times \mathrm{Int}(\gamma_0^\circ)) \cup \{(0,0)\}.$$

For $x_0 \in M$ and for a given compact neighborhood C of x_0 we may assume that there exists a closed convex cone B of $W^* \times V^*$ such that $A_x \subset B$ for any $x \in C$ and

$$B \subset (W^* \times \mathrm{Int}(\gamma_0^\circ)) \cup \{(0,0)\}.$$

We may assume $x_0 = 0 \in W$. We choose an isomorphism $W \simeq \mathbb{R}^d$ so that $]-1,1[^d \subset C$. Then we apply a change of coordinates as in Lemma 4.26, with $E' = W$, $E'' = V$, $C_1 = B$, and we are reduced to Lemma 4.25 applied to the vector space $W \times V$ and the cone $\gamma = \{0\} \times \gamma_0$. □

A Separation Theorem

The next result is a slight generalization of Tamarkin's Theorem [12, Th. 3.2]. In this statement and its proof, we write $\hat{\pi}$ instead of $\hat{\pi}_E$ for short.

Theorem 4.28 (The separation theorem). *Let A, B be two closed strict γ-cones in $T^*M \times V^*$. Let $F \in \mathsf{D}^{\mathrm{b}}_{\hat{\pi}^{-1}(A)}(\mathbf{k}_E; U_\gamma)$ and $G \in \mathsf{D}^{\mathrm{b}}_{\hat{\pi}^{-1}(B)}(\mathbf{k}_E; U_\gamma)$. Assume that $A \cap B \subset T^*_M M \times \{0\}$ and that the projection $q_2\colon M \times V \to V$ is proper on the set $\{(x, v_1 - v_2);\ (x, v_1) \in \operatorname{supp} G, (x, v_2) \in \operatorname{supp} F\}$. Then*

$$\mathrm{R}q_{2*}\mathscr{H}om^*(l_\gamma(F), l_\gamma(G)) \simeq 0,$$

where l_γ is defined in Proposition 4.21. In particular $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_E; U_\gamma)}(F, G) \simeq 0$.

Proof. We set $L = \mathscr{H}om^*(l_\gamma(F), l_\gamma(G))$ and $L' = \mathrm{R}q_{2*}L$. By (62) we have $L \in \mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\mathbf{k}_E)^{\perp,r}$. By adjunction between $\mathrm{R}q_{2*}$ and q_2^{-1} we deduce $L' \in \mathsf{D}^{\mathrm{b}}_{Z_{\gamma_0}}(\mathbf{k}_V)^{\perp,r}$. It remains to check that $\mathrm{SS}(L') \subset Z_{\gamma_0}$.

By Theorem 4.27 we have $\mathrm{SS}(F) \subset \hat{\pi}^{-1}(A)$ and $\mathrm{SS}(G) \subset \hat{\pi}^{-1}(B)$. Then Proposition 4.13 gives $\mathrm{SS}(L) \subset \hat{\pi}^{-1}(A) \mathbin{\hat{\star}} (\hat{\pi}^{-1}(B))^a$. Applying Lemma 4.7 we get

$$\mathrm{SS}(L) \cap (T^*_M M \times T^*V) \subset T^*_E E.$$

Using Lemma 4.10, the hypothesis implies that q_2 is proper on $\operatorname{supp} L$. We deduce $\mathrm{SS}(L') \subset T^*_V V$ and thus $L' \simeq 0$.

Proposition 4.21 and Lemma 4.18 give the first two isomorphisms in the sequence

$$\begin{aligned}\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_E;U_\gamma)}(F,G) &\simeq \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)}(F,G)\\ &\simeq \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)}(\mathbf{k}_\gamma, L) \simeq \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_V)}(\mathbf{k}_{\gamma_0}, L') \simeq 0,\end{aligned}$$

which proves the last assertion. □

Kernels

We consider $E = M \times V$, $\gamma = M \times \gamma_0$ and a kernel $K \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{E\times E})$. We introduce the coordinates $(x, y, x', y'; \xi, \eta, \xi', \eta')$ on $T^*(E \times E)$ and we make the following hypothesis

$$\mathrm{SS}(K) \subset \{\eta + \eta' = 0\}. \tag{74}$$

We recall that $L_\gamma \simeq \mathbf{k}_{\gamma^+} \circ \cdot$, where $\gamma^+ \subset E \times E$ is defined in (55).

Proposition 4.29. *Let* $K \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{E\times E})$ *which satisfies* (74)*. Then* $K \circ \mathbf{k}_{\gamma^+} \simeq \mathbf{k}_{\gamma^+} \circ K$*. In particular* $K \circ \cdot$ *sends* $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M^{\gamma,l})$ *into itself. Moreover* $\mathrm{SS}(K) \overset{a}{\circ} \{\eta < 0\} \subset \{\eta < 0\}$ *and* $\mathrm{SS}(K) \overset{a}{\circ} \{\eta \geq 0\} \subset \{\eta \geq 0\}$.

Proof. We define the projection $\sigma: M \times V \times M \times V \to M \times M \times V$ as the product of $\mathrm{id}_{M\times M}$ with $\sigma_0: V \times V \to V$, $(y, y') \mapsto y - y'$. Then the hypothesis (74) and Corollary 2.8 give $K \simeq \sigma^{-1}(K')$, where $K' = \mathrm{R}\sigma_*(K)$. We also have by definition $\mathbf{k}_{\gamma^+} \simeq \sigma^{-1}(\mathbf{k}_{M\times M\times\gamma_0})$. The base change formula applied to the Cartesian square

$$\begin{array}{ccc} V\times V\times V & \xrightarrow{\;q_{13}\;} & V\times V \\ {\scriptstyle \sigma_0\circ q_{12}\times\sigma_0\circ q_{23}}\downarrow & & \downarrow{\scriptstyle \sigma_0} \\ V\times V & \xrightarrow{\;s\;} & V \end{array}$$

gives the first and third isomorphisms below:

$$K \circ \mathbf{k}_{\gamma^+} \simeq \sigma^{-1}(K' \star \mathbf{k}_{M\times M\times\gamma_0}) \simeq \sigma^{-1}(\mathbf{k}_{M\times M\times\gamma_0} \star K') \simeq \mathbf{k}_{\gamma^+} \circ K.$$

The last assertion follows from the hypothesis (74). □

5 The Tamarkin Category

We particularize the preceding results to the case where $V = \mathbb{R}$ and $\gamma_0 = \{t \in \mathbb{R}; t \geq 0\}$. Hence, with the notations of (57), we have $U_\gamma = \{\tau > 0\}$. As in Sect. 3 we denote by $T^*_{\{\tau>0\}}(M \times \mathbb{R})$ the open subset $\{\tau > 0\}$ of $T^*(M \times \mathbb{R})$ and we define the map

$$\rho: T^*_{\{\tau>0\}}(M \times \mathbb{R}) \to T^*M, \quad (x, t; \xi, \tau) \mapsto (x; \xi/\tau). \tag{75}$$

We also use Notation 4.22. Moreover, for a closed subset A of T^*M we set

$$\mathsf{D}^{\mathrm{b}}_A(\mathbf{k}^\gamma_M) := \mathsf{D}^{\mathrm{b}}_{\rho^{-1}(A)}(\mathbf{k}_{M\times\mathbb{R}}; \{\tau > 0\}).$$

Lemma 5.1. *Let $A \subset T^*M$ and $F \in \mathsf{D}^{\mathrm{b}}_A(\mathbf{k}^\gamma_M)$. Let $A' \subset T^*M \times \mathbb{R}$ be given by $A' = \{(x; \xi, \tau); \tau > 0,\ (x; \xi/\tau) \in A\}$ and consider F as an object of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}^{\gamma,l}_M)$. Assume that π_M is proper on A. Then $\overline{A'}$ is a strict γ-cone and* $\mathrm{SS}(F) \subset \hat{\pi}^{-1}(\overline{A'})$. *In particular* $\mathrm{supp}(F) \subset \pi_M(A) \times \mathbb{R}$.

Proof. The properness hypothesis gives $\overline{A'} = A' \cup (\pi_M(A) \times \{\tau = 0\})$ and this implies the first assertion. Then Theorem 4.27 gives $\mathrm{SS}(F) \subset \hat{\pi}^{-1}(\overline{A'}) \cup T^*_{M\times\mathbb{R}}(M \times \mathbb{R})$. Hence, if $(x, t; 0, 0) \notin \hat{\pi}^{-1}(\overline{A'})$, we have $\mathrm{SS}(F|_{U\times\mathbb{R}}) \subset T^*_{U\times\mathbb{R}}(U \times \mathbb{R})$ for some neighborhood U of x. But $L_\gamma F \simeq F$ and we deduce $F|_{U\times\mathbb{R}} = 0$, which proves $(x, t; 0, 0) \notin \mathrm{SS}(F)$. So we get $\mathrm{SS}(F) \subset \hat{\pi}^{-1}(\overline{A'})$. □

Example 5.2. (i) Let $M = \mathbb{R}$ endowed with the coordinate x and consider the set

$$Z = \{(x, t) \in M \times \mathbb{R}; -1 \leq x \leq 1, 0 \leq 2t < -x^2 + 1\}.$$

Consider the sheaf $\mathbf{k}_Z$ and denote by $(x, t; \xi, \tau)$ the coordinates on $T^*(M \times \mathbb{R})$. The set $\mathrm{SS}(\mathbf{k}_Z)$ is given by

$$\begin{gathered}\{t = 0, -1 \leq x \leq 1, \tau > 0, \xi = 0\} \cup \{2t = -x^2 + 1, \xi = x\tau, \tau > 0\} \\ \cup\{x = -1, t = 0, 0 \leq -\xi \leq \tau, \tau > 0\} \cup \{x = 1, t = 0, 0 \leq \xi \leq \tau, \tau > 0\} \\ \cup \overline{Z} \times \{\xi = \tau = 0\}.\end{gathered}$$

It follows that, denoting by $(x; u = \xi/\tau)$ the coordinates in T^*M, $\rho(\mathrm{SS}(\mathbf{k}_Z) \cap (T^*M \times \dot{T}^*\mathbb{R}))$ is the set

$$\begin{gathered}\{u = 0, -1 \leq x \leq 1\} \cup \{u = x, -1 \leq x \leq 1\} \\ \cup\{x = -1, -1 \leq u \leq 0\} \cup \{x = 1, 0 \leq u \leq 1\}.\end{gathered}$$

(ii) Let $a \in \mathbb{R}$ and consider the set $Z = \{(x, t) \in M \times \mathbb{R}; t \geq ax\}$. Then $\rho(\mathrm{SS}(\mathbf{k}_Z))$ in T^*M is the set $\{(x; u); u = a\}$.

(iii) If G is a sheaf on M and $F = G \boxtimes \mathbf{k}_{s\geq 0}$, then $\rho(\mathrm{SS}(F)) = \mathrm{SS}(G)$.

The Separation Theorem

Using Lemma 5.1 we get the following particular case of Theorem 4.28:

Theorem 5.3 (See [12, Th. 3.2]). *Let A and B be two compact subsets of T^*M and assume that $A \cap B = \emptyset$. Then, for any $F \in \mathsf{D}^{\mathrm{b}}_A(\mathbf{k}^\gamma_M)$ and $G \in \mathsf{D}^{\mathrm{b}}_B(\mathbf{k}^\gamma_M)$, we have* $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}^\gamma_M)}(F, G) \simeq 0$.

6 Localization by Torsion Objects

In [12], Tamarkin introduces the notion of torsion objects, but does not study the category of such objects systematically. Hence, most of the results of this section are new.

In this section we set for short $Z = (T^*M) \times \mathbb{R} \times \{\tau \geq 0\}$, a closed subset of $T^*(M \times \mathbb{R})$. Recall that $\mathsf{D}^{\mathrm{b}}_Z(\mathbf{k}_{M\times\mathbb{R}})$ is the subcategory of $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times\mathbb{R}})$ such that $\mathrm{SS}(F) \subset Z$. By Proposition 4.9 we have $F \in \mathsf{D}^{\mathrm{b}}_Z(\mathbf{k}_{M\times\mathbb{R}})$ if and only if the morphism (43) is an isomorphism, which reads

$$F \star_{np} \mathbf{k}_{M\times[0,+\infty[} \xrightarrow{\sim} F. \tag{76}$$

Define the map

$$T_c \colon M \times \mathbb{R} \to M \times \mathbb{R}, \quad (x, t) \mapsto (x, t + c).$$

For $F \in \mathsf{D}^{\mathrm{b}}_Z(\mathbf{k}_{M\times\mathbb{R}})$ we deduce easily from (76)

$$F \star_{np} \mathbf{k}_{M\times[c,+\infty[} \xrightarrow{\sim} T_{c*}F. \tag{77}$$

The inclusions $[d, +\infty[\subset [c, +\infty[$, for $c \leq d$, induce natural morphisms of functors from $\mathsf{D}^{\mathrm{b}}_Z(\mathbf{k}_{M\times\mathbb{R}})$ to itself

$$\tau_{c,d} \colon T_{c*} \to T_{d*}, \quad c \leq d.$$

We have the identities:

$$T_{c*} \circ T_{d*} \simeq T_{(c+d)*}, \quad c, d \in \mathbb{R}, \tag{78}$$

$$T_{e*}(\tau_{c,d}(\bullet)) = \tau_{e+c,e+d}(\bullet) = \tau_{c,d}(T_{e*}(\bullet)), \quad c \leq d, e \in \mathbb{R}, \tag{79}$$

$$\tau_{c,d} \circ \tau_{d,e} = \tau_{c,e}, \quad c \leq d \leq e. \tag{80}$$

Definition 6.1 (Tamarkin). An object $F \in \mathsf{D}^{\mathrm{b}}_Z(\mathbf{k}_{M\times\mathbb{R}})$ is called a torsion object if $\tau_{0,c}(F) = 0$ for some $c \geq 0$ (and hence all $c' \geq c$).

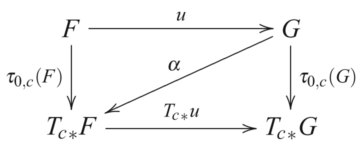
Let $F \in \mathsf{D}^{\mathrm{b}}_{Z}(\mathbf{k}_{M\times\mathbb{R}})$ and assume that F is supported by $M \times [a,b]$ for some compact interval $[a,b]$ of $\mathbb{R}$. Then F is a torsion object.

Remark 6.2. One can give an alternative definition of the torsion objects by using the classical notion of ind-objects (see [6] for an exposition). An object $F \in \mathsf{D}^{\mathrm{b}}_{Z}(\mathbf{k}_{M\times\mathbb{R}})$ is torsion if and only if the natural morphism $F \to$ "$\varinjlim_{c}$" $T_{c*}F$ is the zero morphism.

We let $\mathcal{N}_{\mathrm{tor}}$ be the full subcategory of $\mathsf{D}^{\mathrm{b}}_{Z}(\mathbf{k}_{M\times\mathbb{R}})$ consisting of torsion objects.

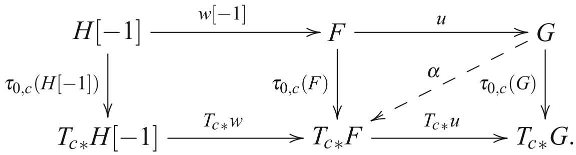
Lemma 6.3. *Let* $F \xrightarrow{u} G \xrightarrow{v} H \xrightarrow{w} F[1]$ *be a distinguished triangle in* $\mathsf{D}^{\mathrm{b}}_{Z}(\mathbf{k}_{M\times\mathbb{R}})$.

(i) *If H belongs to $\mathcal{N}_{\mathrm{tor}}$, then there exist $c \geq 0$ and $\alpha \colon G \to T_{c*}F$ such that $\tau_{0,c}(F) = \alpha \circ u$.*
(ii) *If there exist $c \geq 0$ and $\alpha \colon G \to T_{c*}F$ making the diagram*



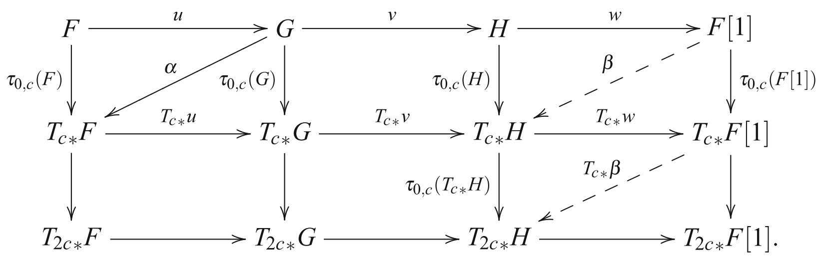
commutative, then $H \in \mathcal{N}_{\mathrm{tor}}$.

Proof. (i) Choose $c \geq 0$ such that $\tau_{0,c}(H) \simeq 0$ and consider the diagram with solid arrows



Since $\tau_{0,c}(H[-1]) \simeq 0$, we have $\tau_{0,c}(F) \circ w[-1] = 0$. Since $\mathrm{Hom}\,(\bullet, T_{c*}F)$ is a cohomological functor we deduce the existence of α.

(ii) We apply T_{c*} twice and obtain morphisms of distinguished triangles:



By hypothesis $\tau_{0,c}(H) \circ v = T_{c*}v \circ T_{c*}u \circ \alpha = 0$. As above, we deduce the existence of β such that $\tau_{0,c}(H) = \beta \circ w$. Applying the morphism of functors $\tau_{0,c} \colon \mathrm{id} \to T_c$ to β we find

$$\tau_{0,c}(T_{c*}H) \circ \beta = T_{c*}\beta \circ \tau_{0,c}(F[1]).$$

We deduce:

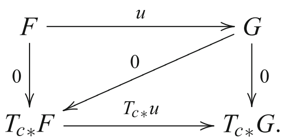
$$\begin{aligned}\tau_{0,c}(T_{c*}H) \circ \tau_{0,c}(H) = \tau_{0,c}(T_{c*}H) \circ \beta \circ w = T_{c*}\beta \circ \tau_{0,c}(F[1]) \circ w \\ = T_{c*}\beta \circ \alpha[1] \circ u[1] \circ w = 0.\end{aligned}$$

Using (78) we obtain $\tau_{0,2c}(H) \simeq 0$ so that $H \in \mathcal{N}_{\mathrm{tor}}$.

□

Theorem 6.4. *The subcategory $\mathcal{N}_{\mathrm{tor}}$ is a strictly full triangulated subcategory of* $\mathsf{D}^{\mathrm{b}}_Z(\mathbf{k}_{M\times\mathbb{R}})$.

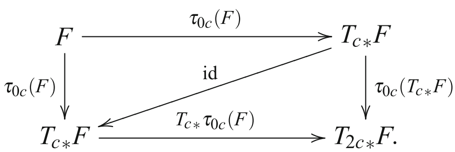
Proof. It is clear that an object isomorphic to a torsion object is itself a torsion object and that $\mathcal{N}_{\mathrm{tor}}$ is stable by the shift functor. Hence it remains to check that if $F \to G \to H \xrightarrow{+1}$ is a distinguished triangle with $F, G \in \mathcal{N}_{\mathrm{tor}}$ then $H \in \mathcal{N}_{\mathrm{tor}}$. We choose $c \geq 0$ such that $\tau_{0,c}(F) = 0$ and $\tau_{0,c}(G) = 0$ and we apply Lemma 6.3(ii) to the diagram



□

Corollary 6.5. *For any $F \in \mathsf{D}^{\mathrm{b}}_Z(\mathbf{k}_{M\times\mathbb{R}})$ and any $c \geq 0$, the cone of $\tau_{0,c}(F)$ is a torsion object.*

Proof. We apply Lemma 6.3(ii) to the commutative diagram



□

The subcategory $\mathsf{D}^{\mathrm{b}}(\mathbf{k}^{\gamma,l}_M)$ of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times\mathbb{R}})$ is contained in $\mathsf{D}^{\mathrm{b}}_Z(\mathbf{k}_{M\times\mathbb{R}})$. So we can define torsion objects in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}^{\gamma,l}_M)$ or in the equivalent category $\mathsf{D}^{\mathrm{b}}(\mathbf{k}^{\gamma}_M)$. We let $\mathcal{N}^{\gamma}_{\mathrm{tor}}$

be the subcategory of torsion objects in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M^\gamma)$. Then Theorem 6.4 implies that $\mathscr{N}_{\mathrm{tor}}^\gamma$ is a strictly full triangulated subcategory.

Definition 6.6. The triangulated category $\mathscr{T}(\mathbf{k}_M)$ is the localization of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M^\gamma)$ by $\mathscr{N}_{\mathrm{tor}}^\gamma$. In other words, $\mathscr{T}(\mathbf{k}_M) = \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M^\gamma)/\mathscr{N}_{\mathrm{tor}}^\gamma$.

By Corollary 6.5, $\tau_{0,c}(G)$ becomes invertible in $\mathscr{T}(\mathbf{k}_M)$ for any $G \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M^\gamma)$. Hence for a morphism $u\colon F \to G$ in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M^\gamma)$ and for $c \geq 0$ we can define $\tau_{0,c}(G)^{-1} \circ u\colon F \to G$ in $\mathscr{T}(\mathbf{k}_M)$. The family of $\tau_{c,c'}(G)$'s defines an inductive system $\{T_{c*}G\}_c$ and we have $\tau_{0,c'}(G)^{-1} \circ \tau_{c,c'}(G) \circ u = \tau_{0,c}(G)^{-1} \circ u$ for $c' \geq c$. This defines a natural morphism:

$$\varinjlim_{c \to +\infty} \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M^\gamma)}(F, T_{c*}G) \to \mathrm{Hom}_{\mathscr{T}(\mathbf{k}_M)}(F, G). \tag{81}$$

Proposition 6.7. *For any $F, G \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M^\gamma)$ the morphism (81) is an isomorphism.*

Proof. (i) Let us first show that (81) is surjective. A morphism $u\colon F \to G$ in $\mathscr{T}(\mathbf{k}_M)$ is given by $F \xrightarrow{v} G' \xleftarrow{s} G$, where the cone of s is a torsion object. By Lemma 6.3(i) there exist $c \geq 0$ and $\alpha\colon G' \to T_{c*}G$ such that $\tau_{0,c}(G) = \alpha \circ s$:

$$\begin{array}{ccccc} F & \xrightarrow{v} & G' & \xleftarrow{s} & G \\ & & & \searrow^{\alpha} & \downarrow{\scriptstyle \tau_{0,c}(G)} \\ & & & & T_{c*}G. \end{array}$$

Hence we obtain $u = \tau_{0,c}(G)^{-1} \circ \alpha \circ v$ in $\mathscr{T}(\mathbf{k}_M)$. In other words u is the image of $\alpha \circ v$ by (81).

(ii) Now we show that (81) is injective. We consider $u\colon F \to T_{c*}G$ in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M^\gamma)$ such that $\tau_{0c}(G)^{-1} \circ u = 0$ in $\mathscr{T}(\mathbf{k}_M)$. Then $u = 0$ in $\mathscr{T}(\mathbf{k}_M)$ and this means that there exists $s\colon T_{c*}G \to G'$ such that the cone of s is a torsion object and $s \circ u = 0$ in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M^\gamma)$. By Lemma 6.3(i) there exist $d \geq 0$ and $\alpha\colon G' \to T_{(c+d)_*}G$ such that $\tau_{c,c+d}(G) = \alpha \circ s$:

$$\begin{array}{ccccc} F & \xrightarrow{u} & T_{c*}G & \xrightarrow{s} & G' \\ & & {\scriptstyle \tau_{c,c+d}(G)}\downarrow & \swarrow_{\alpha} & \\ & & T_{(c+d)_*}G. & & \end{array}$$

We obtain $\tau_{c,c+d}(G) \circ u = \alpha \circ s \circ u = 0$ which means that the image of u in the left hand side of (81) is zero, as required.

□

Recall the functor Ψ_γ in (69).

Corollary 6.8. *The composition* $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M) \xrightarrow{\Psi_\gamma} \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times\mathbb{R}}; U_\gamma) \to \mathscr{T}(\mathbf{k}_M)$ *is a fully faithful functor.*

Proof. For $F, G \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$, the proof of Proposition 4.24 gives as well

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)}(G, F) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times\mathbb{R}})}(G \boxtimes \mathbf{k}_{[0,+\infty[}, F \boxtimes \mathbf{k}_{[c,+\infty[})$$

for any $c \geq 0$. Then the result follows from Proposition 6.7. □

Strict Cones and Torsion

For a connected manifold M and $F \in \mathsf{D}^{\mathrm{b}}_Z(\mathbf{k}_{M\times\mathbb{R}})$ we give a condition on $\mathrm{SS}(F)$ which implies that F is torsion over any compact subset as soon as it is torsion at one point.

We first give a preliminary result on $M \times I \times \mathbb{R}$. We set $E = \mathbb{R}^2$ and we take coordinates $(s, t; \sigma, \tau)$ on T^*E. We fix $\alpha > 0$ and define the cone $\gamma_\alpha = \{(s,t); t \geq \alpha|s|\}$ in E. We set $U_\alpha = E \times \mathrm{Int}\gamma_\alpha^\circ$. We recall Proposition 4.9, reformulated using (44): for $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times E})$, we have $\mathrm{SS}(F) \subset T^*M \times \overline{U}_\alpha$ if and only if

$$F \star_{np} \mathbf{k}_{M\times\gamma_\alpha} \simeq \mathrm{R}s_{E*}\mathrm{R}\Gamma_{M\times E\times\mathrm{Int}\gamma_\alpha}(q_1^{-1}F) \xrightarrow{\sim} F, \tag{82}$$

where $s_E \colon M \times E \times E \to M \times E$ is the sum of E.

Proposition 6.9. *Let I be an interval of $\mathbb{R}$, M a manifold and $q\colon M \times I \times \mathbb{R} \to M \times \mathbb{R}$ the projection. Set $\gamma = I \times [0, +\infty[$. Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times I\times\mathbb{R}})$. We assume that there exists a closed strict γ-cone $A \subset (T^*I) \times \mathbb{R}$ such that $\mathrm{SS}(F) \subset T^*M \times \hat{\pi}^{-1}(A)$. Then, for any $s_1 < s_2 \in I$, $\mathrm{R}q_*(F \otimes \mathbf{k}_{M\times[s_1,s_2[\times\mathbb{R}})$ and $\mathrm{R}q_*(F \otimes \mathbf{k}_{M\times]s_1,s_2]\times\mathbb{R}})$ are torsion objects of $\mathsf{D}^{\mathrm{b}}_Z(\mathbf{k}_{M\times\mathbb{R}})$.*

Proof. (i) We only consider $G := \mathrm{R}q_*(F \otimes \mathbf{k}_{M\times[s_1,s_2[\times\mathbb{R}})$, the other case being similar. We may restrict ourselves to a relatively compact subinterval of I containing s_1 and s_2. Hence we may assume that $\mathrm{SS}(F)$ is contained in $T^*M \times \{\tau \geq a|\sigma|\}$ for some $a > 0$. Then, applying Lemma 4.26 and changing a if necessary, we may assume that $I = \mathbb{R}$.

(ii) We set $\alpha = a^{-1}$ so that $\gamma_\alpha^\circ = \{\tau \geq a|\sigma|\}$ and $\mathrm{SS}(F) \subset T^*M \times \overline{U}_\alpha$. Since $\mathrm{SS}(\mathbf{k}_{M\times[s_1,s_2[\times\mathbb{R}}) \subset T^*_M M \times T^*\mathbb{R} \times T^*_{\mathbb{R}}\mathbb{R}$, Corollary 2.7 gives $F \otimes \mathbf{k}_{M\times[s_1,s_2[\times\mathbb{R}} \simeq \mathrm{R}\Gamma_{M\times]s_1,s_2]\times\mathbb{R}}(F)$ and the formula (82) gives

$$G \simeq \mathrm{R}q_*\mathrm{R}s_{E*}\mathrm{R}\Gamma_{M\times D}(q_1^{-1}F),$$

where $D = (E \times \mathrm{Int}\gamma_\alpha) \cap \{(s,t,s',t'); s_1 < s + s' \leq s_2\}$. We consider the commutative diagram

$$\begin{array}{ccccc} & & M \times E \times E & \xrightarrow{s_E} & M \times E \\ & \swarrow^{q_1} & \downarrow{\scriptstyle \mathrm{id}_M \times \tilde{q}} & & \downarrow{\scriptstyle q} \\ M \times E & \xleftarrow{\tilde{q}_1} & M \times E \times \mathbb{R} & \xrightarrow{\tilde{s}} & M \times \mathbb{R}, \end{array}$$

where $\tilde{q}(s,t,s',t') = (s,t,t')$, $\tilde{q}_1(x,s,t,t') = (x,s,t)$ and $\tilde{s}(x,s,t,t') = (x, t+t')$. The adjunction between $\mathrm{R}(\mathrm{id}_M \times \tilde{q})_!$ and $(\mathrm{id}_M \times \tilde{q})^!$ gives

$$\begin{aligned} G &\simeq \mathrm{R}\tilde{s}_* \mathrm{R}(\mathrm{id}_M \times \tilde{q})_* \mathrm{R}\mathscr{H}om\,(\mathbf{k}_{M\times D}, (\mathrm{id}_M \times \tilde{q})^! \tilde{q}_1^{-1} F)[-1] \\ &\simeq \mathrm{R}\tilde{s}_* \mathrm{R}\mathscr{H}om\,(\mathbf{k}_M \boxtimes \mathrm{R}\tilde{q}_! \mathbf{k}_D, \tilde{q}_1^{-1} F)[-1]. \end{aligned} \tag{83}$$

(iii) Through the isomorphism (82) the morphism $\tau_c(F)$ is induced by the morphism $\mathbf{k}_{T_c(E\times \mathrm{Int}\gamma_\alpha)} \to \mathbf{k}_{E\times \mathrm{Int}\gamma_\alpha}$, where $T_c(s,t,s',t') = (s,t,s',t'+c)$. Using (83) it follows that $\tau_c(G)$ is induced by the morphism $u_c\colon \mathbf{k}_{T_c(D)} \to \mathbf{k}_D$. Hence it is enough to see that the image of u_c by $\mathrm{R}\tilde{q}_!$ is the zero morphism. In the remainder of the proof we show that $\mathrm{R}\tilde{q}_!\mathbf{k}_D$ and $\mathrm{R}\tilde{q}_!\mathbf{k}_{T_c(D)}$ have disjoint supports for c big enough.

(iv) For a given point $(s,t,t') \in E \times \mathbb{R}$ we have $\tilde{q}^{-1}(s,t,t') \cap D = \emptyset$ if $t' < 0$ and otherwise

$$\begin{aligned} \tilde{q}^{-1}(s,t,t') \cap D &= \{s';\ s_1 - s < s' \le s_2 - s,\ t' \ge \alpha|s'|\} \\ &=]s_1 - s, s_2 - s] \cap [-\alpha^{-1}t', \alpha^{-1}t']. \end{aligned}$$

This is $\emptyset$ or a half closed interval when t' is not in $I_s := [-\alpha(s_2 - s), -\alpha(s_1 - s)[$. It follows that $\mathrm{supp}(\mathrm{R}\tilde{q}_!\mathbf{k}_D)$ is contained in $D' := \{(s,t,t');\ t' \in \overline{I_s}\}$. The support of $\mathrm{R}\tilde{q}_!\mathbf{k}_{T_c(D)}$ is contained in $T'_c(D')$, with $T'_c(s,t,t') = (s,t,c+t')$. Since I_s is of length $\alpha(s_2 - s_1)$ (independent of s) we obtain $D' \cap T'_c(D') = \emptyset$ for $c > \alpha(s_2 - s_1)$.

□

From now on, we consider a connected manifold M and $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times\mathbb{R}})$. We set $\gamma = M \times [0, +\infty[$ and we make the hypothesis

$$\mathrm{SS}(F) \subset \hat{\pi}^{-1}(A) \text{ for some closed } \gamma\text{-strict cone } A \subset (T^*M) \times \mathbb{R}. \tag{84}$$

In particular $F \in \mathsf{D}^{\mathrm{b}}_{\{\tau\ge 0\}}(\mathbf{k}_{M\times\mathbb{R}})$.

Lemma 6.10. *Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times\mathbb{R}})$ satisfying (84). We assume that there exists $x \in M$ such that $F|_{\{x\}\times\mathbb{R}}$ is a torsion object in $\mathsf{D}^{\mathrm{b}}_{\{\tau\ge 0\}}(\mathbf{k}_{\mathbb{R}})$. Then there exists a neighborhood U of x such that $F|_{U\times\mathbb{R}}$ is a torsion object in $\mathsf{D}^{\mathrm{b}}_{\{\tau\ge 0\}}(\mathbf{k}_{U\times\mathbb{R}})$.*

Proof. (i) We may assume that M is an open set in some vector space V and $x = 0$. We take coordinates $(x,t;\xi,\tau)$ on $T^*(M\times\mathbb{R})$. We may also assume that $\mathrm{SS}(F) \subset \{\tau \ge a||\xi||\}$ for some $a > 0$ and that M contains the open ball of radius 1, say B. We set $I =]-1,1[$ and take coordinates $(s;\sigma)$ on T^*I. We define the homotopy $h\colon B\times I\times\mathbb{R}\to B\times\mathbb{R}$, $(x,s,t)\mapsto(sx,t)$. For $s_0\in I$ we set $h_{s_0}=h(\cdot,s_0,\cdot)$.

(ii) We check that $h^{-1}(F|_{B\times\mathbb{R}})$ satisfies the hypothesis of Proposition 6.9. We have $h_\pi(x,s,t;\xi,\tau)=(sx;t\xi,\tau)$ and $h_d(x,s,t;\xi,\tau)=(x,s,t;s\xi,\langle x,\xi\rangle,\tau)$. Hence $\operatorname{Ker} h_d$ is contained in $\{\tau=0\}$. Since $\mathrm{SS}(F)\cap\{\tau=0\}$ is contained in the zero-section, F is non-characteristic for h and we find

$$\mathrm{SS}(h^{-1}(F))\subset\{(x',s',t';\xi',\sigma',\tau');\ \sigma'=\langle x',\xi'\rangle,\ \tau'\geq a||\xi'||/|s'|\}.$$

On $B\times I$ we have $|s'|\leq 1$ and $|\langle x',\xi'\rangle|\leq||\xi'||$. We deduce $\mathrm{SS}(h^{-1}(F))\subset\{\tau'\geq a|\sigma'|\}$ on $B\times I\times\mathbb{R}$, as required.

(iii) We apply Proposition 6.9 to $h^{-1}(F)$ on $B\times I\times\mathbb{R}$ with $s_1=0$, $s_2=1/2$. For $J\subset I$ we set $G_J=\mathrm{R}q_*(h^{-1}(F|_{B\times\mathbb{R}})\otimes\mathbf{k}_{M\times J\times\mathbb{R}})$. We note that $G_{\{s\}}\simeq h_s^{-1}(F|_{B\times\mathbb{R}})$ for any $s\in I$. We have the distinguished triangles on $B\times\mathbb{R}$

$$G_{]0,1/2]}\to G_{[0,1/2]}\to G_{\{0\}}\xrightarrow{+1},\qquad G_{[0,1/2[}\to G_{[0,1/2]}\to G_{\{1/2\}}\xrightarrow{+1},$$

where $G_{]0,1/2]}$ and $G_{[0,1/2[}$ are torsion by Proposition 6.9. Since h_0 is the contraction $B\times\mathbb{R}\to\{0\}\times\mathbb{R}$ the hypothesis implies that $G_{\{0\}}$ is torsion. Hence $G_{[0,1/2]}$ is torsion by the first distinguished triangle and then $G_{\{1/2\}}$ also is torsion by the second one. Since $h_{1/2}$ is a diffeomorphism from $B\times\mathbb{R}$ to $U\times\mathbb{R}$, where U is the ball of radius $1/2$ we deduce that $F|_{U\times\mathbb{R}}$ is torsion.

□

Lemma 6.11. *Let $F\in\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times\mathbb{R}})$ satisfying (84). We assume that there exists $x_0\in M$ such that $F|_{\{x_0\}\times\mathbb{R}}$ is a torsion object in $\mathsf{D}^{\mathrm{b}}_{\{\tau\geq0\}}(\mathbf{k}_{\mathbb{R}})$. Then $F|_{\{x\}\times\mathbb{R}}$ also is a torsion object in $\mathsf{D}^{\mathrm{b}}_{\{\tau\geq0\}}(\mathbf{k}_{\mathbb{R}})$ for all $x\in M$.*

Proof. We set $I=]-1,1[$ and we choose an immersion $i\colon I\to M$ such that $i(0)=x_0$ and $i(1/2)=x$. Then $i^{-1}F$ satisfies the hypothesis of Proposition 6.9 on $I\times\mathbb{R}$. We let $q\colon I\times\mathbb{R}\to\mathbb{R}$ be the projection. Then $F|_{\{i(s)\}\times\mathbb{R}}\simeq\mathrm{R}q_*(i^{-1}F\otimes\mathbf{k}_{\{s\}\times\mathbb{R}})$ for any $s\in I$. Now we have the distinguished triangles

$$\mathrm{R}q_*(i^{-1}F\otimes\mathbf{k}_{]0,1/2]\times\mathbb{R}})\to\mathrm{R}q_*(i^{-1}F\otimes\mathbf{k}_{[0,1/2]\times\mathbb{R}})\to i^{-1}F|_{\{x_0\}\times\mathbb{R}}\xrightarrow{+1},$$
$$\mathrm{R}q_*(i^{-1}F\otimes\mathbf{k}_{[0,1/2[\times\mathbb{R}})\to\mathrm{R}q_*(i^{-1}F\otimes\mathbf{k}_{[0,1/2]\times\mathbb{R}})\to i^{-1}F|_{\{x\}\times\mathbb{R}}\xrightarrow{+1}$$

and we conclude as in part (iii) of the proof of Lemma 6.10. □

Theorem 6.12. *Let M be a connected manifold and let $F\in\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times\mathbb{R}})$ satisfying (84). Then the following assertions are equivalent:*

(i) *there exists $x_0\in M$ such that $F|_{\{x_0\}\times\mathbb{R}}$ is a torsion object in $\mathsf{D}^{\mathrm{b}}_{\{\tau\geq0\}}(\mathbf{k}_{\mathbb{R}})$,*

(ii) *for any relatively compact open subset $U\subset M$ the restriction $F|_{U\times\mathbb{R}}$ is a torsion object in $\mathsf{D}^{\mathrm{b}}_{\{\tau\geq0\}}(\mathbf{k}_{U\times\mathbb{R}})$.*

Proof. We only need to prove that (i) implies (ii). By Lemmas 6.10 and 6.11 we can find a finite cover of $\overline{U}$, say $\{U_i\}$, $i = 1, \dots, n$, such that $F|_{U_i\times\mathbb{R}}$ is torsion. We conclude with the remark that, for any two open subsets $V, W \subset M$, if $F|_{V\times\mathbb{R}}$ and $F|_{W\times\mathbb{R}}$ are torsion, then so is $F|_{(V\cup W)\times\mathbb{R}}$. Indeed we apply Lemma 6.3 to the triangle $F_{(V\cap W)\times\mathbb{R}} \to F_{V\times\mathbb{R}} \oplus F_{W\times\mathbb{R}} \to F_{(V\cup W)\times\mathbb{R}} \xrightarrow{+1}$ and the commutative square

$$\begin{array}{ccc} F_{(V\cap W)\times\mathbb{R}} & \longrightarrow & F_{V\times\mathbb{R}} \oplus F_{W\times\mathbb{R}} \\ {\scriptstyle \tau_{0,c}=0}\downarrow & \overset{0}{\swarrow} & \downarrow{\scriptstyle \tau_{0,c}=0} \\ T_{c*}(F_{(V\cap W)\times\mathbb{R}}) & \longrightarrow & T_{c*}(F_{V\times\mathbb{R}} \oplus F_{W\times\mathbb{R}}). \end{array}$$

□

7 Tamarkin's Non Displaceability Theorem

We will explain here Tamarkin's non displaceability theorem which gives a criterion in order that two compact subsets of T^*M are non displaceable.

In this section we consider a Hamiltonian isotopy $\Phi\colon T^*M \times I \to T^*M$ satisfying (29), that is, there exists a compact set $C \subset T^*M$ such that $\varphi_s|_{T^*M\setminus C}$ is the identity for all $s \in I$.

Let $\tilde{\Phi}\colon \dot{T}^*(M \times \mathbb{R}) \times I \to \dot{T}^*(M \times \mathbb{R})$ be the homogeneous Hamiltonian isotopy given by Proposition 3.2 and $\tilde{\Lambda} \subset T^*(M \times \mathbb{R} \times M \times \mathbb{R} \times I)$ the conic Lagrangian submanifold associated to $\tilde{\Phi}$ in (26). Let $\tilde{K} \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M\times\mathbb{R}\times M\times\mathbb{R}\times I})$ be the quantization of $\tilde{\Phi}$ given in Theorem 3.1.

Invariance by Hamiltonian Isotopy

For $J \subset I$ a relatively compact subinterval of I, we introduce the kernel

$$K^J = \mathrm{R}q_{1234!}(\tilde{K} \otimes \mathbf{k}_{M\times\mathbb{R}\times M\times\mathbb{R}\times J}) \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times\mathbb{R}\times M\times\mathbb{R}}),$$

where q_{1234} is the projection on the first four factors. We remark that $\tilde{K}$ and K^J satisfy the hypothesis (74). Hence, by Proposition 4.29, composition with K^J defines a functor

$$\Psi_J\colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}^{\gamma}_M) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}^{\gamma}_M), \qquad F \mapsto K^J \circ F. \tag{85}$$

We note that $K^{\{s\}} \simeq \tilde{K}|_{M\times\mathbb{R}\times M\times\mathbb{R}\times\{s\}}$. We set for short $\Psi_s = \Psi_{\{s\}}$. We have $\Psi_0 \simeq \mathrm{id}$.

Theorem 7.1. *Let $\Phi\colon T^*M \times I \to T^*M$ be a Hamiltonian isotopy satisfying* (29). *For $s \in I$ and $J \subset I$ a relatively compact subinterval let $\Psi_J, \Psi_s\colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M^{\gamma}) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M^{\gamma})$ be the functors defined in* (85). *Then for A a closed subset of T^*M and $F \in \mathsf{D}^{\mathrm{b}}_A(\mathbf{k}_M^{\gamma})$ we have*

(i) *$\Psi_s(F) \in \mathsf{D}^{\mathrm{b}}_{\varphi_s(A)}(\mathbf{k}_M^{\gamma})$ for any $s \in I$,*
(ii) *$\Psi_{[a,b[}(F)$ and $\Psi_{]a,b]}(F)$ are torsion objects for any $a < b \in I$,*
(iii) *for $s \in I$, $s \geq 0$, there exist distinguished triangles*

$$\Psi_{]0,s]}(F) \to \Psi_{[0,s]}(F) \to F \xrightarrow{+1}, \quad \Psi_{[0,s[}(F) \to \Psi_{[0,s]}(F) \to \Psi_s(F) \xrightarrow{+1}$$

and similar ones for $s \leq 0$. In particular we have a natural isomorphism $F \simeq \Psi_s(F)$ in $\mathscr{T}(\mathbf{k}_M)$ for any $s \in I$.

Proof. (i) We set $\tilde{\Lambda}_s = \tilde{\Lambda} \circ T_s^* I$. This is the graph of $\tilde{\varphi}_s$. Hence

$$\mathrm{SS}(\Psi_s(F)) \cap \{\tau > 0\} \subset \tilde{\Lambda}_s \circ \rho^{-1}(A) = \tilde{\varphi}_s(\rho^{-1}(A)) = \rho^{-1}(\varphi_s(A)),$$

which proves the first statement.

(ii)–(iii) (a) We set $\tilde{F} = \tilde{K} \circ F$ which belongs to $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M\times I}^{\gamma})$ by Proposition 4.29. We have $\mathrm{SS}(\tilde{F}) \cap \{\tau > 0\} \subset \tilde{\Lambda} \circ \rho^{-1}(A)$. As in Lemma 5.1 we define $A' \subset T^*M \times \mathbb{R}$ by $A' = \{(x;\xi,\tau); \tau > 0,\ (x;\xi/\tau) \in A\}$. Then $\overline{A'}$ is a strict γ-cone. It follows that there exists a closed strict γ-cone $B \subset T^*(M \times I) \times \mathbb{R}$ such that $\tilde{\Lambda} \circ \rho^{-1}(A) \subset \hat{\pi}^{-1}(B) \cap \{\tau > 0\}$. Then Lemma 5.1 gives $\mathrm{SS}(\tilde{F}) \subset \hat{\pi}^{-1}(B) \cup T^*_{M\times I\times\mathbb{R}}(M \times I \times \mathbb{R})$. In particular $\tilde{F}|_{M\times J\times\mathbb{R}}$ satisfies the hypothesis of Proposition 6.9 for any relatively compact subinterval $J \subset I$.

(b) We let $q\colon M \times I \times \mathbb{R} \to M \times \mathbb{R}$ be the projection. For a relatively compact subinterval $J \subset I$ we have $\Psi_J(F) \simeq \mathrm{R}q_*(\tilde{F} \otimes \mathbf{k}_{M\times J\times\mathbb{R}})$. Then (ii) follows from Proposition 6.9. The triangles in (iii) are induced by the excision triangles associated with the inclusions $\{0\} \subset [0,s]$ and $\{s\} \subset [0,s]$. Then (ii) gives $F \xleftarrow{\sim} \Psi_{[0,s]}(F) \xrightarrow{\sim} \Psi_s(F)$ in $\mathscr{T}(\mathbf{k}_M)$. □

Application to Non Displaceability

Recall that two compact subsets A and B of T^*M are called mutually non displaceable if, for any Hamiltonian isotopy $\Phi\colon T^*M \times I \to T^*M$ satisfying (29) and any $s \in I$, $A \cap \varphi_s(B) \neq \emptyset$. A compact subset A is called non displaceable if A and A are mutually non displaceable. Let A and B be two compact subsets of T^*M, let $F \in \mathsf{D}^{\mathrm{b}}_A(\mathbf{k}_M^{\gamma,l})$ and $G \in \mathsf{D}^{\mathrm{b}}_B(\mathbf{k}_M^{\gamma,l})$. Let $q_2\colon M \times \mathbb{R} \to \mathbb{R}$ be the projection. Recall that $\mathscr{H}om^*(F,G) \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M^{\gamma,r})$ by (62). We deduce by adjunction that $\mathrm{R}q_{2*}\mathscr{H}om^*(F,G) \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}^{\gamma,r})$. We shall consider the following hypothesis:

$$\mathrm{R}q_{2*}\mathscr{H}om^*(F,G) \text{ is not torsion.} \tag{86}$$

Theorem 7.2 (The Non Displaceability Theorem of [12, Th. 3.1]). *Let A and B be two compact subsets of T^*M. Assume that there exist $F \in \mathsf{D}^{\mathrm{b}}_A(\mathbf{k}_M^{\gamma,l})$ and $G \in \mathsf{D}^{\mathrm{b}}_B(\mathbf{k}_M^{\gamma,l})$ satisfying the hypothesis (86). Then A and B are mutually non displaceable in T^*M.*

Proof. Assume Φ is a Hamiltonian isotopy such that $\varphi_{s_0}(B) \cap A = \emptyset$. We consider $\tilde{\Phi}\colon \dot{T}^*(M\times\mathbb{R})\times I \to \dot{T}^*(M\times\mathbb{R})$ and $\tilde{K} \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M\times\mathbb{R}\times M\times\mathbb{R}\times I})$ as in the introduction of this section.

We define $F', G' \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}^{\gamma,l}_{M\times I})$ by $F' = F \boxtimes \mathbf{k}_I$ and $G' = \tilde{K} \circ G$. We let $q_{23}\colon M \times \mathbb{R} \times I \to \mathbb{R} \times I$ be the projection. We have $F \simeq F'|_{M\times\mathbb{R}\times\{s\}}$ and we set $G_s = G'|_{M\times\mathbb{R}\times\{s\}}$. By Lemma 5.1 and Corollary 4.15, we have $\mathscr{H}om^*(F', G')|_{M\times\mathbb{R}\times\{s\}} \simeq \mathscr{H}om^*(F, G_s)$. By Lemma 5.1 q_{23} is proper on the support of $\mathscr{H}om^*(F', G')$ and we get

$$(\mathrm{R}q_{23*}\mathscr{H}om^*(F', G'))|_{M\times\mathbb{R}\times\{s\}} \simeq \mathrm{R}q_{2*}\mathscr{H}om^*(F, G_s).$$

Since $\mathrm{SS}(G_s) \subset \rho^{-1}(\varphi_s(B))$, Theorem 5.3 implies $\mathrm{R}q_{2*}\mathscr{H}om^*(F, G_{s_0}) = 0$.

By Proposition 4.13 and Lemma 4.7, the microsupport of $\mathscr{H}om^*(F', G')$ is contained in $\hat{\pi}^{-1}(C)$ for some strict γ-cone C. Hence a similar inclusion holds for the microsupport of $\mathrm{R}q_{23*}\mathscr{H}om^*(F', G')$. Then Theorem 6.12 implies that $\mathscr{H}om^*(F, G_s)$ is torsion for all $s \in I$. In particular $\mathscr{H}om^*(F, G)$ is torsion, which contradicts the hypothesis (86). ☐

Corollary 7.3. *Let A and B be two compact subsets of T^*M. Assume that there exist $F \in \mathsf{D}^{\mathrm{b}}_A(\mathbf{k}^{\gamma}_M)$ and $G \in \mathsf{D}^{\mathrm{b}}_B(\mathbf{k}^{\gamma}_M)$ such that* $\mathrm{Hom}_{\mathscr{T}(\mathbf{k}_M)}(F, G) \neq 0$. *Then A and B are mutually non displaceable in T^*M.*

Proof. By Proposition 6.7, there exists $c \in \mathbb{R}$ such that the morphism induced by $\tau_{c,d}(G)$, $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}^{\gamma}_M)}(F, T_{c*}G) \to \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}^{\gamma}_M)}(F, T_{d*}G)$ is non zero for all $d \geq c$. But Lemma 4.18 gives

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}^{\gamma}_M)}(F, T_{c*}G) \simeq H^0_{[0,+\infty[}(\mathbb{R}; \mathrm{R}q_{2*}\mathscr{H}om^*(F, T_{c*}G)).$$

On the other hand we can see that $\mathrm{R}q_{2*}\mathscr{H}om^*(F, T_{c*}G) \simeq T_{c*}\mathrm{R}q_{2*}\mathscr{H}om^*(F, G)$ and that $\tau_{c,d}(G)$ induces $\tau_{c,d}(\mathrm{R}q_{2*}\mathscr{H}om^*(F, G))$ through this isomorphism. Hence $\mathrm{R}q_{2*}\mathscr{H}om^*(F, G)$ is non torsion and we can apply Theorem 7.2. ☐

Let A be a closed conic subset of T^*M. We know by Corollary 6.8 that the functor

$$j_M\colon \mathsf{D}^{\mathrm{b}}_A(\mathbf{k}_M) \to \mathscr{T}(\mathbf{k}_M), \quad F \mapsto F \boxtimes \mathbf{k}_{[0,+\infty[} \tag{87}$$

is fully faithful. Applying Corollary 7.3 with $F = G = j_M(\mathbf{k}_M) \in \mathscr{T}(\mathbf{k}_M)$ and $A = B = T^*_M M$, we get

Corollary 7.4. *Assume M is compact. Then M is non displaceable in T^*M.*

In [12], Tamarkin applies the non displaceability Theorem 7.2 to prove that the following sets are non displaceable.

Set $X = \mathbb{P}(\mathbb{C})^n$ endowed with his standard real symplectic structure. Consider the sets $A := \mathbb{P}(\mathbb{R})^n$ and $B := \mathbb{T} = \{z = (z_0, \dots, z_n); |z_0| = \dots |z_n|\}$. Then A and B are non displaceable and A and B are mutually non displaceable.

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A-Polynomial, B-Model, and Quantization

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Abstract Exact solution to many problems in mathematical physics and quantum field theory often can be expressed in terms of an algebraic curve equipped with a meromorphic differential. Typically, the geometry of the curve can be seen most clearly in a suitable semi-classical limit, as $\hbar \to 0$, and becomes non-commutative or "quantum" away from this limit. For a classical curve defined by the zero locus of a polynomial $A(x, y)$, we provide a construction of its non-commutative counterpart $\hat{A}(\hat{x}, \hat{y})$ using the technique of the topological recursion. This leads to a powerful and systematic algorithm for computing $\hat{A}$ that, surprisingly, turns out to be much simpler than any of the existent methods. In particular, as a bonus feature of our approach comes a curious observation that, for all curves that "come from geometry," their non-commutative counterparts can be determined just from the first few steps of the topological recursion. We also propose a K-theory criterion for a curve to be "quantizable," and then apply our construction to many examples that come from applications to knots, strings, instantons, and random matrices. The material contained in this chapter was presented at the conference *Mirror Symmetry and Tropical Geometry* in Cetraro (July 2011) and is based on the work: Gukov and Sułkowski, "A-polynomial, B-model, and quantization", JHEP 1202 (2012) 070.

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1 Introduction

In recent years, it has been realized that a solution to a variety of different problems in theoretical and mathematical physics—matrix models, four-dimensional supersymmetric gauge theory, quantum invariants of knots and 3-manifolds, and topological strings—leads to what sometimes is referred to as the "quantization of an algebraic curve."

To be more precise, the classical phase space which is quantized in this problem is the two-dimensional complex plane parametrized by the coordinates u and v

$$(u, v) \in \mathbb{C} \times \mathbb{C}\,, \tag{1}$$

and equipped with the canonical holomorphic symplectic form

$$\omega = \frac{i}{\hbar} du \wedge dv\,. \tag{2}$$

In this space, a polynomial $A(u, v)$ defines an algebraic curve

$$\mathscr{C}: \quad A(u, v) = 0\,, \tag{3}$$

which is automatically Lagrangian with respect to the holomorphic symplectic form (2). A close cousin of this problem (that we consider in parallel) is obtained by taking A to be a polynomial in the $\mathbb{C}^*$-valued variables

$$x = e^u \qquad , \qquad y = e^v\,. \tag{4}$$

In either case, the problem is to quantize the classical phase space $\mathbb{C} \times \mathbb{C}$ (resp. $\mathbb{C}^* \times \mathbb{C}^*$) with the symplectic form (2) and a classical "state" defined by the zero locus of the polynomial A.

Classically, u and v have the Poisson bracket $\{v, u\} = \hbar$ that follows directly from (2). Quantization turns u and v into operators, $\hat{u}$ and $\hat{v}$, which satisfy the commutation relation

$$[\hat{v}, \hat{u}] = \hbar\,. \tag{5}$$

Therefore, quantization deforms the algebra of functions on the phase space into a non-commutative algebra of operators. In particular, it maps a polynomial function $A(u, v)$ (resp. $A(x, y)$) into an operator $\hat{A}$:

$$\hat{A} = \hat{A}_0 + \hbar \hat{A}_1 + \hbar^2 \hat{A}_2 + \ldots\,, \tag{6}$$

where $\hat{A}_0 \equiv A$. Since $\hat{u}$ and $\hat{v}$ (resp. $\hat{x}$ and $\hat{y}$) do not commute, there is no unique way to write the perturbative expansion (6). After all, changing the order of operators changes the powers of $\hbar$. In practice, however, one often makes a choice of polarization, i.e. a choice of what one regards as canonical coordinates and conjugate momenta. For example, in most of the present paper we make a simple choice consistent with (5):

$$\hat{u} = u \qquad , \qquad \hat{v} = \hbar \partial_u \equiv \hbar \frac{\partial}{\partial u}, \tag{7}$$

where u plays the role of a "coordinate" and v is the "momentum." With this or any other choice, one has a natural ordering of operators in (6), such that in every term momenta appear to the right of the coordinates. This leads to a "canonical" form of the perturbative expansion (6) that we will try to follow in the present paper.

Starting with the classical curve (3) defined by the zero locus of $A(u, v)$ or $A(x, y)$, our goal will be to construct the quantum operator $\hat{A}$, in particular, to study the structure of its perturbative expansion (6). A priori, it is not even clear if a solution to this problem exists and, if it does, whether it is unique. We will answer these questions in affirmative and describe a systematic method to produce "quantum corrections" $\hat{A}_k$, for $k \geq 1$, solely from the data of $A(u, v)$ (resp. $A(x, y)$) by drawing important lessons from applications where this problem naturally appears:

1. **SUSY gauge theory:** In $\mathcal{N} = 2$ supersymmetric gauge theory, the curve (3) is known as the Seiberg-Witten curve [48], and $\hbar$ is related to the Ω-deformation [44].
2. **Chern–Simons theory:** In Chern–Simons theory with a Wilson loop, the polynomial $A(x, y)$ is a topological invariant called the A-polynomial and plays a role similar to that of the Seiberg–Witten curve in $\mathcal{N} = 2$ gauge theory [32]. The parameter $\hbar$ is the coupling constant of Chern–Simons theory.
3. **Matrix models:** In matrix models, the curve (3) is called the spectral curve, and $\hbar = 1/N$ controls the expansion in (inverse) matrix size [12].
4. **Topological strings:** In topological string theory [2, 18], every curve of the form (3) defines a (non-compact) Calabi–Yau threefold geometry in which strings propagate, namely a hypersurface in $(\mathbb{C}^*)^2 \times \mathbb{C}^2$:

$$A(x, y) = zw\,. \tag{8}$$

   The parameter $\hbar$ is the string coupling constant.
5. **$\mathcal{D}$-modules:** There is also a mathematical theory of $\mathcal{D}$-modules [35, 36, 38], which studies modules over rings of differential operators, and in particular operators with properties analogous to those which we expect from $\hat{A}$. Some connections of this theory to the above mentioned physics systems were analyzed in [15, 17, 18].

In all these applications, the primary object of interest is the partition function, $Z(u)$, or, to be more precise, a collection of functions $Z^{(\alpha)}(u)$ labeled by a choice of root $v^{(\alpha)} = v^{(\alpha)}(u)$ to Eq. (3):

$$Z^{(\alpha)}(u) = Z(u, v^{(\alpha)}(u))\,. \tag{9}$$

The right-hand side of this expression is the partition function $Z(u, v)$, which is a globally defined function on the Riemann surface (3) and which does *not* depend on the choice of α. The existence of such a globally defined partition function is less obvious in some of the above mentioned applications compared to others. In our discussion below, we find it more convenient and often more illuminating to work with $Z(u, v)$ rather than with a collection of functions $Z^{(\alpha)}(u)$.

From the viewpoint of quantization, the partition function Z is simply the wave-function associated to a classical state (3). It obeys a Schrödinger-like equation

$$\hat{A} Z = 0\,, \tag{10}$$

and has a perturbative expansion of the form

$$Z = \exp\left(\frac{1}{\hbar} S_0 + \sum_{n=0}^{\infty} S_{n+1}\, \hbar^n\right). \tag{11}$$

The quantum operator $\hat{A}$ in (10) is precisely the operator obtained by a quantization of $A(u, v)$ or $A(x, y)$, and the Schrödinger-like equation (10) will be our link relating its perturbative expansion (6) to that of the partition function (11).

Indeed, recently a number of powerful methods have been developed that allow to compute perturbative terms S_n in the $\hbar$-expansion. In particular, insights from matrix models suggest that the perturbative expansion of the partition function (11) should be thought of as a large N expansion of the determinant expectation value in random matrix theory

$$Z = \Big\langle \det(u - M) \Big\rangle. \tag{12}$$

This expectation value is computed in some ensemble of matrices M of size $N = \hbar^{-1}$, with respect to the matrix measure $\mathscr{D}M\, e^{-\mathrm{Tr} V(M)/\hbar}$, where $V(M)$ is a potential of a matrix model. Then, by exploring the relation between perturbative expansions of $\hat{A}$ and Z, we argue that having a systematic procedure for computing one is essentially equivalent to having a similar procedure for the other. In particular, by shifting the focus to $\hat{A}$, we obtain the following universal formula for the first quantum correction $\hat{A}_1$:

$$\hat{A}_1 = \frac{1}{2}\left(\frac{\partial_u A}{\partial_v A}\partial_v^2 + \frac{\partial_u T}{T}\partial_v\right) A\,, \tag{13}$$

expressed in terms of the classical A-polynomial and the "torsion" $T(u)$ that determines the subleading term in the perturbative expansion (11) of the partition function:

$$S_1 = -\frac{1}{2}\log T(u)\,. \tag{14}$$

Usually, the torsion is relatively easy to compute, even without detailed knowledge of the higher-order quantum corrections to (6) or (11) which typically require more powerful techniques. For instance, in the examples coming from knot theory the torsion $T(u)$ is a close cousin of the "classical" knot invariant called the Alexander polynomial.

Furthermore, it is curious to note that, generically, for curves in $\mathbb{C}^* \times \mathbb{C}^*$ the leading quantum correction (13) completely determines the entire quantum operator $\hat{A}$ when all $\hbar$-corrections can be summed up to powers[1] of $q = e^{\hbar}$:

$$\hat{A} = \sum_{(m,n) \in \mathscr{D}} a_{m,n} \, q^{c_{m,n}} \, \hat{x}^m \, \hat{y}^n \,, \tag{15}$$

in other words, when $\hat{A}$ can be written as a (Laurent) polynomial in $\hat{x}$, $\hat{y}$, and q. Here, $\mathscr{D}$ is a two-dimensional lattice polytope; in many examples $\mathscr{D}$ is simply the Newton polygon of $A(x, y)$. Indeed, the coefficients $a_{m,n}$ are simply the coefficients of the classical polynomial, $A = \sum a_{m,n} x^m y^n$, which is obtained from (15) in the limit $q \to 1$. On the other hand, the exponents $c_{m,n}$ can be determined by requiring that (13) holds for all values of x and y (such that $A(x, y) = 0$):

$$\sum_{(m,n) \in \mathscr{D}} a_{m,n} \, c_{m,n} \, x^m y^n = \frac{1}{2} \left(\frac{\partial_u A}{\partial_v A} \partial_v^2 + \frac{\partial_u T}{T} \partial_v \right) A \,. \tag{16}$$

For curves of low genus this formula takes even a more elementary form (79) which, as we explain, is very convenient for calculations of $\hat{A}$. In Sect. 3 we will illustrate how this works in some simple knot theory examples, and in Sects. 6 and 7 in several examples from the topological string theory.

2 Topological Recursion Versus Quantum Curves

In this section, we collect the necessary facts about the perturbative structure of the partition function (11) and the Schrödinger-like equation (10) that, when combined together, can tell us how the polynomial $A(u, v)$ or $A(x, y)$ gets quantized,

$$A \rightsquigarrow \hat{A} \,. \tag{17}$$

To the leading order in the $\hbar$-expansion, $\hat{A}$ is obtained from A simply by replacing u and v by the quantum operators $\hat{u}$ and $\hat{v}$. Then, with the choice of polarization as

[1] It seems that all polynomials $A(x, y)$ that come from geometry have this property. Why this happens is a mystery.

in (7) the Schrödinger-like equation (10) implies the following leading behavior of the wave-function (11):

$$\begin{aligned} S_0 &= \int v du \qquad \text{for curves in } \mathbb{C} \times \mathbb{C}\,, \\ &= \int \log y \frac{dx}{x} \qquad \text{for curves in } \mathbb{C}^* \times \mathbb{C}^*\,. \end{aligned} \tag{18}$$

In fact, in any approach to quantization this should be the leading behavior of the semi-classical wave function associated to the classical state $A = 0$. What about the higher-order terms S_n with $n \geq 1$?

In the introduction we mentioned several recent developments that shed light on the perturbative (and, in some cases, even non-perturbative) structure of the partition function (11). One of such recent developments is the topological recursion of Eynard–Orantin [28] and its extension to curves in $\mathbb{C}^* \times \mathbb{C}^*$ called the "remodeling conjecture" [9, 41]. These techniques are ideally suited for understanding the analytic structure of the quantization (17).

2.1 Topological Recursion

The starting point of the topological recursion [28] is the choice[2] of a parametrization, i.e. a choice of two functions of a local variable p,

$$\begin{cases} u = u(p) \\ v = v(p) \end{cases} \tag{19}$$

where $u(p)$ is assumed to have non-degenerate critical points. (In particular, for curves of genus zero, both $u(p)$ and $v(p)$ can be rational functions. We are not going to assume this, however, and, unless noted otherwise, much of our discussion below applies to curves of arbitrary genus.) Then, from this data alone one can recursively determine the perturbative coefficients S_n of the partition function (11) via a systematic procedure that we explain below.

For example, as we already noted in (18) the leading term S_0 is obtained by integrating a 1-form differential $\phi = vdu$ along a path on the curve $A(u, v) = 0$. When expressed in terms of the local coordinate p, this integral looks like

$$S_0 = \int^p \phi = \int^p v(p) du(p)\,, \tag{20}$$

and sometimes is also referred to as the anti-derivative of ϕ. Then, the next-to-the-leading term S_1 is determined by the two-point function, or the so-called

[2] As will be explained in Sect. 2.3, this choice is related to the choice of polarization.

annulus amplitude. For a curve $\mathscr{C}$ of genus zero it can be expressed in terms of the parametrization data (19) by the following formula[3]

$$S_1 = -\frac{1}{2}\log\frac{du}{dp}\,, \tag{21}$$

whose origin and generalization to curves of arbitrary genus will be discussed in Sect. 2.5. We recall that, according to (14), the term S_1 contains information about the "torsion" $T(u)$ and generically is all one needs in order to determine the quantum curve $\hat{A}$ when it has a nice polynomial form (15).

In a similar manner, the topological recursion of Eynard–Orantin [28] can be used to determine all the other higher-order terms S_n, $n \geq 2$. Starting with the parametrization (19), one first defines a set of symmetric degree-n meromorphic differential forms $W_n^g = W_n^g(p_1, p_2, \ldots, p_n)$ on $\mathscr{C}^n$ via a systematic procedure that we shall review in a moment. Then, by taking suitable integrals and residues one obtains respectively the desired S_n's, as well as their "closed string" analogs known as the genus-g free energies F_g:

$$\boxed{u(p) \text{ and } v(p) \quad \rightsquigarrow \quad W_n^g \quad \rightsquigarrow \quad S_n \text{ and } F_g} \tag{22}$$

Specifically, motivated by the form of a determinant in (12), or a definition of the Baker–Akhiezer function in [27, 28], we construct S_n's as the following linear combinations of the integrated multilinear meromorphic differentials:

$$S_n(p) = \sum_{2g-1+k=n} \frac{1}{k!} \underbrace{\int_{\tilde{p}}^{p} \cdots \int_{\tilde{p}}^{p}}_{k \text{ times}} W_k^g(p_1', \ldots, p_k')\,, \tag{23}$$

where each differential form W_k^g of degree k is integrated k times.[4] The base point of integration $\tilde{p}$ is chosen such that $u(\tilde{p}) \to \infty$ [28]. In turn, the multilinear differentials W_n^g are obtained by taking certain residues around critical points of the "Morse function" $u(p)$, i.e. solutions to the equations

$$du(p)|_{p_i^*} = 0 \qquad \Leftrightarrow \qquad \partial_v A|_{p_i^*} = 0\,, \tag{24}$$

[3]Notice, our prescription here and also in Eq. (23) differs from that in [16]. As will be explained below, these differences are important for overcoming the obstacles in [16] and reproducing the "quantum" q-corrections in the quantization of the A-polynomial (17).

[4]For curves of genus one or higher one should consider more general Baker–Akhiezer function, which in addition includes non-perturbative corrections represented by certain θ-functions [27]. As the examples which we consider concern mostly curves of genus zero, we do not analyse such corrections explicitly.

where the standard shorthand notation $\partial_v \equiv \frac{\partial}{\partial v}$ is used. Following [28], we shall refer to these points as the "branch points" of the curve $\mathscr{C}$ in parametrization (19). For each point p in the neighborhood of a branch point p_i^* there is a unique, conjugate point $\bar{p}$, such that

$$u(p) = u(\bar{p})\,. \tag{25}$$

The next essential ingredient for the topological recursion is the differential 1-form[5] called the "vertex":

$$\begin{aligned} \omega(p) &= \big(v(\bar{p}) - v(p)\big) du(p) && \text{for curves in } \mathbb{C}\times\mathbb{C}\,, \\ &= \big(\log y(\bar{p}) - \log y(p)\big) \frac{dx(p)}{x(p)} && \text{for curves in } \mathbb{C}^*\times\mathbb{C}^*\,, \end{aligned} \tag{26}$$

and the 2-form $B(p,q)$ known as the Bergman kernel. The Bergman kernel $B(p,q)$ is defined as the unique meromorphic differential with exactly one pole, which is a double pole at $p = q$ with no residue, and with vanishing integral over A_I-cycles $\oint_{A_I} B(p,q) = 0$ (in a canonical basis of cycles (A_I, B^I) for $\mathscr{C}$). Thus, for curves of genus zero the Bergman kernel takes a particularly simple form

$$B(p,q) = \frac{dp\,dq}{(p-q)^2}\,, \tag{27}$$

and its form for curves of higher genus is presented in Sect. 2.5. A closely related quantity is a 1-form, defined in a neighborhood of a branch point q_i^*

$$dE_q(p) = \frac{1}{2}\int_q^{\bar{q}} B(\xi, p)\,.$$

Finally, the last important ingredient is the recursion kernel $K(q,p)$,

$$K(q,p) = \frac{dE_q(p)}{\omega(q)}\,. \tag{28}$$

Having defined the above ingredients we can present the recursion itself. When expressed in variables (u,v), the recursion has the same form for curves in $\mathbb{C}\times\mathbb{C}$ as it does for curves in $\mathbb{C}^*\times\mathbb{C}^*$. It determines higher-degree meromorphic differentials $W_n^g(p_1,\ldots,p_n)$ from those of lower degree. The initial data for the recursion are one- and two-point correlators of genus zero, the former vanishing by definition and the latter given by the Bergman kernel:

[5]For reasons that will become clear later, we choose a sign opposite to the conventions of [28].

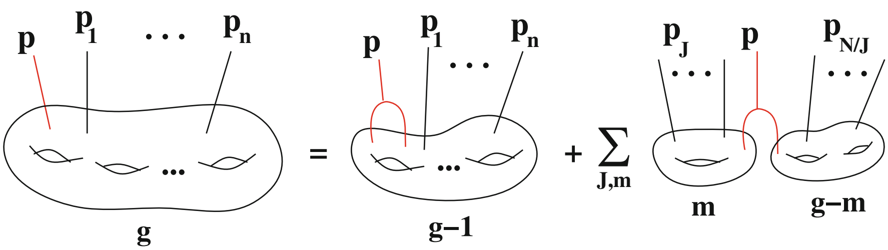
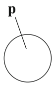
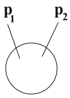


Fig. 1 A graphical representation of the Eynard–Orantin topological recursion



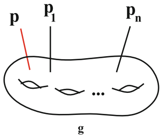
$$: \; W_1^0(p) = 0\,, \tag{29}$$



$$: \; W_2^0(p_1, p_2) = B(p_1, p_2)\,. \tag{30}$$

It is also understood that $W_n^{g<0} = 0$.

The other differentials are defined recursively as follows. For a set of indices J denote $\mathbf{p}_J = \{p_i\}_{i \in J}$. Then, for $N = \{1, \ldots, n\}$ and the corresponding set of points $\mathbf{p}_N = \{p_1, \ldots, p_n\}$ define



$$: W_{n+1}^g(p, \mathbf{p}_N) = \sum_{q_i^*} \mathrm{Res}_{q \to q_i^*} K(q, p) \Big(W_{n+2}^{g-1}(q, \bar{q}, \mathbf{p}_N) + \\ + \sum_{m=0}^{g} \sum_{J \subset N} W_{|J|+1}^m(q, \mathbf{p}_J) W_{n-|J|+1}^{g-m}(\bar{q}, \mathbf{p}_{N/J}) \Big), \tag{31}$$

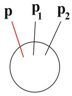
where $\sum_{J \subset N}$ denotes a sum over all subsets J of N, *cf.* Fig. 1. These correlators have many interesting properties. For example, any $W_n^g(p_1, \ldots, p_n)$ is a symmetric function of p_i. Furthermore, apart from the special case of $g = 0$ and $n = 2$, the poles of $W_n^g(p_1, \ldots, p_n)$ in variables p_i appear only at the branch points. In addition, the A_I-cycle integrals with respect to any p_i vanish, $\oint_{p_i \in A_I} W_n^g(p_1, \ldots, p_n) = 0$. For a detailed discussion of these and many other features of W_n^g see [28].

Let us briefly illustrate how the recursion procedure works. First, from the recursion kernel (28) and from the Bergman kernel (30) one finds the genus-1 one-point correlator

$$W_1^1(p) = \sum_{q_i^*} \mathrm{Res}_{q \to q_i^*} K(q, p) W_2^0(q, \bar{q})\,. \tag{32}$$

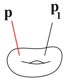
Then, the following series (with $g+n=3$) is determined

$$W_3^0(p,p_1,p_2) =$$



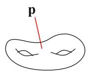
$$\begin{aligned} &= \sum_{q_i^*} \mathrm{Res}_{q\to q_i^*} K(q,p) \\ &\quad \times \Big(W_2^0(q,p_1) W_2^0(\bar{q},p_2) + W_2^0(\bar{q},p_1) W_2^0(q,p_2) \Big), \end{aligned} \tag{33}$$

$$W_2^1(p,p_1) =$$



$$\begin{aligned} &= \sum_{q_i^*} \mathrm{Res}_{q\to q_i^*} K(q,p) \\ &\quad \times \Big(W_3^0(q,\bar{q},p_1) + 2 W_1^1(q) W_2^0(\bar{q},p_1) \Big), \end{aligned} \tag{34}$$

$$W_1^2(p) =$$



$$= \sum_{q_i^*} \mathrm{Res}_{q\to q_i^*} K(q,p) \Big(W_2^1(q,\bar{q}) + W_1^1(q) W_1^1(\bar{q}) \Big). \tag{35}$$

Next, one finds a series $W_4^0, W_3^1, W_2^2, W_1^3$ with $g+n=4$, and so on. In the end, from each such series one can determine one more S_n using (23). For example, as will be discussed in Sect. 2.5, S_1 is obtained by integrating the Bergman kernel:

$$S_1(p) = \frac{1}{2} \lim_{p_1 \to p_2 = p} \int \left(B(p_1,p_2) - \frac{du(p_1)\,du(p_2)}{(u(p_1)-u(p_2))^2} \right), \tag{36}$$

and for curves of genus zero this formula reproduces the expression (21) proposed earlier. At the next step, from the series of the multilinear differentials (33)–(35) one finds the next term in the perturbative series (11):

$$S_2(p) = \int^p W_1^1(p_1) + \frac{1}{3!} \int^p \int^p \int^p W_3^0(p_1,p_2,p_3), \tag{37}$$

and so on.

While not of our immediate concern in this paper, for completeness we also recall a definition of genus-g free energies F_g. For $g \geq 2$ they come[6] from the corresponding W_1^g:

$$F_g = \frac{1}{2g-2} \sum_{q_i^*} \mathrm{Res}_{q\to q_i^*} S_0(q) W_1^g(q), \tag{38}$$

[6]Notice, compared to the conventions of [28] we introduce an extra minus sign in our definition of F_g in order to account for the sign of W_1^g originating from the sign in (26).

where $S_0(q) = \int^q v(p)du(p)$, while F_0 and F_1 are defined independently in a more intricate way presented in [28]. Among various interesting properties of F_g the most important one is their invariance under symplectic transformations of the spectral curve.

Finally, since the relation between S_n and W_k^g will be crucial for computing $\hat{A}$ from the classical curve $A = 0$ and its parametrization, let us briefly explain our motivation behind (23). Recall, that the correlators $W_k^g(p_1, \dots, p_k)$ in (23) were originally introduced [28] in a way which generalizes and, when an underlying matrix model exists, reproduces connected contributions to the matrix model expectation value

$$\Big\langle \mathrm{Tr}\Big(\frac{1}{u(p_1)-M}\Big)\cdots \mathrm{Tr}\Big(\frac{1}{u(p_k)-M}\Big)\Big\rangle_{\mathrm{conn}} = \sum_{g=0}^{\infty} \hbar^{2g-2+k} \frac{W_k^g(p_1,\dots,p_k)}{du(p_1)\dots du(p_k)}$$

in an ensemble of matrices M of size $N = \hbar^{-1}$.[7] Integrating both sides with respect to all variables and then setting $p_1 = \dots = p_k = p$, we get

$$\Big\langle \Big(\mathrm{Tr}\log\big(u(p)-M\big)\Big)^k\Big\rangle_{\mathrm{conn}} = \sum_{g=0}^{\infty} \hbar^{2g-2+k} \int^p \cdots \int^p W_k^g(p_1',\dots,p_k').$$

Dividing both sides by $k!$ and summing over k we get

$$\Big\langle \det(u-M)\Big\rangle_{\mathrm{conn}} = \sum_{n=0}^{\infty} \hbar^{n-1} S_n(p),$$

with $S_n(p)$ defined in (23). Whereas the left hand side represents the connected expectation value, the right hand side plays the role of the free energy, so that

$$Z = \Big\langle \det(u-M)\Big\rangle = e^{\frac{1}{\hbar}\sum_{g=0}^{\infty} \hbar^n S_n(p)}.$$

This result is in agreement with (11) and (12) and provides the motivation for the definition (23). From the matrix model point of view, the free energies F_g defined in (38) encode the total partition function

$$\langle 1 \rangle = \int \mathscr{D}M e^{-\frac{1}{\hbar}\mathrm{Tr}V(M)} = e^{\sum_{g=0}^{\infty} \hbar^{2g-2} F_g}. \tag{39}$$

From a string theory viewpoint, this partition function would correspond to closed string amplitudes. In fact, in many instances relevant to Seiberg–Witten theory

[7] Strictly speaking, this equation holds for $k > 2$ and there are some corrections to the lowest order terms with $k = 1$ and $k = 2$ [28].

or topological strings, matrix models which encode corresponding partition functions (39) have been explicitly constructed in [3, 24–26, 37, 46, 49].

2.2 Quantum Curves and Differential Hierarchies

Our next goal is to compare the results of the topological recursion to the structure of the "quantum curve"

$$\hat{A} \simeq 0 , \tag{40}$$

where we used a shorthand notation "$\simeq$" to write (10) in a form that makes a connection with its classical limit $A(x, y) = 0$ manifest, *cf.* [19]. In general, the Schrödinger-like equation (10) and its abbreviated form (40) is either a q-difference equation (for curves in $\mathbb{C}^* \times \mathbb{C}^*$) or an ordinary differential equation (for curves in $\mathbb{C} \times \mathbb{C}$). In either case, we need to write it as a power series in $\hbar$, which was the expansion parameter in the topological recursion.

In practice, one needs to substitute the perturbative expansions (6) and (11) into the Schrödinger-like equation (10),

$$\left(\hat{A}_0 + \hbar \hat{A}_1 + \hbar^2 \hat{A}_2 + \dots\right) \exp\left(\frac{1}{\hbar}\sum_{n=0}^{\infty} S_n\, \hbar^n\right) = 0 , \tag{41}$$

and collect all terms of the same order in $\hbar$-expansion. This requires some algebra (see [21] and Appendix 1), but after the dust settles one finds[8] a nice hierarchy of "loop equations"

$$\boxed{\sum_{r=0}^{n} \mathfrak{D}_r A_{n-r} = 0 ,} \tag{42}$$

expressed in terms of symbols A_{n-r} of the operators $\hat{A}_{n-r}$ and in terms of differential operators $\mathfrak{D}_r$. Specifically, each $\mathfrak{D}_r$ is a differential operator of degree $2r$; it can be written as a degree-$2r$ polynomial in $\partial_v \equiv \frac{\partial}{\partial v}$, whose coefficients are

[8] Once again, we point out that, when expressed in terms of variables u and v, most of our formulas have the same form on *any* complex symplectic twofold with the holomorphic symplectic 2-form (2). In particular, the hierarchy of differential equations (42) written in variables (u, v) looks identical for curves in $\mathbb{C} \times \mathbb{C}$ and in $\mathbb{C}^* \times \mathbb{C}^*$. Of course, the reason is simple: it is not the algebraic structure, but, rather, the symplectic structure that matters in the quantization problem. For this reason, throughout the paper we write most of our general formulas in variables (u, v) with understanding that, unless noted otherwise, they apply to curves in arbitrary complex symplectic twofold with the holomorphic symplectic 2-form (2).

polynomial expressions in functions $S_k(u)$ and their derivatives. For example, the first few differential operators look like

$$\mathfrak{D}_0 = 1\,, \tag{43a}$$

$$\mathfrak{D}_1 = \frac{S_0''}{2}\partial_v^2 + S_1'\partial_v\,, \tag{43b}$$

$$\mathfrak{D}_2 = \frac{(S_0'')^2}{8}\partial_v^4 + \frac{1}{6}\big(S_0''' + 3S_0''S_1'\big)\partial_v^3 + \frac{1}{2}\big(S_1'' + (S_1')^2\big)\partial_v^2 + S_2'\partial_v\,, \tag{43c}$$

$$\vdots$$

and yield the corresponding equations, at each order $\hbar^n$ in (42):

$$\hbar^0: \qquad A = 0\,, \tag{44}$$

$$\hbar^1: \qquad \Big(\frac{S_0''}{2}\partial_v^2 + S_1'\partial_v\Big)A + A_1 = 0\,, \tag{45}$$

$$\vdots$$

$$\hbar^n: \qquad \mathfrak{D}_n A + \mathfrak{D}_{n-1}A_1 + \ldots + A_n = 0\,, \tag{46}$$

$$\vdots$$

The first equation is equivalent to the classical curve equation (3), provided $S_0' \equiv \frac{dS_0}{du} = v$ which, in turn, leads to the expression (18) for $S_0(u)$. The second equation (45) is also familiar from (13) and (16), where the second order differential operator $\mathfrak{D}_1$ acting on $A_0 \equiv A$ was expressed in terms of the "torsion" $T(u)$. If we know the partition function Z, then, at each order $\hbar^n$, the above equations uniquely determine the correction $\hat{A}_n$; or *vice versa*: from the knowledge of the total $\hat{A}$, at each order $\hbar^n$, we can determine S_n (up to an irrelevant normalization constant).

More generally, the operators $\mathfrak{D}_r$ are defined via the generating function

$$\sum_{r=0}^{\infty}\hbar^r\mathfrak{D}_r = \exp\left(\sum_{n=1}^{\infty}\hbar^n\mathfrak{d}_n\right)\,, \tag{47}$$

where

$$\mathfrak{d}_n = \sum_{r=1}^{n+1}\frac{S_{n+1-r}^{(r)}}{r!}(\partial_v)^r\,. \tag{48}$$

For example, the explicit expressions for small values of n

$$\mathfrak{d}_1 = \frac{1}{2} S_0'' \partial_v^2 + S_1' \partial_v \,,$$

$$\mathfrak{d}_2 = \frac{1}{6} S_0''' \partial_v^3 + \frac{1}{2} S_1'' \partial_v^2 + S_2' \partial_v \,,$$

$$\mathfrak{d}_3 = \frac{1}{4!} S_0^{(4)} \partial_v^4 + \frac{1}{3!} S_1''' \partial_v^3 + \frac{1}{2} S_2'' \partial_v^2 + S_3' \partial_v \,,$$

lead to the formulas (43). More details and a derivation of the above hierarchy are given in Appendix 1.

Our goal in the rest of the paper is to combine the steps in Sects. 2.1 and 2.2 into a single technique that can produce a quantum operator $\hat{A}$ starting with a parametrization of the classical curve (3), much as in the topological recursion:

$$u(p) \text{ and } v(p) \qquad \rightsquigarrow \qquad \hat{A} \,. \tag{49}$$

Basically, one can use the output of (22) as an input for (44)–(46) (written more compactly in (42)) to produce a perturbative expansion (6).

2.3 *Parametrizations and Polarizations*

The quantization procedure (17) on one hand, and the topological recursion (22) on the other come with certain inherent ambiguities which are not unrelated.

In quantization, one needs to split the coordinates on the phase space into "canonical coordinates" and "conjugate momenta." This choice, called the choice of polarization, means that one needs to pick a foliation of the phase space by Lagrangian submanifolds parametrized by a maximal set of mutually commuting "coordinates" (with the remaining variables understood as their conjugate momenta). In the problem at hand, the (complex) phase space is two-dimensional, with the symplectic form (2),

$$\omega = \frac{i}{\hbar} du \wedge dv \,, \tag{50}$$

so that the ambiguity associated with the choice of polarization is described by one functional degree of freedom, say, a choice of function $f(u, v)$ that one regards as a "coordinate." Thus, in most of the present paper we make a natural[9] choice (7) treating u as the "coordinate" and v as the momentum. Any other choice is related to this one by a canonical transformation

[9]In most applications.

$$v = \frac{\partial \mathscr{W}}{\partial u}\,, \qquad V = -\frac{\partial \mathscr{W}}{\partial U} \tag{51}$$

that depends on a single function $\mathscr{W}(u, U)$. By definition, the transformation $(u, v) \mapsto (U, V)$ preserves the symplectic form ω. For example, $U = v$ and $V = -u$ corresponds to $\mathscr{W}(u, U) = uU$.

Similarly, as we reviewed in Sect. 2.1, the ambiguity in the topological recursion is also described by a single function $u(p)$ that enters the choice of parametrization (19). (The functional dependence of $v(p)$ is then determined, up to a discrete action of the Galois group permuting branches $v^{(\alpha)}$, by the condition $A(u, v) = 0$.) Indeed, starting with different parametrizations of the same classical curve (3) and following (49) one arrives at different expressions for $\hat{A}$. To make a contact with the choice of polarization, let us point out that part of its ambiguity is already fixed in the topological recursion (since $u(p)$ is a function of a single variable, whereas $\mathscr{W}(u, U)$ in (51) is a function of two variables). However, a transformation from $u(p)$ to $U(p)$ can be understood as a particular symplectic transformation $(u, v) \mapsto (U, V)$, such that $U = f(u)$ and $V = v/f'(u)$. For example, a simple choice of $f(u) = u + c$ with a constant c corresponds to

$$U = u + c \qquad , \qquad V = v\,, \tag{52}$$

and does not affect $\hat{A}$. On the other hand, a similar "shift transformation" of the momentum v,

$$\hat{v} = \hbar \partial_u \;\rightarrow\; \hat{v} = \hbar \partial_u + c\hbar \tag{53}$$

is equivalent to $Z(u) \rightarrow e^{cu} Z(u)$ and, therefore, transforms the quantum operator $\hat{A}$ as

$$\hat{A}(\hat{x}, \hat{y}) \;\rightarrow\; \hat{A}(\hat{x}, q^c \hat{y})\,. \tag{54}$$

This transformation plays an important role in our applications since it controls a (somewhat ambiguous) constant term in S_1'.

We also note that, with the choice of uniformization (19) and in the polarization where p is the "coordinate" the quantum curve factorizes to the leading order in $\hbar$

$$\hat{A} = \prod_{\alpha} \left(\hbar \partial_p + f^{(\alpha)}(p)\right) + \mathscr{O}(\hbar)\,. \tag{55}$$

Then, to the leading order in $\hbar$, various branches of the partition function (9) are annihilated by the first order operators $(\hbar \partial_p + f^{(\alpha)}(p))$, so that

$$Z^{(\alpha)} = e^{-\frac{1}{\hbar}\int f^{(\alpha)}(p)dp}\Big(1 + \mathscr{O}(\hbar)\Big)\,.$$

2.4 Relation to Algebraic K-Theory

Now we come to a very important point, which could already have been emphasized much earlier in the paper:

> Not every curve $\mathscr{C}$ defined by the zero-locus of a polynomial A is "quantizable"!

Namely, one can always produce a non-commutative deformation of the ring of functions on $\mathbb{C} \times \mathbb{C}$ or $\mathbb{C}^* \times \mathbb{C}^*$, which obeys (5) with $\hbar$ as a formal parameter and, therefore, at least formally gives (17). However, in physics, one is usually interested in the actual (not formal) deformation of the algebra of functions with a parameter $\hbar$ and, furthermore, it is important to know whether a state associated with a particular Lagrangian submanifold in the classical phase space exists in the Hilbert space of the quantum theory.

In the present case, this means that not every Lagrangian submanifold defined by the zero locus of $A(x, y)$ corresponds to an actual state in the Hilbert space of the quantum theory; the ones which do we call[10] "quantizable." Specifically, whether the solution to the quantization problem exists or not depends on the complex structure[11] of the curve $\mathscr{C}$, i.e. on the coefficients of the polynomial $A(x, y)$ that defines it.

Following [32], we explain this important point in a simple example of, say, the figure-8 knot. Relegating further details to the next section, let us take a quick look at the classical curve

$$\mathscr{C} \ : \qquad x^4 - (1 - x^2 - 2x^4 - x^6 + x^8)y + x^4 y^2 = 0 \tag{56}$$

defined by the zero locus of the A-polynomial of the figure-8 knot (see Table 1). This polynomial equation has a number of special properties, including integrality of coefficients, symmetries (with respect to $x \to 1/x$ and $y \to 1/y$), and so on. More importantly, the classical curve (56) is quantizable.

---

[10] Notice, a priori this definition of "quantizability" has nothing to do with the nice property (15) exhibited by many quantum operators $\hat{A}$ that come from physical problems; one can imagine a perfectly quantizable polynomial $A(x, y)$ in the sense described here, for which the quantum corrections (6) can *not* be summed up into a finite polynomial of x, y, and q. We plan to elucidate the relation between these two properties in the future work.

[11] At first, this may seem a little surprising, because the quantization problem is about symplectic geometry and not about complex geometry of $\mathscr{C}$. (Figuratively speaking, quantization aims to replace all classical objects in symplectic geometry by the corresponding quantum analogs.) However, our "phase space," be it $\mathbb{C} \times \mathbb{C}$ or $\mathbb{C}^* \times \mathbb{C}^*$, is very special in a sense that it comes equipped with a whole $\mathbb{C}\mathbf{P}^1$ worth of complex and symplectic structures, so that each aspect of the geometry can be looked at in several different ways, depending on which complex or symplectic structure we choose. This hyper-Kähler nature of our geometry is responsible, for example, for the fact that a curve $\mathscr{C}$ "appears" to be holomorphic (or algebraic). We put the word "appears" in quotes because this property of $\mathscr{C}$ is merely an accident, caused by the hyper-Kähler structure on the ambient space, and is completely irrelevant from the viewpoint of quantization. What is important to the quantization problem is that $\mathscr{C}$ is Lagrangian with respect to the symplectic form (2).

Table 1 Classical A-polynomial and its quantization in prominent examples

Model	Classical curve	Quantum operator
Airy	$v^2 - u$	$\hat{v}^2 - \hat{u}$
tetrahedron	$1 + y + xy^f$	$1 + q^{-1/2}\hat{y} + q^{(f+1)/2}\hat{x}\hat{y}^f$
$c = 1$ model	$u^2 - v^2 + 2t$	$\hat{u}^2 - \hat{v}^2 + 2t + \hbar$
conifold	$1 + x + y + e^t xy^{-1}$	$1 + q^{1/2}\hat{x} + q^{-1/2}\hat{y} + e^t \hat{x}\hat{y}^{-1}$
(p, q) minimal model	$v^p - u^q$	?
figure-8 knot	$(1 - x^2 - 2x^4 - x^6 + x^8)y$ $-x^4 - x^4y^2$	$(1 - q^4\hat{x}^4)$ $\times (1 - q^2\hat{x}^2 - (q^2 + q^6)\hat{x}^4 - q^6\hat{x}^6 + q^8\hat{x}^8)\hat{y}$ $-q^3(1 - q^6\hat{x}^4)\hat{x}^4 - q^5(1 - q^2\hat{x}^4)\hat{x}^4\hat{y}^2$

Preserving most of the nice properties of (56) we can make a tiny change to the polynomial $A(x, y)$ to obtain a close cousin of $\mathscr{C}$:

$$\mathscr{C}' \ : \qquad x^4 - (x^{-2} - x^2 - 2x^4 - x^6 + x^{10})y + x^4y^2 = 0 \tag{57}$$

To a naked eye, there is almost no difference between the curves $\mathscr{C}$ and $\mathscr{C}'$; indeed, every obvious property of one is manifest in the other and vice versa. Nevertheless, the curve (56) defined by the true A-polynomial of the figure-8 knot is quantizable, whereas the counterfeit (57) is not. Why?

The reason, as explained in [32], is that all periods of the 1-form $\mathrm{Im}\,\phi$ must vanish

$$\oint_\gamma \Big(\log|x| d(\arg y) - \log|y| d(\arg x) \Big) = 0 \,, \tag{58}$$

and, furthermore, the periods of the 1-form $\mathrm{Re}\,\phi$ should be integer (or, at least, rational) multiples of $2\pi i$ or, equivalently,

$$\frac{1}{4\pi^2} \oint_\gamma \Big(\log|x| d \log|y| + (\arg y) d(\arg x) \Big) \in \mathbb{Q} \tag{59}$$

for all $\gamma \in H_1(\mathscr{C}, \mathbb{R})$. Indeed, these two conditions guarantee that $Z = \exp\big(\frac{1}{\hbar} S_0 + \ldots\big) = \exp\big(\frac{1}{\hbar}\int^p \phi + \ldots\big)$ is well-defined and, therefore, they represent the necessary conditions for $A(x, y) = 0$ to be quantizable.[12] It is not difficult to verify that these conditions are met for the curve (56) but not for the curve (57).

Notice, the constraints (58)–(59) are especially severe for curves of high genus. Moreover, these constraints have an elegant interpretation in terms of algebraic K-theory and the Bloch group of $\overline{\mathbb{Q}}$. To explain where this beautiful connection comes

[12]Notice, various choices discussed in Sect. 2.3 lead to expressions for ϕ which differ by (non-holomorphic) exact terms. For more details on change of polarization see e.g. [33].

from, we start with the observation that the left-hand side of (58) is the image of the symbol $\{x, y\} \in K_2(\mathscr{C})$ under the regulator map[13]

$$r \ : \ K_2(\mathscr{C}) \to H^1(\mathscr{C}, \mathbb{R}) \qquad (60)$$
$$\{x, y\} \mapsto \eta(x, y)$$

evaluated on the homology class of a closed path γ that avoids all zeros and poles of x and y. Indeed, the left-hand side of (58) is the integral of the real differential 1-form on $\mathscr{C}$ (with zeros and poles of x and y excluded),

$$\eta(x, y) = \log|x| d(\arg y) - \log|y| d(\arg x)\,, \qquad (61)$$

which, by definition, is anti-symmetric,

$$\eta(y, x) = -\eta(x, y)\,, \qquad (62)$$

obeys the "Leibniz rule,"

$$\eta(x_1 x_2, y) = \eta(x_1, y) + \eta(x_2, y)\,, \qquad (63)$$

and, more importantly, is closed

$$d\eta(x, y) = \mathrm{Im}\left(\frac{dx}{x} \wedge \frac{dy}{y}\right) = 0\,. \qquad (64)$$

For curves, the latter condition is almost trivial and immediately follows from dimensional considerations, which is another manifestation of the "accidental" extra structure discussed in the Footnote 11. In higher dimensions, however, the condition (64) is very non-trivial and holds precisely when $\mathscr{C}$ is Lagrangian with respect to (real / imaginary part of) the symplectic form (2).

We have learnt that the differential 1-form $\eta(x, y)$ is closed. However, to meet the condition (58) and, ultimately, to reformulate this condition in terms of algebraic K-theory we actually want $\eta(x, y)$ to be exact. In order to understand when this happens, it is important to describe $\eta(x, y)$ near those points on $\mathscr{C}$ where rational functions $x, y \in \mathbb{C}(\mathscr{C})^*$ have zeros or poles. Let p be one of such points and let $\mathrm{ord}_p(x)$ (resp. $\mathrm{ord}_p(y)$) be the order of x (resp. y) at p. Then, we have

$$\frac{1}{2\pi} \oint \eta(x, y) = \log|(x, y)_p| \qquad (65)$$

where the integral is over a small circle centered at p and

[13] Defined by Beilinson [6] after Bloch [7].

$$(x, y)_p = (-1)^{\mathrm{ord}_p(x)\,\mathrm{ord}_p(y)} \frac{x^{\mathrm{ord}_p(y)}}{y^{\mathrm{ord}_p(x)}}\Big|_p \tag{66}$$

is the *tame symbol* at $p \in \mathscr{C}$.

One general condition that guarantees vanishing of (65) is to have $\{x, y\} = 0$ in $K_2(\mathbb{C}(\mathscr{C})) \otimes \mathbb{Q}$. Then, all tame symbols (66) are automatically torsion and $\eta(x, y)$ is actually exact, see e.g. [39]. Motivated by this, we propose the following criterion for quantizability:

$$\boxed{\mathscr{C} \text{ is quantizable} \quad \Longleftrightarrow \quad \{x, y\} \in K_2(\mathbb{C}(\mathscr{C})) \text{ is a torsion class}} \tag{67}$$

This criterion is equivalent [11] to having

$$x \wedge y = \sum_i r_i z_i \wedge (1 - z_i) \qquad \text{in } \wedge^2 (\mathbb{C}(\mathscr{C})^*) \otimes \mathbb{Q} \tag{68}$$

for some $z_i \in \mathbb{C}(\mathscr{C})^*$ and $r_i \in \mathbb{Q}$. When this happens, one can write

$$\eta(x, y) = d\left(\sum_i r_i D(z_i)\right) = dD\left(\sum_i r_i [z_i]\right) \tag{69}$$

in terms of the Bloch–Wigner dilogarithm function,

$$D(z) \;:=\; \log|z| \arg(1 - z) + \mathrm{Im}(\mathrm{Li}_2(z))\,, \tag{70}$$

which obeys the famous 5-term relation

$$D(x) + D(y) + D(1 - xy) + D\Big(\frac{1 - x}{1 - xy}\Big) + D\Big(\frac{1 - y}{1 - xy}\Big) = 0 \tag{71}$$

and $dD(z) = \eta(z, 1 - z)$. Note, the exactness of $\eta(x, y)$ is manifest in (69), which makes it clear that our proposed condition (67) incorporates (58). (The check that (67) also incorporates (59) is similar and we leave it as an exercise to the reader.)

In our example of the A-polynomial for the figure-8 knot, we already claimed that the curve (56) is quantizable. Indeed, the condition (68) in this example reads

$$x \wedge y = z_1 \wedge (1 - z_1) - z_2 \wedge (1 - z_2) \tag{72}$$

where

$$x^2 = z_1 z_2\,, \qquad y = \frac{z_1^2}{1 - z_1} = \frac{1 - z_2}{z_2^2}\,, \tag{73}$$

so that z_1 and z_2 satisfy the “gluing condition” $(z_1 - 1)(z_2 - 1) = z_1^2 z_2^2$.

In practice, the condition (68) is much easier to deal with and, of course, the appearance of the dilogarithm is not an accident. Its role in the quantization problem and the interpretation of (67) based on Morse theory will be discussed elsewhere [22].

2.5 The First Quantum Correction

As we emphasized earlier, the subleading term S_1 contains a lot more information than meets the eye; e.g. generically it determines much of the structure of the quantum curve, if not all of it. Therefore, we devote an entire subsection to the discussion of S_1 and the first quantum correction to $\hat{A}$ that it determines via (45).

In general, the correction S_1 is defined as the integrated two-point function with equal arguments

$$S_1(p) = \frac{1}{2}\int^p \int^p \omega_2(p_1, p_2)\,.$$

The two-point function can be expressed in terms of the Bergman kernel with a double pole removed [28]

$$\omega_2(p_1, p_2) = B(p_1, p_2) - \frac{du(p_1)du(p_2)}{\big(u(p_1) - u(p_2)\big)^2}\,.$$

Generally, for curves of arbitrary genus, the Bergman kernel is given by a derivative of a logarithm of the theta function of odd characteristic θ_{odd} associated to the classical curve $\mathscr{C}$ [28, 41]

$$B(p_1, p_2) = \partial_{p_1}\partial_{p_2}\log\theta_{odd}\big(u(p_1) - u(p_2)\big)\,,$$

and it has only one (second-order) pole at equal values of the arguments. For curves of genus zero this pole is the only ingredient of the Bergman kernel, see (27), and in that case the above two-point function was used in (36) to get (21).

Let us discuss now how this result is modified for curves of higher genus. For curves of genus one the Bergman kernel can be expressed as[14]

$$B(p_1, p_2) = \Big(\wp(p_1 - p_2; \tau) + \frac{\pi}{\mathrm{Im}\,\tau}\Big)dp_1 dp_2\,. \tag{74}$$

[14]More generally, one can consider a generalized Bergman kernel [28], which differs from an ordinary Bergman kernel by a dependence on an additional parameter κ. In most applications, including matrix models, one can set $\kappa = 0$, which leads to the ordinary Bergman kernel given above.

The Weierstrass function $\wp$ has the expansion

$$\wp(z;\tau) = \frac{1}{z^2} + \frac{g_2}{20} z^2 + \frac{g_3}{28} z^4 + \mathcal{O}(z^6)\,, \tag{75}$$

where τ and g_2, g_3 denote, respectively, the modulus and the standard invariants of an elliptic curve. Using this expansion we get

$$\int^{p_1}\int^{p_2} \omega_2(p_1,p_2) = -\log\frac{u(p_1)-u(p_2)}{p_1-p_2} + \frac{\pi}{\mathrm{Im}\,\tau} p_1 p_2 - \frac{g_2}{240}(p_1-p_2)^4 + \mathcal{O}\big((p_1-p_2)^6\big).$$

In the limit $p_1 \to p_2 = p$ the first term reproduces the genus zero result (21), while the other contributions in the expansion of the function $\wp(p_1 - p_2;\tau)$ vanish. In consequence, we are left with the quadratic correction to the genus zero result

$$S_1(p) = \frac{1}{2}\int^{p_1}\int^{p_2} \omega_2(p_1,p_2) = -\frac{1}{2}\log\frac{du}{dp} + \frac{\pi}{2\mathrm{Im}\,\tau} p^2. \tag{76}$$

As we already mentioned, for curves of higher genus the Bergman kernel also has only one double pole at coinciding arguments. This implies that S_1 for any genus will have similar structure as we found for genus one, i.e. it will include the term (21) plus some corrections.

The Bergman kernel, or the two-point function, are expressed above in terms of uniformizing parameters p. Sometimes it is convenient to express them in terms of the coordinate u which enters the algebraic equation (3) and the branch points $a_i = u(p_i^*)$ determined in (24). For a curve of genus one there are four branchpoints $a_1, \ldots, a_4$, and the corresponding two-point function has been found, using matrix model techniques, in [5]. This result can also be obtained, see [8], using properties of elliptic functions and rewriting the Bergman kernel given above, so that[15]

$$B(u_1,u_2) = \frac{1}{2(u_1-u_2)^2} + \frac{(a_3-a_1)(a_4-a_2)}{4\sqrt{\sigma(u_1)}\sqrt{\sigma(u_2)}}\frac{E(k)}{K(k)} + \\ + \frac{1}{4(u_1-u_2)^2}\left(\sqrt{\frac{(u_1-a_1)(u_1-a_4)(u_2-a_2)(u_2-a_3)}{(u_1-a_2)(u_1-a_3)(u_2-a_1)(u_2-a_4)}} + \right. \\ \left. + \sqrt{\frac{(u_1-a_2)(u_1-a_3)(u_2-a_1)(u_2-a_4)}{(u_1-a_1)(u_1-a_4)(u_2-a_2)(u_2-a_3)}}\right),$$

[15]Taking the common denominator of the two square roots, the dependence on branch points in numerator can be expressed in terms of symmetric functions of a_i, which leads to the formula presented in [16].

where

$$\sigma(u) = (u - a_1)(u - a_2)(u - a_3)(u - a_4)$$

and

$$k^2 = \frac{(a_1 - a_4)(a_2 - a_3)}{(a_1 - a_3)(a_2 - a_4)}$$

is the modulus of the complete elliptic functions of the first and second kind, $K(k)$ and $E(k)$, related to the parameter of the torus in (75) as $\tau = i\,K(1-k)/K(k)$.

In particular, the above expression for Bergman kernel was used in [9, 41] to determine several terms in the u-expansion of the two-point function, as well as a few lower order correlators W_n^g for mirror curves of genus one, for local $\mathbb{P}^2$ and local $\mathbb{P}^1 \times \mathbb{P}^1$. Nonetheless, these results are not sufficient to determine corrections $\hat{A}_1$ or $\hat{A}_2$ to the corresponding putative quantum curves, as the hierarchy of Eqs. (42) requires the knowledge of the exact dependence of S_k on both u and v. We plan to elucidate this point in future work.

3 Quantum Curves and Knots

As we already mentioned in the introduction, in applications to knots and 3-manifolds the polynomial $A(x, y)$ is a classical topological invariant called the A-polynomial. (For this reason, we decided to keep the name in other examples as well and, for balance, changed the variables to those used in the literature on matrix models and topological strings.) In this context, the quantum operator $\hat{A}$ is usually hard to construct (see [30, 31] for first indirect calculations and [19] for the most recent and systematic ones); therefore, any insight offered by an alternative method is highly desirable.

The study of such an alternative approach was pioneered in a recent work [16], which focused on the computation of the perturbative partition function (11) using the topological recursion of Eynard and Orantin [28]. Starting with a rather natural[16] prescription for the perturbative coefficients S_n in terms of W_n^g, the authors of [16] were able to match the perturbative expansion of the Chern–Simons partition function e.g. for the figure-8 knot complement [21] up to order $n = 4$, provided certain *ad hoc* renormalizations are made. It was also pointed out in [16] that such renormalizations are non-universal, i.e. knot-dependent. Motivated by these observations, we start with a different prescription for the S_n's described

[16]The choice of the prescription in [16] automatically incorporates the symmetries of the $SL(2,\mathbb{C})$ character variety, in particular, the symmetry of the A-polynomial under the Weyl reflection $x \mapsto x^{-1}$ and $y \mapsto y^{-1}$.

in Sect. 2.1, which appears to avoid the difficulties encountered in [16] and to reproduce the $SL(2,\mathbb{C})$ Chern–Simons partition function in all examples that we checked. In addition, we shift the focus to the A-polynomial itself, and describe how its quantization (17) can be achieved in the framework of the topological recursion.

3.1 Punctured Torus Bundle $-L^2R^2$

We start with a simple example of a hyperbolic 3-manifold M that can be represented as a punctured torus bundle over $\mathbf{S}^1$ with monodromy $\varphi = -L^2R^2$, where

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad , \quad R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{77}$$

are the standard generators of the mapping class group of a punctured torus, $\Gamma \cong PSL(2,\mathbb{Z})$. This 3-manifold has a number of nice properties. For example, it was considered in [23] as an example of a hyperbolic 3-manifold whose $SL(2,\mathbb{C})$ character variety has ideal points for which the associated roots of unity are not ± 1.

For this 3-manifold M, the A-polynomial has a very simple form[17]

$$A(x,y) = 1 + ix + iy + xy\,, \tag{78}$$

and its zero locus, $A(x,y) = 0$, defines a curve of genus zero. According to our criterion (67), this curve should be quantizable. Indeed, this can be shown either directly by verifying that all tame symbols $(x,y)_p$ are roots of unity or, alternatively [51], by noting that the polynomial $A(x,y)$ is *tempered*, which means that all of its face polynomials have roots at roots of unity. Either way, we conclude that the genus zero curve defined by the zero locus of (78) is quantizable in the sense of Sect. 2.4.

Therefore, we can apply the formula (21) from Sect. 2.1 to compute the one-loop correction $S_1(u)$ or, equivalently, the torsion $T(u)$. In fact, we can combine (16) and (21) to produce the following general formula

$$\boxed{\sum_{(m,n)\in\mathcal{D}} a_{m,n}\, c_{m,n}\, x^m y^n = \frac{1}{2}\left(\frac{du}{dp}\right)^{-2} \left(\frac{d^2u}{dp^2}\,\partial_v - \frac{du}{dp}\frac{dv}{dp}\,\partial_v^2\right) A} \tag{79}$$

[17] In fact, this polynomial occurs as a geometric factor in the moduli space of flat $SL(2,\mathbb{C})$ connections for infinitely many distinct incommensurable 3-manifolds [23] that can be constructed e.g. by Dehn surgery on one of the two cusps of the Neumann–Reid manifold (= the unique 2-cover of $m135$ with $H_1 = \mathbb{Z}/2 + \mathbb{Z}/2 + \mathbb{Z} + \mathbb{Z}$). Indeed, the latter is a two cusped manifold with strong geometric isolation, which means that Dehn surgery on one cusp does not affect the shape of the other and, in particular, does not affect the A-polynomial. As a result, all such Dehn surgeries have the same A-polynomial $A(x,y) = 1 + ix + iy + xy$ as the manifold $m135$.

that allows to determine the exponents $c_{m,n}$ of the q-deformation (15) directly from the data of the classical A-polynomial $A = \sum a_{m,n} x^m y^n$ and a parametrization (19).

In our present example, we can choose the following parametrization:

$$x(p) = -\frac{1+ip}{i+p} \quad , \quad y(p) = p \,, \tag{80}$$

suggested by the form of (78). Substituting it into (79) uniquely determines the values of the q-exponents $c_{m,n}$ and, therefore, the quantum operator (15):

$$\hat{A} = 1 + iq^{1/2}\hat{x} + iq^{-1/2}\hat{y} + q\hat{x}\hat{y} \,. \tag{81}$$

In order to fully appreciate how simple this derivation of $\hat{A}$ is (compared to the existent methods and to the full-fledged topological recursion) it is instructive to follow through the steps of Sects. 2.1 and 2.2 that, eventually, lead to the same result (81).

First, one needs to go through all the steps of the topological recursion. Relegating most of the details to Sect. 7, where (78) will be embedded in a larger class of similar examples (and dealing with various singular limits as presented in Sect. 6), we summarize here only the output of (22):

$$\begin{aligned} S_0' &= \log \frac{x-i}{ix-1} \,, \\ S_1' &= \frac{i-x}{2x+2i} \,, \\ S_2' &= \frac{x(5i-12x-5ix^2)}{12(1+x^2)^2} \,, \\ &\vdots \end{aligned}$$

which should be used as an input for (42). Indeed, from the first few equations in (44)–(46) one finds the perturbative expansion (6) of the quantum operator $\hat{A}$:

$$\begin{aligned} \hat{A}_1 &= \frac{1}{2}\big(i\hat{x} - i\hat{y} + 2\hat{x}\hat{y}\big) \,, \\ \hat{A}_2 &= \frac{1}{8}\big(i\hat{x} + i\hat{y} + 4\hat{x}\hat{y}\big) \,, \\ \hat{A}_3 &= \frac{1}{48}\big(i\hat{x} - i\hat{y} + 8\hat{x}\hat{y}\big) \,, \\ &\vdots \end{aligned}$$

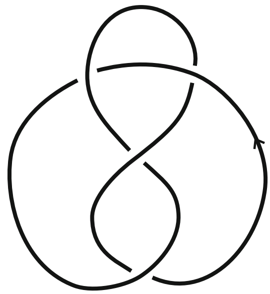


Fig. 2 Figure-8 knot

It does not take long to realize that the perturbative terms $\hat{A}_n$ come from the $\hbar$-expansion of the "quantum polynomial" (81) with $q = e^{\hbar}$. Pursuing the topological recursion further, one can verify this to arbitrary order in the perturbative $\hbar$-expansion, thus, justifying that $\hat{A}$ can be written in a nice compact form (15).

Hence, our present example provides a good illustration of how all these steps can be streamlined in a simple computational technique (49) which, for curves of genus zero, can be summarized in a single general formula (79).

3.2 Figure-8 Knot

The lesson in our previous example extends to more interesting knots and 3-manifolds, sometimes in a rather trivial and straightforward manner and, in some cases, with small new twists. The main conceptual point is always the same, though: at least in all examples that "come from geometry," *the full quantum curve $\hat{A}$ is completely determined by the first few terms in the $\hbar$-expansion, which can be easily obtained using the tools of the topological recursion.*

For example, let us consider the figure-8 knot complement, $M = \mathbf{S}^3 \setminus K$, for which the story is a little less trivial. The figure-8 knot is shown in Fig. 2. Much like our first example in this section, M is a hyperbolic 3-manifold that also can be represented as a punctured torus bundle with the monodromy

$$\varphi = RL = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} ,$$

where L and R are defined in (77). Even though the classical curve (56) for the figure-8 knot is hyper-elliptic, one can still easily find the torsion $T(u)$ needed for (16). In fact, for curves associated[18] with knots and 3-manifolds the torsion $T(u)$ is exactly what low-dimensional topologists call the Ray–Singer (or Reidemeister) torsion of a 3-manifold M. To be more precise, the function $T(u)$ is the torsion of

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[18] That is defined by the zero locus of the A-polynomial.

M twisted by a flat $SL(2,\mathbb{C})$ bundle $E_\rho \to M$ determined by the representation $\rho : \pi_1(M) \to SL(2,\mathbb{C})$ or, at a practical level, by the point $\rho = (x,y)$ on the classical curve $\mathscr{C}$.

In particular, $T(u)$ is a topological invariant of $M = \mathbf{S}^3 \setminus K$ and, therefore, can be computed by the standard tools. For instance, when ρ is Abelian, the torsion $T(u)$ is related to the Alexander–Conway polynomial $\nabla(K;z)$ [42, 50]:

$$\sqrt{T} = \frac{\nabla(K; x - x^{-1})}{x - x^{-1}} \tag{82}$$

that, for every knot K, can be computed by recursively applying a simple skein relation[19]

(83)

and the normalization $\nabla(\bigcirc) = 1$. Similarly, when ρ is non-Abelian (and irreducible) the torsion looks like

$$T(x) = \sqrt{\Delta(x)}\,, \tag{84}$$

where $\Delta(x)$ is the Alexander polynomial of M twisted by the flat $SL(2,\mathbb{C})$ bundle E_ρ, cf. [29]. For example, for the figure-8 knot that we are interested in here, it has the form [33, 47]:

$$\Delta_{\mathbf{4}_1}(x) = -x^{-4} + 2x^{-2} + 1 + 2x^2 - x^4\,. \tag{85}$$

Now we are ready to plug this data into our universal formula (16) and compute the quantum operator $\hat{A}$ or, at least, its first-order approximation. The computation is fairly straightforward; indeed, from (84) and (85) we find

$$\frac{\partial_u T}{T} = \frac{2(-1 + x^2 - x^6 + x^8)}{1 - 2x^2 - x^4 - 2x^6 + x^8} \tag{86}$$

and, by solving (56) we get $y^{(\alpha)}(x) = \frac{1-x^2-2x^4-x^6+x^8}{2x^4} \pm \frac{1-x^4}{2x^2}\sqrt{-\Delta(x)}$ which immediately gives the second part of the input data for (16), namely

$$\frac{\partial_u A}{\partial_v A} = -\frac{dv}{du} = \frac{2(2x^{-2} - 1 + 2x^2)}{\sqrt{-\Delta(x)}}\,. \tag{87}$$

[19] For example, $\nabla_{\mathbf{3}_1}(z) = 1 + z^2$ for the trefoil knot and $\nabla_{\mathbf{4}_1}(z) = 1 - z^2$ for the figure-8 knot. Note, that our definition of $T(u)$ is actually the inverse of the Ray–Singer torsion, as defined in the mathematical literature. This unconventional choice turns out to be convenient in other applications, beyond knots and 3-manifolds.

Then, once we plug these ingredients into (16) we come to our first surprise: we find that there is no way to satisfy (16) with constant real numbers $c_{m,n}$ if for $\mathscr{D}$ we simply take the Newton polygon of the classical curve (56). In other words, the figure-8 knot is a good illustration of the following phenomenon (that rarely happens in simple examples, but seems to be fairly generic in more complicated ones): one may need to enlarge the domain $\mathscr{D}$ in order to solve (16). For the figure-8 knot, the minimal choice is

$$A(x,y) = (1-x^4)x^4 - (1-x^4)(1-x^2-2x^4-x^6+x^8)y + (1-x^4)x^4y^2 \quad (88)$$

and differs from (56) by an extra factor $1 - x^4$. Now, with this $A(x, y)$, the formula (13) produces the set of coefficients $c_{m,n}$ or, equivalently, their "generating function"

$$\hat{A}_1 = (3-9\hat{x}^4)\hat{x}^4 - (-2\hat{x}^2 - 12\hat{x}^4 + 24\hat{x}^8 + 10\hat{x}^{10} - 12\hat{x}^{12})\hat{y} + (5-7\hat{x}^4)\hat{x}^4\hat{y}^2 , \quad (89)$$

which almost uniquely determines the full quantum A-polynomial for the figure-8 knot in Table 1:

$$\hat{A} = q^3(1-q^6\hat{x}^4)\hat{x}^4 - (1-q^4\hat{x}^4)(1-q^2\hat{x}^2-(q^2+q^6)\hat{x}^4-q^6\hat{x}^6+q^8\hat{x}^8)\hat{y} + q^5(1-q^2\hat{x}^4)\hat{x}^4\hat{y}^2 .$$

Indeed, if one knows that $\hat{A}$ is in the general form (15), then the above expression for $\hat{A}_1$ determines almost all of the coefficients in $\hat{A}$, except for the factor $q^2 + q^6$ which is easily fixed by going to the next order in the recursion.

3.3 Torus Knots and Generalizations

For a (m, n) torus knot, the classical curve (3) is defined by a very simple polynomial [13]:

$$A(x, y) = y - x^{mn} . \quad (90)$$

In fact, this curve is a little "too simple" to be an interesting example for quantization since it has only two monomial terms, whose relative coefficient in the quantum version

$$\hat{A}(\hat{x}, \hat{y}) = \hat{y} - q^c \hat{x}^{mn} \quad (91)$$

can be made arbitrary by a suitable canonical transformation, as discussed in Sect. 2.3. (Indeed, one can attain arbitrary values of c even with the simple shift transformation (53).) Another drawback of (90) is that, for general m and n, it describes a singular curve.

Both of these problems can be rectified by passing to a more general class of examples,

$$A(x, y) = y + P(x)\,, \tag{92}$$

where $P(x)$ can be either a polynomial or, more generally, an arbitrary function of x. Then, the A-polynomial (90) of (m, n) torus knots (and its quantization (91)) can be recovered as a limiting case of this larger family, $P(x) \to -x^{mn}$. Another important advantage of choosing generic $P(x)$ is that we can use (79) to find $\hat{A}$.

In practice, in order to implement the algorithm summarized in (49) and (79), it is convenient to exchange the role of x and y. Hence, we will work with the "mirror" version of (92):

$$A(x, y) = x + P(y)\,, \tag{93}$$

where $P(y)$ can be an arbitrary function of y. In general, the curve defined by the zero locus of this function is a multiple cover of the x-plane. It admits different parametrizations which, therefore, lead to different expressions for $\hat{A}$ (related by canonical transformations discussed in Sect. 2.3). However, one can always make a natural choice of parametrization with

$$\begin{cases} x(p) = -P(p) \\ y(p) = p \end{cases} \tag{94}$$

Substituting this into (16) (or, equivalently, into (79)) we find

$$\sum_{(m,n)\in\mathscr{D}} a_{m,n}\, c_{m,n}\, x^m y^n = \frac{x}{2} - \frac{y}{2}\frac{dP(y)}{dy} \tag{95}$$

which, for generic $P(y)$, immediately determines the quantization of (93):

$$\boxed{\hat{A} = q^{1/2}\hat{x} + P\left(q^{-1/2}\hat{y}\right).} \tag{96}$$

Notice, in spite of the suggestive notation, $P(y)$ does not need to be a polynomial in this class of examples. For instance, choosing $P(y)$ to be a rational function,

$$P(y) = \frac{1 + iy}{i + y} \tag{97}$$

from (96) we find the quantum curve,

$$q^{1/2}\hat{x} + \frac{q^{1/2} + i\,\hat{y}}{iq^{1/2} + \hat{y}} \;\simeq\; 0\,, \tag{98}$$

which, after multiplying by $iq^{1/2} + \hat{y}$ on the left and using the commutation relation $\hat{y}\hat{x} = q\hat{x}\hat{y}$, agrees with the earlier result (81).

4 Examples with $\hat{A} = A_{\text{classical}}$

In certain examples, it turns out that the quantum curve can be obtained from the classical one simply by replacing u and v by $\hat{u}$ and $\hat{v}$ with no additional $\hbar$ corrections (and with our standard ordering conventions, *cf.* Sect. 1). There are examples of such special curves in $\mathbb{C} \times \mathbb{C}$ as well as in $\mathbb{C}^* \times \mathbb{C}^*$; e.g. from (96) it is easy to see that $A(x, y) = x + 1/y$ is one example. In this section, for balance, we consider curves with this property defined by a polynomial equation $A(u, v) = 0$ in $\mathbb{C} \times \mathbb{C}$. In particular, we discuss in detail a family of examples related to the Airy function,[20] in order to explain how our formalism works for curves embedded in $\mathbb{C} \times \mathbb{C}$.

The Airy function (and its cousins) can be defined by a contour integral,

$$Z_{\text{Ai}}(u) = \int_\gamma \frac{dz}{2\pi i}\, e^{-\frac{1}{\hbar}S(z)}\,, \qquad S(z) = -uz + \frac{z^3}{3} \tag{99}$$

over a contour γ that connects two asymptotic regions in the complex z-plane where the "action" S behaves as $\operatorname{Re} S(z) \to +\infty$. For such a contour γ, we have the following Ward identity:

$$0 = \frac{1}{2\pi i} \int_\gamma d\left[e^{-\frac{1}{\hbar}S(z)}\right] = \frac{1}{\hbar} \int_\gamma \frac{dz}{2\pi i} \left(u - z^2\right) e^{-\frac{1}{\hbar}S(z)}$$

which we can write in the form of the differential equation

$$\left(\hat{v}^2 - u\right) Z_{\text{Ai}}(u) = 0 \tag{100}$$

where we used the definition of $Z_{\text{Ai}}(x)$ and

$$\hat{v}^2 Z_{\text{Ai}}(x) = (\hbar\partial_u)^2 \int_\gamma \frac{dz}{2\pi i}\, e^{-\frac{1}{\hbar}S(z)} = \int_\gamma \frac{dz}{2\pi i}\, z^2\, e^{-\frac{1}{\hbar}S(z)}\,. \tag{101}$$

This simple, yet instructive, example is a prototype for a large class of models where quantum curves are identical to the classical ones, i.e. $\hat{A} = A(u, v)$. Indeed, let us consider a contour integral,

$$Z(u) = \int_\gamma \frac{dz}{2\pi i}\, e^{-\frac{1}{\hbar}S(z)}\,, \qquad S(z) = -uz + P(z)$$

[20] In this model, computation of W_n^g and their generating functions are also presented in [28].

where γ is a suitable contour in the complex z-plane, and $P(z)$ is a Laurent polynomial. Then, following the same arguments as in the example of the Airy function, we obtain the following Ward identity

$$\int_\gamma \frac{dz}{2\pi i}\left(u - P'(z)\right) e^{-\frac{1}{\hbar}S(z)} = 0$$

which translates into a differential equation $\hat{A}Z(u) = 0$ with

$$\hat{A} = P'(\hat{v}) - \hat{u}\,. \tag{102}$$

The special choice of $P'(z) = z^p$ gives rise to $(p, 1)$ minimal model coupled to gravity. In this case, the corresponding partition function has an interpretation of the amplitude of the FZZT brane [40], and in the dual matrix model this partition function is indeed computed as the expectation value of the determinant (12). Recall, that a double scaling limit of hermitian matrix models with polynomial potentials describes (p, q) minimal models coupled to gravity, characterized by singular spectral curves [14]:

$$A(u, v) = v^p - u^q = 0\,. \tag{103}$$

In the simpler case of $q = 1$ discussed here the classical Riemann surface $P'(v) - u = 0$, given by the $\hbar \to 0$ limit of the quantum curve (102), represents the semi-classical target space of the minimal string theory. Below we discuss in detail how the above $\hat{A}$ arises from our formalism in the Airy case, $p = 2$.

4.1 Quantum Airy Curve

For a minimal model with $(p, q) = (2, 1)$ the classical curve (103) looks like

$$A(u, v) = v^2 - u = 0\,. \tag{104}$$

It has two branches labeled by $\alpha = \pm$,

$$v = S_0' = \pm\sqrt{u} = v^{(\pm)}\,, \tag{105}$$

and exchanged by the Galois transformation[21]

$$v \to -v\,.$$

[21] By definition, the action of the Galois group preserves the form of the curve (104).

This model provides an excellent example for illustrating how the hierarchy of differential equations (Sect. 2.2) and the topological recursion (Sect. 2.1) work. Because we already know the form of the quantum curve in this example, we start by deriving the $\hbar$ expansion of the Airy function using the hierarchy (42). Then, we will show that this expansion is indeed reproduced by the topological recursion. In examples considered later we will also illustrate the reverse process: from the knowledge of S_k (computed from the topological recursion) we will determine the form of the quantum curve.

In our calculations, we will use global coordinates, such as v or p, and avoid using the coordinate u (that involves a choice of branch of the square root) except for writing the final result. In particular, from the equation of the Airy curve (104) we find the relation

$$v' = \frac{dv}{du} = -\frac{\partial_u A(u,v)}{\partial_v A(u,v)} = \frac{1}{2v} \tag{106}$$

that will be useful below.

4.1.1 Differential Hierarchy

First, we solve the hierarchy of equations that follow from the quantum curve (100):

$$\hat{A} Z_{\mathrm{Ai}} = \big(\hbar^2 \partial_u^2 - u\big) Z_{\mathrm{Ai}} = 0\,. \tag{107}$$

To solve this equation in variable u, already in the first step one would have to make a choice of the branch (105). This would influence then all higher order equations in the differential hierarchy, and eventually lead to two well-known variants of the Airy function. Instead, we express the coefficients S_k in a universal way in terms of v, so that a particular solution in terms of u can be obtained by evaluating v in the final expression on either branch (105).

The first equation in the differential hierarchy is already given in (105), i.e. $v = S_0'$. The second equation (45) takes form

$$S_1' \partial_v A(u,v) + \frac{1}{2} S_0'' \partial_v^2 A(u,v) = 0\,,$$

and implies

$$S_1' = -\frac{v'}{2v} = -\frac{1}{4v^2}.$$

Solving further Eqs. (42) we find

$$S_2' = \frac{-1-8vv'}{32v^5} = -\frac{5}{32v^5}\,, \quad S_3' = -\frac{5(1+10vv')}{128v^8} = -\frac{15}{64v^8}\,, \quad S_4' = -\frac{1105}{2048v^{11}}\,.$$

We can integrate these results taking advantage of (106) to find

$$S_k = \int \frac{S_k'}{v'} dv\,. \tag{108}$$

In particular, the first few terms look like

$$S_0 = \frac{2}{3}v^3\,, \quad S_1 = -\frac{1}{2}\log\,v\,, \quad S_2 = \frac{5}{48v^3}\,, \quad S_3 = \frac{5}{64v^6}\,, \quad S_4 = \frac{1105}{9216v^9}\,. \tag{109}$$

Finally, using (105) we can evaluate these expressions on either of the two branches $v^{(\pm)} = \pm\sqrt{u}$ to find two asymptotic expansions of the Airy function (99) (often denoted Bi and Ai),

$$Z_{\mathrm{Ai}}^{(\pm)}(u) = \frac{1}{u^{1/4}}\exp\Big(\pm \frac{2u^{3/2}}{3\hbar} \pm \frac{5\hbar}{48u^{3/2}} + \frac{5\hbar^2}{64u^3} \pm \frac{1105\hbar^3}{9216u^{9/2}} + \ldots \Big)\,, \tag{110}$$

which indeed satisfy the second order equation (107).

4.1.2 Topological Recursion

Now we reconsider the Airy curve from the topological recursion viewpoint. The classical curve can be parametrized as

$$\begin{cases} u(p) = p^2 \\ v(p) = p \end{cases}$$

The conjugate point is simply $\bar{p} = -p$, and there is one branch-point at $p = 0$. All ingredients of the recursion can be found in the exact form, in particular the anti-derivative and the recursion kernel take the following form

$$S_0(p) = \int^p \phi = \frac{2}{3}p^3\,, \qquad K(q,p) = \frac{1}{4q(p^2-q^2)}\,.$$

The annulus amplitude gives

$$S_1 = -\frac{1}{2}\log\frac{du}{dp} = -\frac{1}{2}\log(2v)\,,$$

which correctly reproduces S_1 found in (109) (up to an irrelevant constant).

Now we apply the topological recursion to find the higher order terms S_k with $k \geq 2$. These terms are computed as functions on the curve, i.e. as functions of the parameter p, and can be expressed as rational functions of u and v. In particular we find

$$W_1^1(p) = -\frac{1}{16p^4}, \qquad W_3^0(p_1, p_2, p_3) = -\frac{1}{2p_1^2 p_2^2 p_3^2},$$

which implies

$$S_2 = \int^p W_1^1(p)dp + \frac{1}{6}\iiint^p W_3^0(p_1, p_2, p_3)dp_1 dp_2 dp_3 = \frac{5}{48v^3}.$$

In higher orders, we get

$$S_3 = \frac{5}{64v^6}, \qquad S_4 = \frac{1105}{9216v^9}.$$

These results agree with the expansion (109) obtained from the differential hierarchy. It is clear that, had we not known the form of the quantum curve to start with, we could compute the coefficients S_k using the topological recursion and then apply the hierarchy of differential equations (42). This would reveal that all quantum corrections $\hat{A}_k$ vanish, and the quantum curve indeed takes the form (107) and coincides with the classical curve.

Let us also illustrate the factorization of the quantum curve (55) to the leading order in $\hbar$. In the polarization where p is the "coordinate," the curve (107) takes the form

$$\hat{A} = \big(\hbar\partial_p - 2p^2\big)\big(\hbar\partial_p + 2p^2\big) + \mathcal{O}(\hbar).$$

Then, to the leading order, the two branches of the partition function are annihilated by the operators $(\hbar\partial_p \mp 2^2 p)$ and the solutions to these equations represent the two variants of the Airy function (110):

$$Z = e^{\pm\frac{2p^3}{3\hbar}}\big(1 + \mathcal{O}(\hbar)\big) = e^{\pm\frac{2u^{3/2}}{3\hbar}}\big(1 + \mathcal{O}(\hbar)\big).$$

5 $c = 1$ Model

The aim of this section is to analyze the so-called $c = 1$ model. As in the previous section, however, it is instructive to start with a more a general class of models associated with the contour integral

$$Z(u) = \int_\gamma \frac{dz}{2\pi i}\, z^{\frac{t}{\hbar}}\, e^{-\frac{1}{\hbar}S(z)}, \qquad S(z) = -uz + \frac{z^{n+1}}{n+1}.$$

This integral satisfies the following Ward identity

$$\int_\gamma \frac{dz}{2\pi i} \left(\frac{t}{z} + u - z^n \right) z^{\frac{t}{\hbar}} e^{-\frac{1}{\hbar} S(z)} = 0$$

that leads to the quantum curve

$$\hat{A} = t + \hat{v} \left(\hat{u} - \hat{v}^n \right) .$$

In the special case $n = 1$ this reproduces the quantum curve of the $c = 1$ model:

$$\hat{A} = t + \hat{v}\hat{u}$$

where we used the freedom of shifting u by an arbitrary function of v to implement a change of polarization $\hat{u} \to \hat{u} + \hat{v}$, *cf.* Sect. 2.3. (Note that this shift does not affect the commutation relations of $\hat{u}$ and $\hat{v}$.) Another convenient choice of polarization is implemented by a canonical transformation

$$\hat{u} \to \frac{1}{\sqrt{2}} \left(\hat{u} - \hat{v} \right) , \qquad \hat{v} \to \frac{1}{\sqrt{2}} \left(\hat{u} + \hat{v} \right)$$

and leads to a perhaps more familiar representation of the quantum curve for the $c = 1$ model:

$$\hat{A} = \left(\hat{u} + \hat{v} \right) \left(\hat{u} - \hat{v} \right) + 2t = \hat{u}^2 - \hat{v}^2 + 2t + \hbar \, . \tag{111}$$

In what follows we consider this last form of the quantum curve. Note, in this case the underlying classical curve is embedded in $\mathbb{C} \times \mathbb{C}$ by the equation

$$A(u, v) = u^2 - v^2 + 2t = 0 \, , \tag{112}$$

and has two branches $v^{(\alpha)}$ labeled by $\alpha = \pm$,

$$v^{(\pm)}(u) = \pm \sqrt{u^2 + 2t} \, . \tag{113}$$

These branches are mapped to each other by a Galois transformation

$$v \to -v \, ,$$

that does not change the form of the curve (112). We also note that

$$v' = \frac{dv}{du} = -\frac{\partial_u A(u, v)}{\partial_v A(u, v)} = \frac{u}{v} \, . \tag{114}$$

The solution of the $c = 1$ model is well known. In particular, the associated closed string free energies, for $g \geq 2$, are given by

$$F_g = \frac{B_{2g}}{2g(2g-2)} \frac{1}{t^{2g-2}} \,. \tag{115}$$

Below we reexamine this model in the new formalism, in particular from the viewpoint of open (rather than closed) string invariants. Since the quantum curve (111) has only the first order quantum correction $\hat{A}_1 = 1$, we start by verifying that it is indeed correctly reproduced by the annulus amplitude (21) in our formalism. Then, we follow the strategy employed in the previous section and show that higher order amplitudes S_k, determined by the quantum curve equation, agree with those given by the topological recursion. Equivalently, this guarantees that, had we computed S_k first by applying the topological recursion to the classical curve (112) and then determined the quantum curve using the hierarchy (42), we would indeed find $\hat{A}$ given in (111).

5.1 Differential Hierarchy

The differential hierarchy (42) starts with the equation which, as usual, specifies the disk amplitude; integrating (113) we find that it takes the form

$$S_0 = \pm\Big(\frac{1}{2}u\sqrt{u^2+2t} + t\log(u+\sqrt{u^2+2t})\Big) \,. \tag{116}$$

The second equation in the differential hierarchy (45) implies that

$$S_1' = \frac{A_1 - v'}{2v} = \frac{A_1 v - u}{2v^2} \,, \tag{117}$$

with $A_1 = 1$. The first (and the only) quantum correction $A_1 = 1$ follows directly from (111) as well as from the annulus amplitude which we compute below in (120).

To find the higher order amplitudes S_k we take advantage of the fact that all higher order corrections to the quantum curve (111) vanish. Therefore, using the fact that the first correction $A_1 = 1$ is annihilated by all $\mathfrak{D}_{r>0}$, all higher order equations in the hierarchy (42) take a simple form $\mathfrak{D}_n A = 0$. Moreover, noting that the classical curve is quadratic in v, the hierarchy of differential equations reduces to

$$\begin{aligned}
0 &= S_2'\partial_v A + \frac{1}{2}\big((S_1')^2 + S_1''\big)\partial_v^2 A \,, \\
0 &= S_3'\partial_v A + \Big(\frac{S_2''}{2} + S_1'S_2'\Big)\partial_v^2 A \,, \\
0 &= S_4'\partial_v A + \frac{1}{2}\Big((S_2')^2 + S_3'' + 2S_1'S_3'\Big)\partial_v^2 A \,, \\
&\ \ \vdots
\end{aligned}$$

and solving these equations we get

$$\begin{aligned} S_1' &= \frac{v-u}{2v^2}\,, \\ S_2' &= \frac{-5u^2+4uv+v^2}{8v^5}\,, \\ S_3' &= -\frac{(u-v)(3u-v)(2u+v)}{16v^8}\,, \\ S_4' &= -\frac{(u-v)(1105u^3+145u^2v-389uv^2-21v^3)}{128v^{11}}\,. \end{aligned} \tag{118}$$

We stress that given here are global solutions; in order to restrict to a particular branch one needs to substitute $v = v^{(\pm)}$ using (113). Making such a choice of branch and expanding in u we find

$$\begin{aligned} S'_{1,\pm} &= \pm\frac{1}{2\sqrt{2t}} - \frac{u}{4t} \mp \frac{u^2}{4(2t)^{3/2}} + \frac{u^3}{8t^2} \pm \frac{3u^4}{16(2t)^{5/2}} - \frac{u^5}{16t^3} + \ldots \\ S'_{2,\pm} &= \pm\frac{1}{8(2t)^{3/2}} + \frac{u}{8t^2} \mp \frac{13u^2}{16(2t)^{5/2}} - \frac{u^3}{8t^3} \pm \frac{115u^4}{64(2t)^{7/2}} + \frac{3u^5}{32t^4} + \ldots \\ S'_{3,\pm} &= \mp\frac{5}{16(2t)^{5/2}} + \frac{5u}{64t^3} \pm \frac{75u^2}{32(2t)^{7/2}} - \frac{15u^3}{64t^4} \mp \frac{875u^4}{128(2t)^{9/2}} + \frac{45u^5}{128t^5} + \ldots \\ S'_{4,\pm} &= \mp\frac{21}{128(2t)^{7/2}} - \frac{23u}{128t^4} \pm \frac{1215u^2}{256(2t)^{9/2}} + \frac{19u^3}{32t^5} \mp \frac{29387u^4}{1024(2t)^{11/2}} - \frac{265u^5}{256t^6} + \ldots \end{aligned}$$

Integrating these results term by term gives the u expansion of S_k. One can also find the global representation of S_k in terms of u and v using the integral (108) and the result (114); we determine such a global representation below.

5.2 *Topological Recursion*

Now we show how the above results can be reproduced using the topological recursion. The curve (112) can be parametrized as

$$\begin{cases} u(p) = 2pt - \frac{1}{4p} \\ v(p) = 2pt + \frac{1}{4p} \end{cases} \tag{119}$$

Note, this implies that a local parameter p can be expressed as

$$p = \frac{u+v}{4t}\,.$$

Having fixed the parametrization, we can compute the annulus amplitude (21) and find that its derivative in this case is

$$S_1' = \frac{v-u}{2v^2}\,. \tag{120}$$

Comparing this with (117) we confirm that the first quantum correction to the A-polynomial indeed reads

$$A_1 = 1\,,$$

in complete agreement with (111).

The other ingredients of the topological recursion are as follows. There are two branch points $du(p_*) = 0$ at

$$p_* = \pm\frac{i}{2\sqrt{2t}}\,. \tag{121}$$

Conveniently, there is a global expression for the conjugate point

$$\overline{p} = -\frac{1}{8tp}\,, \tag{122}$$

and the recursion kernel reads

$$K(q,z) = \frac{4q^3}{(1-8q^2t)(q-z)(1-8qtz)}\,. \tag{123}$$

The correlators contributing to (23) take form

$$\begin{aligned}
W_1^1(p) &= \frac{64p^3t}{(1+8p^2t)^4}\,,\\
W_1^2(p) &= \frac{86016t(p^7-24p^9t+64p^{11}t^2)}{(1+8p^2t)^{10}}\,,\\
W_1^3(p) &= \frac{2883584p^{11}t(135-8784p^2t+133376p^4t^2-562176p^6t^3+552960p^8t^4)}{(1+8p^2t)^{16}}\,,
\end{aligned}$$

and so on. Hence, using (23) we get the global representation

$$S_2 = \frac{2t(2t-9(u+v)^2)}{6(2t+(u+v)^2)^3}\,, \qquad S_3 = \frac{20t(u+v)^4(2t-(u+v)^2)}{(2t+(u+v)^2)^6}\,,$$

and derivatives of these functions with respect to u indeed agree with our earlier results (118). Therefore, the results of the topological recursion are in excellent agreement with (111). Again, had we not known the quantum curve to start with, we could reverse the order of the computation and from the knowledge of the coefficients S_k determine

$$\hat{A} = u^2 - (\hbar\partial_u)^2 + 2t + \hbar\,. \tag{124}$$

Finally, we illustrate the factorization property (55) of the quantum curve in p-polarization. In this polarization, the curve (111) gives rise to first order differential operators $\left(\hbar\partial_p \mp \frac{(1+8tp^2)^2}{16p^3}\right)$ which (to the leading order in $\hbar$) annihilate the two branches of the partition function:

$$Z^{(\alpha)} = e^{\pm\frac{1}{\hbar}\left(-\frac{1}{32p^2}+2t^2p^2+t\log p\right)}\Big(1+\mathcal{O}(\hbar)\Big)\,.$$

After substituting $p = (u+v)/4t$ and v given by (113) we indeed reproduce the leading behavior (116).

Let us also mention that from W_1^2 and W_1^3 computed here one can determine the "closed string" free energies (38). This computation reveals that

$$F_2 = -\frac{1}{240t^2}\,, \qquad F_3 = \frac{1}{1008t^4}$$

in excellent agreement with the expected result (115), thereby, providing yet another nice check of the topological recursion formalism.

6 Tetrahedron or Framed $\mathbb{C}^3$

In this section we consider quantization of a classical curve that plays a very important role simultaneously in two different areas: in low-dimensional topology and in topological string theory.

One of the problems in low-dimensional topology is to associate quantum group invariants to 3-manifolds (possibly with boundary). Topological string theory, on the other hand, computes various enumerative invariants of Calabi–Yau threefolds (possibly with extra branes). In both cases, the computation is usually done by decomposing a 3-manifold (resp. a Calabi–Yau threefold) into elementary pieces, for which the invariants are readily available. As basic building blocks, one can take e.g. tetrahedra and patches of $\mathbb{C}^3$, respectively. Indeed, just like 3-manifolds can be built out of tetrahedra, Calabi–Yau threefolds can be constructed by gluing local patches of the $\mathbb{C}^3$ geometry. For this reason, a tetrahedron might be called the "simplest 3-manifold," whereas $\mathbb{C}^3$ might be called "the simplest Calabi–Yau."

Furthermore, in both cases the invariants associated to these basic building blocks involve dilogarithm functions (classical and quantum). In quantum topology, the quantum dilogarithm is the $SL(2)$ invariant associated to an ideal tetrahedron, from which one can construct partition function of $SL(2)$ Chern–Simons theory

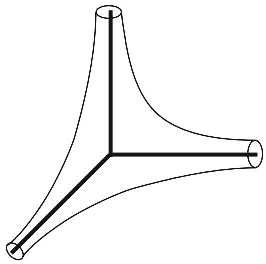

Fig. 3 Mirror curve for $\mathbb{C}^3$ geometry

on a generic 3-manifold [19, 21]. Similarly, the partition function of a local toric Calabi–Yau threefold (with branes) can be computed by gluing several copies of the *topological vertex* associated to each $\mathbb{C}^3$ patch [4] and, in the most basic case, the answer reduces to the quantum dilogarithm function.

As in many other examples discussed in this paper, the exact solution to both of these problems is determined by a quantization of a classical algebraic curve. The complex curve associated to an ideal tetrahedron is simply the zero locus of the A-polynomial $A(x, y) = 1 + x + y$, *cf.* Sect. 3. In topological string theory, this is the mirror curve for the $\mathbb{C}^3$ geometry. More precisely, there is a whole family of such curves labeled by the so-called framing parameter f, such that[22]

$$A(x, y) = 1 + y + xy^f , \tag{125}$$

where $x, y \in \mathbb{C}^*$ and, as usual, $x = e^u$ and $y = e^v$. The curve (125) can be visualized by thickening the edges of the toric diagram of $\mathbb{C}^3$, as shown in Fig. 3. Various choices of framing are related by symplectic transformations $(x, y) \mapsto (xy^f, y)$, under which closed string amplitudes F_g are invariant, while W_n^g and S_k transform as discussed in Sect. 2.3.

For integer values of f, the curve (125) is an f-sheeted cover of the x-plane. There are various possible choice of parametrization of this curve, which can be related by Galois transformations. In following subsections, we find the corresponding quantum curves from several perspectives. First, in Sect. 6.1, we choose one very natural parametrization and determine the corresponding quantum curve for arbitrary f. Then, in Sect. 6.2 we set $f = 2$ and demonstrate how the form of the quantum curve changes under a change of parametrization. Finally, in Sect. 6.3, we discuss some special choices of framing, $f = 0$ and $f = 1$, for which the topological recursion cannot be applied directly, but the answer can nevertheless be obtained by treating f as a continuous parameter and considering limits of our results for generic f.

---

[22] One can invert the curve equation [9, 10] to find the expansion $y(x) = -1 + \sum_{k=1}^{\infty} (-1)^{k(f+1)} \frac{(-kf+k-2)!}{(-kf-1)!k!} x^k$ (where the factorial function with negative argument is understood as the appropriate Γ-function).

6.1 General Framing

We wish to find a quantum version of the curve defined by the zero locus of (125),

$$\mathscr{C}: \quad 1 + y + xy^f \ = 0\,. \tag{126}$$

As we explained earlier, the answer depends on the choice of parametrization. Here we make the most convenient choice

$$\begin{cases} u(p) = \log \frac{-1-p}{p^f} \\ v(p) = \log p \end{cases} \tag{127}$$

such that $x(p)$ and $y(p)$ are rational functions. As one can easily verify, these rational functions have trivial tame symbols (66) at all points $p \in \mathscr{C}$, which means [43] that our K-theory criterion (67) is automatically satisfied and the curve (126) should be quantizable for all values of f.

In fact, we can immediately make a prediction for what the form of the quantum curve should be, by writing the classical curve (125) in the form $A(x, y) = x + P(y)$, with $P(y) = (1+y)y^{-f}$. This is the same form as we considered in (93), and the parametrization (127) is consistent with the one in (94). Therefore, (96) implies that the quantization of (125) is

$$\hat{A} = 1 + q^{-1/2}\hat{y} + q^{(f+1)/2}\hat{x}\hat{y}^f\,. \tag{128}$$

This result, however, is based only on the first quantum correction (21) and the assumption that all higher-other corrections can be summed up into factors of q. Now we wish to show that this is indeed the case by a direct analysis of the higher order corrections.

Our first task is to determine the subleading terms S_n in the wave-function (11) associated to the curve (126). In order to use the topological recursion, we first need to find the branch points. Solving Eq. (24) we find a single branch point at

$$x_* = -\frac{f^f}{(1-f)^{1-f}}\,, \qquad y_* = p_* = \frac{f}{1-f}\,. \tag{129}$$

Note, this result is our first hint that special values of framing $f = 0, 1$ require extra care: one can not simply set $f = 0$ or $f = 1$ from the start since for those values (129) gives $y_* \notin \mathbb{C}^*$. In these exceptional cases, our strategy will be to carry out all computations with f generic, and then set $f = 0$ or $f = 1$ only in the final expressions.

The next ingredient we need is the location of the conjugate point $\overline{p}$ introduced in (25). For the above curve, the value of $\overline{p}$ cannot be found in closed form. However, if we write

$$p = p_* + r\,, \tag{130}$$

we can find the conjugate point as a power series expansion in a local coordinate r

$$\overline{p} = p_* - r + \frac{2(1-f^2)r^2}{3f} - \frac{4(1-f^2)^2 r^3}{9f^2} + \frac{2(1-f)^3(22+57f+57f^2+22f^3)r^4}{135f^3} + \mathcal{O}(r^5)\,.$$

The remaining ingredients of the recursion are the following. Because the curve (126) has genus zero, the Bergman kernel is given by a simple formula (27). We also find ω and $dE_q(p)$ and hence determine the recursion kernel (28). Using local coordinates q and r centered at the branch point (130), the recursion kernel has a q-expansion that starts with

$$K(q,r) = \frac{f^2}{2(1-f)^4 r^2 q} + \frac{f(1+f)}{2(1-f)^3 r^2} + + \frac{f\big(2f^2 r(-1+2r) + 2r(1+2r) + f(3-8r^2)\big)\,q}{6(1-f)^4 r^4} + \mathcal{O}(q^2)\,.$$

Even though we do not make much use of the anti-derivative, we mention that it can be found in the exact form,

$$S_0(r) = -\frac{f}{2}\log\Big(r + \frac{f}{1-f}\Big)^2 + \log\Big(r + \frac{f}{1-f}\Big)\log\Big(\frac{1+(1-f)r}{1-f}\Big) + \mathrm{Li}_2\Big(\frac{f+(1-f)r}{-1+f}\Big),$$

expressed in a local coordinate r, *cf.* (130).

Using all these ingredients, the topological recursion leads to the following results for the amplitudes (23):

$$S_2 = -\frac{f^2\big(-3+(-1+f)f\big) + (-1+f)f\big(3+f(-3+2f)\big)y + (-1+f)^4 y^2}{24(-1+f)\big(f+(-1+f)y\big)^3},$$

$$S_3 = \frac{\left[\begin{array}{l} fy(1+y)\big(-2+8f^2-(-1+f)(1+y) \\ \qquad (2-2y+f(2+7y+f(2-7y+2f(1+y))))\big) \end{array}\right]}{48\big(f+(-1+f)y\big)^6},$$

and so on. We again stress that we obtain these amplitudes in a closed form, defined globally on the curve. Now, in turn, we can apply the hierarchy of Eqs. (42) to determine corrections $\hat{A}_k$ to the curve (125). To this end, we also need the following derivatives

$$\frac{dy}{dx} = -\frac{y^{1+f}}{y + fxy^f}\,,$$
$$\frac{d^2y}{dx^2} = \frac{fy(1+2f)(2y+(1+f)xy^f)}{(y+fxy^f)^3}\,,$$

etc. Substituting the leading (20) and the subleading (21) terms

$$x\partial_x S_0 = \log y\,,$$
$$S_1 = -\frac{1}{2}\log\frac{y-f-fy}{y(y+1)}\,,$$

into the hierarchy (42) we find the first few quantum corrections

$$\hat{A}_1 = -\frac{1}{2}(1+f+2\hat{y}+f\hat{y})\,,$$
$$\hat{A}_2 = \frac{1}{8}\big((1+f)^2+(2+f)^2\hat{y}\big)\,,$$
$$\hat{A}_3 = -\frac{1}{48}\big((1+f)^3+(2+f)^3\hat{y}\big)\,.$$

These corrections clearly arise from the $\hbar$-expansion of $e^{-(f+1)\hbar/2} + e^{-(1+f/2)\hbar}y + xy^f$. Equivalently, choosing a slightly different overall normalization constant, the quantum curve reads

$$\hat{A} = 1 + q^{-1/2}\hat{y} + q^{(f+1)/2}\hat{x}\hat{y}^f\,,$$

in perfect agreement with the original prediction (128).

Let us mention that one can also compute from the topological recursion the coefficients F_g defined in (38). As shown in [10], for the mirror $\mathbb{C}^3$ curve this leads to the $\hbar$-expansion of the square root of the MacMahon function, in agreement with the closed topological string free energy. For more complicated toric manifolds (like generalized conifolds analyzed in Sect. 7) the corresponding constant contributions to the (closed) partition functions turn out to be given by multiplicities of the MacMahon function. They are also reproduced by the topological recursion computation, which in this case can be interpreted in terms of a pant decomposition of the mirror curve, and mirrors A-model localization computation [10].

We can also demonstrate that the form of the above quantum curve is consistent with, and annihilates the B-brane partition function in the topological string theory, if conventions are adjusted appropriately. The B-brane partition function, in arbitrary framing f, in the topological vertex formalism, can be represented as[23]

$$\begin{aligned}\psi_f(x\,q^f) &:= \sum_{\mu}(-1)^{f|\mu|}e^{\frac{f}{2}\hbar\kappa(\mu^t)}s_{\mu^t}(x\,q^f)\,C_{\phi\phi\mu}(e^{\hbar},e^{\hbar})\\ &= \sum_{\mu=0}^{\infty}\frac{(-1)^{(f+1)\mu}e^{\frac{f}{2}\hbar\mu(\mu-1)}e^{\hbar\mu(f+1/2)}x^{\mu}}{(1-e^{\hbar})\cdots(1-e^{\mu\hbar})}\\ &= \sum_{\mu=0}^{\infty}\frac{(-1)^{(f+1)\mu}q^{\frac{\mu}{2}+\frac{f}{2}\mu(\mu+1)}x^{\mu}}{(1-q)\cdots(1-q^{\mu})}\,, \end{aligned} \tag{131}$$

where $|\mu|$ is the total number of boxes in the partition μ. As the Schur function s_{μ^t} with a single argument forces partitions involved to be effectively one-dimensional, in the second line we changed the domain of summation to integers. Also note that a general expression

$$\kappa(\mu) = |\mu| + \sum_i(\mu_i^2 - 2i\mu_i) \tag{132}$$

in our case gives

$$\kappa(\mu) = \mu + \sum_{i=1}^{\mu}(1-2i) = -\mu(\mu-1) \tag{133}$$

and $\kappa(\mu^t) = \mu(\mu-1)$. The functions ψ_f can be interpreted as framed invariants of the unknot on the three-sphere. Let us now write $\psi_f(x\,q^f) = \sum_{\mu=0}^{\infty} a_\mu$, with

$$a_\mu = \frac{(-1)^{(f+1)\mu}q^{\frac{\mu}{2}+\frac{f}{2}\mu(\mu+1)}x^{\mu}}{(1-q)\cdots(1-q^{\mu})}\,. \tag{134}$$

---

[23]We shifted the argument x by q^f to match our conventions with the topological vertex ones. Also note, that for framing f, one has

$$\langle \mathrm{Tr}\,U^m\rangle = \frac{[m+fm-1]!}{m[fm]![m]!}\,,$$

where $[x] = q^{x/2} - q^{-x/2}$ is the q-number. Notice that for $f=0$ it reduces to $\frac{1}{m[m]}$, which is the answer for zero framing leading to the dilogarithm. We do not know a product formula for

$$\sum_{m=1}^{\infty}\frac{[m+fm-1]!}{m[fm]![m]!}x^m\,.$$

Then,

$$\frac{a_{\mu+1}}{a_\mu} = -x\,\frac{(-1)^f q^{\frac{1}{2}+f(\mu+1)}}{(1-q^{\mu+1})}\,, \tag{135}$$

so that

$$(1-q^{\mu+1})a_{\mu+1} = -x(-1)^f q^{\frac{1}{2}+f(\mu+1)} a_\mu \tag{136}$$

Summing over μ, we get

$$\big(1-\hat{y}+q^{f+1/2}\hat{x}(-\hat{y})^f\big)\,\psi_f(x\,q^f) = 0.$$

As we stressed before, there is a freedom of shifting the subleading S_1 term in the partition function by a linear term in u. To match to our conventions we define $Z_f(x) = x^{1/2}\psi_f(x\,q^f)$, and commuting the additional $x^{1/2}$ in the above equation we find that

$$\big(1+q^{-1/2}(-\hat{y})+q^{(f+1)/2}\hat{x}(-\hat{y})^f\big)\,Z_f(x) = 0.$$

Therefore, up to a sign of $\hat{y}$ which also is a matter of convention, we reproduce the quantum curve which we found in (128) in our formalism.

6.2 Framing $f = 2$

So far we discussed mirror curve for $\mathbb{C}^3$ geometry in an arbitrary framing f, but with a special choice of parametrization. Now we do roughly the opposite, and discuss how the form of the quantum curve depends on the choice of parametrization, but with a particular choice of framing $f = 2$,

$$A(x,y) = 1 + y + xy^2\,. \tag{137}$$

This curve has two branches $y^{(\alpha)}$ labeled by $\alpha = \pm$, such that

$$y^{(\pm)} = \frac{-1\pm\sqrt{1-4x}}{2x}\,.$$

We note that these two branches are mapped to each other by the Galois transformation

$$x \mapsto x\,, \qquad y \mapsto \frac{1}{xy} \tag{138}$$

that preserves the form of the curve (137). From the equation of the curve we also have

$$\frac{dy}{dx} = -\frac{A_x}{A_y} = -\frac{y^2}{1+2xy}\,,$$
$$\frac{d^2y}{dx^2} = 2\frac{A_x A_{xy}}{A_y^2} - \frac{A_{xx}}{A_y} - \frac{A_x^2 A_{yy}}{A_y^3} = \frac{2y^3(2+3xy)}{(1+2xy)^3}\,, \tag{139}$$
$$\frac{d^3y}{dx^3} = -\frac{6y^4(5+14xy+10x^2y^2)}{(1+2xy)^5}\,.$$

6.2.1 Topological Recursion

Let us apply the topological recursion to the curve (137). We will consider two different parametrizations related by the symplectic transformation (138). The first parametrization which we consider is the natural one

$$\begin{cases} u(p) = \log x(p) = \log \frac{-1-p}{p^2} \\ v(p) = \log y(p) = \log p \end{cases} \tag{140}$$

It leads to a single branch point $dx(p_*) = 0$ with $p_* = -2$. The conjugate of a point p is

$$\overline{p} = -\frac{p}{1+p}\,.$$

The recursion kernel (28) and the anti-derivative (20) can be found in the closed form (here we use a local parameters q, r, defined such that $p = p_* + q$):

$$K(q,r) = \frac{(2-q)^2(q-1)}{2\big(q^2(-1+r)+r^2-qr^2\big)\log(1-q)}\,,$$
$$S_0(q) = \log(q-2)\,\log\Big(\frac{q-1}{q-2}\Big) + \mathrm{Li}_2(2-q)\,.$$

Computing the annulus amplitude and solving the topological hierarchy we find

$$S_1 = -\frac{1}{2}\log\frac{2+y}{xy^3}\,,$$
$$S_2 = \frac{4-10y-y^2}{24(2+y)^3}\,,$$

$$S_3 = -\frac{5y^2(1+y)}{4(2+y)^6}\,, \tag{141}$$

$$S_4 = \frac{\begin{bmatrix} y(1+y)(4096+y(8448+y(-22592+y(-25344 \\ +y(5122+y(162+7y)))))) \end{bmatrix}}{5760(2+y)^9}.$$

Computing derivatives and using the results (139), we get

$$\begin{aligned} S_1' &= \frac{1}{2} - \frac{xy(3+y)}{(2+y)(1+2xy)}\,, \\ S_2' &= -\frac{xy^2(-32+16y+y^2)}{24(2+y)^4(1+2xy)}\,, \\ S_3' &= -\frac{(5xy^3(-4-2y+3y^2)}{4(2+y)^7(1+2xy)}\,, \\ S_4' &= -\frac{\begin{bmatrix} xy^2(8192+17408y-172672y^2-298624y^3+37460y^4 \\ +144296y^5-13486y^6-226y^7-7y^8) \end{bmatrix}}{5760(2+y)^{10}(1+2xy)}. \end{aligned} \tag{142}$$

Now, let us consider another parametrization, which is related to (140) by the transformation $y \to (xy)^{-1}$ given in (138), so that

$$\begin{cases} u(p) = \log x(p) = \log \frac{-1-p}{p^2} \\ v(p) = \log y(p) = \log \frac{-p}{p+1} \end{cases} \tag{143}$$

In this parametrization the Eq. (137) is also satisfied. Since we did not redefine x, the expressions for the branch point $p_* = -2$ and for the conjugate $\overline{p} = -p/(1+p)$ of a point p are still the same as in the previous parametrization. The recursion kernel and the anti-derivative in the present case read (again, using local coordinates q and r vanishing at the branch point):

$$K(q,r) = \frac{(2-q)^2(1-q)}{2\big(q^2(-1+r)+r^2-qr^2\big)\log(1-q)}\,,$$

$$S_0(q) = -\big(\log(q-2)\big)^2 + \frac{1}{2}\log(q-1)\,\log\Big(\frac{(q-2)^2}{q-1}\Big) - \mathrm{Li}_2(2-q)\,.$$

Using the new parametrization we compute the annulus amplitude and solve topological hierarchy to find

$$S_1 = -\frac{1}{2}\log\frac{-(1+y)^2(2+y)}{xy^3}\,,$$

$$
\begin{aligned}
S_2 &= -\frac{(1+y)(4+18y+13y^2)}{24(2+y)^3}\,, \\
S_3 &= -\frac{5y^2(1+y)^3}{4(2+y)^6}\,, \\
S_4 &= \frac{\begin{bmatrix} y(1+y)(4096+y(16128+y(-3392+y(-67584 \\ +y(-77438+13y(-1686+259y)))))) \end{bmatrix}}{5760(2+y)^9}.
\end{aligned}
\tag{144}
$$

Finally, computing derivatives we get

$$
\begin{aligned}
S_1' &= -\frac{xy(3+2y)}{(2+3y+y^2)(1+2xy)}, \\
S_2' &= \frac{xy^2(32+80y+47y^2)}{24(2+y)^4(1+2xy)}, \\
S_3' &= \frac{5xy^3(1+y)^2(4+6y-y^2)}{4(2+y)^7(1+2xy)}, \\
S_4' &= \frac{xy^2\, f_4(x,y)}{5760(2+y)^{10}(1+2xy)}, \\
&\quad \text{where} \quad f_4(x,y) = -8192 - 48128y + 65152y^2 + 644224y^3 + \\
&\quad +1095340y^4 + 612184y^5 - 38354y^6 - 90974y^7 + 3367y^8.
\end{aligned}
\tag{145}
$$

Not surprisingly, the perturbative coefficients (141) and (144) are different in two different parametrizations that we have considered. However, one can immediately check that they are, in fact, related by the transformation (138). Therefore, as expected, the entire partition function Z also enjoys the action of (138).

6.2.2 Quantum Curves

Once we found the coefficients S_k' of the perturbative expansion, we can plug our results into the hierarchy (42) to produce the quantum corrections $\hat{A}_k$ and, hence, the entire quantum curve $\hat{A}$. As usual, we start with the leading term

$$
S_0' = \log y\,, \tag{146}
$$

which is the same in both parametrizations, and then use higher order amplitudes computed above. We start with the first parametrization (140), in which the derivatives of S_k summarized in (142). From the hierarchy of Eqs. (42) we get

$$\hat{A}_1 = -\Big(\frac{S_0''}{2}\partial_v^2 + S_1'\partial_v\Big)A_0 = -\frac{3}{2} - 2\hat{y}\,,$$
$$\hat{A}_2 = \frac{9}{8} + 2\hat{y}\,,$$
$$\hat{A}_3 = -\frac{9}{16} - \frac{4}{3}\hat{y}\,.$$

These coefficients arise from the $\hbar$-expansion of $e^{-3\hbar/2} + e^{-2\hbar}\hat{y} + \hat{x}\hat{y}^2$ and, therefore, up to an overall normalization, the quantum curve (6) in this case reads

$$\hat{A}(\hat{x},\hat{y}) = 1 + q^{-1/2}\hat{y} + q^{3/2}\hat{x}\hat{y}^2\,, \tag{147}$$

in agreement with (128) for $f = 2$.

We can also consider the second parametrization (143). The leading term S_0' is the same as (146), and the higher order perturbative corrections are given by (145). This time, the hierarchy (42) leads to

$$\hat{A}_1 = -\frac{3}{2} - \hat{y}\,,$$
$$\hat{A}_2 = \frac{9}{8} + \frac{\hat{y}}{2}\,,$$
$$\hat{A}_3 = -\frac{9}{16} - \frac{\hat{y}}{6}\,.$$

These terms (up to an overall normalization) arise from the expansion of the quantum curve

$$\hat{A}(\hat{x},\hat{y}) = 1 + q^{1/2}\hat{y} + q^{3/2}\hat{x}\hat{y}^2, \tag{148}$$

which is different from (147).

Finally, the present example gives us a good opportunity to illustrate how the factorization (55) works for curves in $\mathbb{C}^* \times \mathbb{C}^*$. Indeed, it is easy to see that to the leading order in $\hbar$ the quantum curve factorizes as

$$\hat{A} = 1 + \hat{w} - \frac{p+1}{p^2}\hat{w}^2 + \mathscr{O}(\hbar) = (p - \hat{w})(p + (p+1)\hat{w}) + \mathscr{O}(\hbar)\,, \tag{149}$$

where we used (140) and also introduced $\hat{w} = e^{-\frac{p(p+1)}{p+2}\hbar\partial_p}$. In this factorized expression, the first factor $(p - \hat{w})$ annihilates the wave function

$$Z = e^{-\frac{1}{\hbar}\int dp \frac{p+2}{p(p+1)}\log p}\Big(1 + \mathscr{O}(\hbar)\Big) = e^{\frac{1}{\hbar}\left(\mathrm{Li}_2(-p) + \log p\cdot\log(1+p^{-1})\right)}\Big(1 + \mathscr{O}(\hbar)\Big)\,.$$

The exponent here indeed reproduces the leading order term in the partition function, $S_0 = \int v(p)du(p)$, in the parametrization (140). On the other hand, from the second factor $p + (p+1)\hat{w}$ in (149) one finds S_0 in the second parametrization (143).

6.3 Framing $f = 0, 1$

In the preceding subsections, we found the quantum curves for a tetrahedron (or $\mathbb{C}^3$) model with a generic framing, and also analyzed in excruciating detail the case $f = 2$. The situation becomes more delicate for special values of framing $f = 0, 1$ because in these cases the branch point (129) escapes "to infinity" and the topological recursion can no longer be directly applied. However, as also stressed in [10], one can still obtain meaningful results by treating f as a continuous parameter, and taking the limit $f \to 0, 1$ in the end of the computation.

Let us analyze the case $f = 0$ from this viewpoint first. From the general result (128) we conclude that for $f = 0$ the quantum curve should take the form

$$\hat{A}_{f=0} = 1 + q^{-1/2}\hat{y} + q^{1/2}\hat{x}\,. \tag{150}$$

The partition function Z associated to this operator is given by a version of the quantum dilogarithm (183) and can be written as

$$Z_{f=0} = c \cdot x^{1/2}\psi(-x)\,, \tag{151}$$

where c is some multiplicative factor which is not fixed by the q-difference equation (10). This form of the partition function follows from the application of the differential hierarchy (42) to the quantum curve (150), or can be seen directly as follows. Assuming that the constant normalization factor c contains $\prod_k(-1) = (-1)^{\zeta(0)}$ and changing the signs in each factor of the product (183) we see that

$$\hat{y}Z_{f=0} = q^{1/2}x^{1/2}\prod_{k=1}^{\infty}\frac{1}{-1-xq^{k+1/2}} = q^{1/2}(-1-xq^{1/2})Z_{f=0}\,, \tag{152}$$

which is equivalent to the statement $\hat{A}_{f=0}Z_{f=0} = 0$.

Now, let us compare the perturbative $\hbar$-expansion of the partition function (151) with what one might find from the topological recursion. The leading term is

$$S_0 = \int \frac{\log(-1-x)}{x}dx = i\pi\log x - \mathrm{Li}_2(-x)\,,$$

where the dilogarithm properly reproduces the leading term in (183). The next, subleading contribution given by the annulus amplitudes is

$$S_1 = \frac{i\pi}{2} + \frac{1}{2}\log x\,,$$

and, again, it reproduces the corresponding factor $x^{1/2}$ in (151). The higher order terms S_k arise from the topological recursion as follows. First, notice that all W_n^g with $n \neq 1$ vanish for $f = 0$. This immediately implies that all $S_{2k+1} = 0$ because only W_n^g with even values of n contribute to S_{2k+1}. On the other hand, the correlators with $n = 1$, which remain non-zero in the $f \to 0$ limit, read

$$\begin{aligned} W_1^1(p) &= \frac{1}{24p^2}\,, \\ W_1^2(p) &= -\frac{7(6+6p+p^2)}{5760p^4}\,, \\ W_1^3(p) &= \frac{31(120+240p+150p^2+30p^3+p^4)}{967680p^6}\,. \end{aligned}$$

Integrating these correlators (and including an appropriate integration constant in S_2) we find the following functions of x,

$$\begin{aligned} S_2 &= \frac{1}{24}\mathrm{Li}_0(-x)\,, \\ S_4 &= -\frac{7}{5760}\mathrm{Li}_{-2}(-x)\,, \\ S_4 &= -\frac{31}{967680}\mathrm{Li}_{-4}(-x)\,, \end{aligned}$$

which, as expected, agree with the expansion (184). In topological string theory, this partition function represents a B-brane amplitude in the $\mathbb{C}^3$ geometry.

In the second special limit, $f \to 1$, the situation is a little more subtle due to the divergence of the correlators W_{2k}^g. This, however, does not affect the leading terms S_0 and S_1 which still can be computed by direct methods. The higher-order terms, on the other hand, can be obtained from the hierarchy of Eqs. (42) applied to the quantum curve (128) with $f = 1$:

$$\hat{A}_{f=1} = 1 + q^{-1/2}\hat{y} + q\hat{x}\hat{y}\,. \tag{153}$$

From the topological string point of view, this choice of framing corresponds to an anti-B-brane, whose partition function should be roughly the inverse of that for a B-brane. Curiously, however, the hierarchy (42) applied to the above quantum curve reveals that the $\hbar$-expansion of the free energy contains not only polylogarithms of even order, but also polylogarithms of odd order. This expansion starts with

$$S_0 = \mathrm{Li}_2(-x)\,, \quad S_1 = \log x^{1/2} + \mathrm{Li}_1(-x)\,, \quad S_2 = \frac{11}{24}\mathrm{Li}_0(-x)\,, \quad S_3 = \frac{1}{8}\mathrm{Li}_{-1}(-x)\,,$$

and can be summed up to a generating function

$$Z_{f=1} = \frac{c \cdot x^{1/2}}{\psi(-x)} e^{\sum_{k=0}^{\infty} \frac{\hbar^k}{2^k k!}\mathrm{Li}_{1-k}(-x)} = \frac{c \cdot x^{1/2}}{\psi(-x)} e^{-\log(1+xe^{\hbar/2})}$$
$$= c \cdot x^{1/2} \prod_{k=1}^{\infty} \big(1 + xe^{\hbar(k+1/2)}\big).$$

As a check of this result we make an observation analogous to (152):

$$\hat{y} Z_{f=1} = q^{1/2} x^{1/2} \prod_{k=1}^{\infty} \big(-1 - xq^{k+3/2}\big) = q^{1/2} \frac{Z_{f=1}}{-1 - xq^{3/2}}\,,$$

where we also identified the multiplicative factor c with $\prod_k (-1) = (-1)^{\zeta(0)}$. After multiplying both sides of this expression by the denominator $1 + xq^{3/2}$ we recover the quantum curve equation (153).

7 Conifold and Generalizations

There is a large class of toric Calabi–Yau manifolds, known as the generalized conifolds, whose mirror curves have genus zero. They provide especially simple and attractive examples, for which the corresponding quantum curves can be easily determined using our technique. Toric diagrams for this class of manifolds arise from a triangulation of a "strip," as shown in Fig. 4. The corresponding mirror curves are always linear in one of the variables. Therefore, up to a coordinate change, they can be put in the form

$$A(x, y) = B(x) + yC(x)\,. \tag{154}$$

With a suitable choice of framing, $B(x)$ and $C(x)$ can be written in a simple product form $B(x) = \prod_i (1 + Q_i x)$ and $C(x) = \prod_j (1 + \tilde{Q}_j x)$, where Q_i and $\tilde{Q}_j$ encode the Kähler parameters of the toric Calabi–Yau threefold. For this choice of framing the partition function of generalized conifolds is always a product of quantum dilogarithms, which can be easily recognized from the leading behavior

$$S_0 = \int \log y \frac{dx}{x} = \Big(\sum_j \mathrm{Li}_2(-\tilde{Q}_j x)\Big) - \Big(\sum_i \mathrm{Li}_2(-Q_i x)\Big)\,.$$

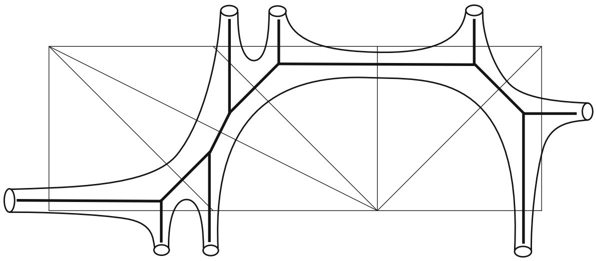

Fig. 4 An example of mirror curve for a generalized conifold

The higher-order $\hbar$-corrections complete the dilogarithms here to quantum dilogarithms in the full partition function, generalizing the expansion (184) in an obvious way. With this particularly nice choice of framing, it is also easy to extend the computation (152) to find corresponding quantum curves.

For general framing, however, a derivation of the quantum curve along these lines is by far non-obvious. It is this point where our results turn out to be very powerful and allow to determine quantum curves in any framing in a straightforward and systematic manner. Writing Eq. (154) with x and y interchanged, as

$$A(x, y) = B(y) + xC(y)\,, \tag{155}$$

essentially represents the same toric geometry and the same algebraic curve. Equivalently, the curve $A(x, y) = 0$ can be described as the zero locus of (93) with $P(y) = B(y)/C(y)$, and from (96) we immediately obtain

$$\hat{A} = B(q^{-1/2}\hat{y}) + q^{1/2}\hat{x}\, C(q^{1/2}\hat{y})\,. \tag{156}$$

Because the latter choice of the generalized conifold equation (linear in x) differs from (154) by the exchange of x and y, the corresponding partition functions are related by a Fourier transform. In particular, we mentioned earlier that for a specific choice of framing[24] the partition function Z is built out of quantum dilogarithms. Since the quantum dilogarithm is self-similar under Fourier transform, it follows that the convolution of a product of quantum dilogarithms is again a product of quantum dilogarithms. Hence, the Fourier transform of the partition function should also be a product of quantum dilogarithms. This can be verified directly using the form of the quantum curve (156) and the hierarchy of Eqs. (42).

As a check of our result (156), we note that for $B(y) = 1 + y$ and $C(y) = y^f$ we get

$$\hat{A}_{\mathbb{C}^3} = 1 + q^{-1/2}\hat{y} + q^{(f+1)/2}\hat{x}\hat{y}^f\,,$$

[24] In which $B(x)$ and $C(x)$ have a product form $B(x) = \prod_i (1 + Q_i x)$ and $C(x) = \prod_j (1 + \tilde{Q}_j x)$.

which correctly reproduces the quantum curve (128) of the $\mathbb{C}^3$ geometry discussed earlier in Sect. 6. As another example one can consider an ordinary conifold, whose mirror curve in zero framing $f = 0$ reads

$$A_{f=0}(x, y) = 1 + x + y + Q\frac{x}{y},$$

where, as usual, Q is the (exponentiated) Kähler parameter. Similarly, for general value of framing f, the mirror curve of the conifold is given by the zero locus of a degree-f polynomial

$$A_f(x, y) = 1 + xy^f + y + Qxy^{f-1}, \tag{157}$$

which is manifestly in the form (155) with $B(y) = 1 + y$ and $C(y) = y^f + Qy^{f-1}$. Therefore, from (156) we conclude that the quantization of this A-polynomial is

$$\hat{A}_f = 1 + q^{-1/2}\hat{y} + q^{(f+1)/2}\hat{x}\hat{y}^f + Qq^{f/2}\hat{x}\hat{y}^{f-1}. \tag{158}$$

Another special choice of framing $f = 2$ leads to the quantum curve (164) which will be analyzed next to high order in topological recursion. Before we proceed to this example, however, let us remind the reader that a particular form of the quantum curve depends not only on the classical equation but also on the choice of parametrization, as discussed in Sects. 2.3 and 6.2, and as will be also discussed below. For example, the quantum curves (156), (158), and (164) all come from the choice of parametrization (94).

Quantum curves for generalized conifolds were also studied recently in [1], Beem et al. (Private communication, 2011). In particular, in [1] a different quantization of the classical curve $A(x, y) = 0$ was related to the Nekrasov–Shatashvili limit [45] of the *refined* topological string partition function, where $\epsilon_1 = 0$ and $\epsilon_2 = \hbar$ (see also [20]). In that framework, the classical curves for generalized conifolds and even more general examples are quantized[25] by simply replacing x and y with $\hat{x}$ and $\hat{y}$ (where all q-factors in $\hat{A}$ can be absorbed in a normalization of $\hat{x}$, $\hat{y}$, or Kähler parameters). In particular, the new interesting phenomena where the numerical coefficients "split" into several powers of q, as in

$$A = 3x^5 + \ldots \quad \rightsquigarrow \quad \hat{A} = (q + q^3 + q^5)x^5 + \ldots$$

or where completely new terms appear upon quantization (as in $\hat{A} = (1 - q^3)x^3 + \ldots$) never happen in the framework of [1]. It is tempting to speculate that such phenomena—that one encounters e.g. in quantization of A-polynomials for some simple knots—can be accounted for by going from the Nekrasov–Shatashvili limit $\epsilon_1 = 0$, $\epsilon_2 = \hbar$ to the limit $\epsilon_1 = -\epsilon_2 = \hbar$.

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[25] We thank Mina Aganagic and Robbert Dijkgraaf for clarifying discussions on this.

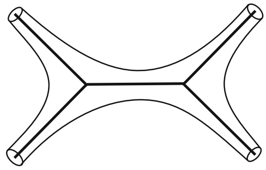

Fig. 5 Mirror curve for the conifold geometry

7.1 Conifold in $f = 2$ Framing

In this section we analyze the ordinary conifold, whose mirror curve is shown in Fig. 5. As in the case of $\mathbb{C}^3$ geometry, we wish to discuss a special choice of framing (namely, $f = 2$) and study how a choice of parametrization affects the form of the quantum curve.

For $f = 2$, the conifold mirror curve (157) takes the form

$$A(x, y) \equiv A_{f=2}(x, y) = 1 + y + xy^2 + Qxy \,, \tag{159}$$

and in the limit $Q \to 0$ reduces to the $\mathbb{C}^3$ mirror curve (137) in the same framing. In fact, the relation between these two models goes much further. For example, the curve defined by the zero locus of (159) has two branches $y^{(\alpha)}$ labeled by $\alpha = \pm$,

$$y^{(\pm)} = \frac{-1 - Qx \pm \sqrt{(1 + Qx)^2 - 4x}}{2x} \,, \tag{160}$$

which, as in the $\mathbb{C}^3$ model, are exchanged by the Galois transformation (138):

$$(x, y) \mapsto \left(x, \frac{1}{xy} \right) . \tag{161}$$

From the equation of the curve we also find the following formulae

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{A_x}{A_y} = -\frac{Qy + y^2}{1 + Qx + 2xy} \,, \qquad (162)\\
\frac{d^2y}{dx^2} &= 2\frac{A_x A_{xy}}{A_y^2} - \frac{A_{xx}}{A_y} - \frac{A_x^2 A_{yy}}{A_y^3} \\
&= \frac{2y(Q + y)\big(Q + Q^2x + (2 + 3Qx)y + 3xy^2\big)}{(1 + Qx + 2xy)^3} \,, \\
\frac{d^3y}{dx^3} &= -\frac{6y(Q + y)}{(1 + Qx + 2xy)^5}\Big(Q^2(1 + Qx)^2 + Q(5 + 11Qx + 6Q^2x^2)y + \\
&\qquad +(5 + 21Qx + 16Q^2x^2)y^2 + 2x(7 + 10Qx)y^3 + 10x^2y^4\Big).
\end{aligned}$$

which will be useful to us later.

7.1.1 Topological Recursion

The curve (159) is quadratic and, therefore, is a double cover of the x-plane. We introduce two parametrizations of this curve which, just like the two branches (160), are permuted by the Galois transformation (161).

The first parametrization is the obvious one

$$\begin{cases} u(p) = \log x(p) = \log \frac{-1-p}{p(p+Q)} \\ v(p) = \log y(p) = \log p \end{cases} \tag{163}$$

and is motivated by writing (159) in the form (93) with $P(y) = (1+y)/(Qy+y^2)$. Indeed, applying our general result (96) to this particular model we immediately obtain

$$\hat{A} = 1 + q^{-1/2}\hat{y} + q^{3/2}\hat{x}\hat{y}^2 + qQ\hat{x}\hat{y}\,, \tag{164}$$

which is also consistent with (158). As we pointed out earlier, however, this result is based only on the elementary computation of the annulus amplitude S_1, and now we wish to verify that computing S_n and $\hat{A}_n$ to higher order does not lead to any modifications and merely confirms the result (164).

The conifold curve (163) has two branch points

$$p_* = -1 \mp \sqrt{1-Q}\,. \tag{165}$$

Notice, in the $Q \to 0$ limit, the branch point with the minus sign reduces to the $\mathbb{C}^3$ branch point $p_* = -2$, whereas the other branch point runs away to $p_* = 0 \notin \mathbb{C}^*$.

The conjugate of a generic point p is given in a global form (the same around both branch points)

$$\overline{p} = \frac{-p-Q}{1+p}\,.$$

The recursion kernel and the anti-derivative can be found in the closed form

$$K(q,z) = \frac{q(1+q)(q+Q)}{2(z-q)(q+Q+z+qz)\log\left(\frac{-q-Q}{q(1+q)}\right)}\,,$$

$$S_0(q) = -\frac{1}{2}\log q\Big(\log q + 2\log\Big(\frac{q+Q}{Q(1+q)}\Big)\Big) + \mathrm{Li}_2(-q) - \mathrm{Li}_2(-q/Q)\,,$$

from which we can compute the annulus amplitude and solve the topological hierarchy. We find

$$S_1 = -\frac{1}{2}\log\Big(\frac{Q+y(2+y)}{xy^2(Q+y)^2}\Big)\,,$$

$$S_2 = \frac{y(1-Q)\big(11Q^2 + 2Q(7-5y)y - y^2(-4+y(10+y))\big)}{24(Q+y(2+y))^3}\,, \tag{166}$$

$$S_3 = \frac{\begin{array}{l}(Q-1)y(1+y)(Q+y)(Q-y^2) \\ \quad (Q^3 - 10y^4 - 6Q^2y(1+3y) + Qy^2(y^2-26y-6))\end{array}}{8(Q+y(2+y))^6}.$$

Now, let us consider another parametrization of the classical curve (159), related to (163) by the transformation (161):

$$\begin{cases} u(p) = \log x(p) = \log \frac{-1-p}{p(p+Q)} \\ v(p) = \log y(p) = \log \frac{-p-Q}{p+1} \end{cases} \tag{167}$$

Since x is not affected by the transformation (161), we find the same two branch points (165):

$$p_* = -1 \mp \sqrt{1-Q}\,,$$

whose behavior in the $Q \to 0$ limit was discussed below Eq. (165).

In the new parametrization (167), the conjugate of a point p is given by the same formula as in the previous parametrization (163):

$$\overline{p} = \frac{-p-Q}{1+p}\,.$$

The recursion kernel and the anti-derivative can be also found in the closed form. The kernel differs by a sign from the kernel in previous parametrization

$$K(q,z) = \frac{q(1+q)(q+Q)}{2(q-z)(q+Q+z+qz)\log\big(\frac{-q-Q}{q(1+q)}\big)}\,,$$

and, as everything else, in the $Q \to 0$ limit reduces to the recursion kernel of the $\mathbb{C}^3$ model. The formula for S_0 can be also written explicitly, even though its form is a little involved.

Computing the annulus amplitude and solving the topological hierarchy we now find

$$S_1 = -\frac{1}{2}\log\big(\frac{(1+y)^2(Q+y(2+y))}{xy^2(Q+y)^2(Q-1)}\big)\,, \tag{168}$$

$$S_2 = \frac{\begin{array}{l}(1+y)(Q+y)\big(Q^3 + Q^2(1+2y(7+5y)) \\ \quad +y^2(4+y(18+13y)) - Qy(6+y(2+y(10+11y)))\big)\end{array}}{24(Q-1)(Q+y(2+y))^3}\,,$$

which should be compared to the analogous formulae (166) obtained in a different parametrization / polarization.

7.1.2 Quantum Curves

Once we found the perturbative amplitudes S_k, we can compute their derivatives and determine the form of the quantum curve from the hierarchy of Eqs. (42). With the first choice of parametrization (163), we get

$$\hat{A}_1 = -\frac{\hat{y}}{2} + Q\hat{x}\hat{y} + \frac{3}{2}\hat{x}\hat{y}^2 ,$$
$$\hat{A}_2 = \frac{1}{8}(\hat{y} + 4Q\hat{x}\hat{y} + 9\hat{x}\hat{y}^2) ,$$
$$\hat{A}_3 = \frac{1}{48}(-\hat{y} + 8Q\hat{x}\hat{y} + 27\hat{x}\hat{y}^2) .$$

It is easy to see that these are precisely the coefficients which arise from the perturbative $\hbar$-expansion of the curve (164):

$$\hat{A}(\hat{x},\hat{y}) = 1 + q^{-1/2}\hat{y} + q^{3/2}\hat{x}\hat{y}^2 + qQ\hat{x}\hat{y} , \tag{169}$$

which, in the $Q \to 0$ limit, reduces to the quantum curve (147) of the $\mathbb{C}^3$ model (in a similar parametrization).

In the second parametrization (167), computing the derivatives of S_k from (168) and substituting the result into the hierarchy of loop equations (42) gives

$$\hat{A}_1 = -1 - \frac{\hat{y}}{2} + \frac{1}{2}\hat{x}\hat{y}^2 ,$$
$$\hat{A}_2 = \frac{1}{2} + \frac{\hat{y}}{8} + \frac{1}{8}\hat{x}\hat{y}^2 ,$$

etc. Up to an overall normalization, these coefficients arise from the $\hbar$-expansion of the quantum curve

$$\hat{A}(\hat{x},\hat{y}) = 1 + q^{1/2}\hat{y} + q^{3/2}\hat{x}\hat{y}^2 + qQ\hat{x}\hat{y} . \tag{170}$$

As expected, in the limit $Q \to 0$ this expression reduces to (148).

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Appendix 1: A Hierarchy of Differential Equations

In this appendix we provide more details on the hierarchy of differential equations (42) arising from the quantum curve equation $\hat{A}Z = 0$. This hierarchy allows to determine the quantum operator $\hat{A}$, order by order in $\hbar$, from the knowledge of the partition function Z it annihilates, or vice versa. We stress that the hierarchy (42) takes the same form for curves embedded in $\mathbb{C} \times \mathbb{C}$ or $\mathbb{C}^* \times \mathbb{C}^*$, even though its derivation in both cases is much different.

We recall that, in the classical limit, we consider curves embedded either in $\mathbb{C}\times\mathbb{C}$ with coordinates (u, v), or in $\mathbb{C}^* \times \mathbb{C}^*$ with coordinates $(x = e^u, y = e^v)$. The classical curve is given by the polynomial equation

$$0 = A \equiv A_0. \tag{171}$$

In the quantum regime we introduce the commutation relation $[\hat{v}, \hat{u}] = \hbar$ and use the representation $\hat{u} = u, \hat{v} = \hbar\partial_u$. For $\mathbb{C}^*$ coordinates we then have $\hat{x} = x = e^u$, $\hat{y} = e^{\hat{v}} = e^{\hbar\partial_u}$ and $\hat{y}\hat{x} = q\hat{x}\hat{y}$, where $q = e^{\hbar}$. In what follows we denote derivatives w.r.t u by $' = \partial_u = x\partial_x$.

To represent the quantum curves corresponding to (171) we use the following expansions, respectively in $\mathbb{C} \times \mathbb{C}$ and $\mathbb{C}^* \times \mathbb{C}^*$ case

$$\hat{A} = \sum_{j=0}^{d} a_j(u,\hbar)\hat{v}^j, \qquad \hat{A} = \sum_{j=0}^{d} a_j(x,\hbar)\hat{y}^j,$$

where, respectively,

$$a_j(u,\hbar) = \sum_{l=0}^{\infty} a_{j,l}(u)\hbar^l, \qquad a_j(x,\hbar) = \sum_{l=0}^{\infty} a_{j,l}(x)\hbar^l.$$

We also reassemble contributions of fixed $\hbar$ order into, respectively,

$$A_l = A_l(u,v) = \sum_{j=0}^{d} a_{j,l}(u)v^j, \qquad A_l = A_l(x,y) = \sum_{j=0}^{d} a_{j,l}(x)y^j. \tag{172}$$

Replacing classical variables in these expansions by quantum operators $\hat{u}, \hat{v}$ or $\hat{x}, \hat{y}$, ordered such that $\hat{v}$ or $\hat{y}$ appear to the right of $\hat{u}$ or $\hat{x}$, defines corrections $\hat{A}_l$ to the quantum curve (6). Using the above notation, the quantum curve equation can be written, respectively in $\mathbb{C} \times \mathbb{C}$ and $\mathbb{C}^* \times \mathbb{C}^*$ case, as

$$\hat{A}Z(u) = \Big(\sum_{j=0}^{d} a_j(u,\hbar)\hat{v}^j\Big)Z(u) = 0, \qquad \hat{A}Z(x) = \Big(\sum_{j=0}^{d} a_j(x,\hbar)\hat{y}^j\Big)Z(x) = 0, \tag{173}$$

where

$$Z = \exp\Big(\frac{1}{\hbar}\sum_{k=0}^{\infty}\hbar^k S_k\Big). \tag{174}$$

Hierarchy in the $\mathbb{C}^*$ Case: q-Difference Equation

The quantum curve equation gives rise to a hierarchy of differential equations which arise as follows. Substituting the partition function (174) into (173) and dividing by $e^{\hbar^{-1}S_0}$ results in

$$0 = \sum_{j,l=0}^{\infty} a_{j,l}\hbar^l e^{jS_0'} \exp\Big(\sum_{n=1}^{\infty}\hbar^n \mathfrak{d}_n(j)\Big), \tag{175}$$

where $\mathfrak{d}_n(j)$ combine terms with a fixed power of $\hbar$ in the expansion of $\sum_k \hbar^k S_k\big(e^{u+j\hbar}\big)$

$$\mathfrak{d}_n(j) = \sum_{r=1}^{n+1}\frac{j^r}{r!}S^{(r)}_{n+1-r}(x). \tag{176}$$

For example

$$\mathfrak{d}_1(j) = \frac{j^2}{2}S_0'' + jS_1',$$
$$\mathfrak{d}_2(j) = \frac{j^3}{6}S_0''' + \frac{j^2}{2}S_1'' + jS_2',$$
$$\mathfrak{d}_3(j) = \frac{j^4}{4!}S_0^{(4)} + \frac{j^3}{3!}S_1''' + \frac{j^2}{2}S_2'' + jS_3',$$

and note that for each n we have $\mathfrak{d}_n(0) = 0$. Let us now expand the exponent in (175) and collect terms with fixed power of $\hbar$

$$\exp\Big(\sum_{n=1}^{\infty}\hbar^n\mathfrak{d}_n(j)\Big) = \sum_{r=0}^{\infty}\hbar^r\mathfrak{D}_r(j), \tag{177}$$

so that, for example,

$$\mathfrak{D}_0(j) = 1,$$
$$\mathfrak{D}_1(j) = \mathfrak{d}_1(j) = \frac{S_0''}{2}j^2 + S_1'j,$$

$$\mathfrak{D}_2(j) = \mathfrak{d}_2(j) + \frac{1}{2}\mathfrak{d}_1(j)^2 = \frac{(S_0'')^2}{8} j^4 + \frac{1}{6}\big(S_0''' + 3S_0''S_1'\big) j^3 + \frac{1}{2}\big(S_1'' + (S_1')^2\big) j^2 + S_2' j,$$

$$\begin{aligned}\mathfrak{D}_3(j) &= \mathfrak{d}_3(j) + \mathfrak{d}_1(j)\mathfrak{d}_2(j) + \frac{1}{6}\mathfrak{d}_1(j)^3 \\ &= \frac{(S_0'')^3}{48} j^6 + \Big(\frac{S_0''S_0'''}{12} + \frac{(S_0'')^2 S_1'}{8}\Big) j^5 \\ &\quad + \frac{1}{24}\big(S_0'''' + 6S_0''S_1'' + 4S_0'''S_1' + 6S_0''(S_1')^2\big) j^4 + \\ &\quad + \frac{1}{6}\big(3S_1''S_1' + (S_1')^3 + S_1''' + 3S_0''S_2'\big) j^3 + \big(\frac{S_2''}{2} + S_1'S_2'\big) j^2 + S_3' j,\end{aligned}$$

$$\begin{aligned}\mathfrak{D}_4(j) &= \mathfrak{d}_4(j) + \mathfrak{d}_1(j)\mathfrak{d}_3(j) + \frac{1}{2}\mathfrak{d}_2(j)^2 \\ &\quad + \frac{1}{2}\mathfrak{d}_1(j)^2\mathfrak{d}_2(j) + \frac{1}{4!}\mathfrak{d}_1(j)^4 \\ &= \frac{(S_0'')^4}{384} j^8 + \frac{1}{48}\big((S_0'')^2 S_0''' + (S_0'')^3 S_1'\big) j^7 \\ &\quad + \ldots + \frac{1}{2}\big((S_2')^2 + S_3'' + 2S_1'S_3'\big) j^2 + S_4' j.\end{aligned}$$

Finally, expanding (175) in total power of $\hbar$ and collecting terms with a fixed such power $\hbar^n$, gives rise to a hierarchy of differential equations

$$0 = \sum_j e^{jS_0'} \sum_{r=0}^{n} a_{j,r} \mathfrak{D}_{n-r}(j). \tag{178}$$

Now we use the fact that the disk amplitude in $\mathbb{C}^* \times \mathbb{C}^*$ case is $S_0 = \int \log(y)\frac{dx}{x}$, so $S_0' = \log(y)$. Therefore $e^{jS_0'} = y^j$ and we can write (178) in terms of corrections A_k to the quantum curve (172). In particular the first equation in the hierarchy $0 = \sum_{j=0}^{d} a_{j,0} y^j = A_0(x, y)$ coincides with the classical curve equation (171). Now, writing $\mathfrak{D}_{n-r}(j) = \sum_m \mathfrak{D}_{n-r,m} j^m$, we can rewrite (178) as

$$\begin{aligned}0 &= \sum_{r=0}^{n} \sum_{j,m} a_{j,r} \mathfrak{D}_{n-r,m} j^m y^j = \sum_{r=0}^{n} \sum_{j,m} a_{j,r} \mathfrak{D}_{n-r,m} (y\partial_y)^m y^j \\ &= \sum_{r=0}^{n} \Big(\sum_m \mathfrak{D}_{n-r,m} (y\partial_y)^m\Big) A_r.\end{aligned}$$

The expression in the last bracket is nothing but the operator $\mathfrak{D}_{n-r}(j)$ from (177) with all j replaced by $y\partial_y = \partial_v$. Therefore we denote this operators by $\mathfrak{D}_{n-r}(\partial_v)$, or simply $\mathfrak{D}_{n-r}$; for example

$$\mathfrak{D}_1 = \frac{S_0''}{2}(y\partial_y)^2 + S_1'(y\partial_y),$$

etc. In terms of these new operators, the hierarchy of Eqs. (178) takes a particularly simple form

$$0 = \sum_{r=0}^{n} \mathfrak{D}_{n-r} A_r, \tag{179}$$

as advertised in (42), and with $\mathfrak{D}_{n-r}$ defined as in (177) with j replaced by ∂_v.

Hierarchy in the $\mathbb{C}$ *Case: Differential Equation*

Now we show that the hierarchy of equations which arises for curves in $\mathbb{C}\times\mathbb{C}$ takes the same form (42) as in $\mathbb{C}^* \times \mathbb{C}^*$ case, even though the explicit derivation of this hierarchy is much different. Now Eq. (173) takes a form

$$0 = \hat{A}Z(u) = \sum_{j=0}^{d}\sum_{l=0}^{\infty} a_{j,l}\hbar^{l+j}\partial_u^j Z(u),$$

and by induction we find that the last term can be written as $\partial_u^j Z = Z(\partial_u + S')^j S'$. Then the factor of Z can be factored out of an entire expression, which results in

$$0 = \sum_{l=0}^{\infty}\Big[a_{0,l}\hbar^l + \sum_{j=0}^{d-1} a_{j+1,l}\hbar^l\Big(\hbar\partial_u + \sum_{k=0}^{\infty}\hbar^k S_k'\Big)^j \sum_{r=0}^{\infty}\hbar^r S_r'\Big]. \tag{180}$$

Recalling that $S_0' = v$, an explicit computation reveals that the last term in this expression can be written as

$$\big(\hbar\partial_u + \hbar S'\big)^j \hbar S' = v^{j+1} + \hbar\Big(S_0''\frac{j(j+1)}{2}v^{j-1} + S_1'(j+1)v^j\Big) + \tag{181}$$

$$+\hbar^2\Big((S_0'')^2\frac{(j-2)(j-1)j(j+1)}{8}v^{j-3} + \big(S_0''' + 3S_0''S_1'\big)\frac{(j-1)j(j+1)}{6}v^{j-2} +$$

$$+\big(S_1'' + (S_1')^2\big)\frac{j(j+1)}{2}v^{j-1} + S_2'(j+1)v^j\Big) + \mathscr{O}(\hbar^3) =$$

$$= \Big[1 + \hbar\Big(\frac{S_0''}{2}\partial_v^2 + S_1'\partial_v\Big) + \hbar^2\Big(\frac{(S_0'')^2}{8}\partial_v^4 + \frac{S_0''' + 3S_0''S_1'}{6}\partial_v^3 + \frac{S_1'' + (S_1')^2}{2}\partial_v^2 + S_2'\partial_v\Big) + \mathcal{O}(\hbar^3)\Big]v^{j+1}.$$

We see that a coefficient at each power $\hbar^r$ above is nothing but $\mathfrak{D}_r$ introduced in (179), i.e. the operator defined in (177) with j replaced by ∂_v. Therefore

$$\big(\hbar\partial_u + \hbar S'\big)^j \hbar S' = \sum_{r=0}^{\infty} \hbar^r \mathfrak{D}_r.$$

Using a definition A_r from (172) we find that (180) takes form

$$0 = \sum_{r,l=0}\sum_{j=0}^{d} a_{j,l}\hbar^l\hbar^r\mathfrak{D}_r v^j = \sum_{r,l}\hbar^{r+l}\mathfrak{D}_r A_l = \sum_{n=0}^{\infty}\hbar^n\Big(\sum_{r=0}^{n}\mathfrak{D}_{n-r}A_r\Big).$$

Therefore at order $\hbar^n$ we get

$$0 = \sum_{r=0}^{n}\mathfrak{D}_{n-r}A_r, \tag{182}$$

with $\mathfrak{D}_{n-r}$ defined as in (177) with j replaced by ∂_v. This is the same equation as in $\mathbb{C}^* \times \mathbb{C}^*$ case (179), and as already advertised in (42).

Appendix 2: Quantum Dilogarithm

In literature several representations of quantum dilogarithm can be found. We use the following one

$$\psi(x) = \prod_{k=1}^{\infty}(1 - xe^{\hbar(k-1/2)})^{-1} = \tag{183}$$

$$= \exp\Big(-\sum_{k=1}^{\infty}\frac{x^k}{k(e^{\hbar k/2} - e^{-\hbar k/2})}\Big) =$$

$$= \sum_{k=0}^{\infty} x^k e^{\frac{\hbar k}{2}} \prod_{i=1}^{k}\frac{1}{1 - e^{i\hbar}},$$

which has the following “genus expansion”

$$\log \psi(x) = \frac{1}{\hbar} S_0(x) + S_1(x) + \hbar S_2(x) + \hbar^2 S_3(x) + \hbar^3 S_4(x) + \hbar^4 S_5(x) + \ldots$$

$$\equiv -\frac{1}{\hbar}\mathrm{Li}_2(x) + \frac{\hbar}{24}\mathrm{Li}_0(x) - \frac{7\hbar^3}{5760}\mathrm{Li}_{-2}(x)$$

$$+ \frac{31\hbar^5}{967680}\mathrm{Li}_{-4}(x) + \ldots = \tag{184}$$

$$= \sum_{k=0}^{\infty} \hbar^{k-1}(1-2^{1-k})\frac{B_k}{k!}\mathrm{Li}_{2-k}(x)\,. \tag{185}$$

Note, all terms with even power of $\hbar$ vanish. For terms $\sim \hbar^{k-1}B_k$ with $k = 3, 5, 7, \ldots$ this is so, because $B_3 = B_5 = B_7 = \ldots = 0$. On the other hand, the term with $k = 1$ is proportional to $(1 - 2^{1-1}) = 0$, hence it vanishes as well. Further details can be found e.g. in [21].

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Spherical Hall Algebra of $\overline{\mathrm{Spec}(\mathbb{Z})}$

M. Kapranov, O. Schiffmann, and E. Vasserot

To Yuri Ivanovich Manin on his 75th birthday

Abstract We study an arithmetic analog of the Hall algebra of a curve, when the curve is replaced by the spectrum of the integers compactified at infinity. The role of vector bundles is played by lattices with quadratic forms. This algebra H consists of automorphic forms with respect to $GL_n(\mathbb{Z}), n > 0$, with multiplication given by the parabolic pseudo-Eisenstein series map We concentrate on the subalgebra SH in H generated by functions on the Arakelov Picard group of $\overline{Spec(Z)}$. We identify H with a Feigin–Odesskii type shuffle algebra, with the function defining the shuffle algebra expressed through the Riemann zeta function. As an application we study relations in H. Quadratic relations express the functional equation for the Eisenstein–Maass series. We show that the space of additional cubic relations (lying an appropriate completion of H and considered modulo rescaling), is identified with the space spanned by nontrivial zeroes of the Riemann zeta function.

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1 Introduction

(1.1) The construction of the Hall algebra of an abelian category $\mathscr{A}$ is known to produce interesting Hopf algebras of quantum group-theoretic nature. A condition usually imposed to ensure that the Hall algebra has a compatible comultiplication, is that $\mathscr{A}$ is hereditary (of homological dimension 1). There are two main types of hereditary abelian categories which have been studied in this respect.

First, if Q is a *quiver*, we can form the category $\mathscr{A} = \mathscr{R}ep_{\mathbb{F}_q}(Q)$ of (finite-dimensional) representations of Q over a finite field $\mathbb{F}_q$. As discovered by Ringel [22], the Hall algebra of $\mathscr{R}ep_{\mathbb{F}_q}(Q)$ is related to the quantized Kac–Moody algebra whose Dynkin diagram is Q. More precisely, it contains $U_q(\mathfrak{n}_+)$, the quantization of the unipotent subalgebra on the positive root generators from the Kac–Moody root system.

Second, if X is a smooth projective *curve* over $\mathbb{F}_q$, we can form the category $\mathscr{A} = \mathscr{C}oh(X)$ of coherent sheaves on X. In this case the Hall algebra contains the spaces of unramified automorphic forms on the groups GL_r, $r \geqslant 1$ over the function field $K = \mathbb{F}_q(X)$, and the multiplication corresponds to forming Eisenstein series [14]. One can also include "orbifold curves" $G\backslash\backslash X$ where G is a finite group of automorphisms of a curve X, see [23]. The algebras obtained in this way include both quantum affine algebras [14, 23] and spherical Cherednik algebras [25].

(1.2) The goal of the present paper is to begin the study of a third, more arithmetic, type of Hall algebras. It is obtained by replacing a curve $X/\mathbb{F}_q$ by the spectrum of the *ring of integers in a number field*, compactified at infinity by the Archimedean valuations. In this paper we consider only the basic example of $\overline{\mathrm{Spec}(\mathbb{Z})} = \mathrm{Spec}(\mathbb{Z}) \cup \{\infty\}$. The role of rank n vector bundles for $\overline{\mathrm{Spec}(\mathbb{Z})}$ is played by free abelian groups L of rank n with a positive definite quadratic form in $L \otimes \mathbb{R}$, see [10, 29, 30] as well as [18, 28] for a more general point of view of Arakelov geometry. The "moduli space" of such bundles is the classical quotient of reduction theory of quadratic forms

$$\mathrm{Bun}_n = GL_n(\mathbb{Z})\backslash GL_n(\mathbb{R})/O_n.$$

Functions on Bun_n are the same as automorphic forms on $GL_n(\mathbb{R})$, see [9] for a detailed study of precisely this situation.

(1.3) To describe our arithmetic analog of the Hall algebra, let $H_n = C_0^\infty(\mathrm{Bun}_n)$ be the space of smooth functions on Bun_n with compact support. The space $H = \bigoplus_n H_n$ has a natural structure of an associative algebra, constructed in Sect. 3. From the point of view of the automorphic form theory, the multiplication in H is given by the parabolic pseudo-Eisenstein series map. If X is a curve over $\mathbb{F}_q$, the analogous map for unramified automorphic forms over the function field $\mathbb{F}_q(X)$ gives the multiplication in the Hall algebra of X, see [14]. So in this paper we study the space H of automorphic forms on all the $GL_n(\mathbb{R})$ as an associative algebra in its own right.

We further concentrate on the subalgebra $SH \subset H$ generated by $H_1 = C_0^\infty(\mathbb{R}_{>0})$. Extending the terminology of [26], we call SH the *spherical Hall algebra* of the "curve" $\overline{\mathrm{Spec}(\mathbb{Z})}$. From the point of view of spectral decomposition [20], SH consists of automorphic forms expressible through the Eisenstein–Selberg series [27], the simplest higher-dimensional analogs of the nonholomorphic Eisenstein–Mass series on the upper half plane. This algebra has an explicit space of generators, but relations among these generators are not directly given.

(1.4) Our first main result describes SH as a Feigin–Odesskii-type shuffle algebra, in a way similar to the results of [26] for the case of curves over a finite field. However, in our case the shuffle algebra is based not on a rational, but on a meromorphic function: the Riemann zeta function $\zeta(s)$. This function, therefore, encodes all the relations among the generators from H_1.

Quadratic relations in SH correspond to the classical functional equation for the Eisenstein–Maass series, in a way similar to the case of function field considered in [14]. One form of writing the relations is in terms of "generating functions" (formal H-valued distributions) $\mathfrak{E}(s)$ depending on $s \in \mathbb{C}$. It has the form

$$\mathfrak{E}(s_1)\mathfrak{E}(s_2) = \frac{\zeta^*(s_1 - s_2)}{\zeta^*(s_1 - s_2 + 1)}\mathfrak{E}(s_2)\mathfrak{E}(s_1),$$

where $\zeta^*(s)$ is the full zeta function of $\overline{\mathrm{Spec}(\mathbb{Z})}$ (the product of $\zeta(s)$ with the Gamma and exponential factors). This is discussed in Sect. 7.

Our second main result, Theorem 8.6, is that the space of the cubic relations (not following from the quadratic ones) is identified with (an appropriate completion of) the space spanned by nontrivial zeroes of $\zeta(s)$. In other words, *the space spanned by the zeroes of $\zeta(s)$ can be realized as a certain algebraic homology space of the associative algebra H*. This is remindful of (but different from) the result of D. Zagier [32] who gave an interpretation of the zeta-space using integrals of Eisenstein–Maass series over anisotropic tori associated to real quadratic fields.

(1.5) After the first draft of this paper was written, we learned that M. Kontsevich and Y. Soibelman [17] have recently considered the algebra H as well. Their interest was in studying wall-crossing formulas in $\mathscr{B}un$, so our results practically do not intersect. We are grateful to M. Kontsevich and Y. Soibelman for explaining their work and providing us with the preliminary version of [17].

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2 Vector Bundles on $\overline{\mathrm{Spec}(\mathbb{Z})}$

By a *vector bundle on* $\overline{\mathrm{Spec}(\mathbb{Z})}$ we will mean a triple $E = (L, V, q)$, where V is a finite-dimensional $\mathbb{R}$-vector space, q is a positive definite quadratic form on V, and $L \subset V$ is a $\mathbb{Z}$-lattice of maximal rank. In this case, V becomes a Banach space with norm $\|v\| = \sqrt{q(v)}$.

The *rank* of E is defined as $\mathrm{rk}(E) = \dim_{\mathbb{R}}(V) = \mathrm{rk}_{\mathbb{Z}}(L)$. A *morphism* $f : E' = (L', V', q') \longrightarrow E = (L, V, q)$ of vector bundles on $\overline{\mathrm{Spec}(\mathbb{Z})}$ is, by definition, a linear operator $f : V' \to V$ such that, first, $f(L') \subset L$ and, second, $\|f\| \leqslant 1$, i.e., we have $q(f(v')) \leqslant q'(v')$ for each $v' \in V'$. In this way we get a category which we denote $\mathscr{B}un$. All the Hom-sets in $\mathscr{B}un$ are finite.

We denote by $\mathscr{O} = (\mathbb{Z}, \mathbb{R}, x^2)$ the *trivial bundle of rank 1*.

The *dual bundle* to E is defined as $E^{\vee} = (L^{\vee}, V^*, q^{-1})$, where q^{-1} is the inverse quadratic form on the dual space. The *tensor product* of two bundles is defined as

$$E \otimes E' = (L \otimes_{\mathbb{Z}} L', V \otimes_{\mathbb{R}} V', q \otimes q'), \quad (q \otimes q')(v \otimes v') := q(v)q'(v').$$

In particular, we have the bundle $\underline{\mathrm{Hom}}(E, E') = E^{\vee} \otimes E'$. The corresponding quadratic form on $\mathrm{Hom}_{\mathbb{R}}(V, V')$ takes $f : V \to V'$ into $\mathrm{tr}(f^t \circ f)$, where the transpose is taken with respect to q, q'. We leave to the reader the proof of the following:

Proposition 2.1. *Let* $E_i = (L_i, V_i, q_i)$, $i = 1, 2, 3$, *be three vector bundles on* $\overline{\mathrm{Spec}(\mathbb{Z})}$. *Then*

$$\mathrm{Hom}_{\mathscr{B}un}(E_1, \underline{\mathrm{Hom}}(E_2, E_3)) \subset \mathrm{Hom}_{\mathscr{B}un}(E_1 \otimes E_2, E_3)$$

as subsets in $\mathrm{Hom}_{\mathbb{R}}(V_1 \otimes V_2, V_3)$. □

Note the particular case of $E_1 = \mathscr{O}$. The proposition in this case reduces to the inequality

$$\|f\| \leqslant \sqrt{\mathrm{tr}(f^t \circ f)}$$

for any linear operator $f : V_2 \to V_3$. We also see why the inclusion in the proposition is not, in general, an equality. Indeed, for $E_1 = \mathscr{O}$, the Hom-set on the left consists of integer points in the domain $\mathrm{tr}(f^t \circ f) \leqslant 1$, which is an ellipsoid. But the Hom-set on the right consists of integer points in the domain $\|f\| \leqslant 1$ which is not an ellipsoid, if $\dim(V_2), \dim(V_3) > 1$.

We also have the symmetric and exterior product functors

$$S^r(E) = (S^r_{\mathbb{Z}}(L), S^r_{\mathbb{R}}(V), S^r(q)), \quad S^r(q)(v_1 \bullet \cdots \bullet v_r) := q(v_1) \cdots q(v_r),$$

$$\Lambda^r(E) = (\Lambda^r_{\mathbb{Z}}(L), \Lambda^r_{\mathbb{R}}(V), \Lambda^r(q)), \quad \Lambda^r(q)(v_1 \wedge \cdots \wedge v_r) := \det \|B(v_i, v_j)\|.$$

Here $\bullet$ is the product in the symmetric algebra, while B is the symmetric bilinear form such that $q(v) = B(v, v)$.

Let Bun_n be the set of isomorphism classes of rank n vector bundles on $\overline{\text{Spec}(\mathbb{Z})}$. This set is a classical double quotient of the theory of automorphic forms:

$$\text{Bun}_n \xleftarrow{\sim} GL_n(\mathbb{Z})\backslash GL_n(\mathbb{R})/O_n. \tag{1}$$

Explicitly, the double coset of $g_\infty \in GL_n(\mathbb{R})$ corresponds to the isomorphism class of the bundle $(\mathbb{Z}^n, \mathbb{R}^n, (g_\infty^t)_*^{-1}(q_{\text{st}}))$, where

$$q_{\text{st}}(x_1, \dots, x_n) = \sum_{i=1}^n x_i^2$$

is the standard quadratic form on $\mathbb{R}^n$ and $(g_\infty^t)_*^{-1}(q_{\text{st}})(x) = q_{\text{st}}((g_\infty^t)^{-1}(x))$ is the quadratic form corresponding to the symmetric matrix $(g_\infty^t)^{-1} \cdot g_\infty^{-1}$.

We will also need an adelic version of (1). Let $\mathfrak{A}^f = \prod_p^{\text{res}} \mathbb{Q}_p$ be the ring of finite adeles of the field $\mathbb{Q}$, let $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p \subset \mathfrak{A}^f$ be the profinite completion of $\mathbb{Z}$, and $\mathfrak{A} = \mathbb{R} \times \mathfrak{A}^f$ be the full ring of adeles. Then $K_n := O_n \times \prod_p GL_n(\mathbb{Z}_p)$ is a maximal compact subgroup of $GL_n(\mathfrak{A})$.

Proposition 2.2. *The embedding of $GL_n(\mathbb{R})$ into $GL_n(\mathfrak{A})$ induces a bijection*

$$\text{Bun}_n \simeq GL_n(\mathbb{Z})\backslash GL_n(\mathbb{R})/O_n \xrightarrow{\alpha} GL_n(\mathbb{Q})\backslash GL_n(\mathfrak{A})/K_n.$$

Proof. The statement is of course well known. We describe the inverse map explicitly for later use. Let $g = (g_\infty, (g_p)) \in GL_n(\mathfrak{A})$, so $g_\infty \in GL_n(\mathbb{R})$ and $g_p \in GL_n(\mathbb{Q}_p)$, with $g_p \in GL_n(\mathbb{Z}_p)$ for almost all p. We associate to g a vector bundle $E_g = (L_g, V_g, q_g)$ on $\overline{\text{Spec}(\mathbb{Z})}$ by putting:

$$L_g = \mathbb{Q}^n \cap \bigcap_p g_p^t(\mathbb{Z}_p^n), \quad V_g = \mathbb{R}^n, \quad q_g = (g_\infty^t)_*^{-1}(q_{\text{st}}).$$

It is clear that $E_{\gamma g k} \simeq E_g$ for $\gamma \in GL_n(\mathbb{Q})$, $k \in K_n$, so we get a map

$$GL_n(\mathbb{Q})\backslash GL_n(\mathfrak{A})/K_n \xrightarrow{\beta} \text{Bun}_n.$$

By construction, $\beta\alpha = \text{Id}$; the fact that $\alpha\beta = \text{Id}$ follows since L_g is a free abelian group.

Example 2.3. Take $n = 1$. The set Bun_1 formed by isomorphism classes of line bundles, will be also denoted by $\text{Pic}(\overline{\text{Spec}(\mathbb{Z})})$. This set is a group under tensor multiplication. It is identified with $\mathbb{R}_+^\times$, the multiplicative group of positive real numbers. Explicitly, given $E = (L, V, q)$ with $\dim_{\mathbb{R}}(V) = 1$, we associate to it the number $\deg(E) = 1/\sqrt{q(l_{\min})} \in \mathbb{R}_+$, where $l_{\min}$ is one of the two generators of L.

Conversely, for $a \in \mathbb{R}_+$ we denote by $\mathscr{O}(a) = (\mathbb{Z}, \mathbb{R}, a^{-2} \cdot q_{\mathrm{st}})$ the corresponding line bundle with $\deg(\mathscr{O}(a)) = a$. The convention, compatible with (1) for $n = 1$, is chosen so that for $a \gg 0$ the bundle $\mathscr{O}(a)$ has many "global sections", i.e., lattice points l such that $q(l) \leqslant 1$.

Example 2.4. For any n, taking the top exterior power together with the isomorphism of Example 2.3, gives a map

$$\mathrm{Bun}_n \xrightarrow{\det} \mathrm{Pic}(\overline{\mathrm{Spec}(\mathbb{Z})}) \xrightarrow{\sim} \mathbb{R}_+.$$

Explicitly, $E = (L, V, q)$ is sent into $1/\operatorname{Vol}(V/L)$, the inverse of the covolume of L with respect to the Lebesgue measure defined by q. We will denote this inverse covolume by $\deg(E)$ and call it the degree of E. We denote by $\mathrm{Bun}_{n,a}$ the set of isomorphism classes of bundles of rank n and degree a.

Consider the case $n = 2$ and take $a = 1$. In this case

$$\mathrm{Bun}_{2,1} = SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R}) / SO_2$$

is identified with the quotient $SL_2(\mathbb{Z}) \backslash \mathbb{H}$, where $\mathbb{H} \subset \mathbb{C}$ is the upper half-plane $\mathrm{Im}(z) > 0$. More explicitly, consider the standard quadratic form on $\mathbb{C}$ given by $q_{\mathrm{st}}(z) = |z|^2$. Then, for $\tau \in \mathbb{H}$, the lattice $\mathbb{Z} + \mathbb{Z}\tau$ has, with respect to q_{st}, the covolume equal to $\mathrm{Im}(\tau)$. We therefore associate to τ the bundle

$$E_\tau = \left(\mathbb{Z} + \mathbb{Z}\tau,\ \mathbb{C},\ q_{\mathrm{st}} / \operatorname{Im}(\tau)^{1/2}\right) \in \mathrm{Bun}_{2,1}\,.$$

Lemma 2.5. *For $\gamma \in SL_2(\mathbb{Z})$ we have $E_{\gamma(\tau)} \simeq E_\tau$, and this establishes an identification $SL_2(\mathbb{Z}) \backslash H \to \mathrm{Bun}_{2,1}$.*

Proof. It is clear that $E_\tau \simeq E_{\tau+1}$. Let us show that $E_{-1/\tau} \simeq E_\tau$. Note that

$$\operatorname{Vol}(\mathbb{C}/L_{-1/\tau}) = \operatorname{Im}(-1/\tau) = \operatorname{Im}\left(\frac{-\overline{\tau}}{|\tau|^2}\right).$$

Notice also that multiplication by τ defines an isomorphism of lattices

$$L_{-1/\tau} \xrightarrow{\tau} L_\tau.$$

The determinant of the multiplication by τ being $|\tau|^2$, we conclude that this multiplication defines an isomorphism

$$\left(L_{-1/\tau},\ \mathbb{C},\ q_{st} / \operatorname{Im}(-1/\tau)^{1/2}\right) \longrightarrow \left(L_\tau,\ \mathbb{C},\ q_{\mathrm{st}} / \operatorname{Im}(\tau)^{1/2}\right)$$

of vector bundles over $\overline{\mathrm{Spec}(\mathbb{Z})}$.

Let now

$$0 \to E' = (L', V', q') \xrightarrow{i} E = (L, V, q) \xrightarrow{j} E'' = (L'', V'', q'') \to 0 \qquad (2)$$

be a sequence of vector bundles on $\overline{\mathrm{Spec}(\mathbb{Z})}$ and their morphisms.

Definition 2.6. We say that a sequence (2) is *short exact* (in $\mathcal{B}un$), if the following hold:

(1) The induced sequences of vector spaces and abelian groups are short exact.
(2) The form q' is equal to $i^*(q)$, the pullback of q via i, defined by

$$(i^*q)(v') = q(i(v')), \quad v' \in V'.$$

(3) The form q'' is equal to $j_*(q)$, the pushforward of q via j, defined by

$$(j_*q)(v'') = \min_{j(v)=v''} q(v), \quad v'' \in V''.$$

An *admissible monomorphism* (resp. *admissible epimorphism*) in $\mathcal{B}un$ is a morphism which can be included into a short exact sequence as i (resp. j).

Let us call a *subbundle* in E an equivalence class of admissible monomorphisms $E' \to E$ modulo isomorphisms of the source. For such a subbundle E' we have the quotient bundle $E/E' \in \mathcal{B}un$.

Proposition 2.7. *Let $E = (L, V, q)$ be a vector bundle on $\overline{\mathrm{Spec}(\mathbb{Z})}$. The following sets are in bijection:*

(i) Rank r subbundles $E' \subset E$.
(ii) Rank r primitive sublattices, i.e., subgroups $L' \subset L$ such that L/L' has no torsion.
(iii) $\mathbb{Q}$-linear subspaces $W' \subset L \otimes_{\mathbb{Z}} \mathbb{Q}$ of dimension r.

Proof. The bijection between (ii) and (i) takes a primitive sublattice L' into $E' = (L', V', q')$, where $V' = L' \otimes_{\mathbb{Z}} \mathbb{R}$ and $q' = q|_{V'}$. The bijection between (iii) and (ii) takes a subspace W' into the sublattice $L' = L \cap W'$.

Proposition 2.8. *Let $E = (L, V, q)$ be a vector bundle on $\overline{\mathrm{Spec}(\mathbb{Z})}$. For any $r \in \mathbb{Z}_+$ and $a \in \mathbb{R}_+$, the set of subbundles $E' \subset E$ with $\mathrm{rk}(E') = r$ and $\deg(E') \geqslant a$, is finite.*

Proof. Let $W = L \otimes_{\mathbb{Z}} \mathbb{Q}$. Consider first the case $r = 1$. If $E' \subset E$ corresponds to a one-dimensional subspace $W' \subset W$, then $\deg(E) = 1/\sqrt{q(w')}$, where $w' \in W' \cap L = L'$ is one of two generators of this free abelian group of rank 1. Since the number of w' such that $q(w') \leqslant a$ is finite, our statement follows.

Consider now the case of arbitrary r and use the Plücker embedding of the Grassmannian $G(r, W)$ into $\mathbb{P}(\Lambda^r(W))$. If $W' \subset W$ is an r-dimensional subspace with $L' = W' \cap L$, then $\Lambda^r(W') \subset \Lambda^r(W)$ is a one-dimensional subspace, and

$\Lambda^r(W') \cap \Lambda^r_{\mathbb{Z}}(L) = \Lambda^r_{\mathbb{Z}}(L')$ is a free abelian group of rank 1 and a primitive sublattice in $\Lambda^r_{\mathbb{Z}}(L)$. Further, $\Lambda^r_{\mathbb{R}}(V)$ is equipped with the quadratic form $\Lambda^r(q)$, and $\deg(E') = 1/\sqrt{\Lambda^r(q)(\xi')}$, where E' is the subbundle corresponding to W' and $\xi' \in \Lambda^r_{\mathbb{Z}}(L')$ is one of the two generators. We thus reduce to the case of subbundles of rank 1.

Let $\overline{E'} = (L', V', q')$ and, $\overline{E''} = (L'', V'', q'')$ be two vector bundles on $\overline{\mathrm{Spec}(\mathbb{Z})}$. We define $\mathrm{Ext}^1(E'', E')$ to be the set of admissible short exact sequences (2) modulo automorphisms of such sequences identical on E' and E''.

Proposition 2.9. *The set* $\mathrm{Ext}^1(E'', E')$ *has a natural structure of a* C^∞*-manifold isomorphic to the torus* $(\mathbb{R}/\mathbb{Z})^{n'n''}$*, where* $n' = \mathrm{rk}(E')$ *and* $n'' = \mathrm{rk}(E'')$.

Proof. For any short exact sequence as in (2), the induced short exact sequence of lattices necessarily splits. Let us fix a splitting $L = L' \oplus L''$ and the induced splitting $V = V' \oplus V''$ of $\mathbb{R}$-vector spaces, so that i and j become the canonical embedding into and the projection from the direct sum. Let Λ be the set of positive definite quadratic forms q on V such that $i^*q = q'$ and $j_*q = q''$. By definition, Λ is a closed subset in the space of all positive definite quadratic forms on V and so has a natural topology.

The group $\mathrm{Hom}_{\mathbb{Z}}(L'', L')$ is identified with the group of automorphisms of the split exact sequence

$$0 \to L' \xrightarrow{i} L' \oplus L'' \xrightarrow{j} L'' \to 0$$

identical on L', L''. Therefore this group acts on Λ, and we have $\mathrm{Ext}^1(E'', E') = \Lambda / \mathrm{Hom}_{\mathbb{Z}}(L'', L')$.

Lemma 2.10. *The map* $\Lambda \xrightarrow{\mathrm{res}} \mathrm{Hom}_{\mathbb{R}}(V' \otimes V'', \mathbb{R})$ *which sends* q *into the induced pairing between the summands* V' *and* V''*, is a homeomorphism. This map takes the action of the group* $\mathrm{Hom}_{\mathbb{Z}}(L'', L')$ *on* Λ *into its action on* $\mathrm{Hom}_{\mathbb{R}}(V' \otimes V'', \mathbb{R})$ *by translations.*

Proof. Fix a basis $e_1, \ldots, e_{n'}$ of V', orthonormal with respect to q, and a basis $v_1, \ldots, v_{n''}$ of V''. Let B', B'' be the symmetric bilinear forms on V', V'' corresponding to q', q'', and let q be a quadratic form on V with corresponding symmetric bilinear form B. Then the condition $q \in \Lambda$ means:

$$B(e_i, e_j) = \delta_{ij} = B'(e_j, e_j),$$
$$B\left(v_p - \sum_{\mu=1}^{n'} B(v_p, e_\mu) \cdot e_\mu,\ v_q - \sum_{\nu=1}^{n'} B(v_q, e_\nu) \cdot e_\nu\right) = B''(v_p, v_q). \tag{3}$$

Indeed, the minimum in the definition of j_*q is given by the orthogonal projection to V' with respect to B, and the left hand side of the second formula above involves exactly such projections.

Denote by X the matrix $\|B(v_p, e_\mu)\|$ of size $n'' \times n'$, and let Y be the matrix $\|B(v_p, v_q)\|$ of size $n'' \times n''$. From the first condition in (3) we see that a quadratic form q with $i^*q = q'$ is completely determined by the datum of X and Y, while the second equation implies that $Y = B'' - X \cdot X^t$, where $B'' = \|B''(v_p, v_q)\|$. Therefore $q \in \Lambda$ is indeed completely defined by X, which is the matrix representative of $\mathrm{res}(q)$. The action of elements of $\mathrm{Hom}_{\mathbb{Z}}(L'', L')$ in the matrices X is the action by translation. This proves the lemma and Proposition 2.9.

Remark 2.11. More generally, one can consider data $\mathscr{F} = (L, V, q)$ similar to the above but where L is any finitely generated abelian group, $V = L \otimes_{\mathbb{Z}} \mathbb{R}$ and q is a positive definite quadratic form on V. They correspond to *coherent sheaves on* $\overline{\mathrm{Spec}(\mathbb{Z})}$ *locally free at infinity*. We get in this way a category $\mathscr{C}oh_{\neq\infty}(\overline{\mathrm{Spec}(\mathbb{Z})})$, with admissible short exact sequences defined similarly to Definition 2.6. A more systematic theory should enlarge $\mathscr{C}oh_{\neq\infty}(\overline{\mathrm{Spec}(\mathbb{Z})})$ by allowing a meaningful concept of sheaves with torsion at ∞. This will be done in a subsequent paper. For example, sheaves supported at ∞ can be described in terms of two positive definite quadratic forms $q \leqslant q'$ on one $\mathbb{R}$-vector space V, much in the same way as representing a finite abelian p-group as quotient of two free $\mathbb{Z}_p$-modules of the same rank. The role of elementary divisors is then played by the logarithms $\log \lambda_i(q : q') \in \mathbb{R}_+$, of the eigenvalues of q with respect to q'.

3 The Hall Algebra

Let

$$Y_n = GL_n(\mathbb{R})/O_n$$

be the space of quadratic forms on $\mathbb{R}^n$. It is a C^∞-manifold of dimension $n(n+1)/2$. It is well known that for large N the congruence subgroup

$$GL_n(\mathbb{Z}, N) = \{\gamma \in GL_n(\mathbb{Z}) : \gamma \equiv 1 \mod N\}$$

acts on Y_n freely, so $GL_n(\mathbb{Z}, N)\backslash Y_n$ is a C^∞-manifold. The set Bun_n is the quotient of this manifold by the finite group $GL_n(\mathbb{Z}/N)$ and therefore has a structure of a C^∞-orbifold. In particular, we can speak about C^∞-functions on Bun_n. They are C^∞-functions on $GL_n(\mathbb{R})$, left invariant under $GL_n(\mathbb{Z})$ and right invariant under O_n, i.e., C^∞-automorphic forms in the classical sense. Let

$$H_n = C_0^\infty(\mathrm{Bun}_n) = C_0^\infty\big(GL_n(\mathbb{Z})\backslash GL_n(\mathbb{R})/O_n\big)$$

be the space of C^∞-functions on Bun_n with compact support. In other words, an element of H_n is a C^∞-function $f : GL_n(\mathbb{R}) \to \mathbb{C}$ which is left $GL_n(\mathbb{Z})$-invariant, right O_n-invariant and such that $\mathrm{Supp}(f) \subset GL_n(\mathbb{Z}) \cdot K$ where $K \subset GL_n(\mathbb{R})$ is a compact subset.

Consider the direct sum

$$H = \bigoplus_{n=0}^{\infty} H_n, \quad H_0 = \mathbb{C}.$$

Let $f \in H_m, g \in H_n$. We define their *Hall product* $f * g$ to be the function $\mathrm{Bun}_{m+n} \to \mathbb{C}$ given by the formula

$$(f * g)(E) = \sum_{E' \subset E} \deg(E')^{n/2} \deg(E/E')^{-m/2} \cdot f(E') g(E/E'), \tag{4}$$

where the sum is over all subbundles $E' \subset E$ of rank m.

Proposition 3.1. *(a) For every E the sum in (4) is actually finite, so $f * g$ is a well defined function.*
(b) $f * g$ is again a C^∞-function with compact support.
(c) The operation $f * g$ makes H into a graded associative algebra, with unit $1 \in H_0$.

We will call the algebra H the *Hall algebra of* $\overline{\mathrm{Spec}(\mathbb{Z})}$. In this paper will be particularly interested in the subalgebra $SH \subset H$ generated by $H_1 = C_0^\infty(\mathbb{R}_+)$. We will call SH the *spherical Hall algebra*, adopting the terminology of [26], where a similar algebra was studied for the case of a curve over a finite field.

Remark 3.2. (a) The quantity

$$\langle E/E', E' \rangle = \deg(E')^{n/2} \deg(E/E')^{-m/2} = \sqrt{\deg \underline{\mathrm{Hom}}(E/E', E')}$$

is the analog of the Euler form used by Ringel [22] to twist the multiplication in the Hall algebra of representations of a quiver. In our case, as well as in the case of curves over a finite field [14, 26], twisting by this form simplifies the form of commutation relations.
(b) One can get larger algebras by relaxing the condition of compact support to that of sufficiently rapid decay at infinity. More generally, there are interesting cases when f and g do not have rapid decay, but $f * g$ still makes sense as a convergent series.
(c) Hall algebras of exact categories were considered in [5, 12]. Note that Definition 2.6 defines on $\mathscr{B}un$ a structure remindful of that of an exact category but lacking additivity (Hom-sets do not form abelian groups), Such structures were axiomatized in [3] under the name "proto-exact categories".

Proof of Proposition 3.1. (a) Since f is with compact support, there is $A > 1$ such that $f(E') = 0$ unless $\deg(E') \in [1/A, A]$. By Proposition 2.8 all but finitely many $E' \subset E$ have $\deg(E') < A$, so that the sum in (4) is indeed finite.
(b) To see that $f * g$ is smooth, suppose that E_1 and E_2 are close to each other in Bun_{m+n}. Then the corresponding lattices L_1 and L_2 are identified in a canonical

fashion. Therefore the sets of subbundles $E_1' \subset E_1$ and $E_2' \subset E_2$ of rank m, are identified, and so we have a bijection between the sets of summands in $(f * g)(E_1)$ and $(f * g)(E_2)$. Next, the number of nonzero summands in both sums is bounded by the same number by the continuity of f and g, so we can view $f * g$ as a sum of finitely many C^∞-functions.

To see that $f * g$ has compact support, let $\Sigma_1 \subset \mathrm{Bun}_m$ be a compact set supporting f, and Σ_2 be a compact set supporting g. For any $E_1 \in \Sigma_1$, $E_2 \in \Sigma_2$ the set of $E \in \mathrm{Bun}_{m+n}$ that can fit into a sequence (2), is a compact topological space. Indeed, it is the image of a continuous map $\mathrm{Ext}^1(E_2, E_1) \to \mathrm{Bun}_{m+n}$, whose source is a compact torus. Let F be the total space of the fibration over $\Sigma_1 \times \Sigma_2$ with fiber over (E_1, E_2) being $\mathrm{Ext}^1(E_2, E_1)$. Then F is compact, while the support of $f * g$ is contained in the image of F under a natural continuous map into Bun_{m+n}.

(c) To prove associativity, let $f \in H_{n_1}$, $g \in H_{n_2}$, $h \in H_{n_3}$. Then for $E \in \mathrm{Bun}_{n_1+n_2+n_3}$ we have

$$((f * g) * h)(E) = \sum_{E_1 \subset E_2 \subset E} d_1^{\frac{n_2+n_3}{2}} d_2^{\frac{-n_1+n_3}{2}} d_3^{\frac{-n_1-n_2}{2}} \cdot f(E_1) \cdot g(E_2/E_1) \cdot h(E/E_2),$$

where E_1 runs over subbundles of E of rank $n_1 + n_2$, and E_1 runs over subbundles of E_2 of rank n_1, and we have denoted

$$d_1 = \deg(E_1),\ d_2 = \deg(E_2/E_1),\ d_3 = \deg(E/E_2).$$

On the other hand

$$(f*(g*h))(E) = \sum_{\substack{E_1 \subset E \\ E_2' \subset E/E_1}} \delta_1^{\frac{n_2+n_3}{2}} \delta_2^{\frac{-n_1+n_3}{2}} \delta_3^{\frac{-n_1-n_2}{2}} \cdot f(E_1) \cdot g(E_2') \cdot h\big((E/E_1)/E_2'\big),$$

where we have denoted

$$\delta_1 = \deg(E_1),\ \delta_2 = \deg(E_2'),\ \delta_3 = \deg((E/E_1)/E_2').$$

Let F be the set over which the first sum is extended, i.e., the set of admissible filtrations $E_1 \subset E_2 \subset E$ with $\mathrm{rk}(E_1) = n_1$ and $\mathrm{rk}(E_2) = n_1 + n_2$. Similarly, let F_2 be the set over which the second sum is extended, i.e., the set of pairs (E_1, E_2'), where $E_1 \subset E$ is a subbundle of rank n_1, and $E_2' \subset E/E_1$ is a subbundle of rank n_2. We have a map $\rho : F \to F'$ sending $(E_1 \subset E_2 \subset E)$ into $(E_1, E_2' := E_2/E_1)$. The summand corresponding to any $\phi \in F$ is equal to the summand corresponding to $\rho(\phi) \in F'$. So our statement reduces to the following.

Lemma 3.3. *The map ρ is a bijection.*

Proof. An element of F has the form

$$(L_1, V_1, Q_1) \subset (L_2, V_2, q_2) \subset (L, V, q) = E,$$

where $L_1 \subset L_2 \subset L$ is a filtration by primitive sublattices, and $q_i = q|_{V_i}$, $i = 1, 2$. An image of such an element by ρ is the pair $\big((L_1, V_1, q_1),\ (L_2', V_2', q_2')\big)$, where (L_1, V_1, q_1) is as above, while $L_2' = L_2/L_1 \subset L/L_1$, and $q_2' = \pi'_*(q_2)$, with $\pi' : V_2 \to V_2/V_1 = V_2'$ being the canonical projection.

On the other hand, a general element of F' is a pair $\big((L_1, V_1, q_1),\ (L_2', V_2', q_2')\big)$, where (L_1, V_1, q_1) is as above, while $L_2' \subset L/L_1$ is an arbitrary primitive sublattice of rank n_2, and $V_2' = L_2' \otimes \mathbb{R}$ and q_2' is the restriction to V_2' of the quotient quadratic form $\pi_*(q)$ for the projection $\pi : V \to V/V_1$, i.e., $q_2' = (i')^*(\pi_* q)$. We have therefore a Cartesian square of $\mathbb{R}$-vector spaces

$$\begin{array}{ccc} V_2 & \xrightarrow{\quad i \quad} & V \\ {\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ V_2' = V_2/V_1 & \xrightarrow{\quad i' \quad} & V/V_1 \end{array}$$

with π, π' surjective and i, i' injective. We claim that $\pi'_* i^*(q) = i'^* \pi_*(q)$, and hence $\rho(F) = F'$. This is a particular case of the following base change property for quadratic forms.

Proposition 3.4. *Let*

$$\begin{array}{ccc} U_2 & \xrightarrow{\ i_1\ } & U \\ {\scriptstyle j'}\big\downarrow & & \big\downarrow{\scriptstyle j} \\ U_2' & \xrightarrow{\ i_2\ } & U' \end{array}$$

be a Cartesian square of $\mathbb{R}$-vector spaces, such that i_1, i_2 are injective and j, j' are surjective. Then for any positive definite quadratic form q on U we have the equality $j'_* i_1^* q = i_2^* j_* q$ of quadratic forms on U_2'.

Proof. Let $u_2' \in U_2'$. Then

$$(j'_* i_1^* q)(u_2') = \min_{u_2 :\, j'(u_2) = u_2'} (i_1^* q)(u_2) = \min_{u_2 :\, j'(u_2) = u_2'} q(i_1(u_2)).$$

Since the square is Cartesian, i_1 identifies $(j')^{-1}(u_2')$ with $j^{-1}(i_2(u_2'))$, so the last minimum is equal to

$$\min_{u :\, j(u) = i_2(u_2')} q(u) = (j_* q)(i_2(u_2')).$$

This finishes the proof of Lemma 3.3 as well as Proposition 3.1.

Remark 3.5. One can extend the definition of the Hall algebra to the category $\mathscr{C}oh_{\neq\infty}(\overline{\text{Spec}(\mathbb{Z})})$ defined as in Remark 2.11, using the concept of admissible exact sequences outlined there. The algebra thus obtained will be a semidirect product of H and the Hall algebra of the category of finite abelian groups, similarly to [15], §2.6.

4 The Mellin Transform

A standard tool in the theory of quantum affine algebras is the use of generating functions, i.e., passing from a collection of coefficients $(c_\alpha)_{\alpha\in\mathbb{Z}^n}$ to the Laurent series

$$F(t) \;=\; \sum_{\alpha\in\mathbb{Z}^n} c_\alpha t^\alpha, \quad t=(t_1,\dots,t_n)\in(\mathbb{C}^*)^n, \quad t^\alpha=\prod t_\nu^{\alpha_\nu}. \tag{5}$$

This is just the Fourier transform on the free abelian group $\mathbb{Z}^n$, but understood in a more pragmatic way: we do not necessarily restrict to unitary characters (they form the real torus $|t_i|=1$) but pay attention to the domains of convergence in the space $(\mathbb{C}^*)^n$ of all characters.

A typical free abelian group to which the above is applied is, in the Hall algebra approach, $\text{Pic}(X)/\{\text{torsion}\} = \mathbb{Z}$, where X is a smooth projective curve over $\mathbb{F}_q$, see [14, 26], In the present paper the corresponding role is played by the group $\text{Pic}(\overline{\text{Spec}(\mathbb{Z})}) = \mathbb{R}_+$. The Fourier transform on $\mathbb{R}_+^n$ is known as the *Mellin transform*. We now give a summary of its properties from the same pragmatic standpoint as above.

Unitary characters of $\mathbb{R}_+^n$ have the form

$$a=(a_1,\dots,a_n)\longmapsto a^s \;=\; \prod a_\nu^{s_\nu}, \qquad s_\nu\in i\,\mathbb{R}\subset\mathbb{C},\; a_\nu^{s_\nu}=e^{s_\nu\log(a_\nu)},$$

and the Haar measure is $d^*a = \prod da_\nu/a_\nu$. Accordingly, the Mellin transform of a function (or a distribution) $f(a)$ on $\mathbb{R}_+^n$ is the integral

$$F(s)=(\mathscr{M}f)(s) \;=\; \int_{a\in\mathbb{R}_+^n} f(a)a^s d^*a. \tag{6}$$

Here, a priori, $s\in i\,\mathbb{R}^n$, but we are interested in allowing the s_i to vary in the complex domain, i.e., in considering not necessarily unitary characters. The group isomorphism

$$\exp:\mathbb{R}^n \xrightarrow{\sim} \mathbb{R}_+^n. \tag{7}$$

transforms the Mellin integral into the standard Fourier integral on $\mathbb{R}^n$.

Example 4.1 (Paley–Wiener Theorem). If $f(a)$ has compact support, then $(\mathscr{M}f)(s)$ converges for any $s \in \mathbb{C}^n$, i.e., $\mathscr{M}f$ is an entire function, analogously to the case of a Laurent series in (5) being a Laurent polynomial. Recall that an entire function $F(s)$, $s \in \mathbb{C}^n$, is called a *Paley–Wiener function*, if there is a constant $B > 0$ and, for every $N > 0$ there is a $c_N > 0$ such that

$$|F(s)| \leqslant c_N(1 + \|s\|)^{-N} e^{B\cdot\|\operatorname{Re}(s)\|}.$$

This means, in particular, that F has a faster than polynomial decay on each vertical subspace $\{\sigma_0 + i\mathbb{R}^n, \sigma_0 \in \mathbb{R}^n\}$, while allowed to have exponential growth on any horizontal subspace. We denote by $\mathscr{PW}(\mathbb{C}^n)$ the space of Paley–Wiener functions on $\mathbb{C}^n$. The Paley–Wiener theorem says:

Proposition 4.2. *The Mellin transform $\mathscr{M}$ identifies $C_0^\infty(\mathbb{R}_+^n)$ with $\mathscr{PW}(\mathbb{C}^n)$.*

Proof. The classical formulation, see, e.g., [21], Vol. II, Thm. IX.11, is for the Fourier transform of compactly supported functions on $\mathbb{R}^n$. The case of the Mellin transform reduces to this via exp.

An important point about series (5) is that one (meromorphic) function can have different Laurent expansions in different regions, while the region of convergence of each expansion is "logarithmically convex", i.e., is the preimage of a convex open set $\Delta \subset \mathbb{R}^n$ under the map

$$\lambda : (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n, \quad (t_i) \mapsto (\log|t_i|).$$

We now review the corresponding features of Mellin expansions. Unlike in the case of Laurent series, these features are less familiar, and a precise treatment involves L. Schwartz's theory of Fourier transform for distributions.

For a C^∞-manifold or orbifold M we denote by $\mathscr{D}ist(M) = C_0^\infty(M)'$ the space of distributions on M. Let $\mathscr{S}(\mathbb{R}^n)$ be the space of Schwartz functions on $\mathbb{R}^n$, and $\mathscr{D}(\mathbb{R}^n) = \mathscr{S}(\mathbb{R}^n)' \subset \mathscr{D}ist(\mathbb{R}^n)$ be the dual space of tempered distributions, see [21], Vol. I, §V.3. Recall that a C^∞-function lies in $\mathscr{D}(\mathbb{R}^n)$ if and only if it has at most polynomial growth.

We define $\mathscr{S}(\mathbb{R}_+^n)$ and $\mathscr{D}(\mathbb{R}_+^n)$, the spaces of Schwartz functions and tempered distributions on $\mathbb{R}_+^n$, by means of the group isomorphism exp of (7). For $f \in \mathscr{D}(\mathbb{R}_+^n)$ we define $\mathscr{M}f$ to be the tempered distribution on $i\mathbb{R}^n$ given by the Fourier–Schwartz transform of $f \circ \exp$.

For a distribution $f \in \mathscr{D}ist(\mathbb{R}_+^n)$ we denote by $\operatorname{Temp}(f)$ and call the *tempering set* of f, the set of $\sigma \in \mathbb{R}^n$ such that $f(a)\cdot a^\sigma$ is a tempered distribution. It is known (see [21], Vol. II, Lemma after Th. IX. 14.1) that $\operatorname{Temp}(f)$ is a convex subset in $\mathbb{R}^n$. We say that f is *temperable*, if $\operatorname{Temp}(f)$ has non-empty interior. For any convex open set $\Delta \subset \mathbb{R}^n$ we denote by $U_\Delta = \{s \in \mathbb{C}^n \mid \operatorname{Re}(s) \in \Delta\}$ the corresponding tube domain.

Proposition 4.3. *Let f is a temperable distribution on $\mathbb{R}^n_+$, and Δ be the interior of* $\mathrm{Temp}(f)$. *Then $F(s) = (\mathscr{M}f)(s)$ is an analytic function in U_Δ, which has an at most polynomial growth on each vertical subspace $\sigma_0 + i\mathbb{R}^n$, $\sigma_0 \in \Delta$.*

Proof. To see holomorphy, it is enough to assume that 0 is an interior point of $T(f)$ and to show that $\mathscr{M}f$ is holomorphic in an open neighborhood of $i\mathbb{R}^n$. For a sequence of signs $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$, $\varepsilon_i \in \{\pm 1\}$ let $(\mathbb{R}^n_+)_\varepsilon \subset \mathbb{R}^n_+$ be the domain given by conditions $a_i^{\epsilon_i} > 1$, and $\mathbb{C}^n_\varepsilon \subset \mathbb{C}^n$ be given by the condition $\varepsilon_i \,\mathrm{Re}(s_i) < 0$. Let $\mathscr{M}_\varepsilon f$ be the partial Mellin integral of f, taken over $(\mathbb{R}^n_+)_\varepsilon$. If $s \in \mathbb{C}^n_\varepsilon$, then the function a^s decays exponentially at the infinity of $(\mathbb{R}^n_+)_\varepsilon$. Therefore, if f is a tempered distribution on $\mathbb{R}^n_+$ (i.e., if $0 \in \mathrm{Temp}(f)$), then $\mathscr{M}_\varepsilon f$ extends to a holomorphic function in $\mathbb{C}^n_\varepsilon$. If, moreover, 0 is an interior point of $\mathrm{Temp}(f)$, then $\mathscr{M}_\varepsilon f$ is holomorphic in additive translates $\mathbb{C}^n_\varepsilon + \sigma$ for σ running in an open neighborhood of 0 in $\mathbb{R}^n$. Therefore $\mathscr{M}f = \sum_\varepsilon \mathscr{M}_\varepsilon f$ is holomorphic for $\mathrm{Re}(s)$ running in some open neighborhood of 0, as claimed.

To see that $\mathscr{M}f$ has at most polynomial growth on each $\sigma_0 + i\,\mathbb{R}^n$, it is again enough to treat the case $\sigma_0 = 0$. The restriction of $\mathscr{M}f$ to $i\,\mathbb{R}^n$ is a tempered distribution, the Fourier–Schwartz transform of $f \circ \exp$. As it is also a real analytic function, it must be of polynomial growth.

Next, we discuss the inverse Mellin transform, i.e., the analog of the formula which finds each coefficient c_α in (5) as an integral of $F(t)$ times a monomial. Formally, the inverse Mellin integral is defined by

$$f(a) = (\mathscr{N}_\Delta F)(a) = \frac{1}{(2\pi i)^n} \int_{s \in \sigma_0 + i\mathbb{R}^n} F(s) a^{-s} ds. \quad \sigma_0 \in \Delta,\ a \in \mathbb{R}^n_+, \qquad (8)$$

In our case, this integral should again be understood using Schwartz's theory. More precisely, we have:

Proposition 4.4. *Let $\Delta \subset \mathbb{R}^n$ be a convex open set and $F(s)$ be an analytic function in U_Δ with at most polynomial growth on each vertical subspace. Choose $\sigma_0 \in \Delta$ and define $f(a) = (\mathscr{N}_\Delta F)(a)$ as $a^{-\sigma_0}$ times the inverse Fourier transform of g as a tempered distribution on $\sigma_0 + i\ \mathbb{R}^n \simeq \mathbb{R}^n$ (the Fourier transform being transplanted to $\mathbb{R}^n_+$ via* $\exp$*). Then $\mathscr{N}_\Delta F$ is independent on $\sigma_0 \in \Delta$, and is a temperable distribution on $\mathbb{R}^n_+$ such that $\Delta \subset$* $\mathrm{Temp}(f)$ *and $\mathscr{M}f = F$.*

We will call $\mathscr{N}_\Delta(F)$ the *coefficient function* of F in U_Δ. Thus the existence of the coefficient function presupposes that F grows at most polynomially on each vertical subspace in U_Δ. As usual with the Fourier transform, the coefficient function of the product of analytic functions is the convolution (on the group $\mathbb{R}^n_+$) of the coefficient functions of the factors.

Proof. To show independence, it is enough to assume $0 \in \Delta$ and compare the integrals (8) over $i\ \mathbb{R}^n$ and $\sigma_0 + i\ \mathbb{R}^n$ for σ_0 being close to 0 in Δ. Both functions $F(s)$ and $F(s + \sigma_0)$ are tempered distributions on $i\ \mathbb{R}^n \simeq \mathbb{R}^n$ and so have Fourier–Schwartz transforms. Moreover, $F(s + \sigma_0)$ the sum of a Taylor series involving derivatives of $F(s)$ (evaluated on $i\,\mathbb{R}^n$). So the Fourier transform of $F(s+\sigma_0)$ is product of the Fourier transform of $F(s)$ and an exponential factor. This factor is accounted for by the change in a^s in the integral (8), showing the independence. The remaining claims follow from the inversion theorem for the Fourier–Schwartz transform.

Let us note the particular case $\Delta = \mathbb{R}^n$.

Corollary 4.5. *The Mellin transforms $\mathscr{M}$ and $\mathscr{N}$ defines mutually inverse isomorphisms between the following two spaces:*

- $\mathscr{D}(\mathbb{R}^n_+)_{\mathrm{abs}}$*, the space of* absolutely tempered distributions*, i.e., of distributions $f(a)$ such that $f(a)a^s$ is tempered for each $s \in \mathbb{C}^n$.*
- $\mathscr{O}(\mathbb{C}^n)_{\mathrm{pol}}$*, the space of entire functions in $\mathbb{C}^n$ with at most polynomial growth on each vertical subspace.*

□

Note that an absolutely tempered distribution has actually exponential decay at the infinity of $\mathbb{R}^n_+$.

For future reference we recall two elementary properties of the Mellin/Fourier transform. We denote by $\delta_c \in \mathscr{D}(\mathbb{R}_+)$ the delta function at $c \in \mathbb{R}_+$.

Proposition 4.6. *(a) Let $F(s)$ be analytic in U_Δ, with the coefficient function $f(a) = \mathscr{N}_\Delta(F)$. Then for any $\nu = 1, \ldots, n$ we have*

$$\mathscr{N}_\Delta(s_\nu F(s)) \;=\; -a_\nu \frac{d}{da_\nu} f(a).$$

(b) Let Δ be an interval $(c, c') \subset \mathbb{R}$, so U_Δ is a strip in $\mathbb{C}$. Let $h(s)$ be analytic in U_Δ, with coefficient function $k(a)$, $a \in \mathbb{R}_+$. Consider the function of two variables

$$F(s_1, s_2) = h(s_1 - s_2), \quad (s_1, s_2) \in U_{\tilde{\Delta}} = \{c < \mathrm{Re}(s_1 - s_2) < c'\}$$

Then the coefficient function of F is found by

$$(\mathscr{N}_{\tilde{\Delta}} F)(a_1, a_2) \;=\; \delta_1(a_1 a_2) \cdot k(a_1).$$

□

Example 4.7. Let $\zeta(s)$ be the Riemann zeta function, and

$$\zeta^*(s) \;=\; \pi^{-s/2}\Gamma(s/2)\zeta(s) \tag{9}$$

be the zeta function of $\overline{\mathrm{Spec}(\mathbb{Z})}$. It is a meromorphic function on $\mathbb{C}$ with simple poles at 0 and 1, satisfying $\zeta^*(s) = \zeta^*(1-s)$.

The function $\Gamma(s)$ has exponential decay on each vertical line $\sigma_0 + i\,\mathbb{R}$, as follows from the Stirling formula. The function $\zeta(s)$ has at most polynomial growth on each vertical line, see [4], Chap. 9. Therefore $\zeta^*(s)$ has exponential decay on each vertical line and therefore has a well defined coefficient function in each of the three strips of holomorphy: $\mathrm{Re}(s) > 1$, $0 < \mathrm{Re}(s) < 1$ and $\mathrm{Re}(s) < 0$. The coefficient function in $\mathrm{Re}(s) > 1$ is given by the classical formula of Riemann, see [4], §1.7:

$$(\mathscr{N}_{\mathrm{Re}(s)>1}\zeta^*)(a) \;=\; \theta(a^2) - 1, \quad \theta(b) \;:=\; \sum_{n=-\infty}^{\infty} e^{-n^2\pi b}. \tag{10}$$

It can be obtained by forming the convolution (on $\mathbb{R}_+$) of the distribution $\sum_{n=1}^{\infty} \delta_{1/n}$, of the function $2e^{-a^2}$ and of the distribution $\delta_{1/\sqrt{\pi}}$. These three distributions are the coefficient functions for $\zeta(s) = \sum 1/n^s$, for $\Gamma(s/2) = 2\int_0^\infty e^{-a^2} a^s d^*a$ and for $\pi^{-s/2}$ respectively. The coefficient functions in the two other strips are obtained by moving the contour past the poles of $\zeta^*(s)$ at $s = 1$ and $s = 0$ with residues $\pm 1/\sqrt{\pi}$:

$$(\mathscr{N}_{0<\mathrm{Re}(s)<1}\zeta^*)(a) = \theta(a^2) - 1 - \frac{1}{a\sqrt{\pi}},$$

$$(\mathscr{N}_{\mathrm{Re}(s)<0}\zeta^*)(a) = \theta(a^2) + 1/\sqrt{\pi} - 1 - \frac{1}{a\sqrt{\pi}}.$$

5 The Zeta Function Shuffle Algebra

We recall the formalism of shuffle algebras of Feigin–Odesskii [7], see [15, 26] for a more systematic discussion in the rational function case. We denote by $\mathfrak{S}_n$ the symmetric group of permutations of $\{1, \dots, n\}$.

Let $\varphi(s)$ be a meromorphic function on $\mathbb{C}$. For any $m, n > 0$ let $Sh(m, n)$ be the set of (m, n)-*shuffles*, i.e., permutations $w \in \mathfrak{S}_{m+n}$ such that $w(i) < w(j)$ whenever $i < j$ and either both $i, j \in [1, m]$ or both $i, j \in [m+1, m+n]$. For any $w \in Sh(m, n)$ consider the following meromorphic function on $\mathbb{C}^{m+n}$:

$$\varphi_w(s_1, \dots, s_{m+n}) \;=\; \prod_{\substack{i\in[1,m] \\ j\in[m+1,m+n], \\ w(i)>w(j)}} \varphi(s_i - s_j). \tag{11}$$

Let $\mathscr{O}(\mathbb{C}^n) \subset \mathscr{M}er(\mathbb{C}^n)$ be the spaces of all entire and meromorphic functions on $\mathbb{C}^n$ (defined to be equal to $\mathbb{C}$ for $n = 0$). On the direct sum $\bigoplus_n \mathscr{M}er(\mathbb{C}^n)$ we introduce the *shuffle multiplication*

$$ⓢ_{m,n} : \mathscr{M}er(\mathbb{C}^m) \otimes \mathscr{M}er(\mathbb{C}^n) \longrightarrow \mathscr{M}er(\mathbb{C}^{m+n}), \quad F \otimes F' \mapsto F ⓢ F', \tag{12}$$

by the formula

$$
\begin{aligned}
&(F \circledS F')(s_1, \dots, s_{m+n}) \\
&\quad = \sum_{w \in Sh(m,n)} w\Big(F(s_1, \dots, s_m) F'(s_{m+1}, \dots, s_{m+n}) \Big) \cdot \varphi_w(s_1, \dots, s_{m+n}). \quad (13)
\end{aligned}
$$

The following is then straightforward, as in [7].

Proposition 5.1. *The shuffle multiplication* $\circledS$ *makes* $\bigoplus_n \mathscr{M}er(\mathbb{C}^n)$ *into a graded associative algebra, with unit* $1 \in \mathscr{M}er(\mathbb{C}^0)$. □

Assume further that the function φ satisfies the equation $\varphi(-s)\varphi(s) = 1$, and, moreover, is represented in the form

$$
\varphi(s) = \lambda(s)^{-1}\lambda(-s) \quad (14)
$$

for some meromorphic function $\lambda(s)$. For $n \geqslant 0$ let $\mathscr{M}er(\mathbb{C}^n)^{\mathfrak{S}_n}$ be the space of symmetric meromorphic functions on $\mathbb{C}^n$. On the direct sum $\bigoplus_n \mathscr{M}er(\mathbb{C}^n)^{\mathfrak{S}_n}$, we introduce the *symmetric shuffle multiplication*

$$
\begin{aligned}
\star_{m,n} : \mathscr{M}er(\mathbb{C}^m)^{\mathfrak{S}_m} \otimes \mathscr{M}er(\mathbb{C}^n)^{\mathfrak{S}_n} &\longrightarrow \mathscr{M}er(\mathbb{C}^{m+n})^{\mathfrak{S}_{m+n}}, \\
F \otimes F' &\mapsto F \star F', \quad (15)
\end{aligned}
$$

by the formula

$$
\begin{aligned}
&(F \star F')(s_1, \dots, s_{m+n}) \\
&\quad = \sum_{w \in Sh(m,n)} w\Bigg(F(s_1, \dots, s_m) F'(s_{m+1}, \dots, s_{m+n}) \prod_{\substack{1 \leqslant i \leqslant m \\ m+1 \leqslant j \leqslant m+n}} \lambda(s_i - s_j) \Bigg). \quad (16)
\end{aligned}
$$

Proposition 5.2. *(a) The multiplication* $\star$ *makes* $\bigoplus_n \mathscr{M}er(\mathbb{C}^n)^{\mathfrak{S}_n}$ *into a graded associative algebra with unit.*

(b) The correspondence

$$
F(s_1, \dots, s_n) \longmapsto F(s_1, \dots, s_n) \prod_{i<j} \lambda(s_i - s_j)
$$

defines an injective algebra homomorphism

$$
\Big(\bigoplus_n \mathscr{M}er(\mathbb{C}^n)^{\mathfrak{S}_n}, \star \Big) \hookrightarrow \Big(\bigoplus_n \mathscr{M}er(\mathbb{C}^n), \circledS \Big).
$$

(c) *Assume that* $\lambda(s)$ *has no poles except, possibly, a first order pole at* $s = 0$. *Then the graded subspace* $\bigoplus_n \mathscr{O}(\mathbb{C}^n)^{\mathfrak{S}_n}$ *is a subalgebra with respect to* $\star$.

Proof. Parts (a) and (b) are proved straightforwardly, as in [7]. For (c), let us indicate why

$$\star_{1,1} : \mathscr{O}(\mathbb{C}) \times \mathscr{O}(\mathbb{C}) \longrightarrow \mathscr{M}er(\mathbb{C}^2)$$

takes values in $\mathscr{O}(\mathbb{C}^2)$ (the general case is similar). Writing $\lambda(s) = cs^{-1} + h(s)$ with h entire, we have, for $f, g \in \mathscr{O}(\mathbb{C})$:

$$\begin{aligned}(f \star g)(s_1, s_2) &= \lambda(s_1 - s_2) f(s_1) g(s_2) + \lambda(s_2 - s_1) f(s_2) g(s_1) \\ &= \frac{c}{s_1 - s_2}\big[f(s_1)g(s_2) - f(s_2)g(s_1)\big] \ + \ (\text{entire}),\end{aligned}$$

and the expression in square brackets, being an entire antisymmetric function, vanishes on the diagonal $s_1 = s_2$.

Definition 5.3. (a) We call the *shuffle algebra* associated to φ the subalgebra $\mathscr{SH}(\varphi) \subset \bigoplus_{n\geqslant 0} \mathscr{M}er(\mathbb{C}^n)$ generated by the space $\mathscr{O}(\mathbb{C}) \subset \mathscr{M}er(\mathbb{C}^1)$. We call the *symmetric shuffle algebra* associated to λ the subalgebra $\mathscr{SSH}(\lambda) \subset \bigoplus_{n\geqslant 0} \mathscr{M}er(\mathbb{C}^n)^{\mathfrak{S}_n}$ generated by $\mathscr{O}(\mathbb{C})$.

(b) The *Paley–Wiener shuffle algebra* $\mathscr{SH}(\varphi)_{\mathscr{PW}}$, resp. the *Paley–Wiener symmetric shuffle algebra* $\mathscr{SSH}(\lambda)_{\mathscr{PW}}$, is defined as the subalgebra in $\mathscr{SH}(\varphi)$, resp. $\mathscr{SSH}(\lambda)$, generated by the subspace $\mathscr{PW}(\mathbb{C}) \subset \mathscr{O}(\mathbb{C})$.

Thus, if φ and λ are related by (14), then $\mathscr{SH}(\varphi)$ is isomorphic to $\mathscr{SSH}(\lambda)$ and $\mathscr{SH}(\varphi)_{\mathscr{PW}}$ to $\mathscr{SSH}(\lambda)_{\mathscr{PW}}$ If, further, λ satisfies the condition (c) of Proposition 5.2, then $\mathscr{SSH}(\lambda)$ is a subalgebra of $\bigoplus \mathscr{O}(\mathbb{C}^n)^{\mathfrak{S}_n}$.

We now specialize $\varphi(s)$ to be the following meromorphic function:

$$\Phi(s) \ = \ \zeta^*(s)/\zeta^*(s+1). \tag{17}$$

It is known as the *global Harish–Chandra function* (or the *scattering matrix*) for $\overline{\mathrm{Spec}(\mathbb{Z})}$, cf. [16, §7]. The functional equation for $\zeta(s)$ implies that $\Phi(-s)\Phi(s) = 1$. We also consider the function

$$\Lambda(s) = \zeta^*(-s)(s-1)(-s-1). \tag{18}$$

It has just one simple pole at $s = 0$, with $\mathrm{res}_{s=0}\, \Lambda(s) = 1$, and zeroes at nontrivial zeroes of $\zeta(s)$ as well as at $s = -1$. We also have the identity

$$\Phi(s) = \Lambda(s)^{-1}\Lambda(-s). \tag{19}$$

Here is the main result of this paper, which will be proved in Sect. 6.

Theorem 5.4. *The Mellin transform* $\mathcal{M} : SH_1 = C_0^\infty(\mathbb{R}_+) \xrightarrow{\sim} \mathcal{PW}(\mathbb{C})$ *extends to an isomorphism of algebras* $SH \to \mathcal{SH}(\Phi)_{\mathcal{PW}} \simeq \mathcal{SSH}(\Lambda)_{\mathcal{PW}}$.

The bigger algebra $\mathcal{SSH}(\Lambda) \simeq \mathcal{SH}(\Phi)$ can be thus seen as a natural completion of SH.

6 The Constant Term and Its Mellin Transform

The sum over shuffles appearing in the definition of the shuffle algebra turns out to match quite exactly the sum over shuffles appearing in the classical formula for the constant term of a (pseudo-)Eisenstein series, cf. [20], II.1.7. In this section we perform a detailed comparison and obtain a proof of Theorem 5.4. Our comparison can be organized into 5 steps:

(A) Taking the constant term of an automorphic form on GL_n with respect to the Borel subgroup B_n, defines a map $\mathrm{CT}_n : H_n \to C^\infty(\mathbb{R}^n_+)$.

(B) We denote by $\widetilde{\mathrm{CT}}_n$ the twist of CT_n by the analog of the Euler form (Iwasawa Jacobian) to match the formula (4) for the Hall product. It is then adjoint to the Hall multiplication map

$$*_{1^n} = *_{1,\dots,1} : H_1^{\otimes n} \longrightarrow H_n$$

with respect to natural positive definite Hermitian scalar products on both sides. This adjointness implies that the restriction of $\widetilde{\mathrm{CT}}_n$ to $SH_n = \mathrm{Im}(*_{1^n})$ is an embedding $SH_n \to C^\infty(\mathbb{R}^n_+)$.

(C) The standard principal series intertwiners for GL_n give rise to integral operators

$$M_w : C_0^\infty(\mathbb{R}^n_+) \longrightarrow C^\infty(\mathbb{R}^n_+), \quad w \in \mathfrak{S}_n,$$

whose domain of definition can be extended to include more general functions. The formula for the constant term of a pseudo-Eisenstein series then says:

$$\widetilde{\mathrm{CT}}_{n'+n''}(f' * f'') = \sum_{w \in Sh(n',n'')} M_w(\widetilde{\mathrm{CT}}_{n'}(f') \otimes \widetilde{\mathrm{CT}}_{n''}(f'')),$$
$$f' \in H_{n'},\ f'' \in H_{n''}. \tag{20}$$

(D) For $f \in SH_n$ we define $Ch_n(f)$ to be the Mellin transform of $\widetilde{\mathrm{CT}}_n(f)$. It is verified to represent a meromorphic function on $\mathbb{C}^n$. Taken together, the maps Ch_n define then an embedding of vector spaces $Ch : SH \to \bigoplus_n \mathcal{M}er(\mathbb{C}^n)$.

(E) Finally, one sees that the Mellin transform takes M_w to the operator on $\mathscr{M}er(\mathbb{C}^n)$ taking a function $F(s_1, \ldots, s_n)$ to

$$(wF)(s_1, \ldots, s_n) \cdot \prod_{\substack{i<j \\ w(i)>w(j)}} \Phi(s_i - s_j),$$

and so Ch takes the Hall product into the shuffle product, by comparing (20) with (13).

We now implement each step in detail.

A. The Constant Term

We will use both the real and the adelic interpretation of the component H_n of H:

$$H_n \;=\; C_0^\infty\big(GL_n(\mathbb{Z})\backslash GL_n(\mathbb{R})/O_n\big) \;=\; C_0^\infty\big(GL_n(\mathbb{Q})\backslash GL_n(\mathfrak{A})/GL_n(\hat{\mathscr{O}})\big).$$

Let $B = B_n$ be the lower triangular Borel subgroup in GL_n and U be the unipotent radical of B. For $f \in H_n$ its *constant term* is the function $\mathrm{CT}(f)$ on $\mathbb{R}_+^n$ defined in either interpretation by:

$$\begin{aligned}\mathrm{CT}_n(f)(a_1, \ldots, a_n) &= \int_{u\in U(\mathbb{Z})\backslash U(\mathbb{R})} f\big(u \cdot \mathrm{diag}(a_1, \ldots, a_n)\big)du \\ &= \int_{u_{\mathfrak{A}}\in U(\mathbb{Q})\backslash U(\mathfrak{A})} f\big(u_{\mathfrak{A}} \cdot \mathrm{diag}(a_1, \ldots, a_n)\big)du_{\mathfrak{A}}, \quad a_i \in \mathbb{R}_+.\end{aligned} \tag{21}$$

Here du, resp. $du_{\mathfrak{A}}$, is the Haar measure on $U(\mathbb{R})$, resp. $U(\mathfrak{A})$, normalized so that $U(\mathbb{Z})\backslash U(\mathbb{R})$, resp. $U(\mathbb{Q})\backslash U(\mathfrak{A})$, has volume 1. Clearly, $\mathrm{CT}_n(f)$ is a C^∞-function on $\mathbb{R}_+^n$, bounded by $\max|f(g)|$.

Proposition 6.1. *For every $f \in H_n$ there is $c \in \mathbb{R}_+$ such that* $\mathrm{Supp}(\mathrm{CT}_n(f))$ *is contained in the domain*

$$a_1 \leqslant c,\; a_1a_2 \leqslant c,\; \cdots,\; a_1 \ldots a_{n-1} \leqslant c,\; \frac{1}{c} \leqslant a_1 \cdots a_n \leqslant c.$$

Proof. For $(a_1, \ldots, a_n) \in \mathbb{R}_+^n$ and $u \in U(\mathbb{R})$ let $V(a_1, \ldots, a_n; u)$ be the vector bundle on $\overline{\mathrm{Spec}(\mathbb{Z})}$ associated to the class of $u \cdot \mathrm{diag}(a_1, \ldots, a_n)$ in the double quotient. This bundle has a canonical admissible filtration

$$V_1 \subset V_2 \subset \cdots \subset V_n \;=\; V(a_1, \ldots, a_n; u)$$

with $\mathrm{rk}(V_i) = i$ and $V_i / V_{i-1} \simeq \mathscr{O}(a_i)$. But given any vector bundle V on $\overline{\mathrm{Spec}(\mathbb{Z})}$, there is $c \in \mathbb{R}_+$ such that for any admissible filtration $V_1 \subset \cdots \subset V_n = V$ with $\mathrm{rk}(V_i) = i$, the numbers $a_i = \deg(V_i / V_{i-1})$ satisfy the conditions of Proposition 6.1. This follows from Proposition 2.8, and we can clearly find a common c for bundles varying in a compact subset of Bun_n.

B. Twisted Constant Term and Its Adjointness

Let

$$dg = \frac{\prod_{i,j=1}^{n} dg_{ij}}{\det(g)^n}, \quad d^*a = \prod_{i=1}^{n} \frac{da_i}{a_i}$$

be the standard Haar measures on $GL_n(\mathbb{R})$ and $\mathbb{R}_+^n$. We introduce notation for the factors in the Iwasawa decomposition:

$$GL_n(\mathbb{R}) = U \cdot \mathbb{R}_+^n \cdot O_n, \quad g = u \cdot a \cdot k, \quad a = (a_1, \ldots, a_n).$$

We write $a = a(g)$, $a_\nu = a_\nu(g)$ etc. as functions of $g \in GL_n(\mathbb{R})$. Let dk be the Haar measure on O_n of volume 1.

The Haar measure dg on $GL_n(\mathbb{R})$ has, in Iwasawa coordinates, the well known form

$$dg = \delta(a) du \cdot dk \cdot d^*a, \tag{22}$$

where the *Iwasawa Jacobian* $\delta(a)$ is defined by

$$\delta(a) = \delta_n(a) = \prod_{1 \leqslant i < j \leqslant n} \frac{a_j}{a_i} = \prod_{i=1}^{n} a_i^{-n+2i-1}. \tag{23}$$

See, e.g., [31], §4.1, Exercise 20 for upper-triangular matrices. We also write $\delta_n(g) = \delta_n(a(g))$ for $g \in GL_n(\mathbb{R})$.

Let us make H_n and $C_0^\infty(\mathbb{R}_+^n) \supset H_1^{\otimes n}$ into pre-Hilbert spaces via the positive definite Hermitian scalar products

$$(f_1, f_2)_H = \int_{GL_n(\mathbb{Z}) \backslash GL_n(\mathbb{R})} f_1(g) \overline{f_2(g)} dg, \quad (\varphi_1, \varphi_2) = \frac{1}{2^n} \int_{\mathbb{R}_+^n} \varphi_1(a) \overline{\varphi_2(a)} d^*a.$$

More generally, in each case the scalar product makes sense whenever only one of the arguments has compact support. Define the *twisted constant term* of $f \in H_n$ to be the function

$$\widetilde{\mathrm{CT}}_n(f)(a_1,\dots,a_n) \;=\; \mathrm{CT}(f)(a_1,\dots,a_n)\cdot\delta(a)^{1/2}. \tag{24}$$

Proposition 6.2. *The map $\widetilde{\mathrm{CT}}_n : H_n \to C^\infty(\mathbb{R}^n_+)$ is adjoint to $*_{1^n} : H_1^{\otimes n} \to H_n$, i.e., we have*

$$(*_{1^n}(\varphi), f)_H \;=\; (\varphi, \widetilde{\mathrm{CT}}_n(f)), \quad \varphi \in H_1^{\otimes n},\; f \in H_n.$$

Proof. This is standard, we provide details for convenience of the reader. For $\varphi \in C_0^\infty(\mathbb{R}^n_+)$ we define a function $\tilde{\varphi}$ on $U\backslash GL_n(\mathbb{R})/O_n$ by

$$\tilde{\varphi}(g) \;=\; \varphi(a_1(g),\dots,a_n(g))\cdot\delta(g)^{-1/2}.$$

Translating the (iterated) formula (4) for the Hall product, into group-theoretical terms, we have

$$(*_{1^n}(\varphi))(g) \;=\; \sum_{\gamma\in B_n(\mathbb{Z})\backslash GL_n(\mathbb{Z})} \tilde{\varphi}(\gamma g)$$

(a pseudo-Eisenstein series). The adjointness then follows from the expression of dg in terms of the Iwasawa factorization:

$$\begin{aligned}
(*_{1^n}(\varphi), f)_H &= \int_{g\in GL_n(\mathbb{Z})\backslash GL_n(\mathbb{R}} \overline{f(g)} \sum_{\gamma\in B(\mathbb{Z})\backslash GL_n(\mathbb{Z})} \tilde{\varphi}(\gamma g)dg\\
&= \int_{x\in B(\mathbb{Z})\backslash GL_n(\mathbb{R})} \overline{f(x)}\tilde{\varphi}(x)dx\\
&\overset{\text{def}}{=} \int_{x\in B(\mathbb{Z})\backslash GL_n(\mathbb{R})} \overline{f(x)}\varphi(x)\delta(x)^{-1/2}dx\\
&= \frac{1}{2^n}\int_{y\in U(\mathbb{Z})\backslash GL_n(\mathbb{R})} \overline{f(y)}\varphi(y)\delta(y)^{-1/2}dy\\
&= \frac{1}{2^n}\int_{z\in U(\mathbb{R})\backslash GL_n(\mathbb{R})}\int_{u\in U(\mathbb{Z})\backslash U(\mathbb{R})} \overline{f(uz)}\varphi(z)\delta(z)^{-1/2}dudz\\
&\overset{(22)}{=} \frac{1}{2^n}\int_{a\in\mathbb{R}^n_+} \overline{\mathrm{CT}_n(f)(a)}\varphi(a)\delta(a)^{+1/2}d^*a \;=\; (\varphi, \widetilde{\mathrm{CT}}_n(f)).
\end{aligned}$$

Corollary 6.3. *The map* $\widetilde{\mathrm{CT}}_n : SH_n \to C_0^\infty(\mathbb{R}^n_+)$ *is injective.*

Proof. By definition of SH as the subalgebra generated by H_1, a non-zero element $f \in SH_n$ has the form $f = *_{1^n}(\varphi)$ for some $\varphi \in H_1^{\otimes n}$. We can regard φ as an element of $C_0^\infty(\mathbb{R}^n_+)$. To prove that $\widetilde{\mathrm{CT}}_n(*_{1^n}(\varphi)) \neq 0$, we notice that by adjointness and by the positivity of the scalar product on H, we have

$$\big(\varphi, \widetilde{\mathrm{CT}}_n(*_{1^n}(\varphi))\big) = \big(*_{1^n}(\varphi), *_{1^n}(\varphi)\big)_H = (f, f)_H > 0.$$

C. The Principal Series Intertwiners

We use the intertwiners in their adelic form, as this form accounts for the appearance of the factors involving the Riemann zeta in the function $\Phi(s)$ defining the shuffle algebra, see (17).

Let A_n be the diagonal subgroup in GL_n. We have the identification

$$\mathbb{R}^n_+ = U(\mathfrak{A})A_n(\mathbb{Q})\big\backslash GL_n(\mathfrak{A})\big/K_n, \quad K_n = O_n \prod_p GL_n(\mathbb{Z}_p).$$

For $w \in \mathfrak{S}_n$ let $U_w = U \cap (w^{-1}Uw)$. Using the above identification, we define the operator

$$M_w : C_0^\infty(\mathbb{R}^n_+) \longrightarrow C^\infty(\mathbb{R}^n_+), \quad (M_w\varphi)(g) = \int_{u\in(U(\mathfrak{A})\cap U_w(\mathfrak{A}))\backslash U(\mathfrak{A})} \varphi(wug)du,$$

cf. [20], II.1.6. More generally, $M_w(\varphi)$ can be defined if, for any g, the function $u \mapsto \varphi(wug)$ on the domain of integration has sufficiently fast decay (for example, has compact support). Here is an example, to be used later.

We consider the following domain in $\mathbb{C}^n$:

$$\mathbb{C}^n_> = \big\{s = (s_1, \dots, s_n) : s_\nu - s_{\nu+1} > 1, \nu = 1, \dots, n\big\}, \tag{25}$$

where we put $s_{n+1} = 0$. For $w \in \mathfrak{S}_n$ put

$$\Phi_w(s) = \prod_{\substack{1\leqslant i<j\leqslant n \\ w(i)>w(j)}} \Phi(s_i - s_j). \tag{26}$$

Proposition 6.4. *If* $s = (s_1, \dots, s_n) \in \mathbb{C}^n_>$, *then applying* M_w *to the function* $a \mapsto a^s$ *gives a convergent integral, and it is found as follows:*

$$M_w(a^s) = a^{w(s)}\Phi_w(s).$$

Proof. This is a version of the classical Gindikin–Karpelevich formula. More precisely, the value of the adelic intertwiner is found as the Euler product of the values of similarly defined local intertwiners (involving the integration over the p-adic or real group). Each local integral is found by Gindikin–Karpelevich to contribute the factor

$$\prod_{\substack{1\leq i<j\leq n\\ w(i)>w(j)}} \frac{\zeta_p(s_i - s_j)}{\zeta_p(s_i - s_j + 1)},$$

where ζ_p is the pth Euler factor of the Riemann zeta, or the Gamma factor for $p = \infty$.

For $\varphi' \in C^\infty(\mathbb{R}_+^{n'})$ and $\varphi'' \in C^\infty(\mathbb{R}_+^{n''})$ we define $\varphi' \otimes \varphi'' \in C^\infty(\mathbb{R}_+^{n'+n''})$ by

$$(\varphi \otimes \varphi'')(a_1, \dots, a_{n'+n''}) \;=\; \varphi'(a_1, \dots, a_{n'})\varphi''(a_{n'+1}, \dots, a_{n'+n''}). \tag{27}$$

We will use similar notation in other situations without special explanation.

Having now defined all the ingredients of the equality (20), we explain how it is proved. This is again a standard argument, using the Bruhat decomposition of a Grassmannian into cells labelled by shuffles, cf. [20], II.1.7 for the case of any parabolic subgroup in any reductive group.

To give some details in our particular case, let $n = n' + n''$ and $P_{n',n''} \subset GL_n$ be the parabolic (block-lower-triangular) subgroup corresponding to (n', n''). We denote $U_{n',n''}$ its unipotent radical and $A_{n',n''} = GL_{n'} \times GL_{n''}$ the Levi subgroup. Then the Iwasawa decomposition implies that

$$\begin{aligned}\big(GL_{n'}(\mathbb{Q})\backslash GL_{n'}(\mathfrak{A})/K_{n'}\big) \times \big(GL_{n'}(\mathbb{Q})\backslash GL_{n'}(\mathfrak{A})/K_{n'}\big)\\ \xrightarrow{\sim}\; \big(U_{n',n''}(\mathfrak{A})A_{n',n''}(\mathbb{Q})\big)\big\backslash GL_n(\mathfrak{A})\big/K_n.\end{aligned} \tag{28}$$

Given $f' \in H_{n'}$, $f'' \in H_{n''}$, let f be the function on the right hand side of (28) corresponding to the function

$$(g', g'') \;\longmapsto\; |\det(g')|^{n''/2} \cdot |\det(g'')|^{-n'/2} \cdot f'(g')f''(g'')$$

on the left hand side. Here $|a|$ is the adelic norm of a. The Hall product $f' * f''$ is then given by the parabolic pseudo-Eisenstein series

$$(f' * f'')(g) \;=\; \sum_{\gamma\in P_{n',n''}(\mathbb{Q})\backslash GL_n(\mathbb{Q})} f(\gamma g).$$

Now, writing

$$\widetilde{\mathrm{CT}}_n(f' * f'')(g) \;= \int_{u\in U(\mathbb{Q})\backslash U(\mathfrak{A})} \sum_{\gamma\in P_{n',n''}(\mathbb{Q})\backslash GL_n(\mathbb{Q})} f(\gamma u g)\delta_n(g)^{1/2}du,$$

we notice that the Grassmannian $\mathrm{Gr}(n', \mathbb{Q}^n) = P_{n',n''}(\mathbb{Q})\backslash GL_n(\mathbb{Q})$ splits, under the right $U(\mathbb{Q})$-action, into $\binom{n}{n'}$ orbits (Schubert cells)

$$\Sigma_w = P_{n',n''}(\mathbb{Q})\backslash wU(\mathbb{Q}), \quad w \in Sh(n', n'').$$

Notice that for $w \in Sh(n', n'')$ we have $U_w = U \cap w^{-1} P_{n',n''} w$. This means that we can write each $\gamma \in \Sigma_w$ uniquely in the form $\gamma = P_{n'n''}(\mathbb{Q}) \cdot w \cdot v$ for $v \in U_w(\mathbb{Q})\backslash U(\mathbb{Q})$ and so

$$\begin{aligned} \widetilde{\mathrm{CT}}_n(f' * f'')(g) &= \sum_{w \in Sh(n',n'')} \int_{u \in U(\mathbb{Q})\backslash U(\mathfrak{A})} \sum_{v \in U_w(\mathbb{Q})\backslash U(\mathbb{Q})} f(wvug)\delta_n(g)^{1/2} du \\ &= \sum_{w \in Sh(n',n'')} \int_{\tilde{u} \in U_w(\mathbb{Q})\backslash U(\mathfrak{A})} f(w\tilde{u}g)\delta_n(g)^{1/2} d\tilde{u}, \end{aligned}$$

and we identify the integral over $\tilde{u}$ corresponding to w, with $M_w(\widetilde{\mathrm{CT}}_{n'}(f') \otimes \widetilde{\mathrm{CT}}_{n''}(f''))$. Note that this argument shows, in particular, that M_w is indeed applicable in this case as the domain of integration reduces to a compact one (since all we did was re-partition the integral for $\widetilde{\mathrm{CT}}_n(f' * f'')(g)$, which was over a compact domain to begin with). We leave the rest to the reader.

Let us note a version of the above statement for the constant term of the n-tuple Hall product. The proof is similar.

Proposition 6.5. *Let $\varphi_1, \ldots, \varphi_n \in C_0^\infty(\mathbb{R}_+)$ and $\varphi = \varphi_1 \otimes \ldots \otimes \varphi_n \in C_0^\infty(\mathbb{R}_+^n)$. Then*

$$\widetilde{\mathrm{CT}}_n(*_{1^n}(\varphi)) = \sum_{w \in \mathfrak{S}_n} M_w(\varphi). \square$$

D. The Mellin Transform of the Constant Term

For $f \in H_n$ we set $Ch_n(f) = \mathscr{M}(\widetilde{\mathrm{CT}}_n(f))$.

Proposition 6.6. *The Mellin integral for $Ch_n(f)$ converges to an analytic function in the region $\mathbb{C}^n_>$.*

Proof. The Mellin transform of $\widetilde{\mathrm{CT}}_n(f)(a) = \delta_n(a)^{1/2}\mathrm{CT}_n(a)$ differs from $\mathscr{M}(\mathrm{CT}_n(a))$ by a shift of variables, and our statement is equivalent to saying that $\mathscr{M}(\mathrm{CT}_n(a))$ converges for

$$\mathrm{Re}(s_1 - s_2) > 0,\ \mathrm{Re}(s_2 - s_3) > 0,\ \cdots, \mathrm{Re}(s_{n-1} - s_n) > 0,\ \mathrm{Re}(s_n) > 0.$$

To see this, note that by Proposition 6.1 and of boundedness of $\mathrm{CT}_n(f)$, the integral is bounded by

$$\text{const} \int_{a_1=0}^{c} \int_{a_1 a_2=0}^{c} \cdots \int_{a_1\ldots a_n=0}^{c} a_1^{s_1-s_2}(a_1a_2)^{s_2-s_3}\cdots(a_1\ldots a_n)^{s_n} \times d^*a_1 d^*(a_1a_2)\cdots d^*(a_1\ldots a_n).$$

Since $\int_0^c a^s d^*a$ converges for $\mathrm{Re}(s) > 0$, the claim follows.

Proposition 6.7. *For any $f \in SH_n$, the function $Ch_n(f)$ extends to a meromorphic function on $\mathbb{C}^n$.*

Before giving the proof, we recall the properties of a classical type of Eisenstein series due to Selberg [27].

For any $s \in \mathbb{C}$ we denote by $\mathfrak{E}(s)$ the following function on Bun_1:

$$\mathfrak{E}(s) : E \longmapsto \deg(E)^s = \exp(s \cdot \ln(\deg(E))). \tag{29}$$

The (formal) Hall product

$$\mathfrak{E}(s_1) * \cdots \mathfrak{E}(s_n) = *_{1^n}(a_1^{s_1}\ldots a_n^{s_n}) \tag{30}$$

is a series of functions on Bun_n, known as the (primitive) *Eisenstein–Selberg series*, see [27] and [13], §8.3.

Proposition 6.8. *(a) The series* (30) *converges for $s = (s_1,\ldots,s_n) \in \mathbb{C}^n_>$, to a C^∞-function on* Bun_n.

(b) For any $g \in \mathrm{Bun}_n$ the function $\big(\mathfrak{E}(s_1)*\cdots*\mathfrak{E}(s_n)\big)(g)$ extends to a meromorphic function in the s_i, with position and order of poles independent on g.

(c) The twisted constant term of $\big(\mathfrak{E}(s_1) * \cdots * \mathfrak{E}(s_n)\big)(g)$ as a function on g is given by

$$\widetilde{\mathrm{CT}}_n\big(\mathfrak{E}(s_1)*\cdots*\mathfrak{E}(s_n))(a_1,\ldots,a_n\big) = \sum_{w\in\mathfrak{S}_n} a_1^{s_{w(1)}}\cdots a_n^{s_{w(n)}} \prod_{\substack{i<j\\ w(i)>w(j)}} \Phi(s_i-s_j).$$

Proof. For (a), see, e.g., [13], §8.5, Remark, and take into account the Ringel twist in the definition of $*$ which translates the shifts by $1/2$ into shifts by 1. See also [9], Proposition 10.4.3 for a slightly weaker statement.

For (b), see [13], §8.6-7.

Finally, (c) follows by the formula (20) applied to the function a^s, $s \in \mathbb{C}^n_>$ (the application is legal because of the decay conditions) and then using Proposition 6.4.

Proof of Proposition 6.7. It is enough to assume that $f = f_1 * \cdots * f_n$, where $f_\nu \in H_1 = C_0^\infty(\mathbb{R}_+)$. Let $F_\nu = \mathscr{M}(f_\nu) \in \mathscr{PW}(\mathbb{C})$ be the Mellin transform of f_ν. Then $f_\nu = \mathscr{N}(F_\nu)$, and the inverse Mellin integral (understood as in Proposition 4.4) can be taken along any vertical line $\mathrm{Re}(s) = \sigma_\nu$.

Let us now choose $\sigma_1,\ldots,\sigma_n$ such that $\sigma_{\nu+1} - \sigma_\nu > 1$ for each $\nu = 1,\ldots,n-1$ and $\sigma_n > 1$. The equalities $\mathscr{N}(F_\nu) = f_\nu$ then imply that

$$f(g) = \frac{1}{(2\pi i)^n} \int_{\mathrm{Re}(s_\nu)=\sigma_\nu} F_1(s_1)\cdots F_n(s_n)\big(\mathfrak{E}(-s_1) * \cdots * \mathfrak{E}(-s_n)\big)(g) ds_1 \cdots ds_n.$$

Substituting the formula for the twisted constant term of $\big(\mathfrak{E}(-s_1) * \cdots * \mathfrak{E}(-s_n)\big)(g)$ from Proposition 6.8(c) into the integral for $f(g)$, we represent $\widetilde{\mathrm{CT}}_n(f)$ as the inverse Mellin transform of the function

$$F(s_1, \ldots, s_n) = \sum_{w \in \mathfrak{S}_n} F_1(s_{w(1)}) \cdots F_n(s_{w(n)}) \prod_{\substack{i<j \\ w(i)>w(j)}} \Phi(s_j - s_i),$$

which is analytic in the region $\mathrm{Re}(s_{\nu+1}) - \mathrm{Re}(s_\nu) > 1$. Further, if we take $\sigma_1, \ldots, \sigma_n$ such that $\sigma_{\nu+1} - \sigma_\nu > 1$, $\sigma_n > 1$, then F is bounded on the vertical subspace $\mathrm{Re}(s_\nu) = s_\nu$. Indeed, each F_i, being a Paley–Wiener function, decays exponentially at the imaginary infinity. On the other hand, the lemma below shows that $\Phi(s)$ is bounded on vertical lines $\mathrm{Re}(s) = \sigma_0 > 1$. Therefore we can apply the Mellin inversion (Proposition 4.4) to F and obtain that $Ch_n(f) = \mathscr{M}(\widetilde{\mathrm{CT}}_n(f)) = F(s_1, \ldots, s_n)$ and so it is meromorphic.

Lemma 6.9. *For every $\sigma_0 > 1$, the function $\Phi(\sigma_0 + it)$ is bounded, as a function of $t \in \mathbb{R}$, and decays as $|t| \to \infty$.*

Proof. Indeed, for $s = \sigma_0 + it$, $\sigma_0 > 1$ we have

$$\zeta(s)/\zeta(s+1) = \sum_{n=1}^{\infty} \varphi(n) n^{-s-1},$$

where $\varphi(n) = |(\mathbb{Z}/n)^\times|$ is the Euler function. This is bounded by

$$\sum n \cdot n^{-\sigma_0 - 1} = \zeta(\sigma_0).$$

Further, $\Gamma(\frac{s}{2})/\Gamma(\frac{s+1}{2})$ decays at infinity as $s^{-1/2}$, as it follows from the Stirling formula.

E. Intertwiners and the Constant Term

We now study the action of the intertwiners M_w on the Mellin transform of the constant term.

Proposition 6.10. *For $\varphi \in C_0^\infty(\mathbb{R}_+^n)$ and any $w \in \mathfrak{S}_n$ we have*

$$\mathscr{M}(M_w(\varphi))(s) = \mathscr{M}(\varphi)(w(s)) \cdot \Phi_w(s).$$

Proof. Write φ as the inverse Mellin integral of a Paley–Wiener function F over any vertical subspace $\sigma + i\mathbb{R}^n$ inside $\mathbb{C}^n_>$, and apply Proposition 6.4.

At this point, we can finish the proof of Theorem 5.4. It remains only to prove that Ch is a homomorphism of algebras, i.e., that

$$Ch_n(f' * f'') \ = \ Ch_{n'}(f') \circledS Ch_{n''}(f''), \quad n = n' + n'' \tag{31}$$

for any $f' \in SH_{n'}$ and $f'' \in SH_{n''}$. Using the formula (20) for the left hand side and the definition of the shuffle product $\circledS$ for the right hand side, we write this as an equality of two sums over shuffles

$$\begin{aligned}\sum_{w\in Sh_{n',n''}} &\mathscr{M}\big(M_w(\widetilde{\mathrm{CT}}_{n'}(f') \otimes \widetilde{\mathrm{CT}}_{n''}(f''))\big)(s) \\ &= \sum_{w\in Sh_{n',n''}} \mathscr{M}\big(\widetilde{\mathrm{CT}}_{n'}(f') \otimes \widetilde{\mathrm{CT}}_{n''}(f'')\big)(w(s)) \cdot \Phi_w(s).\end{aligned} \tag{32}$$

As f', f'' belong to the subalgebra SH, we can write them as

$$f' = *_{1^n}(\varphi'), \quad f'' = *_{1^n}(\varphi'')$$

for some $\varphi' \in C_0^\infty(\mathbb{R}^{n'}_+)$, $\varphi'' \in C_0^\infty(\mathbb{R}^{n''}_+)$. By Proposition 6.5, we have

$$\widetilde{\mathrm{CT}}_{n'}(f') \ = \ \sum_{w'\in\mathfrak{S}_{n'}} M_{w'}(\varphi'),$$

and similarly for $\widetilde{\mathrm{CT}}_{n'}(f')$. Substituting this to the LHS of the putative equality (32), we find that it is equal to

$$\sum_{w\in\mathfrak{S}_n} \mathscr{M}(M_w(\varphi)) \ \overset{6.10}{=} \ \sum_{w\in\mathfrak{S}_n} \mathscr{M}(\varphi)(w(s)) \cdot \Phi_w(s), \quad \varphi = \varphi' \otimes \varphi''. \tag{33}$$

On the other hand, writing $s \in \mathbb{C}^n$ as (s', s'') with $s' \in \mathbb{C}^{n'}$, $s'' \in \mathbb{C}^{n''}$, we have

$$\mathscr{M}\big(\widetilde{\mathrm{CT}}_{n'}(f') \otimes \widetilde{\mathrm{CT}}_{n''}(f'')\big)(s) \ = \ \mathscr{M}(\widetilde{\mathrm{CT}}_{n'}(f'))(s') \cdot \mathscr{M}(\widetilde{\mathrm{CT}}_{n''}(f''))(s''),$$

and so the summand in the RHS of (32) corresponding to $w \in Sh_{n',n''}$, is equal by Proposition 6.5, to $\Phi_w(s)$ times

$$\sum_{\substack{w' \in \mathfrak{S}_{n'} \\ w'' \in \mathfrak{S}_{n''}}} \mathscr{M}(\varphi')(w'(s')) \cdot \mathscr{M}(\varphi'')(w''(s'')) \cdot \Phi_{w'}(s')\Phi_{w''}(s'')$$

$$= \sum_{\substack{w' \in \mathfrak{S}_{n'} \\ w'' \in \mathfrak{S}_{n''}}} \mathscr{M}_{w' \times w''}(\varphi)((w' \times w'')(s)) \cdot \Phi_{w' \times w''}(s),$$

and further summation over w gives the same result as (33). □

7 Quadratic Relations and Eisenstein Series

Let

$$S = \bigoplus_{n=0}^{\infty} S_n, \quad S_0 = \mathbb{C},$$

be a graded associative algebra over $\mathbb{C}$. The space of degree n relations among elements of degree 1 is then

$$R_n \;=\; \mathrm{Ker}\{S_1^{\otimes n} \longrightarrow S_n\} \;\subset\; S_1^{\otimes n}. \tag{34}$$

Here we are interested in quadratic relations ($n = 2$) for the algebra SH generated by $SH_1 = H_1 = C_0^{\infty}(\mathbb{R}_+)$. Because of the analytic nature of elements of H it is not reasonable to look for relations inside the algebraic tensor product $H_1 \otimes H_1$ and we consider a completion of it, namely the space

$$H_1 \hat{\otimes} H_1 \;:=\; \mathscr{D}(\mathbb{R}_+^2)_{\mathrm{abs}}$$

of absolutely tempered distributions on $\mathbb{R}_+^2$, see Corollary 4.5.

Proposition 7.1. *If $f \in H_1 \hat{\otimes} H_1$, then the series*

$$\hat{*}_{1,1}(f)(E) \;=\; \sum_{E' \subset E} \deg(E')^{1/2} \deg(E/E')^{-1/2} f(\deg(E'), \deg(E/E')), \; E \in \mathrm{Bun}_2,$$

converges absolutely, defining a distribution $\hat{*}_{1,1}(f)$ on Bun_2. *The resulting linear map $\hat{*}_{1,1} : H_1 \hat{\otimes} H_1 \to \mathscr{D}ist(\mathrm{Bun}_2)$ extends the Hall multiplication $*_{1,1} : H_1 \otimes H_1 \to H_2$.*

Proof. The points $(\alpha, \beta) = (\deg(E'), \deg(E/E'))$ lie on the hyperbola $\alpha\beta = \deg(E)$. An absolutely tempered distribution decays exponentially at the infinity of $\mathbb{R}^2_+$, in particular at the infinity of any such hyperbola. Now the number of subbundles in $E = (L, V, q)$ of given degree $\alpha = 1/a$ is one half the number of primitive vectors in L of norm a. This number of all lattice vectors of norm a grows linearly with a, so exponential decay of f ensures the convergence.

Remark 7.2. It is possible that one can extend H to a bigger algebra, consisting of some analogs of absolutely tempered distributions on the Bun_n, which have sufficient decay at the infinity. Note that the concept of a tempered distribution on a semisimple Lie group was introduced by Harish-Chandra [11].

We will therefore understand quadratic relations in SH is a wider sense, as elements of the space

$$\hat{R}_2 = \text{Ker}(\hat{*}_{1,1}) \subset H_1 \hat{\otimes} H_1. \tag{35}$$

Let also $\mathscr{R}_2$ be the space of entire functions $F \in \mathscr{O}(\mathbb{C}^2)_{\text{pol}}$ such that

$$F(s_1, s_2) + \Phi(s_1 - s_2) F(s_2, s_1) = 0. \tag{36}$$

Proposition 7.3. *The Mellin transform identifies $\hat{R}_2$ with $\mathscr{R}_2$.*

Proof. This follows from an instance of Eq. (31) for $m = n = 1$ but applied to absolutely tempered distributions instead of functions with compact support. The proof in the new case is the same, given the decay (to define the Hall product) and the analyticity of the Mellin transform.

Note that $\mathscr{R}_2$ is a module over the ring $\mathscr{O}(\mathbb{C}^2)^{\mathfrak{S}_2}_{\text{pol}}$ of symmetric entire functions of polynomial growth on vertical planes.

Example 7.4. Let $P(s) = s(s-1)(s+1)$. Then the function

$$F_{1,1}(s_1, s_2) = P(s_1 - s_2)\zeta^*(s_1 - s_2)$$

belongs to $\mathscr{R}_2$. Further, for any $\lambda_1, \lambda_2 \in \mathbb{R}_+$ the function

$$F_{\lambda_1,\lambda_2}(s_1, s_2) = (\lambda_1^{s_1}\lambda_2^{s_2} + \lambda_1^{s_2}\lambda_2^{s_1}) F_{1,1}(s_1, s_2)$$

again lies in $\mathscr{R}_2$ by the remark above. Let

$$\nabla_a = P\Big(a\frac{d}{da}\Big) = a^3\frac{d^3}{da^3} - a^2\frac{d^2}{da^2}.$$

The inverse Mellin transform of $F_{1,1}$ is, in virtue of Proposition 4.6 and the Riemann formula (10), equal to

$$\Psi_{1,1}(a_1, a_2) = \delta_1(a_1 a_2) \cdot \nabla_{a_1}\theta(a_1^2) \in \hat{R}_2,$$

and the inverse Mellin transform of F_{λ_1,λ_2} is the distribution

$$\Psi_{\lambda_1,\lambda_2}(a_1,a_2) = \Psi_{1,1}(a_1/\lambda_1, a_2/\lambda_2) \; + \; \Psi_{1,1}(a_1/\lambda_2, a_2/\lambda_1) \; \in \; \hat{R}_2.$$

This gives a 2-parameter family of quadratic relations in SH.

Remark 7.5. This 2-parameter family of relations is analogous to the family of relations

$$\begin{aligned}[\mathcal{O}(m+1)] * [\mathcal{O}(n)] - q[\mathcal{O}(n)] * [\mathcal{O}(m+1)] \;&=\; q[\mathcal{O}(m)] * [\mathcal{O}(n+1)] \\ &\quad - [\mathcal{O}(n+1)] * [\mathcal{O}(m)]\end{aligned}$$

in the Hall algebra of the category of vector bundles on $\mathbb{P}^1_{\mathbb{F}_q}$, see [14], §5.2 or [2], Lemma 16.

We now explain the relation of the above quadratic relations with the functional equation for *Eisenstein–Maas series*

$$\mathbf{E}(\tau,s) \;=\; \frac{1}{2}\sum_{(m,n)=1} \frac{\mathrm{Im}(\tau)^s}{|m+n\tau|^{2s}}, \qquad \tau \in \mathbb{H}, \; \mathrm{Re}(s) > 1,$$

see [9], §3.1. It is classical that $\mathbf{E}(\tau,s)$ extends to a function meromorphic in the entire s-plane and satisfying the functional equation

$$\mathbf{E}(\tau,s) \;=\; \frac{\zeta^*(2s-1)}{\zeta^*(2s)}\,\mathbf{E}(\tau,1-s).$$

Further, the poles of $\mathbf{E}(\tau,s)$ are all among the poles of the ratio of the ζ^*-functions, in particular, they do not depend on τ.

On the other hand, recall (29) the function

$$\mathfrak{E}(t) : \mathrm{Bun}_1 \to \mathbb{C}, \qquad E \longmapsto \deg(E)^t.$$

Here $t \in \mathbb{C}$ is a fixed complex number. This function does not lie in $H_1 = C_0^\infty(\mathbb{R})$. Nevertheless, the correspondence $t \mapsto \mathfrak{E}(t)$ can be seen as a kind of H-valued distribution ("operator field") on $\mathbb{C}$ (or, rather, on $i\mathbb{R} \subset \mathbb{C}$). That is, for any Paley–Wiener function $G(t)$ we have a well defined element

$$\int_{i\mathbb{R}} \mathfrak{E}(t)G(t)dt \;\in\; H_1.$$

This simply the function $E \mapsto f(\deg(E)^{-1})$, where $f = \mathcal{N}(G) \in C_0^\infty(\mathbb{R}_+)$.

Proposition 6.8(a) implies that for $\mathrm{Re}(t_1 - t_2) > 0$ the Hall product $\mathfrak{E}(t_1) * \mathfrak{E}(t_2)$ defined as a formal series, converges to a real analytic function on Bun_2. This function essentially reduces to the series $\mathbf{E}(\tau,s)$ above. Indeed, let E_τ be the bundle of rank 2 and degree 1 corresponding to τ as in Example 2.4. Rank 1 subbundles $E' = E'_{m,n}$ in E_τ are parametrized by pairs $(m,n) \in \mathbb{Z}^2$ of coprime integers, taken modulo simultaneous change of sign. Explicitly, the primitive sublattice $L'_{m,n}$ of $E'_{m,n}$ is spanned by $m + n\tau$, and we have

$$\deg(E'_{m,n}) = \frac{\mathrm{Im}(\tau)^{1/2}}{|m+n\tau|}, \quad \deg(E_\tau/E'_{m,n}) = \frac{|m+n\tau|}{\mathrm{Im}(\tau)^{1/2}}.$$

Therefore

$$(\mathfrak{E}(t_1) * \mathfrak{E}(t_2))(E_\tau) \;=\; \mathbf{E}(\tau, (t_1 - t_2 + 1)/2). \tag{37}$$

This means that the product $\mathfrak{E}(t_1) * \mathfrak{E}(t_2)$ extends to a meromorphic function of t_1, t_2 (with values in the space of functions on Bun_2) and we can write a formula looking like "quadratic commutation relations" in H:

$$\mathfrak{E}(t_1) * \mathfrak{E}(t_2) \;-\; \Phi(t_1 - t_2)\, \mathfrak{E}(t_2) * \mathfrak{E}(t_1) \;=\; 0. \tag{38}$$

The two summands in (38) are given by series converging in different regions, having no points in common, and the relations should be understood via analytic continuation. This way of understanding commutation relations is quite standard in the theory of vertex operators [8]. In our situation it is modified as follows.

In order to translate the relations (38) into actual elements of $\hat{R}_2$, we can rewrite them in the form "free of denominators"

$$*_{1,1}\left\{P(t_1-t_2)\cdot\zeta^*(t_1-t_2+1)\cdot a_1^{t_1}a_2^{t_2} \;-\; P(t_1-t_2)\cdot\zeta^*(t_1-t_2)\cdot a_1^{t_2}a_2^{t_1}\right\} \;=\; 0. \tag{39}$$

Here we write $\mathfrak{E}(t_1) \otimes \mathfrak{E}(t_2)$ as the function $(a_1, a_2) \mapsto a_1^{t_1}a_2^{t_2}$ on $\mathrm{Bun}_1 \times \mathrm{Bun}_1 = \mathbb{R}^2_+$. We then "compare coefficients" in both sides of this equality at any $\lambda_1^{t_1}\lambda_2^{t_2}$, $\lambda_\nu \in \mathbb{R}_+$, by multiplying with $\lambda_1^{-t_1}\lambda_2^{-t_2}$ and integrating (performing the inverse Fourier-Schwartz transform) along any vertical 2-plane, which we can choose separately for each summand. This gives a family of distributions $\Psi_{\lambda_1,\lambda_2}(a_1,a_2) \in \hat{R}_2$ which is the same as in Example 7.4.

We can thus say that quadratic relations such as (38) are built into the very definition of the shuffle algebra.

8 Wheels, Cubic Relations, and Zeta Roots

A. Wheels

Let $\lambda(s)$ be a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 0$ and no other singularities. In this section we sketch a general approach to higher order relations in the symmetric shuffle algebra $\mathscr{SSH}(\lambda)$ and illustrate it on the case of cubic relations in the shuffle algebra completion of the spherical Hall algebra SH, which corresponds to

$$\lambda(s) \;=\; \Lambda(s) \;=\; \zeta^*(-s)(s-1)(-s-1).$$

Our approach is based on studying the following additive patterns of roots of λ which were introduced in [6] and used in the case when λ is rational.

Definition 8.1. A *wheel* of length n for λ is a sequence $(s_1, \dots, s_n)$ of distinct complex numbers such that

$$\lambda(s_2 - s_1) = 0,\; \lambda(s_3 - s_2) = 0,\; \cdots, \lambda(s_n - s_{n-1}) = 0,\; \lambda(s_1 - s_n) = 0.$$

Wheels $(s_1, \dots, s_n)$ and $(s_1 + c, \dots, s_n + c)$ for $c \in \mathbb{C}$, will be called *equivalent*.

In other words, equivalence classes of wheels are the same as ordered sequences

$$(z_1, \dots, z_n) \in (\mathbb{C}^*)^n, \quad \lambda(z_i) = 0, \quad \sum_{i=1}^{n} z_i = 0, \quad \sum_{i=p}^{q} z_i \neq 0, \quad (p,q) \neq (1,n).$$

Example 8.2. All wheels for $\Lambda(s)$ have length 3 or more. The sequences corresponding to wheels of length 3 have, up to permutation, the form

$$(z_1, z_2, z_3) \;=\; (\rho, 1-\rho, -1),$$

where ρ runs over nontrivial zeroes of $\zeta(s)$. Indeed, zeroes of Λ are of the form $s = \rho$ together with one more zero $s = -1$. So there are no pairs of them summing up to 0 and the only triples summing to up 0 are as stated.

B. Relations and Bar-Complexes

Let S be a graded associative algebra as in Sect. 7. A systematic way of approaching relations in S is via the bar-complexes

$$B_n^\bullet = B_n^\bullet(S) \;=\; \left\{ S_1^{\otimes n} \to \cdots \to \bigoplus_{i+j+k=n} S_i \otimes S_j \otimes S_k \to \bigoplus_{i+j=n} S_i \otimes S_j \to S_n \right\}.$$

Here $i, j, k, \dots$ run over positive integers. The grading is such that $S_1^{\otimes n}$ is in degree $(-n)$, while S_n is in degree (-1). The differential is given by

$$d(s_1 \otimes \cdots s_p) \;= \sum_{i=1}^{p-1} (-1)^{i-1} s_1 \otimes \ldots \otimes s_{i-1} \otimes s_i s_{i+1} \otimes s_{i+2} \otimes \ldots \otimes s_p,$$

so that the condition $d^2 = 0$ follows from the associativity of S. It is well known that

$$H^{-i}(B_n^\bullet(S)) = \mathrm{Tor}_i^S(\mathbb{C}, \mathbb{C})_n,$$

the part of the Tor-group which has degree n w.r.t. the grading induced from that on S. In particular, the rightmost cohomology has the meaning of the space of generators in degree n, and the previous one is interpreted as the space of relations which have degree n with respect to the grading on the generators (which, a priori, can be present in any degree).

As in (34), let R_n be the space of degree n relations among generators in degree 1. For instance, quadratic relations are found as $R_2 = H^{-2}(B_2^\bullet)$. The next case of cubic relations corresponds to the complex

$$B_3^\bullet = \{S_1 \otimes S_1 \otimes S_1, \xrightarrow{d_{-3}} (S_2 \otimes S_1) \oplus (S_1 \otimes S_2) \xrightarrow{d_{-2}} S_3\}.$$

We treat this case directly. Denote

$$R_{12} = R_2 \otimes S_1, \quad R_{23} = S_1 \otimes R_2 \quad \subset \quad S_1 \otimes S_1 \otimes S_1.$$

We have then an inclusion $R_{12} + R_{23} \subset R_3$ of subspaces in $S_1^{\otimes 3}$. The left hand side of this inclusion is, by definition, the space of those cubic relations which follow algebraically from the quadratic ones. Thus the quotient

$$R_3^{\mathrm{new}} = R_3/(R_{12} + R_{23})$$

can be seen as the space of "new", essentially cubic, relations.

Proposition 8.3. *Assume that the multiplication map $S_1 \otimes S_1 \to S_2$ is surjective. Then R_3^{new} is identified with $H^{-2}(B_3^\bullet)$, the middle cohomology space of $B_3^\bullet$.*

Proof. Denote for short

$$V = S_1^{\otimes 3}, \quad A = R_{12}, \quad B = R_{23}, \quad C = R_3,$$

so that $A, B \subset C \subset V$. Under our assumption, the complex $B_3^\bullet$ can be written as

$$V \xrightarrow{\delta_{-3}} (V/A) \oplus (V/B) \xrightarrow{\delta_{-2}} V/C,$$

with δ_{-3} being the difference of the two projections, and δ_{-2} being the sum of the two projections. It is a general fact that in such a situation the middle cohomology is identified with $C/(A+B)$. Explicitly, if $(v+A, w+B) \in \mathrm{Ker}(\delta_{-2})$, then $v+w \in C$. The image of $v + w$ in $C/(A + B)$ depends only on the class of $(v + A, w + B)$ in $\mathrm{Ker}(\delta_{-2})/\mathrm{Im}(\delta_{-3})$. We leave the rest to the reader.

C. Localization of the Bar-Complexes

We now apply the above to the two graded algebras

$$\mathscr{S}\mathscr{S}\mathscr{H}(\lambda) \;\subset\; \mathscr{S} := \Big(\bigoplus_n \mathscr{O}(\mathbb{C}^n)^{\mathfrak{S}_n},\; \star\Big).$$

By definition, these algebras coincide in degrees 0 and 1, and $\mathscr{S}\mathscr{S}\mathscr{H}(\lambda)$ is the subalgebra in $\mathscr{S}$ generated by the degree 1 part which is $\mathscr{S}_1 = \mathscr{O}(\mathbb{C})$. Accordingly, the space of relations of any degree n among degree 1 generators in $\mathscr{S}$ and $\mathscr{S}\mathscr{S}\mathscr{H}(\lambda)$ are the same. As in Sect. 7, we will look at relations as elements of the completed tensor product. That is, for any two Stein manifolds M and N we write

$$\mathscr{O}(M)\hat{\otimes}\mathscr{O}(N) \;:=\; \mathscr{O}(M\times N)$$

and understand $\mathscr{S}_1^{\hat{\otimes} n} = \mathscr{O}(\mathbb{C}^n)$ accordingly. The version of the bar-complex of $\mathscr{S}$ using $\hat{\otimes}$, has the form

$$\mathbf{B}_n^\bullet = \Big\{ \mathscr{O}(\mathbb{C}^n) \to \cdots \to \bigoplus_{i+j+k=n} \mathscr{O}(\mathbb{C}^n)^{\mathfrak{S}_i\times\mathfrak{S}_j\times\mathfrak{S}_k} \\ \to \bigoplus_{i+j=n} \mathscr{O}(\mathbb{C}^n)^{\mathfrak{S}_i\times\mathfrak{S}_j} \to \mathscr{O}(\mathbb{C}^n)^{\mathfrak{S}_n} \Big\}$$

Notice that each term of this complex is a module over the ring $\mathscr{O}(\mathbb{C}^n)^{\mathfrak{S}_n}$ of symmetric entire functions, and the differentials, coming from multiplication in $\mathscr{S}$, are $\mathscr{O}(\mathbb{C}^n)^{\mathfrak{S}_n}$-linear. This means that $\mathbf{B}_n^\bullet$ is the complex of global section of a complex of vector bundles $\mathscr{B}_n^\bullet$ on the Stein manifold $\mathrm{Sym}^n(\mathbb{C})$. Explicitly, for $i_1+\ldots+i_p = n$ we denote by

$$\pi_{i_1,\ldots,i_p} : \mathrm{Sym}^{i_1}(\mathbb{C})\times\cdots\mathrm{Sym}^{i_p}(\mathbb{C}) \longrightarrow \mathrm{Sym}^n(\mathbb{C})$$

the symmetrization map (a finite flat morphism). Then

$$\mathscr{B}_n^{-p} \;=\; \bigoplus_{i_1+\ldots+i_p=n} (\pi_{i_1,\ldots,i_p})_*\; \mathscr{O}_{\prod \mathrm{Sym}^{i_\nu}(\mathbb{C})}, \tag{40}$$

in particular, $\mathscr{B}_n^\bullet$ is a complex of holomorphic vector bundles on $\mathrm{Sym}^n(\mathbb{C})$. This allows us to approach the cohomology of $\mathbf{B}_n^\bullet$ (and, in particular, relations in $\mathscr{S}$) in a more geometric way, by studying the cohomology of the fibers

$$\mathscr{B}_{n,T}^\bullet \;=\; \mathscr{B}_n^\bullet \otimes_{\mathscr{O}_{\mathrm{Sym}^n(\mathbb{C})}} \mathscr{O}_T$$

of the complex $\mathscr{B}_n^\bullet$ over various points $T \in \text{Sym}^n(\mathbb{C})$. Now, our main technical result is as follows.

Theorem 8.4. *Let $T = \{s_1^0, \dots, s_n^0\} \in \text{Sym}^n(\mathbb{C})$ be an unordered collection of distinct points. Suppose that no subset of T (in any order) is a wheel. Then $\mathscr{B}_{n,T}^\bullet$ is exact everywhere except the leftmost term, where the cohomology is one-dimensional.*

Recall that similar exactness of all the bar-complexes $B_n^\bullet(S)$ for a graded algebra S means that S is quadratic Koszul. The wheels represent therefore local obstructions to Koszulity for $\mathscr{S}$.

D. Cubic Relations in SH and Zeta Roots

Before giving the proof of Theorem 8.4, let us explain how to apply it to the case of cubic relations for $\lambda = \Lambda$. Let ρ be a nontrivial zero of $\zeta(s)$. Denote by $W_\rho \subset \text{Sym}^3(\mathbb{C})$ the subset of points $\{s_1, s_2, s_3\}$ such that, after some renumbering of the s_i we have $s_2 - s_1 = \rho$, $s_3 - s_2 = 1 - \rho$ (such a renumbering is then unique). Let W be the union of the W_ρ over all nontrivial zeroes ρ of $\zeta(s)$. The following is then straightforward.

Proposition 8.5. *(a) Each W_ρ is a complex submanifold in* $\text{Sym}^3(\mathbb{C})$*, isomorphic to $\mathbb{C}$, the symmetric function $s_1 + s_2 + s_3$ establishing an isomorphism.*

(b) For $\rho \neq \rho'$ we have $W_\rho \cap W_{\rho'} = \varnothing$.

(c) A point $\{s_1, s_2, s_3\} \in$ $\text{Sym}^3(\mathbb{C})$ *lies in W, if and only if it is a wheel (in some numbering).* □

Theorem 8.6. *Let $\lambda(s) = \Lambda(s)$.*

(a) The multiplication map $\mathscr{S}_1 \hat{\otimes} \mathscr{S}_1 \to \mathscr{S}_2$ is surjective, so, by Proposition 8.3, the space

$$H^{-2}(\mathbf{B}_3^\bullet) = H^0(\text{Sym}^3(\mathbb{C}), \underline{H}^{-2}(\mathscr{B}_3^\bullet))$$

is identified with the space of new cubic relations in $\mathscr{S}$ as well as in in $\mathscr{S}\mathscr{S}\mathscr{H}(\Lambda)$.

(b) The support of the coherent sheaf $\underline{H}^{-2}(\mathscr{B}_3^\bullet)$ is equal to $W = \bigsqcup W_\rho$. If ρ is a simple root of $\zeta(s)$, then $\underline{H}^{-2}(\mathscr{B}_3^\bullet) \simeq \mathscr{O}_{W_\rho}$ in a neighborhood of W_ρ.

Remark 8.7. From the point of view of this section, a cubic relation in $\mathscr{S}\mathscr{S}\mathscr{H}$ is an entire function $F(s_1, s_2, s_3) \in \mathscr{O}(\mathbb{C}^3) = \mathscr{S}_1^{\hat{\otimes} 3}$ mapped to the zero element of $\mathscr{S}_3$ by the symmetric shuffle multiplication. On the other hand, from the more immediate point of view of Sect. 7, a cubic relation in the spherical Hall algebra SH is a distribution $f(a_1, a_2, a_3)$ on $\mathbb{R}_+^3 = (\text{Bun}_1)^3$, mapped to the zero distribution on Bun_3 by the Hall multiplication. The relation between f and F is that of the

Mellin transform. Note that whenever $f(a_1, a_2, a_3)$ is a relation, then so is the rescaling $f(\alpha a_1, \alpha a_2, \alpha a_3)$ for any $\alpha \in \mathbb{R}_+$. Taking a weighted average of such rescalings, i.e., a convolution

$$\int_0^\infty f(\alpha a_1, \alpha a_2, \alpha a_3)\varphi(\alpha)d^*\alpha$$

corresponds, on the Mellin transform side, to multiplying $F(s_1, s_2, s_3)$ by a function of the form $\psi(s_1 + s_2 + s_3)$. Since $s_1 + s_2 + s_3$ is a global coordinate on each W_ρ, Theorem 8.6 admits the following striking interpretation: *the space of new cubic relations in SH modulo rescaling is identified with the space spanned by nontrivial zeroes of* $\zeta(s)$.

This fact is also true (with a similar proof) for the Hall algebras corresponding to arbitrary compactified arithmetic curves (= spectra of rings of integers in number fields) as well as (with an easier, more algebraic proof) for Hall algebras of smooth projective curves $X/\mathbb{F}_q$. Note that for $X = \mathbb{P}^1$ there are no new cubic relations [2, 14], while for X elliptic, new cubic relations were found in [24]. Our results show that presence of cubic relations is a general phenomenon, holding for all curves $X/\mathbb{F}_q$ of genus $\geqslant 1$.

We will give a detailed proof of Theorem 8.4 and a sketch of proof of Theorem 8.6, which will be taken up and generalized in a subsequent paper.

E. Permuhohedra and the Proof of Theorem 8.4

Our approach, similar to that of [1,19], uses the *permutohedron*, which is the convex polytope

$$P_n = \operatorname{Conv}\big(\mathfrak{S}_n \cdot (1, 2, \dots, n)\big) \subset \mathbb{R}^n$$

of dimension $(n-1)$. Thus vertices of P_n are the $n!$ vectors $(i_1, \dots, i_n)$ for all the permutations. It is well known that faces of P_n are in bijection with sequences $(I_1, \dots, I_p)$ of subsets of $\{1, \dots, n\}$ which form a disjoint decomposition. We denote $[I_1, \dots, I_p]$ the case corresponding to $(I_1, \dots, I_p)$. Subfaces of $[I_1, \dots, I_p]$ correspond to sequences obtained by *refining* $(I_1, \dots, I_p)$, i.e., by replacing each I_ν, in its turn, by a sequence $(J_{\nu,1}, \dots, J_{\nu,q_\nu})$ of subsets of I_ν forming a disjoint decomposition. Thus, as a polytope,

$$[I_1, \dots, I_p] \simeq P_{|I_1|} \times \cdot \times P_{|I_p|}, \quad \dim[I_1, \dots, I_p] = n - p.$$

Let $C^\bullet(P_n)$ be the cochain complex of P_n with complex coefficients. The basis of $C^m(P_n)$ is formed by the $\mathbf{1}_F$, the characteristic functions of the m-dimensional faces. We choose an orientation for each face. Then

$$d(\mathbf{1}_F) = \sum_{F' \supset F} \varepsilon_{FF'} \cdot \mathbf{1}_{F'}.$$

Here the sum is over $(m+1)$-dimensional faces F' containing F, and $\varepsilon_{FF'} = \pm 1$ is the sign factor read from the orientations of F and F'.

On the other hand, (40) gives a natural basis of $\mathscr{B}_{n,T}^{n-1-m}$ labeled by the disjoint union of the preimages

$$\pi^{-1}_{i_1,\dots,i_p}(\{s_1^0, \dots, s_n^0\}), \quad i_1 + \cdots i_p = n.$$

For a subset $I \subset \{1, \dots, n\}$ let $T_I = \{s_i^0 | i \in I\} \subset T$. Elements of each $\pi^{-1}_{i_1,\dots,i_p}(\{s_1^0, \dots, s_n^0\})$ are precisely the

$$(T_{I_1}, \cdots, T_{I_p}) \in \mathrm{Sym}^{i_1}(\mathbb{C}) \times \cdots \times \mathrm{Sym}^{i_p}(\mathbb{C})$$

for all sequences of subsets $(I_1, \dots, I_p)$, forming a disjoint decomposition of $\{1, \dots, n\}$. Denoting by $e_{I_1,\dots,I_p}$ the corresponding basis vector in $\mathscr{B}_{n,T}^{n-1-m}$, we get an isomorphism of graded vector spaces

$$\mathscr{B}_{n,T}^{\bullet} \xrightarrow{\sim} C^{\bullet}(P_n)[n], \quad e_{I_1,\dots,I_p} \mapsto \mathbf{1}_{[I_1,\dots,I_p]}. \tag{41}$$

To see the differential in $\mathscr{B}_{n,T}^{\bullet}$ from this point of view, consider the matrix

$$\mathfrak{L} = \|\lambda_{ij}\|, \quad \lambda_{ij} = \lambda(s_i^0 - s_j^0), \ 1 \leqslant i, j \leqslant n, \ i \neq j.$$

Let $F \subset F'$ be a codimension 1 embedding of faces of P_n. That is, $F' = [I_1, \dots, I_p]$ and F is a minimal refinement of F', i.e., is obtained by replacing some I_ν by (I', I'') where I', I'' are nonempty sets forming a disjoint decomposition of I_ν. We put

$$\lambda_{FF'} = \prod_{\substack{i' \in I' \\ i'' \in I''}} \lambda_{i'i''}.$$

It is immediately so see that the $\lambda_{FF'}$ satisfy the multiplicativity property for any pair of composable codimension 1 embeddings:

$$\lambda_{FF'} \lambda_{F'F''} = \lambda_{FF'}, \quad F \subset F' \subset F''.$$

This implies that by putting

$$d_{\mathfrak{L}}(\mathbf{1}_F) = \sum_{F' \supset F} \lambda_{FF'} \cdot \varepsilon_{FF'} \cdot \mathbf{1}_{F'},$$

we obtain a differential $d_{\mathfrak{L}}$ in $C^{\bullet}(P_n, \mathbb{C})$ with square 0. This is a certain perturbation of the cochain differential for P_n. We then see easily:

Proposition 8.8. *The isomorphism* (41) *defines an isomorphism of complexes*

$$B^{\bullet}_{n,T} \longrightarrow \big(C^{\bullet}(P_n), d_{\mathfrak{L}}\big)[n].$$

□

Note that the perturbed differential $d_{\mathfrak{L}}$ can be written for any system $\mathfrak{L} = \|\lambda_{ij}\|_{i \neq j}$ of complex numbers. Conceptually, $\mathfrak{L}$ is a $\mathbb{C}$-valued function on the root system of type A_{n-1}. We simply refer to $\mathfrak{L}$ as a matrix.

By a *wheel* for $\mathfrak{L}$ we mean a sequence of $i_1, \dots, i_m$ of indices such that

$$\lambda_{i_1,i_2} = \lambda_{i_2,i_3} = \cdots = \lambda_{i_{p-1},i_p} = \lambda_{i_p,i_1} = 0.$$

Theorem 8.4 is now a consequence of the following result.

Proposition 8.9. *Let* $\mathfrak{L} = \|\lambda_{ij}\|_{i \neq j}$ *be an n by n matrix without wheels. Then* $\big(C^{\bullet}(P_n), d_{\mathfrak{L}}\big)$ *is exact outside of the leftmost term, where the cohomology (kernel) is one-dimensional.*

Proof. For a face $F = [I_1, \dots, I_p]$ of P_n we put

$$\lambda_F = \prod_{\mu<\nu} \prod_{\substack{i \in I_\mu \\ j \in I_\nu}} \lambda_{ij}. \tag{42}$$

Then for an embedding $F \subset F'$ of codimension 1 we have

$$\lambda_F = \lambda_{F'} \cdot \lambda_{FF'}.$$

This means that we have a morphism of complexes

$$\Psi : \big(C^{\bullet}(P), d_{\mathfrak{L}}\big) \longrightarrow \big(C^{\bullet}(P), d\big), \quad \Psi(\mathbf{1}_F) = \lambda_F \cdot \mathbf{1}_F,$$

where d is the usual cochain differential. As P_n is a convex polytope, $\big(C^{\bullet}(P), d\big)$ is exact outside the leftmost term, with $H^0 = \mathbb{C}$. We now analyze the kernel and cokernel of Ψ. For a face F of P_n as before we call the *depth* of P the number of factors in (42) which are zero. In other words, we put

$$Z = \{(i,j) : i \neq j,\ \lambda_{ij} = 0\} \subset \{1, \dots, n\}^2. \tag{43}$$

Then the depth of F is the number

$$\mathrm{dpt}(F) = \#\big\{(i,j) \in Z : \exists\, \mu < \nu : i \in I_\mu, j \in I_\nu\big\}. \tag{44}$$

Note that if F is a subface of F', then $\mathrm{dpt}(F) \geqslant \mathrm{dpt}(F')$. Therefore we have a descending chain of polyhedral subcomplexes

$$P^{(r)} = \bigcup_{\mathrm{dpt}(F)\geqslant r} F \subset P_n, \quad r \geqslant 0.$$

Lemma 8.10. *(a) The complex $Coker(\Psi)$ is isomorphic to the relative cochain complex $C^\bullet(P_n, P^{(1)})$.*
(b) The complex $\mathrm{Ker}(\Psi)$ *has a filtration with quotients isomorphic to the relative cochain complexes $C^\bullet(P^{(r)}, P^{(r+1)})$, $r \geqslant 1$.*

Proof. The matrix of Ψ is diagonal in the chosen bases, and $\mathrm{Im}(\Psi) \subset C^\bullet(P_n)$ is spanned by the $\mathbf{1}_F$, $F \in P^{(1)}$, which shows (a). As for (b), for each $r \geqslant 0$ we have the cochain subcomplex $C^\bullet(P_n)^{\geqslant r} \subset C^\bullet(P)$ spanned by $\mathbf{1}_F$ with $\mathrm{dpt}(F) \geqslant r$, with $C^\bullet(P_n)^{\geqslant 1} = \mathrm{Ker}(\Psi)$. The quotient $C^\bullet(P_n)^{\geqslant r}/C^\bullet(P_n)^{\geqslant r+1}$ is identified with $C^\bullet(P^{(r)}, P^{(r+1)})$ in a way similar to (a). □

Note that the weights of faces of P_n and the polyhedral subcomplexes $P^{(r)}$ are defined entirely in terms of the subset Z in (43) which can be, a priori, arbitrary. Now, absense of wheels in $\mathfrak{L}$ (or, what is the same, in Z) means that after an appropriate renumbering of $\{1, \dots, n\}$, any $(i, j) \in Z$ satisfies $i < j$. Such renumbering does not change the combinatorial type of any of the $P^{(r)}$. Proposition 8.9 is therefore a consequence of the following purely combinatorial fact.

Proposition 8.11. *Let $Z \subset \{(i, j)| \, 1 \leqslant i < j \leqslant n\}$ be any subset of positive roots for A_{n-1}. Then each polyhedral complex $P^{(r)}$ is either empty or contractible.*

Proof. For a permutation $\sigma \in \mathfrak{S}_n$ let

$$O(\sigma) = \big\{(i < j)| \, \sigma(i) < \sigma(j)\big\}$$

be the set of order preserving pairs of σ. Thus the weak Bruhat order on $\mathfrak{S}_n$ is given by

$$\sigma \leqslant \tau \quad \text{iff} \quad O(\tau) \subseteq O(\sigma).$$

Now, fir a face $F \subset P_n$ we have

$$\mathrm{dpt}(F) = \min_{[\sigma]\in \mathrm{Vert}(F)} |O(\sigma) \cap Z|. \tag{45}$$

Indeed, for $F = [\sigma]$ a vertex this is precisely the definition (44), while for $F = [I_1, \dots, I_p]$ the minimum in the RHS of (45) is achieved for σ arranging each I_ν in the decreasing order and is equal to $\mathrm{dpt}(F)$.

Let $D = |Z|$. Then for $r > d$ we have $P^{(r)} = \emptyset$, while for $r \leqslant d$ we have that $P^{(r)}$ contains at least the vertex $[e]$ corresponding to the unit permutation. Further, by (45), the set $\mathrm{Vert}(P^{(r)}) \subset \mathfrak{S}_n$ is a "left order ideal" with respect to the weak Bruhat order: with each τ, it contains all $\sigma \leqslant \tau$. This implies that $P^{(r)}$ contracts onto $[e]$.

This finishes the proof of Theorem 8.4.

F. Proof of Theorem 8.6 (Sketch)

(a) It is enough to prove that the map of the fibers $\mathscr{B}^{-2}_{2,T} \to \mathscr{B}^{-1}_{2,T}$ over any $T = \{s^0_1, s^0_2\} \in \mathrm{Sym}^2(\mathbb{C})$ is surjective. If $s^0_1 \neq s^0_2$, it follows from Theorem 8.4, as there are no wheels of length 2. Assume now that $s^0_1 = s^0_2 = s^0$. The fiber of $p_{1,1*}\mathscr{O}_{\mathbb{C}^2}$ at $\{s^0, s^0\}$ is then $\mathscr{O}(\mathbb{C})/\mathfrak{m}^2_{s^0}$, the space of first jets of sections of $\mathscr{O}_{\mathbb{C}}$ at s^0. Since $\Lambda(s)$ has a first order pole at 0 with residue 1, for any analytic function $f(s_1, s_2)$ we have

$$\lim_{s_1,s_2\to s^0} (\hat{\star}_{1,1}F)(s_1, s_2) \;=\; \frac{1}{2}\frac{d}{dt}\bigg|_{t=0} F(s^0 + t, s^0 - t).$$

This implies that the subspace $\mathfrak{m}_{s^0}/\mathfrak{m}^2_{s_0}$ of jets vanishing at s, will map surjectively onto the fiber of $\mathscr{O}_{\mathrm{Sym}^2(\mathbb{C})}$ at $\{s^0, s^0\}$.

(b) For $T = \{s_1, s_2, s_3\} \in \mathrm{Sym}^3(\mathbb{C})$ let $\mathbb{C}_T$ be the skyscraper sheaf at T. We have a spectral sequence

$$E_2^{ij} = \mathrm{Tor}_i^{\mathrm{Sym}^3(\mathbb{C})}(\underline{H}^j(\mathscr{B}^\bullet_3), \mathbb{C}_T) \implies H^{j-i}(\mathscr{B}^\bullet_{3,T}). \tag{46}$$

We analyze it backwards, using the information about the abutment to say something about E_2 and then about the $\underline{H}^j(\mathscr{B}^\bullet_3)$. Some parts of this analysis involve straightforward computations which we omit, highlighting the conceptual points only.

First, let $\Delta \subset \mathrm{Sym}^3(\mathbb{C})$ be the locus of T such that $s_i = s_j$ for some $i \neq j$. Note that $W \cap \Delta = \emptyset$. Theorem 8.4 implies that for $T \notin W \cup \Delta$ the abutment of (46) is zero for $j - i > -3$ and this implies that both $\underline{H}^{-2}$ and $\underline{H}^{-1}$ of $\mathscr{B}^\bullet_3$ are zero outside $W \cup \Delta$.

Next, $\mathscr{B}^{-1}_3 = \mathscr{O}_{\mathrm{Sym}^3(\mathbb{C})}$, so $d_{-2}(\mathscr{B}^{-1}_3)$ is a sheaf of ideals there and therefore $\underline{H}^{-1}(\mathscr{B}^\bullet_3)$ is the structure sheaf of an analytic subspace $\mathscr{W} \subset \mathrm{Sym}^3(\mathbb{C})$. By the above the support of $\mathscr{W}$ is contained in $W \cup \Delta$.

Next, we analyze (46) in the case when $T \in W$. The permutohedron P_3 is a hexagon, so for $T \notin \Delta$ the complex $\mathscr{B}^\bullet_{3,T}$ is, by Proposition 8.8, the perturbed cochain complex of this hexagon corresponding to the matrix $\mathfrak{L} = \|\lambda_{ij}\| = \|\Lambda(s_i - s_j)\|$. If $T \in W$, then, after renumbering, we have $\lambda_{12} = \lambda_{23} = \lambda_{31} = 0$, while other $\lambda_{ij} \neq 0$. >From this it is an elementary computation to find the dimensions of the cohomology spaces of $\mathscr{B}^\bullet_{3,T}$ to be

$$h^{-3} = 3,\ h^{-2} = 3,\ h^{-1} = 1.$$

This, shows that $\mathscr{W}$ contains W. Further, let ρ be a nontrivial zero of $\zeta(s)$ of multiplicity ν and $\mathscr{W}_\rho$ be the part of $\mathscr{W}$ supported on W_ρ. We can then analyze the last map in complex $\mathscr{B}^\bullet_3$ near $T = \{\rho + c, 1 - \rho + c, -1 + c\} \in W_\rho$ directly, using the family of perturbed differentials $d_{\mathfrak{L}} : C^1(P_3) \to C^2(P_3)$ with $\mathfrak{L} = \|\Lambda(s_i - s_j)\|$ depending on $\{s_1, s_2, s_3\}$ near W_ρ. This is again an elementary computation which yields that $\mathscr{W}_\rho$ is isomorphic to the νth infinitesimal neighborhood of W_ρ in an embedded surface. In particular, if ρ is a simple root, then $\mathscr{W}_\rho = W_\rho$ as an analytic subspace.

This means that $\mathscr{W} = \underline{H}^{-1}(\mathscr{B}_3^\bullet)$ is given locally in $\mathrm{Sym}^3(\mathbb{C}) - \Delta$ by two equations and so $\dim \mathrm{Tor}_1^{\mathrm{Sym}^3(\mathbb{C})}(\underline{H}^{-1}(\mathscr{B}_3^\bullet), \mathbb{C}_T) = 2$ for any $T \in W$. From the equality $h^{-2}(\mathscr{B}_{3,T}^\bullet) = 3$ and the spectral sequence (46) we then conclude that $\dim(\underline{H}^{-2}(\mathscr{B}_3^\bullet) \otimes \mathbb{C}_T) = 1$, and so $W \subset \mathrm{supp}(\underline{H}^{-2}(\mathscr{B}_3^\bullet))$. The statement that $\underline{H}^{-2}(\mathscr{B}_3^\bullet) = \mathscr{O}_{W_\rho}$ near W_ρ for a simple root ρ, uses an additional local calculation which we omit. We also omit the analysis of the case $T \in \Delta$ which shows that the support of $\underline{H}^{-2}(\mathscr{B}_3^\bullet)$ does not meet Δ. □

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Wall-Crossing Structures in Donaldson–Thomas Invariants, Integrable Systems and Mirror Symmetry

Maxim Kontsevich and Yan Soibelman

1 Introduction

1.1 DT-Invariants and Poisson Manifolds

The paper is devoted to the notion of *wall-crossing structure* and its constructions and applications in various situations. It is motivated by our previous work on Mirror Symmetry and Donaldson–Thomas invariants (see [30, 31, 35, 36]) where examples of wall-crossing structures appeared for the first time.

We consider two types of wall-crossing structures in this paper: the one related to the theory of Donaldson–Thomas invariants (DT-invariants for short) and the one related to Mirror Symmetry. Since our main motivation is the former, let discuss it in detail.

It was proposed in [30], Sect. 1.5 (see also [36], Sect. 7.2) that so-called numerical Donaldson–Thomas invariants counting semistable objects in three-dimensional Calabi–Yau categories (DT-invariants of $3CY$ categories for short) introduced in loc.cit. encode a geometric object, which is a (formal) Poisson manifold. Construction of the Poisson manifold relies on the wall-crossing formulas (WCF for short) introduced in loc.cit.

Conversely, we suggested there that the DT-invariants can be recovered from that formal Poisson manifold. Therefore, collections of DT-invariants satisfying WCF are in one-to-one correspondence with a certain class of formal Poisson manifolds.

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Wall-crossing formulas of this type appear in physics (as samples of the numerous literature on the subject we mention [2, 9, 11, 18–21]).

WCF can be naturally realized as identities in the group of (formal) Poisson automorphisms of the Poisson torus naturally associated with the Grothendieck group of the category. In the categorical framework wall-crossing formulas depend on a choice of stability condition on the category. It was explained in [30], Sect. 2 that in fact wall-crossing formulas arise in a more general framework of stability data on graded Lie algebras. The above-discussed case of $3CY$ categories endowed with stability condition corresponds to the case of torus Lie algebras endowed with the stability data given by the DT-invariants of the category.

On the other hand, in our work on Mirror Symmetry (see [31, 35]) a different class of wall-crossing formulas appeared (later it was considered in [24, 25] in the higher-dimensional case). They can be realized as identities in the group of (formal) volume-preserving transformations of a complex torus. This type of formulas is related to the counting of pseudo-holomorphic discs and hence does not depend on the stability condition on the relevant Fukaya category.

Similarities between two types of wall-crossing formulas mentioned above lead to a question: is there a structure which makes the formulas similar? The answer is positive. We call it wall-crossing structure (WCS for short) in this paper.

Besides of the general formalism of WCS we also discuss several new situations in which they appear, most notably, the case of complex integrable systems "with central charge" (see [30] and Sect. 4.2 below). Those include Hitchin integrable systems or Seiberg–Witten integrable systems. To make a link with the above-mentioned categorical version of DT-invariants, we observe that in many cases the (universal cover of the) base of complex integrable system can be thought of as a subspace in the space of stability conditions on some $3CY$ category. In the case of Hitchin integrable system this category is the Fukaya category of the local Calabi–Yau threefold associated with the spectral curve. An interesting fact is that using the geometry of the base of the integrable system we can construct a collection of integers which are similar to DT-invariants (e.g. they satisfy WCF). It is natural to expect that they coincide with the DT-invariants of the above-mentioned Fukaya category.

More precisely, in this paper we are going to discuss *three* different ways to produce collections of integer numbers which enjoy WCF (and subsequently define WCS).

The first construction of the invariants is based on the count of certain gradient trees on the base of the integrable system (they can be called "tropical DT-invariants" because of that).

The second construction extracts DT-invariants from the geometry of the formal neighborhood of a singular curve ("wheel of projective lines") in a certain algebraic variety. This variety is a compactification of the mirror dual to the total space of the integrable system.

Finally, for a "good" non-compact Calabi–Yau threefold one can define DT-invariants of its Fukaya category (we expect they can be defined for any Calabi–Yau threefold). Moreover the moduli space of deformations of the Calabi–Yau threefold is naturally a base of a complex integrable system with central charge. Hypothetically the base can be embedded into the space of stability conditions on the Fukaya category.

We conjecture (see Conjecture 1.2.1) that all three approaches agree in the cases when they all can be applied.

In the case of non-compact Calabi–Yau threefolds, the formal Poisson manifold mentioned at the beginning of this subsection is a "completion at infinity" of an algebraic Poisson variety. In some examples this variety can be realized as the moduli space of local systems on a punctured curve.

1.2 Three Constructions

Here we describe the above-mentioned three approaches (constructions) with more details.

1. For a complex integrable system $\pi : M \to B$ with central charge the following geometry arises on its base B.

   Denote by $B^0 \subset B$ an open dense subset parametrizing non-degenerate fibers of π. There is a local system $\underline{\Gamma} \to B^0$ with fibers $\underline{\Gamma}_b = H_1(\pi^{-1}(b), \mathbf{Z})$. The local system carries an integer skew-symmetric pairing. In general the pairing can be degenerate, hence M is a Poisson manifold only.

   There is a well-defined central charge $Z \in \Gamma(B^0, \underline{\Gamma}^\vee \otimes \mathbf{C})$, where $\underline{\Gamma}^\vee$ is the dual local system. The central charge can be thought of as a local embedding $B^0 \to \underline{\Gamma}_b^\vee \otimes \mathbf{C}$.

   Then (under some conditions on B) we assign to every generic point $b \in B^0$ and every $\gamma \in \underline{\Gamma}_b$ an integer number $\Omega_b^{trop}(\gamma) \in \mathbf{Z}$ which we informally call tropical DT-invariant. Our construction uses tropical trees on B with external vertices at the smooth part of $B^{sing} = B - B^0$, as well as wall-crossing formulas from [30]. The construction is reminiscent to the attractor flow story in supergravity (see [9]) recast in mathematical terms in [36]. Edges of the tropical trees are also gradient lines of the functions on B given by $b \mapsto |Z_b(\gamma)|^2, \gamma \in \underline{\Gamma}_b$.
2. Here we assume for simplicity that the skew-symmetric form on $\underline{\Gamma}$ is non-degenerate, hence M is a holomorphic symplectic manifold (this assumption is not necessary and will be relaxed in the main body of the paper). Let $\omega^{2,0}$ denote the holomorphic symplectic form. Also assume that the above integrable system is endowed with a holomorphic Lagrangian section $s : B \to M$. Then we can assign to the above data a filtered associative algebra of finite type over $\mathbf{C}$ (in fact over $\mathbf{Z}$). Roughly speaking, it is the algebra of endomorphisms (in the Fukaya category of $(M, Re(\omega^{2,0}))$ of the Lagrangian submanifold $s(B)$ with filtration coming from areas of pseudo-holomorphic discs. Let $M^\vee$ denotes the affine scheme of finite type, which is the spectrum of this algebra. It can be thought of as a mirror dual to the symplectic manifold $(M, Re(\omega^{2,0}))$. Actual geometric

construction goes along the lines of [31] and uses the corresponding WCF. Then $M^\vee$ is a complex symplectic manifold of the same complex dimension $2n$. Hypothetically, for a reductive group G and the corresponding Hitchin integrable system on a smooth projective curve, the space $M^\vee$ can be identified with the moduli space of ${}^L G$-local systems on the same curve (Betti realization), where ${}^L G$ is the Langlands dual group.

We will reconstruct the collection of integer numbers satisfying WCF using the geometry of $M^\vee$. Namely, with $M^\vee$ one can canonically associate a $\mathbf{Z}PL$-space $Sk := Sk(M^\vee)$ called the *skeleton* which sits in the Berkovich spectrum of $M^\vee$ (see [31] and Sect. 6.5. below for the details). Hypothetically, each point $b \in B^0$ gives rise to a piecewise linear embedding $i_b : \mathbf{C}^* \simeq \mathbf{R}^2 - \{0\} \to Sk$. In the case of Hitchin integrable systems it can be interpreted in terms of the asymptotic behavior of the monodromy of a connection depending on small parameter.

From the point of view of the geometry of $M^\vee$ this embedding can be interpreted such as follows. We have a (partial) Poisson compactification $\overline{M^\vee}$ of $M^\vee$ by normal crossing divisors and a singular curve C which is a "wheel" of projective lines $\mathbf{CP}^1$ in $\overline{M^\vee} - M^\vee$, and such that locally near C the space $\overline{M^\vee}$ is isomorphic to a toric Poisson variety endowed with a wheel of one-dimensional toric strata. The embedding i_b gives rise to an element $Z_b \in H^1(U_\varepsilon(C) \cap M^\vee, \mathbf{Z}) \otimes \mathbf{C}$, where $U_\varepsilon(C)$ is a small tubular neighborhood of C. In other words, we have a linear functional $Z_b : H_1(U_\varepsilon(C) \cap M^\vee, \mathbf{Z}) \simeq \Gamma_b \to \mathbf{C}$.

Simple arguments from the deformation theory show that, after a choice of Z_b, deformations of the above local toric model are parametrized by collections $(\Omega_b^{MS}(\gamma))_{\gamma \in \Gamma_b - \{0\}}$ satisfying the Support Property from [30]. Moreover, varying the point $b \in B^0$ we arrive to the collection of numbers satisfying WCF.

3. There is a class of algebraic complex integrable systems with central charge associated with non-compact Calabi–Yau threefolds. The base of the integrable system associated with a Calabi–Yau threefold X is (roughly) isomorphic to the moduli space of deformations of X. It looks plausible that all Hitchin systems arise in this way (see [12] for the $A - D - E$ case). Hypothetically, any point $b \in B^0$ gives rise to a stability condition on the category $\mathscr{F}(X)$, the Fukaya category of X. According to the general theory of [30, 34] with a stability condition on $\mathscr{F}(X)$ one can associate a collection $\Omega_b^{cat}(\gamma)$ of "categorical" DT-invariants of $\mathscr{F}(X)$.

Conjecture 1.2.1. $\Omega_b^{trop}(\gamma) = \Omega_b^{MS}(\gamma) = \Omega_b^{cat}(\gamma)$.

This conjecture should be the guiding line for the paper.

Remark 1.2.2. For Hitchin system with the group $SL(n, \mathbf{C})$ Gaiotto, Moore and Neitzke proposed an interpretation of the invariants $\Omega_b^{cat}(\gamma)$ as counting invariants of certain "networks" on the spectral curve of the Hitchin system (for $n = 2$ they are geodesics of the quadratic differential defined by the point $b \in B^0$). The trees in the definition of $\Omega_b^{trop}(\gamma)$ are different from the networks, since they are subsets of the base B rather than of the spectral curve. We do not have a "counting" interpretation for the numbers $\Omega_b^{MS}(\gamma)$.

The concept of wall-crossing structure (WCS for short) underlying all three constructions is discussed in the next section, starting with WCS in a vector space. Intuitively, WCS is given by a collection of group elements parametrized by pairs of points outside of codimension one "walls" and satisfying some consistency conditions. We observe that WCS can be described in different ways, in particular, as a sheaf of sets.

Furthermore, WCS is determined by simpler data, which we call *initial data* for WCS. In the case of complex integrable systems with central charge discussed later in the paper, the initial data are ultimately related to the behavior of the affine structure on the base near the discriminant set B^{sing}. The above conjecture can be reformulated as an equivalence of three wall-crossing structures defined in three different ways.

1.3 Content of the Paper

After the detailed discussion of the Conjecture 1.2.1 let us briefly explain other topics which we discuss in the paper.

Sections 2 and 3 are devoted to the concept of wall-crossing structure and examples. We introduce several useful notions like e.g. support of WCS (this concept is related to the Support Property from [30] which in turn controls the support of DT-invariants). We introduce the notion of attractor flow (the latter goes back to supergravity, see [9–11]) and define initial data in terms of trees with edges which are trajectories of the attractor flow. The initial data can be thought of as a space of "boundary values" which are assigned to "free ends" of attractor trees.

We start Sect. 4 with a brief discussion of complex integrable systems from the point of view of Hodge theory. In fact we consider not only polarized integrable systems but semipolarized as well. In the latter case fibers are semiabelian varieties with polarized quotients. In the case of Hitchin the semipolarized integrable systems appears when the Higgs field has singularities. Then we introduce the notion of a complex integrable system with the central charge. We also explain the construction of WCS and initial data for complex integrable systems with central charge. The initial data are related to the behavior of the integrable system at the discriminant set.

The approach to DT-invariants via wheels of projective lines is the subject of Sect. 5. The idea which we have already discussed above is that DT-invariants can be interpreted as "coordinates" on the moduli space of deformations of the formal neighborhood of a wheel of projective lines in a Poisson toric variety.

The relationship of WCS and SYZ picture of Mirror Symmetry is discussed in Sect. 6. Among other things we argue that the mirror dual to the total space of an integrable system with central charge is an affine scheme of finite type over $\mathbf{Z}$. We also stress the role of canonical B-field, which is a 2-torsion. We explain how the set up of Sect. 5 appears in this framework. Roughly speaking the mirror dual is a log Calabi–Yau whose skeleton is isomorphic to the base of the integrable system. Wheel of lines is related to a choice of central charge.

Deformation theory of non-compact Calabi–Yau threefolds is the subject of Sect. 7. We discuss there not only the smoothness of the moduli space but also some plausible assumptions under which the Fukaya category and Donaldson–Thomas invariants are well-defined. Moduli spaces of such deformations serve as bases of the corresponding integrable systems. This gives a generalization of the work [12] where the case of local Calabi–Yau threefolds associated with ALE spaces was considered. We should also mention the papers [13, 14] which initiated mathematical works on the relationship between Calabi–Yau threefolds and complex integrable systems. In that case the authors considered *compact* Calabi–Yau threefolds, differently from [12]. The corresponding complex integrable systems and their relationship to WCS are discussed in Sect. 9.

Section 8 is essentially devoted to $GL(r)$ Hitchin integrable systems with possibly irregular singularities (although it also contains other interesting topics like deformation theory of complex Lagrangian manifolds in Sect. 8.2). In the case of Hitchin integrable systems many structures discussed earlier for general complex integrable systems admit non-trivial interpretations in terms of (irregular) spectral curves. Our approach to the notion of irregular spectral curve is non-standard (in particular it is quite different from the one in [4]). Roughly speaking, the spectral curve is defined as an effective divisor in the Poisson surface obtained from the compactified cotangent bundle of the initial curve by a series of blow-ups. The existence of the smooth locus in the base of complex integrable system formed by such spectral curves is related to the existence of a solution to additive Deligne–Simpson problem (see Sect. 8.3). In that case the general machinery of Sect. 4 can be applied.

We remark that Sect. 8.6 contains several interesting conjectures about the mirror dual to the total space of $GL(r)$ Hitchin integrable system which are related to different topics which we do not discuss here. In particular, the conjectures about extension of the (twistor) family over $\mathbf{C}^*$ of mirror duals to the whole line $\mathbf{C}$ relate those mirror duals to WKB asymptotics of flat section of connections with a small parameter and to the corresponding theory of resurgent functions. This relationship has a flavor of "non-linear Hodge theory of infinite rank" and deserves further study.

In Sect. 9 we discuss WCS, attractor flow and DT-invariants in the framework of compact Calabi–Yau threefolds. As we have already mentioned, the corresponding complex integrable systems were studied by Donagi and Markman. They are nonpolarized. In this framework one still expects the WCS but the initial data are determined by the values of DT-invariants not only at conifold points (i.e. B^{sing} in the above notation) but also by their values at the so-called attractor points. The value of DT-invariants at a generic conifold point should be equal to 1. This restriction is not completely clear from first principles. The values of DT-invariants at the attractor points are arbitrary integers.

Section 10 is devoted to a version of WCS for the Lie algebra of volume-preserving vector fields on an algebraic torus. This WCS arises naturally in Mirror Symmetry, in the study of SYZ picture of mirror dual families of collapsing

Calabi–Yau manifolds (see [24, 25, 31]). The formalism of WCS in this case is a bit more complicated than the one developed in the main body of the paper. Nevertheless, there are several surprising similarities between them. It is natural to suggest that certain (not yet discovered) unified structure is hidden behind.

In the appendix we describe a cocycle with coefficients in $\mathbf{Z}/2\mathbf{Z}$ associated with a skew-symmetric form. It is used in the definition of the canonical B-field in Sect. 6.

Finally, we should say that the main goal of our paper is to describe the general picture of the rich geometry of Wall Crossing Structures. We have tried to formulate (sometimes in the form of conjectures and assumptions) of what *should be true*. As a result, besides of proven theorems the paper contains many ideas and new projects. On the other hand, many aspects of the story are not discussed (or just touched) in this paper, in particular quantum versions of the results or the relation to canonical bases in cluster algebras, etc.

2 Wall-Crossing Structures

Wall-crossing formulas presented in [30] are identities in certain pronilpotent groups of automorphisms. It is convenient to axiomatize the corresponding structure, which appears in a completely different situations. In particular it generalizes the notion of stability data on a graded Lie algebra introduced in the loc.cit.

In this section Γ denotes a fixed finitely-generated free abelian group, i.e. $\Gamma \simeq \mathbf{Z}^k$ for some $k \in \mathbf{Z}_{\geq 0}$. The associated real vector space is $\Gamma_{\mathbf{R}} := \Gamma \otimes \mathbf{R}$. We will denote by $\mathfrak{g}$ a fixed Γ-graded Lie algebra over $\mathbf{Q}$,

$$\mathfrak{g} = \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma .$$

2.1 Wall-Crossing Structures on a Vector Space

2.1.1 Nilpotent Case

Let us assume that the set

$$\mathrm{Supp}\, \mathfrak{g} := \{\gamma \in \Gamma \mid \mathfrak{g}_\gamma \neq 0\} \subset \Gamma$$

is finite and is contained in an open half-space in $\Gamma_{\mathbf{R}}$. In particular, all elements of $\mathrm{Supp}\, \mathfrak{g}$ are non-zero, i.e. $\mathfrak{g}_0 = 0$. Under our assumption the Lie algebra $\mathfrak{g}$ is nilpotent. Let us denote by G the corresponding nilpotent group. The exponential map $\exp : \mathfrak{g} \to G$ is a bijection of sets.

The finite union of hyperplanes $\gamma^\perp \subset \Gamma^*_{\mathbf{R}}$ ("wall associated with γ") will be denoted by $\mathrm{Wall}_{\mathfrak{g}}$. Its complement has a finite number of connected components which are open convex domains in $\Gamma^*_{\mathbf{R}}$. These components are exactly open strata in the natural stratification of $\Gamma^*_{\mathbf{R}}$ associated with the finite collection of hyperplanes $(\gamma^\perp)_{\gamma\in\mathrm{Supp}\,\mathfrak{g}}$. Notice that different elements $\gamma \in \mathrm{Supp}(\mathfrak{g})$ can give the same hyperplane,

$$\gamma_1^\perp = \gamma_2^\perp \Longleftrightarrow \gamma_1 \parallel \gamma_2.$$

Definition 2.1.1. A (global) wall-crossing structure ((global) WCS for short) for $\mathfrak{g}$ is an assignment

$$(y_1, y_2) \to g_{y_1,y_2} \in G$$

for any $y_1, y_2 \in \Gamma^*_{\mathbf{R}} - \mathrm{Wall}_{\mathfrak{g}}$ which is locally constant in y_1, y_2, satisfies the cocycle condition

$$g_{y_1,y_2} \cdot g_{y_2,y_3} = g_{y_1,y_3} \quad \forall y_2, y_2, y_3 \in \Gamma^*_{\mathbf{R}} - \mathrm{Wall}_{\mathfrak{g}}$$

and such that in the case when the straight interval connecting y_1 and y_2 intersects only one of hyperplanes $\gamma^\perp$ then

$$\log(g_{y_1,y_2}) \in \bigoplus_{\gamma':\gamma'\parallel\gamma} \mathfrak{g}_{\gamma'}\,.$$

It follows from the definition that we can associate with any stratum τ of codimension 1 (which is an open domain in $\gamma^\perp$ for some $\gamma \in \mathrm{Supp}\,\mathfrak{g}$) a "jump"

$$g_\tau := g_{y_1,y_2}$$

where points y_1, y_2 are such that $y_1(\gamma) > 0$, $y_2(\gamma) < 0$, and the interval connecting y_1 with y_2 intersects τ and no other strata of codimension ≥ 1 (hence this interval does not intersect other hyperplanes in our collection, except $\gamma^\perp$). Obviously, a WCS is uniquely determined by the collection of jumps $(g_\tau)_{\mathrm{codim}\,\tau=1}$, satisfying the cocycle condition for each stratum of codimension 2.

2.1.2 Description in Terms of Sheaves and Groups

Notice that the complement $\Gamma^*_{\mathbf{R}} - \mathrm{Wall}_{\mathfrak{g}}$ contains two distinguished components U_+, U_- (which are different iff $\mathfrak{g} \neq 0$) consisting of points $y \in \Gamma^*_{\mathbf{R}}$ such that $y(\gamma) > 0$ (resp. $y(\gamma) < 0$) for all $\gamma \in \mathrm{Supp}\,\mathfrak{g}$. Hence with any global WCS $\sigma = (g_{y_1,y_2})$ we can associate an element

$$g_{+,-} := g_{y_+,y_-} \in G\,, \quad y_\pm \in U_\pm.$$

We will prove later in this section (see Theorem 2.1.6) that the map $\sigma \mapsto g_{+,-}$ provides a bijection between the set of wall-crossing structures and G (considered as a set).

For any point $y \in \Gamma^*_{\mathbf{R}}$ we have a decomposition of $\mathfrak{g}$ (considered as a vector space) into the direct sum of three vector spaces

$$\mathfrak{g} = \mathfrak{g}_-^{(y)} \oplus \mathfrak{g}_0^{(y)} \oplus \mathfrak{g}_+^{(y)}$$

corresponding to components $\mathfrak{g}_\gamma$ such that $y(\gamma) \in \mathbf{R}$ is negative, zero or positive respectively. Obviously all these subspaces are Γ-graded Lie subalgebras of $\mathfrak{g}$. We denote by $G_-^{(y)}, G_0^{(y)}, G_+^{(y)}$ the corresponding nilpotent subgroups of G. Then it is easy to see that the multiplication map

$$G_-^{(y)} \times G_0^{(y)} \times G_+^{(y)} \to G \quad , (g_-, g_0, g_+) \mapsto g_- \cdot g_0 \cdot g_+$$

is a bijection. Hence any element $g \in G$ can be uniquely decomposed as the product

$$g = g_-^{(y)} g_0^{(y)} g_+^{(y)}.$$

We denote by $\pi_y : G \to G_0^{(y)} = G_-^{(y)} \backslash G / G_+^{(y)}$ the canonical projection to the double coset. In the above notation we have $\pi_y(\mathfrak{g}) = \mathfrak{g}_0^{(y)}$. We claim that there exists a sheaf of sets on $\Gamma^*_{\mathbf{R}}$ with the stalk over $y \in \Gamma^*_{\mathbf{R}}$ given by $G_0^{(y)}$.

This is a particular case of the following general construction. Suppose we are given:

(a) a topological space M;
(b) a set S;
(c) an assignment to any point $m \in M$ of a set S_m and a surjection $\pi_m : S \to S_m$, such that for any two elements $s_1, s_2 \in S$ the set $\{m \in M | \pi_m(s_1) = \pi_m(s_2)\}$ is open in M.

Then the above data give rise to a sheaf of sets $\mathscr{S}$ on M in the following way. Its étalé space $\mathscr{S}^{\text{ét}}$ consists of pairs $\{(m, s') | m \in M, s' \in S_m\}$. A base of topology is given by the sets $W_{s,U} = \{(m, s') | m \in U, s' = \pi_m(s)\}$, where s runs through the set S and U runs through the set of open subsets of M.

One can easily prove the following result.

Lemma 2.1.2. *The projection $\mathscr{S}^{\text{ét}} \to M, (m, s') \mapsto m$ is a local homeomorphism.*

Using the standard equivalence between sheaves of sets and local homeomorphisms (étale maps), we obtain

Corollary 2.1.3. *The above construction gives rise to a sheaf of sets $\mathscr{S}$ such that the stalk $\mathscr{S}_m$ is equal to S_m for any $m \in M$.*

Let us apply this lemma to the case $M = \Gamma^*_{\mathbf{R}}, S = G, S_y = G_0^{(y)}, y \in M$ and the map $\pi_y : G \to G_0^{(y)} = G_-^{(y)}\backslash G/G_+^{(y)}, g \mapsto g_0^{(y)}$ given by the canonical projection to the double coset. It is easy to see that the openness condition from c) is satisfied.

Definition 2.1.4. The corresponding sheaf of sets is called the sheaf of wall-crossing structures and is denoted by $WCS_{\mathfrak{g}}$.

Sheaf $WCS_{\mathfrak{g}}$ is constructible with respect to the natural stratification of $\Gamma^*_{\mathbf{R}}$ given by the finite arrangement of hyperplanes $\gamma^{\perp} \subset \Gamma^*_{\mathbf{R}}$, where $\gamma \in \operatorname{Supp} \mathfrak{g}$.

Notice that if $y \in \Gamma^*_{\mathbf{R}} - \mathrm{Wall}_{\mathfrak{g}}$ then the stalk at y is $G_0^{(y)} = \{1\}$. If the point y belongs to a stratum of codimension one (i.e. y lies on exactly one wall $\gamma^{\perp}$) then the Lie algebra of the corresponding stalk is $G_0^{(y)}$ where $Lie(G_0^{(y)}) = \oplus_{\gamma' \| \gamma} \mathfrak{g}_{\gamma'}$. Finally, the stalk at $y = 0$ is the whole group G.

It will be important for the future to study the space of sections of $WCS_{\mathfrak{g}}$ in the following situation. Let $l \subset \Gamma^*_{\mathbf{R}}$ be a straight line intersecting U_+ and U_-. We endow l with the direction from U_+ to U_- and require that it does not intersect strata of codimension bigger or equal than 2. Let $y_1, \dots, y_n$ be the ordered along l set of intersection points with walls. Then the set of sections $\Gamma(l, WCS_{\mathfrak{g}})$ is just the product $\prod_{i=1}^n G_0^{(y_i)}$. The natural map

$$\bigoplus_{i=1}^{n} \mathfrak{g}_0^{(y_i)} \to \mathfrak{g}$$

is a bijection, hence any element $g \in G$ can be uniquely decomposed into the ordered product of elements of $G_0^{(y_i)}$. We conclude that the set of sections $\Gamma(l, WCS_{\mathfrak{g}})$ can be identified naturally with G.

Let us consider the following three sets:

(a) $S_1 = G$;
(b) $S_2 = \Gamma(\Gamma^*_{\mathbf{R}}, WCS_{\mathfrak{g}})$;
(c) S_3 being the set of all wall-crossing structures on $\Gamma^*_{\mathbf{R}}$.

There are three maps $S_1 \to S_2 \to S_3 \to S_1$ given such as follows:

1. the map $S_1 \to S_2$ sends $g \in G$ to the section s_g such that $s_g(y) = g_0^{(y)}$;
2. the map $S_2 \to S_3$ sends a section $s \in \Gamma(\Gamma^*_{\mathbf{R}}, WCS_{\mathfrak{g}})$ to the unique WCS such that for any straight interval connecting two points y_1, y_2, intersecting a hyperplane $\gamma^{\perp}$ at one point y_0 and not intersecting other hyperplanes, and such that

$$y(\gamma_1) < 0,\ y_2(\gamma) > 0$$

the transformation g_{y_1,y_2} coincides with $s(y_0) \in G_0^{(y_0)} \subset G$,
3. the map $S_3 \to S_1$ sends a WCS σ to the corresponding element $g_{+,-} := g_{+,-}(\sigma)$.

It is clear that the maps $S_1 \to S_2$ and $S_3 \to S_1$ are well-defined. It is not obvious that the map $S_2 \to S_3$ indeed takes values in the set of wall-crossing structures.

Proposition 2.1.5. *The map $S_2 \to S_3$ is well-defined.*

Proof. Our description of the map defines jumps g_τ for all strata of codimension one. We need to check the cocycle condition in codimension 2. Let ρ be any stratum of codimension 2. It lies in a finite collection of $k \geq 2$ different hyperplanes $(\gamma_i^\perp)_{i=1,\ldots,k}$. Near ρ we have $2k$ open strata and $2k$ strata of codimension 1. Among $2k$ open strata we have two distinguished ones containing points y_1, y_2 respectively, where

$$y_1(\gamma_i) < 0, y_2(\gamma_i) > 0 \quad \forall i = 1, \ldots, k .$$

There are two paths (up to homotopy) connecting y_1 and y_2 contained in the union of $2k$ open and $2k$ codimension one strata near τ and intersecting each of the hyperplanes $(\gamma_i^\perp)_{i=1,\ldots,k}$ at one point. We want to prove that the composition of jumps along one path coincides with the similar composition along another one. It follows from the definition of the sheaf $WCS_{\mathfrak{g}}$ that both compositions coincide with $g_0^{(y)}$ where y is any point in ρ. Hence the cocycle condition is satisfied. ∎

Theorem 2.1.6. *The above three maps are bijections, and their composition is the identity map.*

We see that sets S_1, S_2, S_3 are canonically identified with each other.

Proof. We split the proof into three lemmas.

Lemma 2.1.7. *The map $S_1 \to S_2$ is a bijection.*

This map is obviously injective, because the $s_g(0) = g$ for any $g \in G$. It is surjective because the point $0 \in \Gamma_{\mathbf{R}}^*$ belongs to the closure of any stratum (which is a conical set). Therefore any section is uniquely determined by its value at 0. This proves first lemma.

Lemma 2.1.8. *The composition $S_1 \to S_2 \to S_3 \to S_1$ is the identity map.*

Given $g \in G$ let us choose a line $l \subset \Gamma_{\mathbf{R}}^*$ such that it intersects both sets U_+ and U_- and does not intersect strata of codimension greater or equal than 2. Let us endow the line with the direction from U_+ to U_-. Let us denote by $y_1, \ldots, y_m$ the ordered points of intersection of l with walls. Then g coincides with the ordered product of $g_0^{(y_i)}$ and hence $g = g_{+,-}$. This proves the second lemma.

Lemma 2.1.9. *The map $S_3 \to S_1$ is injective.*

Observe that for any point $y \in \Gamma_{\mathbf{R}}^*$ which belongs to the stratum of codimension 1 (which is open in a wall) there exists a line l as in the previous lemma and such that $y \in l$. Because of our choice of the line, there exists a unique $i_0, 1 \leq i_0 \leq m$ that $y = y_{i_0}$. We know that the element $g_{+,-}$ determines uniquely all elements $g_0^{(y_i)}$, in particular $g_0^{(y_{i_0})} = g_0^{(y)}$. We conclude that all jumps are determined uniquely by the element $g_{+,-}$. This means that the corresponding WCS is also determined uniquely. This proves the third lemma. Combined together, the three lemmas give the proof of the theorem. ■.

Remark 2.1.10. The set of sections of the sheaf $WCS_{\mathfrak{g}}$ on any open $U \subset \Gamma^*_{\mathbf{R}}$ can be described as the set of locally constant maps from the set of connected components of intersections of codimension one strata with U to corresponding subgroups of G, which satisfy the cocycle condition near points of strata of codimension two. Also, let $U \subset \Gamma^*_{\mathbf{R}}$ be an open convex subset. We define cones $C_\pm(U) \subset \Gamma_{\mathbf{R}}$ as cones generated by $\gamma \in \Gamma$ such that $\pm y(\gamma) > 0$ for all $y \in U$. We denote by $G_\pm(U) = exp(\oplus_{\gamma \in C_\pm(U)} \mathfrak{g}_\gamma)$ the corresponding nilpotent Lie groups. Then $\Gamma(U, \mathrm{WCS}_{\mathfrak{g}}) \simeq G_-(U)\backslash G/G_+(U)$. There is also a description of the set $\Gamma(U, \mathrm{WCS}_{\mathfrak{g}})$ similar to the Definition 2.1.1. Namely, in the Definition 2.1.1 we consider pairs $y_1, y_2 \in U - \mathrm{Wall}_{\mathfrak{g}}$.

2.1.3 Pronilpotent Case

Let $\mathfrak{g} = \oplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma$ be a graded Lie algebra. We do not impose any restrictions on $\mathrm{Supp}(\mathfrak{g})$. In particular we do not assume that the support belongs to a half-space in $\Gamma_{\mathbf{R}}$ (cf. Sect. 2.1.1).

Let $C \subset \Gamma_{\mathbf{R}} := \Gamma \otimes \mathbf{R}$ be a convex cone. We assume that C is *strict*, which means that the closure of C does not contain a line, or, equivalently, C is contained in the positive octant (in some coordinates on $\Gamma_{\mathbf{R}}$). Yet another equivalent condition: there exists $\phi \in \Gamma^*_{\mathbf{R}}$ such that the restriction of ϕ to the cone C is a proper map to $\mathbf{R}_{\geq 0}$.

In this case we define a pronilpotent Lie algebra $\mathfrak{g}_C$ as an infinite product

$$\mathfrak{g}_C := \prod_{\gamma \in C \cap \Gamma - \{0\}} \mathfrak{g}_\gamma$$

and denote by G_C the corresponding pronilpotent group. The exponential map identifies $\mathfrak{g}_C$ and G_C.

Lie algebra $\mathfrak{g}_C$ is the projective limit of nilpotent Lie algebras

$$\mathfrak{g}^{(k)}_{C,\phi} = \bigoplus_{\gamma \in C \cap \Gamma - \{0\}, \phi(\gamma) \leq k} \mathfrak{g}_\gamma = \mathfrak{g}_C / m^{(k)}_{C,\phi},$$

where

$$m^{(k)}_{C,\phi} = \bigoplus_{\gamma \in C \cap \Gamma, \phi(\gamma) > k} \mathfrak{g}_\gamma$$

is the Lie ideal in $\mathfrak{g}_C$, and $\phi \in \Gamma^*_{\mathbf{R}}$ is the above function.

Denote by $G_{C,\phi}^{(k)} = exp(\mathfrak{g}_{C,\phi}^{(k)})$ the corresponding nilpotent group and by $pr_{C,\phi}^{(k)} : G_C \to G_{C,\phi}^{(k)}$ the natural epimorphism of groups.

Then the sheaf of sets $WCS_{\mathfrak{g}_C}$ is defined as the projective limit of the sheaves $WCS_{\mathfrak{g}_{C\phi}^{(k)}}$. It follows from the end of the Remark 2.1.10 that for any open convex subset $U \in \Gamma_{\mathbf{R}}^*$ the set of sections $WCS_{\mathfrak{g}_C}(U)$ admits the following description:

(a) For any $y_1, y_2 \in U$ which do not belong to $(\cup_{\gamma \in C \cap (\Gamma - \{0\})} \gamma^{\perp}) \cap U$ we are given an element $g_{y_1,y_2} \in G_C$ satisfying the cocycle condition.
(b) The projections of these elements to $G_{C,\phi}^{(k)}(U)$ satisfy the condition from Definition 2.1.1.

The latter condition informally means that the "jump" at the generic point of the hyperplane $H = \gamma^{\perp} \subset \Gamma_{\mathbf{R}}^*$, where $\gamma \in C - \{0\}$ belongs to the subgroup $G_{C,H} := exp(\prod_{\gamma' \in C \cap \Gamma, (\gamma')^{\perp} = H} \mathfrak{g}_{\gamma'})$.

In the next definition we extend this picture to the case when the cone C is not fixed in advance and can depend on a point of $\Gamma_{\mathbf{R}}^*$.

Definition 2.1.11. Let $\mathfrak{g} = \oplus_{\gamma \in \Gamma} \mathfrak{g}_{\gamma}$ be the graded Lie algebra, as before. We define the sheaf $WCS_{\mathfrak{g}}$ on $\Gamma_{\mathbf{R}}^*$ such as follows: for any open subset U the set of sections $\Gamma(U, WCS_{\mathfrak{g}})$ consists of a family of elements $g(y, \gamma) \in \mathfrak{g}_{\gamma}$ such that $y \in U, \gamma \in \Gamma - \{0\}$ and $y(\gamma) = 0$ satisfying the following condition:

For any $y \in U$ there exists a neighborhood $U_y \subset U$ and strict convex cone $C_{y,U_y} \subset \Gamma_{\mathbf{R}}$ such that for any $y_1 \in U_y$ the element $g(y_1, \gamma) \neq 0$ iff $\gamma \neq 0$ and $\gamma \in C_{y,U_y}$.

Furthermore, let us fix $\phi \in \Gamma_{\mathbf{R}}^*$ such that its restriction to the closure $\overline{C_{y,U_y}}$ is a proper map to $\mathbf{R}_{\geq 0}$. Then we require that for any $k > 0$ the map

$$y_1 \mapsto pr_{C_{y,U_y}}^{(k)}(exp(\sum_{\gamma} g(y_1, \gamma))), y_1 \in U_y$$

is an element of the set of sections $\Gamma(U_y, WCS_{\mathfrak{g}_{C_{y,U_y},\phi}^{(k)}})$.

Notice that if Supp($\mathfrak{g}$) is finite and contained in an open half-space in $\Gamma_{\mathbf{R}}$ then the above definition agrees with the one given in Sect. 2.1.2 We also have the following pronilpotent analog of the Theorem 2.1.6.

Proposition 2.1.12. *Assume that* Supp($\mathfrak{g}$) $\subset C - \{0\}$, *where C is the cone described at the beginning of this subsection. Then the set of global sections $\sigma = (g(y, \gamma))$ of $WCS_{\mathfrak{g}}$ is in the natural one-to-one correspondence with elements of $g \in G_C$.*

Proof. Follows from the nilpotent case. ■

Having a section $s \in \Gamma(U, WCS_{\mathfrak{g}})$ for an open $U \subset \Gamma_{\mathbf{R}}^*$ we define its support $Supp(s) \subset U \times \Gamma_{\mathbf{R}}$ as a minimal closed, conic in the direction of $\Gamma_{\mathbf{R}}$ set which contains the set of pairs $(y, \gamma), y \in \Gamma_{\mathbf{R}}^*, \gamma \in \Gamma$ such that $y(\gamma) = 0$ and $log(g_0^{(y)})_{\gamma} \in \mathfrak{g}_{\gamma} - \{0\}$.

2.2 Wall-Crossing Structure on a Topological Space

Let us consider the following data:

1. A Hausdorff locally connected topological space M (then we will speak about WCS on M).
2. A local system of finitely-generated free abelian groups of finite rank $\pi : \underline{\Gamma} \to M$.
3. A local system of $\underline{\Gamma}$-graded Lie algebras $\underline{\mathfrak{g}} = \oplus_{\gamma \in \underline{\Gamma}} \underline{\mathfrak{g}}_{\gamma} \to M$ over the field $\mathbf{Q}$.
4. A homomorphism of sheaves of abelian groups $Y : \underline{\Gamma} \to \underline{Cont}_M$, where $\underline{Cont}_M$ is the sheaf of real-valued continuous functions on M.

Equivalently we can interpret Y locally as a continuous map from a domain in M to $\Gamma^*_{\mathbf{R}}$. Then we define the pull-back sheaf $WCS_{\underline{\mathfrak{g}},Y} := Y^*(WCS_{\mathfrak{g}})$, where $WCS_{\mathfrak{g}}$ is the sheaf of sets on $\Gamma^*_{\mathbf{R}}$ constructed in the previous subsection.

Definition 2.2.1. A (global) wall-crossing structure on M is a global section of $WCS_{\mathfrak{g},Y}$. The support of WCS σ is a closed subset of $tot(\underline{\Gamma} \otimes \mathbf{R})$ whose fiber over any point $m \in M$ is described such as follows: it is a strict convex closed cone $Supp_{m,\sigma} \subset \underline{\Gamma}_m \otimes \mathbf{R}$ which is equal to the support of the germ of $WCS_{\underline{\mathfrak{g}},m}$ at the point $Y(m) \in \underline{\Gamma}^*_m \otimes \mathbf{R}$ associated with the section σ.

This definition makes obvious the functoriality of the notion of WCS with respect to pullbacks.

2.3 Examples of Wall-Crossing Structures

(1) Let us fix a free abelian group of finite rank Γ together with a Γ-graded Lie algebra $\mathfrak{g} = \oplus_{\gamma \in \Gamma} \mathfrak{g}_{\gamma}$ and a homomorphism of abelian groups $Z : \Gamma \to \mathbf{C}$ (central charge). Then we take $M = \mathbf{R}/2\pi\mathbf{Z}$, and define on M constant local systems with fibers Γ and $\mathfrak{g}$. We set $Y_{\theta}(\gamma) = Im(e^{-i\theta} Z(\gamma))$, where $\theta \in \mathbf{R}$. Then a WCS associated with this choice is the same as stability data on $\mathfrak{g}$ in the sense of [30]. Family of stability data from [30] parametrized by the topological space M is the same as WCS on $\mathbf{R}/2\pi\mathbf{Z} \times Hom(\Gamma, \mathbf{C})$ with constant local systems $\underline{\Gamma}, \underline{\mathfrak{g}}$ and the above map Y.

(2) Let us fix $\Gamma, \mathfrak{g}, Z \in Hom(\Gamma, \mathbf{C})$ as in Example (1). Let $\hat{M} = \mathbf{R}/2\pi\mathbf{Z} \times Hom(\Gamma, \mathbf{C})$. We endow $\hat{M}$ with constant local systems with fibers Γ and $\mathfrak{g}$ respectively.

The subset $M_Z = \mathbf{R}/2\pi\mathbf{Z} \times \{Z\} \subset \hat{M}$ is isomorphic to the one from the previous example. Then interpreting WCS on M_Z as the pullback sheaf $(Y_{|M_Z})^*(WCS_{\mathfrak{g}})$ we conclude that this WCS can be extended to a neighborhood of Z.

Thus we have a WCS for nearby central charges. Using compactness of the circle $\mathbf{R}/2\pi\mathbf{Z}$ we conclude that for any stability data on $\mathfrak{g}$ with the central charge Z_0 there exists a germ of universal family of stability data with central charges in a neighborhood of Z_0. The above construction gives an alternative to [30] way to define the notion of continuous family of stability data on a graded Lie algebra. As a byproduct we can interpret wall-crossing formulas from [30] in terms of WCS.

(3) Assume that in the Example (1) we have an involution $\eta : \mathfrak{g} \to \mathfrak{g}$ which maps $\mathfrak{g}_\gamma \to \mathfrak{g}_{-\gamma}$. Let us choose a local system on $M = \mathbf{R}/2\pi\mathbf{Z}$ obtained from the trivial local systems from Example (1) by identification $\theta \mapsto \theta + \pi$ on $\mathbf{R}$, $\gamma \mapsto -\gamma$ on Γ and $x \mapsto \eta(x)$ on $\mathfrak{g}$. The corresponding WCS can be identified with symmetric stability data on $\mathfrak{g}$ from [30].

(4) Assume that Γ is endowed with an integer skew-symmetric form $\langle \bullet, \bullet \rangle$. Let us fix central charge Z and set $\mathfrak{g} = \oplus_{\gamma \in \Gamma} \mathbf{Q} \cdot e_\gamma$, where

$$[e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle \gamma_1, \gamma_2 \rangle} \langle \gamma_1, \gamma_2 \rangle e_{\gamma_1 + \gamma_2}.$$

We will call it the *torus Lie algebra*. All previous examples can be specified to this case.

In this case one can encode the WCS as a collection of numbers $\Omega(\gamma)$ which are called DT-invariants for the torus Lie algebras and the central charge Z. The relation with Definition 2.1.11 is as follows:

$$g(e^{i\theta}, \gamma) = \sum_{k|\gamma, k \geq 1} \frac{\Omega(\gamma/k)}{k^2} e_\gamma,$$

where $Z(\gamma) \in e^{i\theta}\mathbf{R}_{>0}$

(5) The quantum version of the previous example deals with the Lie algebra $\mathfrak{g} = \oplus_{\gamma \in \Gamma} \mathbf{Q}(q^{1/2}) \cdot \hat{e}_\gamma$ where

$$[\hat{e}_{\gamma_1}, \hat{e}_{\gamma_2}] = \frac{q^{\langle \gamma_1, \gamma_2 \rangle/2} - q^{-\langle \gamma_1, \gamma_2 \rangle/2}}{q^{1/2} - q^{-1/2}} \hat{e}_{\gamma_1 + \gamma_2}.$$

Here $\hat{e}_\gamma = \frac{\hat{e}_\gamma^{quant}}{q^{1/2} - q^{-1/2}}$ are the normalized generators of the quantum torus $\hat{e}_{\gamma_1}^{quant} \hat{e}_{\gamma_2}^{quant} = q^{\langle \gamma_1, \gamma_2 \rangle/2} \hat{e}_{\gamma_1 + \gamma_2}^{quant}$. We will call it the *quantum torus Lie algebra*.

Remark 2.3.1. In the previous two examples each group G_U contains a subgroup G_U^{adm} of admissible [or quantum admissible in the Example (4)] series (see [34] for the definition). Then in the definition of WCS we can require that $g_{m_1, m_2, U} \in G_U^{adm}$. This leads to the integrality of the corresponding DT-invariants.

(6) Let $\mathscr{C}$ be an ind-constructible $3CY$ category with the class map $cl : K_0(\mathscr{C}) \to \Gamma$ (see [30] for the terminology and notation). Let $G \subset Aut(\mathscr{C}, cl)$ be a subgroup which preserves a connected component $Stab_0(\mathscr{C}, cl) \subset Stab(\mathscr{C}, cl)$ of the set of constructible stability conditions on $\mathscr{C}$. In particular G acts on Γ.

We require that G acts on $Stab_0(\mathscr{C}, cl)$ freely and properly discontinuously, and also that G contains the group $\mathbf{Z}$ generated by the shift functor [1]. Consider $M := Stab_0(\mathscr{C}, cl)$. Notice that there exists a WCS on M with the constant local systems Γ and $\mathfrak{g}$ given in the Examples (4), (5) and $Y = Im(e^{-i\theta}Z)$. Then this WCS descends to M/G. In practice G is a stabilizer of $Stab_0(\mathscr{C}, cl)$ in $Aut(\mathscr{C}, cl)$.

Mirror Symmetry predicts that the moduli space of Calabi–Yau threefolds endowed with a square of a holomorphic volume form carries a local system of $3CY$ categories (Fukaya categories) endowed with stability condition. Thus we expect that there exists the corresponding WCS on an appropriate topological space (see Sects. 7.3 and 9.1 about that). The group G in this case is the fundamental group of the above moduli space.

(7) Let (Q, W) be a quiver with polynomial potential (more generally, we can consider a smooth algebra with potential). Then we constructed in [34] an invertible series A (called quantum DT-series) in the quantum torus with generators $e_\gamma, \gamma \in \Gamma_+ \otimes \mathbf{R} \simeq C := \mathbf{R}^n_{\geq 0}$, where $\Gamma_+ \simeq \mathbf{Z}^n_{\geq 0}$ is the cone of dimension vectors. By the Theorem 2.1.6 it defines a WCS on $\Gamma^*_{\mathbf{R}}$.

Let $Z : \Gamma_+ \to \mathbf{C}$ be a central charge. We assume that $Y = Im(Z)$ is positive on $C - \{0\}$. Consider the straight line $l \subset \Gamma^*_{\mathbf{R}}$ given by $t \mapsto Re(Z) + tIm(Z) \in \Gamma^*_{\mathbf{R}}$. Intersections of this line with the walls (i.e. points $t \in \mathbf{R}$ for which there exists $\gamma \in \Gamma_+ - \{0\}$ such that $Re(Z(\gamma)) + tIm(Z(\gamma)) = 0$) correspond to rays α in the upper-half plane with vertex in the origin. The clockwise factorization formula $A = \prod^{\longrightarrow}_\alpha A_\alpha$ (see [30, 34]) can be interpreted as the previously discussed product formula for the element $g_{+,-} \in G_C$.

(8) In many examples it is natural to consider Γ-graded Lie algebras where Γ is a finitely generated abelian group, possibly with torsion. For example, for the Fukaya category $\mathscr{F}(X)$ of a Calabi–Yau threefold X one should take $\Gamma = H_3(X, \mathbf{Z})$, which can have a non-zero torsion.

In such cases the considerations of the previous and this sections still work, if we replace Γ by Γ^{free} which is the quotient of Γ by the torsion subgroup Γ^{tors}. Then having the Γ-graded Lie algebra $\mathfrak{g}$ we define the graded Lie algebra $\mathfrak{g}_{\Gamma^{free}} = \oplus_{\mu \in \Gamma^{free}} \mathfrak{g}^{free}_\mu$ where $\mathfrak{g}^{free}_\mu = \oplus_{\gamma \, mod \, \Gamma^{tors} = \mu} \mathfrak{g}_\gamma$.

3 Initial Data and Attractor Flow

Definitions and construction of this section, which might look a bit unmotivated, will be used below in Sects. 4.5 and 9.3 when we will discuss the set of "initial values" which determine DT-invariants associated with complex integrable systems.

3.1 Initial Data

Let Γ be a free abelian group of finite rank endowed with a skew-symmetric integer form $\langle \bullet, \bullet \rangle : \bigwedge^2 \Gamma \to \mathbf{Z}$. Let $\mathfrak{g} := \gamma_\Gamma = \oplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma$ be a Γ-graded Lie algebra over a commutative ring of characteristic zero satisfying the condition $[\mathfrak{g}_{\gamma_1}, \mathfrak{g}_{\gamma_2}] = 0$ as soon as $\langle \gamma_1, \gamma_2 \rangle = 0$. E.g. the Lie algebras from examples 4), 5) in the previous subsection satisfy this condition. We denote by Γ_0 the kernel of $\langle \bullet, \bullet \rangle$. Then one has a decomposition $\mathfrak{g} = \mathfrak{g}_{\Gamma_0} \oplus \mathfrak{g}_{\Gamma - \Gamma_0}$, where $\mathfrak{g}_{\Gamma_0} = \oplus_{\gamma \in \Gamma_0} \mathfrak{g}_\gamma$ is a central subalgebra and $\mathfrak{g}_{\Gamma - \Gamma_0} = \oplus_{\gamma \in \Gamma - \Gamma_0} \mathfrak{g}_\gamma$ is its complement. The skew-symmetric form $\langle \bullet, \bullet \rangle$ gives rise to a homomorphism of abelian groups $\iota : \Gamma \to \Gamma^\vee$. Then $\Gamma_0 = Ker\, \iota$. Since a WCS in the abelian case of $\mathfrak{g}_{\Gamma_0}$ is something very simple (just a collection of elements of $\mathfrak{g}_\gamma$ with support in a strict cone, we are going to discuss only the "non-trivial" part which is the induced WCS for $\mathfrak{g}_{\Gamma - \Gamma_0}$. In what follows we assume that if $\gamma \in \Gamma_0$ then $\mathfrak{g}_\gamma = 0$.

Let us consider the "global" version of the above situation. Namely, assume we are given a WCS, say, σ, on a smooth manifold B^0 such that the corresponding local systems $\underline{\Gamma}, \underline{\mathfrak{g}}$ satisfy the same properties as Γ, g above. More precisely, $\underline{\Gamma}$ is endowed with an integer skew-symmetric pairing $\langle \bullet, \bullet \rangle$ such that if for $b \in B^0$ we have $\langle \gamma_1, \gamma_2 \rangle = 0, \gamma_i \in \underline{\Gamma}_b$ then for the corresponding components of $\mathfrak{g}_b$ we have $[\mathfrak{g}_{b,\gamma_1}, \mathfrak{g}_{b,\gamma_2}] = 0$. Let $\underline{\Gamma}_0 \subset \underline{\Gamma}$ be the kernel of $\langle \bullet, \bullet \rangle$. Then assume that if $\gamma \in \Gamma_{0,b}$ then $\mathfrak{g}_{b,\gamma} = 0, b \in B^0$.

Now we will make an additional assumption on B^0 which will be justified later in the case of complex integrable systems (strange notation for B^0 is also borrowed from there). Namely, we assume that the homomorphism of sheaves Y interpreted locally as a continuous map from B^0 to a real vector space, is a smooth submersion. Moreover, we assume that B^0 is endowed with a foliation such that locally near $b \in B^0$ the leaf $M := M_b$ containing b is identified via Y with an affine space over the vector space $\iota(\underline{\Gamma}_{\mathbf{R},b})$.

For each leaf M let us define a smooth manifold $M'_{\mathbf{Z}}$ as the set of pairs $(m, \gamma), m \in M, \gamma \in \underline{\Gamma}_m$ which satisfy the condition that $\gamma \in \underline{\Gamma}_m - \underline{\Gamma}_{0,m}$ and such that $Y(m)(\gamma) = 0$. Notice that $dim\, M'_{\mathbf{Z}} = dim\, M - 1$.

We define a bigger set $M' \supset M'_{\mathbf{Z}}$ as the set of pairs

$$\{(m, v) \in tot(\underline{\Gamma}_{\mathbf{R}}) | Y(m)(v) = 0\}.$$

Clearly $dim\, M' = dim\, M + rk\, \Gamma - 1$.

Definition 3.1.1. The attractor flow on M' is defined by the vector field $\dot{v} = 0, \dot{m} = \iota(v), v \in \underline{\Gamma}_{m,\mathbf{R}}$. It preserves the manifold $M'_{\mathbf{Z}}$ and hence induces the "integer" attractor flow on it.

By our assumptions the vector field does not vanish on $M'_{\mathbf{Z}}$.

Similarly we define $(B^0)'$ and $(B^0)'_{\mathbf{Z}}$ as the union of above-defined sets over all leaves M. The attractor flow extends to the both bigger manifolds.

Assume that we are given a WCS σ on B^0. Therefore we have a piecewise constant map $a : (B^0)'_{\mathbf{Z}} \to tot(\underline{\Gamma})$ which assigns to a point (b, γ) the element $a_b(\gamma) \in \underline{\mathfrak{g}}_{b,\gamma}$, which is the γ-component of the corresponding section of $WCS_{\mathfrak{g},Y}$ (we identify naturally the latter with the logarithm of the corresponding element of the pronilpotent group, see Proposition 2.1.12).

The discontinuity set W_a of the map a (this set is an analog of "walls of first kind" from [30]) belongs to the set of pairs $(b, \gamma) \in Supp_\sigma \cap tot(\underline{\Gamma}) - tot(\underline{\Gamma}_0)$ such that $\gamma = \gamma_1 + \gamma_2$, with $\langle \gamma_1, \gamma_2 \rangle \neq 0$ (this condition implies that all vectors $\gamma, \gamma_1, \gamma_2$ do not belong to $\underline{\Gamma}_{0,b}$, while $(b, \gamma_i) \in Supp_\sigma, i = 1, 2$). Clearly the discontinuity set of the map a is a locally-finite hypersurface in $(B^0)'_{\mathbf{Z}}$ which is locally a pull-back of a $\mathbf{Z}PL$ hypersurface in $\underline{\Gamma}_b^\vee \otimes \mathbf{R}$.

The set $Supp_\sigma$ can be very complicated. For example in the case of SL_2 Hitchin integrable system considered in [19] the fibers of $Supp_\sigma$ should coincide with cones of invariant measures for singular foliations on surfaces given by real parts of certain quadratic differentials.

We believe that in general the support of WCS has a "fractal" structure similar to the following one. Let us consider the set of points $(x, y) \in \mathbf{R}^2$ such that either $x \in \mathbf{R} - \mathbf{Q}, y = 0$ or $x = \frac{p}{q} \in \mathbf{Q}, (p, q) = 1$ and $0 \leq y \leq \frac{1}{q}$. This is a closed subset in $\mathbf{R}^2$, and fibers of the projection $(x, y) \mapsto x$ are compact convex sets.

Although $Supp_\sigma$ could be complicated, it is natural to expect that one can find an "upper bound" on it, which is a closed subset $C^+ \subset (B^0)' \subset tot(\underline{\Gamma}_{\mathbf{R}})$ satisfying the following properties:

(a) Fibers of C^+ under the natural projection $tot(\underline{\Gamma}_{\mathbf{R}}) \to B^0$ are strict convex cones;
(b) The set C^+ is preserved by the "inverse attractor flow" $\dot{v} = 0, \dot{b} = -\iota(v), v \in \underline{\Gamma}_{b,\mathbf{R}}, t > 0$.'

Suppose we know C^+ by some a priori (e.g. geometric) reasons. Then it gives us an "upper bound" for the discontinuity set W_a. More precisely, let us define $\mathrm{Wall}^+ \subset C^+$ as the set of pairs (b, γ) such that $\gamma = \gamma_1 + \gamma_2$ where $\gamma_1, \gamma_2 \in C^+ \cap (B^0)'_{\mathbf{Z}}$ such that $\langle \gamma_1, \gamma_2 \rangle \neq 0$ (in particular it follows that γ_1 and γ_2 are non-parallel vectors). Thus if $Supp_\sigma \subset C^+$ then $\mathrm{W}_a \subset \mathrm{Wall}^+$.

Proposition 3.1.2. *The attractor flow is transversal to* Wall^+ *at the interior of the latter set.*

Proof. It suffices to check the statement on a fixed leaf M. We can work locally and assume that M is an affine space, Γ is a fixed lattice endowed with an integer skew-symmetric form. Let fix $\gamma \in \Gamma$ and consider the attractor flow $y \mapsto y + t\iota(\gamma), t > 0$. On the discontinuity variety we have $\gamma = \gamma_1 + \gamma_2$ and $(y + t\iota(\gamma))(\gamma_m) = 0, m = 1, 2$. Then at the intersection point of the attractor flow with the discontinuity subvariety we have $y(\gamma_1) + t\langle \gamma, \gamma_1 \rangle = 0$ for some (maybe many) $t > 0$. Since $\langle \gamma, \gamma_1 \rangle = -\langle \gamma_1, \gamma_2 \rangle \neq 0$ we conclude that t is determined uniquely. This implies the transversality and finishes the proof. ■

Notice that for each point $(b,\gamma) \in (B^0)'$ there exists maximal possible $t_{max} := t_{max}(b,\gamma) \in (0,+\infty]$ such that the trajectory $\gamma = const, b \mapsto b + \iota(\gamma)t, t \in [0, t_{max})$ does exist. The above trajectory considered for $t \in [0, t_{max})$ will be called *the maximal positive trajectory of the point* (b,γ).

Let us define an open subset $\mathcal{T}_{(B^0)'_{\mathbf{Z}}} \subset (B^0)'_{\mathbf{Z}}$ called the *tail set* consisting of points $(b,\gamma) \in C^+$ such that their maximal positive trajectories with respect to the attractor flow do not intersect the set Wall^+, belong to C^+, and moreover the same properties hold for all nearby points $(b',\gamma') \in (B^0)'$. There is a local system $\underline{\mathfrak{g}}_{loc}$ over $\mathcal{T}_{(B^0)'_{\mathbf{Z}}}$ with the fiber $\underline{\mathfrak{g}}_{b,\gamma}$ over $(b,\gamma) \in \mathcal{T}_{(B^0)'_{\mathbf{Z}}}$.

Tail Assumption. *For any open $U \subset (B^0)'$ the subset of points $(b,\gamma) \in U$ such that their maximal positive trajectories intersect the tail set $\mathcal{T}_{(B^0)'_{\mathbf{Z}}}$, is dense in U.*

Remark 3.1.3. Typically $\underline{\mathfrak{g}}_{loc}$ is trivial of rank one. E.g. in Example (4) from the previous subsection the fiber is $\mathbf{Q}$, while in Example (5) it is $\mathbf{Q}(q^{1/2})$.

Definition 3.1.4. The initial data of a WCS bounded by C^+ is the restriction of the map a to $\mathcal{T}_{(B^0)'_{\mathbf{Z}}}$.

As we will explain below, under some additional conditions the initial data uniquely determine its WCS. Moreover, in some cases one can reconstruct WCS just from the knowledge of initial data. This explains the meaning of this notion.

3.2 Attractor Trees

In what follows we are going to consider metrized rooted trees with finitely many edges oriented toward tails. Here "metrized tree" understood as a length metric space. Internal vertices have outcoming valency at least 2, internal edges have finite length, while tail edges can be infinite. Our convention is that the root vertex has valency 1.

Let us assume the notation of the previous subsection.

Definition 3.2.1. An attractor tree is a metrized rooted tree T endowed with a continuous map $f : T \to M$ to a leaf $M \subset B^0$ and a lift $f' : T - \{Vertices\} \to M'_{\mathbf{Z}}$. We assume that f' maps edges of T to trajectories of the attractor flow, and the metric on each edge of T is given by $|dt|$, where t is the time parameter for attractor flow on its lifting. We assume that all tail edges are maximal positive trajectories of the corresponding internal vertices of T. We assume the balancing condition $\sum_i \gamma_i^{out} = \gamma^{in}$ is satisfied at each internal vertex v. Here γ^{in} is the speed of the f'-lift of the only edge incoming from v, and γ_i^{out} are speeds of the f'-lifts of all outcoming edges. We assume that all γ_i^{out} are pairwise distinct and there exist i_1, i_2 such that $\langle \gamma_{i_1}^{out}, \gamma_{i_2}^{out} \rangle \neq 0$.

To an attractor tree T with a root b and root edge γ we can assign its combinatorial type in the following way. Namely, let us consider an abstract rooted tree $\mathscr{T}$ corresponding to T and a collection of velocities of all its edges, including tails. The velocities can be treated as elements of $\underline{\Gamma}_b$ via the parallel transport along edges of T. Then the *combinatorial type of* T *at* (b,γ) consists of the above abstract tree and the above subset of velocities in $\underline{\Gamma}_b$. Varying (b,γ) we conclude that combinatorial types form a local system over $(B^0)'_{\mathbf{Z}}$ with countable fibers.

It is easy to see that for any combinatorial type at (b,γ) an attractor tree with this combinatorial type is uniquely determined by the collection $\{l_e\}$ of lengths of its inner edges e. Moreover the lengths l_e and the vector $Y(b) \in \underline{\Gamma}^*_{b,\mathbf{R}}$ satisfy a system of linear equations with integer coefficients arising from the following two conditions:

(a) $Y(f(v))(f'(u)) = 0$, where v is a vertex of T and u is a point on an edge adjacent to v and sufficiently close to v;
(b) For any inner edge e connecting vertices v_1 and v_2 we have $Y(f(v_2)) - Y(f(v_1)) = \iota(f'(u))l_e$, where u is any point of e.

For a fixed attractor tree T with the root b and root edge γ let us consider the set of attractor trees with sufficiently close roots, combinatorial types and lengths. This set (which can be thought of as a germ of the universal deformation of T) can be identified with an open domain in the vector subspace of the vector space $\underline{\Gamma}^*_{b,\mathbf{R}} \oplus \mathbf{R}^{\{inner\ edges\}}$ defined by the above system of linear equations. In particular, for any vertex v of T the point $Y(f(v))$ runs through an open domain in a vector subspace $H_v \subset \underline{\Gamma}^*_{b,\mathbf{R}}$ defined over $\mathbf{Q}$. In particular, we see that the set of roots of attractor trees which are close to T and have the same combinatorial type is locally an open domain in a vector subspace of $\underline{\Gamma}^*_{b,\mathbf{R}}$.

Definition 3.2.2. We say that attractor tree T is locally planar (this property will depend on its combinatorial type only) if for each internal vertex v of T the corresponding vectors γ_i^{out} span a two-dimensional vector subspace in $\underline{\Gamma}_v \otimes \mathbf{R}$.

Proposition 3.2.3. *For an attractor tree T with the root b and the root edge γ the set of roots of all sufficiently close attractor trees of the same combinatorial type has codimension ≥ 1 if T is locally planar and has codimension ≥ 2 otherwise (this codimension is the same as $codim(H_b)$ in the above notation). Moreover in the former case any sufficiently close attractor tree is uniquely determined by its root.*

Proof. Let us call a vertex v of T non-planar if the vector γ_i^{out} outcoming from v span a vector space of dimension ≥ 3. Let us prove the second part of the Proposition. For that let us assume that T contains a non-planar vertex v_0. Then the set $Y(f(v_0))$ belongs to a vector subspace $H_{v_0} \subset \underline{\Gamma}^*_{b,\mathbf{R}}$ of codimension ≥ 3 defined over $\mathbf{Q}$ because of the conditions $Y(f(v_0))(\gamma_i^{out}) = 0$. Consider the shortest path $v_0 \leftarrow v_1 \leftarrow \dots . \leftarrow v_n = b$ of vertices of T joined by edges. We will prove by induction that for all $0 \leq i \leq n-1$ we have $codim(H_{v_i}) \geq 3$. Then $codim(H_b) = codim(H_{v_n}) \geq 2$.

The induction step is given by the following lemma.

Lemma 3.2.4. *Consider a germ of the universal deformation of a given attractor tree T. For any edge $e : w_2 \to w_1$ connecting two internal vertices of a variable tree T_s we have the following: if codim$(H_{w_1}) \geq 3$ then codim$(H_{w_2}) \geq 3$.*

Proof of the Lemma. Let $\gamma = f'(u) \in \underline{\Gamma}_b$ denote the velocity of the edge e. Then $Y(f(w_2)) = Y(f(w_1)) - l_e\iota(\gamma)$, where l_e is the length of e. It follows that $Y(f(w_2))$ belongs to a vector subspace of $H_{w_1} + \mathbf{R} \cdot \iota(\gamma)$. If the latter subspace has codim ≥ 3 we are done. Assume that it has codimension 2. Then there exist two linearly independent vector $\mu_1, \mu_2 \in \underline{\Gamma}_b$ such that this vector subspace is equal to $\mu_1^{\perp} \cap \mu_2^{\perp}$. Let us denote by $\{\gamma_j^{out}\}$ the set of velocities of edges outgoing from w_2. Obviously γ belongs to this set. The covector $Y(f(w_2))$ is orthogonal to all vectors $\mu_1, \mu_2, \{\gamma_j^{out}\}$. If the vector subspace in $\underline{\Gamma}_{b,\mathbf{R}}$ generated by the latter set has dimension ≥ 3 the we are done. Hence we can assume that it has dimension 2. Then γ and all γ_j^{out} are linear combinations of μ_1 and μ_2. Since $Y(f(w_2)) \in \text{Wall}^+$ we know that there exist $\gamma_{j_1}^{out}, \gamma_{j_2}^{out}$ such that $\langle \gamma_{j_1}^{out}, \gamma_{j_2}^{out} \rangle \neq 0$. It follows that $\langle \mu_1, \mu_2 \rangle \neq 0$. Thus we have a two-dimensional vector space generated by linearly independent vectors μ_1, μ_2 and the vector γ in this vector space such that $\langle \mu_i, \gamma \rangle = 0, i = 1, 2$ (the latter follows from the fact that $\mu_i(\iota(\gamma)) = 0, i = 1, 2$). We conclude that $\gamma = 0$. This proves the Lemma and the second part of the Proposition.

In order to prove first part we assume that all vertices are locally planar. And then we again proceed by induction by the number of vertices. The statement is obvious for the tree which has only one vertex (root vertex) and one edge (the root edge which coincides with the tail edge). Assume that T is planar and contains at least one internal vertex. Let us choose a vertex v such that the only edges outcoming from v are tails edges. Let us denote by γ^{in}, (γ_i^{out}) the velocities of edges attached to v. Let us denote by T' the tree obtained from T by deleting the vertex v and all outcoming tail edges, and extending the incoming edge by maximal positive trajectory e of the attractor flow with velocity γ^{in}. In this way we obtain a map from the germ of the universal deformation of T to the one of T'. We claim that this is a local homeomorphism. Indeed, the vertex $Y(f(v))$ belongs to the codimension 2 vector subspace of $\Gamma_{\mathbf{R}}^*$ given by $\cap_i (\gamma_i^{out})^{\perp}$. Hence the point $Y(f(v)) - l\iota(\gamma^{in})$ (here l is the length of the incoming edge) varies in the open domain of the hyperplane $(\gamma^{in})^{\perp}$. Therefore the point $Y(f(v))$ is (locally) uniquely determined by the tree T' as the intersection point of the trajectory $Y(f(e)) \subset (\gamma^{in})^{\perp}$ with $\cap_i (\gamma_i^{out})^{\perp}$.

Notice that T' has one less vertex than T. Continuing by induction we reduce the problem to the case of one root vertex and one edge. This completes the proof of Proposition. ■

From now on we assume that we are given an upper bound C^+ as in the previous subsection.

Definition 3.2.5. We say that the attractor tree is bound by C^+ if the image of f' belongs to C^+ (then one can easily see that for any internal vertex v the f'-lift of the only outcoming edge starts on Wall$^+$).

It follows from the properties of C^+ that the attractor tree is bound by C^+ if an only if its tail edges are bound by C^+.

Every attractor tree T has finitely many tail edges which are invariant with respect to the (positive) attractor flow. Let us denote by T^0 the tree obtained by deleting all tail edges. Every edge of such a tree joins two vertices.

Compactness Assumption.

There exists an open dense subset $(B^0)''_{\mathbf{Z}} \subset (B^0)'_{\mathbf{Z}}$ *with the following property: for every* $(b,\gamma) \in (B^0)''_{\mathbf{Z}}$ *there exists a compact subset* $K_{(b,\gamma)} \subset (B^0)'$ *and an open neighborhood* U *of* (b,γ) *such that for every attractor tree* T *with the root and root edge in* U *the corresponding tree* T^0 *belongs to* $K_{(b,\gamma)}$.

Mass Function Assumption.

There exists a morphism of sheaves $X : \underline{\Gamma} \to \underline{Cont}_{(B^0)}$ *such that the pull-back of* X *to* $(B^0)'$ *considered as a continuous function in* $(b,v) \in tot(\underline{\Gamma}_{\mathbf{R}})$ *decreases (non-strictly) along the attractor flow* $v = const, \dot{b} = i(v)$ *as long as* $(b,v) \in C^+$ *and is strictly positive on the set* $C^+ - tot(\underline{\Gamma}_{0,\mathbf{R}})$ *and strictly decreasing along the flow on this set.*

Imposing the above three assumptions (Tail, Compactness and Mass), let us consider the graph $G := G(b,\gamma)$ obtained as the union of all attractor trees with the root at a fixed $(b,\gamma) \in (B^0)''_{\mathbf{Z}}$. Then there are finitely many attractor trees which form the graph and that the obtained graph is acyclic. Assume that the root b runs through the set of generic points satisfying the condition $Y(b) \in \gamma^{\perp}$. The Proposition 3.2.3 implies that $G(b,\gamma)$ is locally planar (with the obvious generalization of the Definition 3.2.2 to graphs). The genericity here means that $Y(b)$ belongs to the complement of the locally finite union of codimension ≥ 2 subspaces of $\Gamma^*_{\mathbf{R}}$.

Proposition 3.2.6. *WCS with fixed* $\underline{\mathfrak{g}}$ *and the support belonging to* $C^+_{\mathbf{Z}} := C^+ \cap (B^0)'_{\mathbf{Z}}$ *is uniquely determined by its initial data.*

Proof. Fix a generic point $(b,\gamma) \in C^+_{\mathbf{Z}}$. Let us consider the maximal acyclic graph G described above. All its tails belong to $\mathcal{T}_{(B^0)'_{\mathbf{Z}}}$ (otherwise we can enlarge the graph). Then we reconstruct the value $a(b,\gamma)$ by induction, starting with the restriction of the function a to $\mathcal{T}_{(B^0)'_{\mathbf{Z}}}$ (initial data) and moving toward the point (b,γ) along the edges of G. Since G is acyclic, for any internal vertex (b',γ') we can uniquely compute $a(b',\gamma')$ from the axioms of WCS in two-dimensional case. Finally we compute $a(b,\gamma)$ by induction. The Proposition is proved. ■

Remark 3.2.7. Since the function a is locally-constant we can reconstruct WCS from the knowledge of a on a dense open subset of $(B^0)'_{\mathbf{Z}}$. We do not claim that the procedure given in the proof of Proposition 3.2.6 produces the data $a(m,\gamma)$ which correspond to a WCS. The reason for that is that the procedure in the proof ensures that the cocycle condition is satisfied for some (but possibly not all) strata of codimension 2 (see the end of Sect. 2.1.1). We need more geometric conditions in order to be sure that all strata of codimension 2 are taken into account.

3.3 Initial Data for WCS in a Vector Space

We assume the set up and the notation of the beginning of the previous subsection, i.e. we have a fixed lattice Γ with a fixed integer skew-symmetric form $\langle \bullet, \bullet \rangle$, a fixed Γ-graded Lie algebra $\mathfrak{g}$, etc. We also fix a closed strict convex cone $C \subset \Gamma_{\mathbf{R}}$. We define $B^0 = \Gamma^*_{\mathbf{R}}$ and $Y = id$. We use the Poisson structure on $\Gamma^*_{\mathbf{R}}$ induced by $\langle \bullet, \bullet \rangle$. We set $C^+ = \{(b, v) \in (B^0)' | b \in B^0, v \in C, b(v) = 0\}$.

Then the Tail Assumption is satisfied. Indeed for any $\gamma \in C \cap (\Gamma - \Gamma_0)$ there are only finitely many $\gamma_1, \gamma_2 \in C \cap (\Gamma - \Gamma_0)$ such that $\gamma = \gamma_1 + \gamma_2, \langle \gamma_1, \gamma_2 \rangle \neq 0$. Then the set of points$\{b | (b, \gamma) \in \mathrm{Wall}^+\}$ is a finite union of $(rk\, \Gamma - 2)$-dimensional hyperplanes in $\gamma^{\perp}$, such that all of them are transversal to the flow $\dot{b} = \iota(\gamma)$.

Therefore for sufficiently large times the attractor flow does not intersect Wall^+. This implies the following

Corollary 3.3.1. $\pi_0(\mathscr{T}_{(B^0)'_{\mathbf{Z}}}) \simeq \{\gamma \in C \cap (\Gamma - \Gamma_0)\}$.

In other words for any $\gamma \in \Gamma - \Gamma_0$ we have a unique connected component of the (integer) tail set, which contains "sufficiently large" parts of rays in the direction of γ.

The Compactness Assumption follows from the finiteness of the set of combinatorial types of attractor trees with given (b, γ) and such that velocities of all edges belong to C.

The Mass Function Assumption is more tricky. In order to construct "mass function" X let us choose coordinates $y_1, \dots, y_{2n}, t_1, \dots, t_m$ in $\Gamma_{\mathbf{R}}$ such that (y_i) are symplectic coordinates and (t_i) are coordinates on the center $\Gamma_{0,\mathbf{R}}$. Let us denote by $x_1, \dots, x_{2n}, s_1, \dots, s_m$ the dual coordinates on $\Gamma^*_{\mathbf{R}}$. Let us also choose a bounded strictly increasing smooth function $f : \mathbf{R} \to \mathbf{R}$ (e.g. $f(x) = arctan(x)$).

Then we define

$$X(b, \gamma) = \sum_{i_1, i_2} f(x_{i_1}) y_{i_2} \omega^{i_1 i_2} + L(\gamma).$$

Here $b = (x_1, \dots, x_{2n}, s_1, \dots, s_m), \gamma = (y_1, \dots, y_{2n}, t_1, \dots, t_m)$ and $L \in \Gamma^*_{\mathbf{R}}$ is a covector independent on b, (ω^{ij}) is the symplectic form on Γ^{symp}.

The condition $\dot{X}(b, \gamma) > 0$ if $\dot{b} = \iota(\gamma)$ is satisfied for $\gamma \in \Gamma - \Gamma_0$ and arbitrary L. Indeed, $\dot{b}$ is given $\dot{x}_{i_1} = \sum_{i_2} y_{i_2} \omega^{i_1 i_2}, \dot{s}_j = 0$. It follows that $\dot{X}(b, \gamma)$ is strictly positive. Moreover for sufficiently large $L \in C^{\vee}$ we will have $X(b, \gamma) > 0$ for any $\gamma \in C - \{0\}$.

In the Remark 3.2.7 we warned the reader that for any initial data there exists at most one corresponding WCS, but its existence is not guaranteed in general. It follows from the Proposition 3.3.2 below that in the case of WCS in a vector space there is no problem with the existence.

From the decomposition $\mathfrak{g} = \mathfrak{g}_{\Gamma_0} \oplus \mathfrak{g}_{\Gamma - \Gamma_0}$ we obtain a similar decomposition $\mathfrak{g}_C := \oplus_{\gamma \in C \cap \Gamma} \mathfrak{g}_\gamma = \mathfrak{g}_{\Gamma_0, C} \oplus \mathfrak{g}_{\Gamma - \Gamma_0, C}$. As we know a WCS on $B^0 = \Gamma^*_{\mathbf{R}}$ with the support in C^+ is the same as an element of the pronilpotent group

$G_{\Gamma-\Gamma_0,C} = exp(\mathfrak{g}_{\Gamma-\Gamma_0,C})$. The initial data is given by the restriction of the map a to the tail set, which defines an embedding $\psi : G_{\Gamma-\Gamma_0,C} \to \prod_{\gamma\in C\cap(\Gamma-\Gamma_0)} \mathfrak{g}_\gamma$.

Proposition 3.3.2. *ψ is a bijection of sets.*

Proof. Let us choose an additive map $\eta : \Gamma \to \mathbf{Z}$ such that $\eta(C - \{0\}) \subset \mathbf{R}_{>0}$. Then $G_{\Gamma-\Gamma_0,C} = \varprojlim_k G_{C,k}$, where each nilpotent group $G_k := G_{C,k}$ is defined similarly to $G_{C,\Gamma-\Gamma_0}$ by taking the exponent of the Lie algebra which is the quotient by the ideal generated by $\mathfrak{g}_\gamma$ with $\eta(\gamma) > k, \gamma \in (\Gamma - \Gamma_0) \cap C$. Then we have a sequence of (compatible with respect to the index k) maps $\psi_k : G_k \to \prod_{\gamma\in C\cap(\Gamma-\Gamma_0),\eta(\gamma)\le k} \mathfrak{g}_\gamma$. We will prove that ψ is a bijection by induction in k. For $k = 0$ there is nothing to prove. Assume that ψ_{k-1} is a bijection. Notice that the fiber of the natural projection $G_k \to G_{k-1}$ is a torsor over the abelian Lie group $exp(\prod_{\gamma\in C\cap(\Gamma-\Gamma_0),\eta(\gamma)=k} \mathfrak{g}_\gamma)$. Similarly the fiber of the natural projection $\prod_{\gamma\in C\cap(\Gamma-\Gamma_0),\eta(\gamma)\le k} \mathfrak{g}_\gamma \to \prod_{\gamma\in C\cap(\Gamma-\Gamma_0),\eta(\gamma)\le k-1} \mathfrak{g}_\gamma$ is a torsor over the $\mathbf{Q}$-vector space $\prod_{\gamma\in C\cap(\Gamma-\Gamma_0),\eta(\gamma)=k} \mathfrak{g}_\gamma$. For a fixed point $g_{k-1} \in G_{k-1}$ we have an isomorphism of torsors. This implies that ψ_k is a bijection. ∎

Remark 3.3.3. Last part of the proof of the above proposition is very transparent in the language of the factorization $G = G_-^{(y)} G_0^{(y)} G_+^{(y)}$ from the previous section: multiplication of the left factor by an element from $Centr(G_0^{(y)})$ is equivalent to the multiplication of the factor from $G_0^{(y)}$ by this element.

The above considerations imply the following alternative description of the initial data for WCS in a vector space. For a $\gamma \in \Gamma - \Gamma_0$ consider the Lie subalgebra $\mathfrak{g}^\gamma = \oplus_{\gamma'\in C\cap(\Gamma-\Gamma_0),\langle\gamma',\gamma\rangle=0}\mathfrak{g}_{\gamma'}$. Then we have a homomorphism of Lie algebras pr from $\mathfrak{g}^\gamma$ onto the abelian Lie algebra $\mathfrak{g}^\gamma_{ab} = \oplus_{\gamma'\in C\cap(\Gamma-\Gamma_0),\gamma'\|\gamma}\mathfrak{g}_{\gamma'}$. The restriction $a(\gamma)$ of $a(y,\gamma)$ to the tail set is given by the γ-component of the element $log(g_0^{(\iota(\gamma))})$. Indeed the element $g_0^{(y)}$ stabilizes for sufficiently large t as long as we follow the attractor flow $y \mapsto y + t\iota(\gamma)$.

In the framework of Sect. 2.3, Example (4) let us fix an isomorphism $\Gamma \simeq \mathbf{Z}^I$. Then consider the initial data given by: $a_{init}(\gamma) = 0, \gamma \notin \mathbf{Z}_{\ge 1}e_i$ and $a_{init}(ke_i) = \frac{1}{k^2}$ otherwise. Here $(e_i)_{i\in I}$ is the standard basis in $\mathbf{Z}^I$. By the above Proposition 3.2.6 we have a unique WCS with these initial data and support in the cone $C = \mathbf{R}^I_{\ge 0}$.

Recall that for a lattice with a basis and an integer skew-symmetric form we can construct a quiver Q with the set of vertices I and the number of arrows $i \to j$ equal to $\langle e_i, e_j\rangle$. Let us choose a generic potential W for this quiver.

Conjecture 3.3.4. The group element $g = 1 + \ldots$ corresponding to the above WCS coincides with the DT-series from [30].

Similarly, if we replace above $\frac{1}{k^2}$ by $\frac{q^{1/2}-q^{-1/2}}{k(q^{k/2}-q^{-k/2})}$ and take the Lie algebra from Example (5) (quantum torus) then conjecturally we obtain the quantum DT-series from [30], Sect. 8 (up to multiplication by a central series in the generators $\hat{e}_\gamma, \gamma \in C \cap \Gamma_0 - \{0\}$, see [30] for the notation). Recall that it is related to the theory of (quantum) cluster varieties. Finally, we remark that the canonical group element g corresponding to the WCS derived from a $3CY$ category in the formal way by means of the transformation between positive and negative chambers does not have to coincide with the motivic DT-series of that category (but the latter defines *some* WCS and the corresponding canonical group element). The comparison is sometimes possible in case if the category has a "good" set of generators (e.g. three-dimensional spherical generators) which can serve as "initial data" in the categorical framework.

4 Geometry of Complex Integrable Systems

Complex integrable system is usually understood as a holomorphic generically surjective map $\pi : (X, \omega^{2,0}) \to B$ of a complex analytic symplectic manifold of dimension $2n$ to a complex analytic manifold of dimension n such that generic fibers are holomorphic Lagrangian submanifolds. A generic fiber is acted locally transitively by an abelian Lie algebra of dimension n. In many interesting situations generic fibers are Zariski open subsets in complex abelian varieties. We can compactify these fibers, and obtain a fibration by abelian varieties over an open dense subset $B^0 \subset B$. This will be the situation discussed in Sect. 4.1.1. In Sect. 4.1.2 we will generalize the story to semiabelian case. The behavior of an integrable system near the discriminant locus $B^{sing} = B - B^0$ is more complicated, although in the generic point of the discriminant one can find an explicit local model (see Sect. 4.6). Our philosophy is that all the information necessary for the construction of a wall-crossing structure (which is our principal goal) is already encoded in the geometry of B^0 (see e.g. Completeness Assumption in Sect. 4.4). Hence in what follows we will use a slightly nonstandard terminology. Complex integrable systems in the usual sense recalled above we will call *full complex integrable systems*. Hence a full integrable system can have e.g. singular fibers. Complex integrable systems in our sense has semiabelian fibers such that the first integer homology of fibers form a local system of lattices.

4.1 Integrable Systems and Variations of Hodge Structure

4.1.1 Case of Pure Hodge Structure

Let $(X^0, \omega^{2,0})$ be a complex analytic symplectic manifold of complex dimension $2n$.[1] Assume we are given an holomorphic map $\pi : X^0 \to B^0$ such that for any $b \in B^0$ the fiber $\pi^{-1}(b)$ is a complex Lagrangian submanifold of X^0, which is in

[1]Many of the results below can be generalized to the case of smooth algebraic varieties.

fact a torsor over an abelian variety endowed with a covariantly constant integer polarization. We will call such data a *(polarized) complex integrable system.*

Let $\underline{\Gamma}$ be a local system of free abelian groups over B^0 with a fiber $\underline{\Gamma}_b := H_1(\pi^{-1}(b), \mathbf{Z}), b \in B^0$. Polarization gives rise to a covariantly constant skew-symmetric bilinear form $\langle \bullet, \bullet \rangle : \bigwedge^2 \underline{\Gamma} \to \underline{\mathbf{Z}}_{B^0}$ which induces a covariantly constant symplectic form on $\underline{\Gamma} \otimes \mathbf{Q}$. In this case we will speak about local system of symplectic lattices.

Then map $\gamma \mapsto \int_\gamma \omega^{2,0}$ gives rise to a morphism of sheaves of abelian groups

$$\alpha := \int \omega^{2,0} : \underline{\Gamma} \to \Omega^{1,cl}_{B^0,hol},$$

where $\Omega^{1,cl}_{B^0,hol}$ denotes the sheaf of holomorphic closed 1-forms on B^0.

Let $U \subset B^0$ be a simply connected domain. Let us choose a basis $(\gamma_1, \dots, \gamma_{2n})$ of $\underline{\Gamma}(U)$. Then the homomorphism α gives rise to a collection of holomorphic closed 1-forms $\alpha_i = \int_{\gamma_i} \omega^{2,0}, 1 \le i \le 2n$ which can be written on U as $\alpha_i = dz_i, 1 \le i \le 2n, z_i \in \mathcal{O}(U)$. The collection of functions $(z_1, \dots, z_{2n})$ defines a holomorphic map $Z : U \to \mathbf{C}^{2n}$. Let $\omega_{ij} = \langle \gamma_i, \gamma_j \rangle$ and $(\omega^{ij})_{i,j} \in Mat(2n, \mathbf{Q})$ be the matrix inverse to $(\omega_{ij})_{i,j}$. It is easy to see that:

(1) $\sum_{i,j} \omega^{ij} dz_i \wedge dz_j = \sum_{i,j} \omega^{ij} \alpha_i \wedge \alpha_j = 0$.
(2) the $(1,1)$-form $\sqrt{-1} \sum_{i,j} \omega^{ij} dz_i \wedge d\bar{z}_j = \sqrt{-1} \sum_{i,j} \omega^{ij} \alpha_i \wedge \overline{\alpha}_j$ is positive (hence it defines a Kähler metric on U).

It follows from (2) that Z is an immersion. It follows from (1) that $Z(U)$ is a Lagrangian submanifold.

Recall the well-known fact that near each point $b \in B^0$ the structure of polarized integrable system is determined by the triple $(\underline{\Gamma}, \langle \bullet, \bullet \rangle, \alpha)$ satisfying (1) and (2).

Indeed, suppose we are given a symplectic lattice $(\Gamma, \langle \bullet, \bullet \rangle)$ and a holomorphic Lagrangian embedding of a neighborhood U of b to $\Gamma^\vee \otimes \mathbf{C}$ defined up to a shift, such that for any $b_1 \in U$ and non-zero $v \in T_{b_1} B^0$ we have $Im\langle dZ_{b_1}(v), \overline{dZ_{b_1}(v)} \rangle > 0$.

Then the polarized integrable system $\pi : (\pi^{-1}(U), \omega^{2,0}) \to U$ is isomorphic (as a polarized integrable system) to the "canonical local model" which is the polarized integrable system with the fiber over $b \in U$ given by $\Gamma \backslash (\Gamma \otimes \mathbf{C}) / (T_{Z(b)} Z(U))^\perp$ endowed with an obvious symplectic form and polarization (we are going discuss the symplectic form in a more general case below in Sect. 4.1.2).

Alternatively the local model is given by the quotient of T^*U by the action of Γ given by $(b, v) \mapsto (b, v + \alpha_b(\gamma)), b \in U, \gamma \in \Gamma, v \in T^*_b U$.

Remark 4.1.1. Locally, on the total space of the polarized integrable system one has an action of a real compact torus $\Gamma \backslash (\Gamma \otimes \mathbf{R})$ by holomorphic symplectomorphisms preserving fibers, and each fiber is a torsor over this torus.

Notice that the local model for a polarized integrable system is endowed with a holomorphic Lagrangian section (zero section). The isomorphism between our integrable system and the local model is not unique. It is determined by a choice of holomorphic Lagrangian section over U. Gluing together local models we obtain a new polarized integrable system $\pi' : X' \to B^0$ with a Lagrangian section $B^0 \to X'$, which is canonically associated with the triple $(\underline{\Gamma}, \langle \bullet, \bullet \rangle, \alpha)$ satisfying (1) and (2). For any $b \in B^0$ the fiber $(\pi')^{-1}(b)$ has a structure of abelian group (zero is given by the Lagrangian section). The fiber $\pi^{-1}(b)$ is a torsor over $(\pi')^{-1}(b)$. Isomorphism classes of polarized integrable systems with fixed $(\underline{\Gamma}, \langle \bullet, \bullet \rangle, \alpha)$ are in one-to-one correspondence with elements of the group $H^1(B^0, \Omega^{1,cl}_{B^0}/\alpha(\underline{\Gamma}))$.

The data $(\underline{\Gamma}, \langle \bullet, \bullet \rangle, \alpha)$ satisfying the conditions (1), (2) are equivalent to the following data:

(a) A variation of polarized pure Hodge structure on B^0 of weight -1 given by $(R^1\pi_*\underline{\mathbf{Z}}_{X^0})^\vee$. The Hodge filtration is given by

$$0 = F^{-2} \subset F^{-1} \simeq (R^1\pi_*(\mathcal{O}_{X^0}))^* \subset F^0 = \underline{\Gamma} \otimes \mathcal{O}_{B^0}.$$

(b) An isomorphism of vector bundles $\Psi : T_{B^0} \to (F^0/F^{-1})^* \simeq F^{-1}$. This isomorphism satisfies certain conditions which can be derived from the conditions (1), (2).

Remark 4.1.2. We can consider complex integrable systems with fibers which are compact complex tori without polarization. In this case the local model is determined by a submanifold $Z(U) \subset \Gamma^\vee \otimes \mathbf{C}$ such that $dim\, U = \frac{1}{2} rk\, \Gamma$ and for any $b \in U$ we have $T_{Z(b)}Z(U) \cap \Gamma^*_{\mathbf{R}} = 0$, where Γ is a lattice of even rank without skew-symmetric integer form.

4.1.2 Case of Mixed Hodge Structure

Definition 4.1.3. A semipolarized complex integrable system is given by a holomorphic fibration of a complex analytic symplectic manifold $\pi : X^0 \to B^0$ where fibers are Lagrangian submanifolds which are semiabelian varieties with polarized abelian quotients.

Our considerations in polarized case can be generalized to the semipolarized one.

Namely we have a local system of lattices $\underline{\Gamma} \to B^0$ which is endowed with an integer skew-symmetric bilinear form $\langle \bullet, \bullet \rangle : \bigwedge^2 \underline{\Gamma} \to \underline{\mathbf{Z}}_{B^0}$, possibly degenerate. Similarly to the pure case it is given by $\underline{\Gamma} = (R^1\pi_*\underline{\mathbf{Z}}_{X^0})^\vee$.

This gives rise to an exact short sequence of local systems

$$0 \to \underline{\Gamma}_0 \to \underline{\Gamma} \to \underline{\Gamma}^{symp} \to 0,$$

where $\underline{\Gamma}_0$ is the kernel of the skew-symmetric bilinear form $\langle \bullet, \bullet \rangle$ and $\underline{\Gamma}^{symp}$ is the symplectic quotient.

The local model is now given by a lattice Γ endowed with a skew-symmetric form $\langle \bullet, \bullet \rangle : \bigwedge^2 \Gamma \to \underline{\mathbf{Z}}_{B^0}$ and a local embedding $Z : U \to \Gamma^\vee \otimes \mathbf{C}$, where U is a small neighborhood of a point $b \in B^0$.

Thus we have a local embedding $Z : U \to \Gamma^\vee \otimes \mathbf{C}$ such that the composition $T_bU \stackrel{dZ}{\to} \Gamma^\vee \otimes \mathbf{C} \to \Gamma_0^\vee \otimes \mathbf{C}$ is surjection for any $b \in U$ and fibers of the corresponding submersion $U \to \Gamma_0^\vee \otimes \mathbf{C}$ are complex Lagrangian submanifolds of the symplectic leaves in the Poisson manifold $\Gamma^\vee \otimes \mathbf{C}$ (they are affine symplectic spaces parallel to the fibers of the natural map $\Gamma^\vee \otimes \mathbf{C} \to \Gamma_0^\vee \otimes \mathbf{C}$). Then $Z(U)$ is a family of Lagrangian submanifolds over a domain in $\Gamma_0^\vee \otimes \mathbf{C}$. The positivity condition is also satisfied: the restriction of the pseudo-hermitian form $i^{-1}\langle v, \overline{v}\rangle_*$ to $(dZ_b)(T_bU) \cap (\Gamma^{symp})^\vee \otimes \mathbf{C}, b \in U$ is positive. Thus for any $Z_0 \in \Gamma_0^\vee \otimes \mathbf{C}$ we have the corresponding Lagrangian submanifold $U_{|Z_0}$ which is endowed with a local system $\Gamma^{symp} \to U_{|Z_0}$ of symplectic lattices satisfying the positivity property. In other words, locally we have a family of (polarized) complex integrable systems parametrized by $\Gamma_0^\vee \otimes \mathbf{C}$.

The same double coset formula as in the polarized case describes fibers of the canonical local model of a semipolarized integrable system. The symplectic form on $\cup_{b\in U}\{b\} \times (\Gamma \backslash \Gamma \otimes \mathbf{C}/(T_{Z(b)}Z(U))^\perp$ is described such as follows. Its pull-back to $U \times (\Gamma \otimes \mathbf{C})$ is the restriction of the canonical 2-form on $(\Gamma^\vee \otimes \mathbf{C}) \times (\Gamma \otimes \mathbf{C})$ obtained by the skew-symmetrization of the canonical pairing between Γ and $\Gamma^\vee$.

Alternatively, we observe that T^*U is a Γ-covering of the local model, hence the canonical symplectic structure on T^*U descends to the local model giving the above symplectic structure.

Similarly to the pure case we have locally a natural action of the connected abelian Lie group $\Gamma \backslash Ker(\Gamma \otimes \mathbf{C} \to \Gamma^{symp} \otimes \sqrt{-1}\mathbf{R})$ (which is a product of a real torus and real vector space) by holomorphic symplectomorphisms, such that fibers of the integrable system are torsors over this group.

Semipolarized integrable system gives rise to the following data:

(1) A local system $\underline{\Gamma} \to B^0$ of free abelian groups endowed with a skew-symmetric pairing $\langle \bullet, \bullet \rangle : \bigwedge^2 \underline{\Gamma} \to \underline{\mathbf{Z}}_{B^0}$. We denote by $\underline{\Gamma}_0$ the kernel of this pairing.
(2) A weight filtration $W_\bullet$

$$W_{-3} = 0 \subset W_{-2} \simeq \underline{\Gamma}_0 \subset W_{-1} = \underline{\Gamma}$$

(notice that $W_\bullet$ is canonically determined by the pair $(\underline{\Gamma}, \langle \bullet, \bullet \rangle)$).
(3) A Hodge filtration

$$F^{-2} = 0 \subset F^{-1} \subset F^0 = \underline{\Gamma} \otimes \mathcal{O}_{B^0}.$$

(4) An isomorphism of holomorphic vector bundles $\Psi : T_{B^0} \simeq (F^0/F^{-1})^* \subset (F^0)^*$.

The data (1)–(4) are required to satisfy the following properties:

(i) $gr_{-2}^{W_\bullet}(\underline{\Gamma})$ is a variation of pure Hodge structure of weight -2 concentrated in bidegree $(-1, -1)$.
(ii) $gr_{-1}^{W_\bullet}(\underline{\Gamma})$ is concentrated in bidegrees $(-1, 0)$ and $(0, -1)$.
(iii) The pairing $\langle \bullet, \bullet \rangle$ induces a polarization on $gr_{-1}^{W_\bullet}(\underline{\Gamma})$.

(iv) Locally and isomorphism Ψ can be written as $\Psi = dZ$, where Z is a local embedding of B^0 to $\Gamma^\vee \otimes \mathbf{C}$ as in the pure case.

Conversely, the data (1)–(4) satisfying the properties (i)–(iv) define a semipolarized integrable system $X' \to B^0$ endowed with a holomorphic Lagrangian section $B^0 \to X'$. An integrable system without Lagrangian section is determined by the associated integrable system with Lagrangian section (see Sect. 4.1.1) and a cohomology class in $H^1(B^0, \Omega^{1,cl}_{B^0}/\alpha(\underline{\Gamma}))$.

Let now $\hat{X}^0 \to B^0$ be a fibration obtained from the initial semipolarized integrable system $X^0 \to B^0$ by the fiberwise quotient by the natural action of the torus $\Gamma_0 \otimes \mathbf{C}^* \simeq (\mathbf{C}^*)^{r_0}, r_0 = rk\,\Gamma_0$. Then $\hat{X}^0$ is a Poisson manifold. Its symplectic leaves are total spaces of polarized integrable systems with bases which are submanifolds of B^0 obtained by fixing (locally) the value $Z_{|\Gamma_0}$ (e.g. in the case of Hitchin systems with regular singularities we fix residues of the Higgs field at singularities).

In other words we obtain a family of polarized integrable systems (with canonical Kähler metrics on the base) which is parametrized (locally) by a domain in an affine space parallel to $\Gamma_0^\vee \otimes \mathbf{C}$.

Remark 4.1.4. Traditionally people speak about integrable systems as Poisson manifolds with symplectic leaves fibered by Lagrangian abelian varieties. The notion of semipolarized integrable system gives rise to such a structure (if we forget about polarization). But in a sense it is more precise. Namely, we have a variation of mixed Hodge structure (not visible in the traditional approach). Furthermore, the space of symplectic leaves carries locally a structure of an affine vector space. In practice the holonomy of the local system $\underline{\Gamma}_0$ is finite (see Lemma 4.4.1 below). Also, under some mild assumptions (which are usually satisfied in practice) an integrable system in the traditional sense gives an integrable system in our sense by means of a simple topological construction, see Sect. 4.2 below.

4.2 Integrable Systems with Central Charge

We are going to consider semipolarized integrable systems. Recall that the map Z is defined locally up to a shift.

Definition 4.2.1. A central charge for a semipolarized integrable system $\pi : X^0 \to B^0$ is a holomorphic section $Z \in \Gamma(B^0, \underline{\Gamma}^\vee \otimes \mathcal{O}_{B^0})$ such that the local isomorphism $\Psi : T_{B^0} \to (F^0/F^{-1})^*$ composed with the natural embedding $(F^0/F^{-1})^* \to (F^0)^* = \Gamma(B^0, \underline{\Gamma}^\vee \otimes \mathcal{O}_{B^0})$ coincides with dZ (cf. (iv) in (4) in the previous subsection).

In other words Z is a homomorphism $\underline{\Gamma} \to \mathcal{O}_{B^0}$. Clearly dZ defines the only non-trivial F^{-1}-term of the Hodge filtration. For an integrable system with central charge we can locally embed B^0 as a submanifold in a *vector* space (not just an affine space as before). Not every integrable system has a central charge.

Theorem 4.2.2. *If a semipolarized integrable system with holomorphic Lagrangian section has central charge then* $[\omega^{2,0}] = 0$.

Proof. We keep the notation of Sect. 4.1. Locally we can identify (in C^∞ sense) the total space $\pi^{-1}(U)$ with the product of U and $\mathbb{T}_\Gamma := Ker(\Gamma \otimes \mathbf{C} \to \Gamma^{symp} \otimes \sqrt{-1}\mathbf{R})/\Gamma$. Then the holomorphic symplectic form $\omega^{2,0}$ is $\mathbb{T}_\Gamma$-invariant and its restriction to the tangent space at any point $(b, 0)$ of the Lagrangian section is described such as follows. We have: $T_{(b,0)}(\pi^{-1}(U)) \simeq Lie(\mathbb{T}_\Gamma) \oplus T_b U \subset (\Gamma \otimes \mathbf{C}) \oplus (\Gamma^\vee \otimes \mathbf{C})$. The canonical skew-symmetric form on the latter space gives $\omega^{2,0}$ after restriction to $T_{(b,0)}(\pi^{-1}(U))$. Using the central charge Z we define a complex-valued C^∞ 1-form β on $\pi^{-1}(U)$ as the $\mathbb{T}_\Gamma$-invariant 1-form, whose restriction to $T_{(b,0)}(\pi^{-1}(U))$ is the restriction of the 1-form on $(\Gamma \otimes \mathbf{C}) \oplus (\Gamma^\vee \otimes \mathbf{C})$ given by the pairing with $(0, Z(b))$. The direct calculation shows that $d\beta = \omega^{2,0}$. ∎

The above Theorem gives an obstruction to the existence of central charge.

We remark that the condition in the above Theorem that the integrable system has Lagrangian section can be relaxed. Namely, let us recall that an integrable system without Lagrangian section is determined by the associated system with Lagrangian section and the "twist", which is the cohomology class in $H^1(B^0, \Omega^{1,cl}_{B^0}/\alpha(\underline{\Gamma}))$. There is a morphism of sheaves of abelian groups $\mathbb{T}_{\underline{\Gamma}} \to \Omega^{1,cl}_{B^0}/\alpha(\underline{\Gamma})$ over B^0 (here $\mathbb{T}_{\underline{\Gamma}} = Ker(\underline{\Gamma} \otimes \mathbf{C} \to (\underline{\Gamma}^{symp} \otimes \sqrt{-1}\mathbf{R})/\underline{\Gamma})$) given by

$$0 \to \mathbb{T}_{\underline{\Gamma}} \to (\underline{\Gamma} \otimes \mathbf{C})/\underline{\Gamma} \xrightarrow{\alpha} \Omega^{1,cl}_{B^0}/\alpha(\underline{\Gamma}).$$

Then one can easily generalize the above proof of the Theorem 4.2.2 to the case when the above twist belongs to the image of an element from $H^1(B^0, \mathbb{T}_{\underline{\Gamma}})$. This generalization is useful for Hitchin integrable systems (see Sect. 8).

Let us now discuss the condition $[\omega^{2,0}] = 0$ in several examples.

4.2.1 *K*3 Surfaces

Let $\pi : X \to \mathbf{P}^1$ be an elliptic fibration of a $K3$ surface, and let $X^0 \to B^0$ be the polarized integrable system obtained by throwing away singular fibers of π. Then $[\omega^{2,0}_{X^0}] \neq 0$, hence the integrable system does not have a central charge. More generally complex integrable systems with the total space being a compact hyperkähler manifold do not have central charge.

4.2.2 Integrable Systems from Dimer Models

In [22] the authors defined a class of integrable systems $X^0 \to B^0$ for which X^0 is birationally symplectomorphic to the torus $(\mathbf{C}^*)^{2n}$ endowed with the constant symplectic form $\sum_{i,j} \omega^{ij} dlog\, z_i \wedge dlog\, z_j$. Then $[\omega^{2,0}_{X^0}] \neq 0$, hence such integrable systems do not have central charge.

4.2.3 Systems Birationally Equivalent to Those on Cotangent Spaces

Let $\pi : X^0 \to B^0$ be a semipolarized integrable system such X^0 is birationally symplectomorphic to a cotangent space T^*M for some complex manifold M (e.g. this is the case for Hitchin systems). We expect that under some mild conditions the central charge does exist. More precisely we can pull-back from T^*M the Liouville form pdq, thus obtaining a meromorphic 1-form λ on X^0, such that $d\lambda = \omega^{2,0}$. The restriction of λ to a semiabelian fiber $\pi^{-1}(b)$ is closed at generic points. In particular we can define residues of $\lambda_{|\pi^{-1}(b)}$ at smooth components of the divisor of poles of λ. One can easily show that the residues are locally constant with respect to $b \in B^0$. Hence we obtain a finite collection of residues. If all of them are equal to zero then we can define the central charge $Z : \gamma' \mapsto \int_{\gamma'} \lambda$, where γ' is any loop in $\pi^{-1}(b)$ which sits in the complement to the divisor of poles and represents a class $\gamma \in H_1(\pi^{-1}(b), \mathbf{Z})$. Since all residues are zero, the integral does not depend on the choice of representative γ'. This class of integrable systems contains so-called Seiberg–Witten integrable systems (see [13]).

4.2.4 Hitchin Integrable Systems

We will discuss this class of examples at length later in the paper. Let us just mention now that for a large class of $GL(n)$ Hitchin integrable systems with singularities (possibly irregular) one can define central charge. Hopefully it can be done for Hitchin systems associated with any reductive group.

This example can be put in the framework of log-families of Lagrangian submanifolds in non-compact Calabi–Yau threefolds (in the particular case of Hitchin systems we have log-families of spectral curves). We are going to discuss this class of examples later in Sects. 7, 8.

4.3 Families of Integrable Systems Without Central Charge

Suppose we are given an analytic family of *full* complex integrable systems $\pi_t : (X_t, \omega_t^{2,0}) \to B_t$, where $t \in U$ and U is a complex analytic manifold. We assume that there exist open dense subsets $X_t^0 \subset X_t$ and $B_t^0 \subset B_t$ such that the restriction of π_t to X_t^0 gives rise to a polarized integrable system $\pi_t : (X_t^0, \omega_t^{2,0}) \to B_t^0$. We assume that $\cup_{t\in U} X_t$ forms a locally trivial bundle over U in the topological sense (we *do not* assume that $\cup_{t\in U} X_t^0$ forms a locally trivial bundle over U). We also assume that for every $t \in U$ we have: $H^1(X_t, \mathbf{Q}) = 0$.

Let $B = \cup_{t\in U} B_t$ and $B^0 \subset B$ be the open dense subset $\cup_{t\in U} B_t^0$. We define a local system of lattices $\underline{\Gamma} \to B^0$ with fibers $\underline{\Gamma}_b = H_2(X_t, \pi_t^{-1}(b), \mathbf{Z}), b \in B_t^0$. The long exact sequence of the pair $(X_t, \pi_t^{-1}(b))$ gives rise to a short exact sequence

$$0 \to \underline{\Gamma}_{0,b} \to \underline{\Gamma}_b \to \underline{\Gamma}_b^{symp} \to 0,$$

where $\underline{\Gamma}_{0,b} = H_2(X_t, \mathbf{Z})/Im(H_2(\pi_t^{-1}(b), \mathbf{Z}))$ and $\underline{\Gamma}_b^{symp} = Ker(H_1(\pi_t^{-1}(b), \mathbf{Z}) \to H_1(X_t, \mathbf{Z}))$ is a sublattice of finite index in a symplectic lattice $H_1(\pi_t^{-1}(b), \mathbf{Z})$, hence itself symplectic. The we see that $\underline{\Gamma}$ carries a covariantly constant integer skew-symmetric pairing with the kernel $\underline{\Gamma}_0$. Local system $\underline{\Gamma}_0$ is constant along fibers of the projection $B^0 \to U$.

Integration of $\omega_t^{2,0}$ gives a linear functional $Z : \underline{\Gamma} \to \mathbf{C}$. The restriction of Z to $\underline{\Gamma}_0$ is constant along B_t^0 for any $t \in U$. Hence it defines (locally near $t \in U$) a map

$$\chi_0 : U \to \underline{\Gamma}_{0,b}^{\vee} \otimes \mathbf{C} = Ker(H^2(X_t, \mathbf{C}) \to H^2(\pi_t^{-1}(b), \mathbf{C})), \;\; t \mapsto [\omega_t^{2,0}]$$

for an arbitrary $b \in B_t^0$. We assume that it is in fact a local open embedding. Thus U can be thought of as base of the universal family of integrable systems. This can be compared with Moser theorem which says that the universal family of real symplectic structures on a symplectic manifold (X, ω) which are close to ω is parametrized by an open neighborhood of $[\omega] \in H^2(X, \mathbf{R})$.

The assumption that χ_0 is a locally open embedding implies that $(B^0, \underline{\Gamma}, \langle \bullet, \bullet \rangle, Z)$ defines a semipolarized integrable system with the base B^0. We warn the reader that the total space of this integrable system is bigger (it has even a bigger dimensions if $dim(U) > 0$) than $\cup_{t \in U} X_t^0$, where $X_t^0 = \pi_t^{-1}(B_t^0)$.

4.4 Finiteness of the Monodromy

Finally we discuss the monodromy of the local system $\underline{\Gamma}_0$, which is the kernel of the skew-symmetric form. We assume that the integrable system has central charge denoted by Z. Then we obtain a locally well-defined map $Z : B^0 \to \Gamma_0^{\vee} \otimes \mathbf{C}, Z \mapsto Z_{|\Gamma_0}$, where Γ_0 is a fiber of $\underline{\Gamma}_0$. We will assume that our integrable system $\pi : X^0 \to B^0$ is algebraic, i.e. in the definition of complex integrable system we have:X^0 is a smooth algebraic symplectic variety, B^0 is a smooth algebraic variety and π is a regular map.

Lemma 4.4.1. *For an algebraic semipolarized integrable system the monodromy of the local system* $\underline{\Gamma}_0$ *is finite.*

Proof. As we discussed in the previous subsection the local system $\underline{\Gamma}_0$ gives rise to a variation of pure Hodge structure of weight -2. Hence it admits a polarization. The monodromy preserves positive quadratic form (polarization) and is given by integer transformations. Hence its elements have finite order. ■

It follows from lemma that we have a map $p_{B^0} : B^0 \to (\Gamma_0^{\vee} \otimes \mathbf{C})/G$, where G is a finite group. Fibers of p_{B^0} are smooth algebraic varieties endowed with Kähler metric.

4.5 Local Model Near the Discriminant

We assume that our semipolarized integrable system $X^0 \to B^0$ has central charge Z, and it is an open dense in a full complex integrable system $X \to B$.

Let us make the following

A_1-Singularity Assumption.

We will assume that $D = B - B^0$ is an analytic divisor. We will also assume that there exists an analytic divisor $D^1 \subset D$ such that $\dim D^1 \le \dim B^0 - 2$, the complement $D^0 := D - D^1$ is smooth, and such that our VMHS together with the central charge Z (see Definition 4.2.1) has the following local model near D^0:

1) *There exist local coordinates $(z_1, .., z_n, w_1, \ldots, w_m)$ near a point of D^0 such that z_1 is small and $D^0 = \{z_1 = 0\}$.*
2) *The map $Z : B^0 \to \mathbf{C}^{2n+m} \simeq \Gamma^\vee \otimes \mathbf{C}$ is a multi-valued map given in coordinates by*

$$(z_1, \ldots, z_n, w_1, \ldots, w_m) \mapsto (z_1, \ldots, z_n, \partial_1 F_0, \ldots, \partial_n F_0, w_1, \ldots, w_m),$$

where $\partial_i = \partial/\partial z_i$, and F_0 is given by the formula

$$F_0 = \frac{1}{2\pi i}\frac{z_1^2}{2} log\, z_1 + G(z_1, .., z_n, w_1, \ldots, w_m),$$

and G is a holomorphic function. The Poisson structure on $\mathbf{C}^{2n+m}$ in the standard coordinates $(x_1, \ldots, x_{2n+m})$ is given by the bivector $\sum_{1\le i\le n} \partial/\partial x_i \wedge \partial/\partial x_{i+n}$.
3) *The function F_0 (called prepotential) satisfies also a positivity condition coming from the condition $i\langle dZ, \overline{dZ}\rangle > 0$, which is satisfied for the restriction of dZ to symplectic leaves $S_{c_1,\ldots,c_m} := \{(z_1, \ldots, z_n, w_1, \ldots, w_m) | w_i = c_i\}$.*
4) *The monodromy of the local system $\underline{\Gamma}$ about D^0 has the form $\mu \mapsto \mu + \langle \mu, \gamma\rangle\gamma$, where γ is such that the pairing $\langle \gamma, \bullet\rangle \in \underline{\Gamma}^\vee$ is a primitive covector.*

The map Z is defined up to a linear change of coordinates

$$(z_1, \ldots, z_n, z_{n+1}, \ldots, z_{2n}, w_1, \ldots, w_m)$$
$$\mapsto (z_1, \ldots, z_n, z_{n+1} + z_1, z_{n+2}, \ldots, z_{2n}, w_1, \ldots, w_m),$$

where $z_{n+i} = \partial_i F_0$, $1 \le i \le n$. This can be interpreted as a local system of complex vector spaces endowed with a skew-symmetric form and a section. A choice of branch of F_0 allows us to identify the fibers of this local system with the standard $(\mathbf{C}^{2n+m})^*$ endowed the skew-symmetric form which is the product of the standard symplectic form in $(\mathbf{C}^{2n})^*$ with the trivial form in $(\mathbf{C}^m)^*$.

Remark 4.5.1. The A_1-Singularity Assumption allows only the simplest possible singularity at the discriminant D. There are other possibilities for such singularities.

They are related to simply-laced Dynkin diagrams (Kodaira classification). We do not discuss more complicated singularities in this paper for two reasons:

i) in many examples (e.g. $GL(r)$ Hitchin systems discussed later) they do not appear;
ii) we do not understand fully the local geometry of other singularities.

4.6 WCS for Integrable Systems with Central Charge

We expect that for a large class of semipolarized integrable systems with central charge (including all algebraic ones) there exists a canonical WCS.

More precisely, suppose we are given a semipolarized integrable system $\underline{\Gamma} \to B^0$ with the central charge Z. The local system $\underline{\Gamma}$ gives rise to a canonical local system of torus (or quantum torus) Lie algebras $\underline{\mathfrak{g}}$ [see Sect. 2.3, Examples (4) and (5)].

We will assume that the monodromy of $\underline{\Gamma}_0 \to B^0$ is a finite group G (as we have seen this is true in the case when B^0 is algebraic). Central charge Z defines a submersion $p_{B^0} : B^0 \to (\Gamma_0^\vee \otimes \mathbf{C})/G$. Fix a non-singular point Z_0 of the orbifold $(\Gamma_0^\vee \otimes \mathbf{C})/G$ and let $M = B^0_{Z_0} = p_{B^0}^{-1}(Z_0)$. The restriction of the local system $\underline{\Gamma}_0$ to M is trivial. Let us also fix $\theta \in \mathbf{R}$. Set $Y = Im(e^{-i\theta}Z)$. Then Y defines a local embedding of every fiber M of p_{B^0} into $\Gamma^*_{\mathbf{R}}$ as an affine symplectic leaf which is parallel to $(\Gamma^{symp}_{\mathbf{R}})^*$.

In order to construct WCS we would like to use the approach of Sect. 3. Recall that in Sect. 3 we gave a definition of the attractor tree bound by a family of cones C^+. Finiteness of the number of attractor trees was guaranteed by several assumptions, including the Mass Function Assumption. In this subsection we are going to reverse the logic. More precisely, in the hypothetic WCS we have an a priori idea of what are the relevant attractor trees.

Definition 4.6.1. We call attractor tree good if it is a locally planar attractor tree in a fiber M of p_{B^0} such that its tail edges hit transversally the discriminant D at the locus D^0, and the velocity of any tail edge is proportional to the corresponding vector γ (see part 4 of the A_1-Singularity Assumption, Sect. 4.5).

Then one defines the family of convex cones C^+_{min} as the minimal closed subset of $(B^0)'$ whose fibers under the natural projection to B^0 are closed convex cones and such that C^+_{min} contains all velocities of the above-described good attractor trees. It is not a priori clear that those conic fibers are *strict* cones.

We claim that the function $X = Re(e^{-i\theta}Z)$ plays the role of the mass function (see Sect. 3.3) and simultaneously gives a restriction on the velocities of good attractor trees.

Lemma 4.6.2. *Let us restrict X to a fiber M and identify the latter locally with a symplectic leaf in $\Gamma^*_{\mathbf{R}}$. Let us also fix $v \in \Gamma_{\mathbf{R}} - \Gamma_{0,\mathbf{R}}$. Then*

$$\frac{d}{dt}_{|t=0} X(m + t\iota(v), v) = -||\iota(v)||^2 < 0,$$

where $\iota : \Gamma_{\mathbf{R}} \to \Gamma_{\mathbf{R}}^*$ was defined in Sect. 3.1 and we understand the non-zero vector $\iota(v)$ as an element of $T_m M$ (recall that M carries natural Kähler metric).

Proof. Follows from definitions. ■

Corollary 4.6.3. *The function X strictly decreases along the attractor flow. Moreover $X(m, v) > 0$ at all inner points of a tail edge of a good attractor tree.*

Proof. It suffices to show that $X(m, v)$ approaches 0 along a tail edge of a good attractor tree. ■

Corollary 4.6.4. *The function X is positive on edges of good attractor trees.*

Proof. By previous Corollary the result holds for tail edges. For other edges it follows by induction using the balancing conditions $\sum_i \gamma_i^{out} = \gamma^{in}$. ■

Consider now all functions $\hat{X}(b, v)$ where $(b, v) \in (B^0)', Y(b)(v) = 0$ which satisfy the properties:

(a) $\hat{X}(b, v)$ is linear in v and strictly decreases along the attractor flow.
(b) $\hat{X}(b, v)$ is strictly positive at inner points of tail edges of good attractor trees.

Every such function defines a closed subset in $(B^0)'$ which is conic and convex in the direction of $\Gamma_{\mathbf{R}}$. Namely, we take $C^+_{\hat{X}} = \{(b, v) | \hat{X}(b, v) \geq 0\}$. We define $C^+ := \cap_{\hat{X}} C^+_{\hat{X}}$, where the intersection is taken over all such functions. The Corollaries 4.6.3 and 4.6.4 hold for the functions $\hat{X}(b, v)$. Therefore $C^+_{min} \subset C^+$.

Conjecture 4.6.5. C^+ is a strict convex cone in the direction of $\Gamma_{\mathbf{R}}$.

One can check that conjecture holds if $\underline{\Gamma}_0 = 0$. We will discuss a motivation of a similar conjecture in the framework of Mirror Symmetry in Sect. 10.3.

Definition 4.6.6. Canonical initial data associate with the tail of a good attractor tree with the velocity $k\gamma$ ($k \geq 1$ and γ is primitive), the element $\frac{1}{k^2} e_{k\gamma}$ of the torus Lie algebra.

The canonical initial data are motivated by interpretation of WCS via DT-invariants (see Example (4), Sect. 2.3 and end of Sect. 3.3).

Conjecture 4.6.7. Assuming Conjecture 4.6.5, Compactness Assumption, Tail Assumption and Mass Assumption, there is a unique WCS on $\mathbf{R}/\pi\mathbf{Z} \times B^0$ with the support in C^+ and the canonical initial data.

In terms of DT-invariants for that WCS we have $\Omega_b(\gamma) = \Omega_b(-\gamma)$ for all generic $b \in B^0$.

In practical terms this means that starting with DT-invariants equal to 1 which we assign to the smooth locus of the discriminant divisor $B - B^0$ we can assign by induction DT-invariants $\Omega_b(\gamma) \in \mathbf{Z}$ for any pair (b, θ) which does not belong to a wall and any $\gamma \in \Gamma_b$ such that $\theta = Arg(Z_b(\gamma))$. The collection $(\Omega_b(\gamma))$ satisfy the wall-crossing formulas from [30]. Algorithm for the construction follows from general considerations of Sects. 3.2. Namely, for a fixed (b, γ) we determine $\theta = Arg(Z_b(\gamma))$. Then we consider all good attractor trees on $\mathbf{R}/\pi\mathbf{Z} \times B^0$ such that their root vertex is (b, θ) and the root edge is γ.

Our assumptions imply that for generic b there are finitely many such trees. They form a graph without oriented cycles (because $X = Re(e^{-i\theta} Z_b)$ is monotone along attractor trajectories). Hence we can order edges of the graph is such a way that the lowest numbering receive vertices in $B - B^0$. Then we move from the lowest order vertices to the vertex b using the WCF from [30] in order to calculate the DT-invariant for the outcoming edges of the graph. Using it last time for the root edge γ we obtain $\Omega_b(\gamma)$. Finally, varying θ and Z_0 we arrive to the WCS on $S^1_\theta \times B^0$.

4.7 *Metric on the Base*

Recall the notation and assumptions of Sects. 4.4, and 4.5. We will assume that the monodromy of the local system $\underline{\Gamma}_0$ is finite. Then we make the following

Completeness Assumption.
The map p_{B^0} extends uniquely to a complex analytic map $p_B : B \to (\Gamma_0^\vee \otimes \mathbf{C})/G$. Fibers of p_B are metric completions of the fibers of p_{B^0}.

Under the Completeness Assumption we can work with non-singular B^0 and then extend arising structures to the whole space B uniquely. We expect that the Completeness Assumption holds in all realistic examples of full complex integrable systems.

For any $Z_0 \in (\Gamma_0^\vee \otimes \mathbf{C})/G$ let us denote $p_{B^0}^{-1}(Z_0)$ by $B^0_{Z_0}$ and $p_B^{-1}(Z_0)$ by B_{Z_0}. The Completeness Assumption means that the Kähler metric on $B^0_{Z_0}$ extends to B_{Z_0} making it into a length space (the metric is singular on $D \cap B_{Z_0}$).

Since B_{Z_0} is a metric completion of $B^0_{Z_0}$, it can be canonically reconstructed from the latter as a topological space. Furthermore the complex structure on B_{Z_0} can be reconstructed from the one on $B^0_{Z_0}$. Indeed we can extend it to the set $B_{Z_0} - D^1$ using the local model described in the previous subsection. Then we extend the complex structure to the whole space B_{Z_0} by the Hartogs principle, taking the direct image of the sheaf on analytic function $\mathcal{O}_{B_{Z_0} - D^1}$. These considerations explain that the WCS and the initial data can be canonically reconstructed from our semipolarized integrable system on B^0.

Next, for every $Z_0 \in (\Gamma_0^\vee \otimes \mathbf{C})/G$ such that $Re(Z_0) = 0$ (i.e. $Z_0 \in (\Gamma_0^\vee \otimes i\mathbf{R})/G$) we define a real 1-form α_{Z_0} on $B^0_{Z_0}$ by the following formula

$$\alpha_{Z_0} = \sum_{i,j} \omega^{ij} Re(z_i) d\, Im(z_j).$$

Here, locally on $B^0_{Z_0} \subset B^0$, we define functions $z_i(b) := Z_b(\gamma_i)$. In this definition $(\gamma_i)_{i=1,\dots,2n}, n = dim_{\mathbf{C}} B$ is a basis of a covariantly constant subspace $V^{sympl} \subset \underline{\Gamma}_b \otimes \mathbf{Q}$, $b \in B^0_{Z_0}$ such that V is complementary to $\underline{\Gamma}_{0,b} \otimes \mathbf{Q}$, and $(\omega^{ij})_{1 \le i,j \le 2n}$ is the inverse matrix to the symplectic pairing $\omega_{ij} = \langle \gamma_i, \gamma_j \rangle$.

It is immediate to check that α_{Z_0} is well-defined and closed.

In general, we expect that the following holds.

Potential Assumption. *There is a smooth function H_{Z_0} on $B^0_{Z_0}$ such that $dH_{Z_0} = \alpha_{Z_0}$. Moreover, the function H_{Z_0} extends by continuity to B_{Z_0}, and gives a bounded below and proper map from B_{Z_0} to $\mathbf{R}$.*

Let us give some motivations for the Potential Assumption under the A_1-Singularity Assumption from the Sect. 4.5.

One can check that the integral of α_{Z_0} around a small loop around the divisor $D^0 \cap B_{Z_0} \subset B_{Z_0}$ vanishes, and that an antiderivative of α_{Z_0} (defined a priori on $B^0_{Z_0}$ near any point of $D^0 \cap B_{Z_0}$) extends continuously to $D^0 \cap B_{Z_0}$.

It is natural to expect that $H^1(B_{Z_0}, \mathbf{C}) = H^1(B_{Z_0} - D^1, \mathbf{C})$ (at least it holds if B_{Z_0} is nonsingular analytic space because the removing of complex analytic subset of codimension at least 2 does not change the fundamental group). There are good reasons to expect that B_{Z_0} itself is simply connected (in fact contractible, see below). Therefore, we conclude that α_{Z_0} is exact, i.e. it can be written as $\alpha_{Z_0} = dH_{Z_0}$ for some real-valued function H_{Z_0} on $B^0_{Z_0}$. This function is strictly convex in the affine structure given by $Im(Z)$ because its tensor of second derivatives coincides with the metric tensor of the Riemannian metric $g_{B^0_{Z_0}}$ associated with the canonical Kähler metric on $B^0_{Z_0}$. By the A_1-Singularity Assumption the function H_{Z_0} continuously extends to $D^0 \cap B_{Z_0}$.

The boundedness an properness of H_{Z_0} can be checked in special cases. For example, let us consider Seiberg–Witten integrable system, which is a polarized integrable system with central charge, fibers being the elliptic curves $y + 1/y - x^2 = 2u$ parametrized by $u \in B^0_{Z_0} = B^0 = \mathbf{C} - \{-1, 1\}$. Then the metric completion is $B_{Z_0} = B = \mathbf{C}$. The function $H := H_{Z_0}$ can be expressed as $H = K - 2Im(F)$, where

$$K = \sqrt{-1} \sum_{i,j} \omega^{ij} z_i \overline{z}_j$$

is the potential of the Kähler metric on B^0 and F is a holomorphic function on B^0 such that

$$dF = \sum_{i,j} \omega^{ij} z_i z_j \,.$$

One can check that $dF = const \cdot du$.

In this example K is a transcendental function which is non-negative and grows as $u \to \infty$ as $C|u| \cdot log|u|$, where $C > 0$. We see that the summand K dominates $2Im(F)$, and hence H is bounded from below and proper. We expect that the general case is similar to this example.

Remark 4.7.1. In general, the convexity of the function H_{Z_0} on $B^0_{Z_0}$ and its boundedness from below on B_{Z_0} supports the idea that it has a unique global minimum $b^{min}_{Z_0} \in B_{Z_0}$ for any Z_0 satisfying the condition $Re(Z_0) = 0$.

Notice that the condition $Re(Z_0) = 0$ implies that the map $b \mapsto Re(Z_b) \in (\underline{\Gamma}_b/\underline{\Gamma}_{0,b})^\vee \otimes \mathbf{R}$ identifies locally $B^0_{Z_0}$ with an open domain in a symplectic vector space. Therefore we can define a canonical Euler vector field Eu_{Z_0} on $B^0_{Z_0}$. We conjecture that the flow associated with Eu_{Z_0} extends to a continuous flow on B_{Z_0} which contracts as $t \to -\infty$ the space B_{Z_0} to the point $b^{min}_{Z_0}$. In particular this conjecture implies that B_{Z_0} is contractible.

If $b^{min}_{Z_0} \in B^0_{Z_0}$ then the arising local geometry is related to the theory of cluster transformations. We will discuss it elsewhere.

In the example of SL_2 Hitchin system with regular singularities on a smooth projective curve C, fixing purely imaginary Z_0 we obtain a complex integrable system over the base B_{Z_0} which consists of quadratic differentials with fixed residues at their singularities, which are poles of second order. Then the point $b^{min}_{Z_0}$ can be identified with the unique Strebel differential on the curve C.

5 Formal Neighborhood of a Wheel of Projective Lines and Stability Data

Given a lattice Γ endowed with an integer skew-symmetric form $\langle \bullet, \bullet \rangle$, we can consider stability data on the graded Lie algebra $\mathfrak{g} = \oplus_{\gamma \in \Gamma - \Gamma_0} \mathbf{Q} \cdot e_\gamma$ where $[e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle \gamma_1, \gamma_2 \rangle} \langle \gamma_1, \gamma_2 \rangle e_{\gamma_1 + \gamma_2}$ and $\Gamma_0 = Ker \langle \bullet, \bullet \rangle$. The aim of this section is to encode these data in terms of formal Poisson varieties endowed with some additional structures. Small variation of the central charge $Z : \Gamma \to \mathbf{C}$ corresponds to an isomorphic Poisson variety.

5.1 *Wheels of Lines, Wheels of Cones and Toric Varieties*

Let Y be a complex toric variety of dimension n. Then it is stratified by the orbits of the action of the torus $T \simeq (\mathbf{C}^*)^n$.

Definition 5.1.1. Wheel of lines in Y is a cyclically ordered collection of one-dimensional T-orbits $F_i, i \in \mathbf{Z}/m\mathbf{Z}, m \geq 3$ such that each

a) $\overline{F}_i \simeq \mathbf{P}^1$ for any i.
b) $\overline{F}_{i-1} \cap \overline{F}_i = \{p_i\}$, where p_i is a point;
c) $\overline{F}_i \cap \overline{F}_j = \emptyset$ if $|i - j| > 1$;

Let $\Gamma = Hom(T, \mathbf{C}^*)$ be the group of characters. Each intersection point p_i is a zero-dimensional T-invariant stratum, hence it defines a closed strict rational convex cone $C_i \subset \Gamma^\vee \otimes \mathbf{R}$ of full dimension with interior points corresponding to 1-parameter subgroups which attract points of the open dense T-orbit of Y to points p_i. Clearly the collection of cones C_i forms a wheel of cones in the following sense.

Definition 5.1.2. We will call wheel of cones a cyclically ordered collection of real closed polyhedral strict convex cones of dimension n, $C_i \subset \Gamma^\vee \otimes \mathbf{R}, i \in \mathbf{Z}/m\mathbf{Z}, m \geq 3$ such that:

a) $C_i \cap C_{i+1}$ is a face of codimension one in C_i and C_{i+1}, $i \in \mathbf{Z}/m\mathbf{Z}$.
b) $int(C_i) \cap int(C_j) = \emptyset$ if $i \neq j$ (here *int* means the interior).

If Y is smooth then all C_i are isomorphic to octants, i.e. to $\mathbf{R}^n_{\geq 0}$ (modulo the action of $GL(n, \mathbf{Z})$). Let us denote $C_i \cap C_{i+1}, i \in \mathbf{Z}/m\mathbf{Z}$ by $C_{i,i+1}$.

We will call the wheel of cones *admissible* if it satisfies the following two assumptions:

Connectedness Assumption.
For any $i \in \mathbf{Z}/m\mathbf{Z}$ *the cone* C_i *is the convex hull of* $C_{i-1,i} \cup C_{i,i+1}$ *and for any* $i, j, |i-j| > 1$ *the convex hull of the pair* $C_{i,i+1}, C_{j,j+1}$ *contains all cones* $C_{i+1}, \dots, C_j$ *or all cones* $C_{j+1}, \dots, C_i$.

We call it the Connectedness Assumption because it implies the following property of the wheel of cones:

for any closed half-space $\alpha \subset \Gamma_{\mathbf{R}}$ such that $0 \in \partial\alpha$ the set of indices i for which $C_{i,i+1}$ belongs to α, forms an interval in $\mathbf{Z}/m\mathbf{Z}$.

Non-degeneracy Assumption.
$\cap_i C_i^\vee = \{0\}$, *where* $C_i^\vee = \{x \in \Gamma_{\mathbf{R}} | y(x) \geq 0, y \in C_i\}$ *is the dual cone.*

One can check that wheel of cones (and hence toric varieties) satisfying the above two assumptions do exist in any dimension $n \geq 3$. Indeed, let us consider a compact rational polyhedron $P \subset \mathbf{R}^{n-2}$ which contains the origin in the interior. Let $v_i, i \in \mathbf{Z}/3\mathbf{Z}$ be three cyclically ordered vectors in $\mathbf{R}^2$ which generate $\mathbf{R}^2$ and satisfy the condition $v_1 + v_2 + v_3 = 0$. Then we define the cone C_i as the convex hull of the set $\{\mathbf{R}_{\geq 0}(v \oplus p) | p \in P, v = v_i \text{ } or \text{ } v = v_{i+1}\}$. One can check that in this way we obtain the wheel of cones.

We denote by $x^\gamma \in \mathcal{O}(T)$ the monomial corresponding to the vector $\gamma \in \Gamma$. Then $x^{\gamma_1} x^{\gamma_2} = x^{\gamma_1+\gamma_2}$. We identify the open orbit of T with T itself.

For a wheel of lines $(F_i)_{i \in \mathbf{Z}/m\mathbf{Z}}$ we denote by $\hat{Y} := \hat{Y}_{\cup_i \overline{F}_i}$ the completion of Y along $\cup_i \overline{F}_i$. The following result will be used in the next subsection.

Theorem 5.1.3. *For an admissible wheel of cones one has:*

a) $H^0(\hat{Y}, \mathcal{O}_{\hat{Y}}) = \mathbf{C}$.
b) $H^1(\hat{Y}, \mathcal{O}_{\hat{Y}}) = \prod_{\gamma \in \cup_{1 \leq i \leq m}(C_{i,i+1})^\vee \cap \Gamma} \mathbf{C} \cdot x^\gamma$.
c) $H^i(\hat{Y}, \mathcal{O}_{\hat{Y}}) = 0$ *for* $i \geq 2$.

Proof. We are going to compute cohomology groups using a covering of $\hat{Y}$, which consists of formal neighborhoods U_i of $F_i \cup \{p_i\} \cup F_{i+1}$. Then $U_{i,i+1} = U_i \cap U_{i+1}$

is the formal neighborhood of F_{i+1} and $U_i \cap U_j = \emptyset$ for $|i-j| > 1$. Notice that the schemes F_i and $F_i \cup \{p_i\} \cup F_{i+1}, i \in \mathbf{Z}/m\mathbf{Z}$ are open affine subschemes of $\cup_i \overline{F}_i$.

The Čech cochain complex associated with this covering has the form

$$\bigoplus_{i \in \mathbf{Z}/m\mathbf{Z}} \mathcal{O}(U_i) \to \bigoplus_{i \in \mathbf{Z}/m\mathbf{Z}} \mathcal{O}(U_{i,i+1}).$$

This makes (c) clear, since there are no non-trivial Čech i-cochains where $i \geq 2$.

Next we observe that the algebra of functions on the formal neighborhood of p_i is $\hat{\mathcal{O}}(U_i) \simeq \prod_{\gamma \in C_i^\vee \cap \Gamma} \mathbf{C} \cdot x^\gamma$. Similarly we define the completed vector space $\hat{\mathcal{O}}(U_{i,i+1}) = \prod_{\gamma \in C_{i,i+1}^\vee \cap \Gamma} \mathbf{C} \cdot x^\gamma$. We will proceed by replacing the algebras of functions $\mathcal{O}(U_i)$, $\mathcal{O}(U_{i,i+1})$ by the corresponding completed vector spaces of formal series. After that we will explain how return to the actual algebras of functions.

In order to compute H^0 we observe that by the Non-degeneracy Assumption we see that only monomial $x^0 = 1$ appears in $H^0(\hat{Y}, \mathcal{O}_{\hat{Y}})$, hence (a) holds. More generally, for any γ we can compute the input of x^γ to H^0 and H^1. For that we draw a planar polygon with vertices corresponding to p_i and edges corresponding to F_i. We are interested in those points p_i and lines F_i for which the monomial x^γ appears in the algebras $\mathcal{O}(U_i)$ and $\mathcal{O}(U_{i,i+1})$ respectively. The union of the relevant vertices and edges is an open subset of the polygon. Hence it can be one of the following subsets:

i) empty subset;
ii) full polygon;
iii) an open interval which consists of a chain of $1 \leq k \leq m$ consecutive open edges and $k-1$ vertices;
iv) a disjoint union of at least two open intervals from (iii).

Case (i) is clear since x^γ does not appear in the cohomology at all. The case (ii) by the Non-degeneracy Assumption corresponds to $\gamma = 0$, which gives the input to both H^0 and H^1.

In the case (iii) we have trivial input to H^0 and one-dimensional input of x^γ to H^1. Case (iv) is impossible by the Connectedness Assumption. Hence we have proved (b) by replacing the algebras of functions by their completions. In order to finish the proof it suffices to check that the following complex of vector spaces is acyclic

$$\bigoplus_{i \in \mathbf{Z}/m\mathbf{Z}} \hat{\mathcal{O}}(U_i)/\mathcal{O}(U_i) \to \bigoplus_{i \in \mathbf{Z}/m\mathbf{Z}} \hat{\mathcal{O}}(U_{i,i+1})/\mathcal{O}(U_{i,i+1}).$$

It follows from definitions that $\mathcal{O}(U_i)$ consists of such series $f = \sum_{\gamma \in C_i^\vee \cap \Gamma} f_\gamma x^\gamma \in \hat{\mathcal{O}}(U_i)$ that $Supp(f)$ is finite on rays parallel to the rays in $C_i^\vee$ corresponding to $(n-1)$-dimensional faces $C_{i-1,i}$ and $C_{i,i+1}$. Similarly $\mathcal{O}(U_{i,i+1}) \subset \hat{\mathcal{O}}(U_{i,i+1})$ consists of such series $g = \sum_{\gamma \in C_{i,i+1}^\vee \cap \Gamma} g_\gamma x^\gamma$ that $Supp(g)$ is finite on lines parallel to $C_{i,i+1}^\perp \subset C_{i,i+1}^\vee$.

In order to describe quotient spaces in the above complex let us introduce some notation. Let L be an oriented rational line in $\Gamma_{\mathbf{R}}$ with a positive primitive generator $l \in L \cap \Gamma$. Let $V \subset \Gamma_{\mathbf{R}}/L$ be a closed strict polyhedral cone. We denote by π the natural projection $\Gamma_{\mathbf{R}} \to \Gamma_{\mathbf{R}}/L$. We denote by $\mathscr{F}(V,l)$ the vector space which is the quotient of the vector space of such series $f = \sum_{\gamma\in\pi^{-1}(V)\cap\Gamma} f_\gamma x^\gamma$ that $f_{\gamma-nl} = 0$ for sufficiently large n by the subspace of series for which for a given γ we have $f_{\gamma\pm nl} = 0$ for all sufficiently large $n \geq 0$. If we choose a splitting of the projection $\Gamma \to \Gamma \cap L$ then

$$\mathscr{F}(V,l) \simeq \{ \sum_{\mu\in(\Gamma/\Gamma\cap L)\cap V} a_\mu x^\mu | a_\mu \in \mathbf{C}[[x^l]]/\mathbf{C}[x^l] \simeq \mathbf{C}((x^l))/\mathbf{C}[x^l,(x^l)^{-1}]\},$$

where x^l is the monomial corresponding to the generator l.

Let us return to the proof. Let l_i be a generator of $C_{i,i+1}^{\perp} \cap \Gamma \simeq \mathbf{Z}$ which is positive as a functional on C_i. Let V_i be the image of $C_{i,i+1}^{\vee}$ under the projection $\Gamma_{\mathbf{R}} \to \Gamma_{\mathbf{R}}/\mathbf{R}\cdot l_i$. Then $\hat{\mathscr{O}}(U_i)/\mathscr{O}(U_i) \simeq \mathscr{F}(V_{i-1}, l_{i-1}) \oplus \mathscr{F}(V_i, -l_i)$ and similarly $\hat{\mathscr{O}}(U_{i,i+1})/\mathscr{O}(U_{i,i+1}) \simeq \mathscr{F}(V_i, l_i) \oplus \mathscr{F}(V_i, -l_i)$. From this explicit description it is easy to see that the differential in the quotient complex is an isomorphism. This completes the proof. ■

5.2 *Deformations of Formal Poisson Manifolds*

Let us choose an admissible wheel of cones $(C_i)_{i\in\mathbf{Z}/m\mathbf{Z}}$. Suppose that the character lattice Γ is endowed with an integer skew-symmetric form $\langle\bullet,\bullet\rangle : \bigwedge^2\Gamma \to \mathbf{Z}$ with the kernel $\Gamma_0 \subset \Gamma$. Then the toric variety Y and its completion $\hat{Y}$ defined in the previous subsection carry T-invariant Poisson structures. Both schemes are stratified (by the closures of T-orbits in the case of Y and by their completions in the case of $\hat{Y}$).

Conversely, suppose we have a free abelian group Γ endowed with a skew-symmetric integer form $\langle\bullet,\bullet\rangle$ and a wheel of cones satisfying the Connectedness and Non-degeneracy Assumptions. Recall that a toric variety is given by a T-torsor together with a choice of fan. In what follows we are going to use the canonical T-torsor $\mathscr{T}_{can} := \mathscr{T}_{can}(\Gamma,\langle\bullet,\bullet\rangle)$ which is the spectrum of the algebra with $\mathbf{C}$-linear basis $e_\gamma, \gamma \in \Gamma$ and the multiplication rule $e_\gamma e_\mu = (-1)^{\langle\gamma,\mu\rangle} e_{\gamma+\mu}$ (the choice of this torsor is motivated by applications to the theory of DT-invariants). Let us choose the fan which consists of cones C_i and their faces. We will denote the corresponding toric variety by Y_{can}. Its completion along $\overline{F}_{can} = \cup_i \overline{F}_{i,can}$ will be called *canonical local toric model associated with the wheel of cones* $(C_i)_{i\in\mathbf{Z}/m\mathbf{Z}}$ (or simply *local toric model*) and denoted by $\hat{Y}_{can}$. Notice that all one-dimensional T-orbits $F_{i,can} \subset \hat{Y}_{can}$ are endowed with distinguished coordinates given by e_{γ_i}, where $\gamma_i \in C_i^{\vee} \cap \Gamma$ is a primitive vector such that $C_{i,i+1} \subset \gamma_i^{\perp}$.

In case of the local toric model we have a distinguished collection of rational functions which are central with respect to the Poisson bracket and which are parametrized by Γ_0. Namely, denote by $D_Y \subset Y$ the canonical toric divisor (i.e. the complement to the open T-orbit), and by $D_{\hat{Y}} \subset \hat{Y}$ its completion along $\overline{F}$. Let $\mathcal{O}_{\hat{Y}}(*D_{\hat{Y}})$ be the sheaf of rational functions having poles at $D_{\hat{Y}}$. An element $e_\gamma, \gamma \in \Gamma$ defines a section s_γ of this sheaf. In particular the map $c : \Gamma_0 \to \Gamma(\hat{Y}, \mathcal{O}_{\hat{Y}}(*D_{\hat{Y}})^\times), \gamma \mapsto s_\gamma$ defines a homomorphism of Γ_0 to the abelian group of invertible functions having poles at $D_{\hat{Y}}$. The image of c (equivalently the collection $(s_\gamma)_{\gamma\in\Gamma_0}$) is by definition our distinguished collection.

Next we would like to describe stratified formal Poisson varieties which are locally isomorphic to the above local toric model and endowed with additional data called decoration. We need some preparations for that.

Definition 5.2.1. A pair $(\mathcal{X}, D)$ consisting of a (possibly formal) normal scheme $\mathcal{X}$ with a reduced divisor D such that all singularities of $\mathcal{X}$ are contained in D is called a local formal toric pair if the following condition is satisfied: for any closed point $x \in D$ the pair $(\hat{\mathcal{X}}_x, \hat{D}_x)$ obtained by the completion at x is isomorphic to a similar pair $(\hat{\mathcal{X}}_x^{tor}, \hat{D}_x^{tor})$, where $\mathcal{X}_x^{tor}$ is a toric variety and D_x^{tor} is the canonical toric divisor (i.e. the complement to the open orbit).

For any local formal toric pair the divisor D is canonically stratified, where the stratification is induced by the canonical stratification of the corresponding canonical toric divisors D_x^{tor}. All open strata are smooth.

With a zero-dimensional stratum $\{x\} \subset D$ we associate a lattice Γ_x and a torsor $\mathcal{T}_x$ over the torus $Hom(\Gamma_x, \mathbf{C}^*)$. Namely, Γ_x is the lattice of characters of the torus of the corresponding local formal toric model. The torsor $\mathcal{T}_x$ is defined such as follows. Choose an isomorphism $\phi_x : (\hat{\mathcal{X}}_x, \hat{D}_x) \simeq (\hat{\mathcal{X}}_x^{tor}, \hat{D}_x^{tor})$. It induces an isomorphism of completed local algebras

$$\mathcal{O}_{\hat{\mathcal{X}}_x} \simeq \prod_{\gamma\in\Gamma_x\cap C_x} \mathbf{C}\cdot x^\gamma \simeq \mathcal{O}_{\hat{\mathcal{X}}_x^{tor}},$$

where $C_x \subset \Gamma_x \otimes \mathbf{R}$ is a strict rational polyhedral cone.

Notice that there exists a natural homomorphism of groups

$$G_x := Aut(\mathcal{O}_{\hat{\mathcal{X}}_x^{tor}}, J_{\hat{D}_x^{tor}}) \to Hom(\Gamma_x, \mathbf{C}^*),$$

where $J_{\hat{D}_x^{tor}} = \prod_{\gamma\in\Gamma_x\cap int(C_x)} \mathbf{C}\cdot x^\gamma$ is the ideal of $\hat{D}_x^{tor}$.

It is easy to describe the above homomorphism at the level of Lie algebras. Namely,

$$Lie(G_x) = \prod_{\gamma\in\Gamma_x\cap C_x} (\Gamma_x^\vee \otimes \mathbf{C})\cdot x^\gamma,$$

where we interpret elements of $\Gamma_x^\vee \otimes \mathbf{C}$ as toric vector fields. Then the homomorphism is defined by taking the x^0-component of the vector field. It follows that $Hom(\Gamma_x, \mathbf{C}^*) \simeq G_x/\overline{[G_x, G_x]}$ (quotient of the topological group G_x by the closure of its commutant). The torsor $\mathcal{T}_x$ is the $Hom(\Gamma_x, \mathbf{C}^*)$-torsor associated with the natural G_x-torsor consisting of isomorphisms ϕ_x.

Similarly, for any one-dimensional closed stratum $\overline{F} \simeq \mathbf{CP}^1$ of D which contains exactly two 0-dimensional strata x_0, x_∞ one can define a lattice $\Gamma_{\overline{F}}$ and a torsor $\mathcal{T}_{\overline{F}}$ over $Hom(\Gamma_{\overline{F}}, \mathbf{C}^*)$. It is easy to see that there is a canonical isomorphism $(\Gamma_{\overline{F}}, \mathcal{T}_{\overline{F}}) \simeq (\Gamma_x, \mathcal{T}_x)$ where $x = x_0$ or $x = x_\infty$.

Definition 5.2.2. Suppose we are given a lattice Γ endowed with a skew-symmetric form $\langle \bullet, \bullet \rangle : \bigwedge^2 \Gamma \to \mathbf{Z}$.

A decorated formal scheme is defined by the following data:

i) A formal Poisson scheme $\hat{X}$ of pure dimension $n = rk\,\Gamma$, endowed with a normal Poisson divisor $D_{\hat{X}} \subset \hat{X}$ such that the pair $(\hat{X}, D_{\hat{X}})$ is a local formal toric pair.
ii) The reduced (non-formal) scheme associated with $\hat{X}$ is a wheel of parametrized projective lines $\overline{F} = \cup_{k \in \mathbf{Z}/m\mathbf{Z}} i_k(\mathbf{P}^1)$ where each i_k is an embedding of the projective line such that $p_k = i_k(\infty) = i_{k+1}(0)$ is the only intersection point of $\overline{F}_k = i_k(\mathbf{P}^1)$ and $\overline{F}_{k+1} = i_{k+1}(\mathbf{P}^1)$. Moreover all points p_k and lines $\overline{F}_k$ are strata of the canonical stratification of $D_{\hat{X}}$.
iii) A homomorphism $c_{\hat{X}} : \Gamma_0 \to \Gamma(\hat{X}, \mathcal{O}_{\hat{X}}(*D_{\hat{X}})^\times)$.
iv) For any $k \in \mathbf{Z}/m\mathbf{Z}$ isomorphisms $\Gamma_{p_k} \simeq \Gamma$ and $\mathcal{T}_{p_k} \simeq \mathcal{T}_{can}$.

We require that the data (i)–(iv) satisfy the following conditions:

a) for any $k \in \mathbf{Z}/m\mathbf{Z}$ there exists an isomorphism ϕ_k of the pair $(\hat{X}_{p_k}, \hat{D}_{\hat{X}, p_k})$ (completions at the point p_k) with the completions at the corresponding point of the corresponding local formal toric pair (see Definition 5.2.1), which identifies the Poisson structures and compatible with the homomorphism $c_{\hat{X}}$.
b) for any $k \in \mathbf{Z}/m\mathbf{Z}$ the composition of the isomorphisms

$$(\Gamma_{p_k}, \mathcal{T}_{p_k}) \simeq (\Gamma, \mathcal{T}_{can}) \simeq (\Gamma_{p_{k+1}}, \mathcal{T}_{p_{k+1}})$$

coincides with the composition of the isomorphisms

$$(\Gamma_{p_k}, \mathcal{T}_{p_k}) \simeq (\Gamma_{\overline{F}_k}, \mathcal{T}_{\overline{F}_k}) \simeq (\Gamma_{p_{k+1}}, \mathcal{T}_{p_{k+1}}).$$

Remark 5.2.3. The local toric model $\hat{Y}_{can}$ carries a natural structure of decorated formal Poisson scheme.

Next we would like to describe the deformation theory of decorated formal Poisson schemes. For that we need to study local symmetries of the local toric model preserving the structure of a decorated formal Poisson scheme. A toy-model of the result can be illustrated by the following Proposition.

Proposition 5.2.4. *Let $(\Gamma, \langle\bullet,\bullet\rangle)$ be a lattice endowed with an integer skew-symmetric form, $C \subset \Gamma_{\mathbf{R}}$ be a closed rational strict convex cone such that $int\, C \neq \emptyset$.*

Let $Y_C = Spec(\oplus_{\gamma\in C\cap\Gamma}\mathbf{Q}\cdot e_\gamma)$, where $e_\gamma e_\mu = (-1)^{\langle\gamma,\mu\rangle}\langle\gamma,\mu\rangle e_{\gamma+\mu}$ be the corresponding toric variety, and let $\hat{Y}_C$ be its completion at $0 \in C$, i.e. $\hat{Y}_C = Spf(\prod_{\gamma\in C\cap\Gamma}\mathbf{Q}\cdot e_\gamma)$. Consider the group of such automorphisms of $\hat{Y}_C$ that:

1) *they preserve the completion of the toric stratification;*
2) *they preserve the Poisson structure induced by $\langle\bullet,\bullet\rangle$;*
3) *they preserve all elements $e_\gamma, \gamma \in \Gamma_0 = Ker\,\langle\bullet,\bullet\rangle$ considered as rational functions on $\hat{Y}_C$;*
4) *they are equal to id on the torsor $\mathscr{T}_{y_0}$, where y_0 is the only zero-dimensional toric stratum.*

Then this group is a pronilpotent proalgebraic with the Lie algebra isomorphic to $\prod_{\gamma\in C\cap(\Gamma-\Gamma_0)}\mathbf{Q}\cdot e_\gamma$, which acts via $\{e_\gamma,\bullet\}$.

We are not going to prove the Proposition, since we are not going to use it. Let us explain informally its meaning. An automorphism of an affine Poisson variety or its completion can act non-trivially on the Poisson center. Inner Poisson derivations act trivially on the center. This explains the condition (3). The condition (4) allows us to exclude infinitesimal symmetries identical on the center but not inner which are given by $\{log\, e_\gamma,\bullet\}$.

In a similar way we conclude that the sheaf of Lie algebras of infinitesimal symmetries of the local model $\hat{Y}_{can}$ is naturally isomorphic to $\mathfrak{g}_{can} = \mathscr{O}_{\hat{Y}_{can}}/\mathscr{O}^{center}_{\hat{Y}_{can}}$ endowed with obvious Poisson bracket $\{\bullet,\bullet\}$. Here $\mathscr{O}^{center}_{\hat{Y}_{can}} = Ker\,\{\bullet,\bullet\}$.

Theorem 5.2.5. *We have:*

1) $H^0(\hat{Y}_{can},\mathfrak{g}_{can}) = 0.$
2) $H^1(\hat{Y}_{can},\mathfrak{g}_{can}) \simeq \prod_{\gamma\in\cup_i C^\vee_{i,i+1}\cap(\Gamma-\Gamma_0)}\mathbf{Q}\cdot e_\gamma.$
3) $H^{\geq 2}(\hat{Y}_{can},\mathfrak{g}_{can}) = 0.$

Proof. Analogous to the proof of Theorem 5.1.3. The only difference is that elements $e_\gamma, \gamma \in \Gamma_0$ are now excluded from considerations. ■

The sheaf of pronilpotent Lie algebras $\mathfrak{g}_{can}$ defines the deformation functor $Def_{\mathfrak{g}_{can}}$ from the category of commutative (not necessarily Arin) unital algebras over $\mathbf{Q}$ to the category of groupoids: $Def_{\mathfrak{g}_{can}}(R)$ is the groupoid of $exp(\mathfrak{g}_{can}\hat{\otimes}R)$-torsors over $\hat{Y}_{can}$. Notice that such torsors can be identified with decorated formal Poisson schemes $\hat{X}$. The above theorem implies the following result.

Proposition 5.2.6. *The deformation functor $Def_{\mathfrak{g}_{can}}$ is representable by an affine scheme $\mathscr{M} := \mathscr{M}_{\Gamma,\langle\bullet,\bullet\rangle,(C_i)_{i\in\mathbf{Z}/m\mathbf{Z}}}$ (with the trivial stacky structure) which is (non-canonically) isomorphic to the infinite-dimensional affine space over $\mathbb{A}^\infty = \varprojlim_N \mathbb{A}^N$ over $\mathbf{Q}$.*

There is an alternative description of $\mathcal{M}$. Namely, let G_i denotes the proalgebraic group of automorphisms of the formal neighborhood U_i and $G_{i,i+1}$ be a similar group for $U_{i,i+1}$ [we always assume that automorphisms are compatible with identifications (i)–(iv)]. We have a chain of embeddings:

$$\hookleftarrow G_1 \hookrightarrow G_{1,2} \hookleftarrow G_2 \hookrightarrow G_{2,3} \hookleftarrow G_3 \hookrightarrow \dots .$$

Then the product group $G = G_{1,2} \times G_{2,3} \times G_{3,4} \times \dots$ is endowed (as a scheme) with the free action of the group $H = G_1 \times G_2 \times G_3 \times \dots \subset G \times G$ (namely the factors of $G_i \times G_{i+1}$ act on $G_{i,i+1}$ by left and right multiplication respectively). By the above considerations with Lie algebras we conclude that the following holds.

Proposition 5.2.7. *The scheme $\mathcal{M}$ is the scheme of orbits of the above action of H on G. It is also isomorphic to the double coset $\mathcal{M} \simeq G_{diag}\backslash(G \times G)/H$.*

Remark 5.2.8. Suppose $(C_i')_{i\in\mathbf{Z}/m'\mathbf{Z}}$ be another admissible wheel of cones such that for any j there exists i such that $C_j' \subset C_i$. Then there is natural embedding $\mathcal{M}_{\Gamma,\langle\bullet,\bullet\rangle,(C_i)_{i\in\mathbf{Z}/m\mathbf{Z}}} \to \mathcal{M}_{\Gamma,\langle\bullet,\bullet\rangle,(C_i')_{i\in\mathbf{Z}/m'\mathbf{Z}}}$. Furthermore if under this embedding $\hat{X}$ is mapped into $\hat{X}'$ then $\hat{X}'$ is obtained from $\hat{X}$ in the following way:

a) first we make a finite sequence of blow-ups of $\hat{X}$ with centers at some strata (such blow-ups will be automatically stratified);
b) then in the resulting formal scheme we choose a wheel of lines each of which is a one-dimensional stratum, and take the completion along the wheel.

Notice that this procedure depends on purely combinatorial data.

5.3 *Relation to Stability Data*

Let $(\Gamma, \langle\bullet,\bullet\rangle)$ be a lattice endowed with an integer skew-symmetric form $\langle\bullet,\bullet\rangle$. Stability data on the torus Lie algebra $\mathfrak{g} = \mathfrak{g}_{\Gamma-\Gamma_0}$ are given by a central charge $Z : \Gamma \to \mathbf{C}$ and a collection of numerical DT-invariants $\Omega(\gamma) \in \mathbf{Q}, \gamma \in \Gamma - \Gamma_0$ (see [30] or Sect. 2.3, Example (1)). The Support Property ensures that $\overline{\cup_{\gamma\in Supp(\Omega)}\mathbf{R}\cdot\gamma} \cap (Ker\, Z_{\mathbf{R}} - \{0\}) = \emptyset$, where $Z_{\mathbf{R}} : \Gamma_{\mathbf{R}} \to \mathbf{C}$ is the $\mathbf{R}$-linear extension of Z. We will assume in this subsection that $rk\, Z_{\mathbf{R}} = 2$. In this case $Z_{\mathbf{R}}^*((\mathbf{R}^2)^*) \subset \Gamma_{\mathbf{R}}^*$ is an oriented two-dimensional vector space.

Definition 5.3.1. An admissible wheel of cones $(C_i)_{i\in\mathbf{Z}/m\mathbf{Z}}$ is compatible with the central charge $Z : \Gamma \to \mathbf{C}, rk\, Z_{\mathbf{R}} = 2$ if

a) for any $i \in \mathbf{Z}/m\mathbf{Z}$ the intersection $l_i = (C_{i,i+1} - \partial C_{i,i+1}) \cap Z_{\mathbf{R}}^*((\mathbf{R}^2)^*)$ is an open ray;
b) rays $l_i, i \in \mathbf{Z}/m\mathbf{Z}$ go in the clockwise order with respect to the orientation in $Z_{\mathbf{R}}^*((\mathbf{R}^2)^*)$.

Proposition 5.3.2. *In the above notation there exists an admissible wheel of cones $C_i, i \in \mathbf{Z}/m\mathbf{Z}$ compatible with Z and such that $Supp(\Omega) \subset \cup_i C_{i,i+1}^{\vee} - \{0\} \subset \Gamma_{\mathbf{R}} - Ker\, Z_{\mathbf{R}}$.*

Proof. Let us choose an isomorphism $\Gamma_{\mathbf{R}} \simeq \mathbf{R}^n = \mathbf{R}^2 \oplus \mathbf{R}^{n-2}$ in such a way that $Z_{\mathbf{R}}$ becomes a projection $(x_1, \ldots, x_n) \mapsto x_1 + ix_2$. Recall the example of the admissible wheel of cones from Sect. 5.1. In that example $\cup_i C_{i,i+1}^{\vee} - \{0\}$ contains an open neighborhood of $\mathbf{R}^2 - \{0\}$ in $\Gamma_{\mathbf{R}}$ and is disjoint from $\mathbf{R}^{n-2} = Ker\, Z_{\mathbf{R}}$. It follows from the Support Property that for sufficiently large $t > 0$ the set $\delta_t(\cup_i C_{i,i+1}^{\vee})$ contains $Supp(\Omega)$, where $\delta_t(x_1, x_2, \ldots, x_n) = (x_1, x_2, tx_3, \ldots, tx_n)$. Then the cones $(\delta_t^*)^{-1}(C_i), i \in \mathbf{Z}/m\mathbf{Z}$ obtained by application of the map which is inverse of conjugate to δ_t form an admissible wheel of (non-rational) cones. Then taking a small perturbation we obtain an admissible wheel of rational cones. This completes the proof. ■

We will need a stronger statement proof of which is analogous but lengthy and hence omitted.

Proposition 5.3.3. *For stability data on $\mathfrak{g}_{\Gamma-\Gamma_0}$ as above there exist an admissible wheel of cones $(C_i)_{i\in\mathbf{Z}/m\mathbf{Z}}$ as in the Proposition 5.3.2 as well as the following data:*

1) *a cyclic decomposition $\mathbf{R}^2 = V_{1,1} \cup \ldots \cup V_{1,k_1} \cup V_{2,1} \cup \ldots \cup V_{2,k_2} \cup \ldots \cup V_{m,1} \cup \ldots \cup V_{m,k_m}, m \geq 3, k_i \geq 1$, where $V_{i,j}$ are closed strict sectors such that two consecutive sectors have a common edge, $Z^{-1}(\partial V_{i,j} - \{0\}) \cap \Gamma = \emptyset$;*
2) *a cyclically ordered collection of closed strict convex cones $C(V_{i,j}) \subset \Gamma_{\mathbf{R}}$ compatible with Z and such that $Z(C(V_{i,j})) \subset V_{i,j}$, the set $Supp(\Omega)$ belongs to $\cup_{i,j} C(V_{i,j})$, and for any i, j the set $C(V_{i,j}) - \{0\}$ belongs to $int(C_{i,i+1}^{\vee})$.*

Let us make a choice of sectors and cones as in the Proposition. Then our stability data on $\mathfrak{g}_{\Gamma-\Gamma_0}$ give rise to the collection of elements

$$g_{i,j} \in exp(\prod_{\gamma \in C(V_{i,j}) \cap (\Gamma - \Gamma_0)} \mathbf{Q} \cdot e_\gamma) \subset G_{i,i+1},$$

where the latter group was defined in the end of the previous subsection. Let us associate with our stability data a point of $\mathscr{M}$ represented by the coset of the element $(g_{1,1}g_{1,2}\ldots g_{1,k_1}, g_{2,1}\ldots g_{2,k_2}, \ldots, g_{m,1}\ldots g_{m,k_m})$.

Theorem 5.3.4. *For given $\Gamma, \langle \bullet, \bullet \rangle, Z : \Gamma \to \mathbf{C}$ with $rk\, Z_{\mathbf{R}} = 2$ the above map provides a bijection from the set of stability data on $\mathfrak{g}_{\Gamma-\Gamma_0}$ with fixed central charge Z to the set $\varinjlim \mathscr{M}_{\Gamma,\langle\bullet,\bullet\rangle,(C_i)_{i\in\mathbf{Z}/m\mathbf{Z}}}$, where the inductive limit is taken with respect to subdivision maps described in the Remark 5.2.8 over all admissible wheel of cones (C_i) compatible with Z.*

Sketch of the proof. In what follows all chains of cones (C_i) will be compatible with the central charge Z. The proof will consist of several steps.

Step 1. The moduli space $\mathscr{M}_{\Gamma,\langle\bullet,\bullet\rangle,(C_i)_{i\in\mathbf{Z}/m\mathbf{Z}}} := \mathscr{M}_{(C_i)}$ can be defined for admissible wheels of *non-rational cones* via the double coset construction.

Step 2. For a special choice of cones C_i we can identify the space of stability data on the Lie algebra $\mathfrak{g}_{\Gamma-\Gamma_0}$ with the central charge Z and $Supp(\Omega) \subset \cup_i C_{i,i+1}^{\vee}$ with $\mathscr{M}_{(C_i)}$. It can be done along the lines of the example with polyhedron in Sect. 5.1. More precisely, let us fix a convex polygon in $\mathbf{R}^2$ which contains the origin in the interior and has cyclically ordered vertices $v_1, v_2, \ldots, v_m$. Let us fix a convex bounded closed polyhedron $P \subset \mathbf{R}^{n-2}$ which contains the origin in the interior. Let us define a decomposition $(\mathbf{R}^2)^* - \{0\} = \cup_{i\in\mathbf{Z}/m\mathbf{Z}} V_i$ into the union of semiclosed strict sectors defined by the condition $V_i = \{u \in (\mathbf{R}^2)^* | u(v_i) > u(v_{i-1}), u(v_i) \geq u(v_{i+1})\}$. We define strict convex cones $C(V_i) \subset (v_i \oplus P)^{\vee}$ as $\{u \oplus w | u \in V_i, w \in P\}$. Then

$$\cup_i (v_i \oplus P)^{\vee} - \{0\} = \sqcup_i C(V_i).$$

For such a choice we have $C_{i,i+1} = \mathbf{R}_{\geq 0}(v_i \oplus P)$ and C_i is the convex hull of $C_{i-1,i}, C_{i,i+1}$.

Below we will define a bijection between the space of stability data on the Lie algebra $\mathfrak{g}_{\Gamma-\Gamma_0}$ with the central charge Z and $Supp(\Omega) \subset \cup_i C_{i,i+1}^{\vee}$ with $\mathscr{M}_{(C_i)}$.

First, let us introduce a collection of pronilpotent groups $G_i^{(1)} \subset G_i$ such that $Lie(G_i^{(1)}) = \prod_{\gamma\in C(V_i)\cap(\Gamma-\Gamma_0)} \mathbf{Q} \cdot e_\gamma$. Then following [30] we parametrize our stability data by the collection of elements $A_{V_i} \subset G_i^{(1)}, i \in \mathbf{Z}/m\mathbf{Z}$. Namely, we set $A_{V_i} = \prod^{\rightarrow}_{l\subset V_i} A_l$, where the product is taken in the clockwise order over the set of rays in V_i with vertex at 0, each factor is given by the formula $A_l = \prod_{\gamma\in C(V_i)\cap(\Gamma-\Gamma_0), Z(\gamma)\in V_i} T_\gamma^{\Omega(\gamma)}$, and $T_\gamma : e_\mu \mapsto (1-e_\gamma)^{\langle\gamma,\mu\rangle}$.

Then in the double coset description of $\mathscr{M}_{(C_i)}$ we take the orbit of the element $(id_G, A_{V_1}, A_{V_2}, \ldots, A_{V_m}) \in G \times G$. It is easy to see that this gives the desired bijection (cf. Proposition 5.2.5). In fact we have

$$\prod_i G_i^{(1)} \simeq G_{diag} \backslash (G \times G)/H \simeq \mathscr{M}.$$

Step 3. Let us introduce a partial order on the set of wheels $\mathscr{C} = (C_i)$ compatible with Z. Namely we say that $\mathscr{C}' = (C'_j) \leq \mathscr{C} = (C_i)$ if for any i there exists j such that $C'_{j,j+1} \subset C_{i,i+1}$ (equivalently, for any j there exists i such that $C'_j \subset C_i$). The partial order $\leq$ gives rise to the category with objects $\mathscr{C} = (C_i)$ such that $int(C_i) \cap \mathbf{R}^2$ is non-empty, and morphisms defined by the partial order (poset category). Then we observe that if $\mathscr{C}' \leq \mathscr{C}$ then we have a natural embedding $\mathscr{M}_{(C_i)} \to \mathscr{M}_{(C'_i)}$ (notice that we do not need cones to be rational for all that).

Step 4. Assume that $\mathscr{C}' \leq \mathscr{C}$, and for any i there exists j such that $C'_{j,j+1} = C_{i,i+1}$ and also we have $\cup_j C'_j = \cup_i C_i$. Then the embedding from Step 3 is an isomorphism of affine schemes.

Step 5. For an admissible wheel of cones $\mathscr{C} = (C_i)$ there exist $\mathscr{C}' \leq \mathscr{C}$ and $\mathscr{C}'' \geq \mathscr{C}'$ such that the conditions from Step 4 hold and $\mathscr{C}''$ is a wheel of cones from Step 2.

Let us comment on Step 5. In order to find $\mathscr{C}''$ one chooses the vertices $v_i, 1 \leq i \leq m$ of the polygon in Step 2 in such a way that $v_i \notin C_{i,i+1} \cap \mathbf{R}^2$. Then one replaces the polyhedron P from Step 2 by εP, where ε is a sufficiently small positive number.

Step 6. By previous steps an element from $\mathscr{M}_{\mathscr{C}}$ gives an element from $\mathscr{M}_{\mathscr{C}''}$, hence the stability data on $\mathfrak{g}_{\Gamma-\Gamma_0}$ by Step 2. This concludes the sketch of the proof. ■

Assume we are given Γ, $\langle \bullet, \bullet \rangle$. Consider a continuous family of central charges $Z_x, x \in X$, where X is a Hausdorff topological space and such that $rk\, Z_x = 2$ for all $x \in X$. Consider a family $\sigma_x, x \in X$ (non-necessarily continuous) of stability data on $\mathfrak{g}_{\Gamma-\Gamma_0}$ with central charges Z_x. Then we have the following result proof of which is omitted.

Proposition 5.3.5. *The family $\sigma_x, x \in X$ is continuous if and only if there exists an open covering $X = \cup_\alpha U_\alpha$, collection of admissible wheels of cones $\mathscr{C}_\alpha = (C_{\alpha,i})$ and points $m_\alpha \in \mathscr{M}_{\mathscr{C}_\alpha}$ such that for any $x \in U_\alpha$ the stability condition corresponding to m_α and having central charge Z_x is identified by Theorem 5.3.4 with σ_x.*

Remark 5.3.6. In the case $rk\, Z_{\mathbf{R}} = 1$ one can develop a similar theory by replacing the formal neighborhood of a wheel of lines by the one for chains of lines (in some interesting cases just one projective line is enough).

5.4 Toric-Like Compactifications

Let $\mathscr{N}$ be a smooth algebraic variety over a field k of characteristic zero, and $\mathscr{N}_1$ be a normal scheme over k which contains $\mathscr{N}$ as an open subscheme.

Definition 5.4.1. We say that $\mathscr{N}_1$ is a toric-like compactification of $\mathscr{N}$ if the pair $(\mathscr{N}_1, D)$, where $D = \mathscr{N}_1 - \mathscr{N}$ is a reduced divisor, is a local formal toric pair.

In what follows we assume that $k = \mathbf{C}$, although this assumption can be relaxed. Notice that $\mathscr{N}_1$ does not have to be proper. It follows from the definition that D is stratified and its stratification is compatible with the local toric picture.

With each zero-dimensional stratum (point) $x \in D$ we can associate a free abelian group (lattice), which in the obvious notation can be written as $\Gamma_x = H^1((x \to \mathscr{N}_1)^*(\mathscr{N} \to \mathscr{N}_1)_* \mathbf{Z}_{\mathscr{N}})$ where $\mathbf{Z}_{\mathscr{N}}$ is the constant sheaf and the arrows are natural embeddings. In the case $k = \mathbf{C}$ and analytic topology $\Gamma_x = H^1(B_x \cap \mathscr{N}, \mathbf{Z}) \simeq \mathbf{Z}^{dim\, \mathscr{N}}$, where B_x is a small ball with the center in x.

Let $\overline{F}$ be a closed one-dimensional stratum whose complement to the union of zero-dimensional strata is isomorphic to $\mathbf{C}^*$. Then one has a similarly defined lattice

$\Gamma_{\overline{F}} \simeq \mathbf{Z}^{dim \mathcal{N}}$. In the complex analytic case one can define $\Gamma_{\overline{F}} = H^1(U_{\overline{F}} \cap \mathcal{N}, \mathbf{Z})$, where $U_{\overline{F}}$ is a tubular neighborhood of $\overline{F}$. If $x \in \overline{F}$ then by topological reasons we have a canonical isomorphism $\Gamma_x \simeq \Gamma_{\overline{F}}$.

Definition 5.4.2. A chain of lines $\overline{F}_k, \overline{F}_k \neq \overline{F}_{k+1}, 1 \leq k \leq m$ is given by a sequence of one-dimensional strata with parametrization $i_k : \mathbf{P}^1 \simeq \overline{F}_k$ such that $i_{k+1}(0) = i_k(\infty), 0 \leq k \leq m-1$, and $i_k(0), i_k(\infty)$ are the only zero-dimensional strata of $\overline{F}_k$.

We will use the notation $p_k = i_k(0), k = 1, \ldots, m, p_{m+1} = i_m(\infty)$.

The above considerations give us a chain of canonical isomorphisms of lattices $\Gamma_{p_1} \simeq \Gamma_{\overline{F}_1} \simeq \Gamma_{p_2} \simeq \Gamma_{\overline{F}_2} \simeq \ldots \simeq \Gamma_{\overline{F}_m} \simeq \Gamma_{p_{m+1}}$. We denote the identified lattices by Γ. We assume that Γ is endowed with a skew-symmetric pairing $\langle \bullet, \bullet \rangle : \bigwedge^2 \Gamma \to \mathbf{Z}$. We denote the kernel of this form by Γ_0.

Suppose that we are given an automorphism $T : \mathcal{N} \to \mathcal{N}$. Suppose furthermore that there exist open subsets $\mathcal{N} \subset U_1 \subset \mathcal{N}_1$ and $\mathcal{N} \subset U_{m+1} \subset \mathcal{N}_1$ such that $p_1 \in U_1, p_{m+1} \in U_{m+1}$ and such that T extends to an isomorphism $\overline{T} : U_1 \to U_{m+1}$. Then $\overline{T}$ induces an isomorphism $\overline{T}_{1,m+1} : \Gamma_{p_1} \simeq \Gamma_{p_{m+1}}$.

Next we would like to formulate a list of assumptions under which we will construct a point of the moduli space of decorated formal Poisson schemes. Namely we assume that:

(a) $\mathcal{N}$ is endowed with a Poisson structure.
(b) Automorphism T preserves the Poisson structure.
(c) The isomorphism $T_{1,m+1}$ coincides with the one obtained from the chain of isomorphisms of lattices.
(d) We are given a homomorphism of abelian groups $c : \Gamma_0 \to \mathcal{O}(\mathcal{N})^{\times}$ whose image belongs to the Poisson center.
(e) For any $1 \leq i \leq m+1$ there exists an isomorphism ϕ_i of the pair $(\hat{\mathcal{N}}_{1,p_i}, \hat{D}_{p_i})$ (completions at p_i) with the corresponding pair in the local formal toric model. This is an isomorphism of formal Poisson schemes, where the local formal toric model is endowed with the Poisson structure given by the skew-symmetric form $\langle \bullet, \bullet \rangle_i : \bigwedge^2 \Gamma_{p_i} \to \mathbf{Z}$ obtained from the skew-symmetric form $\langle \bullet, \bullet \rangle$ via the canonical isomorphism $\Gamma_{p_i} \simeq \Gamma$.
(f) For any $1 \leq i \leq m+1$ we are given an isomorphism $g_{p_i} : \mathcal{T}_{p_i} \simeq \mathcal{T}_{can}$ such that for any one-dimensional closed stratum $\overline{F} \simeq \mathbf{CP}^1$ containing exactly two 0-dimensional strata $\{x_0\}, \{x_\infty\}$ the following two compositions coincide:

$$(\Gamma_{x_0}, \mathcal{T}_{x_0}) \overset{(id, g_{x_0})}{\simeq} (\Gamma, \mathcal{T}_{can}) \overset{(id, g_{x_\infty}^{-1})}{\simeq} (\Gamma_{x_\infty}, \mathcal{T}_{x_\infty})$$

and

$$(\Gamma_{x_0}, \mathcal{T}_{x_0}) \simeq (\Gamma_{\overline{F}}, \mathcal{T}_{\overline{F}}) \simeq (\Gamma_{x_\infty}, \mathcal{T}_{x_\infty}).$$

(g) For any $1 \le i \le m+1$ there exists ϕ_i (see e)) such that the pull-back $\phi_i^*(c(\gamma))$ is a function of weight γ on the open toric stratum.
(h) Let $C_i \subset \Gamma_{p_i}^\vee \otimes \mathbf{R}$, $1 \le i \le m$ be closed strict rational convex cones arising from the toric-like stratification. Then (after identification $T_{1,m} : C_1 \simeq C_m$) we obtain an admissible wheel of cones.
(i) The composition

$$\mathscr{T}_{p_1} \overset{g_{p_1}}{\simeq} \mathscr{T}_{can} \overset{g_{p_{m+1}}^{-1}}{\simeq} \mathscr{T}_{p_{m+1}}$$

coincides with the Poisson isomorphism induced by $\overline{T}$.

Under the above assumptions (a)–(i) let us consider the disjoint union of completions $\sqcup_{1 \le i \le m} \hat{\mathscr{N}}_{1,\overline{F}_i}$ and then identify the formal neighborhood of $\infty_i \in \overline{F}_i$ with the one of $0_{i+1} \in \overline{F}_{i+1}$ for $1 \le i \le m-1$ using the embeddings to $\mathscr{N}$, and finally the formal neighborhood of $\infty_m \in \overline{F}_m$ with the one of $0_1 \in \overline{F}_1$ using the isomorphism $\overline{T}_{1,m+1}$. In this way we obtain an admissible wheel of cones endowed with additional data giving us a point in the moduli space $\mathscr{M}_{\Gamma,\langle\bullet,\bullet\rangle,(C_i)_{1 \le i \le m+1}}$.

6 WCS and Mirror Symmetry

Considerations in this section will be mostly heuristic. We are going to explain how the ideas of Mirror Symmetry in Strominger–Yau–Zaslow (SYZ for short) torus fibration picture can be combined with previous considerations of this paper. This will give us a WCS which conjecturally should coincide with the one constructed in Sect. 4.

6.1 Reminder on Fukaya Categories

Let (X,ω) be a compact smooth symplectic manifold of dimension $2n$ and $\mathbf{B} \in H^2(X, \mathbf{R}/2\pi\mathbf{Z}) \simeq Hom(H_2(X,\mathbf{Z}), \mathbf{R}/2\pi\mathbf{Z})$ be the B-field. It is expected that for a sufficiently large $\lambda > 0$ the triple $(X, \lambda\omega, \mathbf{B})$ gives rise to a $\mathbf{Z}/2\mathbf{Z}$-graded A_∞-category $\mathscr{F}(X, \lambda\omega, \mathbf{B})$ called the Fukaya category. Some objects of the Fukaya category are pairs (L, E) where $L \subset X$ is an oriented Lagrangian submanifold endowed with a spin structure such that $\mathbf{B}_{|L} = 0$, and E is a $U(1)$-local system on L. Space of morphisms $Hom((L_1, E_1), (L_2, E_2))$ is labeled by intersection points of L_1 and L_2. In order to define the A_∞-structure one needs to choose an almost complex structure on X. Then higher composition maps are given by a properly defined count of pseudo-holomorphic discs D "weighted" by $e^{-\int_D(-\lambda\omega + i\mathbf{B})}$. Not every Lagrangian submanifold L can support an object of the Fukaya category. A sufficient condition for that is the absence of pseudo-holomorphic discs of Maslov index 2 such that $\partial D \subset L$. More advanced picture which handles this problem was developed in [16]. Convergence of series which defines higher composition maps is another big issue, which is not proved by this time. Typically people avoid this problem by working over Novikov ring of series in the above-mentioned weight. In this case the approach can be made rigorous (see [16]). Furthermore, in the presence of a top degree almost complex form one can define a $\mathbf{Z}$-graded version of the Fukaya category. In what follows we will assume that for "sufficiently large ω" we can define the Fukaya category $\mathscr{F}(X, \omega, \mathbf{B})$ over the field of complex numbers. In some sense this category depends holomorphically on $[\omega] + i\mathbf{B} \in H^2(X, \mathbf{C}/2\pi i\mathbf{Z})$.

Under appropriate conditions one can define a version of $\mathscr{F}(X, \omega, \mathbf{B})$ called *wrapped Fukaya category* in case when X is non-compact and endowed with a proper map $H : X \to [0, +\infty)$ (see [1]). In that case Lagrangian submanifolds supporting objects are non-compact, having "good" behavior "at infinity". The space $Hom((L_1, E_1), (L_2, E_2))$ is defined by means of intersection points $L_1 \cap exp(t\varphi_H)(L_2)$, where $\varphi_H = \{H, \bullet\}$ is the Hamiltonian vector field corresponding to H.

6.2 SYZ Picture and Integrable Systems

Now suppose that we have a real integrable system $\pi : (X, \omega) \to B$ with a Lagrangian section $s : B \to X$. Then over an open dense subset B^0 of a smooth n-dimensional manifold B we have a Lagrangian torus fibration with marked zero points in the fibers. If X is non-compact then we assume that X is endowed with a proper function $H : X \to [0, +\infty)$ mentioned in the previous subsection, and that the function is a pull-back of a similar function on B. We will impose the condition $c_1(T_X) = 0 \in H^2(X, \mathbf{Z})$, although it is not necessary for some of our considerations.

The open submanifold B^0 carries a $\mathbf{Z}$-affine structure with local affine coordinates $x_i, 1 \leq i \leq n$ in a neighborhood of $b \in B^0$ which are determined (up to a shift) by the condition $dx_i = \int_{\gamma_i} \omega$, where $\{\gamma_i\}_{1 \leq i \leq n}$ is a basis in $\Gamma_b = H_1(\pi^{-1}(b), \mathbf{Z})$. The vector fields $\partial/\partial x_i$ generate a covariantly constant lattice $T_{\mathbf{Z}} \subset TB^0$. We will assume that B carries a metric g_B which is complete and which is Riemannian on B^0. Then it gives rise to an isomorphism $T^*B^0 \simeq TB^0$ of the tangent and cotangent bundles. In particular we have a lattice $T^\vee_{\mathbf{Z}} \subset T^*B^0$. Notice that $\pi^{-1}(B^0)$ is canonically symplectomorphic to the total space of the Lagrangian torus fibration $(T^*B^0/T^*_{\mathbf{Z}}, \omega_{T^*B^0/T^*_{\mathbf{Z}}})$. The latter is in turn symplectomorphic to the total space of the "rescaled" torus bundle $(T^*B^0/\varepsilon T^*_{\mathbf{Z}}, \varepsilon^{-1}\omega_{T^*B^0/T^*_{\mathbf{Z}}})$. Using the metric we endow $\pi^{-1}(B^0)$ with an almost complex structure J_ε compatible with the rescaled symplectic form $\varepsilon^{-1}\omega$ and the pull-back of the metric g_B. The structure J_ε is singular on $X - \pi^{-1}(B^0)$. We can replace it by a non-singular almost complex structure $\overline{J}_\varepsilon$ compatible with $\varepsilon^{-1}\omega$, and such that $\overline{J}_\varepsilon = J_\varepsilon$ outside of a small $\delta = \delta(\varepsilon)$- neighborhood of $X - \pi^{-1}(B^0)$ such that $lim_{\varepsilon \to 0}\delta(\varepsilon) = 0$.

Then as $\varepsilon \to 0$ the $\overline{J}_\varepsilon$-holomorphic curves converge to singular surfaces whose π-images are graphs in B with edges which are g_B-gradient lines of affine functions (see a discussion of this result of Fukaya and Oh in [35]). At a vertex v of the gradient graph the balancing condition is satisfied: $\sum_i \gamma_v^i = 0$, where γ_v^i denote adjacent to v edges which are identified with the corresponding integer affine functions.

In what follows we will assume that $dim_{\mathbf{R}}(B - B^0) \geq 2$ (cf. [35]). This condition is closely related to the condition $c_1(T_X) = 0$. Then for a generic point $b \in B^0$ there is no gradient tree as above with the root at b and external vertices at $B - B^0$. Such trees correspond to limits of $\overline{J}_\varepsilon$-holomorphic discs with boundaries on $\pi^{-1}(b)$. Indeed, the union of roots of such trees is a union of countably many hypersurfaces in B. We can call them "walls". The reader should not mix them with walls in WCS for complex integrable systems (see Sect. 10 for discussion of these walls). Informally we can think that B^0 locally looks as a locally finite union of convex polyhedral domains separated by walls and each polyhedral domain P gives a family of objects in the Fukaya category parametrized by the tube domain $Log^{-1}(P) \subset (\mathbf{C}^*)^n$, where $Log : (\mathbf{C}^*)^n \to \mathbf{R}^n$ is the tropical map $(z_1, \dots, z_n) \mapsto (log|z_1|, \dots, log|z_n|)$. Then $Arg(z_i)$ correspond to $U(1)$-local systems on fibers of π. According to the Mirror Symmetry philosophy there is a complex variety $X^\vee$ (mirror dual to X) containing all parametrized families of objects of the Fukaya category as open subsets. Crossing a wall which separates polyhedral domains corresponds to a change of coordinates on $X^\vee$. If g_B is locally given in affine coordinates by the Hessian matrix $(\partial^2 H/\partial x_i \partial x_j)$ for some convex function H, then edges of gradient graphs will be $\mathbf{Z}$-affine segments in the dual affine structure on B^0.

6.3 The Case of Complex Integrable Systems

First, let us assume that we are given a polarized (full) integrable system $\pi : (X, \omega^{2,0}) \to B$ endowed with a holomorphic Lagrangian section $s : B \to X$. Notice that $codim_{\mathbf{R}}(B - B^0) \geq 2$ automatically. Let us fix $\zeta \in \mathbf{C}^*$ and take $\omega_\zeta = Re(\zeta^{-1}\omega^{2,0})$ as the real symplectic form on X. As the B-field we take $\mathbf{B}_\zeta = Im(\zeta^{-1}\omega^{2,0}) + \mathbf{B}_{can}$, where $\mathbf{B}_{can} \in H^2(X, \pi\mathbf{Z}/2\pi\mathbf{Z}) \simeq H^2(X, \mathbf{Z}/2\mathbf{Z})$ is a "canonical" B-field defined in the Appendix.

Remark 6.3.1. Our choice of $\mathbf{B}_{can}$ is motivated by the appearance of the factor $(-1)^{\langle \gamma, \mu \rangle}$ in the theory of DT-invariants.

Recall (see Sect. 4.7) that under some natural assumptions there exists a proper continuous function $H_\zeta : B^0 \to [0, +\infty)$ which gives the metric on the base.

Tube domains for mirror duals $X_\zeta^\vee$ in coordinate-free language belong to $Hom(\Gamma_b, \mathbf{C}^*), b \in B^0, \Gamma_b = H_1(\pi^{-1}(b), \mathbf{Z})$. Hence they are endowed with a symplectic structure associated with the polarization. The above-discussed changes of coordinates preserve the symplectic structure. Hence each $X^{\vee}_{\zeta}, \zeta \in \mathbf{C}^*$ is a holomorphic symplectic manifold. One can explain the symplectic structure in a different way. Indeed the polarization on fibers of $X \to B$ gives rise to a canonical holomorphic line bundle $\mathscr{L}$ on X whose restriction on fibers $X \to B$ is ample (more precisely it is defined outside of the preimage of the discriminant locus, but we expect that it extends to the whole space). The cohomology class $c_1(\mathscr{L}) \in H^2(X, \mathbf{Z})$ can be interpreted as an element of the second Hochschild cohomology of the above wrapped Fukaya category $\mathscr{F}(X, \omega_{\zeta}, \mathbf{B}_{\zeta})$. Equivalently it is a second Hochschild cohomology class of the derived category of coherent sheaves on the mirror dual $X^{\vee}_{\zeta}$. One can argue that this cohomology class is represented by a non-degenerate holomorphic Poisson bivector field. Its inverse is our holomorphic symplectic form.

Now we are ready to consider the semipolarized case. To simplify the exposition we fix $\zeta = 1$ and omit the B-field $\mathbf{B}$ from the notation.

Namely, let $\pi : (X, \omega^{2,0}) \to B$ be a semipolarized integrable system with central charge Z and holomorphic Lagrangian section. We assume that the monodromy of the local system $\underline{\Gamma}_0 \to B^0$ is trivial so all its fibers can be identified with the fixed lattice Γ_0 (this can always be achieved by taking a finite cover, see Lemma 4.4.1). Then we obtain a holomorphic family of complex integrable systems $(X_{Z_0}, \omega^{2,0}_{X_{Z_0}}) \to B_{Z_0}$ parametrized by $Z_0 \in Hom(\Gamma_0, \mathbf{C})$. Our discussion of the Fukaya categories make plausible the proposal that the holomorphic family of the Fukaya categories gives rise to a holomorphic family of mirror duals $X^{\vee}_{Z_0} := (X_{Z_0}, Re(\omega^{2,0}_{X_{Z_0}}))^{\vee}$ parametrized by $Hom(\Gamma_0, \mathbf{C})$. We argue that $X^{\vee}_{Z_0}$ carry a holomorphic symplectic form.

The total space of this family will be denoted by $X^{\vee}$.

In fact it is a pull-back via the map $exp : Hom(\Gamma_0, \mathbf{C}) \to Hom(\Gamma_0, \mathbf{C}^*)$ of an *algebraic* family of smooth complex symplectic varieties $X^{\vee,alg} \to Hom(\Gamma_0, \mathbf{C}^*)$.

Here is an informal argument in favor of that. For each $Z_0 = Z_{|\Gamma_0}$ we have the corresponding polarized integrable system $(X_{Z_0}, \omega^{2,0}_{X_{Z_0}}) \to B_{Z_0}$ as discussed before. Consider the real affine space $\alpha_{X_0} \subset Hom(\Gamma_0, \mathbf{C})$ defined by the condition $Re(Z_0) = X_0$. This gives rise to a family of smooth symplectic manifolds $(X_{Z_0}, Re(\omega^{2,0}_{X_{Z_0}}))$, where $Z_0 \in \alpha_{X_0}$. The cohomology class $[Re(\omega^{2,0}_{X_{Z_0}})]$ is locally constant along α_{X_0}. Then by a Moser-type theorem (suitably adopted to non-compact case) we conclude that all symplectic manifolds $(X_{Z_0}, Re(\omega^{2,0}_{X_{Z_0}}))$ can be (non-canonically) identified up to symplectic isotopy. Notice that the B-field depends on $Im(Z_0)$. Therefore the corresponding Fukaya categories (and therefore their mirror duals) depend only on Z modulo $Hom(\Gamma_0, 2\pi i\mathbf{Z})$. The corresponding Fukaya categories (and hence their mirror duals) are periodic, with respect to the shifts, hence form a holomorphic family over the torus $Hom(\Gamma_0, \mathbf{C}^*)$. The algebraicity of this family is a conjecture, which we will discuss in a separate subsection below in Sect. 6.5. A priori the dual variety is just a complex manifold without an algebraic structure. The latter comes from additional considerations related to the *wrapped* Fukaya category.

From the above discussion we conclude that the total space of the family of mirror duals is a complex Poisson variety $X^{\vee,alg}$ (which we will call the mirror dual to our complex integrable system) endowed with a Poisson map to $Hom(\Gamma_0, \mathbf{C}^*)$ (hence fibers of this map are symplectic leaves). We will use the term "mirror dual" being applied to the holomorphic family $X^\vee \to Hom(\Gamma_0, \mathbf{C})$.

There is an alternative approach to the construction of $X^{\vee,alg}$ which we are going to explain below. Let $B_{\mathbf{R}} \subset B$ be the closure of the set $\{b \in B^0 | Re\,(Z_b)_{|\Gamma_0} = 0\}$. Let $X_{\mathbf{R}} = \pi^{-1}(B_{\mathbf{R}}) \subset X$. Then $X_{\mathbf{R}}$ is a coisotropic submanifold. Therefore it carries a foliation with symplectic quotient which we will denote by $X'_{\mathbf{R}}$. We have the corresponding real integrable system $((X'_{\mathbf{R}}, \omega_{X'_{\mathbf{R}}}) \to B_{\mathbf{R}}$. The fiber over $b \in B^0_{\mathbf{R}} := B_{\mathbf{R}} \cap B^0$ is isomorphic to the compact real torus $(\Gamma_b \otimes \mathbf{R})/\Gamma_b$. Notice that the cohomology class $[\omega_{X'_{\mathbf{R}}}] = 0$ (i.e. our real integrable system is exact). We can define the Fukaya category $\mathscr{F}(X'_{\mathbf{R}}, \omega_{X'_{\mathbf{R}}}, \mathbf{B}'_{can})$, where $\mathbf{B}'_{can} \in H^2(X'_{\mathbf{R}}, \pi\mathbf{Z}/2\pi\mathbf{Z})$ is the canonical B-field associated with the integer skew-symmetric form on Γ_b. Since $(X'_{\mathbf{R}}, \omega_{X'_{\mathbf{R}}})$ is an exact symplectic manifold and the pairing of $exp(\mathbf{B}'_{can})$ with the class of any pseudo-holomorphic curve belongs to $exp(\pi i \mathbf{Z}) = \{-1, +1\} \subset \mathbf{Q}$, we conclude that $\mathscr{F}(X'_{\mathbf{R}}, \omega_{X'_{\mathbf{R}}}, \mathbf{B}'_{can})$ is defined over $\mathbf{Q}$ (again, we ignore here the convergence problem). The torus $(\Gamma_0 \otimes \mathbf{R})/\Gamma_0$ acts on $X'_{\mathbf{R}}$ in the Hamiltonian way preserving Lagrangian torus fibers of the projection to $B_{\mathbf{R}}$. By Mirror Symmetry this corresponds to the holomorphic map $(X'_{\mathbf{R}})^\vee \to Hom(\Gamma_0, \mathbf{C}^*)$.

Conjecture 6.3.2. The complex Poisson manifold $(X'_{\mathbf{R}})^\vee$ is algebraic and endowed with the map to $Hom(\Gamma_0, \mathbf{C}^*)$. It is canonically isomorphic to the complex algebraic Poisson variety $X^{\vee,alg}$ with its map to $Hom(\Gamma_0, \mathbf{C}^*)$.

So far we have been discussing semipolarized integrable systems with fixed holomorphic symplectic form. Let us consider the $\mathbf{C}^*$-family of holomorphic symplectic forms $\omega^{2,0}_\zeta = \omega^{2,0}/\zeta$ on X. Then the corresponding mirror dual Poisson varieties $X^{\vee,alg}_\zeta, \zeta \in \mathbf{C}^*$ form a local system of quasi-affine algebraic varieties over $\mathbf{C}^*$ (we will discuss it in Sect. 6.5). More precisely, recall that mirror duals were constructed first by fixing $Z_0 = Z_{|\Gamma_0}$ and then by looking at the result as a family over either $Hom(\Gamma_0, \mathbf{C})$ (this gives us a holomorphic family) or $Hom(\Gamma_0, \mathbf{C}^*)$ (this gives us an algebraic family). After the rescaling the symplectic form to $\omega^{2,0}/\zeta$, the central charge Z gets replaced by Z/ζ, hence $Z_0 = Z_{|\Gamma_0}$ gets replaced by Z_0/ζ. Hence in the construction of mirror duals we fix Z_0/ζ. Taking the union of mirror duals (for fixed ζ) we obtain the mirror dual Poisson variety $X^{\vee,alg}_\zeta$. In fact they form a holomorphic local system of algebraic varieties over $\mathbf{C}^*$. This can be proved by using of the Moser-type arguments (in case when we deal with polarized integrable systems having central charge one can identify the fibers directly). In a similar way we obtain a holomorphic family over $\mathbf{C}^*$ of complex Poisson manifolds $X^\vee_\zeta$ each of which is endowed with a holomorphic Poisson morphism to $Hom(\Gamma_0, \mathbf{C})$. Clearly the total space $X^\vee$ of the latter family is the universal cover of the total space of the former family $X^{\vee,alg}$.

Taking the fiber $X_1^{\vee,alg} := X_{\zeta=1}^{\vee,alg}$ we obtain a Poisson variety endowed with a Poisson automorphism $T : X_1^{\vee,alg} \to X_1^{\vee,alg}$, which is equal to *id* on the algebra of central functions (the latter is isomorphic to $\mathcal{O}(Hom(\Gamma_0, \mathbf{C}^*))$) (cf. with Sect. 5.4).

Finally, let us remark that if the monodromy of the local system $\underline{\Gamma}_0 \to B^0$ is a finite group G then the above considerations still work and give us the mirror dual Poisson variety $X^{\vee,alg}$ together with a Poisson morphism to $Hom(\Gamma_0, \mathbf{C}^*)/G$ (here Γ_0 can be thought of as a fiber of the pull-back of $\underline{\Gamma}_0$ to the universal cover).

6.4 Wall-Crossing Structure from the Point of View of Mirror Symmetry

Recall g_B-gradient trees from Sect. 6.2. In the case of semipolarized integrable systems with central charge and holomorphic Lagrangian section we have Kähler metrics on the bases of the corresponding polarized integrable systems, hence edges of the gradient trees are straight segments in the dual $\mathbf{Z}$-affine structure (see Sect. 6.2). In terms of the central charge, the dual affine structure for the symplectic form $Re(\omega^{2,0}/\zeta), |\zeta| = 1$ is given by $Y_\theta := Im(e^{-i\theta} Z)$ with fixed restriction of Y_θ to Γ_0, where $\zeta = e^{i\theta} \in \mathbf{C}^*$. As we briefly recalled in Sect. 6.2, the SYZ approach to Mirror Symmetry (see more on that in [35]) gives rise to an inductive procedure of constructing walls and changes of coordinates, starting with certain data assigned to generic points of the discriminant $B - B^0$. Namely, for a point $b \in B^0$ sufficiently close to a generic point of the discriminant, one counts limiting pseudo-holomorphic discs whose π-image on the base is a short gradient segment connecting the point b with a point of $B - B^0$.

The inductive procedure is a priori different from the one discussed above in Sect. 4. Nevertheless in this case one can prove by induction (moving along the oriented gradient tree from the discriminant to a given point) that the walls and the changes of coordinates in Mirror Symmetry story of Sect. 6.2 coincide with those in Sect. 4. In particular, the changes of coordinates preserve the Poisson structure on $X_{Z_0}^\vee, Z_0 \in Hom(\Gamma_0, \mathbf{C})$ and depend algebraically on the point of $Hom(\Gamma_0, \mathbf{C}^*)$. They can be interpreted as Poisson transformations of $X^\vee$ identical on the Poisson center.

The above discussion gives an alternative approach to WCS constructed in Sect. 4. The initial data for which $\Omega(\gamma) = 1$ (see Sect. 4) for A_1-singularities correspond to the count of pseudo-holomorphic discs in the standard A_1-singularity model (see e.g. [7, 31]).

6.5 Algebraicity of the Mirror Dual

For simplicity we are going to discuss the case of polarized integrable systems. We hope that our arguments can be extended to the semipolarized case. In particular, the basis described below should be a basis in the algebra over $\mathscr{O}(Hom(\Gamma_0, \mathbf{C}^*))$. One can speculate that it coincides with the canonical bases expected in the theory of cluster varieties (see [17]).

As we discussed before, the mirror dual $X^\vee$ to an exact *real* integrable system $\pi : (X, \omega) \to B$ endowed with Lagrangian section $s : B \to X$ and the B-field which is a 2-torsion, is an algebraic variety defined over $\mathbf{Q}$. More precisely, we expect that $X^\vee$ is a quasi-affine (maybe formal) scheme of finite type over $\mathbf{Z}$. In case if there exists a proper continuous function $H : B \to [0, +\infty)$ which is (strictly) convex with respect to the $\mathbf{Z}$-affine structure on B^0 and has "good" behavior at the discriminant $B - B^0$, we expect that $X^\vee$ will be a Zariski open in the spectrum of a finitely generated algebra R, which can be described such as follows.

For any $t \in \mathbf{R}$ let $L_t = exp(t\varphi_H(s(B))) \subset X$ be a Lagrangian submanifold obtained from the zero section $s(B)$ by the Hamiltonian shift along the vector field $\varphi_H = \{H, \bullet\}$. Morally all $L_t, t \in \mathbf{R}$ should correspond to isomorphic objects in the wrapped Fukaya category. More precisely, for $t_1 < t_2$ let us consider the basis of the Floer complex $CF(L_{t_1}, L_{t_2})$ given by intersection points $L_{t_1} \cap L_{t_2}$. We will assume that this intersection belongs to $\pi^{-1}(B^0)$. Convexity of H implies that Maslov indices of all intersection points are zero. Hence the Floer differential is trivial. The composition

$$m_{t_1,t_2,t_3} : CF(L_{t_1}, L_{t_2}) \otimes CF(L_{t_2}, L_{t_3}) \to CF(L_{t_1}, L_{t_3})$$

sends the tensor product of two basis elements to a *finite* $\mathbf{Z}$-linear combination of basis elements (this follows from the "energy considerations" with the function H). Hence we obtain a directed $\mathbf{Z}$-linear non-unital A_∞-precategory (see [35]) with objects $L_t, t \in \mathbf{R}$ and $Hom(L_{t_1}, L_{t_2})$ well-defined for $t_1 < t_2$ only.

Assume now that H has a unique global minimum $b_{min} \in B^0$. It gives a common intersection point of all $L_t, t \in \mathbf{R}$, hence a canonical element $i_{t_1,t_2} \in Hom(L_{t_1}, L_{t_2})$ which satisfies the property $i_{t_1,t_2} i_{t_2,t_3} = i_{t_1,t_3}$. In this way we identify all objects L_t of our precategory. Then we define the algebra R as the algebra of the endomorphisms of any of them. The fact that R is finitely generated is not entirely obvious. One can hope that it follows from more careful considerations with filtrations on R coming from the function H.

By Mirror Symmetry the zero section $s(B)$ corresponds to the sheaf $\mathscr{O}_{X^\vee}$ (or maybe a line bundle on $X^\vee$) because $Hom((s(B), \mathbf{C}), (\pi^{-1}(b), \rho)) \simeq \mathbf{C}$ for any $b \in B^0$ and any $U(1)$-local system ρ on $\pi^{-1}(b)$.

We conclude that $R \simeq \mathscr{O}(X^\vee)$, and parameterizations of open subsets of $X^\vee$ by the tube domains give rise to embeddings of R into the algebras of Laurent series obtained by completions of $\mathscr{O}(Hom(\Gamma_b, \mathbf{C}^*))$ with respect to closed strict convex cones. In the case if our real integrable system comes from a complex polarized one with the central charge, the algebraic symplectomorphism $T : X_1^\vee := X^\vee \to X^\vee$ corresponds to an automorphism of R. Being the mirror dual, the variety $X^\vee \subset Spec(R)$ carries an algebraic volume element $\Omega_{X^\vee}$. We expect that there is a **Z**PL map of B^0 to the skeleton $Sk(X^\vee, \Omega_{X^\vee})$, where the skeleton is defined for the class of logarithmic Calabi–Yau manifolds in a way slightly more general than in [31].

Definition 6.5.1 (cf. [23]). By a logarithmic Calabi–Yau manifold (log CY manifold for short) we will understand a complex non-compact algebraic manifold $\mathscr{Y}^0$ endowed with a nowhere vanishing algebraic top degree form $\Omega_{\mathscr{Y}^0}$ which admits a compactification $\mathscr{Y}$ by a simple normal-crossing divisor D such that $\Omega_{\mathscr{Y}^0}$ has poles of order at most 1 on $D = \cup_i D_i$ (i.e. $\Omega_{\mathscr{Y}^0}$ is a log-form) and there exists a point in $\mathscr{Y} - \mathscr{Y}^0$ and local coordinates $(z_1, \dots, z_n)$ such that $\Omega_{\mathscr{Y}^0} = \bigwedge_{1 \le i \le n} \frac{dz_i}{z_i}$ where $n = dim_{\mathbf{C}} \mathscr{Y}^0$.

Having a log CY manifold $\mathscr{Y}^0$ one can assign to it a **Z**PL topological space $Sk(\mathscr{Y}^0)$ of dimension n with linear structure called the *skeleton of* $\mathscr{Y}^0$. The construction basically copies the one from [35]. Namely, for any compactification $\mathscr{Y}$ as in the above definition we define $Sk(\mathscr{Y}^0, \mathscr{Y})$ as the complement to $\{0\}$ of the set of such $\sum_i \lambda_i D_i, \lambda_i \in \mathbf{R}_{\ge 0}$ that if $\lambda_i > 0$ then $\Omega_{\mathscr{Y}^0}$ has pole at D_i and $\cap_{i | \lambda_i > 0} D_i \neq \emptyset\}$. Different choices of $\mathscr{Y}$ give rise to **Z**PL-isomorphisms of $Sk(\mathscr{Y}^0, \mathscr{Y})$. Hence we can use the notation $Sk(\mathscr{Y}^0)$, and call it the skeleton of $\mathscr{Y}^0$. Integer points of $Sk(\mathscr{Y}^0)$ correspond to certain divisorial valuations on the algebra of rational functions on $\mathscr{Y}^0$.

The notion of log CY manifold is analogous to the notion of maximally degenerate proper Calabi–Yau manifold over a non-archimedean field (see [31,35]). More precisely, for a proper Calabi–Yau manifold over a non-archimedean field we defined in [31], Sect. 6.6 the notion of its skeleton. It is a compact **Z**PL topological space of dimension less or equal then the dimension of the Calabi–Yau manifold. The dimensions are equal if and only if the Calabi–Yau manifold is maximally degenerate.

Definition 6.5.2. A log CY $\mathscr{Y}^0$ of complex dimension n is called good if its skeleton $Sk(\mathscr{Y}^0)$ coincides with the closure of the set of points of $Sk(\mathscr{Y}^0)$ each of which has a neighborhood homeomorphic to a real n-dimensional ball.

In the language of compactifications by simple normal crossing divisors (s.n.c. divisors for short) this means that for a subset of divisors $D_{i_1}, \dots, D_{i_k}, k \le n-1$ of the compactifying divisor D such that $\Omega_{\mathscr{Y}^0}$ has poles of degree 1 at each of them and such that $D_{i_1} \cap D_{i_2} \cap \dots \cap D_{i_k} \neq \emptyset$, there is another subset $D_{i_{k+1}}, \dots, D_{i_n}$ of divisors such that $D_{i_j} \subset D$ and $\Omega_{\mathscr{Y}^0}$ has poles of degree 1 at each of them, which altogether satisfy the property $D_{i_1} \cap D_{i_2} \cap \dots \cap D_{i_n} = \{pt\}$.

Also, if in the above notation $k = n-1$ the intersection $D_{i_1} \cap D_{i_2} \cap \dots \cap D_{i_{n-1}}$ is isomorphic to $\mathbf{CP}^1$ and the iterated residue of $\Omega_{\mathscr{Y}^0}$ at the intersection coincides with the form dz/z in the chosen coordinate z on $\mathbf{CP}^1$. This implies that $Sk(\mathscr{Y}^0)$ is an oriented topological pseudomanifold of dimension n with possible singularities in codimension ≥ 2.

We remark that typically a log CY is good except of rather pathological examples.

Definition 6.5.3. For a given good log CY $\mathscr{Y}^0$ its s.n.c. compactification $\mathscr{Y}$ is very good if each intersection $D_{i_1} \cap D_{i_2} \cap \ldots \cap D_{i_{n-1}}$ as above contains exactly two 0-dimensional strata (points $z = 0, \infty \in \mathbf{CP}^1$).

Then we claim that any very good compactification $\mathscr{Y}$ defines a natural $\mathbf{Z}$-linear structure on $Sk(\mathscr{Y}^0)$ with singularities in codim ≥ 2. For this we use the isomorphisms of $\Gamma_x \simeq \Gamma_{\overline{F}}$, where $x \in \{0, \infty\} \subset \overline{F} \simeq \mathbf{CP}^1$ (see Sect. 5.2). These isomorphisms allow us to identify canonical $\mathbf{Z}$-linear structures on n-dimensional octants corresponding to the strata 0 and ∞. The definition of very good s.n.c. compactification and the construction of the corresponding $\mathbf{Z}$-linear structure on $Sk(\mathscr{Y}^0)$ can be generalized in a straightforward way to the case of toric-like compactifications.

Assume that we are given a proper toric-like compactification $\mathscr{Y}$ of a good log CY $\mathscr{Y}^0$ such that the form $\Omega_{\mathscr{Y}^0}$ has poles of degree 1 at all components of the divisor $\mathscr{Y} - \mathscr{Y}^0$. This compactification is very good in the above sense. One can describe the corresponding singular $\mathbf{Z}$-linear structure using the language of non-archimedean analytic geometry as in [35]. Namely, let us extend scalars and consider $\mathscr{Y}^0$ as an algebraic variety over the non-archimedean field $\mathbf{C}((t))$. Then we have a continuous map from the Berkovich spectrum of the $\mathbf{C}((t))$-analytic space $(\mathscr{Y}^0)^{an}$ to $Sk(\mathscr{Y}^0)$. It is a non-archimedean n-dimensional torus fibration in the sense of [31], Sect. 4 (see also [35]) outside of the codimension ≥ 2 subset of $Sk(\mathscr{Y}^0)$. Hence it defines a $\mathbf{Z}$-affine structure outside of this subset (see [31], Sects. 4.1, 6.6 for more details). Since our variety was in fact defined over $\mathbf{C}$ (i.e. the corresponding family is constant with respect to the parameter t), we see that the $\mathbf{Z}$-affine structure is in fact linear (i.e. it is a vector structure).

Let us return to our integrable system. Now for a point $b \in B^0$ which does not belong to walls we define a valuation ν_b on R by taking the pull-back of the valuation on Laurent series on the completed algebra of functions on the torus given by the formula $val(\sum_\gamma c_\gamma e_\gamma) = min\{Y_b(\gamma)|c_\gamma \neq 0\}$. We expect that the corresponding map $\nu : B^0 - \{walls\} \to Sk(X^\vee, \Omega_{X^\vee})$ extends to walls giving the desired $\mathbf{Z}PL$ map of B to the skeleton. We keep the same notation ν for the extended map. One can hope that it is a homeomorphism.

6.6 Relation to Chains of Lines

Suppose we have a polarized complex integrable systems with central charge and a holomorphic Lagrangian section. Adding the parameter $\zeta \in \mathbf{C}^*$ to the considerations of the previous subsection we obtain a local system $X_\zeta^\vee$ of symplectic algebraic quasi-affine varieties endowed with algebraic volume forms $\Omega_{X_\zeta^\vee}$ (notice that since $\Gamma_0 = \{0\}$ there is no difference between $X_\zeta^\vee$ and $X_\zeta^{\vee,alg}$). By functoriality of the skeleton we obtain a local system of skeleta $Sk(X_\zeta^\vee) := Sk(X_\zeta^\vee, \Omega_{X_\zeta^\vee})$. The symplectomorphism $T : X_1^\vee \to X_1^\vee$ gives rise to a $\mathbf{Z}PL$ map $T : Sk(X_1^\vee) \to Sk(X_1^\vee)$.

For any point $b \in B^0$ we have a map $\psi_b : \widetilde{\mathbf{R}^2 - \{0\}} = \widetilde{\mathbf{C}}^* \to Sk(X_1^\vee) - \{0\}$ such that the deck transformation $log\,\zeta \mapsto log\,\zeta + 2\pi i$ of the universal covering $\widetilde{\mathbf{C}}^*$ goes to the automorphism T. Namely, $\psi_b(log\,\zeta)$ is defined as the image of the point $b \in B^0$ under the above-defined map ν for the polarized complex integrable system $(X, \frac{\omega^{2,0}}{\zeta}) \to B$ (we identify $Sk(X_\zeta^\vee)$ with $Sk(X_1^\vee)$ using the holonomy of the local system of skeleta over $\mathbf{C}^*$).

This map is piecewise linear with respect to the natural affine structure on $\widetilde{\mathbf{R}^2 - \{0\}}$. Let us assume we have chosen a very good compactification of $X_1^\vee$. Then it defines a $\mathbf{Z}$-linear structure on $Sk(X_1^\vee)$ with conical singularities in codimension ≥ 2. It follows that for *generic* $b \in B^0$ the image of ψ_b is disjoint from the locus of singularities of the $\mathbf{Z}$-linear structure. We will assume that this is the case. Moreover we are going to assume that the point $\psi_b(0) = \psi_b(log\,1)$ does not belong to the locus of nonlinearity of the $\mathbf{Z}PL$ map $T : Sk(X_1^\vee) \to Sk(X_1^\vee)$ in the above $\mathbf{Z}$-linear structure (it suffices to assume that $\psi_b(0)$ does not belong to any rational hyperplane).

Consider the image under the map ψ_b of the set $\{log\,\zeta | 0 \leq |Im\,log\,\zeta| \leq 2\pi\}$. This is the fundamental domain for the natural $\mathbf{Z}$-action on the universal covering $\tilde{\mathbf{C}}^*$. We will make the following assumption.

Monodromy Assumption. *Let $f(t), t \in [0,1]$ be a path $t \mapsto log\,\zeta = 2\pi i t$. Then the holonomy of the $\mathbf{Z}$-linear structure from $f(0)$ to $f(1)$ coincides with the map on the tangent spaces induced by T.*

Notice that the tangent map $d\psi_b$ can be thought of as an $\mathbf{R}$-linear map $\mathbf{C} \to T^{\mathbf{Z}}_{\psi_b(log\,\zeta), Sk(X_1^\vee)} \otimes \mathbf{R} := \Gamma^\vee_{\psi_b(log\,\zeta)} \otimes \mathbf{R}$ in the obvious notation. Dualizing we obtain an $\mathbf{R}$-linear map $Z_{\psi_b(log\,\zeta)} : \Gamma_{\psi_b(log\,\zeta)} \to \mathbf{C}$. The Monodromy Assumption implies that after the identification $\Gamma_{\psi_b(0)} = \Gamma_{\psi_b(2\pi i)}$ induced by the holonomy of the $\mathbf{Z}$-linear structure, the transformation T identifies the "central charges" $Z_{\psi_b(0)}$ and $Z_{\psi_b(2\pi i)}$.

The image of the restriction of ψ_b to the fundamental domain is compact modulo the natural global action of the group $\mathbf{R}_{>0}$ on $Sk(X_1^\vee) - \{0\}$. Hence we can cover an open neighborhood of the image by a chain of rational convex polyhedral cones, which are disjoint from the locus of singularities of the $\mathbf{Z}$-linear structure. Furthermore, we can shrink the cones and sequence of cones $C_1, \ldots, C_{m+1}, C_{m+1} = T(C_1)$ such that after the identification of C_1 and C_{m+1} by T we obtain an admissible wheel of cones (cf. Sect. 5.4, condition (h)). Arguments here are similar to those in the proof of Proposition 5.3.2. This change of the collection of cones can be interpreted in terms of a sequence of blow-ups and blow-downs at toric strata of a toric-like compactification of $X_1^\vee$. Hence in a certain toric-like compactification of $X_1^\vee$ we obtain a chain of lines as in Sect. 5.4. We also expect that in our situation there is a natural choice of the additional data needed for obtaining a decorated formal Poisson scheme (see Definition 5.2.2).

The above considerations can be generalized to the semipolarized case.

Recall that construction of Sect. 5.4 gives rise to certain stability data of algebro-geometric origin on the graded Lie algebra $\mathfrak{g}_{\Gamma-\Gamma_0}$. The above considerations relate that construction to Mirror Symmetry.

6.7 Extension to $\zeta = 0$

Suppose we are given a polarized complex integrable system $\pi : (X, \omega^{2,0}) \to B$ endowed with a holomorphic Lagrangian section $s : B \to X$. Let $\underline{\Gamma} \to B^0$ be the corresponding local system of lattices over the complement to the discriminant. Suppose we are given a class $\beta \in H^1(B^0, \underline{\Gamma}^\vee \otimes (\mathbf{R}/2\pi\mathbf{Z}))$ which comes from the class $\beta_X \in H^2(X, \mathbf{R}/2\pi\mathbf{Z})$ which vanishes on $s(B)$ and on fibers of π. Let us consider the holomorphic family of the Fukaya categories $\mathscr{F}(X, Re(\omega^{2,0}/\zeta), \mathbf{B} = Im(\omega^{2,0}/\zeta) + \beta_X)$. Then mirror duals $X_\zeta^\vee, \zeta \in \mathbf{C}^*$ form a holomorphic family of symplectic algebraic varieties over $\mathbf{C}$. We denote the holomorphic symplectic form on $X_\zeta^\vee$ by $\omega_{X_\zeta^\vee}$.

Definition 6.7.1. Dual integrable system is a complex integrable system $Y \to B$ such that its restriction to B^0 is obtained by:

a) taking dual abelian varieties to fibers of π;
b) replacing (a) by the torsor corresponding to β.

Notice that the definition of the dual integrable system does not depend on a choice of polarization. The dual to a polarized integrable system is not polarized in the proper sense. It is "fractionally polarized", i.e. the corresponding cohomology class is rational, and its positive integer multiple gives a polarization. The above definition is not quite satisfactory since we ignore the discriminant locus $B - B^0$.

Notice that there is a holomorphic Lagrangian section $B^0 \to Y$ of the dual integrable system. Let us assume that it extends to the section $B \to Y$.

Conjecture 6.7.2. The above family $X_\zeta^\vee$ of mirror duals endowed with holomorphic symplectic forms $\zeta\omega_{X_\zeta^\vee}$ extends to $\zeta = 0$ holomorphically in such a way that the fiber at $\zeta = 0$ is holomorphically symplectomorphic to the total space of the dual integrable system.

In a similar way we can consider the case of semipolarized integrable system $\pi : X \to B$ with central charge, holomorphic Lagrangian section (in order to relate our considerations to DT-invariants we can take $\beta = \mathbf{B}_{can}$). In that case we start with mirror duals to the integrable systems $X_{Z_0/\zeta} \to B_{Z_0/\zeta}$, where $Z_0/\zeta \in Hom(\Gamma_0, \mathbf{C})$ is fixed, and then combine them into a local system of complex Poisson manifolds $X_\zeta^\vee$ endowed with holomorphic Poisson maps to $Hom(\Gamma_0, \mathbf{C})$. Recall that by Conjecture 6.3.2 the manifolds $X_\zeta^\vee$ are expected to be pull-backs via the exponential map $Hom(\Gamma_0, \mathbf{C}) \to Hom(\Gamma_0, \mathbf{C}^*)$ of Poisson algebraic varieties defined over $\mathbf{Z}$ and fibered over $Hom(\Gamma_0, \mathbf{C}^*)$. Then the above conjecture is formulated such as follows.

Conjecture 6.7.3. The local system of Poisson manifolds $X_\zeta^\vee$ over $\mathbf{C}^*$ extends holomorphically (after rescaling the Poisson structure by $1/\zeta$) to $\zeta = 0$. This extension is compatible with the Poisson morphism to $Hom(\Gamma_0, \mathbf{C})$. Furthermore, the fiber at $\zeta = 0$ is the total space of a (fractionally) semipolarized integrable system $X^{dual} \to B$ whose restriction to B^0 has semiabelian Lagrangian fibers with abelian quotients which are dual abelian varieties to the corresponding abelian quotients of fibers of π.

This conjecture gives in particular a mathematical interpretation of the picture of twistor family for the total space of the Hitchin system proposed in [19].

Two conjectures below are formulated for simplicity in the polarized case. There are versions of them in the case of semipolarized integrable systems with central charge and holomorphic Lagrangian section.

Conjecture 6.7.4. Let us fix a point $b \in B^0$ in the base of a complex integrable system $\pi : X^0 \to B^0$ with abelian fibers endowed with a complex Lagrangian section $B^0 \to X^0$. Let us fix a point $e^{i\theta} \in S^1$ such that the pair $(e^{i\theta}, b)$ does not belong to the wall in $M = S^1 \times B^0$. Then the constant family of complex symplectic manifolds $X^\vee_{te^{i\theta}}$ over an open ray $l_\theta = \mathbf{R}_{>0}e^{i\theta}$ can be extended to a C^∞ family over the closed ray $\mathbf{R}_{\geq 0}e^{i\theta}$ in such a way that the fiber at $t = 0$ is a real integrable system over Sk_θ. Here Sk_θ is the skeleton of $(X^0)^\vee_{\zeta=e^{i\theta}}$.

Conjecture 6.7.5. For any $e^{i\theta} \in S^1$ the corresponding Sk_θ is $\mathbf{Z}PL$-manifold isomorphic to B which is endowed with the affine structure derived from the symplectic form $Re(e^{-i\theta}\omega^{2,0})$ on X^0.

Let us discuss their relationship of the Conjectures 6.7.4, and 6.7.5 with the Conjecture 6.7.3 which predicts holomorphic extension at $\zeta = 0$ of the local system of holomorphic Poisson manifolds $X_\zeta^\vee$. Assuming such an extension let us consider a holomorphic section $\zeta \mapsto f(\zeta)$ of the extended family over a small disc $|\zeta| < \varepsilon$. Let us assume that $f(0) \in \pi^{-1}(B^0)$. Then the restriction of f to l_θ gives rise to a real analytic path $f_\theta(t)$ in $X^\vee_{e^{i\theta}}$. Recall that there is an analytic cover map $X_\zeta^\vee \to X_\zeta^{\vee,alg}$, where the latter is (again conjecturally) a quasi-affine algebraic variety over $\mathbf{Q}$.

Conjecture 6.7.6. For generic θ the analytic path f_θ define a valuation val_θ on the algebra $\mathcal{O}(X^{\vee,alg}_{e^{i\theta}})$. This valuation depends only on the projection of $f(0)$ to B^0 and defines a point in Sk_θ. After a continuous extension to B we obtain in this way a homeomorphism $B \simeq Sk_\theta$ for any θ.

The monodromy of the local system of skeleta Sk_θ around S^1 is given by the $\mathbf{Z}PL$-automorphism T discussed before in Sect. 6.6. In terms of chains of lines this means that there is a finite decomposition $\cup_{1\leq k\leq m+1} V_k = S^1$ in the union of strict semiclosed sectors such that two consecutive ones have a common ray, and such that the limiting points at $t = 0$ of the above analytic paths $f_\theta(t)$ are the same as long as $e^{i\theta} \in V_k$. Furthermore, on the intersection $V_1 \cap V_{m+1}$ we have an identification of the corresponding skeleta given by the automorphism $\overline{T}_{1,m+1}$.

7 Wall-Crossing Structures and DT-Invariants for Non-compact Calabi–Yau Threefolds

There is a class of non-compact Calabi–Yau threefolds which gives rise to complex integrable systems with central charge (and hence to the corresponding wall-crossing structures). Such Calabi–Yau manifolds admit compactifications by a normal crossing divisor where the holomorphic volume form has poles of degree at least one (such a variety is not a log CY since we allow poles of order strictly bigger than one). This class of "good" Calabi–Yau threefolds include those of the type $\{uv + P(x, y) = 0\}$ where $P(x, y)$ is a polynomial such that the equation $P(x, y) = 0$ defines a smooth affine curve. Presumably this class includes non-compact Calabi–Yau threefolds associated with Hitchin systems (possibly with irregular singularities) for all gauge groups generalizing [12].

7.1 Deformation Theory of Non-compact Calabi–Yau Threefolds

Let $\overline{X}$ be a complex projective variety of complex dimension 3, and $D = \cup_i D_i \subset \overline{X}$ be a normal crossing divisor such that algebraic variety $X := \overline{X} - D$ has a nowhere vanishing top degree form $\Omega_X^{3,0}$ which extends to $\overline{X}$ with poles of order $n_i \geq 1$ on D_i. We will also fix an ample line bundle $\mathscr{L}$ on $\overline{X}$ which defines a projective embedding of $\overline{X}$. In this subsection we are going to discuss the deformation theories related to the pair $(\overline{X}, D)$ or the triple $(\overline{X}, D, \mathscr{L})$ (later we will see that these deformation theories are equivalent, see Proposition 7.1.3). We impose the following assumptions which will later guarantee that the global moduli stack $\mathscr{M}_{(\overline{X},D,\mathscr{L})}$ of deformations will be a smooth orbifold :

A1 $H^{2,0}(\overline{X}) = 0$.

A2 There exists a component D_{i_0} such that $ord_{D_{i_0}} \Omega_X^{3,0} \geq 2$ and such that the restriction homomorphism $H^1(\overline{X}, \mathbf{Z}) \to H^1(D_{i_0}, \mathbf{Z})$ is an isomorphism.

Loosely speaking we are going to consider deformations of $\overline{X}$ which preserve the isomorphism class of the restriction of the ample line bundle $\mathscr{L}_{|D_{i_0}}$ (see Assumption $A2$) as well as some "decoration" of D, which consists of jets of order $n_i - 1$ at each D_i for which $n_i = ord_{D_i} \Omega_X^{3,0} > 1$ (we skip a more precise formulation of the latter condition).

We will see that the Assumption $A2$ implies that the deformed varieties carry $(3, 0)$-forms with same orders of poles at the deformed smooth components D_i. The condition $K_{\overline{X}} = -\sum_i n_i D_i$ is preserved under deformations.

Let us now describe precisely the corresponding moduli problems. We will work in analytic topology. Let $T_{\overline{X},D} := T_{\overline{X},D,\Omega_X^{3,0}}$ be the sheaf of holomorphic vector fields on $\overline{X}$ satisfying the property that the contraction of such a vector field with $\Omega_X^{3,0}$ has poles of order 1 on each D_i (i.e. the contraction is a logarithmic form). Then $T_{\overline{X},D}$ is a sheaf of Lie subalgebras of the sheaf of graded Lie algebras of polyvector fields $\Lambda_{\overline{X},D}$ (here k vector fields are placed in degree $1-k$ for $k=0,1,2,3$) which satisfy the property that their contractions with $\Omega_X^{3,0}$ are logarithmic forms. Recall that a sheaf of L_∞-algebras over a field of characteristic zero (e.g. Lie algebras or DGLAs) gives rise to the corresponding formal deformation theory (see e.g. [29,37]). We will denote the formal moduli space associated with an L_∞-algebra $\mathfrak{g}$ by $\mathcal{M}_\mathfrak{g}$.

Let us consider the following deformation theories:

(a) The one associated with the DGLA $\mathfrak{g}_0 = R\Gamma(\overline{X}, T_{\overline{X},D})$. These are deformations of the pair $(\overline{X}, D)$ preserving certain decoration on D.
(b) The one associated with the differential graded Lie algebra (DGLA for short) $\mathfrak{g}_1 = R\Gamma(\overline{X}, \Lambda_{\overline{X},D}, div_{\Omega_X^{3,0}})$, where $div := div_{\Omega_X^{3,0}}$ is the divergence operator associated with the holomorphic volume form $\Omega_X^{3,0}$. This deformation theory does not have a straightforward geometric interpretation.
(c) The one associated with the DGLA subalgebra

$$\mathfrak{g}_2 = R\Gamma(\overline{X}, T_{\overline{X},D} \xrightarrow{div_{\Omega_X^{3,0}}} \mathcal{O}_{\overline{X},D}).$$

These are deformations of the pair $(\overline{X}, D)$ preserving the same decorations as in a), but also this time preserving the section $(\Omega_X^{3,0})^{-1}$ of the anticanonical bundle $-K_{\overline{X}}$.

Here $\mathcal{O}_{\overline{X},D}$ is the sheaf of functions on $\overline{X}$ such that being multiplied by $\Omega_X^{3,0}$ they have pole of order at most one at D (i.e. they are degree zero polyvector fields from $\Lambda_{\overline{X},D}$).

Proposition 7.1.1. *The moduli space $\mathcal{M}_{\mathfrak{g}_1}$ is naturally isomorphic to a formal submanifold of the formal neighborhood of* $0 \in H^3(X, \mathbf{C})$. *The moduli space $\mathcal{M}_{\mathfrak{g}_2}$ is a formal submanifold of $\mathcal{M}_{\mathfrak{g}_1}$.*

Proof. DGLA $\mathfrak{g}_1$ maps quasi-isomorphically to a DGLA

$$\begin{aligned}\mathfrak{g}_1' &:= R\Gamma(\overline{X}, (i_{X\to\overline{X}})_*(Sym^\bullet(T_X[1])[-1], div_{\Omega_X^{3,0}}))\\ &\simeq R\Gamma(X, (Sym^\bullet(T_X[1])[-1], div_{\Omega_X^{3,0}})).\end{aligned}$$

This follows form the classical result of Grothendieck that the complex of differential log-forms on $\overline{X}$ endowed with de Rham differential embeds quasi-isomorphically to $(i_{X\to\overline{X}})_*(\Omega_X^\bullet, d_{dR})$. Similarly the homomorphism of DGLAs

$$R\Gamma(X, (Sym^\bullet(T_X[1])[-1], div_{\Omega_X^{3,0}})) \to R\Gamma(X_{an}, (Sym^\bullet(T_{X_{an}}[1])[-1], div_{\Omega_X^{3,0}}))$$

relating Zariski and analytic topologies, is a quasi-isomorphism. The embedding of the abelian DGLA $\underline{\mathbf{C}}_X[2]$ to the sheaf of DGLAs$(Sym^\bullet(T_{X_{an}}[1])[-1], div_{\Omega_X^{3,0}})$

which maps 1_X to $(\Omega_X^{3,0})^{-1}$ is also a quasi-isomorphism. This implies that the corresponding moduli space is a formal neighborhood of a point in the affine space $H^3(X, \mathbf{C})$.

There is an obvious embedding of the complex of sheaves corresponding to $\mathfrak{g}_2$ into the one corresponding to $\mathfrak{g}_1$. We need to check that it induces an embedding at the level of hypercohomology. Contracting both complexes with $\Omega_X^{3,0}$ we convert polyvector fields into logarithmic forms. Then $\mathfrak{g}_2$ is quasi-isomorphic to $\mathbb{H}^\bullet(\Omega^2_{\overline{X}}(log\, D) \to \Omega^3_{\overline{X}}(log\, D), d)$. By Hodge theory this is embedded into the hypercohomology $\mathbb{H}^\bullet(\Omega^\bullet_{\overline{X}}(log\, D), d)$. Since $\mathfrak{g}_1$ is quasi-isomorphic to an abelian DGLA, the same is true for $\mathfrak{g}_2$, and moreover $\mathscr{M}_{\mathfrak{g}_2}$ is a formal submanifold of $\mathscr{M}_{\mathfrak{g}_1}$ (see e.g. [27], Proposition 4.11(ii)). ■

Consider the natural L_∞-morphism $\mathfrak{g}_2 \to \mathfrak{g}_0$.

Proposition 7.1.2. *Under the Assumption A2 this morphism induces an isomorphism of the moduli space* $\mathscr{M}_{\mathfrak{g}_2} \to \mathscr{M}_{\mathfrak{g}_0}$.

Proof. Hodge theory implies that the morphism $\mathfrak{g}_2 \to \mathfrak{g}_0$ induces an epimorphism on cohomology. Then it is easy to show that $\mathfrak{g}_0$ is quasi-isomorphic to an abelian DGLA (see e.g. [27], Proposition 4.11(iii)). Thus we have a surjection $\mathscr{M}_{\mathfrak{g}_2} \to \mathscr{M}_{\mathfrak{g}_0}$ which is a smooth fibration. The tangent space to a fiber is isomorphic to $H^0(\overline{X}, \mathscr{O}_{\overline{X},D})$. By Assumption $A2$ it is trivial. This proves the Proposition. ■

Now we would like to discuss the formal deformation theory which takes into account the ample line bundle $\mathscr{L}$.

Let $At(\mathscr{L})$ denotes the sheaf of Lie algebras of infinitesimal symmetries of the pair $(\overline{X}, \mathscr{L})$ (Atiyah algebra of $\mathscr{L}$). It fits into an exact short sequence

$$0 \to \mathscr{O}_{\overline{X}} \to At(\mathscr{L}) \to T_{\overline{X}} \to 0.$$

Let us denote by $At_{i_0}(\mathscr{L})$ the subsheaf of $At(\mathscr{L})$ of infinitesimal symmetries vanishing at the divisor D_{i_0} from the Assumption $A2$. Then we have a short exact sequence

$$0 \to \mathscr{O}_{\overline{X}}(-D_{i_0}) \to At_{i_0}(\mathscr{L}) \to T_{\overline{X},D_{i_0}} \to 0,$$

where $T_{\overline{X},D_{i_0}} \subset T_{\overline{X}}$ is a sheaf of vector fields vanishing identically on D_{i_0}. One can check that $T_{\overline{X},D}$ is a subsheaf of $T_{\overline{X},D_{i_0}}$. Let us define a sheaf of DGLAs $\mathfrak{g}_3$ as the fiber product of the morphisms

$$At_{i_0}(\mathscr{L}) \to T_{\overline{X},D_{i_0}} \leftarrow T_{\overline{X},D}$$

over $T_{\overline{X},D_{i_0}}$. Then we have an exact sequence of sheaves

$$0 \to \mathscr{O}_{\overline{X}}(-D_{i_0}) \to \mathfrak{g}_3 \to T_{\overline{X},D} \to 0.$$

The sheaf of DGLAs $\mathfrak{g}_3$ controls the deformation theory of $\overline{X}$ (together with decorations on D) endowed with a line bundle such that the restriction of the line bundle to D_{i_0} is identified with $\mathscr{L}_{D_{i_0}}$.

Proposition 7.1.3. *The natural map $\mathscr{M}_{\mathfrak{g}_3} \to \mathscr{M}_{\mathfrak{g}_0}$ is an isomorphism.*

Proof. There are natural maps to the tangent sheaf $T_{\overline{X}}$ of both $At(\mathscr{L})$ and $T_{\overline{X},D}$. The fiber product of these two maps is a sheaf of Lie algebras which we will denote by $At_{\overline{X},D}(\mathscr{L})$. Let $\mathfrak{g}_4 = R\Gamma(\overline{X}, At_{\overline{X},D}(\mathscr{L}))$ be the corresponding DGLA. The moduli space $\mathscr{M}_{\mathfrak{g}_4}$ is smooth because $\mathscr{M}_{\mathfrak{g}_0}$ is smooth and there is a formal bundle $q : \mathscr{M}_{\mathfrak{g}_4} \to \mathscr{M}_{\mathfrak{g}_0}$ with fibers which are smooth with tangent spaces isomorphic to $H^1(\overline{X}, \mathscr{O}_{\overline{X}})$ (for that we need the condition $h^{2,0}(\overline{X}) = 0$ which is the Assumption $A1$). Restricting $At_{\overline{X},D}(\mathscr{L})$ to D_{i_0} we obtain the map $p : \mathscr{M}_{\mathfrak{g}_4} \to \widehat{Pic(D_{i_0})}_{\mathscr{L}}$ which is the formal neighborhood of $\mathscr{L}$ in the Picard group thought of as the moduli space of line bundles. This is an epimorphism at the level of tangent spaces since the map $H^1(\overline{X}, \mathscr{O}_{\overline{X}}) \to H^1(D_{i_0}, \mathscr{O}_{D_{i_0}})$ is an isomorphism by Assumption $A2$. The morphisms p and q give rise to an isomorphism $\mathscr{M}_{\mathfrak{g}_4} \simeq \mathscr{M}_{\mathfrak{g}_0} \times \widehat{Pic(D_{i_0})}_{\mathscr{L}}$. Notice that the fiber of the map p over $\mathscr{L}_{|D_{i_0}}$ is isomorphic to the moduli space $\mathscr{M}_{\mathfrak{g}_3}$. This proves the desired isomorphism $\mathscr{M}_{\mathfrak{g}_3} \simeq \mathscr{M}_{\mathfrak{g}_0}$. ∎

The above propositions imply that there are three canonically isomorphic formal moduli spaces: $\mathscr{M}_{\mathfrak{g}_3} \to \mathscr{M}_{\mathfrak{g}_0} \simeq \mathscr{M}_{\mathfrak{g}_2}$. The deformation theory associated with $\mathfrak{g}_3$ is convenient for the construction below of the global moduli space.

Assume $\mathscr{L}$ is an ample bundle. Let us choose $N \geq 1$ such $\mathscr{L}^{\otimes N}$ gives a projective embedding of $\overline{X}$. Then we can consider *non-formal* deformation theory corresponding to the above formal deformation theory associated with $\mathfrak{g}_3$. More precisely, consider the scheme $\mathscr{M}'$ which parametrizes the following data:

(1) Smooth projective subvarieties $\overline{X}' \subset \mathbf{P}^{m-1}, m = rk\, H^0(\overline{X}, \mathscr{L}^{\otimes N})$ such that $rk\, H^0(\overline{X}', \mathscr{O}(k)) = rk\, H^0(\overline{X}, \mathscr{L}^{\otimes kN})$ for all sufficiently large $k > 0$. We also assume that $\overline{X}'$ satisfy $A1, A2$.
(2) Normal crossing divisors $D' = \cup_i D_i' \subset \overline{X}'$ together with a bijection between the set of irreducible components of D' and D.
(3) A chosen holomorphic volume element $\Omega^{3,0}_{X'}$ on $X' := \overline{X}' - D'$ such that $n_i' := ord_{D_i'} \Omega^{3,0}_{X'} = n_i$.
(4) For components D_i with $n_i \geq 2$ chosen isomorphisms $D_i \simeq D_i'$ preserving stratifications by other divisors as well as an isomorphism $\cup_{n_i \geq 2} D_i \simeq \cup_{n_i' \geq 2} D_i'$ which is required to induce the above isomorphisms of individual divisors.
(5) Chosen isomorphisms $\mathscr{O}(1)_{|D_{i_0}} \simeq \mathscr{L}^{\otimes N}_{|D_{i_0}}$.
(6) For any i such that $n_i \geq 2$ and any $x \in D_i$ a chosen isomorphism of the formal neighborhood of x with the formal neighborhood of the corresponding (see (4)) $x' \in D_i'$, preserving stratifications by other divisors as well as holomorphic volume elements, and defined up to the action of the formal completion of the Lie subalgebra $T^{div}_{\overline{X},D} \subset T_{\overline{X},D}$ of vector fields with zero divergence. Moreover, we require that the above formal isomorphisms can be chosen in such a way that they depend locally algebraically in Zariski topology on $x \in \cup_{n_i \geq 2} D_i$.

Remark 7.1.4. The last condition can be formulated in terms of jet spaces.

The scheme $\mathcal{M}'$ is smooth by the above-discussed formal deformation theory. The group $GL(m)$ acts on $\mathcal{M}'$ with finite stabilizers because $\Gamma(X', T_{\overline{X}',D}) = 0$. Hence the quotient of $\mathcal{M}'$ by this action is a smooth Deligne–Mumford stack (orbifold). Let us denote it by $\mathcal{M}$. This will be the base of our complex integrable system.

Remark 7.1.5. Locally in analytic topology a neighborhood of a point $m \in \mathcal{M}$ corresponding to $(\overline{X}', D')$ is embedded as a maximal isotropic submanifold in the Poisson manifold $H^3(X')$ by taking the cohomology class $[\Omega^{3,0}_{X'}]$ (the period map). The Poisson structure on the third cohomology comes from the observation that it is dual to $Hom(\Gamma', \mathbf{C})$, where $\Gamma' = H_3(X', \mathbf{Z})/tors$ carries an integer skew-symmetric intersection form.

Also we remark that one can generalize the above considerations by allowing $\Omega^{3,0}_X$ to extend to some components D_i without zeros and poles.

7.2 WCS and Integrable Systems Associated with the Moduli Space

By Remark 7.1.5 we see that $\mathcal{M}$ carries a local system $\underline{\Gamma}$ with the fiber over $u \in \mathcal{M}$ given by $H_3(X', \mathbf{Z})/tors$, where X' is the corresponding non-compact Calabi–Yau threefold. The intersection form endows $\underline{\Gamma}$ with an integer skew-symmetric form $\langle \bullet, \bullet \rangle$ while the period map can be interpreted as a central charge $Z : \gamma \mapsto \int_\gamma \Omega^{3,0}_X$. It gives rise to a holomorphic family of homomorphisms $Z_u : \Gamma_u \to \mathbf{C}, u \in \mathcal{M}$, so we have a local embedding of $\mathcal{M}$ into $\Gamma_u^\vee \otimes \mathbf{C}$ such that the image of Z is a family of Lagrangian submanifolds in symplectic leaves of the Poisson structure on $\Gamma_u^\vee \otimes \mathbf{C}$ dual to the 2-form $\langle \bullet, \bullet \rangle$. As we discussed previously, this family of Lagrangian manifolds is parametrized by the kernel of $\langle \bullet, \bullet \rangle$ and each Lagrangian manifold is the base of a complex integrable system.

Proposition 7.2.1. *The mixed Hodge structure on* $H^3(X, \mathbf{C})$ *can have non-trivial components in* $H^{1,2}, H^{2,1}, H^{2,2}$ *only.*

Proof. Since X is smooth we have $W_3 H^3(X) = H^3(X)$. Hence it suffices to show that $F^3 H^3(X) = 0$. Recall that the latter space can be defined as $H^3(R\Gamma(\overline{X}, 0 \to 0 \to 0 \to \Omega^3_{\overline{X}}(log\, D))$ which is equal to $H^0(\overline{X}, \Omega^3_{\overline{X}}(log\, D)) \simeq H^0(\overline{X}, \mathcal{O}_{\overline{X},D}) = 0$ since $n_{i_0} > 1$ by Assumption $A2$. ■

After twisting by the Tate motive $\mathbf{Z}(1)$ we obtain a variation of mixed Hodge structure. It satisfies all the conditions from Sect. 4.1.2 except of (possibly) the condition (3) (iii). By general reasons explained there it gives us a complex integrable system with fibers being semiabelian varieties, a central charge and holomorphic Lagrangian section. If the positivity condition (3) (iii) is satisfied, then our complex integrable system is semipolarized.

Let us now discuss the positivity condition more precisely. The tangent space to the base of each of the integrable systems is isomorphic to the image of $H^1(\overline{X}, \Omega^2_{\overline{X}})$ in $H^1(\overline{X}, \Omega^2_{\overline{X}}(log\, D))$. Then we need positivity of the restriction of the pseudo-hermitian form on the latter space to the image of the former one. It is convenient to dualize the above embedding. The dual space can be identified with the image of the composition

$$H^3(\overline{X}, D) \to H^3(\overline{X}) \to \mathscr{H}^3(\overline{X}, \Omega^0_{\overline{X}} \to \Omega^1_{\overline{X}}),$$

where $H^3(\overline{X}, D) \simeq H^3(X)^*$ is the cohomology of pair with complex coefficients and $\mathscr{H}^3(\overline{X}, \Omega^0_{\overline{X}}) = H^2(\overline{X}, \Omega^1_{\overline{X}}) \simeq (H^1(\overline{X}, \Omega^2_{\overline{X}}))^*$ since $h^{3,0}(\overline{X}) = 0$. One can identify the image of the map $H^3(\overline{X}, D) \to H^3(\overline{X})$ with $Ker(H^3(\overline{X}) \to \oplus_i H^3(D_i))$ using the long exact sequence of the cohomology of pair. Using Hodge decompositions of $H^3(\overline{X})$ and $H^3(D_i)$ we conclude that the positivity condition is equivalent to the following

Positivity Assumption A3. *For any non-zero differential form* α *representing an element of* $Ker\left(H^1(\overline{X}, \Omega^2_{\overline{X}}) \to \oplus_i H^1(D_i, \Omega^2_{D_i})\right)$ *we have* $\int \alpha \wedge \overline{\alpha} > 0$.

Remark 7.2.2. Assumption $A3$ holds e.g. for the compactification of the total space of Hitchin system on $\mathbf{P}^1$ or in the case of Hitchin systems related to *ALE* spaces as in [12].

We recall that having a Kähler metric we can enlarge B^0 to the "full" base B defining the latter as the completion of B^0 with respect to the metric. We do not know how to define the integrable system with the base B, but this is not necessary for the construction of attractor trees and WCS.

Let us discuss the Assumption $A3$. The vector space $\bigwedge^2 H^3(\overline{X}, \mathbf{Q})$ contains an element $\overline{\delta}$ which is the Künneth component of the diagonal. Then the vector space $\bigwedge^2(W_3 H^3(X, \mathbf{Q})) = Im\left(\bigwedge^2 H^3(\overline{X}, \mathbf{Q}) \to \bigwedge^2 H^3(X, \mathbf{Q}))\right)$ contains the image of $\overline{\delta}$ which we will denote by δ. Then $A3$ is essentially equivalent to the claim that $\overline{\delta}$ is non-degenerate and together with the Lagrangian vector subspace $Im\left(H^{2,1}(\overline{X}) \to W_3 H^3(X, \mathbf{Q}) \otimes \mathbf{C}\right)$ it defines the hermitian metric with non-degenerate skew-symmetric part on the ambient vector space

$$Im\left(H^3(\overline{X}, \mathbf{C}) \to H^3(X, \mathbf{C})\right).$$

Since the latter is the tangent space to the base of our integrable system, we conclude that it carries the Kähler metric.

In order to apply the algorithm of construction of WCS from Sect. 4.6 we also need a family of cones. It is more natural to discuss that piece of data in the framework of the Support Property for the DT-invariants of the Fukaya category of X. But the very existence of the Fukaya category and an appropriate stability condition is a non-trivial question. We are going to discuss it below.

7.3 Fukaya Categories for Non-compact Calabi–Yau Threefolds and Stability Conditions

We start with general remarks. Our definition of WCS is motivated by the theory of Donaldson–Thomas invariants for the Fukaya category. Namely, for a "good" non-compact Calabi–Yau threefold X one should have a well-defined Fukaya category endowed with a stability condition, whose central charge is the period map of the holomorphic 3-form. As we have already discussed in this section, this should give us a family of polarized integrable systems whose bases sweep the moduli space of deformations of X (equivalently, a semipolarized integrable system). The base of each polarized integrable system is endowed with a Kähler metric. Furthermore, the theory of DT-invariants from [30, 34] says that tangent spaces of points of the base should carry strict convex cones (this is equivalent to the so-called Support Property from [30]). We propose some sufficient conditions for the above picture to be true. In particular we impose the Assumptions $A1$–$A3$ which give part of the desired structures. In this subsection we discuss the conditions under which the Fukaya category and a stability condition do exist. This gives additional to Assumptions $A1$–$A3$ sufficient conditions for X to belong to a "good" class. On the other hand we expect that *for any Calabi–Yau threefold X, compact or non-compact, one should be able to define collections of integers $\Omega_u(\gamma)$ parametrized by the open subspace in the moduli space of deformations of X, and which coincide with the DT-invariants of the Fukaya category of X endowed with an appropriate stability condition in the case when the latter can be defined.*

The content of this subsection is purely motivational.

(1) In order to have a well-defined Fukaya category $\mathscr{F}(X)$ we need to ensure that holomorphic discs cannot touch the divisor D.
(2) In order to have a stability condition on $\mathscr{F}(X)$ we need to ensure compactness of the space of special Lagrangian submanifolds (SLAGs) in a fixed homology class.

Having (1) and (2) we can speak about virtual Euler characteristic of the moduli spaces of SLAGs, hence to define DT-invariants. They should satisfy the wall-crossing formulas from [30]. For that we need to ensure the Support Property from [30].

Suppose we have ensured (2). We claim that the Support Property is satisfied by the Assumption $A2$. Indeed we can choose logarithmic forms $\alpha_i, 1 \le i \le n$ on X which are representatives of a basis in $H^3(X, \mathbf{R})$. Then for points $x \in X$ which are sufficiently close to D we have

$$||\alpha_i(x)|| \le C(\alpha_i)\sqrt{\frac{|\Omega^{3,0}(x)|^2}{|\omega^{1,1}(x)|^3}},$$

for any $1 \le i \le n$. Here $\omega^{1,1}$ is a chosen Kähler form on $\overline{X}$, and we take any norm $|| \bullet ||$ of the functional $\alpha_i(x)$. The inequality follows from the fact that the form $\Omega^{3,0}$ has poles of order at least one at any component D_i of D, and there exists a component D_{i_0} where it grows faster. Then considerations from Remark 1 of [30] can be applied in the non-compact case in the same way as in the compact one. This gives the Support Property and strict convex cones in WCS constructed in the previous subsection.

Now we turn to a discussion of (1). Let (X, ω) be a real symplectic manifold, possibly non-compact. We fix an almost complex structure J which is integrable outside of a compact subset and compatible with ω. In other words, X is a complex manifold "at infinity" and $\omega(v, J(v)), v \in T_x X$ defines an almost Kähler form. If we want the Fukaya category to be $\mathbf{Z}$-graded we ask for a differential form Ω_X such that Ω_X has type $(n, 0)$ in the complex structure defined by J and does not have zeros on X.

In order to ensure that no pseudo-holomorphic discs "go to infinity" one can impose one of the following sufficient conditions:

(a) There exists smooth proper function $f : X \to [0, +\infty)$ with the property that for sufficiently large $c > 0$ the hypersurface $f = c$ is smooth (i.e. it does not contain critical points of f), and the Levy form of this hypersurface is non-negative (it suffices to require that $\overline{\partial}\partial(f) \ge 0$ outside of a compact). The desired property of pseudo-holomorphic discs follows from the maximum principle.
(b) There is a compact manifold $\overline{X}$ containing X and such that outside of a compact it is an embedding of complex manifolds, and such that $D = \cup_i D_i := \overline{X} - X$ is a normal crossing divisor which satisfies the following positivity condition:

if a rational curve $C \subset \overline{X}$ contains a smooth component belonging to some D_i then its intersection index with D is non-negative.

If this condition is satisfied then a family of pseudo-holomorphic discs S_t in X cannot converge as $t \to \infty$ to a degenerate disc S such that $S \cap D \neq \emptyset$. Indeed the intersection index of S_t with D is zero. The same should be true for S since the intersection index is a homological invariant. But S must have a component intersecting some D_i with strictly positive index, and possibly other components which are rational curves belonging to D. Therefore the intersection index of S and D is strictly positive. This contradiction shows that S cannot exist.

Condition (b) looks weaker than the condition (a) since it deals with rational curves only. There is a class of examples where both (a) and (b) are satisfied. For that class the function f is an extension to X of the function

$$f(x) = \sum_i n_i log\left(\frac{1}{dist(x, D_i)}\right) + r(x),$$

where $D = \sum_i n_i D_i$ as a divisor, $x \mapsto dist(x, Z)$ is the distance function to a set Z, and $r(x)$ is a smooth function on $\overline{X}$. Then $\overline{\partial}\partial(f) = \alpha - \sum_i n_i \delta_{D_i}$, where α is a smooth $(1, 1)$-form on $\overline{X}$ which is non-negative outside of D, not equal to zero on D, and δ_{D_i} is the delta-distribution for the component D_i. Since the integral of the LHS is zero for any holomorphic curve in a neighborhood of D, we conclude that the intersection index of $D = \sum_i n_i D_i$ with such a curve is non-negative. Notice also that such a curve does not have to be rational.

Summarizing, if (a) or (b) are satisfied then one can hope that there exists well-defined Fukaya category $\mathscr{F}(X, \omega)$.

Now we turn to a discussion of the condition (2), which should lead to a well-defined count of SLAGs. These considerations are purely heuristic. The idea is to consider a flow on Lagrangian submanifolds of X defined by the differential one form $\mu(L) := dArg(\Omega_X)_{|L}$. The direct computation shows that the function $vol(L) = \int_L (\Omega_X)_{|L}$ decreases along the flow (it is the gradient flow with respect to the L^2-metric on differential 1-forms). Stable points of the flow are SLAGs. Thus in order to achieve compactness of the space of SLAGs it is sufficient to ensure that there exists a real-valued function H with the following property:

if a compact Lagrangian L belongs to the set $H \leq c$ where c is sufficiently big, then the trajectory of L along the above flow also belongs to the same set.

Reason for that: if H has a local maximum at $x \in L$ then a small shift of L along the flow makes the maximum smaller. We discuss below sufficient conditions for H to be a desired function. We expect that under some conditions on the volume form there exists H with the desired property.

Since the above condition is local, one can assume that L is a real Lagrangian manifold in a flat Kähler vector space with coordinates $(z_i, \overline{z}_i)$, $1 \leq i \leq n$ given by $\overline{z}_i = z_i + \sqrt{-1}P(z_1, \ldots, z_n)$, where P is a real formal power series which starts with terms of degree greater or equal than 3.

The top degree form is given by $\Omega = dz_1 \wedge \ldots \wedge dz_n (1 + \sum_{1 \leq i \leq n} (a_i + \sqrt{-1} b_i) z_i + O(z^2))$, where $a_i, b_i \in \mathbf{R}$.

We are looking for a function H which can be written as

$$H = c + \sum_i (H_i z_i + \overline{H}_i \overline{z}_i) + \sum_{i,j} (H_{ij} z_i z_j + \overline{H}_{ij} \overline{z}_i \overline{z}_j) + \sum_{ij} r_{ij} z_i \overline{z}_j + O(z^4),$$

where $H_{ij} = H_{ji}, r_{ij} = r_{ji}$.

Local expression for $Arg\, \Omega_{|L} = \frac{\Omega_{|L}}{\overline{\Omega}_{|L}}$ is $\sum_i b_i z_i - \frac{1}{2} \sum_{ij} \partial_{iij}(P) z_j$, where ∂_{ijk} denotes the partial derivative with respect to z_i, z_j, z_k.

Normal vector field to L (i.e. the vector field of the flow) is given by

$$n = (\sqrt{-1})^{-1} \sum_i b_i (\partial/\partial z_i - \partial/\partial \overline{z}_i)$$

$$-\frac{1}{2} \sum_{ij} \partial_{iij}(P)(\partial/\partial z_i - \partial/\partial \overline{z}_i).$$

One can check that the partial derivative $\partial H/\partial n = (dH, n)$ is non-negative if

$$\sum_i Im(H_i) + 2\sum_i Re(H_{ii}) + \sum_i Re(r_{ii}) \leq 0.$$

This inequality can be written in a more invariant way. Notice that the function $log(\frac{\Omega}{dz_1\wedge\ldots\wedge dz_n})$ is well-define up to a constant. Therefore its Poisson bracket (with respect to the Poisson structure given by the Kähler form) is well-defined. Let us denote by $(c_i)_{1\leq i\leq n}$ the (non-negative) spectrum of the quadratic form $H^{(2)}$ defined by the quadratic part of H (with respect to the Hermitian metric given by the Kähler metric). Let $\Delta = \sum_{ij} \partial/\partial z_i \partial/\partial \overline{z}_i$ be the Laplace operator defined by the Kähler metric. Then the sufficient condition looks such as follows:

$$\Delta(H) + Re\{log\left(\frac{\Omega}{dz_1\wedge\ldots\wedge dz_n}\right), H\} - \sum_i c_i \geq 0.$$

Moreover,if the inequality is strict for sufficiently large values of H, then all SLAGs belong to a compact subset of X. One can check that the strictness of the inequality is not always achieved. A counter example is given by $X = \mathbf{C}^*$ and $\Omega = dz/z$. Then we have infinitely many SLAGs (circles) which "go to infinity" on the corresponding cylinder. In this case the above inequality becomes an equality. One can hope that if poles of Ω at D have order greater or equal than 2, then the above inequality is strict. In that case one can hope to have a stability condition on the Fukaya category of X and well-defined count of SLAGs (hence the corresponding DT-invariants in the case when X is a $3CY$ manifold).

8 Hitchin Integrable Systems for $GL(r)$

In this section C will denote a connected smooth projective irreducible curve over $\mathbf{C}$. Although we are going to discuss Hitchin systems with the gauge group $GL(r)$, we hope that our constructions admit generalizations to arbitrary reductive groups.

8.1 Reminder on Non-singular Case

Let us recall some basics on Hitchin systems with non-singular Higgs fields assuming that the genus of the curve C is bigger than 1. Recall that $GL(r)$-Higgs bundle is a rank r vector bundle over C endowed with a morphism $\phi : E \to E \otimes K_C$ (equivalently, a morphism $T_C \to End(E)$) called the Higgs field. Here $K_C = T_C^*$ is the canonical sheaf of C. The moduli space $\mathscr{M}_{Higgs}(r, d)$ of stable Higgs bundles of rank r and degree d on C is the total space of a polarized complex integrable system. The base $B = \prod_{1\leq i\leq r} \Gamma(C, K_C^{\otimes i})$ carries a structure of vector space.

The projection map $\pi : \mathcal{M}_{Higgs}(r, d) \to B$ assigns to a pair (E, ϕ) a collection $(Tr\,\phi, Tr\,\phi^2, \dots, Tr\,\phi^r) \in B$. This map has the following geometric meaning. Higgs bundle is the same as a coherent sheaf E_1 on T^*C, which is pure and supported on a compact curve $S \subset T^*C$, called the spectral curve of (E, ϕ). The purity means that E_1 has no non-trivial subsheaves with zero-dimensional support. The direct image of E_1 under the canonical projection $T^*C \to C$ is isomorphic to E. Generically S is smooth and E_1 is the direct image of a line bundle on S. For given $x \in C$ points of the intersection $S \cap T^*_xC$ correspond to eigenvalues of the linear map ϕ_x understood as an endomorphism of E_x associated with a non-zero tangent vector to C. The point $\pi(E, \phi) \in B$ can be thought of as a collection of coefficients of the characteristic equation $p(x, y) := det(\phi_x - y \cdot id_E) = 0$ which defines the spectral curve S.

We denote by $B^0 \subset B$ the locus of smooth connected spectral curves. The fiber $\pi^{-1}(b), b \in B^0$ is a torsor over the Jacobian $Jac(S_b)$ of the corresponding spectral curve S_b. It consists of line bundles on S_b of a certain degree. Therefore B^0 can thought of as a space of smooth connected projective curves S in an open complex symplectic variety (T^*C, ω_{T^*C}) with the homology class $r[C] \in H_2(T^*C, \mathbf{Z})$. Our integrable system depends on degree d, but the associated integrable system over B^0 with holomorphic Lagrangian section (see Sect. 4.1.1) does not depend on d and has as fibers Jacobians $Jac(S_b)$.

The above construction can be generalized. Namely, instead of T^*C we can consider an arbitrary complex symplectic surface Y and smooth compact curves $S \subset Y$. More generally, one can replace Y by a higher-dimensional complex quasiprojective symplectic manifold (more generally, Kähler manifold) and consider smooth compact complex Lagrangian submanifolds in it. Then B^0 is the analytic space of such Lagrangian submanifolds. One can show that B^0 is smooth (see next subsection). The fiber of the integrable system over the Lagrangian submanifold $L \in B^0$ is the Albanese variety $Pic_0(L)^*$. Later we are going to generalize the above picture to the case of singular (possibly irregular) Higgs fields ϕ.

8.2 Smoothness of the Moduli Space of Deformations of Complex Lagrangian Submanifolds

The reader can skip this subsection, since its results will not be used in the paper. We also remark that more general smoothness results in the dg-setting were obtained in [3]. Nevertheless we present a different approach which has some benefits on its own.

In order to demonstrate smoothness we need the following result.

Proposition 8.2.1. *Let $L \subset X$ be a compact complex Lagrangian submanifold in a quasiprojective symplectic manifold X. Then the moduli space of deformations of L is smooth (i.e. the deformation theory is unobstructed).*

Proof. The proof will consist of several steps.

Step 1. We start with general remarks about deformations in the case of characteristic zero. If we study the formal deformation theory which is controlled by an L_∞-algebra with (possibly) non-trivial cohomology in strictly positive degrees, then the corresponding deformation functor from the category of Artin algebras to sets is represented by a pro-Artin scheme, say, Y.

Let us now fix $k > 0$ and consider the formal deformation theory with the deformation functor on Artin algebras given by $R \mapsto Hom(Spec(\mathbf{C}[t]/(t^k) \otimes R), Y)$. The corresponding moduli space is the formal neighborhood of the map to the basis point in the formal scheme $Hom(Spec(\mathbf{C}[t]/(t^k)), Y)$. More generally we can consider any map $f \in Hom(Spec(\mathbf{C}[t]/(t^k)), Y)$ and study its formal deformation theory, thus getting a formal scheme Y_f.

Let us now recall the following result due to Z.Ran (see [42]): *Y smooth if and only if for any k, f the tangent space to Y_f at the point f is a free $\mathbf{C}[t]/(t^k)$-module.*

Remark 8.2.2. This tangent space can be identified with space of such maps $Spec(\mathbf{C}[t]/(t^k) \otimes \mathbf{C}[s]/(s^2)) \to Y$ that their restriction to $Spec(\mathbf{C}[t]/(t^k)$ coincides with f. The structure of $\mathbf{C}[t]/(t^k)$-module on the tangent space then comes from the natural action of the monoid $\mathbf{C}[t]/(t^k)$ endowed with the operation of multiplication. More precisely an element $a(t)$ of the monoid acts on $\mathbf{C}[t]/(t^k) \otimes \mathbf{C}[s]/(s^2)$ as $t \mapsto t, s \mapsto a(t)s$. The proof of Ran's result in one direction is straightforward: if Y is smooth then the tangent space to the scheme $Hom(Spec(\mathbf{C}[t]/(t^k)), Y)$ is a free $\mathbf{C}[t]/(t^k)$-module for any $k \geq 1$.

Step 2. Another general remark is that for any finite-dimensional Artin algebra R one can speak about smooth projective varieties over $Spec(R)$. The degeneration of Hodge-to-de Rham spectral sequence holds for such varieties. Indeed the de Rham cohomology forms a free R-module. Hodge cohomology coincides with de Rham cohomology at the marked point of $Spec(R)$. Then at the generic point of $Spec(R)$ Hodge cohomology can only drop. But this is impossible, because the spectral sequence implies that the rank of the de Rham cohomology must also drop, but it is constant.

Step 3.

Lemma 8.2.3. *Deformation theory of L as a complex Lagrangian submanifold coincides with its deformation theory as a complex submanifold.*

Proof of Lemma. Suppose R is a finite-dimensional Artin algebra. Consider a family $L_s, s \in Spec(R)$ of complex submanifolds of X. Then the restriction of the holomorphic symplectic form $\omega_X^{2,0}$ to each L_s is a closed 2-form. Let us assume that it is non-trivial. Then we have a non-trivial family of de Rham cohomology classes, which is equal to zero at the marked point of $Spec(R)$. By Step 2 we arrive to the contradiction, since the family of such cohomology classes must be constant.

Step 4. Let us take $R = Spec(\mathbf{C}[t]/(t^k))$ and consider a family of complex *Lagrangian* submanifolds L_s over $Spec(R)$ which coincides with the given L at the marked point. Let us consider their first order infinitesimal deformations as *submanifolds*, forgetting the Lagrangian structures. We are allowed to do that by Step 3. But the first order deformations of a manifold are given by sections of the normal bundles. Since L is Lagrangian the normal bundle can be identified with the space of 1-forms on L. Hence the tangent space can be identified with the space of (global) 1-forms on $L \times Spec(\mathbf{C}[t]/(t^k))$. By Step 2 this space does not jump. This concludes the proof of the Proposition. ■

8.3 Hitchin Systems with Irregular Singularities

Typically Hitchin systems on C are studied for at most logarithmic singularities of the Higgs fields. The irregular case is less developed.

For any point $x_0 \in C$ we denote by K_{x_0} the field of Laurent series at x_0, i.e. $K_{x_0} \simeq \mathbf{C}((t))$, where t is a coordinate on the formal disc centered at x_0. The algebraic closure $\overline{K}_{x_0}$ is the field of Puiseux series: $\overline{K}_{x_0} \simeq \cup_{N \geq 1}\mathbf{C}((t^{1/N}))$. The Galois group $Gal(\overline{K}_{x_0}/K_{x_0})$ is a topological group isomorphic to $\hat{\mathbf{Z}}$. Its topological generator acts on $\overline{K}_{x_0}$ as $t^{1/N} \mapsto e^{\frac{2\pi i}{N}} t^{1/N}$. Denote by $\mathcal{O}_{\overline{K}_{x_0}} \subset \overline{K}_{x_0}$ the ring of integers, $\mathcal{O}_{\overline{K}_{x_0}} \simeq \cup_{N \geq 1}\mathbf{C}[[t^{1/N}]]$.

Definition 8.3.1. A singular term at the point $x_0 \in C$ is an orbit of the Galois action of $\hat{\mathbf{Z}}$ on the vector space $\overline{K}_{x_0}/\mathcal{O}_{\overline{K}_{x_0}}$.

In a local coordinate $t = x - x_0$ one can represent a singular term as a finite sum $c = \sum_{\lambda \in \mathbf{Q}_{<0}} c_\lambda (x - x_0)^\lambda$ considered modulo the action on coefficients of the finite cyclic group $\mathbf{Z}/N\mathbf{Z}$ given by $c_\lambda \mapsto c_\lambda e^{2\pi i \lambda}$, where the number N called *the ramification index of the singular term* is defined as the minimal $N \geq 1$ such that all exponents λ with $c_\lambda \neq 0$ belong to $\frac{1}{N}\mathbf{Z}$.

Definition 8.3.2. An irregular data on a smooth projective curve C is given by a tuple $\{x_i\}_{i \in I}, \{r_{i,\alpha}\}, \{N_{i,\alpha}\}, (c_i^\alpha)_{\alpha \in Q_i}, r)$ where:

a) $x_i \in C, i \in I$ is a finite collection of distinct points ;
b) $r \geq 1$ is an integer number called rank;
c) a finite collection of distinct singular terms $c_i^\alpha, \alpha \in Q_i$ at every point x_i; to each term we assign the multiplicity $r_{i,\alpha} \geq 1$, and we require that the singular term has the ramification index $N_{i,\alpha}$.

We require that for each x_i the following identity holds $\sum_\alpha r_{i,\alpha} N_{i,\alpha} = r$.

Remark 8.3.3. In the case of $GL(r)$-Hitchin system without singularities we can add dummy marked points x_i and set all $c_i^\alpha = 0$ and $r_{i,\alpha} = r$ for all marked points x_i.

Let us compactify T^*C to $\overline{T^*C} = T^*C \cup C_\infty$ where $C_\infty \simeq C$ is the divisor at infinity. The canonical holomorphic symplectic form ω_{T^*C} has pole of order 2 at C_∞. Let us consider various smooth projective surfaces W together with regular maps $f : W \to \overline{T^*C}$ which are birational. We also demand that the pull-back $\omega_W = f^*(\omega_{T^*C})$ has only poles but does not have zeros, and such that the complement to the set of poles of ω_W is isomorphic (via the morphism f) to T^*C. Equivalently, such W is obtained by a sequence of blow-ups $\dots \to W_2 \to W_1 \to W_0 = \overline{T^*C}$, where $W_i = Bl_{p_i}(W_{i-1})$, and each point $p_i \in W_{i-1}$ is either a smooth point of a divisor in W_{i-1} where the form $\omega_{W_{i-1}}$ has pole of order at least 2, or the point p_i is the intersection of two divisors where the form has poles of order at least 1 (notice that in our case the situation when the orders of both poles are equal to 1 is impossible). Such surfaces naturally form a projective system. The set $Irr_1(f^{-1}(C_\infty))$ of irreducible components at which the symplectic forms ω_W have poles of order 1 forms an inductive system of sets. It is easy to see that the inductive limit of this system can be identified with the set of singular terms. More precisely, to a *non-zero* singular term c represented by a formal germ $\hat{c}$ at $x_0 \in C$ we associate a unique divisor $D \in Irr_1(f^{-1}(C_\infty))$ on an appropriate blow-up, such that the graph of $d\hat{c}$ intersects the divisor D at smooth point as long as $x \to x_0$. If $c = 0$ the corresponding divisor is the exceptional divisor of the blow-up of $\overline{T^*C}$ at the point $(x_0)_\infty \in C_\infty \simeq C$ corresponding to x_0.

Given an irregular data $(\{x_i\}, \{r_{i,\alpha}\}, \{N_{i,\alpha}\}, \{c_i^\alpha\}, r)$ we consider a minimal surface W as above such that the divisors corresponding to all singular terms are contained in W. It is easy to see by induction in the number of blow-ups that all elements of $Irr_1(f^{-1}(C_\infty))$ are disjoint smooth rational curves each of which contains only one double point of the divisor $f^{-1}(C_\infty)$. The complement to this double point is an affine line. Thus to a singular term $\sigma = (x_0, c)$ we have associated an affine line, which we will denote by $\mathbb{A}^1(\sigma)$.

Lemma 8.3.4. *This affine line carries a naturally defined coordinate (i.e. it is canonically identified with* $\mathbb{A}^1$*).*

Proof. Consider the 1-form, which is the pull-back of the Liouville 1-form ydx from T^*C. Take any rational curve transversal to $\mathbb{A}^1(\sigma)$. Then the residue of the 1-form at the intersection point does not change if we vary the transversal curve. Indeed, by Stokes theorem the comparison of residues reduces to the computation of the integral of the symplectic form over a two-dimensional chain. The latter can be made arbitrarily small. Hence the residue is well-defined and gives the desired coordinate. ■

Definition 8.3.5. An additive refined irregular data on C consist of:

a) an irregular data $(\{x_i\}, \{r_{i,\alpha}\}, \{N_{i,\alpha}\}, \{c_i^\alpha\}, r)$;
b) for any pair of indices i, α a finite subset $\Sigma_{i,\alpha} \subset \mathbb{A}^1(x_i, c_i^\alpha) \simeq \mathbb{A}^1$ such that $|\Sigma_{i,\alpha}| \geq 1$;
c) a map $\Lambda_{i,\alpha} : \Sigma_{i,\alpha} \to \{Partitions\} - \{0\}$ such that $\sum_{z \in \Sigma_{i,\alpha}} |\Lambda_{i,\alpha}(z)| = r_{i,\alpha}$, where for a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ we use the notation $|\lambda| = \lambda_1 + 2\lambda_2 + 3\lambda_3 + \dots$.

We will often skip the adjective "additive" and will speak simply about refined irregular data. In Sect. 8.4 we are going introduce multiplicative refined irregular data.

The pair $(\Sigma_{i,\alpha}, \Lambda_{i,\alpha})$ can be thought of as a conjugacy class in $gl(r_{i,\alpha}, \mathbf{C})$. More precisely, $\Sigma_{i,\alpha}$ corresponds to the set of eigenvalues, and the partition $\Lambda_{i,\alpha}(z), z \in \Sigma_{i,\alpha}$ describes multiplicities of the corresponding Jordan blocks.

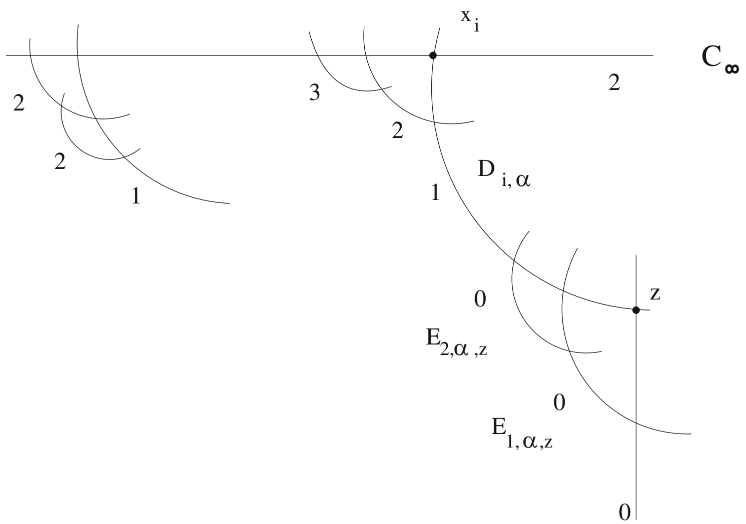
In the case of Hitchin system with regular singularities all $c_i^\alpha = 0$, all $r_{i,\alpha} = r$, and the above conjugacy classes in $gl(r, \mathbf{C})$ can be thought of as the conjugacy classes of the residues of the Higgs field with has poles of order 1 at the marked points.

With a refined irregular data we are going to associate a pair consisting of a complex symplectic surface Y which contains T^*C as a Zariski dense open subset and homology class $\beta \in H_2(Y, \mathbf{Z})$ (the fundamental class of the spectral curve).

Let us describe the construction. Let W be the above-described smooth projective surface constructed for a given irregular data. Now we would like to take the refinement into account. For each pair of indices i, α and any $z \in \Sigma_{i,\alpha} \subset \mathbb{A}^1 := \mathbb{A}^1(x_i, c_i^\alpha) \subset W$ we will make a sequence of blow-ups of W starting with the blow-up at z. More precisely, let $D_{i,\alpha}$ be the closure of $\mathbb{A}^1$. We consider a sequence of blow-ups such that the center of each blow-up is a point on the strict transform of $D_{i,\alpha}$ corresponding to the point z. The number of elements in the sequence of blow-ups is $max\{k|(\Lambda_{i,\alpha}(z))_k \neq 0\}$, where $(\lambda)_k$ denotes the k-th component of the partition λ.

We denote by $\overline{Y}$ the resulting projective surface. The complement to the divisor of poles of the holomorphic symplectic form is denoted by Y. If the irregular data is non-empty then Y is strictly bigger than T^*C, since it contains a chain $E_{1,i,\alpha,z}, E_{2,i,\alpha,z}, \ldots$ of rational curves associated with triples $(i, \alpha, z), z \in \Sigma_{i,\alpha}$. Here $E_{1,i,\alpha,z}$ are exceptional divisors of the blow-ups at which the symplectic form is regular. The numeration is chosen in such a way that $E_{1,i,\alpha,z}$ appears at the first blow-up in case if at least one c_i^α is non-zero. If all $c_i^\alpha = 0$ and z is the intersection of the closure of vertical fiber $T^*_{x_i}C$ with an appropriate component of the divisor $W - T^*C$ we add to the chain the closure $\overline{T^*_{x_i}C}$ and call it $E_{1,i,\alpha,z}$.

We illustrate the above discussion by the figure below. Notice that the numbers $0, 1, 2, 3$ on the figure refer to the order of poles of the symplectic 2-form. The figure contains divisors (and their intersection points) obtained at all steps of our sequence of blow-ups. In order to see the final diagram of them, one should keep all the divisors on the figure as well as only those intersection points which can be reached from the "southwest corner" without crossing lines.



The homology class $\beta \in H_2(Y, \mathbf{Z})$ is uniquely determined by the following intersection indices:

(a) $\beta \cdot [E_{k,i,\alpha,z}] = (\Lambda_{i,\alpha}(z))_k \in \mathbf{Z}_{\geq 1}$;
(b) for generic $x \in C$ we require $\beta \cdot [\overline{T^*_{x_i} C}] = r$.

Now, having Y and β we construct the integrable system in the natural way explained in the end of Sect. 8.1. The base B consists of compact effective divisors in Y of class β. We call those divisors *irregular spectral curves* (cf. [4] in the non-refined case). We will assume that the subspace $B^0 \subset B$ of smooth connected curves is nonempty. This assumption seems to be satisfied almost always, e.g. in case when the genus $g(C) > 0$ or in case if $g(C) = 0$ and Hitchin system with regular singularities, provided the additive Deligne–Simpson problem (see e.g. [38]) has a solution.

The fiber over $b \in B^0$ is the Jacobian of the spectral curve S_b. The corresponding complex integrable system is polarized and has a holomorphic Lagrangian section (zero section). One can also consider a version of this construction, when one takes as fibers the torsors over the Jacobians parametrizing line bundles of a given degree $d \in \mathbf{Z}$. In this case the corresponding integrable system does not have in general a Lagrangian section.

In the case when all partitions are of the type (1) (i.e. $|\lambda_{i,\alpha}(z)| = 1$ for all i, α, which is an analog of Hitchin system with regular singularities and semisimple monodromy) we extend the above integrable system to a semipolarized one by varying points $z \in \Sigma_{i,\alpha}$. The projection of any smooth spectral curve to W is a smooth curve intersecting transversally the divisor of 1-st order poles of ω_W. The lattice Γ can be identified with the integer first homology of the punctured spectral curve (punctures are intersection points with the preimage of the curve C_∞). This is an example of the integrable system associated with a log-family of Lagrangian subvarieties (in this case curves) in the cotangent bundle (see [28]). In Sect. 8.4.

below we are going to describe an analog of the above construction in the case when the restriction $|\lambda_{i,\alpha}(z)| = 1$ on the partitions is dropped.

Proposition 8.3.6. *The base of the integrable system associated with refined irregular data is an affine space.*

Proof. Consider a unique line bundle $\mathscr{L} \to \overline{Y}$ which is trivialized on C_∞ and such that $c_1(\mathscr{L}) = \beta$. More precisely we require that $P.D.(c_1(\mathscr{L})) = (i_{Y\to\overline{Y}})_*(\beta)$ where $i_{Y\to\overline{Y}}$ is the natural embedding and $P.D.$ denotes the Poincaré dual. Then the restriction of $\mathscr{L}$ to $\overline{Y} - Y$ is trivial and trivialized. Consider the space of sections $s \in \Gamma(\overline{Y}, \mathscr{L})$ such that $s_{|\overline{Y}-Y} = 1$. This is an affine space. On the other hand it can be identified with the base B by taking the divisor of zeros of s. This proves that B is an affine space. ■.

Example 8.3.7. In case of $GL(r)$-Hitchin system with logarithmic singularities and semisimple monodromy the above construction can be also described such as follows.

The base of the corresponding polarized integrable system is obtained via the above-described procedure. We make blow-ups at all $(x_i)_\infty \in C_\infty$ corresponding to $x_i \in C \simeq C_\infty$. For each $1 \le i \le n$ we fix r distinct points on the exceptional divisor $D_i \simeq \mathbb{A}^1$ corresponding to the eigenvalues of the residues of the Higgs field at x_i.

Let us consider curves $\overline{\Sigma}$ in $\overline{Y}$ which satisfy the following properties:

a) $\overline{\Sigma}$ intersects each D_i with the multiplicity 1 at each of the chosen r marked points.
b) $\overline{\Sigma}$ intersects each vertical fiber of T^*C with intersection index r.
c) $\overline{\Sigma}$ does not intersect the preimage of C_∞ under the blow-up.

The space of such curves forms an affine space which is the base of our Hitchin system. The total space of the latter is birational to the twisted cotangent bundle to the moduli space of vector bundles $Bun_{GL(r),x_1,\dots,x_n}$ on C endowed with a choice of a full flag at each point x_i. The parameters of the twist correspond to eigenvalues of the residues of Higgs fields at the marked points. We consider only the stable locus in the moduli space of Higgs bundles with logarithmic singularities.

In case when we do not mark points on $D_i, 1 \le i \le n$ (i.e. we do not fix the eigenvalues) we obtain a log family of spectral curves as in [28] which forms the base of a semipolarized integrable system.

8.4 The Betti Version

Recall that Zariski open part $B^0 \subset B$ can be identified with the set of possible irregular data for which the corresponding spectral curve is smooth. Recall that an irregular data includes points $z \in \Sigma_{i,\alpha}$ representing eigenvalues of the residue of the Higgs field at marked points x_i. Let us vary each point z along the affine line where it naturally belongs in such a way that different points do not coincide. Morally this variation corresponds to adding the local system $\underline{\Gamma}_0$ to the picture (indeed fixing the eigenvalues of the monodromy as well as singular terms at marked points corresponds to a choice of symplectic leaf). In this subsection we are going to discuss the Betti version of the story (cf. also [4]).

Let us recall the formal classification of irregular connections over the formal punctured disc or, equivalently, over the field $K := \mathbf{C}((t))$. Let E be an r-dimensional K-vector space endowed with a connection $\nabla : E \to E \otimes \Omega^1_K$. Then as a D-module (E, ∇) admits a canonical decomposition

$$(E, \nabla) \simeq \oplus_\alpha (E_{c^\alpha}, \nabla^{c^\alpha})$$

over a finite collection of singular terms (c^α). We denote the ramification index of c^α by N_α. More precisely the above decomposition can be described such as follows. For each α let us choose a representative $\overline{c}^\alpha \in \mathbf{C}((t^{1/N_\alpha}))$ of the singular term c^α. Then each D-module $(E_{c^\alpha}, \nabla^{c^\alpha})$ is isomorphic to the direct image of the canonical N_α-covering $Spec(\mathbf{C}((t^{1/N_\alpha}))) \to Spec(\mathbf{C}((t)))$ of the D-module which is the tensor product of a vector bundle of rank r_α endowed with a connection with regular singularities and a rank one D-module M_{c^α}, which is also a vector bundle with a connection ∇^α, such that the generator $m_\alpha \in M_{c^\alpha}$ satisfies the condition $\nabla^\alpha(m) = m \otimes d(\overline{c}^\alpha)$. We also have $r = \sum_\alpha r_\alpha N_\alpha$.

Let C be a smooth projective curve with marked points x_i, $1 \leq i \leq n$ and (E, ∇) be an algebraic vector bundle with connection on the punctured curve $C - \{x_i\}_{1 \leq i \leq n}$. Let us choose a formal coordinate at each point x_i. Then the formal expansions of (E, ∇) at the marked points give rise to a collection of singular terms (c_i^α) as well as to a collection of positive integers $r_{i,\alpha}, N_{i,\alpha}$ derived from the above formal classification. In this way we obtain the *canonical irregular data associated with* (E, ∇) (it is canonical in the sense that it does not depend on the choice of formal coordinates at the marked points).

Irregular Riemann–Hilbert correspondence gives a topological description ("Betti side") of the complex analytic stack of algebraic vector bundles with connection with prescribed irregular data ("de Rham side"). Namely with the pair (E, ∇), $rk\, E = r$ we associate a local system of rank r (i.e. a locally constant sheaf in analytic topology) on $C - \{x_i\}_{1 \leq i \leq n}$ endowed with the so-called Stokes structure. The local system consists of analytic germs of flat sections of (E, ∇). We are going to recall the notion of Stokes structure on $\mathbf{C}$ following [27]. It describes the local picture on a general curve C.

For any marked point x_i and a generic ray $x_i + \varepsilon e^{i\varphi}$ for sufficiently small $\varepsilon > 0$ we have a filtration of the local system of flat sections by the exponential growth of a section restricted to the ray. Terms of the filtration can be identified with intersection points of our ray with the closed real analytic curve

$$\theta \mapsto exp(Re(c_i^\alpha(x_i + \varepsilon e^{i\theta})))e^{i\theta},$$

i.e. when $\theta = \varphi$.

Those filtrations are subject to the conditions described in [27]. They are called a *Stokes structure at* x_i. In particular, normalization of each curve is a circle $S^1_{x_i,\alpha}$ with the winding number about x_i equals to $N_{i,\alpha}$. On the union $\cup_\alpha S^1_{x_i,\alpha}$ we have the associated graded local system. The rank of this local system is $r_{i,\alpha}$. Collection of Stokes structures for all points $x_i, 1 \leq i \leq n$ is called the *Stokes structure for* (E,∇).

Let us fix an irregular data. Then the result of Malgrange [39] says that there is a one-to-one correspondence between algebraic connections on $C - \{x_i\}_{1\leq i\leq n}$ producing our irregular data and local systems on $C - \{x_i\}_{1\leq i\leq n}$ endowed with the Stokes structure with the singular terms and discrete parameters $r_{i,\alpha}, N_{i,\alpha}$ derived from the irregular data.

Recall that in Sect. 8.3 we also defined refined irregular data for the moduli space of Higgs bundles ("Dolbeault side"). Now we would like to introduce a similar notion on the Betti side.

Definition 8.4.1. Multiplicative refined irregular data are defined exactly in the same way as additive irregular data with the only change that $\Sigma_{i,\alpha} \subset \mathbf{C}^* \subset \mathbf{C} = \mathbb{A}^1(\mathbf{C})$.

A local system endowed with a Stokes structure defines a multiplicative refined irregular data. Namely, the set $\Sigma_{i,\alpha}$ is defined as the set of eigenvalues of the auxiliary local systems on circles $S^1_{x_i,\alpha}$ and the partitions correspond to the sizes of Jordan blocks.

Definition 8.4.2. Let us fix a multiplicative refined irregular data σ on C. Denote by $M_{Betti}(\sigma)$ the Artin stack over $\mathbf{C}$ of local systems of rank r on $C - \{x_i\}_{1\leq i\leq n}$ endowed with Stokes structure and such that the corresponding multiplicative refined irregular data coincides with σ (in particular the rank of the local system on $S^1_{x_i,\alpha}$ is equal to the number $r_{i,\alpha}$ from σ).

We denote by $M^{simp}_{Betti}(\sigma)$ the algebraic space over $\mathbf{C}$ of isomorphism classes of objects of $M_{Betti}(\sigma)$ which are simple as objects of the abelian category of local systems endowed with Stokes structure. Equivalently, the corresponding holonomic D-module on $C - \{x_i\}_{1\leq i\leq n}$ is simple. The space $M^{simp}_{Betti}(\sigma)$ is smooth. By analogy with the case of regular singularities we expect that it carries a symplectic structure. Moreover we expect that it is a quasi-affine scheme. In order to explain the latter point it is convenient to consider a larger Artin stack $M'_{Betti}(\sigma)$ obtained by weakening of some conditions in the definition of $M_{Betti}(\sigma)$. Recall that in the definition of $M_{Betti}(\sigma)$ we required that the monodromies of local systems along the circles $S^1_{x_i,\alpha}$ belong to certain conjugacy classes $C_{i,\alpha} := C_{i,\alpha}(\sigma) \subset GL(r_{i,\alpha},\mathbf{C})$. In the definition of $M'_{Betti}(\sigma)$ we relax this condition and say that the monodromies belong to the closures $\overline{C}_{i,\alpha}$.

Let us denote by $M^{coarse}_{Betti}(\sigma)$ the affine scheme $Spec(\mathscr{O}(M'_{Betti}(\sigma)))$.

Question 8.4.3. Is $M^{simp}_{Betti}(\sigma)$ an open subscheme of $M^{coarse}_{Betti}(\sigma)$?

Positive answer will imply that $M^{simp}_{Betti}(\sigma)$ is a quasi-affine scheme.

Here are some arguments in favor the positive answer to the Question 8.4.3 in the case of regular singularities. First we observe that there are many functions on $M'_{Betti}(\sigma)$ given by traces of holonomies along closed loops. We claim that the sizes of Jordan blocks of monodromies around points x_i can be also detected. Let us illustrate the claim by an example. Let $y \in C - \{x_i\}_{1\le i\le n}$ be a base point. Then our local system gives rise to an r-dimensional representation ρ of the fundamental group $\pi_1(C - \{x_i\}_{1\le i\le n}, y)$. Let us assume for simplicity that for some point x_{i_0} the monodromy $\rho(l_{i_0})$ is unipotent of order $k \ge 1$, where l_{i_0} is a based loop which is freely homotopic to a small loop surrounding x_{i_0}. Then $(\rho(l_{i_0}) - id)^k = 0$. Hence $Tr\left(\rho(l_{i_0}) - id)^k \rho(l)\right) = 0$ for any $l \in \pi_1(C - \{x_i\}_{1\le i\le n}, y)$. This equation gives identities between traces of monodromies. Similar considerations can be applied to $\bigwedge^i \rho, 1 \le j \le k$. In this way we recover information about sizes of Jordan blocks. We expect that similar arguments work in general case.

The conclusion is that conjecturally for each multiplicative refined irregular data σ we have a smooth symplectic quasi-affine variety $M^{simp}_{Betti}(\sigma)$. This variety depends on continuous parameters which are eigenvalues of monodromies on $S^1_{x_i,\alpha}$. Let us allow the eigenvalues to vary in such a way that they do not coincide. The total space should be a Poisson manifold. We can also enumerate the eigenvalues in the following way.

Definition 8.4.4. A combinatorially refined irregular data is an irregular data endowed with a collection of integers $s_{i,\alpha} \ge 1$ and a collection of maps $\Psi_{i,\alpha}$: $\{1, \dots, s_{i,\alpha}\} \to \{Partitions\} - \{0\}$ such that $\sum_{1\le j\le s_{i,\alpha}} |\Psi_{i,\alpha}(j)| = r_{i,\alpha}$.

For a given combinatorially refined irregular data τ we construct a larger moduli space $M^{simp,en}_{Betti}(\tau)$ (enumerated version of $M^{simp}_{Betti}(\sigma)$) whose set of $\mathbf{C}$-points is the disjoint union of $M^{simp}_{Betti}(\sigma)(\mathbf{C})$, over the set pairs (σ, f), $f = (f_{i,\alpha})$, where each $f_{i,\alpha}$ is a bijection $\Sigma_{i,\alpha} \simeq \{1, \dots, s_{i,\alpha}\}$. Here we assume that the map $\Lambda_{i,\alpha} \circ f^{-1}_{i,\alpha} = \Psi_{i,\alpha}$ where $\Psi_{i,\alpha}$ are the maps from the definition of the combinatorially refined irregular data τ.

It is easy to see that the space $M^{simp,en}_{Betti}(\tau)$ is fibered over the hypersurface $\mathscr{H}_{Betti}(\tau)$ in $\prod_{i,\alpha}((\mathbf{C}^*)^{s_{i,\alpha}} - Diag)$ given by the equation

$$\prod_{i,\alpha} \prod_{1\le j\le s_{i,\alpha}} \lambda^{r_{i,\alpha,j}}_{i,\alpha,j} = (-1)^{\sum_{i,\alpha}(N_{i,\alpha}-1)r_{i,\alpha}}.$$

This equation comes from the fact that the product of determinants of monodromies around marked points x_i is equal to 1. The appearance of the factor of -1 in the RHS is due to the presence of coverings in the definition of Stokes structure.

Question 8.4.5. Is $M^{simp,en}_{Betti}(\tau)$ a smooth Poisson quasi-affine scheme, with the space of symplectic leaves identified with $\mathscr{H}_{Betti}(\tau)$?

We have not discussed above the origin of the symplectic structure on $M^{simp}_{Betti}(\sigma)$ (hence the Poisson structure on $M^{simp,en}_{Betti}(\tau)$). The case of semisimple monodromy and Laurent series (i.e. no fractional powers appear) was studied in [4].

One can also define an affine scheme $M_{Betti}^{coarse,en}(\tau) \supset M_{Betti}^{simp,en}(\tau)$ endowed with a surjective map to the complex torus $\overline{\mathscr{H}_{Betti}}(\tau)$ which is the closure of $\mathscr{H}_{Betti}(\tau)$ in the ambient torus (in other words it is a shifted subtorus in $\prod_{i,\alpha}(\mathbf{C}^*)^{s_{i,\alpha}}$ given by the above equation).

In order to define $M_{Betti}^{coarse,en}(\tau)$ let us consider the moduli stack $M_{Betti}^{\prime,en}(\tau)$ of the following structures: local systems on $C - \{x_i\}_{1\leq i\leq n}$ endowed with the Stokes structure and decompositions of the associated local systems on $S^1_{x_i,\alpha}$ into the direct sums labeled by $j, 1 \leq j \leq s_{i,\alpha}$. For each direct summand the monodromy has only one eigenvalue $\lambda_{i,\alpha,j} \in \mathbf{C}^*$. The unipotent part of the monodromy is dominated by the partition $\Psi_{i,\alpha}(j)$. The latter means that the conjugacy class of the unipotent part of the monodromy belongs to the closure of the unipotent conjugacy class corresponding to $\Psi_{i,\alpha}(j)$. Finally we define $M_{Betti}^{coarse,en}(\tau) := Spec(\mathscr{O}(M_{Betti}^{\prime,en}(\tau)))$. The main point of this definition is to allow the eigenvalues $\lambda_{i,\alpha,j}$ to coincide for different values of j.

Remark 8.4.6. 1) The space $M_{Betti}^{coarse,en}(\tau)$ seems to be an analog of X-variety in the theory of cluster varieties. Presumably one can also define an analog of A-variety (see [17]).

2) Rescaling $c_i^\alpha \mapsto c_i^\alpha/\zeta$, $\zeta \in \mathbf{C}^*$ gives rise to non-linear local systems over $\mathbf{C}^*$ of all versions of M_{Betti}. Taking the fiber over $\zeta = 1$ we see that it is endowed with an automorphism given by the monodromy. This automorphism corresponds to the Coxeter automorphism in the theory of cluster algebras.

8.5 Semipolarized Irregular Systems

First we would like to describe an additive analog $M_{Dol}^{sm,irred,en}(\tau)$ of the space $M_{Betti}^{sm,en}(\tau)$. Here the notation *sm*, *irred* means *smooth, irreducible* correspondingly and refers to spectral curves. Let us comment on the notation. Suppose that the genus $g(C) > 1$, and we are dealing with regular Hitchin system (i.e. there are no marked points and the irregular data is empty). Then by Corlette–Simpson result there is a one-to-one correspondence between simple local systems on C and stable Higgs bundles of degree zero. Recall that Higgs bundles can be identified with coherent sheaves on T^*C with pure one-dimensional compact support. Under this identification line bundles of a certain degree give a Zariski open subset in the moduli space of stable Higgs bundles of degree zero. This observation motivates our notation in the general case.

Recall (see Sect. 8.3) that with an (additive) refined irregular data σ we can associate a complex integrable system with central charge and zero section. Strictly speaking we consider only an open part $B^0(\sigma)$ of the base consisting of smooth irreducible spectral curves in the surface associated with σ. Fibers are Jacobians of spectral curves. Part of the refined irregular data consists of sets of points $\Sigma_{i,\alpha} \subset \mathbb{A}^1(x_i, c_i^\alpha)$. In the case of Hitchin systems with regular singularities they are eigenvalues of the residues of the Higgs field at marked points.

Let us fix a combinatorially refined irregular data τ. Similarly to the case of $M^{simp,en}_{Betti}(\tau)$ we define the moduli space $M^{sm,irred,en}_{Dol}(\tau)$ by allowing enumerated points $z = z_{i,\alpha,j} \in \Sigma_{i,\alpha}$ to vary along the corresponding affine lines $\mathbb{A}^1(\mathbf{C}) = \mathbf{C}$ in such a way that they do not collide. The resulting space $M^{sm,irred,en}_{Dol}(\tau)$ is the total space of a family of polarized integrable systems parametrized by the hypersurface $\mathcal{H}_{Dol} := \mathcal{H}_{Dol}(\tau)$ in $\prod_{i,\alpha}(\mathbf{C}^{s_{i,\alpha}} - Diag)$ singled out by the equation

$$\sum_{i,\alpha} \sum_{1\le j\le s_{i,\alpha}} z_{i,\alpha,j}\, r_{i,\alpha,j} = 0.$$

This equation follows from the condition that sum of the residues of the Liouville form restricted to the spectral curve vanishes.

Denote by $B^0 = B^0(\tau)$ the total space of the fibration over $\mathcal{H}_{Dol}$ whose fibers are bases of the above polarized integrable systems. The projection $p : B^0 \to \mathcal{H}_{Dol}$ is a smooth morphism of smooth algebraic varieties. Fiber $p^{-1}(h), h \in \mathcal{H}_{Dol}$ can be identified by the previous considerations with the moduli space of smooth irreducible spectral curves in the appropriate surface $\overline{Y} = \overline{Y}(h)$. We will construct a local system of lattices $\underline{\Gamma} \to B^0$ endowed with a covariantly constant integer skew-symmetric pairing $\langle \bullet, \bullet \rangle : \bigwedge^2 \underline{\Gamma} \to \underline{\mathbf{Z}}_{B^0}$ and a central charge $Z : \underline{\Gamma} \to \mathcal{O}^{an}_{B^0}$ in such a way that we obtain a semipolarized integrable system with central charge. Moreover the local system $\underline{\Gamma}_0 := Ker\, \langle \bullet, \bullet \rangle$ will be trivial, i.e. $\underline{\Gamma}_0 \simeq \Gamma_0 \otimes \underline{\mathbf{Z}}_{B^0}$, where Γ_0 is a fixed lattice. The restriction $Z_{|\underline{\Gamma}_0}$ will be identified with the composition $B^0 \to \mathcal{H}_{Dol} \hookrightarrow Hom(\Gamma_0, \mathbf{C})$.

Let us explain how to define the dual local system $\underline{\Gamma}^\vee$ and the central charge. Recall that an irregular spectral curve S contains pairwise disjoint effective divisors $D_{i,\alpha,z_{i,\alpha,j}}$, where $deg\, D_{i,\alpha,z_{i,\alpha,j}} = |\Lambda_{i,\alpha}(z_{i,\alpha,j})| = |\Psi_{i,\alpha}(j)|$. These divisors are intersections of S with the chain of rational curves $E_{k,i,\alpha,z=z_{i,\alpha,j}}$ defined in Sect. 8.2.

Then we define the fiber of $\underline{\Gamma}^\vee$ over S as the abelian group of 1-chains on S whose boundaries are $\mathbf{Z}$-linear combinations of divisors $D_{i,\alpha,z_{i,\alpha,j}}$ considered modulo the boundary of 2-chains. Clearly the abelian groups depend continuously on parameters and hence define a local system. Also for any $b \in B^0$ we have a short exact sequence

$$0 \to H_1(S_b, \mathbf{Z}) \to \underline{\Gamma}^\vee_b \to \Gamma^\vee_0 \to 0,$$

where

$$\Gamma^\vee_0 = Ker(f : \oplus_{i,\alpha} \mathbf{Z}^{s_{i,\alpha}} \to \mathbf{Z}),$$

where $f : (n_{i,\alpha,j})_{1\le j\le s_{i,\alpha}} \mapsto \sum_{i,\alpha,j} n_{i,\alpha,j} r_{i,\alpha,j}$.

Dualizing we obtain local systems $\underline{\Gamma}$ and its trivial local subsystem $\underline{\Gamma}_0$. The skew-symmetric pairing on $\underline{\Gamma}$ is the pull-back of the symplectic structure on $H_1(S, \mathbf{Z})$.

Next we are going to describe the central charge. One observes that the fiber over $b \in B^0$ of the local system $\underline{\Gamma}$ can be identified (up to torsion) with the quotient $H_1(S_b - D, \mathbf{Z})/P_b$, where $D = \sqcup_{i,\alpha} D_{i,\alpha}$ is the union of the exceptional irreducible divisors in the surface $\overline{Y}$, where the pull-back of the symplectic form ω_{T^*C} to $\overline{Y} \supset T^*C$ does not have poles, and P_b is the subgroup generated by 1-chains $\gamma_{i,\alpha}$ sitting in a small neighborhood in S_b of the intersection $S_b \cap D_{i,\alpha}$ and such that the linking number of $\gamma_{i,\alpha}$ and $D_{i,\alpha}$ is equal to zero. One observes that integrals of the pull-back of the Liouville form ydx over the elements of $H_1(S_b - D, \mathbf{Z})/P_b$ are well-defined. This gives us the central charge Z.

The group of automorphisms of the tuple $(B^0, \underline{\Gamma}, \langle \bullet, \bullet \rangle, Z)$ contains a finite subgroup $\prod_m Sym_{k_m}$. The latter is the product of symmetric groups with each factor acting on points $z_{i,\alpha,j}$ which belong to the same affine line indexed by (i, α) in such a way that it permutes those points $z_{i,\alpha,j}$ which are endowed with the same partition. The quotient by the group of automorphisms will be a semipolarized integrable system with central charge and non-trivial local system of lattices $\underline{\Gamma}_0$.

8.6 Conjectures About Mirror Duals for Hitchin Systems

Fix a combinatorially refined irregular data τ. The above considerations give rise to a family of polarized integrable systems parametrized by a variety $\mathcal{H}_{Dol}(\tau)$ which is an open dense subset in $Hom(\Gamma_0, \mathbf{C})$, as well as a semipolarized integrable system which we denote by $(X^0(\tau), \omega^{2,0}(\tau)) \to B^0(\tau)$ (all endowed with holomorphic Lagrangian sections). For an individual polarized integrable system corresponding to an irregular data σ we have defined the full base $B(\sigma)$ in terms of irregular spectral curves (see Sect. 8.2), but our integrable systems so far have been defined over the locus $B^0(\sigma)$ of smooth irregular spectral curves.

Conjecture 8.6.1. 1) There exists a full semipolarized complex integrable system $X(\tau) \to B(\tau)$ containing $X^0(\tau) \to B^0(\tau)$ as an open dense complex integrable subsystem.

2) The corresponding individual polarized integrable systems $X(\tau)_{Z_0} \to B(\tau)_{Z_0}$ in the notation of Sect. 4.7 have full bases $B(\tau)_{Z_0} = B(\sigma) \simeq \mathbf{C}^{dim_{\mathbf{C}} B(\sigma)}$, where σ is determined by τ and $Z_0 \in Hom(\Gamma_0, \mathbf{C})$.

3) The mirror dual $X(\tau)^{\vee,alg}$ to the integrable system $X(\tau) \to B(\tau)$ in the sense of Sect. 6.3 is an affine scheme which contains $M_{Betti}^{simp,en}(\tau)$ as Zariski open subset.

Remark 8.6.2. Strictly speaking part (3) of the above conjecture should be corrected. Recall that the algebraic mirror dual in Sect. 6.3 was fibered over the algebraic torus $Hom(\Gamma_0, \mathbf{C}^*)$. The variety $M_{Betti}^{simp,en}(\tau)$ is fibered over the $\mathcal{H}_{Betti}(\tau)$ which is an open subset in a torsor over $Hom(\Gamma_0, \mathbf{C}^*)$. This discrepancy should be corrected by some twist. Probably this twist is related to the choice of the canonical B-field B_{can} which is 2-torsion.

Question 8.6.3. Does the mirror dual coincide with the $Spec(\mathcal{O}(M_{Betti}^{coarse,en}(\tau)))$?

Now we are going to discuss the family over $\mathbf{C}^*$ related to the rescaling $\omega^{2,0} \mapsto \omega^{2,0}/\zeta, \zeta \in \mathbf{C}^*$. In the case of irregular Hitchin systems this rescaling corresponds to the rescaling $c_i^{\alpha} \mapsto c_i^{\alpha}/\zeta$ of the singular terms. Assuming the above conjecture, we obtain by taking the mirror dual, a holomorphic family of Poisson varieties $X_{\zeta}^{\vee,alg}(\tau)$ over $\mathbf{C}^*$ containing $M_{Betti,\zeta}^{simp,en}(\tau)$ as open subvarieties. They are locally constant in analytic topology on $\mathbf{C}^*$, hence we get a local system of algebraic varieties.

Recall from Sect. 6.3 that we also have a complex analytic mirror dual $X(\tau)^{\vee} \to Hom(\Gamma_0, \mathbf{C})$ which is obtained from $X(\tau)^{\vee,alg} \to Hom(\Gamma_0, \mathbf{C}^*)$ via the exponential map. Introducing the parameter ζ we obtain a complex analytic family $X_{\zeta}(\tau)^{\vee}, \zeta \in \mathbf{C}^*$. It contains the pull-back $\tilde{M}_{Betti,\zeta}^{simp,en}(\tau)$ of the space $M_{Betti,\zeta}^{simp,en}(\tau)$ as an open dense subset. We can also define a larger space $\widetilde{M}_{Betti,\zeta}^{coarse,en}(\tau) \supset \widetilde{M}_{Betti,\zeta}^{simp,en}(\tau)$ (see Remark 8.6.2).

Recall Conjecture 6.7.3 (extension to $\zeta = 0$). In our case it says that the analytic family of Poisson varieties $X_{\zeta}(\tau)^{\vee}$ admits analytic extension to $\zeta = 0$ with the fiber at $\zeta = 0$ isomorphic to $X^{dual}(\tau)$ (see Conjecture 6.7.3). In our case it is reasonable to expect that $X^{dual}(\tau) \simeq X(\tau)$ since Jacobians of spectral curves are principally polarized abelian varieties.

Conjecture 8.6.4. The local system of Poisson varieties $\tilde{M}_{Betti,\zeta}^{coarse,en}(\tau)$ admits an analytic extension to $\zeta = 0$ with the fiber at zero isomorphic to $M_{Dol}^{sm,irred,en}(\tau)$.

The total space of the extended to $\zeta = 0$ family should be a complex algebraic variety if we endow fibers $\tilde{M}_{Betti,\zeta}^{coarse,en}(\tau)$ with algebraic structures coming from the de Rham description of fibers via inverse Riemann–Hilbert correspondence as algebraic vector bundles endowed with irregular ζ-connections. Furthermore, the de Rham description makes the above conjecture almost evident similarly to the well-known case if Hitchin systems without singularities.

Let us describe explicitly the geometric meaning of a germ of holomorphic section of the above analytic family at $\zeta = 0$. Let us fix an additive refined irregular data σ on the curve C. In particular it gives us a positive integer r (the rank). Consider a holomorphic vector bundle E of rank r over $C \times D_{\varepsilon}$, where $D_{\varepsilon} = \{\zeta \in \mathbf{C} | |\zeta| < \varepsilon\}$ and $\varepsilon > 0$ is sufficiently small. We will think of it as a family E_{ζ} of holomorphic vector bundles. Consider a relative along C meromorphic connection ∇ such that:

(a) ∇ has finite order poles at the marked points $x_i, 1 \le i \le n$;
(b) ∇ has the pole of order 1 along $C \times \{\zeta = 0\}$.

In particular for all $\zeta \in \mathbf{C}^*$ we have a meromorphic connection ∇_{ζ} on the vector bundle E_{ζ}.

Using the formal classification of irregular connections we assign to each ∇_{ζ} irregular data with singular terms at the points x_i. We require that they coincide with the given irregular data after rescaling $c_i^{\alpha} \mapsto c_i^{\alpha}/\zeta$ of singular terms (here we forget about the refinement).

Let us choose an additive refined irregular data compatible with τ. In other words we choose points $z_{i,\alpha,j} \subset \mathbb{A}^1(x_i, c_i^{\alpha}) \simeq \mathbf{C}$ such that $z_{i,\alpha,j_1} \neq z_{i,\alpha,j_2}$ for $j_1 \neq j_2$. Moreover we assume that eigenvalues $\lambda_{i,\alpha,j}$ of the monodromy of ∇_ζ along the circle $S^1_{x_i,\alpha}$ have the form

$$\lambda_{i,\alpha,j} = exp(\frac{z_{i,\alpha,j}}{\zeta}).$$

Furthermore we assume that for any i, α, j the sizes of the Jordan blocks with eigenvalues $\lambda_{i,\alpha,j}$ as above form the partition $\Psi_{i,\alpha}(j)$ coming from the combinatorially refined irregular data τ as as long as

$$\frac{z_{i,\alpha,j_1} - z_{i,\alpha,j_2}}{\zeta} \notin 2\pi\sqrt{-1}\mathbf{Z}$$

for $j_1 \neq j_2$. We call the latter condition *non-exceptionality condition*. It depends on the formal type of the irregular connection ∇_ζ at x_i and does not depend on the Stokes structure.

Remark 8.6.5. One can generalize the above story by allowing the curve C and irregular data to depend analytically on ζ. Then we will require that

$$\lambda_{i,\alpha,j} = exp(\frac{z_{i,\alpha,j}}{\zeta} + O(1)).$$

Under the above assumptions the limit $lim_{\zeta\to 0}\zeta\nabla_\zeta$ does exist and defines a meromorphic Higgs field φ (with poles at the points x_i) on the vector bundle $E_0 := E_{|\zeta=0}$. It defines the refined irregular data coinciding with the given one. The closure $S \subset \overline{Y}$ of the non-compact spectral curve S^0 given by $det(\phi - yId) = 0 \subset T^*(C - \{x_i\}_{1\le i\le n}) \subset Y$ is an irregular spectral curve in our sense.

Assume that $S := S_b$ is smooth and irreducible (i.e. it corresponds to a point $b \in B^0$). There is a natural line bundle $\mathscr{L} \to S^0$ corresponding to $(E_0)_{|C-\{x_i\}_{1\le i\le n}}$. Extending it to S (the ambiguity for such an extension is a lattice of finite rank) we obtain a line bundle $\mathscr{L} \to S$, hence a point in $Jac^d(S_b) = Jac^d(S)$ for some degree d belonging to a fiber of the integrable system at $\zeta = 0$. Suppose that for our refined irregular data we have chosen $\zeta \in \mathbf{C}^*$ such that non-exceptionality condition holds for $\zeta, \{z_{i,\alpha,j}\}$. Then our connection ∇_ζ defines a point $f(\zeta)$ in the covering $\tilde{M}^{coarse,en}_{Betti,\zeta}(\tau)$ via taking the Stokes structure of (E_ζ, ∇_ζ). The point $f(\zeta)$ depends holomorphically on ζ such that $\zeta \neq 0$ and $\zeta \neq (z_{i,\alpha,j_1} - z_{i,\alpha,j_2})/2\pi\sqrt{-1}k, k \in \mathbf{Z} - \{0\}$.

Conjecture 8.6.6. 1) Let us assume the Conjecture 8.6.4. Then the map $\zeta \mapsto f(\zeta)$ extends to a germ of an analytic curve at $\zeta = 0$.

2) Moreover the value $f(0)$ is the point of the space $M^{sm,irred,en}_{Dol}(\tau)$ corresponding to the line bundle $\mathscr{L} \to S$.

Our definition of $\tilde{M}_{Betti}^{coarse,en}(\tau)$ is a bit artificial. Probably the following version of it will behave better. Fix an irregular data η (no refinement is chosen). Consider an Artin stack $M_{Betti}(\eta)$ parametrizing local systems on $C - \{x_i\}_{1\le i\le n}$ with Stokes structures compatible with irregular data. We denote by $M_{Betti}^{an}(\eta)$ the corresponding analytic stack. Let us define an analytic stack $\tilde{M}_{Betti}^{an}(\eta)$ by adding the following additional data: for each pair i,α a choice of covariantly constant endomorphism $L_{i,\alpha}$ of the associated local system $\mathscr{E}_{i,\alpha}$ on the circle $S^1_{x_i,\alpha}$ such that $exp(L_{i,\alpha})$ is the monodromy automorphism for $\mathscr{E}_{i,\alpha}$. In the case when all monodromies are semisimple and have distinct eigenvalues the set of choices of "logarithms" $L_{i,\alpha}$ for all i,α is a torsor over the lattice $\mathbf{Z}^{\{(i,\alpha,j)\}}$. This stack can be thought of as a replacement of $\tilde{M}_{Betti}^{coarse,en}(\tau)$ (the enumeration of eigenvalues is lost). We claim that there exists another Artin stack $M_{DR}(\eta)$ such that the (irregular) Riemann–Hilbert correspondence gives an isomorphism of analytic stacks $\tilde{M}_{Betti}^{an}(\eta) \simeq M_{DR}^{an}(\eta)$. In the example of connections with regular singularities the stack $M_{DR}(\eta)$ is the moduli stack of vector bundles on C endowed with meromorphic connections which have first order pole at the marked points. Introducing the parameter $\zeta \in \mathbf{C}^*$ as above by rescaling the singular terms of the irregular data we obtain an algebraic family $M_{DR,\zeta}(\eta)$ of Artin stacks. We believe that it can be extended as an algebraic family of Artin stacks over $\mathbf{C}$ with the fiber at $\zeta = 0$ being the stack $M_{Dol}(\eta)$ of semistable generalized Higgs bundles of type η. The latter are roughly certain coherent sheaves on the Poisson surface $\overline{Y} := \overline{Y}(\eta)$ (see Sect. 8.3) with pure one-dimensional support.

Let us discuss the analog of a combinatorial refinement in this setting. Assume that for each pair i,α we are given a finite unordered collection of partitions $\lambda^{i,\alpha,j}$ (possibly with repetitions) such the sum of weights $\sum_j |\lambda^{i,\alpha,j}|$ is equal to the rank $r_{i,\alpha}$ of $\mathscr{E}_{i,\alpha}$. Then we pose the following condition: *the conjugacy class of the linear operator $L_{i,\alpha}$ (considered as a linear endomorphism of the fixed fiber of $\mathscr{E}_{i,\alpha}$) belongs to the closure of the set of such linear operators that for each of its eigenvalues μ_j the set of Jordan blocks with the eigenvalue μ_j defines a collection of partitions which coincides with the given collection of partitions $\lambda^{i,\alpha,j}$.* This defines a closed substack of $M_{DR}(\eta)$. Presumably one can define a similar substack of $M_{Dol}(\eta)$. Introducing the parameter ζ we obtain as above a family of Artin substacks over $\mathbf{C}$. Finally we remark that Mirror Symmetry naturally gives us an *analytic* family of analytic stacks $M_{DR,\zeta}^{an}(\eta) \simeq \tilde{M}_{Betti,\zeta}^{an}(\eta)$ over $\zeta \in \mathbf{C}^*$ and its limit at $\zeta = 0$ given by $M_{Dol}^{an}(\eta)$. Also the fiber over $\zeta \neq 0$ have some "remnants" of the algebraic structure on $M_{Betti,\zeta}(\eta)$. The algebraic structures on $M_{DR}(\eta)$ and $M_{Dol}(\eta)$ familiar in Geometric Langlands Correspondence seem to play no role in the case of Mirror Symmetry considered before.

8.7 Remarks About SL(r) Case

In the case of $SL(r)$ Hitchin systems the above considerations have to be modified. Namely, in the definition of the irregular data we impose an additional condition: *sum of all branches of the singular terms at each marked point x_i is equal to zero (modulo series which are regular at x_i).*

This condition can be reformulated such as follows. For with each singular term $c_i^\alpha = \sum_{\lambda\in\mathbf{Q}_{<0}} a^\alpha_{\lambda,i}(x-x_i)^\lambda$ we associate its trace

$$Tr(c_i^\alpha) = N_{i,\alpha}\cdot \sum_{\lambda\in\mathbf{Z}_{<0}} a^\alpha_{\lambda,i}(x-x_i)^\lambda \in \mathbf{C}[(x-x_i)^{-1}].$$

Then the above condition says that for each $1\le i\le n$ we have $\sum_\alpha Tr(c_i^\alpha)=0$.

We can impose a similar condition for spectral curves. A spectral curve can be thought of as a graph of a multivalued closed 1-form on $C-\{x_i\}_{1\le i\le n}$. We demand that the sum of all branches of the 1-form vanishes identically. Fix combinatorially refined irregular data τ. We denote by $B_{SL(r)}(\tau)\subset B(\tau)$ the subspace of spectral curves S which satisfy the above condition.

Proposition 8.7.1. *One has $\dim B_{SL(r)}(\tau) = \dim B(\tau) - g(C)$, where $g(C)$ is the genus of C.*

Proof. For any spectral curve in $B(\tau)$ the sum of branches of the corresponding multivalued 1-form is a holomorphic 1-form on C. Also the space of 1-forms $\Omega^1(C)$ acts on $B(\tau)$ by adding the graph of the 1-form. This gives an isomorphism $B(\tau)\simeq B_{SL(r)}(\tau)\times\Omega^1(C)$. The result follows. ■

Remark 8.7.2. In the case of Hitchin systems with regular singularities the above condition means that the sum of eigenvalues of the singular part of the Higgs field at each x_i is equal to zero.

Considering the corresponding local system of *symplectic* lattices $\underline{\Gamma}^{symp}$ (which is the quotient of the bigger local system $\underline{\Gamma}$) we see that the fiber of $\underline{\Gamma}_S^{symp}$ is $Prym(S) := Ker(H_1(S,Z)\to H_1(C,\mathbf{Z}))$. The fibers are polarized but not principally polarized. Another choice would be $Coker(H^1(C,\mathbf{Z})\to H^1(S,\mathbf{Z}))$. The mirror duals to the integrable systems corresponding to these two choices are different and should correspond to $M^{SL(r)}_{Betti}(\tau)$ and $M^{PGL(r)}_{Betti}(\tau)$.

8.8 Relation to Non-compact Calabi–Yau Threefolds

Having a spectral curve $S\subset Y\subset \overline{Y}$ as above and line bundles $\mathscr{L}_i\to Y, i=1,2$ such that the restrictions of $\mathscr{L}_i, i=1,2$ to $Y-S$ are trivialized and such that $\mathscr{L}_1\otimes\mathscr{L}_2\simeq\mathscr{O}_Y(S)$ we can construct a non-compact Calabi–Yau threefold (total space of the conic bundle over Y). Namely, let us fix a section $t\in\Gamma(Y,\mathscr{L}_1\otimes\mathscr{L}_2)$

such that $t_{|S} = 0$. Then we consider a subvariety X of the total space $tot(\mathcal{L}_1 \oplus \mathcal{L}_2)$ which consists of pairs $(l_1, l_2) \in \mathcal{L}_{1,y} \oplus \mathcal{L}_{2,y}, y \in Y$ such that $l_1 \cdot l_2 = t(y)$. Writing it in local coordinates as $x_1 x_2 = f(y_1, y_2)$ we see that X carries a well-defined nowhere vanishing holomorphic volume form $\Omega_X^{3,0}$ locally given by $\frac{dx_1}{x_1} dy_1 dy_2$. Then such X satisfies the assumptions $A1$–$A3$ from Sect. 7. Furthermore the corresponding moduli space of deformations of X discussed in Sect. 7 is essentially the same as the space B of spectral curves discussed in this section. The fiber of the local system $\underline{\Gamma}$ is isomorphic to $H_3(X, \mathbf{Z})$ modulo torsion. An easy calculation with exact sequences of fiber bundles shows that if $\mathcal{L}_1$ is trivial then $H_3(X, \mathbf{Z}) \simeq H_2(Y, S, \mathbf{Z})$. Then the symplectic lattice $\underline{\Gamma}^{symp}$ is smaller than $H_1(S, \mathbf{Z})$, while the kernel lattice $\underline{\Gamma}_0$ is bigger than the one for the lattice described in Sect. 8.5.

In the case when the monodromies along $S^1_{x_i,\alpha}$ are semisimple with non-coinciding eigenvalues one can spell the above discussion in terms of log-families of spectral curves. In particular there is an open dense part in the moduli space of deformations of X which is isomorphic to the space of log-families of smooth curves in T^*C. In order to be in agreement with three-dimensional story we need to go from $GL(r)$ Hitchin integrable systems to $SL(r)$ Hitchin integrable systems. Then, as we discussed above, for each spectral curve $S_b, b \in B^0_{SL(r)}(\tau)$ the lattice Γ_b^{symp} is isomorphic to $H_2(Y, S, \mathbf{Z})^{symp} \simeq Prym(S_b) \simeq H_3(X, \mathbf{Z})/Ker(\langle \bullet, \bullet \rangle)$. Periods of the restriction $ydx_{|S_b}$ of the canonical 1-form can be identified with periods of the holomorphic volume form $\Omega_X^{3,0}$.

Remark 8.8.1. The reader remembers that when discussing singular Hitchin systems we fixed the essentially irregular part of the Higgs field. Coefficients of those fixed Puiseux series as well as the conformal structure on $(C, \{x_i\}_{1 \le i \le n})$ can be thought of "external" parameters for the integrable systems in question. In terms of Calabi–Yau threefold X this means that we have extra parameters arising from the full moduli space of deformations of X.

9 Wall-Crossing Structures for Compact Calabi–Yau Threefolds and Split Attractor Flow

Let X be a compact complex Calabi–Yau threefold endowed with an ample line bundle (polarization). We make a simplifying assumption that $H^1(X, \mathbf{Q}) = 0$ (otherwise the considerations below should be changed slightly). The moduli stack $\mathcal{M} := \mathcal{M}_X$ of complex structures on X is a smooth Deligne–Mumford stack (orbifold). The moduli stack $\mathcal{L} := \mathcal{L}_X$ of pairs $(X_\tau, \Omega^{3,0}_{X_\tau})$ parametrizing pairs (complex structure τ, holomorphic volume form) is a $\mathbf{C}^*$-bundle $p : \mathcal{L}_X \to \mathcal{M}_X$. In what follows we ignore those points of the stacks which have non-trivial stabilizers. Thus we will often abuse the terminology and speak about moduli spaces, not stacks. Locally $\mathcal{L}_X$ is embedded into $H^3(X, \mathbf{C})$ via the period map $(X_\tau, \Omega^{3,0}_{X_\tau}) \mapsto [\Omega^{3,0}_{X_\tau}]$.

It is known (see [13,14]) that $\mathscr{L}_X$ is the base of non-polarized complex integrable system with fibers given by intermediate Jacobians of the underlying Calabi–Yau threefolds. Since X is compact the fibers in general are nonalgebraic. Instead the fibers carry natural pseudo-Kähler metrics of signature $(1, n)$, where $n = \frac{1}{2} rk H^3(X) - 1$. This integrable system can be considered as a special case of the one mentioned in Remark 4.1.2. The corresponding local system of lattices has as fibers $\Gamma_b = H^3(X_\tau, \mathbf{Z}), b \in \mathscr{L}_X$, where $p(b) = \tau$. It also has the central charge $Z_b(\gamma) = \int_\gamma \Omega^{3,0}_{X_\tau}$.

The only difference with the case of polarized integrable systems is that now we do not have the positivity constraint on the skew-symmetric bilinear form on Γ_b.

On the other hand in this case one has a well-defined local system of the Fukaya categories $\mathscr{F}(X_\tau)$ over $\mathscr{M}_X$. It is expected that a choice of point $b \in \mathscr{L}_X$ defines a stability condition on $\mathscr{F}(X_\tau), \tau = p(b)$ with the central charge $Z_b : \Gamma_b \to \mathbf{C}$ as above and for which semistable objects are SLAGs endowed with local systems. Hence we can speak (cf. Sect. 7.3) about DT-invariants $\Omega_b(\gamma), b \in \mathscr{L}_X, \gamma \in \Gamma_b - \{0\}$. We conclude that there is a corresponding WCS (see Sect. 2.3, Example (6)).

Moreover, using this WCS we can construct a non-archimedean symplectic orbifold $\mathscr{X}$ along the lines of Sect. 4.6. More precisely, the construction of Sect. 4.6 gives rise to a family $\mathscr{X}_\zeta, \zeta \in \mathbf{C}^*$ of such orbifolds, but all of them are canonically isomorphic due to the natural $\mathbf{C}^*$-action on $\mathscr{L}_X$.

We do not expect that X is isomorphic to an open domain in an algebraic orbifold. It is not even clear whether it is isomorphic to an open domain in a complex analytic orbifold (the problem arises because of the expected overexponential growth of $\Omega_b(\gamma)$ as $|\gamma| \to \infty$. As a result, the second of the three approaches to DT invariants discussed in the Introduction (namely the one with the wheels of lines, see Sect. 6.6) cannot be applied.

Also the first named approach (via attractor flow and trees, see Sect. 3) should be modified. More precisely, the initial WCS should in addition to what was discussed in Sect. 4.6 (which are the values 1 for DT-invariants at generic conifold points) depend on infinitely many integer parameters, which are values of DT-invariants at so-called attractor points (see e.g. [9]). More precisely, in the compact case the volume of X is finite, so it can be used to normalize the central charge. The normalized function (considered as a function on the total space of the local system $\underline{\Gamma} \to \mathscr{M}_X$) has countably many minimal points. Their lifts to $\mathscr{L}_X$ are called attractor points (see Sects. 9.2, and 9.3 about the details).

At this time we do not know how to extract those additional data directly from the geometry of the above-described integrable system over $\mathscr{L}_X$. In a sense the additional data live "at the infinity" of the moduli space $\mathscr{L}_X$. Since we are lacking the second approach to DT-invariants in the compact case, we have modify the Conjecture 1.2.1 and only claim that the DT-invariants coming from the Fukaya categories can be canonically reconstructed from the values of the "tropical" DT-invariants at the attractor points.

Also, the string theory suggests (see e.g. [2]) that there exists a complex analytic contact orbifold $\mathscr{Y}$ with $dim\, \mathscr{Y} = dim\, \mathscr{X} + 1$ (which is called in physics the twistor space for the quaternion-Kähler moduli space of hypermultiplets). The orbifold $\mathscr{X}$

is a formal germ of a divisor at infinity of $\mathscr{Y}$. The structure of $\mathscr{Y}$ is still a mystery, but one can hope that $\mathscr{Y}$ can be used for the description of DT-invariants along the lines of the second approach (via the wheels of lines).

9.1 Split Attractor Flow and Black Holes: Motivation from Supergravity

As a physics motivation for our considerations we will briefly explain the concept of the split attractor flow from the theory of supersymmetric black holes.

Let $\mathscr{M}_{CFT}$ be the "moduli space" of unitary $N = 2$ superconformal field theories. It is believed that in case if there are no chiral fields of dimension $(2, 0)$ then $\mathscr{M}_{CFT} \simeq \mathscr{M}_A \times \mathscr{M}_B$, where for CFTs associated with a $3CY$ manifold X the moduli space $\mathscr{M}_A$ is the space of complexified Kähler structures on X while $\mathscr{M}_B$ is the moduli space of complex structures on X.

Recall critical superstring theory in ten dimensions can degenerate to a family of superconformal field theories with central charge $\hat{c} = 6$ over a four-dimensional flat space-time. The latter is $\mathbf{R}^4$ endowed with a singular metric satisfying Einstein equation with matter. The metric has singularities at black holes. Assuming time invariance we obtain a metric g on $\mathbf{R}^3 \setminus \{x_1, \dots, x_n\}$, where x_i are positions of stationary black holes. This family can be interpreted as a map $h : \mathbf{R}^3 \setminus \{x_1, \dots, x_n\} \to \mathscr{M}_{CFT}$ which satisfies together with the metric g a complicated system of equations.

Let us assume that our CFT is of geometric origin and comes from a $3CY$ manifold X such that $H^{1,0}(X) = H^{2,0}(X) = 0$. Assume that the Kähler component of h is constant. Then according to [9, 10] (see also [11]) the set of pairs (h, g) is in one-to-one correspondence with the set of maps

$$\phi : \mathbf{R}^3 \setminus \{x_1, \dots, x_n\} \to \mathscr{M}_X$$

(here $\mathscr{M}_X := \mathscr{M}_B(X)$ is the moduli space of complex structures on X) coming from the following ansatz. Namely, the map ϕ is obtained by the projectivization of the map $\hat{\phi} : \mathbf{R}^3 \setminus \{x_1, \dots, x_n\} \to \mathscr{L}$, where $\mathbf{R}^3 \setminus \{x_1, \dots, x_n\}$ is endowed with the standard flat Euclidean metric (which is different from the metric g) and $\mathscr{L} := \mathscr{L}_X$ is the Lagrangian cone of the moduli space of deformations of X endowed with a holomorphic volume form (it is locally embedded into $H^3(X, \mathbf{C})$ via the period map). The cone $\mathscr{L}$ is the total space of a $\mathbf{C}^*$-bundle over $\mathscr{M}_X$. We endow $\mathscr{L}$ with an integral affine structure via the local homeomorphism $Im : \mathscr{L} \to H^3(X, \mathbf{R}), (\tau, \Omega^{3,0}_\tau) \mapsto Im([\Omega^{3,0}_\tau])$, where $\tau \in \mathscr{M}_X$ is a complex structure on X, and $\Omega^{3,0}_\tau$ is the corresponding holomorphic volume form. Then the ansatz comes from harmonic maps $\hat{\phi}$ which are locally of the form $Im \circ \hat{\phi}(x) = \sum_{1 \le i \le n} \frac{\gamma_i}{|x - x_i|} + v_\infty$, where $\gamma_i, 1 \le i \le n$ are elements of the charge lattice $\Gamma = H_3(X, \mathbf{Z}) \simeq H^3(X, \mathbf{Z})$ (their meanings are the charges of black holes) and v_∞ is the boundary condition "at infinity" satisfying the constraint $\sum_{1 \le i \le n} \langle \gamma_i, v_\infty \rangle = 0$ (see [9]). This gives us ϕ.

The image of ϕ is an "amoeba-shaped" three-dimensional domain in $\mathcal{M}_X$. Hypothetically, connected components of the moduli space of maps ϕ with given $v_\infty, \gamma_i, 1 \le i \le n$ have some cusps which are in one-to-one correspondence with split attractor trees (see [9]). When we approach to such a cusp the $3d$ amoeba degenerates to a split attractor tree. This is somehow similar to the conventional "tropical" story, when holomorphic rational curves in Gromov–Witten theory degenerate at cusps to the gradient trees on the base of SYZ torus fibration. In fact edges of the split attractor tree are the gradient trajectories of the multivalued function $|F_\gamma|^2 = |\int_\gamma \Omega^{(3,0)}|^2/|\int_X \Omega^{(3,0)} \wedge \overline{\Omega^{(3,0)}}|$ considered as a function on $\mathcal{M}_X$. Any edge is locally a projection of an affine line in $\mathcal{L}$ with the slope $\gamma \in \Gamma$. If the split attractor flow (lifted from $\mathcal{M}_X$ to $\mathcal{L}$) starting at v_∞ in the direction γ hits the wall of marginal stability where $\gamma = \gamma_1 + \gamma_2 + \ldots + \gamma_k, Arg(\int_{\gamma_i} \Omega^{(3,0)}) = Arg(\int_\gamma \Omega^{(3,0)}), 1 \le i \le k$ then all $\gamma_1, \ldots, \gamma_k$ belong (generically) to a two-dimensional plane.

We are going to explain below that using our wall-crossing formulas it is possible to find all $\Omega(\gamma) := \Omega(b,\gamma), b \in \mathcal{L}, \gamma \in \Gamma, \langle Im\, b, \gamma\rangle = 0$ starting with a collection of integers $\Omega(b_\gamma, \gamma)$ at the "generalized attractor points" given by conifold points and points $b_\gamma \in \Lambda$ defined by the equation $Im\, b_\gamma = \gamma$. The points $\mathbf{C}^* b_\gamma \in \mathcal{M}_X$ are external vertices of the split attractor trees. The wall-crossing formulas are used at the internal vertices of the trees for the computation of $\Omega(b,\gamma)$. The numbers $\Omega(b_\gamma, \gamma)$ can be arbitrary.

9.2 *Affine Structure on the Lagrangian Cone*

Let us fix $\gamma \in \Gamma$. The wall of second kind associated with γ (see [30]) is given explicitly by the set

$$\mathcal{L}_\gamma = \{(\tau, \Omega_\tau^{3,0}) \in \mathcal{L} | \langle Im([\Omega_\tau^{3,0}]), \gamma\rangle = 0, \langle Re([\Omega_\tau^{3,0}]), \gamma\rangle > 0\}.$$

(Notice that the condition $\langle Re([\Omega_\tau^{3,0}]), Im([\Omega_\tau^{3,0}])\rangle > 0$ holds on $\mathcal{L}$). In what follows we will locally identify $\mathcal{L}$ with the cone in $H^3(X,\mathbf{C})$ and denote the point corresponding to $(\tau, \Omega_\tau^{3,0})$ simply by $\Omega^{3,0}$. We endow $\mathcal{L}$ with an integer affine structure given locally $\Omega^{3,0} \mapsto Im(\Omega^{3,0}) \in H^3(X,\mathbf{R})$.

We define a multivalued function $F_\gamma : \mathcal{M}_X \to \mathbf{R}_{\ge 0}$ by the formula:

$$F_\gamma(\Omega^{3,0}) = \frac{|\langle \Omega^{3,0}, \gamma\rangle|}{\sqrt{\langle Re(\Omega^{3,0}), Im(\Omega^{3,0})\rangle}}.$$

(The RHS does not depend on a choice of the lift to $\mathcal{L}$).

We define the *volume function* $v : \mathcal{L} \to \mathbf{R}_{>0}$ from the equality

$$\langle Re(\Omega^{3,0}), Im(\Omega^{3,0})\rangle = \frac{-1}{2i}\langle \Omega^{3,0}, \overline{\Omega^{3,0}}\rangle = v(\Omega^{3,0})^2.$$

The moduli space $\mathscr{M}_X$ carries the Weil–Petersson metric. Let us recall its definition. Let us fix a form $\Omega^{3,0}_{X_0}$ such that $v(\Omega^{3,0}_{X_0}) = 1$. The tangent space to $\mathscr{L}$ at a point $\Omega^{3,0}_{X_0}$ can be identified with the term $F^2(H^3(X_0, \mathbf{C}))$ of the Hodge filtration, which is decomposed into the direct sum $H^{3,0}(X_0) \oplus H^{2,1}(X_0)$ by Hodge theory. Hence the tangent space to $\mathscr{M}$ at the point $[X_0]$ is identified with $H^{2,1}(X_0)$. The latter space carries a natural Hermitean norm. This gives the metric on the tangent space $T_{[X_0]}\mathscr{M}_X$.

Theorem 9.2.1. *Let us fix a non-zero $\gamma \in \Gamma$. One can lift the gradient flow of $|F_\gamma|^2$ to a flow on the wall $\mathscr{L}_\gamma$ whose trajectories are straight lines with the slope γ in the affine structure given by $Im(\Omega^{3,0})$. More precisely, the integral curve $\dot{x} = grad\,|F_\gamma|^2(x)$ near the point $x(0) = x_0 \in \mathscr{M}_X$ coincides as unparametrized curve with the image of the straight line $Im(\Omega^{3,0}_t) = Im(\Omega^{3,0}_0) + t\gamma$, where $\langle Im(\Omega^{3,0}_0), \gamma\rangle = 0, \langle Re(\Omega^{3,0}_0), \gamma\rangle > 0$, and $\Omega^{3,0}_0$ belongs to a $\mathbf{C}^*$-fiber over x_0 of the bundle $\mathscr{L} \to \mathscr{M}_X$.*

Proof. For any point $x_0 := [X_0] \in \mathscr{M}$ such that $F_\gamma(x_0) \neq 0$ there is a unique lift $\Omega^{3,0}_0 \in \mathscr{L}_\gamma$ with $v(\Omega^{3,0}_0) = 1$. The tangent space $T_{x_0}\mathscr{M}$ is identified with variations $\Omega^{3,0}_0 \mapsto \Omega^{3,0}_0 + \delta\Omega^{3,0}$ such that $\delta\Omega^{3,0} \in H^{2,1}$.

Let us compute the variation of the function $log\,|F_\gamma(\Omega^{3,0})|^2 = log|\langle\Omega^{3,0}, \gamma\rangle|^2 - log\,\langle Re(\Omega^{3,0}), Im(\Omega^{3,0})\rangle$. We obtain:

$$\delta log\,\langle\Omega^{3,0}, \gamma\rangle + \delta log\,\langle\overline{\Omega^{3,0}}, \gamma\rangle - \delta log\,\langle Re(\Omega^{3,0}), Im(\Omega^{3,0})\rangle$$

$$= 2\frac{Re(\langle\delta\Omega^{3,0}, \gamma\rangle)}{\langle\Omega^{3,0}_0, \gamma\rangle} - 2i\frac{Im(\langle\delta\Omega^{3,0}, \overline{\Omega^{3,0}_0}\rangle)}{\langle\Omega^{3,0}_0, \overline{\Omega^{3,0}_0}\rangle}.$$

The last term vanishes because $\delta\Omega^{3,0}$ belongs to $H^{2,1}$. Since $\langle\Omega^{3,0}_0, \gamma\rangle = \langle Re(\Omega^{3,0}_0), \gamma\rangle > 0$ and $\delta log|F_\gamma|^2 = \delta|F_\gamma|^2/|F_\gamma|^2$ we have:

$$\delta|F_\gamma|^2 = aRe(\langle\delta\Omega^{3,0}, \gamma\rangle)$$

where $a \in \mathbf{R}$ is a real constant depending on $\Omega^{3,0}_0, \gamma$.

Next we observe that by the Hodge decomposition the element $\gamma \in \Gamma = H^3(X, \mathbf{Z}) \subset H^3(X, \mathbf{R})$ can be written as $\gamma = \gamma^{3,0} + \overline{\gamma^{3,0}} + \gamma^{2,1} + \overline{\gamma^{2,1}}$, where the upper index denotes the (p, q)-Hodge component. By orthogonality condition we conclude that $\delta|F_\gamma|^2 = aRe(\langle\delta\Omega^{3,0}, \gamma^{1,2}\rangle) = -aIm(\langle\delta\Omega^{3,0}, \overline{i\gamma^{2,1}}\rangle)$.

The RHS is by definition the pairing of two tangent vectors in $T_{x_0}\mathscr{M}_X \simeq H^{2,1}$ with respect to the Weil–Petersson metric. Hence $grad\,|F_\gamma|$ at x_0 is proportional to $i\gamma^{2,1}$. But $i\gamma^{2,1}$ is the projection of the tangent vector $i(\gamma^{3,0} + \gamma^{2,1})$ at $\Omega^{3,0}_0 \in \mathscr{L}$ whose imaginary part is $\gamma/2 = Im(i\gamma^{3,0} + i\gamma^{2,1})$. This concludes the proof. ■

In what follows we will need to know the behavior of the volume function $v(\Omega^{3,0})$ along the gradient trajectory of the function $|F_\gamma|^2$. We choose a parametrization of the trajectory such that $Im(\dot{\Omega}^{3,0}) = \gamma$ or equivalently that $\dot{\Omega}^{3,0} = 2i(\gamma^{3,0} + \gamma^{2,1})$.

Along the gradient trajectory we have:

$$\frac{d}{dt}(log\, v) = \frac{1}{2}\frac{d}{dt}(log\, v^2) = Re\left(\frac{\langle \Omega^{3,0}, \overline{\dot{\Omega}^{3,0}}\rangle}{\langle \Omega^{3,0}, \overline{\Omega^{3,0}}\rangle}\right)$$
$$= Re\left(\frac{\langle \Omega^{3,0}, \overline{\dot{\Omega}^{3,0}} - \dot{\Omega}^{3,0}\rangle)}{\langle \Omega^{3,0}, \overline{\Omega^{3,0}}\rangle}\right) = Re\left(\frac{\langle \Omega^{3,0}, -2iIm(\dot{\Omega}^{3,0})\rangle}{-2iv^2}\right)$$
$$= \frac{Re(\langle \Omega^{3,0}, \gamma\rangle)}{v^2} = \frac{\langle \Omega^{3,0}, \gamma\rangle}{v^2}.$$

(In the course of the computations we use the Lagrangian property of $\mathscr{L}$ which gives $\langle \Omega^{3,0}, \dot{\Omega}^{3,0}\rangle = 0$, as well as the equality $Im(\dot{\Omega}^{3,0}) = \gamma$).

Therefore $\frac{d}{dt}v = \frac{\langle \Omega^{3,0},\gamma\rangle}{v} = \pm F_\gamma$. Notice that if $\langle \Omega^{3,0}, \gamma\rangle \neq 0$ then similarly to the proof of Theorem 9.2.1 we have

$$\frac{d}{dt}(|F_\gamma|^2)(\Omega^{3,0}) = |F_\gamma|^2(\Omega^{3,0})$$
$$\times\left(2\frac{Re(\langle \dot{\Omega}^{3,0}, \gamma\rangle)}{\langle \Omega^{3,0}, \gamma\rangle} - 2Re(\langle\frac{\dot{\Omega}^{3,0}, i\overline{\Omega^{3,0}}\rangle}{)}\langle \Omega^{3,0}, i\overline{\Omega^{3,0}}\rangle\right).$$

Now we can use the formula $\dot{\Omega}^{3,0} = 2i(\gamma^{3,0} + \gamma^{2,1})$.

Then $2i\gamma^{3,0} = c\Omega^{3,0}$, where $c \in \mathbf{C}$. Hence the input of the summand $2i\gamma^{3,0}$ to the RHS of the above formula is

$$|F_\gamma|^2(\Omega^{3,0})\left(2\frac{Re(c\langle \Omega^{3,0}, \gamma\rangle)}{\langle \Omega^{3,0}, \gamma\rangle} - 2\frac{Re(c\langle \Omega^{3,0}, i\overline{\Omega^{3,0}}\rangle)}{\langle \Omega^{3,0}, i\overline{\Omega^{3,0}}\rangle}\right) = 0.$$

(Notice that $\langle \Omega^{3,0}, \gamma\rangle > 0$ and $\langle \Omega^{3,0}, i\overline{\Omega^{3,0}}\rangle > 0$ by our assumptions, hence the expression in the big brackets simplifies to $2Re(c) - 2Re(c) = 0$).

Therefore in the RHS we have the contribution of the summand $2i\gamma^{2,1}$ only. This gives us

$$\frac{d}{dt}(|F_\gamma|^2) = |F_\gamma|^2 2\frac{Re(\langle 2i\gamma^{2,1}, \overline{\gamma^{2,1}}\rangle)}{\langle \Omega^{3,0}, \gamma\rangle} = 8|F_\gamma|^2\frac{\langle Re(\gamma^{2,1}), Im(\gamma^{2,1})\rangle}{\langle \Omega^{3,0}, \gamma\rangle}.$$

Using the formula $\frac{d}{dt}v = \frac{\langle \Omega^{3,0},\gamma\rangle}{v}$ we conclude that

$$\frac{d^2v}{dt^2} = \frac{4}{v}\langle Re(\gamma^{2,1}), Im(\gamma^{2,1})\rangle.$$

Notice that properties of polarized Hodge structures imply that

$$\langle Re(\gamma^{2,1}), Im(\gamma^{2,1})\rangle \leq 0\,.$$

Recall that the gradient lines are projections of unparametrized straight lines (see Theorem 9.2.1). Then our computations imply the following statement.

Proposition 9.2.2. *The volume function v is concave in parameter t on the straight line $Im(\Omega_t^{3,0}) = Im(\Omega_0^{3,0}) + t\gamma$, where $\langle Im\,\Omega_0^{3,0}, \gamma\rangle = 0$.*

Remark 9.2.3. Notice that in the above computations we could replace γ by any vector in $\Gamma_{\mathbf{R}}$. Recall also that we can identify locally $\mathscr{L}$ with the real symplectic vector space $H^3(X,\mathbf{R})$ via the map $[\Omega^{3,0}] \mapsto Im\,[\Omega^{3,0}]$ which introduces the affine structure on $\mathscr{L}$. The real two-dimensional subspace spanned by $Im([\Omega_0^{3,0}])$ and γ is isotropic. Then the Proposition 9.2.2 can be reformulated in geometric terms as a statement that the function v gives rise to a (positive) concave homogeneous function of degree $+1$ on any real isotropic plane in $H^3(X,\mathbf{R})$.

The above considerations motivate the following definitions.

Definition 9.2.4. Let $\gamma \in \Gamma = H^3(X,\mathbf{Z})$. We call $\Omega_\gamma^{3,0} \in \mathscr{L}$ a γ-attractor point if $Im(\Omega_\gamma^{3,0}) = \gamma$.

Since a lift of the gradient trajectory of $|F_\gamma|^2$ is a straight line in $\mathscr{L}$, the critical points of F_γ can appear only in the limit $t \to \infty$. Hence the limiting point in $\mathscr{M}_X$ is the projection of a γ-attractor point. Thus we see that the projections of attractor points are local minima of the multivalued functions F_γ (in physics literature these projections are called attractor points).

Moreover it is easy to see that critical points of F_γ are all local minima and are either projection of attractor points or belong to the locus $F_\gamma^{-1}(0)$. The equation $F_\gamma = 0$ defines a complex hypersurface in the (universal cover) of the space $\mathscr{M}_X$. Points of this hypersurface are absolute minima of F_γ, and moreover they form a set of points where the function F_γ is not differentiable.

We expect that for a generic gradient line of F_γ on $\mathscr{M}_X$ there are three possibilities:

(1) The gradient line hits the projection of a γ-attractor point.
(2) The gradient line reaches in finite time a point in the boundary of the metric completion of $\mathscr{M}_X$ with respect to the Weil–Petersson metric. This point is called *conifold point*. We will assume that the conifold points form an analytic divisor in the above completion (which is expected to be a complex analytic space).
(3) The gradient line reaches in finite time a point in the locus $F_\gamma^{-1}(0)$.

Then for given γ the universal covering of $\mathscr{M}_X$ splits into a disjoint union of three open domains corresponding to these three possibilities and a closed subset of measure zero.

The divisor of conifold points has (roughly) the following structure which we explain in the framework of complex integrable systems. Consider a polarized integrable system with central charge endowed with conical structure, i.e. with a $\mathbf{C}^*$-action which rescales the central charge and preserves the discriminant. Taking the quotient by the $\mathbf{C}^*$-action we obtain a local model for the divisor of conifold points in the completion of $\mathscr{M}_X$. In particular we expect the typical singularity will be of the type A_1. In terms of X this means that we approach a point in the completion of $\mathscr{M}_X$ where X develops an ordinary double point with local equation $\sum_{1\le i\le 4} x_i^2 = 0$.

We can restate these three possibilities in the language of straight lines on $\mathscr{L}_X$. Let us fix γ and consider in the universal cover of $\mathscr{L}_X$ the ray (or interval) $Im(\Omega_t^{3,0}) = Im(\Omega_0^{3,0}) + t\gamma, t \in [0, t_0)$, where $\langle Im(\Omega_0^{3,0}), \gamma\rangle = 0$, and $t_0 \in (0, +\infty]$ is the maximal possible value of t for which the map $t \mapsto \Omega_t^{3,0}$ is well-defined.

The case (1) means that $t_0 = +\infty$. The restriction of the function v on the ray has strictly positive derivative and the limit of the derivative as $t \to +\infty$ is non-zero: $lim_{t\to+\infty} \frac{dv}{dt} > 0$.

In the case (2) we have $t_0 < +\infty$ and $lim_{t\to t_0} \frac{dv}{dt} = 0$.

We claim that in the case (3) that $t_0 < +\infty$ but the limit of the derivative of the function v is strictly negative as $t \to t_0$. Indeed, the picture in $\mathscr{M}_X$ means that there exists finite t_1 such that the derivative of v at t_1 is equal to zero. It is easy to see that although the gradient line of F_γ stops at such point, we can continue the corresponding ray in $\mathscr{L}$ to some $t > t_1$. In terms of the gradient trajectories this means that we consider another gradient trajectory of the function F_γ and move along it in the opposite direction (i.e. in the direction of increasing values of F_γ) for $t > t_1$. Therefore for $t > t_1$ the derivative $\frac{dv}{dt}$ becomes negative. By concavity of v we conclude that we cannot extend the ray indefinitely, hence $t_0 < +\infty$. We expect that in this case the image of the corresponding interval in $\mathscr{L}_X/\mathbf{R}_{>0}$ is everywhere dense. This property distinguishes the case (3) from the case (2) purely in terms of affine geometry of $\mathscr{L}_X$ (without use of function v).

9.3 Trees and Generalized Attractor Points

Let us discuss abstract attractor trees. Basically, it is the same as the tropical trees discussed previously. The difference is in the notion of attractor point.

Suppose that $\mathscr{L}$ is a smooth C^∞-manifold which admits an open covering $\mathscr{L} = \cup_{i\in I} U_i$ with transition functions belonging to the group $Aut(\Gamma, \langle\bullet,\bullet\rangle)$, where $\Gamma \simeq \mathbf{Z}^{2n}$ is a lattice endowed with the integer non-degenerate skew-symmetric form $\langle\bullet,\bullet\rangle$. We assume that each U_i endowed with the induced $\mathbf{Z}$-linear structure is isomorphic to an open cone in $\mathbf{R}^{2n}$. Notice that because we have a $\mathbf{Z}$-linear structure on $\mathscr{L}$ we can speak about integer points in $\mathscr{L}$. Also the conical structure implies that $\mathbf{R}_{>0}$ acts on $\mathscr{L}$.

Let us define a $(2n-1)$-dimensional manifold $\mathscr{L}'_{\mathbf{Z}}$ as the set of pairs (u, γ), where $u \in \mathscr{L}, \gamma \in T_u\mathscr{L} - \{0\}$ such that in the above $\mathbf{Z}$-linear local coordinates γ is integer and $\langle u, \gamma\rangle = 0$. This definition is similar to the definition of $M'_{\mathbf{Z}}$ from Sect. 3.1. We define the *attractor flow* on $\mathscr{L}'_{\mathbf{Z}}$ by the formula $\dot{u} = \gamma, \dot{\gamma} = 0$.

Definition 9.3.1. We define $\mathscr{L}'^{,attr}_{\mathbf{Z}} \subset \mathscr{L}'_{\mathbf{Z}}$ as a set of points (u, γ) such that the trajectory of the attractor flow starting at (u, γ) exists for all $t \in [0, +\infty)$.

In local coordinates the projection to $\mathscr{L}$ of such a trajectory can be written as $t \mapsto u_t := u + t\gamma$.

Definition 9.3.2. A generalized attractor point is a connected component of the interior $Int(\mathscr{L}'^{,attr}_{\mathbf{Z}}) \subset \mathscr{L}'_{\mathbf{Z}}$.

Below we provide some explanations.

With any $(u, \gamma) \in \mathscr{L}'^{,attr}_{\mathbf{Z}}$ we associate a map $f_{(u,\gamma)} : (0, +\infty) \to \mathscr{L}$ given by $t \mapsto t^{-1}u_t$.

There are two possibilities:

(a) The limit $lim_{t\to+\infty} f_{(u,\gamma)}(t)$ does exist. Then in local coordinates this limit is equal to γ. This condition is open in $\mathscr{L}'_{\mathbf{Z}}$. Generalized attractor points of this type can be identified with γ-attractor points from the previous subsection.
(b) The limit $lim_{t\to+\infty} f_{(u,\gamma)}(t)$ does not exist.

In the case (a) the limit is an integer point in $\mathscr{L}$. It is easy to see that for any integer point $u_0 \in \mathscr{L}$ the set of pairs $(u, \gamma) \in \mathscr{L}'^{,attr}_{\mathbf{Z}}$ such that $lim_{t\to+\infty} f_{(u,\gamma)}(t) = u_0$ is a non-empty open connected subset in $\mathscr{L}'_{\mathbf{Z}}$ (it is a star-shaped domain). Hence we conclude that integer points in $\mathscr{L}$ give generalized attractor points. The inclusion is not a bijection. The complement to the image corresponds to the interior of the domain described in case (b). The latter can be thought of as the set of integer points in the "boundary" of $\mathscr{L}$. Such "integer boundary points" do appear in practice. For example take $\mathscr{L} = \mathscr{L}_X$ and points (u, γ) where u corresponds to the point in the moduli space $\mathscr{M}_X$ close to the cusp and γ belongs to a Lagrangian sublattice in $H_3(X, \mathbf{Z})$ invariant under the monodromy. In the mirror dual picture such classes γ correspond to Chern classes of coherent sheaves on the dual Calabi–Yau with at most one-dimensional support ($D0$–$D2$ branes in the language of physics).

Assume that we are given an open subset $\mathscr{L}'^{,conif}_{\mathbf{Z}} \subset \mathscr{L}'_{\mathbf{Z}}$ which is preserved by the attractor flow for $t \geq 0$ and is disjoint from $\mathscr{L}'^{,attr}_{\mathbf{Z}}$. For example in the situation when $\mathscr{L} = \mathscr{L}_X$ described in the previous subsection we define $\mathscr{L}'^{,conif}_{\mathbf{Z}}$ as the interior of the set of points described in the case (2) there. As we mentioned in Sect. 9.2 this probably means that the projection of the corresponding trajectory of the attractor flow to $\mathscr{L}/\mathbf{R}_{>0}$ is not everywhere dense.

We will be talking about metrized rooted trees below. As in [33] those are trees with lengths assigned to edges. There are internal edges and tail edges. Internal edges have finite (positive) length and tail edges have possibly infinite length. Also, the root vertex is adjacent to exactly one edge.

Definition 9.3.3. A tropical tree in $\mathscr{L}$ is given by

a) A metrized rooted tree T with edges oriented toward tails.
b) A continuous map $\phi : T \to \mathscr{L}$, smooth outside of vertices, with the following properties:

outside of vertices the map $t \mapsto (\phi(t), \phi'(t))$ is a trajectory of the attractor flow on $\mathscr{L}'_{\mathbf{Z}}$;

at each internal vertex v the following balancing condition is satisfied:

$$\sum_{e \in v^{out}} \phi'(e) = \phi'(e^{in}(v)),$$

where v^{out} denote the set of outcoming from v edges and $e^{in}(v)$ is the only edge incoming to v (the derivative $\phi'(e)$ is constant along the edge e);

each tail edge the map ϕ is a trajectory of the attractor flow belonging either to $\mathscr{L}'^{,conif}_{\mathbf{Z}}$ or to $\mathscr{L}'^{,attr}_{\mathbf{Z}}$;

for any vertex v directions ϕ' of all edges in the set v^{out} are different and belong to an open half-space in a rank 2 symplectic sublattice in the tangent space $T_{\phi(v)}\mathscr{L}$;

Having a tropical tree we can (and sometime will) interpret its edges as trajectories in $\mathscr{L}'_{\mathbf{Z}}$.

Let $\phi : T \to \mathscr{L}$ be a rooted tropical tree in $\mathscr{L}$. Abusing the notation we will simply denote it by T. Suppose we are given a volume function $v : \mathscr{L} \to \mathbf{R}_{>0}$ which satisfies the Proposition 9.2.2 (i.e. it is concave along edges). Then the following result holds.

Proposition 9.3.4. *If the volume function increases along tail edges of T then it increases along every edge (we consider orientation of the tree toward tails).*

Let us define the function $F : \mathscr{L}'_{\mathbf{Z}} \to \mathbf{R}$ as the derivative along the attractor flow of the pull-back of v under the natural projection. This function is an analog of the multivalued function F_γ from Sect. 9.2. The function F is invariant under the $\mathbf{R}_{>0}$-action on $\mathscr{L}'_{\mathbf{Z}}$. Proposition 9.3.4 means that edges of any tropical tree belong to the domain $F > 0$. Moreover, the concavity of the function v on edges and the balancing condition imply that the value of F at the root vertex is strictly bigger than the sum of limiting values of F on tail edges (cf. [11]). Such limiting values are strictly positive for edges of the tree hitting attractor points and equal to zero for conifold and "integer boundary points".

Thus we see that the function F imposes the "energy-like" restrictions on tropical trees.

Our considerations with the function F motivates the following *Finiteness Assumption* (cf. assumptions in Sect. 3.2):

For each point $(u, \gamma) \in \mathscr{L}'_{\mathbf{Z}}$ outside of a set of measure zero the number of tropical trees rooted at (u, γ) is finite.

By analogy with Proposition 3.2.6 one can design a procedure which as we expect produces the WCS on $\mathscr{L}$ starting with "initial data" given by integer numbers $\Omega(u,\gamma)$ assigned to generalized attractor points and irreducible components of the divisor of conifold points. We assign arbitrary integers to generalized attractor points and assign integers equal to 1 to conifold points with A_1 singularities. Similarly to Sect. 3.2 this WCS can be understood as an integer-valued function on $\mathscr{L}'_{\mathbf{Z}}$ which discontinuous at polyhedral walls and satisfies the Support Property and WCF from [30].

The procedure is similar to those described in Sects. 3.2 and 4.6.

By the Finiteness Assumption there are finitely many tropical trees in $\mathscr{L}$ rooted at (u,γ). The union of all such trees is a finite directed graph $G := G(u,\gamma)$ without oriented cycles because of monotonicity of the function F. Each inner vertex of G belongs to a symplectic plane. Then we can start with a generalized attractor point or conifold point with A_1 singularity which are tails of T and move backward to (u,γ). We will use the wall-crossing formulas. Recall that they have the following form

$$\overleftarrow{\prod} T_{\gamma^{out}}^{\Omega(P,\gamma_{out})} = \overrightarrow{\prod} T_{\gamma^{in}}^{\Omega(P,\gamma_{in})},$$

where P is a vertex of G and $T_{\gamma^{out}}^{\Omega(P,\gamma_{out})}$ are symplectomorphisms of the two-dimensional symplectic subspace in the tangent space at P corresponding to the edges of G outcoming from P (similarly for incoming edges). Since we know by induction the numbers $\Omega(P,\gamma_{out})$ for outcoming edges, we can calculate $\Omega(P,\gamma_{in})$ from the wall-crossing formula and proceed further toward P. Finally, it gives us the desired number $\Omega(u,\gamma)$.

Remark 9.3.5. As we already mentioned in Remark 3.2.7, there is no guarantee that the result of the application of the above procedure is indeed a WCS. At some strata of codimension 2 the cocycle condition can fail. The geometric structure of walls on $\mathscr{L}$ is very involved, and we do not understand it completely. At the moment we have the following (maybe too optimistic) picture: it is sufficient (and maybe even necessary) to put the constraint $\Omega = 1$ at conifold points (assuming that all conifold points have A_1 singularities). The integer values of Ω at all generalized attractor points can be chosen arbitrarily. Then we obtain a WCS.

9.4 Remarks on the Support of DT-Invariants

Recall that in the definition of WCS we required an existence of a strict convex cone. In the case of Calabi–Yau threefolds this property is called Support Property (see [30]), since it gives a bound on the support of the function $\Omega(u,\gamma)$ (numerical DT-invariants). Heuristic arguments in favor of that given in the Remark 1 [30] were based on the following simple geometric fact. Let η be a closed 3-form on X. Then there exist $C := C_\eta > 0$ such that for any SLAG L we have $|\int_L \eta| \leq C|\int_L \Omega_X^{3,0}|$. Equivalently, we have a constraint on the homology class $\gamma = [L] \in H_3(X,\mathbf{Z})$.

The constant C depends in an essential way on the metric on X. In this subsection we propose an alternative approach to the Support Property based entirely on the affine geometry of $\mathscr{L} = \mathscr{L}_X$. Namely, the volume function v gives us the following constraint on the pair (u, γ) such that $\Omega(u, \gamma) \neq 0$: the derivative of v at u in the direction γ is positive. Recall that this property was deduced from three facts:

(i) the function v is a strictly positive function on $\mathscr{L}$ of homogeneity degree 1 with respect to the $\mathbf{R}_{>0}$ action;
(ii) the function v is concave on germs of two-dimensional isotropic subspaces in $\mathscr{L}$;
(iii) the derivative of v along a trajectory in $\mathscr{L}_{\mathbf{Z}}^{\prime,conif}$ is positive.

We claim that there are infinitely many perturbations of v which still obey (i)–(iii). Namely, let us consider any smooth function δv on $\mathscr{L}$ which is homogeneous of degree 1 and such that $Supp(\delta v)/\mathbf{R}_{>0} \subset \mathscr{L}/\mathbf{R}_{>0}$ is *compact*. Consider the function $v_\varepsilon := v + \varepsilon \delta v$. The compactness of $Supp(\delta v)/\mathbf{R}_{>0}$ implies that the properties (i) and (ii) are satisfied for sufficiently small ε. The property (iii) follows from (ii). More generally, any function v' which satisfies (i) and (ii) and coincides with v outside of a compact (modulo the action of $\mathbf{R}_{>0}$) satisfies also (iii). Here is a reason for that: the condition of monotonicity of such a function is sufficient to check on parts of the trajectories which are close to conifold points, where the function coincides with v. For any function v' satisfying (i)–(iii) let us consider the set $C_{v'} \subset tot(T\mathscr{L})$ consisting of pairs $(u, \dot{u})$ such that $\langle u, \dot{u}\rangle = 0$ and $dv_{|T_u\mathscr{L}}(\dot{u}) \geq 0$. Let us define C_{univ} as the intersection of $C_{v'}$ over all v' satisfying (i)–(iii). It is easy to see that for any $u \in \mathscr{L}$ the intersection $C_{univ} \cap T_u\mathscr{L}$ is a *strict closed convex cone* in the hyperplane $u^\perp := Ker\,\langle u, \bullet\rangle$. The above inductive construction implies that $Supp(\Omega(u, \gamma)) \subset C_{univ}$.

10 Analog of WCS in Mirror Symmetry

10.1 Pair of Lattices, Volume Preserving Transformations and WCS in a Vector Space

In the case of SYZ picture of Mirror Symmetry the construction of mirror dual involves transformations which locally preserve the volume form rather than a Poisson structure. In this case $\mathfrak{g}$ is the Lie algebra of divergence-free vector fields on $Hom(\Gamma, \mathbf{C}^*)$ and there is no distinguished skew-symmetric form on Γ. In notation of Sect. 6.2 the lattice Γ is Γ_b, the first homology group of a fiber of a *real* integrable system at a given point $b \in B^0$. Differently from the case of symplectomorphisms when the dimension of the graded component is equal to 1 (see Sect. 2.3, Example (4)), we now have $dim\,\mathfrak{g}_\gamma = n - 1$, where $n = rk\,\Gamma$ for $\gamma \neq 0$. Explicitly, the Lie algebra of vector fields on the algebraic torus $Hom(\Gamma, \mathbf{C}^*)$ is spanned by elements $x^\gamma \partial_\mu$ where $\gamma \in \Gamma, \mu \in \Gamma^\vee$ satisfying the linear relations

$$x^\gamma \partial_{\mu_1} + x^\gamma \partial_{\mu_2} = x^\gamma \partial_{\mu_1+\mu_2}\ .$$

Derivation ∂_μ is a constant vector field in logarithmic coordinates. The commutator rule is given by

$$[x^{\gamma_1}\partial_{\mu_1}, x^{\gamma_2}\partial_{\mu_2}] = x^{\gamma_1+\gamma_2}\big((\mu_1,\gamma_2)\partial_{\mu_2} - (\mu_2,\gamma_1)\partial_{\mu_1}\big)\ .$$

The subalgebra $\mathfrak{g}$ of divergence-free vector fields is spanned by elements $x^\gamma\partial_\mu$ with $(\mu,\gamma)=0$. It is obviously graded by the lattice Γ. Similarly to the symplectic (and also Poisson) case (see the beginning of Sect. 3.1), the graded complement to $\mathfrak{g}_0$ is a Lie subalgebra $\mathfrak{g}' = \oplus_{\gamma\neq 0}\mathfrak{g}_\gamma$ in $\mathfrak{g}$ (notice that an analogous property *does not* hold for the Lie algebra of all vector fields).

One can generalize the above considerations to the following situation. Suppose we are given two lattices Γ_1, Γ_2 and an integer pairing between them $(\bullet,\bullet) : \Gamma_2 \otimes \Gamma_1 \to \mathbf{Z}$. We do not assume that the pairing is non-degenerate. We denote by $\Gamma_{1,0} \subset \Gamma_1$ and $\Gamma_{2,0} \subset \Gamma_2$ the corresponding kernels of the pairing.

Then we consider the Lie algebra $\mathfrak{g} := \mathfrak{g}_{\Gamma_1,\Gamma_2,(\bullet,\bullet)}$ spanned by elements $x^\gamma\partial_\mu$ where $\gamma\in\Gamma_1$ and $\mu\in\Gamma_2$ such that $(\mu,\gamma)=0$, satisfying the same relations as above. It contains the Lie subalgebra

$$\mathfrak{g}' := \oplus_{\gamma\in\Gamma_1-\Gamma_{1,0}}\mathfrak{g}_\gamma .$$

The previous special case corresponds to $\Gamma_1=\Gamma$, $\Gamma_2=\Gamma^\vee$. In general, $\mathfrak{g}$ can be thought as the Lie algebra of divergence-free vector fields on a torus, preserving a collection of coordinates and commuting with a subtorus action. Explicitly, if we omit the condition $(\mu,\gamma)=0$ of being divergence-free, in some coordinates $(x_1,\dots,x_{a+b+c})$ for $a,b,c\in\mathbf{Z}_{\geq 0}$ we get vector fields of the form

$$\prod_{i=1}^{a+b} x_i^{k_i}\cdot x_j\,\partial/\partial x_j,\ \ (k_i)_{1\leq i\leq a+b}\in\mathbf{Z}^{a+b},\ a+1\leq j\leq a+b+c\,.$$

From the point of view of SYZ picture of Mirror Symmetry we have a real integrable system $X\to B$ with the dimension of the total space X equal to $2b$. Moreover we have chosen a Lagrangian zero section as well as c other Lagrangian sections (more precisely, a homomorphism from $\mathbf{Z}^c$ to the abelian group of Lagrangian sections). Also we assume that there is an a-dimensional space of deformations of the above structure which is an a-dimensional vector subspace in $H^2(X,\mathbf{R})$ defined over $\mathbf{Q}$. The mirror dual $X^\vee$ is the complex manifold of complex dimension b depending on a holomorphic parameters and carrying c line bundles. The Mirror Symmetry preserves the parameter b and exchanges a and c.

Now we are ready to describe the analog of a WCS for Lie algebra $\mathfrak{g}'$.

The main difference with the formalism from 2.1 is that now walls are hyperplanes in $\Gamma^*_{2,\mathbf{R}} := \Gamma_2^\vee\otimes\mathbf{R}$ (and not in the dual space to the grading lattice Γ_1). We define a wall as a hyperplane in $\Gamma^*_{2,\mathbf{R}}$ given by $\mu^\perp$, where $\mu\in\Gamma_2-\Gamma_{2,0}$ (one may assume that μ is primitive). With any wall $H\subset\Gamma^*_{2,\mathbf{R}}$ we associate a graded Lie subalgebra

$$\mathfrak{g}_H := \bigoplus_{\gamma\in\Gamma_1} \mathfrak{g}_{H,\gamma} \subset \mathfrak{g}'$$

spanned by $x^\gamma \partial_\mu$ such that $(\mu,\gamma) = 0$ and $\gamma \in \Gamma_1 - \Gamma_{1,0}$. As in the Poisson case, this Lie algebra is abelian. It is convenient to associate with any γ as above a nonzero constant vector field on the hyperplane H equal to $\iota(\gamma) := (\bullet,\gamma) \in \Gamma_2^\vee \subset \Gamma_{2,\mathbf{R}}^*$. In SYZ picture the trajectories of this vector field are (possible) parts of tropical trees corresponding to analytic discs with the boundary on a small Lagrangian torus, the fiber of SYZ fibration.

Also with any $\mathbf{Q}$-vector subspace $V \subset \Gamma_{2,\mathbf{R}}^*$ which is the intersection of two walls we associate a graded Lie algebra $\mathfrak{g}_V$ (which is *not* a subalgebra of $\mathfrak{g}'$) such as follows. As a Γ_1-graded vector space $\mathfrak{g}_V$ will be equal to the direct sum $\oplus_{H\supset V}\mathfrak{g}_H$ over all walls containing V. The Lie bracket on $\mathfrak{g}_V$ is defined as follows. Let $(x^\gamma\partial_\mu)_H$ where $\gamma \in \Gamma_1 - \Gamma_{1,0}, \mu \in \Gamma_2 - \Gamma_{2,0}$ denotes the element $x^\gamma\partial_\mu \in \mathfrak{g}_H$ considered as an element of $\mathfrak{g}_H \subset \mathfrak{g}_V$, where $H = \mu^\perp$ is a wall containing V. Then we define the Lie bracket by the formula:

$$[(x^{\gamma_1}\partial_{\mu_1})_{H_1}, (x^{\gamma_2}\partial_{\mu_2})_{H_2}] = (x^{\gamma_3}\partial_{\mu_3})_{H_3},$$

in case if $H_i = \mu_i^\perp, i = 1,2,3, \gamma_3 = \gamma_1+\gamma_2, \mu_3 = (\mu_1,\gamma_2)\mu_2 - (\mu_2,\gamma_1)\mu_1$ and $\mu_3 \notin \Gamma_{2,0}$. Otherwise, i.e. if $\mu_3 \in \Gamma_{2,0}$ (and as one can easily see $\mu_3 = 0$), we define the commutator to be equal to zero.

As in Sect. 2.1.3, we consider the pronilpotent case by choosing a strict convex cone $C \subset \Gamma_1 \otimes \mathbf{R}$, and working with $\mathfrak{g}_C := \prod_{\gamma\in\Gamma\cap C-\Gamma_{1,0}} \mathfrak{g}_\gamma$. Then for a given functional $\phi : \Gamma_1 \to \mathbf{Z}$ which is nonnegative and proper on the closure of C, we consider finite-dimensional nilpotent quotients

$$\mathfrak{g}_{C,\phi}^{(k)} = \oplus_{\gamma\in\Gamma_1-\Gamma_{1,0}|\phi(\gamma)\le k}\mathfrak{g}_{C,\gamma} = \mathfrak{g}_C/m_{C,\phi}^{(k)}$$

where $m_{C,\phi}^{(k)} := \prod_{\gamma\in\Gamma_1-\Gamma_{1,0}:\phi(\gamma)>k}\mathfrak{g}_{C,\gamma}$ is an ideal in $\mathfrak{g}_C$.

Similarly we define the Lie algebras $\mathfrak{g}_{H,C,\phi}^{(k)}$. and $\mathfrak{g}_{V,C,\phi}^{(k)}$.

Let us fix finitely many walls $H_i, i \in I$. We define the set $WCS_k(\{H_i\}_{i\in I}, C, \phi)$ of wall-crossing structures for $\mathfrak{g}_{C,\phi}^{(k)}$ which are supported on the union $\cup_{i\in I} H_i$ in the following way. First we observe that the walls $H_i, i \in I$ give rise to the natural stratification of $\Gamma_{2,\mathbf{R}}^*$. Then an element $WCS_k(\{H_i\}_{i\in I}, C, \phi)$ is a map which associates an element g_τ of the group $\exp(\mathfrak{g}_{H_i,C,\phi}^{(k)}), i \in I$, where $\tau \subset H_i$ is a co-oriented stratum of codimension one in $\Gamma_{2,\mathbf{R}}^*$ (notice that τ is an open subset of H_i).

The only condition on this map says that for any generic closed loop $f : \mathbf{R}/\mathbf{Z} \to \Gamma_{2,\mathbf{R}}^*$ surrounding a codimension two stratum $\rho \subset V, codim_{\mathbf{R}} V = 2$, the product of images of the corresponding elements $exp(g_{\tau_{t_i}})$ in $exp(\mathfrak{g}_{V,C,\phi}^{(k)})$ over the finite sequence of intersection points $f(t_i)$ of the loop with walls H_i is equal to the identity.

Now we take the inductive limit of the sets $WCS_k(\{H_i\}_{i\in I}, C, \phi)$ over all finite collections $\{H_i\}_{i\in I}$ of walls and after that we take the projective limit over k. The resulting set $WCS_{\mathfrak{g},C}$ is our analog of WCS relevant to the Mirror Symmetry. Analogously to Sect. 2 we can generalize our considerations and define $WCS_{\mathfrak{g},C}$ as a sheaf of sets on $\Gamma^*_{2,\mathbf{R}}$ and generalize even further assuming that Γ_1 and Γ_2 are local systems of lattices on a topological space, say, M. Instead of the central charge we now have a morphism of sheaves of abelian groups $\Gamma_2 \to \underline{Cont}_M$. It is not clear a priori why the sheaf $WCS_{\mathfrak{g},C}$ is non-trivial, and how to introduce "coordinates" on its stalk at zero. Although the structure we have defined is different from WCS discussed in Sect. 2, we will abuse the language and still call it the wall-crossing structure.

Next we would like to discuss an analog of the initial data.

Proposition 10.1.1. *Let $V = H_1 \cap H_2, H_i = \mu_i^{\perp}, i = 1,2$ be an intersection of two walls and $\gamma \in \Gamma_1 - \Gamma_{1,0}$ be a vector such that $(\mu_i, \gamma) = 0, i = 1,2$. Then the natural map*

$$\bigoplus_{H:V\subset H} \mathfrak{g}_{H,\gamma} \to \left(\mathfrak{g}_V/[\mathfrak{g}_V, \mathfrak{g}_V]\right)_\gamma$$

is injective.

Proof. Follows immediately from the formula for the bracket. ■

Then we can define the analog of the initial data in the following two ways depending on a choice of a sign. For any $\gamma \in \Gamma_1 - \Gamma_{1,0}$ and any $H = \mu^{\perp}$ such that $(\mu, \gamma) = 0$ we will construct elements $a_H^{(k),\pm}(\gamma) \in \mathfrak{g}^{(k)}_{H,C,\phi,\gamma}$. Namely, it follows from the above Proposition that for any stratum $\tau \subset H$ as above such $\iota(\gamma)$ belongs to the closure $\overline{\tau}$ of the stratum τ, the γ-component of $log(g_\tau)$ does not depend on τ. We denote it by $a_H^{(k),+}(\gamma)$. Similarly we define $a_H^{(k),-}(\gamma)$ using the strata τ such that $-\gamma \in \overline{\tau}$. Next we define elements $a_H^{(k),\pm} \in \mathfrak{g}^{(k)}_{H,C,\phi}$ as $\sum_\gamma a_H^{(k),\pm}(\gamma)$. After that we define elements $a^{(k),\pm} \in \oplus_H \mathfrak{g}^{(k)}_{H,C,\phi}$ where the sum is taken over the set of walls (notice that the sum is finite).

Conjecture 10.1.2. The set $WCS_k(C, \phi)$ is identified via passing to initial data $a^{(k),+}$ with the set $\oplus_H \mathfrak{g}^{(k)}_{H,C,\phi}$, where the sum is taken over all walls. Similar statement is true for $a^{(k),-}$.

Finally, taking the projective limit over k we define the elements $a^{\pm}$. These elements play a role of the initial data for the sheaf $WCS_{\mathfrak{g},C}$ in a vector space.

Remark 10.1.3. Notice that for a fixed V there is a homomorphism $\mathfrak{g}_V \to \mathfrak{g}'$ given by the natural inclusions $\mathfrak{g}_H \to \mathfrak{g}'$ for all $H \supset V$. Hence for a small open subset U in $\Gamma_1 \otimes \mathbf{R}$ we have a cocycle with values in $exp(\mathfrak{g}_C)$. In plain terms, it is given by an element $g_{x_1,x_2} \in exp(\mathfrak{g}_C)$ defined for two points $x_1, x_2 \in U$ which do not belong to any wall. Cocycle condition means that $g_{x_1,x_2} g_{x_2,x_3} = g_{x_1,x_3}$. We see that we have a picture similar to the one from Sect. 2. Hence we can use the above transformations in order to glue a Calabi–Yau manifold (possibly over a non-archimedean field) from open coordinate charts.

In the passage from *WCS* to the glued manifold we lose some data. This point is clear when we look at the initial data. Indeed if assume the above conjecture, we see that the direct sum of all $\mathfrak{g}_H$ is "bigger" than $\mathfrak{g}'$. One can speculate that the whole *WCS* contains the information sufficient for reconstruction of both mirror dual Calabi–Yau manifolds. The initial data can be thought of as association of a rational number to any triple (γ, μ, k), where $\gamma \in \Gamma_1 - \Gamma_{1,0}, \mu \in \Gamma_2 - \Gamma_{2,0}, k \in \mathbf{Z}_{\geq 1}$ and γ, μ are primitive vectors such that $(\gamma, \mu) = 0$. The rational number is the coefficient of $x^{k\gamma}\partial_\mu \in \mathfrak{g}_H, H = \mu^\perp$ in the initial data. Notice that the above conditions are symmetric with respect to the exchange of Γ_1 and Γ_2.

Remark 10.1.4. Notice that the Lie algebra $\mathfrak{g}_{\Gamma_1,\Gamma_2,(\bullet,\bullet)}$ does not change if we replace Γ_2 by a sublattice of finite index. Then at first sight it looks like that our WCS depends on $\Gamma_2 \otimes \mathbf{Q}$ only. But one can recover a finer "integer" structure of the WCS. At the level of the corresponding pronilpotent groups we consider subgroups generated by the elements $T_{\gamma,\mu} := exp\left(\sum_{n\geq 1} \frac{x^{n\gamma}\partial_\mu}{n}\right)$. In the case $\Gamma_1 = \Gamma_2$ and $(\bullet,\bullet) = \langle\bullet,\bullet\rangle$ being skew-symmetric, as in the Poisson case considered in Sect. 4, we have similar transformations $T_\gamma := T_{\gamma,\gamma} = exp\,(\{Li_2(x^\gamma), \bullet\})$ (cf. Remark 2.3.1).

10.2 Pair of Local Systems of Lattices from Non-archimedean Point of View

Recall the discussion in Sect. 6.2 of the geometry of the base of the real integrable system which appears in SYZ picture of Mirror Symmetry.

Here we would like to recall the origin of the integrable system following [31, 35]. It can be approached either in the framework of Gromov–Hausdorff collapse or using the language of non-archimedean geometry of Berkovich. In the former approach we have a family $X_t, t \to 0$ of maximally degenerate polarized complex Calabi–Yau manifolds. It was conjectured in [35] that for sufficiently small t the manifold X_t contains an open subset X_t' which is in Gromov–Hausdorff metric close to the total space of the real integrable system $\pi_t : X_t' \to B^0$ over an open smooth manifold $B^0 \subset B$ of some metric space B. The latter is the Gromov–Hausdorff limit of $X_t, t \to 0$ (see [35] for details). The restriction of the metric to B^0 is a smooth Riemannian metric, which locally given by the matrix of second derivatives of a convex function Φ on B^0 which satisfies the real Monge–Ampère equation $det(\partial^2\Phi/\partial x_i \partial x_j) = const.$

For any point $b \in B^0$ we can define the lattice $\Gamma_{1,b} := H_2(X_t, \pi_t^{-1}(b), \mathbf{Z})$ (the latter stabilizes as $t \to 0$ along a ray). This family of lattice gives rise to a local system $\underline{\Gamma}_1 \to B^0$. It can be extended to a local system on $S_t^1 \times B^0$.

It was conjectured in [35] (and hence was assumed in Sect. 6.2) that $codim(B - B^0) \geq 2$. Typically B is a topological manifold homeomorphic to a sphere, complex projective space or a torus.

The approach via non-archimedean geometry gives rise to the non-archimedean integrable system $\pi : \mathcal{X}^{an} \to B$ (see [31]). Here $\mathcal{X}^{an}$ is the compact polarized analytic manifold (in the framework of Berkovich theory) over a non-archimedean field K (typically $K = \mathbf{C}((t))$) corresponding to the collapsing family X_t, and B is a PL space called skeleton of $\mathcal{X}^{an}$.[2] Furthermore, $n := dim_{\mathbf{R}} B = dim_K \mathcal{X}^{an}$, and the projection π locally over B^0 looks as the map $(K^*)^n \to \mathbf{R}^n$ given by $(z_1, \dots, z_n) \mapsto (log|z_1|, \dots, log|z_n|)$. Conjecturally the skeleton coincides with the Gromov–Hausdorff limit, hence the same notation. Thus B carries a (singular) $\mathbf{Z}$-affine structure, which is non-singular on the open smooth manifold B^0. For the "discriminant locus" $B^{sing} = B - B^0$ we have (conjecturally) the condition $dim(B^{sing}) \leq n - 2$.

We are going to assume that the singularity of the singular integral affine structure on B is of the $A_k, k \geq 1$ type at B^{sing}. Here the discriminant B^{sing} is a closed subset of B, which contains an open dense topological submanifold B_2^{sing} such that $dim(B_2^{sing}) = n - 2$ and $dim(B^{sing} - B_2^{sing}) \leq n - 3$ and B is a PL manifold near any point of B_2^{sing}. Furthermore B_2^{sing} locally looks as a topological submanifold in $\mathbf{R}^n$ given in the standard coordinates $(x_1, \dots, x_n)$ by the equations

$$x_1 = f(x_3, \dots, x_n), x_2 = 0,$$

where $f(x_3, \dots, x_n)$ is a continuous function.

The integral affine structure on $B - B_2^{sing}$ in this local model coincides with the standard one on the open set U_+ which is the complement to the closed set $x_1 \geq f(x_3, \dots, x_n), x_2 = 0$. On the open subset U_- which is the complement to the closed subset $x_1 \leq f(x_3, \dots, x_n), x_2 = 0$ the integral affine structure is the standard one in the coordinates $x_1' = x_1 + k \cdot max\{0, x_2\}, x_2' = x_2, \dots, x_n' = x_n$.

One can see that B_2^{sing} belongs to a canonical germ of a hypersurface which is (in the affine structure) an integer $(n-1)$-dimensional hyperplane (outside of B^{sing}) endowed with a locally constant integer vector field. In the local picture the hyperplane is given by the equation $x_2 = 0$, and the vector field is given by $k \cdot sign(f(x_3, \dots, x_n) - x_1)\partial/\partial x_1$.

Recall that the approach with collapse gives a local system of lattices $\underline{\Gamma}_1 \to B^0$. In the non-archimedean picture we have $\underline{\Gamma}_{1,b} = H_{2,Betti}(\mathcal{X}^{an}, \pi^{-1}(b), \mathbf{Z}), b \in B^0$, where $H_{i,Betti}$ denotes properly defined Betti homology of the analytic space $\mathcal{X}^{an}$ over the field $K = \mathbf{C}((t))$.

There is a natural projection $p : \underline{\Gamma}_1 \to T^{\mathbf{Z}}$, where $T^{\mathbf{Z}} := T^{\mathbf{Z}}_{B^0} \subset T_{B^0}$ is the locally covariant lattice which defines the $\mathbf{Z}$-affine structure on B^0. Denoting $\underline{\Gamma}_{1,0} = Ker(p)$ we obtain an exact sequence of lattices

$$0 \to \underline{\Gamma}_{1,0} \to \underline{\Gamma}_1 \to T^{\mathbf{Z}}.$$

[2] In a recent paper [41] the notion of the skeleton was generalized to all K, including the case of mixed and positive characteristic.

In what follows we will assume that the local system $\underline{\Gamma}_{1,0}$ is trivial (this condition is automatically satisfied in most of examples).

Remark 10.2.1. The above geometry (including the integrable systems $\pi_t : X_t' \to B^0$) arises also in the situation when in the non-archimedean integrable system $\pi : \mathscr{X}^{an} \to B$ the total space $\mathscr{X}^{an}$ is non-compact but π is proper. In terms of Gromov–Hausdorff collapse one can think of a family $X_t, t \to 0$ of non-compact complex manifolds endowed with complete Kähler metrics. The limiting metric on B^0 is given by the second derivatives of a convex function but this time not necessarily obeying the real Monge–Ampère equation.

Remark 10.2.2. Let us assume that in the previous Remark the Kähler forms ω_t on X_t have homology classes $(log|t|)^{-1}\beta$, where β belong to the image of $H^2(X_t, \mathbf{Z})$ in $H^2(X_t, \mathbf{R})$ (this image does not depend on t when t is sufficiently small. Let us assume that for any $b \in B^0$ the homomorphism $H_1(\pi_t^{-1}(b), \mathbf{Z}) \to H_1(X_t, \mathbf{Z})$ is equal to zero (e.g. it is sufficient to assume that $H_1(X_t, \mathbf{Z}) = 0$). Then we can define a local system $\underline{\Gamma}_1 \to B^0$ in the following way. Its fiber over $b \in B^0$ is the set of pairs (γ, v), where $\gamma \in T_b^{\mathbf{Z}} = H_1(\pi_t^{-1}(b), \mathbf{Z}) \simeq \mathbf{Z}^n, n = dim_{\mathbf{C}} X_t = dim_{\mathbf{R}} B^0$, and $v \in \mathbf{R}$ is an element of $\mathbf{Z}$-torsor $\mathbf{Z} + lim_{t \to 0} \int_{\delta_t} (log|t|)^{-1}\omega_t$. Here δ_t is a 2-chain in X_t with the boundary in $\pi_t^{-1}(b)$ such that the boundary $\partial\delta_t$ is C^1-close to a closed geodesic in the torus $\pi_t^{-1}(b)$ representing homology class γ. The existence of the limit follows from the condition that ω_t is close to a semiflat metric, for $|t| \ll 1$.

Then we have a short exact sequence

$$0 \to \underline{\Gamma}_{1,0} = \underline{\mathbf{Z}}_{B^0} \to \underline{\Gamma}_1 \to T^{\mathbf{Z}} \to 0.$$

Now we discuss the origin of the local system $\underline{\Gamma}_2$. Let us first assume that the non-archimedean field K carries a discrete valuation $val(K^*) = \mathbf{Z} \subset \mathbf{R}$. In this case the base B^0 carries a sheaf of affine functions with *integer coefficients*. Then we define $\underline{\Gamma}_2$ to be this sheaf. We have a short exact sequence

$$0 \to \underline{\Gamma}_{2,0} = \underline{\mathbf{Z}}_{B^0} \to \underline{\Gamma}_2 \to (T^{\mathbf{Z}})^* \to 0.$$

Then we have the natural pairing $\underline{\Gamma}_1 \otimes \underline{\Gamma}_2 \to T^{\mathbf{Z}} \otimes (T^{\mathbf{Z}})^* \to \underline{\mathbf{Z}}_{B^0}$. In general one should consider Calabi–Yau manifolds over the field $\mathbf{C}((t_1, \ldots, t_m)), m = rk\, \underline{\Gamma}_{2,0}$. This can be thought of as a family of n-dimensional bases B^0 endowed with $\mathbf{Z}$-affine structure and parametrized (locally) by a domain in $\Gamma_{2,0}^{\vee} \otimes \mathbf{R}$. The total space of this family of bases can be identified locally with a domain in $\Gamma_2^{\vee} \otimes \mathbf{R} = \Gamma_{2,\mathbf{R}}^*$ (cf. Sect. 4.6). Therefore we can speak about the WCS on the total space of this family.

Recall that according to the general philosophy recalled in Sect. 6.2 in order to construct the mirror dual to the Fukaya category of the Calabi–Yau manifold near the cusp, we should count holomorphic discs with boundaries on fibers of the SYZ fibration. From the non-archimedean point of view such discs become tropical trees in n-dimensional bases B^0 depending on parameters in $\Gamma_{2,0}^{\vee} \otimes \mathbf{R}$. In the next subsection we are going to discuss such trees for a fixed value of the parameter in $\Gamma_{2,0}^{\vee} \otimes \mathbf{R}$.

10.3 Tropical Trees, Finiteness Assumption and WCS

We keep the notation of the previous subsection. In particular B denotes the base of a non-archimedean integrable system. It is a PL space. We do not assume that the valuation on the non-archimedean field K is integer. This means that we fix a **Z**-affine structure on B^0 corresponding to an arbitrary (not necessarily integer) point in $\Gamma_{2,0}^{\vee} \otimes \mathbf{R}$. In what follows we will not utilize $\underline{\Gamma}_2$. The reader should keep in mind that our objects depend on parameters from $\Gamma_{2,0}^{\vee} \otimes \mathbf{R}$.

Definition 10.3.1. A tropical tree in B is an oriented toward the root metrized tree T (see Sect. 3.2) endowed with a continuous map $f : T - \{Tail\, vertices\} \to B^0$ together with a continuous lift $f' : T - \{Vertices\} \to tot(\underline{\Gamma}_1) - tot(\underline{\Gamma}_{1,0})$ such that each edge lifts to a piece of the trajectory of the attractor flow

$$\dot{b} = \iota(\gamma), \dot{\gamma} = 0, (b, \gamma) \in tot(\underline{\Gamma}_1) - tot(\underline{\Gamma}_{1,0}) \subset tot(\underline{\Gamma}_1 \otimes \mathbf{R}).$$

We assume that at each internal vertex v we have the balancing condition $\sum_i \gamma^{out} = \gamma^{in}$ (cf. Definition 3.2.1), and all γ_i^{out} are pairwise distinct and there are i_1, i_2 such that $\gamma_{i_1}^{out}$ is not parallel to $\gamma_{i_2}^{out}$.

Furthermore we assume that the tail vertices belong to B_2^{sing} and the germs of edges near tail vertices belong to the above-described canonical hypersurface and the speeds of the f'-lifts of the tail edges are proportional (with minus sign) to the canonical locally constant vector field described above.

Next we are going to discuss the analogs of the Finiteness Assumption (cf. Sect. 9.3) as well as the existence of strict convex cones (cf. Conjecture 4.6.5). The idea is to use "tropical metrics" on B^0 which are non-singular as well as limits of such, which can also contain δ-functions supported on "tropical effective divisors". We warn the reader that the conditions discussed below are still not sufficient for the finiteness of the number of trees (which is a tropical analog of Gromov compactness for pseudoholomorphic curves). Our conditions guarantee the existence of strict convex cones, boundedness of lengths and finiteness of the number of tails of tropical trees with the given generic root and the velocity of the root edge. One has to put some extra constraints on the behavior of the affine structure or the "tropical metric" at singularities of codimension ≥ 3 in order to achieve the finiteness. Hopefully it can be done. We expect that such conditions are implicit in [24, 25], where the procedure for construction of the mirror dual Calabi–Yau gives a WCS in our sense.

Recall that in complex geometry it is natural to consider non-negative $(1,1)$-currents as limits of sequences of Kähler metrics on a given manifold. There is an analog of this notion in the non-archimedean geometry. More precisely, there a sheaf of monoids $\mathscr{P}sh \subset \underline{Cont}$ on $\mathscr{X}^{an}$ of continuous plurisubharmonic functions (see [5]). The sheaf of abelian groups $\mathscr{P}h$ is defined as $\mathscr{P}sh \cap (-\mathscr{P}sh)$. The quotient $\mathscr{P}sh/\mathscr{P}h$ is by definition the sheaf of non-negative $(1,1)$-currents. One can define similar sheaves on B in the following way. For example, a continuous of plurisubharmonic function on an open subset of B is a continuous function on this subset such that its pull-back to $\mathcal{X}^{an}$ is a plurisubharmonic functions. A germ of a non-negative $(1,1)$-current at a point $b \in B^0 \subset B$ is the same as a germ of a convex (in the affine structure) function modulo a germ of affine function.

Having a non-negative $(1,1)$-current ϕ on B and a tropical tree T on B we can define the integral $\int_T \phi \in \mathbf{R}_{\geq 0}$, assuming that ϕ is sufficiently regular (e.g. smooth) near the root b_0 of T. The reader can think of the integral as the limit of integrals of non-negative $(1,1)$-currents over holomorphic discs. Furthermore one can show that this integral depends only on the velocity of T at b_0 and hence gives rise to a linear functional on $\underline{\Gamma}_{1,b_0}$. Therefore ϕ defines a half-space where the velocities of the tropical trees at roots must belong (at least at the points where ϕ is smooth and strictly convex). Moreover, for any such point b_0 we can consider a small variation ϕ_ε, e.g. by taking $\phi + \varepsilon\eta$, where η is a an arbitrary smooth function supported in a small neighborhood of b_0. Similarly to Sect. 9.4 we can intersect the corresponding half-spaces for all $\varepsilon > 0$ and obtain a strict convex cone in $T_{b_0}^{\mathbf{Z}} \otimes \mathbf{R}$. But we need a strict convex cone in $\underline{\Gamma}_{1,b_0} \otimes \mathbf{R}$. This can be achieved by considering variations of ϕ of more general type which change its cohomology class $[\phi] \in H^2_{Betti}(\mathcal{X}^{an}) \otimes \mathbf{R}$.

Therefore if there exists a non-negative $(1,1)$-current ϕ which is smooth and strictly convex at any point of B^0, then there exists a family of closed strict convex cones $C_b \subset \underline{\Gamma}_{1,b} \otimes \mathbf{R}, b \in B^0$ such that the velocities of tropical trees rooted at b belong to C_b (cf. Conjecture 4.6.7). We call *Support Property* the existence of such family of cones.

Assuming that such ϕ does exist one can show that the total length of a tropical tree with given root end the velocity of the root edge is bounded. Here the length is measured with the respect to some auxiliary Riemannian metric on B^0 obtained from ϕ by local considerations. This argument is not sufficient to guarantee the analog of Finiteness Assumption, i.e. the finiteness of the number of tropical trees with given root $b \in B^0$ and the velocity of the root edge $\gamma \in \underline{\Gamma}_{1,b} - \underline{\Gamma}_{1,0,b}$. The finiteness can fail if there exists an infinite sequence of tropical trees with given (b,γ) and increasing numbers of tail edges. Hence we need to find a restriction which guarantees the boundedness of the number of tail edges. In order to achieve that we will need a smaller class of non-negative $(1,1)$-currents. Namely they will be smooth and strictly convex on B^0 and satisfy the property that $\int_l \phi \geq 1$ for any arbitrarily small piece l of the trajectory of the canonical locally-constant vector field, such that l hits the discriminant.[3] For such currents $\int_T \phi$ gives an upper bound for the number of tail edges of any tropical tree T. Together with the above-discussed upper bound for the length of T it will imply the finiteness of the number of tropical trees with fixed (b,γ).

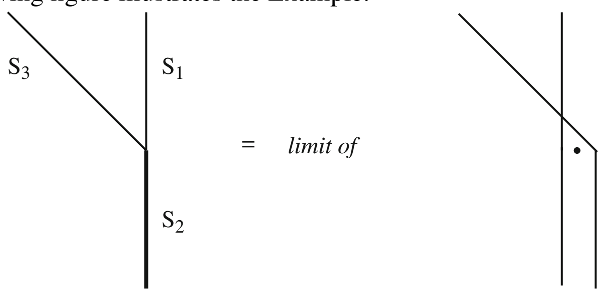
[3] In order to illustrate the latter condition consider delta-currents corresponding to compact curves sitting at the preimage of B_2^{sing} in the total space of the integrable system. The integral of such a current over a holomorphic disc is bounded from below by the intersection index. Our assumption is a "tropical" version of this fact.

First let us make a simplifying assumption that B carries a $\mathbf{Z}PL$-structure which is compatible with $\mathbf{Z}$-affine structure on B^0 and such that $B^{sing} = B - B^0$ is a $\mathbf{Z}PL$ subset of B. In terms of the local model near A_k-singularities discussed above this is equivalent to the fact that the function $f(x_3, \dots, x_n)$ is piecewise-linear with rational slopes. In such a situation we suggest to take as ϕ the $(1, 1)$-current associated with the Gromov–Hausdorff limit S of a family of ample effective divisors $S_t \subset X_t, t \to 0$ (we call S tropical effective divisor). Let us require S contains B^{sing}, does not contain germs of the canonical hyperplanes $x_2 = 0$ (see the description of the local model), and the intersection number of S with any germ of a trajectory of the canonical locally-constant vector field near a generic point of B_2^{sing} is greater or equal than one. More precisely such a germ can be understood as the projection to B of a non-archimedean analytic disc D in $\mathcal{X}^{an}$. In the collapse picture it is represented by a family of complex holomorphic discs $D_t \subset X_t$. Then the above intersection number is defined as the usual intersection number $S_t \cdot D_t$ for $|t| \ll 1$.

Example 10.3.2. Let $n = 2, k = 1$. The tropical effective divisor S is the union of three rays: $S_1 = \{x_1 = 0, x_2 \geq 0\}$, $S_2 = \{x_1 = 0, x_2 \leq 0\}$ and $S_3 = \{x_1 + x_2 = 0, x_2 \geq 0\}$. The rays S_1, S_3 are taken with the multiplicity $+1$, while the ray S_2 is taken with the multiplicity $+2$. Here we use the focus–focus $\mathbf{Z}$-affine structure on $\mathbf{R}^2 - \{(0, 0)\}$ which is the standard affine structure on $\mathbf{R}^2 - \{x_2 = 0, x_1 \geq 0\}$ and has as local affine coordinates $(x_1 + max(0, x_2), x_2)$ near the ray $x_2 = 0, x_1 \geq 0$.

Notice that $S_1 \cup S_2$ (union as sets) is the limit as $\varepsilon \to 0$ of the family of straight lines $x_1 = -\varepsilon, x_2 \in \mathbf{R}$ in $B^0 = \mathbf{R}^2 - \{(0, 0)\}$. Similarly $S_2 \cup S_3$ is the limit of the family of straight lines (in the $\mathbf{Z}$-affine structure) given by $(S_2 + (0, \varepsilon)) \cup (S_3 + (0, \varepsilon))$. There are two types of germs of tropical discs: $D_\pm := D_{\pm,\delta}$. Here D_- is given by $\{-\delta < x_1 \leq 0, x_2 = 0\}$, and D_+ is given by $\{0 \leq x_1 < \delta, x_2 = 0\}$, where $0 < \delta \ll 1$. Then one can easily check that $D_- \cdot (S_1 + S_2) = 1, D_- \cdot (S_2 + S_3) = 0$. Similarly, $D_+ \cdot (S_1 + S_2) = 0, D_+ \cdot (S_2 + S_3) = 1$. Therefore $D_- \cdot S = D_+ \cdot S = 1$, where $S = S_1 + 2S_2 + S_3$.

The following figure illustrates the Example.



If we have a tropical effective divisor S the above constraints on the intersection numbers with germs of trajectories and ϕ is the corresponding non-negative $(1, 1)$-current then the integral $\int_T \phi$ is well-defined for any tropical tree T such the root of T does not belong to S. Furthermore this integral gives an upper bound on the number of tails of T. In order to have an upper bound for *all* tropical trees it is sufficient to find a collection (S_α) of effective tropical divisors satisfying the above constraint and such that $\cap_\alpha S_\alpha \subset B^{sing}$. It is easy to find such a collection using the ampleness of the corresponding divisors in $X_t, t \to 0$.

We conclude that we can achieve (under the appropriate conditions) the Finiteness Assumption as well as the Support Property. Then using the analog of the procedure described in the Sect. 4 we construct the WCS. After that, using a non-negative $(1,1)$-current which is smooth and strictly positive on B^0, we can endow B^0 with the dual $\mathbf{Z}$-affine structure. It can understood as the base of the canonical non-archimedean integrable system which is glued from tube domains in the non-archimedean torus $(\mathbf{C}((t))^*)^n$ (see [31]). The walls of the WCS become curved in the new affine structure. The transformations corresponding to walls can be used in order to modify the total space of the above-mentioned canonical integrable system. As the result, we obtain a non-archimedean integrable system which can be extended to the integrable system with the base B. Its total space can be thought of as the analytic space corresponding to the mirror dual family $X_t^\vee, t \to 0$.

Remark 10.3.3. The approach of [24, 25] gives an example of WCS discussed above. In their case B^{sing} as a $\mathbf{Z}PL$ subset of B. Their notion of "slab" corresponds to the notion of tropical effective divisor discussed above. What we call WCS corresponds to the notion of "scattering diagram" in the loc.cit.

11 Appendix

11.1 Canonical B-Field

Let us consider a fibration $\pi : X^0 \to B^0$ whose fibers are compact tori, endowed with a section. Denote by $\underline{\Gamma}$ the local system $\left(R^1\pi_*(\mathbb{Z})\right)^\vee$ of first homology groups of fibers. We assume that $\underline{\Gamma}$ is endowed with a skew-symmetric pairing $\langle \bullet, \bullet \rangle$, possibly degenerate. The goal of this subsection is to define a canonical cohomology class in $H^2(X^0, \mathbf{Z}/2\mathbf{Z})$ naturally associated with the pairing.

First, let us consider an individual fiber Γ. Skew-symmetric pairing on Γ gives a *symmetric* pairing on $\Gamma \otimes \mathbf{Z}/2\mathbf{Z}$. Hence we can consider the group V of polynomials P of degree at most 2 on $\mathbf{Z}/2\mathbf{Z}$-vector space $\Gamma \otimes \mathbf{Z}/2\mathbf{Z}$ such that $P(0) = 0$ and the bilinear form $(x, y) \mapsto P(x+y) - P(x) - P(y)$ is proportional (with the factor in $\mathbf{Z}/2\mathbf{Z}$) to the form $(x, y) \mapsto \langle x, y \rangle \mod 2$.

We have a short exact sequence

$$0 \to Hom(\Gamma, \mathbf{Z}/2\mathbf{Z}) \to V \to \mathbf{Z}/2\mathbf{Z} \to 0,$$

hence a dual sequence

$$0 \to \mathbf{Z}/2\mathbf{Z} \to V^\vee \to \Gamma \otimes \mathbf{Z}/2\mathbf{Z} \to 0.$$

We have the natural map $\Gamma \to \Gamma \otimes \mathbf{Z}/2\mathbf{Z}$. Let W be the fiber product

$$W = lim(\Gamma \to \mathbf{Z}/2\mathbf{Z} \leftarrow V^{\vee}).$$

Then we have a short exact sequence

$$0 \to \mathbf{Z}/2\mathbf{Z} \to W \to \Gamma \to 0$$

Passing to the classifying spaces we obtain a fibration over the torus $K(\Gamma, 1)$ with the fiber being the Eilenberg–MacLane space $K(\mathbf{Z}/2\mathbf{Z}, 1)$. Going from the local model to the global picture we obtain a fibration with the fiber $K(\mathbf{Z}/2\mathbf{Z}, 1)$ over X^0. Its characteristic class is the desired class in $H^2(X^0, \mathbf{Z}/2\mathbf{Z})$.

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Tropical Eigenwave and Intermediate Jacobians

Grigory Mikhalkin and Ilia Zharkov

Abstract Tropical manifolds are polyhedral complexes enhanced with certain kind of affine structure. This structure manifests itself through a particular cohomology class which we call the eigenwave of a tropical manifold. Other wave classes of similar type are responsible for deformations of the tropical structure.

If a tropical manifold is approximable by a 1-parametric family of complex manifolds then the eigenwave records the monodromy of the family around the tropical limit. With the help of tropical homology and the eigenwave we define tropical intermediate Jacobians which can be viewed as tropical analogs of classical intermediate Jacobians.

1 Tropical Spaces and Tropical Manifolds

In this section we briefly recall basic concepts of tropical spaces relevant for our paper. For more details we refer to [8, 9]. The main assumption we make is that our the tropical space is regular at infinity.

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1.1 Tropical Spaces

A tropical affine n-space $\mathbb{T}^n$ is the topological space $[-\infty,\infty)^n$ (homeomorphic to the nth power of a half-open interval) enhanced with a collection of functions $\mathcal{O}_{\text{pre}} = \{f\}$, $f : U \to \mathbb{T} = [-\infty,\infty)$. Here $U \subset \mathbb{T}^n$ is an open set and f is a function that can be expressed as

$$f(x) = \max_{j\in A}(jx + a_j) \tag{1}$$

for a finite set $A \subset \mathbb{Z}^n$ and a collection of numbers $a_j \in \mathbb{T}$, such that the scalar product jx is well-defined as a number in $\mathbb{T}$ (i.e. is finite or $-\infty$) for any $x \in U$.

The collection of functions $\mathcal{O}_{\text{pre}}$ is a presheaf which gives rise to a sheaf $\mathcal{O}$ of *regular functions* on $\mathbb{T}^n$ (which we will also denote $\mathcal{O}_{\mathbb{T}^n}$ indicating the space where it is defined to avoid ambiguity). $\mathcal{O}$ is called the *structure sheaf* on $\mathbb{T}^n$.

It is convenient to stratify the space $\mathbb{T}^n$ by

$$\mathbb{T}^\circ_I := \{y \in \mathbb{T}^n \ : \ y_i = -\infty, i \in I \text{ and } y_i > -\infty, i \notin I\},$$

where $I \subset \{1,\dots,n\}$. Each T°_I is isomorphic to $\mathbb{R}^{n-|I|}$ and we set $\mathbb{T}_I$ to be its closure in $\mathbb{T}^n$.

To write down a regular function (1) on $\mathbb{R}^n$ all we need is the *integral affine structure* on $\mathbb{R}^n$. This allows us to distinguish functions $\mathbb{R}^n \to \mathbb{R}$ which are affine with linear parts defined over $\mathbb{Z}$. Thus the tropical structure on $\mathbb{T}^n$ can be thought of as an extension of the integral affine structure in $\mathbb{R}^n$ where the overlapping maps are compositions of linear transformations in $\mathbb{R}^n$ defined over $\mathbb{Z}$ with arbitrary translations in $\mathbb{R}^n$.

Given a subset $U \subset \mathbb{T}^N$ we say that a *continuous* map $U \to \mathbb{T}^M$ is *integral affine* if it restricts to an affine map $\mathbb{R}^N \to \mathbb{R}^M$ with integral linear part. We say that a partially defined map $h : \mathbb{T}^N \dashrightarrow \mathbb{T}^M$ is integral affine if it is defined on a subset $U \supset \mathbb{R}^N$ and is integral affine there. Extending h whenever we can by continuity we see that for each $I \subset \{1,\dots,N\}$ h is defined everywhere or nowhere on $\mathbb{T}^\circ_I$.

The automorphisms of a subset $U \subset \mathbb{T}^N$ are invertible integral affine maps $U \to U$. For example, the automorphisms $\text{Aut}(\mathbb{R}^N) \cong \text{GL}_N(\mathbb{Z}) \ltimes \mathbb{R}^N$ form a group of all integral affine transformations of $\mathbb{R}^N$ while $\text{Aut}(\mathbb{T}^N) \cong \mathbb{R}^N$ only consists of translations. We also note that automorphisms of $\mathbb{T}^s \times \mathbb{R}^{N-s}$ translate an s-dimensional affine subspace of $\mathbb{R}^N$ parallel to the $\mathbb{T}^s$ factor to another one with the same property.

A *convex polyhedral domain* D in $\mathbb{T}^N$ is defined as the intersection of a finite collection of half-spaces H_k of the form

$$H_k = \{x \in \mathbb{T}^N \mid jx \le a\} \subset \mathbb{T}^N \tag{2}$$

for some $j \in \mathbb{Z}^N$ and $a \in \mathbb{R}$. The boundary ∂H_k is given by the equation $jx = a$. A *mobile face* E of D is the intersection of D with the boundaries of some of its defining half-spaces given by (2). The adjective *mobile* stands here to distinguish such faces among more general faces of X which we will define later and which are allowed to have support in $\mathbb{T}^N \smallsetminus \mathbb{R}^N$, i.e. be disjoint from $\mathbb{R}^N \subset \mathbb{T}^N$. (They have reduced mobility and are called *sedentary*).

The *dimension* of a convex polyhedral domain D is its topological dimension. Observe that for each mobile face E of D the intersection

$$E^\circ = E \cap \mathbb{R}^N$$

is non-empty. The intersection E° is called the non-infinite part of a mobile face. Each mobile face of D is a convex polyhedral domain itself (although perhaps of smaller dimension).

We say two domains $D \subset \mathbb{T}^N$ and $D' \subset \mathbb{T}^M$ are isomorphic if there is an integral affine map $\mathbb{T}^N \dashrightarrow \mathbb{T}^M$ which restricts to a homeomorphism $D \to D'$ (in particular, it has to be defined everywhere on D).

We say that a convex polyhedral domain $D \subset \mathbb{T}^N$ is *regular at infinity* if for every $I \subset \{1, \dots, N\}$ the intersection $D \cap (\mathbb{T}^\circ_I)$ is either empty or is a $(\dim D - |I|)$-dimensional polyhedral domain in $\mathbb{T}^\circ_I \cong \mathbb{R}^{N-|I|}$.

Definition 1. An n-dimensional polyhedral complex $Y = \bigcup D \subset \mathbb{T}^N$ is the union of a finite collection of convex n-dimensional polyhedral domains D, called the *facets* of Y subject to the following property. For any collection $\{D_j\}$ of facets, their intersection $\bigcap D_j$ is a face of each D_j. Such intersections are called the (mobile) faces of Y. Clearly they are themselves polyhedral domains in $\mathbb{T}^N$.

We say that Y is regular at infinity if all its faces are regular at infinity.

In this paper we assume that all polyhedral complexes are regular at infinity.

Condition 1 (Balancing). *Let E be an $(n-1)$-dimensional mobile face in Y and $D_1, \dots, D_l \subset \mathbb{T}^N$ be the facets adjacent to E. Take the quotient of $\mathbb{R}^N$ by the linear subspace parallel to E°, the non-infinite part of E. The balancing condition requires that*

$$\sum_{k=1}^{l} \epsilon_k = 0, \tag{3}$$

where the ϵ_k are the outward primitive integer vectors parallel to the images of D_k in this quotient.

A polyhedral complex $Y \subset \mathbb{T}^N$ is called *balanced* if all of its $(n-1)$-dimensional faces satisfy the balancing condition.

More generally we can consider spaces that locally look like balanced polyhedral complexes, i.e. admit a covering by open sets U_α enhanced with open embeddings (charts)

$$\phi_\alpha : U_\alpha \to Y_\alpha \subset \mathbb{T}^{N_\alpha}$$

where each $Y_\alpha \subset \mathbb{T}^{N_\alpha}$ is a balanced polyhedral complex. In this paper we assume in addition that each Y_α is regular at infinity.

We may express compatibility of different charts by requiring that the corresponding overlapping maps are induced by integral affine maps $\mathbb{T}^{N_\alpha} \dashrightarrow \mathbb{T}^{N_\beta}$. Or, equivalently, we may use the structure sheaf and enhance each $Y_\alpha \subset \mathbb{T}^{N_\alpha}$ with the sheaf $\mathscr{O}_{Y_\alpha}$ induced from $\mathscr{O}_{\mathbb{T}^{N_\alpha}}$. Its pull-back under ϕ_α is a sheaf on U_α. Two charts ϕ_α and ϕ_β are compatible if the corresponding restrictions to $U_\alpha \cap U_\beta$ agree.

We arrive to the following definition of a tropical space.

Definition 2 (cf. [9]). A tropical space is a topological space X enhanced with a cover of compatible charts $\phi_\alpha : U_\alpha \to Y_\alpha \subset \mathbb{T}^{N_\alpha}$ to balanced polyhedral complexes as above and which satisfies the finite type condition below.

The tropical space X is regular at infinity if it admits charts to polyhedral complexes regular at infinity.

The charts induce a sheaf $\mathscr{O}_X$ on X which we call the structure sheaf of X.

Condition 2 (Finite Type). *The number of charts ϕ_α covering X is finite while each chart is subject to the following property. If $\{x_j \in U_\alpha\}_{j=1}^{\infty}$ is a sequence such that $\phi_\alpha(x_j)$ converges to a point $y \in \mathbb{T}^{N_\alpha}$ then either the sequence $\{x_j\}$ converges inside the topological space X or there exists a coordinate in $\mathbb{T}^{N_\alpha}$ such that its value on y is $-\infty$ while its value on any point in $\phi_\alpha(U_\alpha)$ is finite.*

It is easy to see that this finite type condition is a reformulation of the one from [9].

1.2 Sedentary Points and Faces

Let $D \subset \mathbb{T}^N$ be a polyhedral domain. It is convenient to treat the intersections $D \cap \mathbb{T}_I$ for $I \subset \{1, \ldots, N\}$ also as its faces (at infinity). If we need to distinguish such faces from the mobile ones we have defined before we call these new faces *sedentary*.

Definition 3. We say that

$$E_I := E \cap \mathbb{T}_I$$

is a face of D if E is a mobile face of D. The *sedentarity* of the face E_I is $s = |I|$, while its *refined sedentarity* is I.

Clearly, the mobile faces (defined previously) are the faces of sedentarity 0. If $Y \subset \mathbb{T}^M$ is a polyhedral complex then we define a (possibly sedentary) face of Y as a face of a facet in Y.

We will use the notation $F \prec_j^s E$ when F is a face of E of codimension j and sedentarity s higher. It is also convenient to introduce the following terminology.

Definition 4. A face E of Y is called *infinite* if either it is not compact or it contains a higher sedentary subface. Otherwise E is called finite (even if the sedentarity of E itself is positive).

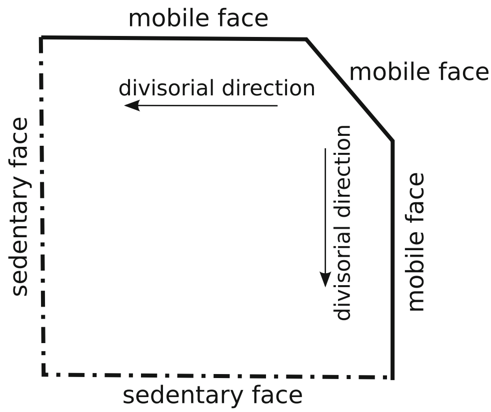


Fig. 1 Mobile and sedentary faces of a polyhedral domain in $\mathbb{T}^N$

Note that even though a face $F \subset Y$ of sedentarity I may be adjacent to several facets, it is always presented as

$$F = E \cap \mathbb{T}^I$$

for a unique mobile face $E \subset Y$ which we call the *parent* of F (as long as Y is regular at infinity). The set of faces of Y with the same parent E is called the *family* of E. In case E is compact the regularity at infinity forces its family to have a very simple combinatorial structure.

Proposition 1. *Let $E \subset Y$ be a compact mobile face containing a face of a maximal sedentarity s. Then its family $\Pi(E)$ forms a lattice poset (under $\prec_j^j$), isomorphic to the face poset of a simplicial cone of dimension s. The maximal sedentary face in the poset is finite.*

Note also that a face F of sedentarity I completely determines the integral affine structure of its parent face E in the neighborhood of $\mathbb{T}_I$. Namely, we have the following proposition.

Proposition 2. *Let $\pi_I : \mathbb{T}^N \to \mathbb{T}^I$ be the projection taking a point $(x_1, \dots, x_N)$ to the point whose j-th coordinate is x_j if $j \notin I$ and $-\infty$ otherwise. The parent face E of F is contained in $\pi_I^{-1}(F)$. Furthermore, for a small open neighborhood $U \supset \mathbb{T}_I$ we have*

$$E \cap U = \pi_I^{-1}(F) \cap U.$$

In other words for a sufficiently small $\epsilon > -\infty$ we have $(x_1, \dots, x_N) \in E$ whenever $\pi_I(x_1, \dots, x_N) \in F$ and $x_j < \epsilon$ for any $j \in I$. Thus the directions parallel to the j-th coordinate in $\mathbb{T}^N$ for $j \in I$ are quite special for E. We orient them toward the $-\infty$-value of the coordinate and call them *divisorial directions*, see Fig. 1. Their positive linear combinations span the *divisorial cone* while all linear combination span the *divisorial subspace* in $\mathbb{R}^N$. The primitive integral vector along a divisorial direction (pointing towards $-\infty$ as the direction itself) is called a *divisorial vector.*

One important observation is that the divisorial vectors are invariant with respect to any integral affine automorphism of $\mathbb{T}^{|I|} \times \mathbb{T}_I^\circ$. Thus they are intrinsically defined for F and so is the divisorial subspace which we denote by W^{div}.

1.3 Tangent Spaces

Let y be a point in the relative interior of a face F of sedentarity I in a balanced polyhedral complex $Y \subset \mathbb{T}^N$. Let $\Sigma(y)$ be the cone in $\mathbb{T}_I^\circ \cong \mathbb{R}^{N-|I|}$ consisting of vectors $u \in \mathbb{T}_I^\circ$ such that $y + \epsilon u \in Y \cap \mathbb{T}_I^\circ$ for a sufficiently small $\epsilon > 0$ (depending on u). We denote the intersection of all maximal linear subspaces contained in $\Sigma(y)$ by $W'(y)$.

Clearly, the cones $\Sigma(y)$ can be canonically identified for all points y in the relative interior of the same face, and so can be the vector spaces $W'(y)$. We say that $y_m \in Y$ is a *nearby mobile point* to y if y_m belongs to the relative interior of the parent face to F.

Definition 5. For a point $y \in Y$ we define $W(y)$, the *wave tangent space* at y, as $W'(y_m)$ for a nearby mobile point y_m. The (conventional) tangent space $T(y)$ at y is defined as the linear span of $\Sigma(y)$ in $\mathbb{T}_I^\circ \cong \mathbb{R}^{N-|I|}$, where I is the refined sedentarity of y.

Note that there are two essential distinctions in defining $T(y)$ and $W(y)$. To define $W(y)$ we always move to a nearby mobile point y_m. The space $W'(y)$ itself, is naturally a quotient of $W(y)$ by the divisorial subspace $W^{div}(y)$.

On the other hand, for $T(y)$ we work in a vector space $\mathbb{T}_I^\circ$, which is naturally the quotient $\mathbb{R}^N / W^{div}(y)$, but we take the linear span of the cone instead of the vector space contained in it.

If we need to specify the space Y for the tangent space $T(y)$ we write $T_Y(y)$, and similarly for $W(y)$. The following proposition is straightforward.

Proposition 3. *An integral affine map $h : \mathbb{T}^N \dashrightarrow \mathbb{T}^M$ induces linear maps $dh^W : W_Y(y) \to W_{h(Y)}(h(y))$ and $dh^T : T_Y(y) \to T_{h(Y)}(h(y))$ whenever h is well-defined on y. We call these maps differentials of h.*

The differentials are natural in the following sense. If $g : \mathbb{T}^M \dashrightarrow \mathbb{T}^L$ is another integral affine map defined on $h(y)$, then the induced differentials satisfy $d(g \circ h) = (dg) \circ (dh)$.

Let $x \in X$ be now a point in a tropical space.

Corollary 1. *The tangent spaces $W_{Y_\alpha}(\phi_\alpha(x))$ (resp. $T_{Y_\alpha}(\phi_\alpha(x))$) for different charts ϕ_α are identified by the differentials of the overlapping maps. The resulting spaces $W(x)$ and $T(x)$ are called the wave tangent space and the (conventional) tangent space to the tropical space X at its point x.*

The tangent spaces $T(x)$ and $W(x)$ carry natural integral structure. We denote the corresponding lattices by $T_{\mathbb{Z}}(x)$ and $W_{\mathbb{Z}}(x)$.

1.4 Polyhedral Structures

Sometimes a tropical space X comes with a structure of an (abstract) polyhedral complex, which is not always the case.

Definition 6. We say that a tropical space X is *polyhedral* if there are finitely many closed subsets $\Delta_j \in X$ (called *facets*) with the following properties.

- For each Δ_j there exists a chart such that $\Delta_j \subset U_\alpha$ and $\phi_\alpha(\Delta_j)$ is a facet of the balanced polyhedral complex $Y_\alpha \subset \mathbb{T}^{N_\alpha}$.
- For any collection $\{\Delta_j\}$ of facets of X and any face Δ_j in this collection the intersection $\bigcap \Delta_j$ is a face of Δ_j.

Note that we may work with tropical polyhedral spaces in the same way as we work with balanced polyhedral complexes in $\mathbb{T}^N$. In particular, we can define in the same way their faces (which will denote by Δ), both mobile and sedentary, parent faces with their families, divisorial directions, and any other notion which is intrinsically defined, that is stable under allowed integral affine maps. For instance, Proposition 1 will read:

Proposition 4. *Let X be a compact polyhedral tropical space. For every face Δ of sedentarity s there is a unique (parent) face Δ_0 of sedentarity 0 such that $\Delta \prec_s^s \Delta_0$. The cells of X with the same Δ_0, the family of Δ_0, form a lattice poset $\Pi(\Delta_0)$ isomorphic to the face poset of a simplicial cone. Every face of X belongs to exactly one family poset Π. The maximal sedentary face $\Delta_{\min}$ in a poset is finite.*

We will denote the k-skeleton of a polyhedral tropical space X (that is the union of $(\leq k)$-dimensional faces) by $\mathrm{Sk}_k(X)$. It is often convenient to take the covering $\{U_\alpha\}$ by open stars of vertices. That is, each U_v is the union of relative interiors of faces of X adjacent to the vertex v. Then the relative interior of a face Δ is contained in every U_v if v is a vertex of Δ.

Another useful feature of a compact polyhedral tropical space is that we can define its first baricentric subdivision. For a finite cell we take an arbitrary point in its interior for its baricenter. For an infinite cell we take for its baricenter the baricenter of its unique most sedentary (necessarily finite, cf. Proposition 4) subface (see Fig. 2). That is, we first choose baricenters of maximal sedentary faces and then name them also as baricenters of any adjacent faces of lower sedentarity. The subdivision of each face of X into simplices is constructed as usual by the flags of its subfaces of *minimal* sedentarity.

The baricentric subdivision of X is *not* a polyhedral tropical space as we defined it. It violates the regularity at infinity property. Nevertheless, it is very convenient to have a triangulation of X. This enables us to define simplicial versions of the (co)homology theories which are very useful for carrying out explicit calculations.

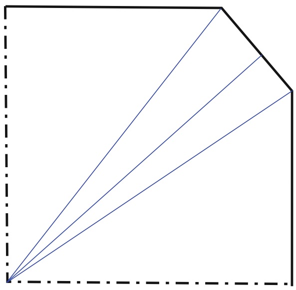


Fig. 2 Baricentric subdivision of an infinite cell. The *dotted faces* have higher sedentarity

1.5 Combinatorial Stratification

Notice that a polyhedral structure on a tropical space (if it exists) is in no way unique. In this subsection we define a combinatorial stratification which is not always polyhedral, but is naturally defined on any tropical space X.

Definition 7. We say that two points $x, x' \in X$ are *combinatorially equivalent* if there exists a path connecting x to x' along which both the dimension of the wave tangent space W and the sedentarity remain constant. *A combinatorial stratum* of the tropical space X is a class of combinatorial equivalence.

We will denote combinatorial strata of X by $\mathcal{E}$ and use the notation $\mathcal{E} \prec \mathcal{E}'$ if the stratum $\mathcal{E}$ lies on the boundary of $\mathcal{E}'$.

Example 1. Consider the circle E_l of length l, otherwise called a *tropical elliptic curve*. E_l is a tropical space: we can present it as a tropical polyhedral space by choosing, e.g., three distinct points so that they split E_l into three facets. This subdivision is not unique as we can move these points around or consider a subdivision into a larger number of facets. The combinatorial stratification for E_l is trivial: it consists of a single stratum E_l.

Let two points $x, y \in U_\alpha \subset X$ belong to one chart $\phi : U_\alpha \to Y$ of X and they sit in some strata $x \in \mathcal{E}_x$ and $y \in \mathcal{E}_y$. If $\mathcal{E}_x = \mathcal{E}_y$, that is if they belong to the same stratum, one can canonically identify the tangent spaces $T(y) = T(x)$ and $W(x) = W(y)$. The identification is natural in the following sense. If the points also belong to another common covering open subset U_β it commutes with the differentials induced by the overlapping map.

In other words, we get flat connections on the bundles T and W over each combinatorial stratum of X.

Furthermore, if $\mathcal{E}_x \prec \mathcal{E}_y$ then one has two natural maps

$$\iota : T_{\mathbb{Z}}(y) \to T_{\mathbb{Z}}(x) \text{ and } \pi : W_{\mathbb{Z}}(x) \to W_{\mathbb{Z}}(y), \tag{4}$$

(note the different directions) defined as follows. If $I(\phi(y)) = I(\phi(x))$ then any face adjacent to $\phi(x)$ is contained in some face adjacent to $\phi(y)$ and ι is given by inclusion. If $I(\phi(y)) \neq I(\phi(x))$ (note that we must have $I(\phi(y)) \subset I(\phi(x))$) then ι is the projection along the divisorial directions indexed by $I(\phi(x)) \smallsetminus I(\phi(y))$. The map π is given by inclusion of the linear spaces spanned by the corresponding parent faces.

Again the maps ι and π are natural in the sense that they commute with the overlapping differentials.

1.6 Tropical Manifolds

First we recall a construction of a balanced polyhedral fan associated to a matroid ([1], see also e.g. [9, 11]).

A matroid $M = (M, r)$ is a finite set M together with a rank function $r : 2^M \to \mathbb{Z}_{\geq 0}$ such that we have the inequalities $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$ and $r(A) \leq |A|$, where $|A|$ is the number of elements in A, for any subsets $A, B \subset M$ as well as the inequality $r(A) \leq r(B)$ whenever $A \subset B$. Subsets $F \subset M$ such that $r(A) > r(F)$ for any $A \supset F$ are called *flats* of M of rank $r(F)$. Matroid M is *loopless* if $r(A) = 0$ implies $A = \emptyset$.

The so-called *Bergman fan* of a loopless matroid M is a polyhedral fan $\Sigma_M \subset \mathbb{R}^{|M|-1}$ constructed as follows. Choose $|M|$ integer vectors $e_j \subset \mathbb{Z}^{|M|-1} \subset \mathbb{R}^{|M|-1}$, $j \in M$ such that $\sum\limits_{j \in M} e_j = 0$ and any $|M| - 1$ of these vectors form a basis of $\mathbb{Z}^{|M|-1}$. To any flat $F \subset M$ we associate a vector

$$e_F := \sum_{j \in F} e_j \in \mathbb{R}^{|M|-1}.$$

For example, $e_M = e_\emptyset = 0$, but $e_F \neq 0$ for any other (proper) flat F. To any flag of flats $F_{i_1} \subset \cdots \subset F_{i_k}$ we associate a convex cone generated by $e_{F_{i_j}}$. We define Σ_M to be the union of such cones, which is, clearly, an $(r(M) - 1)$-dimensional integral simplicial fan. It is easy to check (cf. [1]) that it satisfies the balancing condition, so that Σ_M is a tropical space, called the *Bergman fan* of M.

The matroid M is called *uniform* if $r(A) = |A|$ for any $A \subset M$. Note that the Bergman fan of a uniform matroid is a complete unimodular fan in $\mathbb{R}^{|M|-1}$ with $|M|$ maximal cones.

Definition 8. A tropical space X is called *smooth*, or a *tropical manifold*, if all its charts ϕ_α are open embeddings to $Y_\alpha = \Sigma_M \times \mathbb{T}^s \subset \mathbb{T}^{|M|-1} \times \mathbb{T}^s$ for some loopless matroid M and a number $s \geq 0$. (Here s is the maximal sedentarity in this chart and $n = r(M) - 1 + s$ is the dimension of our tropical manifold X.)

Tropical manifolds can be thought of as tropical spaces without points of multiplicity greater than 1, see [9], thus we use the term *smooth*. Note that smoothness is a property of the tropical space $(X, \mathscr{O}_X)$ alone, it does not involve presentation of X as a polyhedral complex.

2 Homology Groups

2.1 Singular Tropical Homology

Let $x \in X$ be a point in a tropical space. Choose a sufficiently small open set $U \ni x$ and an embedding $\phi : U \to Y \subset \mathbb{T}^N$. Then for points y such that $\phi(y)$ lies in an adjacent face to $\phi(x)$ we have a natural map between lattices in the tangent spaces $\iota : T_{\mathbb{Z}}(y) \to T_{\mathbb{Z}}(x)$, cf. (4).

Definition 9. The group $\mathscr{F}_k(x)$ is defined as the subgroup of the kth exterior power $\Lambda^k(T_{\mathbb{Z}}(x))$ generated by the products $\iota(v_1) \wedge \cdots \wedge \iota(v_k)$ with $v_1, \ldots, v_k \in T_{\mathbb{Z}}(y)$ for a point y such that $\phi(y)$ lies in an adjacent face to $\phi(x)$ of the same sedentarity. It is important that all k elements v_j come from a single adjacent face. The group $\mathscr{F}^k(x)$ is defined as $\mathrm{Hom}(\mathscr{F}_k(x), \mathbb{Z})$.

The discussion at the end of Sect. 1.5 tells us that the groups $\mathscr{F}_k(x)$ and $\mathscr{F}_k(y)$ are canonically identified if x and y belong a single chart U_α and lie in a single stratum $\mathscr{E}$ of X. Furthermore, if for two points x, y, still in the same chart, we have $\mathscr{E}_x \succ \mathscr{E}_y$, then there are natural homomorphisms

$$\iota : \mathscr{F}_k(x) \to \mathscr{F}_k(y). \tag{5}$$

If three points $x, y, z \in U$ lie in the strata with incidence $\mathscr{E}_x \succ \mathscr{E}_y \succ \mathscr{E}_z$ then the three corresponding maps (5) form a commutative diagram. In other words, if we consider the set of strata in the $U_\alpha \subset X$ as a category (under inclusions) then $\mathscr{F}_k$ forms a contravariant functor from strata of U_α to abelian groups (cf. Proposition 6).

We may interpret our data as a system of coefficients suitable to define singular homology groups on X. Namely, we consider the finite formal sums

$$\sum \beta_\sigma \sigma,$$

where each $\sigma : \Delta \to X$ is a singular q-simplex which has image in a single chart U_σ and is such that for each relatively open face Δ' of Δ the image $\sigma(\Delta')$ is contained in a single combinatorial stratum $\mathscr{E}_{\Delta'}$ of X. Slightly abusing the notations we'll identify the source and the image of σ with the singular simplex σ itself and say that $\tau = \sigma|_{\Delta'}$ is a face of σ. Here $\beta_\sigma \in \mathscr{F}_k(\mathscr{E}_\Delta \cap U_\sigma)$.

These chains form a complex $C_\bullet(X; \mathscr{F}_k)$ with the differential ∂ given by the standard singular differential followed by the maps (5). We call such compatible singular chains with coefficients in $\mathscr{F}_k$ *tropical chains*. The groups

$$H_{p,q}(X) = H_q(C_\bullet(X; \mathscr{F}_p), \partial)$$

are called the *tropical homology* groups.

These homology groups is a version of singular homology groups of a topological space X (after imposing the condition of compatibility of singular chains with the charts and combinatorial strata).

A priori the groups $H_{p,q}(X)$ depend on the covering. Indeed, if we refine the covering the tropical chains will be more restrictive. However the usual chain homotopy arguments apply and show that the resulting homology groups are canonically isomorphic. Thus we can conclude that the tropical homology groups are independent of the covering $\{U_\alpha\}$.

In case X has a polyhedral structure one can require the singular chains to be compatible with the polyhedral face structure on X, rather than with its combinatorial structure. Clearly, the homology groups defines by the two complexes are canonically isomorphic. For polyhedral X there are other equivalent ways for constructing tropical homology groups: simplicial, cellular. This is what we are going to consider next.

2.2 Cellular and Simplicial Tropical Homology

We assume X is polyhedral and compact throughout this subsection. The main advantage of dealing with cellular and simplicial chain groups is that they are finitely generated. This will give an effective way to calculate the tropical homology.

Recall that X comes with a subdivision into convex polyhedral domains. We define the cellular chain complex

$$C_q^{cell}(X;\mathscr{F}_p) = \oplus \mathscr{F}_p(\Delta) = \oplus H_q(\Delta, \partial\Delta; \mathscr{F}_p(\Delta)).$$

Here the direct sum is taken over all q-dimensional faces Δ of the subdivision. The homology $H_q(\Delta, \partial\Delta; \mathscr{F}_p(\Delta))$ of the pair with constant coefficients equals $\mathscr{F}_p(\Delta)$ since each q-dimensional face Δ in X is topologically a closed q-disk (recall that X is compact).

Our next step is to define the boundary homomorphism $\partial : C_q^{cell}(X;\mathscr{F}_p) \to C_{q-1}^{cell}(X;\mathscr{F}_p)$. The ∂ is the composition of the maps

$$H_q(\Delta, \partial\Delta; \mathscr{F}_p(\Delta)) \to H_{q-1}(\partial\Delta; \mathscr{F}_p(\Delta)) \to H_{q-1}(\partial\Delta, \partial\Delta \cap \mathrm{Sk}_{q-2}(X); \mathscr{F}_p(\Delta)), \tag{6}$$

the isomorphism

$$H_{q-1}(\partial\Delta, \partial\Delta \cap \mathrm{Sk}_{q-2}(X); \mathscr{F}_p(\Delta)) \to \oplus H_{q-1}(\Delta', \partial\Delta'; \mathscr{F}_p(\Delta)), \tag{7}$$

where the direct sum is taken over all $(q-1)$-dimensional subfaces $\Delta' \prec \Delta$, and

$$\oplus\, H_{q-1}(\Delta', \partial\Delta'; \mathscr{F}_p(\Delta)) \to \oplus H_{q-1}(\Delta', \partial\Delta'; \mathscr{F}_p(\Delta')). \tag{8}$$

In (6) the first homomorphism is the boundary homomorphism of the pair $(\Delta, \partial\Delta)$ and the second one is induced by the inclusion of the pairs $(\Delta, \emptyset) \subset (\Delta, \partial\Delta)$. The isomorphism (7) comes from the excision as the quotient space $\partial\Delta/(\partial\Delta \cap \mathrm{Sk}_{q-2}(X))$ is homeomorphic to a bouquet of $(q-1)$-dimensional spheres, one sphere for each $(q-1)$-dimensional subface $\Delta' \prec \Delta$. Finally, the homomorphism (8) is induced by (5).

The homology groups of the cellular chain complex $(C_\bullet^{cell}(X;\mathscr{F}_p), \partial)$ are called the cellular tropical homology groups $H_\bullet^{cell}(X;\mathscr{F}_p)$. If one has X covered by the open stars of vertices we have the following identification.

Proposition 5. *The cellular tropical homology groups $H_\bullet^{cell}(X;\mathscr{F}_p)$ are canonically isomorphic to the (singular) tropical homology groups $H_\bullet(X;\mathscr{F}_p)$.*

Proof. As in algebraic topology with constant coefficients to prove this isomorphism we need to use cellular homotopy. Let us recall that by the cellular homotopy argument the inclusion $\mathrm{Sk}_q(X) \to X$ induces an epimorphism

$$H_j(\mathrm{Sk}_q(X);\mathscr{F}_p) \to H_j(X;\mathscr{F}_p) \tag{9}$$

for $j \le q$ (which is an isomorphism for $j < q$). Note that even though $\mathscr{F}_p$ is not a constant coefficient system, all cellular homotopy takes place within a single cell, so the classical argument also holds here.

Consider the homomorphism (in singular homology groups) induced by the inclusion of pairs $(\mathrm{Sk}_q(X), \emptyset) \subset (\mathrm{Sk}(X), \mathrm{Sk}_{q-1}(X))$

$$H_q(X;\mathscr{F}_p) \to H_q(\mathrm{Sk}_q(X), \mathrm{Sk}_{q-1}(X);\mathscr{F}_p) = C_q^{cell}(X;\mathscr{F}_p).$$

Its image consists of cycles by the construction of the boundary map in the short exact sequence of the pair and thus it gives us a homomorphism

$$H_q(\mathrm{Sk}_q(X);\mathscr{F}_p) \to H_q^{cell}(X;\mathscr{F}_p). \tag{10}$$

Note that by cellular homotopy the kernel of (10) coincides with the kernel of (9) for $j = q$. To see surjectivity of (10) we consider an element $c \in H_q^{cell}(X;\mathscr{F}_p)$. Subdividing the faces of X into simplices if needed we may represent c by a singular chain in $C_\bullet(\mathrm{Sk}_q(X);\mathscr{F}_p)$, whose boundary ∂c is null-homologous in

$$C_{q-1}^{cell}(\mathrm{Sk}_{q-1}(X), \mathrm{Sk}_{q-2}(X);\mathscr{F}_p).$$

But $H_{q-1}(\mathrm{Sk}_{q-2}(X);\mathscr{F}_p) = 0$ by the dimensional reason and thus ∂c must also vanish in $H_{q-1}^{cell}(\mathrm{Sk}_{q-1}(X);\mathscr{F}_p)$. Thus we may correct c (by adding to it a singular chain in $\mathrm{Sk}_{q-1}(X)$ whose boundary coincides with ∂c) to make it a cycle in $C_\bullet(X;\mathscr{F}_p)$. □

Next observation will be very useful when we define the cap product action by the wave class.

Lemma 1. *Let $\gamma = \sum \beta_\Delta \Delta$ be a cellular cycle in a compact tropical polyhedral space X. Then each β_Δ is divisible by the divisorial vectors of Δ.*

Proof. We only have to check this for infinite cells Δ. Since X is compact, Δ must have a boundary face Δ_q (of sedentarity one higher) for every divisorial direction q. But the coefficient of $\partial\gamma$ at Δ_q comes only from the projection of β_Δ along q. □

There is a simplicial variant of the tropical homology arising from the first baricentric simplicial chains on X. (The baricentric subdivision of X was described at the end of Sect. 1.4). Then we can consider the baricentric simplicial chain complex with coefficients in $\mathscr{F}_p$ as a subcomplex $C_\bullet^{bar}(X;\mathscr{F}_p)$ of $C_\bullet(X;\mathscr{F}_p)$.

Note that the cellular chain complex $C_\bullet^{cell}(X;\mathscr{F}_p)$ can be viewed as a subcomplex of $C_\bullet^{bar}(X;\mathscr{F}_p)$, where all coefficients on simplices of the same cell are taken equal. Applying the standard chain homotopy arguments for constant coefficients one can show that this inclusion

$$C_\bullet^{cell}(X;\mathscr{F}_p) \hookrightarrow C_\bullet^{bar}(X;\mathscr{F}_p)$$

is again a quasi-isomorphism. This allows us to identify both baricentric simplicial and cellular homology with the tropical homology.

Remark 1. In [6] it is shown that in the case when X is a smooth projective tropical manifold that comes as the limit of a complex 1-parametric family the groups $H_{p,q}(X)$ can be obtained from the limiting mixed Hodge structure of the approximating family. In particular, we have the equality

$$h^{p,q}(X_t) = \operatorname{rk} H_{p,q}(X),$$

for the Hodge numbers $h^{p,q}(X_t)$ of a generic fiber X_t from the approximating family.

Remark 2. In Sect. 6.2 we will show that there is a fairly small subcomplex of $C_\bullet^{bar}(X;\mathscr{F}_p)$, called konstruktor, which suffices to calculate the homology groups $H_{p,q}(X)$ in the smooth projective realizable case.

2.3 Tropical Cohomology Groups

Finally we define tropical *cochains* $C^\bullet(X;\mathscr{F}^p)$ to be certain linear functionals on charts/strata compatible $\mathbb{Z}$-singular chains with values in $\bigoplus_{\alpha,\mathscr{E}} \mathscr{F}^p(\mathscr{E}\cap U_\alpha)$. Namely, if a simplex σ lies in $\mathscr{E}\cap U_\alpha$ then we require the value of the cochain to lie in $\mathscr{F}^p(\mathscr{E}\cap U_\alpha)$. If σ also lies in U_β, then its value in $\mathscr{F}^p(\mathscr{E}\cap U_\beta)$ should coincide with its value in $\mathscr{F}^p(\mathscr{E}\cap U_\alpha)$ via the differential of the overlapping map.

Then one can define the differential as the usual coboundary followed by the maps dual to (5)

$$\delta\alpha(\sigma) = \alpha(\partial\sigma) \in \bigoplus_{\tau\subset\sigma} \mathscr{F}^p(\Delta_\tau) \to \mathscr{F}^p(\Delta_\sigma).$$

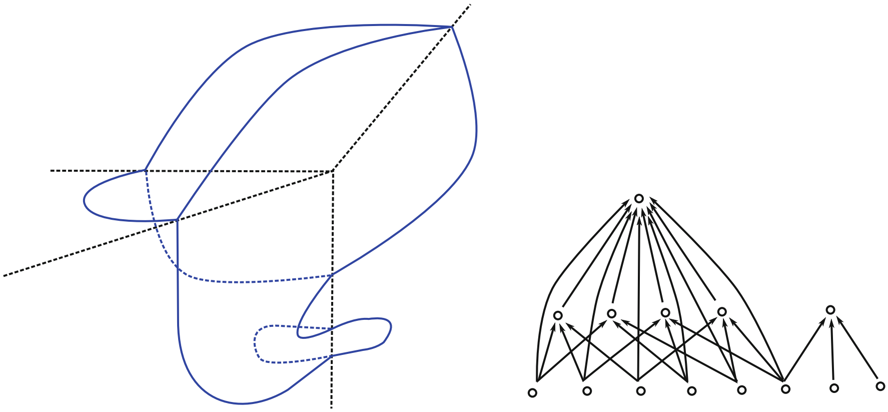


Fig. 3 An open set in a polyhedral complex and the corresponding quiver. Here $\mathscr{F}_1(U) \cong \mathbb{Z}^4$

We can define the *tropical cohomology* groups

$$H^{p,q}(X) = H^q(C^\bullet(X;\mathscr{F}^p), \delta).$$

2.4 Sheaf/Cosheaf (Co)homology

To make connections with sheaf (co)homology theories we use the coefficient systems $\mathscr{F}_p$ to define a constructible cosheaf with respect to the combinatorial stratification of X. With a slight abuse of notations we denote this cosheaf also by $\mathscr{F}_p$. A cosheaf is a suitable notion to take homology, just like sheaf for cohomology.

First we construct the pre-cosheaf in each open chart U_α. Given an open set $U \subset U_\alpha$ we consider the poset formed by the connected components of intersections of the strata of U_α with U. The order is given by adjacency. This poset can be represented by a quiver (oriented graph) $\Gamma(U)$. Each vertex $v \in \Gamma(U)$ corresponds to a connected component of the intersection $U \cap \mathscr{E}$ of the open set U and a stratum $\mathscr{E}$ of U_α. A single stratum can produce several vertices in $\Gamma(U)$, see Fig. 3.

To each vertex v we associate the coefficient group $\mathscr{F}_p(v) = \mathscr{F}_p(\mathscr{E})$. To an arrow from v to w we associate the relevant homomorphism $i_{vw} : \mathscr{F}_p(v) \to \mathscr{F}_p(w)$ from (5). The groups $\mathscr{F}_p(v)$ with maps i_{vw} thus form a representation of the quiver $\Gamma(U)$.

Definition 10. $\mathscr{F}_p(U)$ is the quotient of the direct sum $\bigoplus_{v\in\Gamma(U)} \mathscr{F}_p(v)$ by the subgroup generated by the elements $a - i_{vw}(a)$ for all pairs of connected vertices (v, w), and all $a \in \mathscr{F}_p(v)$.

Note that an inclusion $U \subset V \subset U_\alpha$ induces a morphism between the corresponding quivers $\Gamma(U) \to \Gamma(V)$ with isomorphisms at the corresponding vertices. This map clearly preserves the equivalence relation, and hence descends to the map $\mathscr{F}_p(U) \to \mathscr{F}_p(V)$. Thus, we get a covariant functor from the open sets $U \subset U_\alpha$ (with morphism given by inclusions) to free abelian groups $U \mapsto \mathscr{F}_p(U)$. It is easy to check that all sequences

$$\bigoplus_{i,j} \mathscr{F}_p(U_i \cap U_j) \to \bigoplus_i \mathscr{F}_p(U_i) \to \mathscr{F}_p(U) \to 0, \tag{11}$$

where $U = \bigcup U_i$, are exact. Thus the functor $U \mapsto \mathscr{F}_p(U)$ is a cosheaf (cf., e.g. [2]) on the open set U_α.

To define the sheaf $\mathscr{F}^p$ we need a contravariant functor $U \mapsto \mathscr{F}^p(U)$. Let $\Gamma(U)$ to be the directed graph as before with all arrows reversed. We set $\mathscr{F}^p(U)$ to be the subgroups of $\bigoplus_{v\in\Gamma(U)} \mathscr{F}^p(v)$, where the collections of elements $\{a_v \in \mathscr{F}^p(v)\}$ are compatible with all the morphisms dual to (5). Note that these collections are precisely the ones annihilated by the elements $a - i_{vw}(a)$ from the Definition 10, and thus $F^p(U) = \mathrm{Hom}(\mathscr{F}_p(U), \mathbb{Z})$. Dualizing the exact sequences (11) we see that the functor $U \mapsto \mathscr{F}^p(U)$ is a (constructible) sheaf on U_α.

Finally we can glue together the sheaves and cosheaves defined on all open charts U_α (see, e.g., [5], Ch. II, Exer. 1.22, for the sheaf version). We get a well defined cosheaf $\mathscr{F}_k$ and sheaf $\mathscr{F}^k$ on X as long as we have the isomorphisms $\psi_{\alpha\beta}$ between the charts which satisfy $\psi_{\alpha\beta} \circ \psi_{\beta\gamma} = \psi_{\alpha\gamma}$ (see Proposition 3).

The combinatorial strata of X form a category. Its objects are the strata themselves. There is a unique morphism from $\mathscr{E}$ to $\mathscr{E}'$ if $\mathscr{E} \prec \mathscr{E}$, and no morphisms otherwise. Our reasoning above can be formalized into the following general statement.

Proposition 6. *Suppose X has an open covering $\{U_\alpha\}$ and covariant functors $\mathscr{F}_\alpha$ for each $\mathscr{U}_\alpha$ from the combinatorial strata of $\mathscr{U}_\alpha$ to abelian groups which are compatible on the overlaps in the sense of Proposition 3. Then gluing gives rise to a constructible sheaf on X. Contravariant functors yields a constructible cosheaf on X.*

If the functors $\mathscr{F}_\alpha$ behave naturally with respect to refinements of the covering $\{U_\alpha\}$ the resulting (co)sheaf $\mathscr{F}$ does not depend on the covering.

Finally we can use the sheaf-theoretic or Čech homology and cohomology for cosheaves $\mathscr{F}_p$ and sheaves $\mathscr{F}^p$. The standard algebraic topology techniques identify all these homology theories with the tropical (co)homology.

Proposition 7. *There are natural isomorphisms*

$$H_{p,q} \cong H_q(X, \mathscr{F}_p) \quad \textit{and} \quad H^{p,q} \cong H^q(X, \mathscr{F}^p),$$

where on the right hand side are the sheaf-theoretic (co)homology groups.

3 Tropical Waves

3.1 Waves and Cowaves

There is also another collection of sheaves and cosheaves that can be associated to a tropical space X. Recall that for every point $x \in X$ we defined the wave tangent spaces $W(x)$ in Sect. 1.3.

Definition 11. We define $W_k(x)$ as the exterior power $\Lambda^k W(x)$. We also consider the dual vector space $W^k(x)$.

In any given chart U_α the $W(x)$ can be canonically identified for points in a single stratum $\mathscr{E}$. Thus we may write $W_k(\mathscr{E}\cap U_\alpha) = W_k(x)$ for any point $x \in \mathscr{E}\cap U_\alpha$. For a pair $\mathscr{E} \prec \mathscr{E}'$ of two adjacent strata the map (4) induces the natural homomorphisms

$$\pi : W_k(\mathscr{E}) \to W_k(\mathscr{E}') \quad \text{and} \quad \hat{\pi} : W^k(\mathscr{E}') \to W^k(\mathscr{E}). \tag{12}$$

By Proposition 6 the coefficient system W_k defines a constructible sheaf $\mathscr{W}_k$ on X, whereas the W^k defines a cosheaf $\mathscr{W}^k$ for every integer $k \geq 0$.

Definition 12. Tropical *wave* and *cowave* groups, respectively, are

$$H^q(X;\mathscr{W}_k) \quad \text{and} \quad H_q(X;\mathscr{W}^k). \tag{13}$$

Again, we can think of these groups from the sheaf-theoretic point of view or as stratum-compatible singular (co)homology with coefficients in the systems W_k and W^k.

Example 2. Let us consider a tropical genus 2 curve C with a simple double point. The underlying topological space of C is a wedge of two circle, i.e. it is a graph with a single vertex v and two edges that are glued to v, see Fig. 4.

The tropical structure in the interior of each edge is isomorphic to an open interval of finite length in $\mathbb{R}$ (treated as the tropical torus $\mathbb{T}^\times = \mathbb{T} \setminus \{-\infty\}$). The tropical structure at the vertex v is such that the four primitive vectors divide into 2 pairs of opposite vectors. This means that the chart at v is given by a map to $\mathbb{R}^2$ such that a neighborhood of v in C goes to the union of coordinate axes and the four primitive vectors near v go to the unit tangent vectors to those axes.

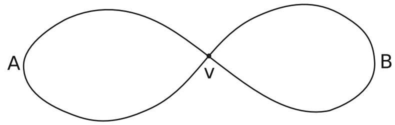
Thus, $\mathscr{F}_1(v) = \mathbb{Z}^2$ and $W_1(v) = 0$. On the other hand every point x in the interior of either edge has the groups $\mathscr{F}_1(x) = \mathbb{Z}$ and $W_1(x) = \mathbb{R}$. The group $\mathscr{F}_0(x)$ is always $\mathbb{Z}$ and $W_0(x) = \mathbb{R}$ for any point x. From the two term cell complex one can easily calculate

$$H_0(C;\mathscr{F}_0) \cong \mathbb{Z}, \quad H_0(C;\mathscr{F}_1) \cong \mathbb{Z}, \quad H_1(C;\mathscr{F}_0) \cong \mathbb{Z}^2, \quad H_1(C;\mathscr{F}_1) \cong \mathbb{Z},$$

and

$$H^0(C;\mathscr{W}_0) \cong \mathbb{R}, \quad H^0(C;\mathscr{W}_1) = 0, \quad H^1(C;\mathscr{W}_0) \cong \mathbb{R}^2, \quad H^1(C;\mathscr{W}_1) \cong \mathbb{R}^2.$$

Fig. 4 Nodal genus 2 curve



In general, $\mathscr{F}_0$ and $\mathscr{W}_0$ are constants, thus for $p = 0$ we recover the ordinary topological homology and cohomology groups.

Proposition 8. *We have* $H_{0,q} = H_q(X;\mathbb{Z})$, $H_q(X;\mathscr{W}^0) = H_q(X;\mathbb{R})$, $H^{0,q} = H^q(X;\mathbb{Z})$, $H^q(X;\mathscr{W}_0) = H^q(X;\mathbb{R})$.

3.2 *Pairing of $\mathscr{F}$ and $\mathscr{W}$*

The importance of the wave classes stems from their action on the tropical homology via a natural bilinear map

$$\cap : H^r(X;\mathscr{W}_k) \otimes H_q(X;\mathscr{F}_p \otimes \mathbb{R}) \to H_{q-r}(X;\mathscr{F}_{p+k} \otimes \mathbb{R})$$

which we are going to define now. On the chain level this map is just the standard cap product between singular chains and cochains coupled with the wedge product on the coefficients $\wedge : W_k \otimes \mathscr{F}_p \to \mathscr{F}_{p+k} \otimes \mathbb{R}$.

Let us clarify the meaning of the wedge multiplication. For a mobile point $x \in X$ the wave tangent space $W(x)$ is naturally a subspace in $T(x)$ hence the product makes sense on the nose. For any sedentary point x the wave space $W(x)$ naturally projects (along the divisorial directions) to $W'(x)$, which is a subspace of $T(x)$. When taking the wedge product we first apply this projection.

In details, let α be a compatible r-cochain with coefficients in W_k and $\gamma = \sum \beta\sigma$ be a tropical q-chain with coefficients in $\mathscr{F}_p$. For each singular simplex σ we denote by $\sigma_{0\ldots r}$ its first r-face (spanned by the first $r+1$ vertices of σ) and by $\sigma_{r\ldots q}$ its last $(q-r)$-face. Then we set

$$\alpha \cap \gamma = \sum (\alpha(\sigma_{0\ldots r}) \wedge \beta)\sigma_{r\ldots q}. \tag{14}$$

Here we push the value of α at the face $\sigma_{0\ldots r}$ to the simplex σ with the sheaf map and then push the value of the result from σ to $\sigma_{r\ldots q}$ using the cosheaf map.

We will need the following local observation. Let us assume we live in a single chart U_α.

Lemma 2. *Let $\mathscr{E}' \prec \mathscr{E}$ be a pair of adjacent strata in U_α. Then the diagram*

$$\begin{array}{ccc} W_k(\mathscr{E}) \otimes \mathscr{F}_p(\mathscr{E}) & \xrightarrow{\ \wedge\ } & \mathscr{F}_{p+k}(\mathscr{E}) \otimes \mathbb{R} \\ \pi \uparrow \quad\quad \downarrow \iota & & \downarrow \iota \\ W_k(\mathscr{E}') \otimes \mathscr{F}_p(\mathscr{E}') & \xrightarrow{\ \wedge\ } & \mathscr{F}_{p+k}(\mathscr{E}') \otimes \mathbb{R} \end{array}$$

is commutative in the sense that for any $\alpha \in W_k(\mathscr{E}')$ *and* $\beta \in \mathscr{F}_p(\mathscr{E})$ *one has* $\iota(\pi(\alpha) \wedge \beta) = \alpha \wedge \iota(\beta)$.

Proof. The wedge product is bilinear with respect to inclusion and quotient (in fact, all) homomorphisms between free abelian groups. □

Proposition 9. *For each* $r \leq q$ *the cap product* (14) *descends to a natural bilinear map in homology*

$$\cap : H^r(X; \mathscr{W}_k) \otimes H_q(X; \mathscr{F}_p \otimes \mathbb{R}) \to H_{q-r}(X; \mathscr{F}_{p+k} \otimes \mathbb{R}).$$

Proof. The statement follows at once from the usual Leibnitz formula

$$(-1)^r \partial(\alpha \cap \gamma) = (\delta\alpha) \cap \gamma + \alpha \cap \partial\gamma.$$

Note that the wedge products in $\delta(\alpha \cap \gamma)$ and $(\partial\alpha) \cap \gamma$ are taken in σ and then pushed to $\mathscr{F}_{p+k}(\sigma_{r\ldots\hat{\imath}\ldots q})$. On the other hand the wedge products in $\alpha \cap \delta\gamma$ are taken in $\sigma_{0\ldots\hat{\imath}\ldots q}$ and then pushed to $\mathscr{F}_{p+k}(\sigma_{r\ldots\hat{\imath}\ldots q})$. But Lemma 2 allows us to identify the results. □

3.3 The Group $H^1(X; \mathscr{W}_1 \otimes \mathbb{R})$ and Deformations of the Tropical Structure of X

In this section we assume that X is compact. Recall that X has a covering by charts $\phi_\alpha : U_\alpha \to Y_\alpha \subset \mathbb{T}^{N_\alpha}$. The transition maps on the overlaps are given by integral affine maps $\psi_{\alpha\beta} : \mathbb{T}^{N_\alpha} \dashrightarrow \mathbb{T}^{N_\beta}$.

As a topological space X can be presented as the quotient of the disjoint union of its covering sets $\bigsqcup_\alpha (U_\alpha)$ by the following equivalence relation. We say two points $x \in U_\alpha$ and $y \in U_\beta$ are equivalent if $\psi_{\alpha\beta} \circ \phi_\alpha(x) = \phi_\beta(y)$. Reflexivity of equivalence says that $\psi_{\alpha\beta} = \psi_{\beta\alpha}^{-1}$ (as partially defined maps). Transitivity translates as the cocycle condition, or as the composition rule, $\psi_{\beta\gamma} \circ \psi_{\alpha\beta} = \psi_{\alpha\gamma}$.

Conversely, given open subsets $\phi_\alpha(U_\alpha) \subset Y_\alpha \subset \mathbb{T}^{N_\alpha}$ and a collection of integral affine maps $\psi_{\alpha\beta}$ satisfying $\psi_{\alpha\beta} = \psi_{\beta\alpha}^{-1}$ and $\psi_{\beta\gamma} \circ \psi_{\alpha\beta} = \psi_{\alpha\gamma}$ we can define a topological space X as the quotient of $\bigsqcup_\alpha \phi_\alpha(U_\alpha)$ by the equivalence given by the ψ's.

X will be a tropical space provided all subsets $\phi_\alpha(U_\alpha)$ remain open in the quotient and X satisfies the finite type condition. Moreover we will get an isomorphic tropical space if the $\psi_{\alpha\beta}$ are changed by a "coboundary" (twisted by automorphisms $\psi_\alpha : \mathbb{T}^{N_\alpha} \dashrightarrow \mathbb{T}^{N_\alpha}$ for some α).

Let τ be a class in $H^1(X; \mathscr{W}_1)$. We can assume that the covering $\{U_\alpha\}$ is fine enough so that τ can be represented by a Čech 1-cocycle $\tau_{\alpha\beta} \in W(U_\alpha \cap U_\beta)$. We can also assume that all U_α and $U_\alpha \cap U_\beta$ are connected. Then $W(U_\alpha \cap U_\beta)$ consists of vectors parallel to all mobile strata in $U_\alpha \cap U_\beta$. We can think of $W(U_\alpha \cap U_\beta)$ as a subspace in $\mathbb{R}^{N_\alpha} \subset \mathbb{T}^{N_\alpha}$ via the map ϕ_α, or in $\mathbb{R}^{N_\beta} \subset \mathbb{T}^{N_\beta}$ via ϕ_β.

By shrinking the U_α if necessary it will also be convenient to assume that slightly larger open subsets $V_\alpha \supset \overline{U_\alpha}$ do not contain any new strata other than those already in the U_α. For instance if X is polyhedral we can take U_α to be the open stars of vertices, "shrunk" a little bit.

Now for $\epsilon > 0$ we modify the overlapping maps $\psi_{\alpha\beta} : \mathbb{T}^{N_\alpha} \dashrightarrow \mathbb{T}^{N_\beta}$ by precomposing them with the translation by $\epsilon\tau_{\alpha\beta}$. Since $\tau_{\alpha\beta} = -\tau_{\beta\alpha}$ the new relation is reflexive. Also since τ is a cocycle the new maps $\psi^{\epsilon\tau}_{\alpha\beta}$ satisfy the composition rule. Thus they define a new equivalence relation and we call the corresponding quotient space $X_{\epsilon\tau}$ the deformation of X.

Proposition 10. *For $\epsilon > 0$, small enough, $X_{\epsilon\tau}$ is a tropical space.*

Proof. We only need to show that $X_{\epsilon\tau}$ is of finite type and each of the U_α is still an open subset in $X_{\epsilon\tau}$. For the latter it is enough to show that each $U_\alpha \cap U_\beta$ is open. But this is clear since by condition that slight enlargements of U_α contain no new strata, no new strata can appear in $U_\alpha \cap U_\beta$ for small enough ϵ. The argument for the finite type condition is similar. □

The deformed tropical space is especially easy to visualize in the polyhedral case. Namely, $X_{\epsilon\tau}$ has the same combinatorial face structure, but the faces of $X_{\epsilon\alpha}$ themselves may have different shapes and sizes. For example if X is one-dimensional, the lengths of the edges of X and $X_{\epsilon\alpha}$ may be different.

4 Straight Classes

4.1 Straight Cycles in Tropical Homology

We start with a natural generalization of balanced polyhedral complexes in $\mathbb{T}^N$ to a situation where a facet can have a weight. A weighted balanced polyhedral complex $Y \subset \mathbb{T}^N$ is a union of a finite number of facets D as before, but now each D is enhanced with an integer weight $w(D) \in \mathbb{Z}$ subject to the following weighted balancing condition for every $(n-1)$-dimensional mobile face $E \subset Y$. As in (3) we consider all facets $D_1, \dots, D_l \subset \mathbb{T}^N$ adjacent to E and take the quotient of $\mathbb{R}^N$ by the linear subspace parallel to E°, the non-infinite part of E. The weighted balanced condition is

$$\sum_{k=1}^{l} w(D_k)\epsilon_k = 0. \tag{15}$$

We say that the weighted balanced complex Y is *effective* if the weights of all its facets are positive.

Just as balanced polyhedral complexes form local models for tropical spaces, effective weighted balanced polyhedral complexes form models for *weighted tropical spaces*.

Definition 13. A weighted tropical space is a topological space X enhanced with a weight function $w : X \dashrightarrow \mathbb{N}$ defined on an open dense set $A \subset X$ and a sheaf $\mathscr{O}_X$ of functions to $\mathbb{T}$ such that there exists a finite covering of compatible charts $\phi_\alpha : U_\alpha \to Y_\alpha \subset \mathbb{T}^{N_\alpha}$ with the following properties.

- Y_α is an effective weighted balanced polyhedral complex in $\mathbb{T}^{N_\alpha}$.
- For the relative interior D° of any facet $D \subset Y_\alpha$ we have $\phi_\alpha^{-1}(D^\circ) \subset A$ while the weight function w is constant on $\phi_\alpha^{-1}(D^\circ)$ and equal to the weight of D.
- For each facet $D \subset Y_\alpha$ there exists a $\mathbb{Z}$-linear transformation $\Phi_D : \mathbb{Z}^{N_\alpha} \to \mathbb{Z}^{N_\alpha}$ of determinant $w(D)$ such that $\mathscr{O}_X|_{D^\circ \cap U_\alpha}$ is induced by $\Phi_D^{-1} \circ \phi_\alpha$.

We may reformulate the last condition of this definition by saying that each facet D comes with a sublattice of index $w(D)$ of the tangent lattice $T_\mathbb{Z}(x)$, $x \in D$. This sublattice is locally constant and does not depend on the choice of charts. Note that not every weighted balanced polyhedral complex in $\mathbb{T}^N$ is a weighted tropical space in this sense as it is not always possible to consistently choose such a sublattice. However no such sublattice for the facets of Z is needed for the following definition.

Definition 14 (cf. [9,11]). Let X be a tropical space. A subspace $Z \subset X$ enhanced with a weight function

$$w : Z \dashrightarrow \mathbb{Z}$$

defined on an open dense set $A \subset Z$ is called a *straight tropical p-cycle* if for every chart $\phi_\alpha : U_\alpha \to Y_\alpha \subset \mathbb{T}^{N_\alpha}$ of X there exists a weighted p-dimensional balanced polyhedral complex $Z_\alpha \subset \mathbb{T}^{N_\alpha}$ such that $\phi_\alpha : Z \cap U_\alpha \to Z_\alpha$ is an open embedding, and for the relative interior D° of any facet $D \subset Z_\alpha$ we have $\phi_\alpha^{-1}(D^\circ) \subset A$. The weight function w is constant on $\phi_\alpha^{-1}(D^\circ)$ and equal to the weight of D.

Proposition 11. *Each straight tropical p-cycle $Z \subset X$ gives rise to a canonical element $[Z] \in H_{p,p}(X)$ in the tropical homology group of X.*

Proof. We choose a sufficiently fine (topological) triangulation of $Z = \bigcup \sigma$ so that each p-simplex σ from the triangulation lies in a single chart U_α and in a single combinatorial stratum $\mathscr{E}$ of X. In particular each σ carries the weight $w(\sigma)$ induced from Z. An orientation of σ defines the canonical volume element $\mathrm{Vol}_\sigma \in \mathscr{F}_p^X(\sigma)$ given by the generator of $\Lambda^p(W_\mathbb{Z}^Z(\sigma)) \cong \mathbb{Z}$. Inverting the orientation of σ will simultaneously invert the sign of Vol_σ. Thus the product $\mathrm{Vol}_\sigma\, \sigma$ is a well-defined tropical chain in $C_p(X; \mathscr{F}_p)$. Then the weighted balancing condition for Z ensures that

$$\gamma_Z = \sum_{\sigma \subset Z} w(\sigma)\, \mathrm{Vol}_\sigma\, \sigma$$

is a cycle in $C_p(X;\mathscr{F}_p)$. Its class is clearly independent of the triangulation and gives the desired element $[Z] \in H_p(X;\mathscr{F}_p) = H_{p,p}(X)$. □

Definition 15. Elements of $H_{p,p}$ realised by straight tropical cycles as in Proposition 11 are called straight homology classes (or, in other existing terminology, *special* or *algebraic*). They form a subgroup

$$H^{straight}_{p,p}(X) \subset H_{p,p}(X).$$

Example 3. Recall that the tropical N-dimensional projective space $\mathbb{TP}^N$ may be obtained by gluing $N+1$ affine charts $\mathbb{T}^N$ with the help of integral affine maps, cf. e.g. [9]. A topological subspace $X \subset \mathbb{TP}^N$ is called a *projective tropical space* if the intersection of X with any such chart is a balanced polyhedral complex. A projective tropical space has a non-trivial straight homology class

$$[H^X_p] \in H^{straight}_{p,p}(X)$$

(called the *hyperplane section*) in any dimension $p = 0,\dots,n = \dim X$.

To see this we start from the case $X = \mathbb{TP}^N$. Consider the equations $x_j = c_j$, $j = p+1,\dots,n$, $c_j \in \mathbb{R}$, in a chart $\mathbb{T}^N \subset \mathbb{TP}^N$. They define a p-dimensional linear space parallel to a coordinate plane. We may take for H_p the topological closure of this linear space in $\mathbb{TP}^N$. Clearly, the homology class $[H_p]$ does not depend on the choice of the $\mathbb{T}^N$-chart or on permutation of coordinates in this chart. Furthermore, H_0 is a point and thus $[H_0] \neq 0$ in $H_{0,0}(\mathbb{TP}^N) \cong \mathbb{Z}$. Note that this also implies that $[H_p] \neq 0$ in $H_{p,p}(\mathbb{TP}^N)$ as we may choose the transverse representatives H_p and $H_{p'}$ so that $H_p \cap H_{p'} = H_{p+p'-N}$, cf. [11]. It is easy to show that any element of $H_{*,*}(\mathbb{TP}^N)$ is generated by $[H_p]$, $p = 0,\dots,N$.

A similar construction can be made for general projective tropical spaces $X \subset \mathbb{TP}^N$. We take $H^X_p = H_{N+p-n} \cap X$ where H_{N+p-n} is chosen to be transverse to X with the help of translations in $\mathbb{R}^N$. But in addition to those hyperplane sections and their powers $H_{*,*}(X)$ may have additional, more interesting, straight classes.

4.2 Straight Cowaves

A notion of straight classes exists also for cowaves. Once again, let $Z \subset X$ be a subspace such that each chart ϕ_α takes Z to a q-dimensional polyhedral complex in $\mathbb{T}^{N_\alpha}$ (which we no longer assume balanced). We refer to such subspace of X as a *straight subspace*.

In this subsection we assume that $\dim W(x) = m$ for some m almost everywhere on Z. In other words we assume that each open facet of Z sits in the m-skeleton of X, but outside of the $(m-1)$-skeleton of X. We call such straight subspaces Z *purely m-skeletal*. For example Z is n-skeletal if no open facets of Z intersect $\mathrm{Sk}_{n-1}(X)$.

Definition 16. A *coweight function* on Z is a function

$$x \mapsto cow(x) \in W^m(x)$$

defined on an open dense set $A \subset Z$. Here we assume that $\dim W(x) = m$ whenever $x \in A$, so we have $W^m(x) \approx \mathbb{Z}$.

This is a dual notion to the weight function. But while the weight function was integer-valued, here we do not have a canonical isomorphism between $W^m(x)$ and $\mathbb{Z}$, it is only canonical up to sign.

Let $x \in A$ be inside of a facet of Z parallel to a q-dimensional affine space L (in a chart ϕ_α). As in the previous subsection, we may consider the volume element $\mathrm{Vol}_L \in W_q(x)$ which is well-defined by the integer lattice in L and a choice of orientation of L. Given this choice we have a well-defined map

$$\lambda \mapsto cow(x)(\lambda \wedge \mathrm{Vol}_L),$$

$\lambda \in W_{m-q}(x)$ and thus an element in $W^{m-q}(x)$, a group that depends only on the open facet of Z containing x. Thus any q-simplex σ embedded to the same facet and parallel to L defines a canonical chain with coefficients in $W^{m-q}(x)$.

In particular, a triangulation of a coweighted purely m-skeletal q-dimensional polyhedral pseudocomplex Y gives rise to a cowave chain in $C_q(X; \mathscr{W}^{m-q})$. Such cowave chains are called *straight*.

As in Proposition 11 we may associate a singular chain with the coefficients in W^{m-q} to Z by using a combinatorial stratification of Z.

Definition 17. A coweighted straight subspace $Z \subset X$ is called *cobalanced* if the resulting chain is a cycle. We may refine this into a local notion by saying that Z is cobalanced at $x \in Z$ if x is disjoint from the support of the boundary of the resulting special cowave cochain.

Note that once an orientation of $W(x)$ is chosen we may identify coweight and weight at x.

Proposition 12. *Suppose that a q-dimensional coweighted straight subspace Z is purely m-skeletal and that $x \in Z$ belongs to a relative interior of a $(q-1)$-dimensional face (in a chart) with* $\dim W(x) = m$. *Then Z is cobalanced at x if and only if Z is balanced at x.*

Proof. Note that x must belong to the same combinatorial stratum of X as its small open neighbourhood in Z, since Z is purely m-skeletal and $\dim W(x) = m$. Thus $\mathscr{W}^m|_Z$ is locally trivial near x and we may translate coweights into weights simultaneously for the whole neighbourhood with the help of an arbitrary orientation of $W(x)$.

At the same time if $\dim W(x) < m$ then the cobalancing condition is different from the balancing condition. We believe that study of straight cowaves might be useful, particularly in the context of mirror symmetry.

5 The Eigenwave

5.1 The Eigenwave ϕ

There is a canonical element $\phi \in H^1(X;\mathcal{W}_1)$ for every compact tropical space X. Unfortunately, it does not have a preferred representative as a singular wave cocycle in the case when X has points of positive sedentarity. Rather we shall represent it in the quotient space $C^1(X;\mathcal{W}_1)/B^1_{div}(X;\mathcal{W}_1)$ where the subspace $B^1_{div}(X;\mathcal{W}_1) \subset C^1(X;\mathcal{W}_1)$ will consist of certain coboundaries. Note that this ambiguity will not cause us any problems with the cap product of ϕ and the homology cycles because, as we will see, taking product with elements in $B^1_{div}(X;\mathcal{W}_1)$ annihilates any singular cycle.

Let $C^0_{div}(X;\mathcal{W}_1) \subset C^0(X;\mathcal{W}_1)$ be the subspace of 0-wave cochains whose values on points $x \in X$ are in $W^{div}(x)$. We let $B^1_{div}(X;\mathcal{W}_1) \subset C^1(X;\mathcal{W}_1)$ consist of the coboundaries of the cochains from $C^0_{div}(X;\mathcal{W}_1)$. The elements $\gamma \in B^1_{div}(X;\mathcal{W}_1)$ are characterized by the property that on any singular 1-simplex τ the values $\gamma(\tau)$ belong to the subspace $W^{div}(\tau) \subset W(\tau)$ spanned by the divisorial subspaces at the boundary points of τ.

We are ready to define the eigenwave class $\phi \in C^1(X;\mathcal{W}_1)/B^1_{div}(X;\mathcal{W}_1)$. Let us first consider the case when all points of X have zero sedentarity, in particular $B^1_{div}(X;\mathcal{W}_1) = 0$. In such case we define the value of ϕ on a singular 1-simplex $\tau : [0,1] \to X$ as $\tau(1) - \tau(0)$. Recall that our singular chains are assumed to be compatible with the combinatorial stratification of X so that $\tau((0,1))$ is contained in a single combinatorial stratum and a single tropical chart. This means that the difference $\tau(1) - \tau(0)$ can be interpreted as a vector in the tangent space to this stratum and therefore in $W(\tau)$.

Returning to the general case, if $x \in X$ is of positive sedentarity we choose a nearby mobile point y_x which maps to x under the projection along divisorial directions. If $x \in X$ is mobile we set $y_x = x$.

Definition 18. The element $\phi \in C^1(X;\mathcal{W}_1)/B^1_{div}(X;\mathcal{W}_1)$ is defined on a 1-simplex $\tau : [0,1] \to X$ as the vector $w_\tau := y_{\tau(1)} - y_{\tau(0)} \in W(\tau)$.

Clearly the ambiguity in w_τ resulting from different choices of y_x is confined to $B^1_{div}(X;\mathcal{W}_1)$. The next proposition asserts that ϕ defines a class in $H^1(X;\mathcal{W}_1)$, which we call the *eigenwave* of X. We denote this class also by ϕ, this should not cause any confusion.

Proposition 13. $\delta\phi = 0$.

Proof. By definition the value of $\delta\phi$ on a 2-simplex σ is the sum of the values of ϕ on the three edges τ_1, τ_2, τ_3 of σ. This is clearly zero (perhaps after applying the maps $\pi : W(\tau) \to W(\sigma)$ in case some of the τ_i land in different strata). □

5.2 Action of the Eigenwave ϕ and Its Powers on Tropical Homology

The k-th cup powers of ϕ are also (higher degree) wave classes $\phi^k \in H^k(X;\mathscr{W}_k)$. One can define the value of ϕ^k on a k-simplex σ modulo the ideal in $W_k(\sigma)$ generated by the W^{div} for all vertices in σ. Namely, for an edge $\tau \prec \sigma$ let $w_\tau \in W(\sigma)$ stand for the vector $y_{\tau(1)} - y_{\tau(0)} \in W(\tau)$ pushed to $W(\sigma)$. Then

$$\phi^k(\sigma) = w_{\sigma_{01}} \wedge \cdots \wedge w_{\sigma_{k-1,k}} =: w_\sigma \in W_k(\sigma). \tag{16}$$

Taking the cap product with $\phi^k = [\phi^k_{sing}]$ gives us the homomorphism:

$$\phi^k \cap : H_q(X;\mathscr{F}_p \otimes \mathbb{R}) \to H_{q-k}(X;\mathscr{F}_{p+k} \otimes \mathbb{R}). \tag{17}$$

In case X is compact and polyhedral we consider its baricentric subdivision and think of the $H_q(X;\mathscr{F}_p)$ as simplicial or cellular homology groups. The advantage is that we can define the cap product with ϕ^k on the cycle level

$$\phi^k : C_q^{cell}(\mathscr{F}_p) \to C_{q-k}^{bar}(\mathscr{F}_{p+k} \otimes \mathbb{R}). \tag{18}$$

Below we give two different descriptions of the map (18) depending on the choice of vertex ordering. The first result is a cycle in $C_{q-k}^{bar}(\mathscr{F}_{p+k} \otimes \mathbb{R})$ while the second one is still in $C_{q-k}^{cell}(\mathscr{F}_{p+k} \otimes \mathbb{R}) \subset C_{q-k}^{bar}(\mathscr{F}_{p+k} \otimes \mathbb{R})$.

We recall the notion of the dual cells in the first baricentric subdivision of a polyhedral complex. Let $\Delta \in X$ be a q-cell. For any *finite* j-dimensional face $\Delta' \prec \Delta$ of the sedentarity $s(\Delta') = s(\Delta)$ its *dual cell* $\hat{\Delta}'_\Delta$ in the baricentric subdivision of Δ is defined as the union of all $(q-j)$-simplices in $bar(\Delta)$ containing the baricenters of Δ and Δ'. We can think of $\hat{\Delta}'_\Delta$ as a simplicial $(q-j)$-chain. The orientations of the pair Δ' and $\hat{\Delta}'_\Delta$ are taken to agree with the original orientation of Δ.

Let $\gamma = \sum \beta_\Delta \Delta$ be a cycle in $C_q^{cell}(X;\mathscr{F}_p)$. Then according to Lemma 1 the coefficients β_Δ for all $\Delta \subset X$ have to be divisible by the divisorial directions of Δ. In particular, the wedge product of β_Δ with any element in $\wedge^k(W(\Delta)/W^{div}(\Delta))$ gives a well-defined element in $\mathscr{F}_{p+k}(\Delta) \otimes \mathbb{R}$. We can also think of $\gamma = \sum_{\sigma \in bar(\Delta)} \beta_\Delta \sigma$ as an element in $C_q^{bar}(X;\mathscr{F}_p)$.

Description 1. We label the vertices of each q-simplex σ in $bar(\Delta)$ according to the dimension of the largest cells whose baricenters they represent (recall that several faces of Δ of different sedentarity may have the same baricenter). In this case the cycle $\phi^k \cap \gamma \in C_{q-k}^{bar}(X;\mathscr{F}_{p+k})$ is supported on the dual subdivision inside the q-skeleton of X.

Precisely, for every k-face Δ' of Δ let $w_{\Delta'} \in W_k(\Delta)$ denote the volume element associated to Δ' as in (16). Clearly, $w_{\Delta'}$ equals the sum of all $w_{\sigma_{0\ldots k}}$ (taken with

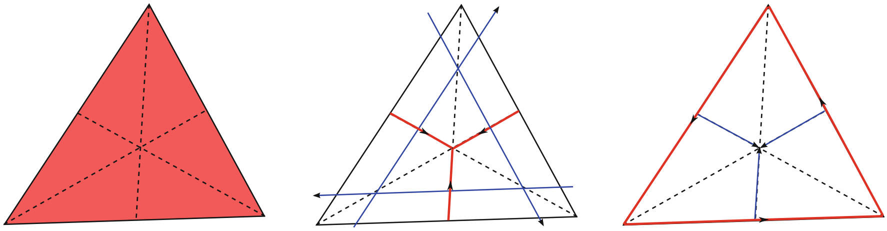


Fig. 5 The two descriptions of the wave action on a 2-cell σ. The support of $\phi_{sing} \cap \sigma$ is *red* and the framing is *blue*

appropriate signs) for the k-simplices $\sigma_{0\dots k}$ forming the baricentric triangulation of Δ'. Then one can easily calculate from the definition of the cap product:

$$\phi^k \cap (\sum_{\sigma \in bar(\Delta)} \beta_\Delta \sigma) = \sum_{\Delta' \prec \Delta} (w_{\Delta'} \wedge \beta_\Delta) \hat{\Delta}'_\Delta, \tag{19}$$

where the sum is taken over all k-dimensional faces of Δ. Note that higher sedentary k-faces don't appear in the sum because β_Δ vanishes when pushed to these higher sedentary faces (Fig. 5).

Description 2. Here we label the vertices of each σ in the opposite order to the description 1. That is the baricenters with the smaller numbers correspond to the larger faces. Now the cycle $\phi^k \cap \gamma \in C^{bar}_{q-k}(X; \mathscr{F}_{p+k})$ is supported on the $(q-k)$-skeleton of X.

Precisely, for every $(q-k)$-face Δ' of Δ let $\hat{w}_{\Delta'} \in W_k(\Delta)$ denote the polyvector corresponding to the integration along the chain $\hat{\Delta}'_\Delta$. Note that the faces $\sigma_{k\dots q}$ lie in the $(q-k)$-faces of Δ. The polyvectors $w_{\sigma_{0\dots k}}$ sum to $\hat{w}_{\Delta'}$ for those simplices $\sigma \in bar(\Delta)$ whose faces $\sigma_{k\dots q}$ give the same simplex in $bar(\Delta')$. Then again from the definition of the cap product we can write:

$$\phi^k \cap (\sum_{\sigma \in bar(\Delta)} \beta_\Delta \sigma) = \sum_{\Delta' \prec \Delta} (\hat{w}_{\Delta'} \wedge \beta_\Delta) \Delta', \tag{20}$$

where the sum is taken now over all $(q-k)$-dimensional faces of Δ.

It is straight forward to check that in both cases the resulting chain

$$\phi^k \cap (\sum_\Delta \sum_{\sigma \in bar(\Delta)} \beta_\Delta \sigma)$$

is a cycle.

Conjecture 1. Let X be a smooth compact tropical variety. Then for $q \geq p$

$$\phi^{q-p} \cap : H_q(X; \mathscr{F}_p \otimes \mathbb{R}) \to H_p(X; \mathscr{F}_q \otimes \mathbb{R})$$

is an isomorphism.

We will prove the conjecture in the realizable case in Sect. 6.2 though we believe that realizability assumption is not necessary. Certain amount of smoothness, on the other hand, is essential. In the non-smooth case even the ranks of $H_q(X;\mathscr{F}_p)$ and $H_p(X;\mathscr{F}_q)$ may not agree. A simple example is provided by the nodal genus 2 curve (see Example 2).

The action of the eigenwave ϕ is trivial on straight tropical (p,p)-classes.

Theorem 1. *If* $\gamma \in H^{straight}_{p,p}(X)$ *then* $\phi \cap \gamma = 0$.

Proof. Any vector parallel to a simplex σ of a special tropical cycle turns to zero after the wedge product with the volume element of σ. □

6 Intermediate Jacobians

6.1 Tropical Tori

Let V be a g-dimensional real vector space containing two lattices Γ_1, Γ_2 of maximal rank, that is $V \cong \Gamma_{1,2} \otimes \mathbb{R}$. Suppose we are given an isomorphism $Q : \Gamma_1 \to \Gamma_2^*$, which is symmetric if thought of as a bilinear form on V.

Definition 19. The torus $J = V/\Gamma_1$ is the *principally polarized tropical torus* with Q being its polarization. The tropical structure on J is given by the lattice Γ_2. If, in addition Q is positive definite, we say that J is an *abelian variety*.

Remark 3. The map $Q : \Gamma_1 \to \Gamma_2^*$ provides an isomorphism of $J = V/\Gamma_1$ with the tropical torus V^*/Γ_2^*. The tropical structure on the latter is provided by the lattice Γ_1^*.

Remark 4. The above data $(V, \Gamma_1, \Gamma_2, Q)$ is equivalent to a non-degenerate real-valued quadratic form Q on a free abelian group $\Gamma_1 \cong \mathbb{Z}^g$. The other lattice $\Gamma_2 \subset V := \Gamma_1 \otimes \mathbb{R}$ is defined as the dual lattice to the image of Γ_1 under the isomorphism $V \to (V)^*$ given by Q.

Let us take the free abelian group $\Gamma_1 = H_q(X;\mathscr{F}_p) \cong \mathbb{Z}^g$ with $p + q = \dim X$, and $p \leq q$. We define the tropical intermediate Jacobian as the torus above together with a symmetric bilinear form Q on $H_q(X;\mathscr{F}_p)$.

The form Q is a certain intersection product on tropical cycles which we define in two ways. The first definition is manifestly symmetric while the second definition descends to homology. And then we show that the two definitions are equivalent.

Unfortunately we are not able to show in this paper that the form is non-degenerate, though we believe that in the smooth and compact case this should be true (cf. Conjecture 2).

6.2 Intersection Product

Let X be a compact tropical space of dimension n. For a singular simplex σ we denote its relative interior by $int(\sigma)$. We abuse the notation $int(\sigma)$ to denote also its image in X.

Definition 20. We say that a tropical chain $\sum \beta_\sigma \sigma \in C_q(X; \mathscr{F}_p)$ is *transversal to the combinatorial stratification of* X (or, simply, transversal) if for any simplex σ and any face $\tau \prec_k \sigma$ we have

- $int(\tau)$ meets strata of X only of dimension $(n-k)$ and higher;
- if τ lies in a sedentary stratum of X then β_σ is divisible by all corresponding divisorial directions.

Definition 21. We say that two transversal tropical chains $\sum \beta_{\sigma'}\sigma' \in C_{q'}(X; \mathscr{F}_{p'})$ and $\sum \beta_{\sigma''}\sigma'' \in C_{q''}(X; \mathscr{F}_{p''})$ form a *transversal pair* if the following holds. For every pair of simplices σ', σ'' from these chains and any choice of their faces $\tau' \prec \sigma', \tau'' \prec \sigma''$, if the interiors $int(\tau'), int(\tau'')$ lie in the same stratum $\mathscr{E}$ then $int(\tau'), int(\tau'')$ are transversal in the usual sense as smooth maps to $\mathscr{E}$.

If a pair of simplices σ', σ'' from the transversal pair have non-empty intersection then all three submanifolds $\sigma', \sigma'', \sigma' \cap \sigma''$ are supported on the same maximal stratum $\mathscr{E}_{\sigma' \cap \sigma''}$ of X (and on no smaller strata). The oriented triple $\sigma', \sigma'', \sigma' \cap \sigma''$ determines an integral volume element $\mathrm{Vol}_{\mathscr{E}_{\sigma' \cap \sigma''}}$ as well as its dual volume form $\Omega_{\mathscr{E}_{\sigma' \cap \sigma''}}$. By transversality, $\sigma' \cap \sigma''$ has dimension $q' + q'' - n$. We can choose a singular chain $\sum \tau$ representing its relative fundamental class agreeing with the orientation of $\sigma' \cap \sigma''$.

Let $\gamma' = \sum \beta_{\sigma'}\sigma' \in C_{q'}(X; \mathscr{F}_{p'})$ and $\gamma'' = \sum \beta_{\sigma''}\sigma'' \in C_{q''}(X; \mathscr{F}_{p''})$ be a transversal pair of tropical chains. We define the following bilinear product with values in the cowave chains:

$$\gamma' \cdot \gamma'' = \sum_{\tau \subset \sigma' \cap \sigma''} \Omega_{\mathscr{E}_\tau}(\beta_{\sigma'} \wedge \beta_{\sigma''}) \cdot \tau \in C_{q'+q''-n}(X; \mathscr{W}^{n-p'-p''}). \qquad (21)$$

Remark 5. Note that $\gamma' \cdot \gamma''$ has no support on infinite simplices $\tau \subset \sigma' \cap \sigma''$ since the divisorial directions in $W^{div}(\tau)$ divide both $\beta_{\sigma'}$ and $\beta_{\sigma''}$.

Remark 6. If $q' + q'' < n$ or $p' + p'' > n$ then $\gamma' \cdot \gamma'' = 0$ for dimensional reasons. In what follows we will tacitly assume this is not the case.

From now on we assume that X is a compact smooth tropical space. Our goal will be to show that in this case the above product descends to homology.

First we show that we can deform all cycles to a transverse position. Since the question is local we can work in a chart $\phi_\alpha : U_\alpha \to Y \subset \mathbb{T}^N$. The next lemma says that we can move a tropical cycle γ off a face E of Y, if it intersects it in higher than expected dimension, not changing it outside the open star $\mathrm{St}(E)$.

Lemma 3. *Let $\gamma \in C_q(X, \mathscr{F}_p)$ be a (singular) tropical cycle in a tropical n-dimensional manifold X and let E be an l-face of Y in a chart $\phi_\alpha : U_\alpha \to Y \subset \mathbb{T}^N$. Then there exists a cycle $\gamma' = \sum \beta_\sigma \sigma \in C_q(X; \mathscr{F}_p)$ homologous to γ and such that for any $(q-k)$-face τ of a simplex σ we have $int(\tau) \cap E = \emptyset$, i.e. τ is not supported on E whenever $k + l < n$. In addition, γ' satisfies to the following properties:*

- $\gamma \cap (X \smallsetminus (U_\alpha \cap \mathrm{St}(E))) = \gamma' \cap (X \smallsetminus (U_\alpha \cap \mathrm{St}(E)))$,
- *the chain $\gamma - \gamma'$ is the boundary of a tropical $(q+1)$-chain supported in $U_\alpha \cap \mathrm{St}(E)$,*
- *if E has positive sedentarity then any simplex σ such that $\sigma \cap E \neq \emptyset$ has its coefficient β_σ divisible by all divisorial vectors corresponding to E.*

Proof. First let us consider the case when E is mobile. Working in a chart we can assume Y is the Bergman fan for some loopless matroid M. Clearly, any matroid M contains a uniform submatroid $M_0 \subset M$ of the same rank $r(M)$ (by submatroid we mean a subset with the restriction of the rank function). Thus, we have a sequence $M_0 \subset \cdots \subset M_{|M|-r(M)} = M$ of submatroids of M such that M_{j+1} is obtained from M_j by adding one element ϵ_{j+1}. We may form a matroid H_j of rank $r(M_j) - 1$ by setting a new rank function r_{H_j} on M_j, $r_{H_j}(A) = r_M(A \cup \epsilon_{j+1}) - 1$ for $A \subset M_j$.

The fan $Y_{M_{j+1}} \subset \mathbb{R}^{|M_j|}$ maps to the fan $Y_{M_j} \subset \mathbb{R}^{|M_j|-1}$ by projection along the coordinate corresponding to the element ϵ_{j+1}. If the matroid H_j has loops this map

$$\tau_j : Y_{M_{j+1}} \to Y_{M_j}$$

is an isomorphism. Otherwise note that the Bergman fan Y_{H_j} is a subfan of Y_{M_j}. Also we denote by $Y'_{M_{j+1}}$ the subfan of $Y_{M_{j+1}}$ containing only those cones whose corresponding flags do *not* have two flats differing just by ϵ_{j+1}, see Fig. 6. Then $\tau_j : Y'_{M_{j+1}} \to Y_{M_j}$ is a one-to-one map linear on the cones, cf. [11]. Indeed, τ_j contracts precisely those cones of $Y_{M_{j+1}}$ which are parallel to $e_{\epsilon_{j+1}}$.

Y_{M_0} is a complete fan in $\mathbb{R}^{r(M)-1}$ and the lemma is trivial since the coefficients $\mathscr{F}_p = \Lambda^p \mathbb{Z}^{r(M)-1}$ are constant on all strata and we may deform γ into a general position (subdividing simplices in γ if needed to keep the chain strata-compatible). Inductively we suppose that the lemma holds for Y_{M_j} and the matroid H_j is loopless and then prove that the lemma holds for $Y_{M_{j+1}}$.

We denote by $\mathrm{St}(e_{\epsilon_{j+1}})$ the complement of $Y'_{M_{j+1}}$ in $Y_{M_{j+1}}$. It really is the open star of $e_{\epsilon_{j+1}}$ (in the coarsest face structure of $Y_{M_{j+1}}$). Note that $\mathrm{St}(e_{\epsilon_{j+1}}) \cong Y_{H_j} \times \mathbb{R}$.

If $E \subset \mathrm{St}(e_{\epsilon_{j+1}})$ we may use the inductive assumption for projections to Y_{H_j} (it has smaller dimension) together with a deformation along a generic vector field parallel to $e_{\epsilon_{j+1}}$.

If $E \not\subset \mathrm{St}(e_{\epsilon_{j+1}})$, that is E is contained in $Y'_{M_{j+1}}$ we have $\dim(\tau_j(E)) = \dim(E) = l$. Consider singular q-simplices from γ with the interiors mapped to $\mathrm{St}(E)$ and such that their closures intersect E. These simplices form a chain γ_E which can be considered as a relative cycle modulo its boundary $\partial\gamma_E$. We have $\partial\gamma_E \cap E = \emptyset$. Furthermore, $\tau_j(\partial\gamma_E)$ is a $(q-1)$-cycle in the $(n-1)$-dimensional tropical

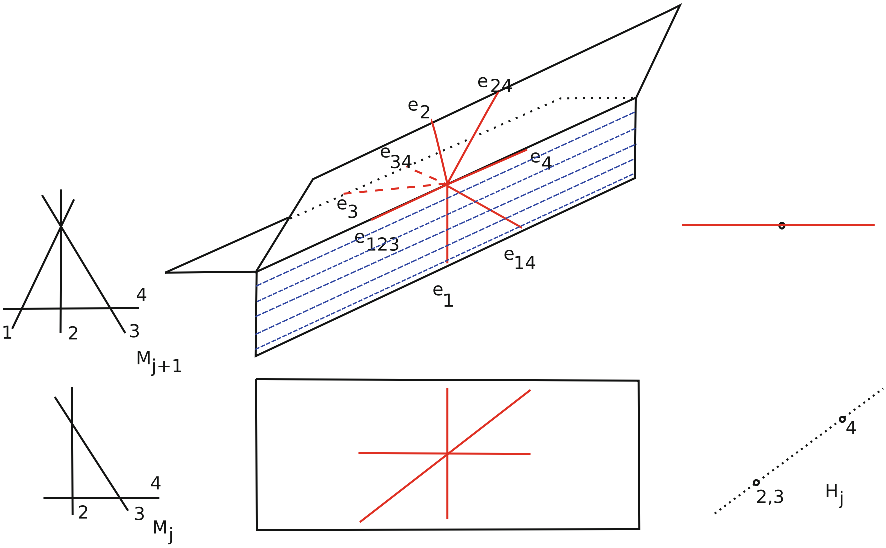


Fig. 6 The matroids M_{j+1}, M_j and H_j and their corresponding fans. The fan $Y'_{M_{j+1}}$ is the unshaded part of $Y_{M_{j+1}}$. The shaded part of $Y_{M_{j+1}}$ is $\mathrm{St}(e_{\epsilon_{j+1}})$

manifold Y_{H_j}. By induction on dimension we may assume that $\tau_j(\partial\gamma_E) \cap \mathrm{St}(e_{\epsilon_{j+1}})$ can be deformed in Y_{H_j} to a cycle with simplices without faces of dimension larger than $q-n+l$ whose relative interiors are contained in E. As $\mathrm{St}(e_{\epsilon_{j+1}}) \cong Y_{H_j} \times \mathbb{R}$ such deformation lifts to $Y_{M_{j+1}}$ and can be extended to a deformation of γ in $Y_{M_{j+1}}$.

By induction on j there exists a tropical chain $b_j \in C_{q+1}(Y_{M_j}; \mathscr{F}_p)$ such that the relative interiors of k-faces of singular simplices of $\gamma'_j = \partial B_j - \tau_j(\gamma)$ are disjoint from E. This assumption holds for any face structure on Y_{M_j}, in particular for the one compatible with Y_{H_j}. Then the relative interiors of all q-dimensional simplices are disjoint from Y_{H_j} and we can form $\tilde{b}_j \in C_{q+1}(Y_{M_{j+1}}; \mathscr{F}_p)$ and $\tilde{\gamma}'_j \in C_{q+1}(Y_{M_{j+1}}; \mathscr{F}_p)$ by applying $\tau_j^{-1}|_{Y_{M_j} \smallsetminus Y_{H_j}}$ to b_j and γ'_j. Note that $\partial\tilde{b}_j - \gamma - \tilde{\gamma}'_j$ must have the coefficients vanishing under τ_j, even though generated from the facets of $Y'_{M_{j+1}}$. Such coefficients must be supported on $\mathrm{St}(e_{\epsilon_{j+1}})$ and thus we may apply the same reasoning as in the case of $E \subset \mathrm{St}(e_{\epsilon_{j+1}})$.

Finally, let us now consider the sedentary case, that is let E be a sedentarity s face of $Y_M \times \mathbb{T}^s$ with $s = |I| > 0$. Let ξ_j be the divisorial vectors, and let $V_J := \wedge_{j\in J}\xi_j$ denote the divisorial $|J|$-polyvector for each $J \subset I$. We will need to deform γ to γ' so that no $(q-s)$-dimensional or smaller-dimensional face of a simplex σ in γ' meets $Y_M \times \{-\infty\}$ (here $\{-\infty\} \in \mathbb{T}^s$ is the point of sedentarity s). In $Y_M \times \mathbb{R}^I$ the groups $\mathscr{F}_p$ split into the direct sum $\oplus_{J\subset I}\mathscr{F}_p^J$, where $\mathscr{F}_p^J$ consists of elements divisible by the polyvector V_J, and no larger $V_{J'}$. (The splitting is not canonical, it depends on a chart). Accordingly, we have a decomposition $\gamma = \sum_{J\subset I}\gamma_J$ into cycles.

If $J \neq I$, that is there exists $j \notin J$, we may push γ_J from E with the help of a vector field parallel to x_j. Note that γ_J remains a cycle after such deformation as ξ_j is not present in the coefficients of γ_J. Thus by induction on sedentarity we may assume $J = I$.

The cycle γ_I has coefficients in $\mathscr{F}_{p-s}^{Y_M} \otimes V_I$, and hence can be interpreted as a relative cycle modulo $\partial\mathbb{T}^I = \mathbb{T}^I \setminus \mathbb{R}^I$ with coefficients in $\mathscr{F}_{p-s}^{Y^M}$ (as V_I vanishes on $\partial\mathbb{T}^I$ and constant otherwise) and $(T^I, \partial T_I)$ is homeomorphic to the pair $\mathbb{R}^{s-1} \times (\mathbb{R}_{\geq 0}, \{0\})$ of a half-space and its boundary. Thus γ_I may be deformed to a product (after simplicial subdivision) of the relative fundamental cycle in the s-dimensional half-space with some $(q-s)$-dimensional singular cycle. In particular, E will not meet any codimension $< s$ face of a q-simplex in a deformed cycle. □

Remark 7. Let $\Sigma = \bigcup \sigma$ be an integral polyhedral fan (with its cones σ oriented). Then using the inclusion homomorphisms (5) we can form the complex $C_k^{(p)} := \bigoplus_{\dim \sigma = k} \mathscr{F}_p(\sigma)$. In case Σ is a matroidal fan the statement of Lemma 3 is equivalent to that the complex $C_\bullet^{(p)}$ has only the highest homology.

Remark 8. When X is not smooth the statement of the Lemma is not true. For example let X be a union of two 2-planes in $\mathbb{R}^4$ intersecting in a point. Consider an unframed path (that is cycle in $C_1(X; \mathscr{F}_0)$ through the vertex which starts in one plane and ends in the other plane. Any deformation of this path will still have to go through the vertex.

Corollary 2. *Let X be a tropical manifold. Then*

1. *Every class in $H_q(X; \mathscr{F}_p)$ is represented by a transversal cycle.*
2. *Every pair of classes in $H_{q'}(X; \mathscr{F}_{p'})$ and $H_{q''}(X; \mathscr{F}_{p''})$ is represented by a transversal pair of cycles.*
3. *If γ_1', γ_2' are two cycles which represent the same class in $H_{q'}(X; \mathscr{F}_{p'})$ and both form transversal pairs with a cycle $\gamma'' \in C_{q''}(X; \mathscr{F}_{p''})$, then there is $b \in C_{q'+1}(X; \mathscr{F}_{p'})$ which form a transversal pair with γ'', and such that $\partial b = \gamma_1' - \gamma_2'$.*

Proof. We may start from any tropical cycle and deform it to a transversal position by applying Lemma 3 stratum by stratum starting from 0-dimensional faces and then higher-dimensional strata. (Note that in a chart the open star of any face can intersect only faces of higher dimension).

Suppose that we have two transversal cycles. Since any stratum $\mathscr{E}$ is a manifold we can make interiors of faces of the simplices from these cycles transversal in $\mathscr{E}$ by a small deformation with the help of the usual Sard's theorem. In any chart this deformation extends to a small deformation in $\mathrm{St}(\mathscr{E})$. Making this procedure stratum by stratum in the order of non-decreasing dimension we make any pair of cycles transversal. A similar argument applies to the relative cycle in the last statement of the corollary. □

If $p' + p'' + q' + q'' = 2n$ we can give a numerical value to the product $\gamma' \cdot \gamma''$ by integrating the $(n - p' - p'')$-form $\Omega_{\mathscr{E}_\tau}(\beta_{\sigma'} \wedge \beta_{\sigma''})$ over the $(q' + q'' - n)$-simplex τ.

Indeed, since $\beta_{\sigma'} \wedge \beta_{\sigma''}$ is divisible by all divisorial directions corresponding to sedentary faces of $\tau = \sigma' \cap \sigma''$, the integration can be carried over in the quotient space to those (infinite) coordinates, thus giving a finite answer. Thus we define

$$\int \gamma' \cdot \gamma'' := \sum_{\tau \subset \sigma' \cap \sigma''} \int_\tau \Omega_{\mathcal{E}_\tau}(\beta_{\sigma'} \wedge \beta_{\sigma''}) \in \mathbb{R}. \tag{22}$$

The most interesting case to us is when $p' + q' = p'' + q'' = n$. Assuming $q' + q'' \geq n$ we can use the eigenwave action on one of the cycles in the pair to make them of complementary dimensions, after which the integration becomes just summing over the intersection points

$$\langle \gamma', \gamma'' \rangle := \int \gamma' \cdot \gamma'' = \sum_{x \in |\gamma'| \cap |\gamma''|} \Omega_x(\beta'_x \wedge \beta''_x),$$

where β'_x, β''_x are the coefficients at σ', σ'' for their intersection points $x \in \sigma' \cap \sigma''$.

Proposition 14. *Let* $\gamma' = \sum \beta_{\sigma'} \sigma' \in C_{q'}(X; \mathcal{F}_{p'})$ *and* $\gamma'' = \sum \beta_{\sigma''} \sigma'' \in C_{q''}(X; \mathcal{F}_{p''})$ *be a transversal pair of tropical cycles with* $p' + q' = p'' + q'' = n$ *and* $q' + q'' \geq n$. *Let* $k := q' - p'' = q'' - p' \geq 0$. *Then*

$$\langle \phi^k \cap \gamma', \gamma'' \rangle = \int \gamma' \cdot \gamma''.$$

Proof. First we need a representative of the cycle $\phi^k \cap \gamma'$ such that it still forms a transversal pair with γ''. We fix first and second baricentric subdivisions of the simplices σ' in γ'. Then by transversality of γ'' we can assume that the intersection of each σ' with γ'' is supported on the star skeleton of σ'. That is $\sigma' \cap |\gamma''|$ consists of the k-simplices of the first baricentric subdivision of σ' spanned by the baricenters of the $q' - k, \ldots, q'$-dimensional faces τ of σ'. We label the k-simplices in the first baricentric subdivision of σ' by the flags of its faces $(\tau_0 \prec \cdots \prec \tau_k)$.

Then the result of the wave action (19) from Description 1 on $\beta_{\sigma'} \sigma'$ gives the following chain (see Fig. 7)

$$\sum_{\tau_0 \prec \cdots \prec \tau_k} (w_{\tau_0 \prec \cdots \prec \tau_k} \wedge \beta_{\sigma'}) \widehat{(\tau_0 \prec \cdots \prec \tau_k)},$$

where $w_{\tau_0 \prec \cdots \prec \tau_k} \in W_k(\Delta_{\sigma'})$ is the polyvector associated to the simplex $(\tau_0 \prec \cdots \prec \tau_k)$, and $\widehat{(\tau_0 \prec \cdots \prec \tau_k)}$ is its star dual in the second baricentric subdivision (cf. definition in Sect. 5.2). When intersected with γ'' only the simplices $(\tau_0 \prec \cdots \prec \tau_k)$ with maximal dimensional flags enter and we see that the result coincides with the definition of $\int \gamma' \cdot \gamma''$. □

Proposition 15. *Let* X *be smooth. Then the intersection product* $\langle \ , \ \rangle$ *on cycles descends to a pairing on homology* $H_q(X; \mathcal{F}_p) \otimes H_p(X; \mathcal{F}_q) \to \mathbb{R}$.

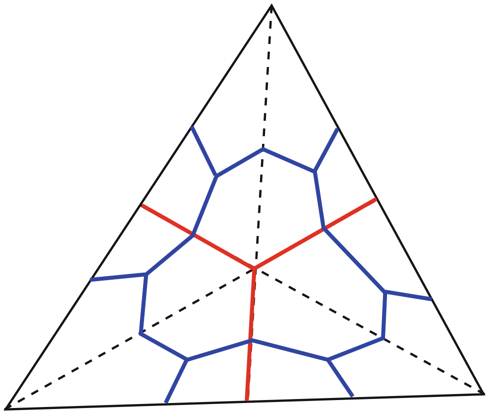


Fig. 7 Intersection in σ': $|\gamma''|$ (*in red*), $|\phi^k \cap \gamma'|$ (*in blue*)

Proof. Suppose that we have two homologous cycles $\gamma_1' \in C_q(X;\mathscr{F}_p)$ and $\gamma_2' \in C_q(X;\mathscr{F}_p)$. Let $b \in C_{q+1}(X;\mathscr{F}_p)$ be the connecting chain, i.e. $\partial b = \gamma_1' - \gamma_2'$. According to Corollary 2 we can assume that each of the three γ_1', γ_2', b forms a transversal pair with a cycle $\gamma'' \in C_p(X;\mathscr{F}_q)$.

It is clear that $\partial(b \cdot \gamma'')$ coincides with the $\gamma_1' \cdot \gamma'' - \gamma_2' \cdot \gamma''$ on the interiors of the maximal strata of X. Thus it is enough to show that $b \cdot \gamma''$ has no boundary on codimension 1 mobile strata of X (according to Lemma 3 the intersection has no support on infinite simplices). This is local so we can work in a chart $\phi_\alpha : U_\alpha \to Y \subset \mathbb{T}^N$.

Let E be a codimension 1 face of Y and let $D_1, \dots, D_k$ be the adjacent facets at E. We choose $v_1, \dots, v_k$, the corresponding primitive vectors such that $\sum_{i=1}^k v_i = 0$ (not just modulo the span of E). Let x be a point in the relative interior of E where b intersects γ'', and let $\tau_1, \dots, \tau_k$ be the intervals in the support of $b \cdot \gamma''$ adjacent to x. Each τ_i lies in D_i. Let $\beta_i' \in \mathscr{F}_p(D_i)$ and $\beta_i'' \in \mathscr{F}_q(D_i)$ be the coefficients of the simplices of b and of γ'', respectively, which intersect at the τ_i.

Since γ'' is a cycle, we have $\sum_i \beta_i'' = 0$. We can write each

$$\beta_i'' = v_i \wedge \bar{\alpha}_i'' + \alpha_i'',$$

where $\bar{\alpha}_i'' \in W_{q-1}(E)$ and $\alpha_i'' \in W_q(E)$.

Recall that our tropical space X is smooth. In particular, this means that the fan at E modulo linear span of E is matroidal. That is, $\sum_{i=1}^k v_i = 0$ is the *only* linear relation among the v_i's. This together with $\sum_i \beta_i'' = 0$ implies that

$$\sum_{i=1}^k \alpha_i'' = 0 \quad \text{and} \quad \bar{\alpha}_1'' = \cdots = \bar{\alpha}_k'' =: \bar{\alpha}''.$$

Similarly, $\sum_i \beta_i' = 0$ since ∂b cannot have support at x. Hence we can write

$$\beta_i' = v_i \wedge \bar{\alpha}' + \alpha_i',$$

with $\sum \alpha'_i = 0$, $\alpha'_i \in W_p(E)$ and $\bar{\alpha}' \in W_{p-1}(E)$. Note that in the product

$$\beta'_i \wedge \beta''_i = (v_i \wedge \bar{\alpha}' + \alpha'_i) \wedge (v_i \wedge \bar{\alpha}'' + \alpha''_i) = v_i \wedge (\bar{\alpha}' \wedge \alpha''_i + \alpha'_i \wedge \bar{\alpha}'')$$

only the cross terms survive. Now we are ready to evaluate $\partial(b \cdot \gamma'')$ at x:

$$\sum_i \Omega_{\Delta_i}[v_i \wedge (\bar{\alpha}' \wedge \alpha''_i + \alpha'_i \wedge \bar{\alpha}'')] = \sum_i \Omega_{\Delta}(\bar{\alpha}' \wedge \alpha''_i + \alpha'_i \wedge \bar{\alpha}'')$$
$$= \Omega_{\Delta}(\bar{\alpha}' \wedge \sum_i \alpha''_i + \sum_i \alpha'_i \wedge \bar{\alpha}'') = 0.$$

□

Finally we restrict to the case when both γ', γ'' are cycles in $C_q(X; \mathscr{F}_p)$ with $p + q = n$. Then $\gamma' \cdot \gamma'' = \gamma'' \cdot \gamma'$. Indeed, assuming the orientation of τ is chosen, taking the product in the opposite order will result in the change of sign of the volume form $\Omega_{\mathscr{E}_\tau}$ according to the parity of p. On the other hand this parity will also affect the coefficients product: $\beta' \wedge \beta'' = (-1)^p \beta'' \wedge \beta'$, both effects cancel in $\Omega_{\mathscr{E}_\tau}(\beta' \wedge \beta'')$. This observation combined with Propositions 14 and 15 lead to the final statement.

Theorem 2. *Let X be compact and smooth. The product on cycles (22) descends to a symmetric bilinear form on $H_q(X; \mathscr{F}_p)$ for any $p + q = n$.*

Conjecture 2. This form is non-degenerate.

Appendix: Konstruktor and the Eigenwave Action in the Realizable Case

Tropical Limit and the Steenbrink–Illusie Spectral Sequence

Suppose X is the tropical limit of a complex projective one-parameter degeneration $\mathscr{X} \to \Delta^*$. Then X is naturally polyhedral. We assume also that X is smooth. In this case the refined stable reduction theorem [7] allows us assume the following (see details in [6]).

- X is unimodularly triangulated. This means that the finite cells are unimodular simplices and the infinite cells are products of unimodular simplices and unimodular cones spanned by the divisorial vectors.
- The finite part of X is identified with the dual Clemens complex of the degeneration with simple normal crossing central fiber $Z = \cup Z_\alpha$. This means that the components of Z are labelled by vertices of zero sedentarity and their intersections $Z_{\alpha_0} \cap \cdots \cap Z_{\alpha_k} =: Z_\Delta$ are labelled by (finite) simplices $\Delta = \{\alpha_0 \ldots \alpha_k\}$ of X of zero sedentarity.

Theorem 3. *Let X be a realizable smooth projective tropical variety. Then for $q \geq p$*

$$\phi^{q-p} : H_q(X; \mathscr{F}_p) \otimes \mathbb{Q} \to H_p(X; \mathscr{F}_q) \otimes \mathbb{Q}$$

is an isomorphism.

In this algebraic setting the eigenwave itself is an integral class in $H^1(X; \mathscr{W}_1)$ (recall that W carries a natural lattice). Hence in the statement we can avoid tensoring the tropical homology groups with $\mathbb{R}$. However its proof relies on the isomorphism in Theorem 4 which we can assert only over $\mathbb{Q}$. Although we believe that the theorem remains true over $\mathbb{Z}$ its proof may be more delicate.

We will prove the theorem by comparing the eigenwave action with the classical monodromy action $T : H_k(X_t, \mathbb{Q}) \to H_k(X_t, \mathbb{Q})$, where X_t is a general fiber in $\mathscr{X}$. The idea that the monodromy can be represented by a cap product with certain cohomology class appeared before in the Calabi-Yau case. The second author [12] proved a related conjecture of Gross [3] that for toric hypersurfaces the monodromy can be described as the fiber-wise rotation by a natural section of the SYZ fibration. Later Gross and Siebert ([4], Sect. 5.1) explored the relation between the monodromy and the cap product in the logarithmic setting.

Notations:

- Δ or Δ' will always denote a finite face of X of sedentarity 0, in particular, a simplex.
- $H_{2l}(\Delta)[-r] = H_{2l}(Z_\Delta, \mathbb{Q})$, Tate twisted by $[-r, -r]$.
- $H_{2l}(k)[-r] = \oplus H_{2l}(\Delta)[-r]$, where Δ runs over all k-simplices in X as above.

First we recall the classical spectral sequence which calculates the limiting mixed Hodge structure of the family $\mathscr{X}$ (see, e.g. [10], Chap. 11). This spectral sequence (from now on referred to as the Steenbrink-Illusie's, or SI for short) has the first term

$$E^1_{r,k-r} = \bigoplus_{i \geq \max\{0,r\}} H_{k+r-2i}(2i-r)[r-i],$$

and it degenerates at E_2 abutting to homology of the smooth fiber X_t of $\mathscr{X}$ with the monodromy weight filtration.

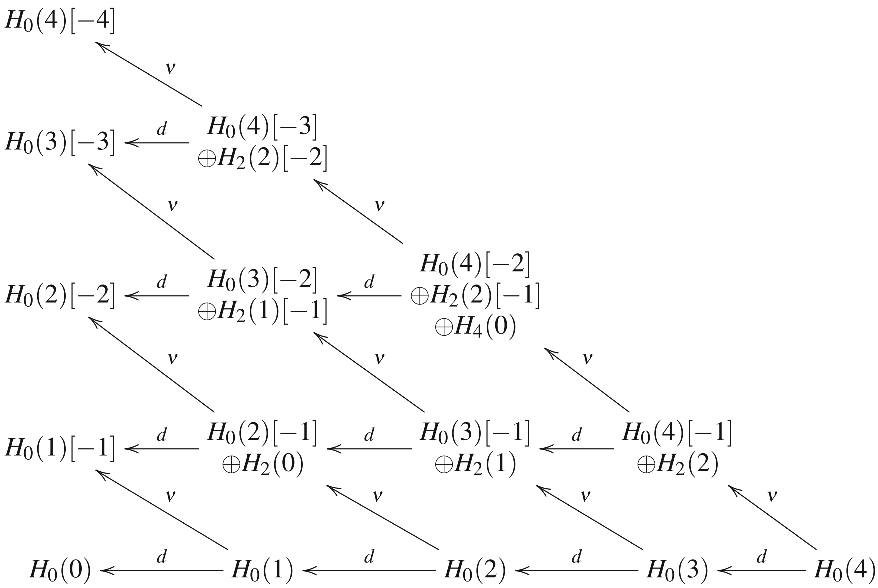
Since all strata in Z are blow ups of projective spaces, the odd rows in Steenbrink-Illusie's E^1 vanish. Removing those and making shifts in the even rows we relabel the terms by

$$\tilde{E}^1_{q,p} := E^1_{q-p,2p} = \bigoplus_{i \geq \max\{0,q-p\}} H_{2q-2i}(2i+p-q)[q-p-i].$$

The first differential $d = d' + d''$ consists of the map d' induced by strata inclusion and the Gysin map d'':

$$d' : H_{2l}(k)[-r] \to H_{2l}(k-1)[-r]$$
$$d'' : H_{2l}(k)[-r] \to H_{2l-2}(k+1)[-r-1].$$

For reader's convenience we write the beginning of the $\tilde{E}^1$ term:



The monodromy operator $\nu = \frac{1}{2\pi i} \log T$ acts along the diagonals by the Tate twist isomorphism $H_{2l}(k)[-r] \to H_{2l}(k)[-r-1]$ or by 0 if the corresponding group is missing (cf. [10], Chap. 11).

Propellers

Next we will give a combinatorial description of the SI groups and the differential in terms of *propellers*—the "local tropical cycles" in X.

Some more notations:

- Recall that $\Delta, \Delta', \Delta''$ always denote finite faces of X of sedentarity 0.
- We write $\Delta \prec_k \Delta'$ or $\Delta' \succ_k \Delta$, if Δ is a face of Δ' of codimension k.
- $\mathrm{Link}_l(\Delta)$ consists of sets $\bar{q} = \{q_1, \ldots, q_l\}$ where each q_i is either a vertex or a divisorial vector, such that the vertices of Δ together with elements of $\bar{q}$ span a face (infinite, in case $\bar{q}$ contains divisorial vectors) adjacent to Δ of dimension l

higher. We denote the corresponding face by $\{\Delta\bar{q}\}$ and often drop the brackets from the notation (e.g., as below) when they become cumbersome.

- $\mathrm{Link}_l^0(\Delta) \subset \mathrm{Link}_l(\Delta)$ consists of those sets $\bar{q} = \{q_1, \dots, q_l\}$ where q_i are allowed to be only vertices (not the divisorial vectors). In this case $\{\Delta\bar{q}\}$ is finite.
- $\mathrm{Vol}_{\Delta\bar{q}}$ is the integral volume element in the (oriented) face $\{\Delta\bar{q}\}$.

Let Δ be an oriented finite cell of sedentarity 0. One can naturally identify (see [6] for details) the homology groups $H_{2l}(\Delta)$ with the space of local tropical relative l-cycles around Δ. That is, we consider formal $\mathbb{Q}$-linear combinations

$$\sum_{\bar{q}\in\mathrm{Link}_l(\Delta)} \rho_{\bar{q}}\{\Delta\bar{q}\}$$

of (possibly infinite) cells $\{\Delta\bar{q}\} \succ_l \Delta$ which are balanced along Δ. We call these local cycles *propellers* and abusing the notation we continue denoting this group by $H_{2l}(\Delta)$ (there is no Tate twist however).

Then one can identify the Gysin map $d'' : H_{2l}(\Delta) \to H_{2l-2}(\Delta')$ with the restriction of the propeller to a consistently oriented finite simplex $\Delta' \succ_1 \Delta$. Put together

$$d''\Big(\sum_{\bar{q}\in\mathrm{Link}_l(\Delta)} \rho_{\bar{q}}\{\Delta\bar{q}\}\Big) = \sum_{q\in\mathrm{Link}_1^0(\Delta)} \Big(\sum_{\bar{r}\in\mathrm{Link}_{l-1}(\Delta q)} \rho_{q\bar{r}}\{\Delta q\bar{r}\}\Big). \qquad (23)$$

The inclusion map $d' : H_{2l}(\Delta) \to H_{2l}(\Delta')$, where $\Delta' = \Delta \setminus v$ is consistently oriented facet of Δ, is somewhat more tricky. Let $c = \sum_{\bar{q}\in\mathrm{Link}_l(\Delta)} \rho_{\bar{q}}\{\Delta\bar{q}\}$ be an element in $H_{2l}(\Delta)$. For any $\bar{q} \in \mathrm{Link}_l(\Delta)$ let $\{\Delta'\bar{q}\} = \{\Delta\bar{q} \setminus v\}$ be the corresponding cell containing Δ'. Then the image of $d'c$ in $H_{2l}(\Delta')$ will be

$$\sum_{\bar{q}\in\mathrm{Link}_l(\Delta)} \rho_{\bar{q}}\{\Delta'\bar{q}\} + \sum_{\bar{r}\in\mathrm{Link}_{l-1}(\Delta)} \rho_{v\bar{r}}\{\Delta\bar{r}\}, \qquad (24)$$

where the coefficients $\rho_{v\bar{r}} \in \mathbb{Q}$ are chosen to make the result balanced along Δ'. There is always a unique such choice (cf. [6]), namely, the $\rho_{v\bar{r}}$ can be read off from the balancing condition for c along $\{\Delta\bar{r}\}$:

$$\sum_q \rho_{q\bar{r}}\overrightarrow{(\Delta' q)} + \rho_{v\bar{r}}\overrightarrow{(\Delta' v)} = 0 \quad \mod \{\Delta'\bar{r}\}, \qquad (25)$$

where $\overrightarrow{(\Delta' q)}$ means the divisorial vector q, or the vector from any vertex of Δ' to q (well defined mod Δ') if q is a vertex, and same for $\overrightarrow{(\Delta' v)}$.

From now on we will not distinguish between the classical geometric Steenbrink-Illusie E_1 complex and its interpretation via complex of propellers. One of the main results in [6] is the following statement.

Theorem 4 ([6]). $\tilde{E}^2_{q,p} \cong H_q(X; \mathscr{F}_p) \otimes \mathbb{Q}$.

Konstruktor

Now we provide another realization of the Steenbrink-Illusie's E_1 complex in terms of specific tropical simplicial chains. The collection of these chains which we call *konstruktor* forms a subcomplex of $C_\bullet^{bar}(X, \mathscr{F}_\bullet)$, and we can refer to Theorem 4 to see that the inclusion is a quasi-isomorphism. A wonderful feature of the konstruktor is that the eigenwave acts on its elements precisely as the monodromy operator ν acts on the terms in the Steenbrink-Illusie's E_1.

Let us fix the first baricentric subdivision of X. We elaborate a little bit on already used notation of the dual cell.

- For a pair $\Delta \succ \Delta'$ of finite simplices of sedentarity 0 in X, and $\bar{q} \in \mathrm{Link}_l(\Delta)$ we let $\hat{\Delta}'_{\Delta\bar{q}}$ denote the dual cell to Δ' in the face $\{\Delta\bar{q}\}$ of X, that is the union of all simplices in the baricentric subdivision containing baricenters of both Δ' and $\{\Delta\bar{q}\}$.
- In the summation formulae to follow we assume the terms with $\hat{\Delta}'_{\Delta\bar{q}}$ are not present if Δ' is not a zero sedentarity finite face of $\{\Delta\bar{q}\}$.

Let Δ be a finite k-simplex of sedentarity 0 in X, and $r \leq k$ a non-negative integer. To any propeller, that is a local tropical l-cycle

$$c = \sum_{\bar{q}\in \mathrm{Link}_l(\Delta)} \rho_{\bar{q}}\{\Delta\bar{q}\} \in H_{2l}(\Delta)$$

we associate a simplicial chain $c[-r] \in C_{k+l-r}^{bar}(X, \mathscr{F}_{l+r})$ as follows (note that $c[0]$ now has other meaning than just c):

$$c[-r] = \sum_{\bar{q}\in \mathrm{Link}_l(\Delta)} \sum_{\substack{\Delta' \prec \Delta \\ \dim \Delta' = r}} (\rho_{\bar{q}} \,\mathrm{Vol}_{\Delta'\bar{q}})\hat{\Delta}'_{\Delta\bar{q}}.$$

The orientation of $\hat{\Delta}'_{\Delta\bar{q}}$ is consistent with the original orientation of Δ and the choice of the volume element $\mathrm{Vol}_{\Delta'\bar{q}}$. Clearly for each r between 0 and k the map

$$(\cdot)[-r] : H_{2l}(k) \to C_{k+l-r}^{bar}(X, \mathscr{F}_{l+r})$$

is an injective group homomorphism. We denote its image in $C_{k+l-r}^{bar}(X, \mathscr{F}_{l+r})$ by $K_l(k)[-r]$.

Definition 22. The *konstruktor* is the subgroup of $C_\bullet^{bar}(X, \mathscr{F}_\bullet)$ generated by the $K_l(k)[-r]$ for all k, l and r . Note that $K_l(k)[-r]$ intersect trivially for different triples k, l, r.

Next we want to show that for each p the $\oplus_r K_{p-r}(\bullet - p + 2r)[-r]$ is indeed a subcomplex of $C_\bullet^{bar}(X, \mathscr{F}_p)$ isomorphic to the SI complex $\tilde{E}_1^{\bullet,p}$. This follows at once from comparing the SI differentials $d = d' + d''$ with the simplicial boundary ∂.

Proposition 16. $\partial(c[-r]) = (d'c)[-r] + (d''c)[-r-1]$.

Proof. For the proof we need two linear algebra identities. Let σ', σ'' be two opposite faces in a unimodular simplex $\sigma = \{\sigma'\sigma''\}$. Then one has

$$\sum_{\tau'' \prec_1 \sigma''} \mathrm{Vol}_{\sigma'\tau''} = \mathrm{Vol}_{\sigma'} \wedge \mathrm{Vol}_{\sigma''} = \sum_{\tau' \prec_1 \sigma'} \mathrm{Vol}_{\tau'\sigma''},$$

where, say, the left equality easily follows from the case when σ' is a vertex. Here all τ' are oriented consistently with σ', and all τ'' with σ''. We will need this identity in the form

$$\sum_{\Delta' \prec_1 \Delta} \mathrm{Vol}_{\Delta'\bar{q}} = \sum_{q \in \mathrm{Link}_1^0(\Delta)} \mathrm{Vol}_{\Delta\bar{q} \smallsetminus q}, \tag{26}$$

where Δ is a finite simplex and $\bar{q} \in \mathrm{Link}_l(\Delta)$. Note that the divisorial vectors (if any) in $\bar{q}$ just multiply both sides of the identity for finite simplices.

The second identity involves a relation among the balancing coefficients $\rho_{v\bar{r}}$ from (24) for $c = \sum \rho_{\bar{q}}\{\Delta\bar{q}\}$. One can show (cf. [6]) that they satisfy a refined version of (25). Namely, for $\Delta' \prec \Delta \prec \{\Delta\bar{q}\}$ we have

$$\sum_q \rho_{q\bar{r}} \overrightarrow{(\Delta' q)} + \sum_{v \in \Delta \smallsetminus \Delta'} \rho_{v\bar{r}} \overrightarrow{(\Delta' v)} = 0 \qquad \mathrm{mod}\ \{\Delta'\bar{r}\}$$

for faces $\Delta' \prec \Delta$ of codimension possibly higher than 1. Multiplying the above by $\mathrm{Vol}_{\Delta'\bar{r}}$ we arrive at

$$\sum_{q \in \mathrm{Link}_1(\Delta)} \rho_{q\bar{r}} \mathrm{Vol}_{\Delta' q\bar{r}} = - \sum_{v \in \Delta \smallsetminus \Delta'} \rho_{v\bar{r}} \mathrm{Vol}_{\Delta' v\bar{r}}\,. \tag{27}$$

Now we are ready to proof the proposition. Let $c = \sum \rho_{\bar{q}}\{\Delta\bar{q}\}$, then we can write

$$c[-r] = \sum_{\substack{\Delta' \prec_{k-r} \Delta \\ \bar{q} \in \mathrm{Link}_l(\Delta)}} (\rho_{\bar{q}} \mathrm{Vol}_{\Delta'\bar{q}}) \hat{\Delta}'_{\Delta\bar{q}}.$$

The topological boundary of each cell $\Delta'_{\Delta\bar{q}}$ consists of two types:

- Type 1: cells in the form $\Delta''_{\Delta\bar{q}}$ for faces $\Delta'' \succ_1 \Delta'$ of $\{\Delta\bar{q}\}$. If the cell $\Delta'_{\Delta\bar{q}}$ includes divisorial directions then its coefficient $\mathrm{Vol}_{\Delta'\bar{q}}$ in $c[-r]$ is divisible by all divisorial vectors. Hence the type 1 part of the boundary $\partial(c[-r])$ is, in fact, supported on the faces $\Delta''_{\Delta\bar{q}}$ for finite Δ''. Thus $\Delta''_{\Delta\bar{q}}$ in the formulae below make sense.
- Type 2: cells in the form $\Delta'_{\Delta\bar{q} \smallsetminus v}$ where v is a vertex or a divisorial vector in $\{\Delta\bar{q}\}$ which is not in Δ'.

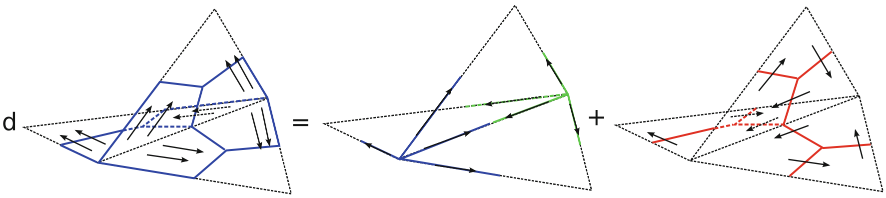


Fig. 8 $d = d' + d'' : H_2(1) \to H_2(0) \oplus H_0(2)[-1]$ (Framing coefficient vectors are not to scale)

Next we show that these two boundary types endowed with the framing correspond to the d'' and d' differentials in the SI complex, respectively, see Fig. 8.

Boundary of type 1:

$$\sum_{\Delta',\bar q}\sum_{q\in\bar q}(\rho_{\bar q}\operatorname{Vol}_{\Delta'\bar q})\widehat{\{\Delta' q\}}_{\Delta\bar q} + \sum_{\bar q}\sum_{\Delta'\prec_1\Delta''\prec\Delta}(\rho_{\bar q}\operatorname{Vol}_{\Delta'\bar q})\hat{\Delta}''_{\Delta\bar q}$$

$$= \sum_{q\in\mathrm{Link}^0_1(\Delta)}\left(\sum_{\substack{\Delta''\prec\Delta q,\ \Delta''\not\prec\Delta\\ \bar r\in\mathrm{Link}_{l-1}(\Delta q)}}(\rho_{q\bar r}\operatorname{Vol}_{\Delta''\bar r})\hat{\Delta}''_{\Delta q\bar r} + \sum_{\substack{\Delta''\prec\Delta\\ \bar r\in\mathrm{Link}_{l-1}(\Delta q)}}(\rho_{q\bar r}\operatorname{Vol}_{\Delta''\bar r})\right)\hat{\Delta}''_{\Delta q\bar r}$$

$$= \sum_{q\in\mathrm{Link}^0_1(\Delta)}\sum_{\substack{\Delta''\prec\Delta q\\ \bar r\in\mathrm{Link}_{l-1}(\Delta q)}}(\rho_{q\bar r}\operatorname{Vol}_{\Delta''\bar r})\hat{\Delta}''_{\Delta q\bar r}.$$

Here in the second summand we used the identity (26) for the pair $\Delta' \prec \Delta''\bar q$. From (23) one can easily see that this coincides with $(d''c)[-r-1]$.

Boundary of type 2:

$$\sum_{q\in\mathrm{Link}_1(\Delta)}\sum_{\substack{\Delta'\prec\Delta\\ \bar r\in\mathrm{Link}_{l-1}(\Delta)}}(\rho_{q\bar r}\operatorname{Vol}_{\tau q\bar r})\hat{\Delta}'_{\Delta\bar r} + \sum_{v\in\Delta}\sum_{\substack{\Delta'\prec\Delta\smallsetminus v\\ \bar q\in\mathrm{Link}_l(\Delta)}}(\rho_{\bar q}\operatorname{Vol}_{\Delta'\bar q})\hat{\Delta}'_{\Delta\bar q\smallsetminus v}$$

$$= \sum_{\substack{v\in\Delta\\ \Delta'\prec\Delta\smallsetminus v}}\left(\sum_{\bar r\in\mathrm{Link}_{l-1}(\Delta)}(\rho_{v\bar r}\operatorname{Vol}_{\tau v\bar r})\hat{\Delta}'_{\Delta\bar r} + \sum_{\bar q\in\mathrm{Link}_l(\Delta)}(\rho_{\bar q}\operatorname{Vol}_{\tau\bar q})\hat{\Delta}'_{\Delta\bar q\smallsetminus v}\right).$$

Here in the first summand we used the identity (27) for each $\Delta', \bar r$ with the sign compensated by the orientation of $\hat{\Delta}'_{\Delta\bar r}$ and the choice of $\operatorname{Vol}_{\tau v\bar r}$. Taking the sum of (24) over all vertices $v \in \Delta$ we easily identify the last expression with $(d'c)[-r]$.

□

Combining the above proposition with Theorem 4 we can conclude that the konstruktor complex can be used to calculate the tropical homology groups $H_q(X; \mathscr{F}_p)$:

Corollary 3. *The inclusion of the konstruktor* $\oplus_r K_{p-r}(\bullet - p + 2r)[-r]$ *into the complex* $C^{bar}_\bullet(X; \mathscr{F}_p)$ *is a quasi-isomorphism for each* p.

Finally, since all infinite cells in the konstruktor chains have coefficients divisible by the divisorial directions we can use the explicit description (17) of the eigenwave action on it. Then unveiling the konstruktor definition we arrive at the following.

Proposition 17. *For any* $c \in H_{2l}(\Delta)$ *one has* $\phi \cap (c[-r]) = c[-r-1]$.

Now we can combine all above observations to prove the claimed isomorphism

$$\phi^{q-p} : H_q(X; \mathscr{F}_p) \to H_p(X; \mathscr{F}_q).$$

Proof (Proof of Theorem 3). The cap product action of the eigenwave ϕ^{q-p} on the homology $H_q(X, \mathscr{F}_p)$ can be induced from its action on the konstruktor, which is a simplicial chain subcomplex. But it agrees there with the classical action of the monodromy ν^{q-p} on the E_1 term of the SI spectral sequence. On the other hand it is well known that the ν^{q-p} induces an isomorphism on the associated graded pieces with respect to the monodromy weight filtration on $H_{p+q}(X_t)$, which are calculated on the E_2 term of the SI spectral sequence. □

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Notes on a New Construction of Hyperkahler Metrics

Andrew Neitzke

Abstract I briefly review a new construction of hyperkahler metrics on total spaces of complex integrable systems, which we described in joint work with Davide Gaiotto and Greg Moore. The key ingredient in the construction is a collection of integers which govern "quantum corrections" to the metric, and which obey the wall-crossing formula of Kontsevich and Soibelman. The construction is not yet mathematically rigorous; I discuss some of what would be required to make it so.

1 Overview

In joint work with Davide Gaiotto and Greg Moore [1] we recently proposed a new connection between hyperkähler geometry and the counting of BPS states in supersymmetric field theory. While the story is motivated by physics, it leads to a concrete new recipe for constructing complete hyperkähler metrics on the total spaces of certain complex integrable systems.

The aim of this note is briefly to describe what this recipe is, and to comment on some of the issues involved in converting it into an actual theorem.

Let us briefly describe some of the highlights.

- We begin with a collection of "integrable system data" described in Sect. 2.1 below. These data include a complex manifold $\mathscr{B}$ containing a divisor D. For example, $\mathscr{B}$ could be the complex plane, and D some collection of points. The

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data also include a local system of lattices Γ over $\mathscr{B}' = \mathscr{B} \setminus D$, from which we build a $2r$-torus bundle $\mathscr{M}'$ over $\mathscr{B}'$, with nontrivial monodromy around D. Finally, we have a "central charge" homomorphism $Z : \Gamma \to \mathbb{C}$, varying holomorphically over $\mathscr{B}'$. From these data we build a simple explicit hyperkähler metric g^{sf} on $\mathscr{M}'$. However, the metric g^{sf} is incomplete, and our main interest is in complete metrics.

- Naively we might hope to complete g^{sf} by adding some degenerate torus fibers over D, thus extending $\mathscr{M}'$ to $\mathscr{M} \supset \mathscr{M}'$, in such a way that g^{sf} will extend to $\mathscr{M}$. However, it seems that this is impossible: roughly speaking, g^{sf} is too homogeneous to have such an extension. Instead, we construct a new metric g on $\mathscr{M}'$, which differs from g^{sf} by certain "quantum corrections."
- The quantum corrections are obtained by solving a certain explicit integral equation, (21) below. The main new ingredient in this equation is a set of integer "invariants" $\Omega(\gamma)$, which should be examples of generalized Donaldson–Thomas invariants in the sense of [2, 3]. In particular, the Kontsevich–Soibelman wall-crossing formula for generalized Donaldson–Thomas invariants, as written in [2], plays an important role in the construction. Indeed the original motivation for this construction was an attempt to understand the physical meaning of the formula of [2].
- Both the metrics g^{sf} and g depend on a real parameter $R > 0$; in the limit as $R \to \infty$, the torus fibers of $\mathscr{M}'$ collapse, in either metric. The corrections $g - g^{\mathrm{sf}}$ are exponentially suppressed in R when we are away from D: so as $R \to \infty$, g looks very close to g^{sf} except in a small neighborhood of the singular fibers. Near the singular fibers the quantum corrections become large, and in particular we expect that the corrected g can be extended over the singular fibers.

  This description of g near the $R \to \infty$ limit should be thought of as an example of a more general picture of the geometry of Calabi–Yau manifolds near their large complex structure limit, proposed by Gross–Wilson [4], Kontsevich–Soibelman [5] and Todorov, motivated by the Strominger–Yau–Zaslow picture of mirror symmetry [6].
- In many examples where our recipe can be applied, it turns out that the hyperkähler metrics in question were already known to exist. The example we have studied in most detail is that of rank-2 Hitchin systems with semisimple ramification [7]. We briefly describe that example in Sect. 9 below.
- Our recipe has not really been tested so far, in the sense that nobody has tried hard to use it to get new explicit information about interesting hyperkähler metrics. We believe that this should be possible: at the very least it should be possible to get a precise asymptotic series for g as $R \to \infty$.

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2 Integrable System Data

2.1 Data

Our construction begins with the following data:

Data 1: A complex manifold $\mathcal{B}$, of dimension r ("Coulomb branch").

Data 2: A divisor $D \subset \mathcal{B}$ ("discriminant locus"). Let $\mathcal{B}' = \mathcal{B} \setminus D$ ("smooth locus"). We use u to denote a general point of $\mathcal{B}'$.

Data 3a: A local system Γ_{g} over $\mathcal{B}'$, with fiber a rank-$2r$ lattice, equipped with a nondegenerate antisymmetric integer-valued pairing $\langle , \rangle$. Abusing notation we will also use $\langle , \rangle$ to denote the inverse pairing on Γ_{g}^* (not necessarily integer-valued.)
Data 3b: A fixed lattice Γ_{f} (possibly trivial). We sometimes think of Γ_{f} as the fiber of a trivial local system of lattices over $\mathcal{B}'$.
Data 3c: A local system Γ of lattices over $\mathcal{B}'$, given as an extension

$$0 \to \Gamma_{\mathrm{f}} \to \Gamma \to \Gamma_{\mathrm{g}} \to 0. \tag{1}$$

The pairing $\langle , \rangle$ on Γ_{g} induces one on Γ which we also denote $\langle , \rangle$. The radical of this pairing is Γ_{f}.

Data 4: A homomorphism $Z : \Gamma \to \mathbb{C}$, varying holomorphically over $\mathcal{B}'$. For any local section γ of Γ we thus get a local holomorphic function Z_γ on $\mathcal{B}'$.

Data 5: A homomorphism $\theta_{\mathrm{f}} : \Gamma_{\mathrm{f}} \to \mathbb{R}/2\pi\mathbb{Z}$.

These data are subject to several conditions:

Condition 1: Z_{γ_f} is a constant function on $\mathcal{B}'$ for any $\gamma_f \in \Gamma_{\mathrm{f}}$. (As a consequence, the Γ^*-valued 1-form $\mathrm{d}Z$ actually descends to Γ_{g}^*; we use this in formulating Condition 2.)

Condition 2: $\langle \mathrm{d}Z \wedge \mathrm{d}Z \rangle = 0$.

Condition 3: For any $u \in \mathscr{B}'$, the $\mathrm{d}Z_\gamma(u)$ span $\mathscr{T}_u^* \mathscr{B}'$.

2.2 Integrable System

The above data are enough to determine an incomplete complex integrable system, i.e. a holomorphic symplectic manifold $\mathscr{M}'$ which is a fibration over a complex base manifold $\mathscr{B}'$, with fibers complex Lagrangian tori. We now describe $\mathscr{M}'$.

For any fiber Γ_u of Γ, let $\{TChar_u(\Gamma, \theta_\mathrm{f})$ be the set of twisted unitary characters of Γ_u, i.e. maps $\theta : \Gamma_u \to \mathbb{R}/2\pi\mathbb{Z}$ obeying

$$\theta_\gamma + \theta_{\gamma'} = \theta_{\gamma+\gamma'} + \pi \langle \gamma, \gamma' \rangle, \tag{2}$$

agreeing with θ_f when restricted to $\Gamma_\mathrm{f} \subset \Gamma_u$. $\mathrm{TChar}_u(\Gamma, \theta_\mathrm{f})$ is topologically a torus $(S^1)^{2r}$. Letting u vary, the $\mathrm{TChar}_u(\Gamma, \theta_\mathrm{f})$ are the fibers of a torus bundle $\mathscr{M}'$ over $\mathscr{B}'$. Any local section γ of Γ then gives a function "evaluation on γ,"

$$\theta_\gamma : \mathscr{M}' \to \mathbb{R}/2\pi\mathbb{Z}. \tag{3}$$

These are the angular coordinates on the torus fibers of $\mathscr{M}'$.

Now we want to construct the complex structure and holomorphic symplectic form on $\mathscr{M}'$. For this purpose note that we have canonical functions

$$Z_\gamma : \mathscr{M}' \to \mathbb{C}, \tag{4}$$

pulled back from the base $\mathscr{B}'$. Differentiating gives a collection of 1-forms $\mathrm{d}\theta_\gamma$ and $\mathrm{d}Z_\gamma$ on $\mathscr{M}'$, which are linear in γ and vanish for $\gamma \in \Gamma_\mathrm{f}$, and hence can be organized into Γ_g^*-valued 1-forms $\mathrm{d}\theta$ and $\mathrm{d}Z$. Then define the complex 2-form

$$\omega_+ = -\frac{1}{2\pi} \langle \mathrm{d}Z \wedge \mathrm{d}\theta \rangle. \tag{5}$$

There is a unique complex structure on $\mathscr{M}'$ for which ω_+ is of type $(2, 0)$. We call this complex structure $J(\zeta = 0)$, for a reason which will emerge momentarily.

The two-form ω_+ gives a holomorphic symplectic structure on $(\mathscr{M}', J(\zeta = 0))$. With respect to this structure, the projection $\pi : \mathscr{M}' \to \mathscr{B}'$ is holomorphic, and the torus fibers $\mathscr{M}'_u = \pi^{-1}(u)$ are compact complex Lagrangian submanifolds.

2.3 Affine Structures

Although we do not use it explicitly in the rest of this note, it may be useful to mention that our data determine an S^1 worth of (symplectic) affine structures on $\mathscr{B}'$. Fix some $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$. Then pick a patch $U \subset \mathscr{B}'$ on which Γ_{g} admits a basis of local sections, $\gamma_1, \ldots, \gamma_{2r}$, in which $\langle , \rangle$ is the standard symplectic pairing. Also choose a local splitting $\rho : \Gamma_{\mathrm{g}} \to \Gamma$ of (1). Then the functions

$$f_i = Re(e^{\mathrm{i}\vartheta} Z_{\rho(\gamma_i)}) \tag{6}$$

are local coordinates on U (possibly after shrinking U). The transition functions on overlaps $U \cap U'$ are valued in $\mathrm{Sp}(2r, \mathbb{Z}) \ltimes \mathbb{R}^{2r}$ (the $\mathrm{Sp}(2r, \mathbb{Z})$ part comes from the choice of basis of Γ_{g}, the $\mathbb{R}^{2r}$ from the choice of splitting ρ.)

3 Semiflat Hyperkähler Metric

We now impose one more condition. Recall that a positive 2-form ω on a complex manifold is a real 2-form for which $\omega(v, Jv) > 0$ for all real tangent vectors v.

Condition 4: $\langle \mathrm{d}Z \wedge \mathrm{d}\bar{Z} \rangle$ is a positive 2-form on $\mathscr{B}'$.

3.1 Semiflat Metric

Fix $R \in \mathbb{R}_+$. $\mathscr{M}'$ carries a canonical 2-form,

$$\omega_3^{\mathrm{sf}} = \frac{R}{4} \langle \mathrm{d}Z \wedge \mathrm{d}\bar{Z} \rangle - \frac{1}{8\pi^2 R} \langle \mathrm{d}\theta \wedge \mathrm{d}\theta \rangle. \tag{7}$$

This form is of type $(1, 1)$ in complex structure $J(\zeta = 0)$ and positive. So the triple $(\mathscr{M}', J(\zeta = 0), \omega_3^{\mathrm{sf}})$ determine a Kähler metric g^{sf} on $\mathscr{M}'$. In fact this metric is hyperkähler. As far as I know, the first place where this was shown is in [8] (albeit in somewhat different notation); see also [9] for a more modern account. Alternatively, though, the hyperkähler property is a consequence of the twistorial construction of the metric which we will give below.

The superscript sf stands for "semi-flat": this terminology first appeared in [4], where it was used to refer to an important special case introduced in [10]. The reason for the name is that g^{sf} is flat when restricted to any torus fiber $\mathscr{M}'_u$, and $\mathscr{M}'_u$ has half the dimension of $\mathscr{M}'$.

3.2 Twistorial Description of the Semiflat Metric

Let us now describe a different, "twistorial" way of constructing the metric g^{sf}; this alternative description is what we will generalize in our construction of the quantum-corrected metric g below.

Any hyperkähler metric on a manifold $\mathscr{M}'$ determines—and is determined by—a collection of holomorphic symplectic structures $(\mathscr{M}', J(\zeta), \varpi(\zeta))$ labeled by $\zeta \in \mathbb{CP}^1$. In the general theory of hyperkähler manifolds all ζ are on the same footing. However, for the hyperkähler manifolds we are describing in this note, the points $\zeta = 0$ and $\zeta = \infty$ will play a distinguished role. It is then convenient to expand $\varpi(\zeta)$ as

$$\varpi(\zeta) = -\frac{i}{2\zeta}\omega_+ + \omega_3 - \frac{i}{2}\zeta\overline{\omega_+} \tag{8}$$

where ω_+, ω_3 are respectively the holomorphic symplectic form and Kähler form, both relative to the complex structure $J(\zeta = 0)$.

In the particular case of the hyperkähler metric g^{sf}, we have written these 2-forms explicitly above in (5) and (7). We will now give an alternative description of the holomorphic symplectic forms $\varpi(\zeta)$ corresponding to g^{sf}, roughly by exhibiting explicit holomorphic Darboux coordinates.

Let T_u denote the complex torus of twisted complex characters of Γ_u. T_u has canonical $\mathbb{C}^\times$-valued functions X_γ ($\gamma \in \Gamma_u$) obeying

$$X_\gamma X_{\gamma'} = (-1)^{\langle\gamma,\gamma'\rangle} X_{\gamma+\gamma'}, \tag{9}$$

and a Poisson structure

$$\{X_\gamma, X_{\gamma'}\} = \langle\gamma,\gamma'\rangle X_{\gamma+\gamma'}. \tag{10}$$

The T_u glue together into a local system over $\mathscr{B}'$ with fiber a complex Poisson torus. Let T denote the pullback of this local system to $\mathscr{M}'$.

Now we consider a section $\mathscr{X}^{\mathrm{sf}}$ of T, depending on an auxiliary parameter $\zeta \in \mathbb{C}^\times$. Locally this just means a collection of "coordinate" functions

$$\mathscr{X}_\gamma^{\mathrm{sf}} : \mathscr{M}' \times \mathbb{C}^\times \to \mathbb{C}^\times \tag{11}$$

(defined by $\mathscr{X}_\gamma^{\mathrm{sf}} = (\mathscr{X}^{\mathrm{sf}})^* X_\gamma$, with γ a local section of Γ). We often write these functions as $\mathscr{X}_\gamma^{\mathrm{sf}}(\zeta)$, leaving the $\mathscr{M}'$ dependence implicit. $\mathscr{X}^{\mathrm{sf}}(\zeta)$ is given by a simple closed formula:

$$\mathscr{X}_\gamma^{\mathrm{sf}}(\zeta) = \exp\left[\pi R \frac{Z_\gamma}{\zeta} + \mathrm{i}\theta_\gamma + \pi R \zeta \bar{Z}_\gamma\right]. \tag{12}$$

Now a direct computation shows[1]

$$\varpi^{\mathrm{sf}}(\zeta) = \frac{1}{8\pi^2 R}\langle \mathrm{d}\log \mathcal{X}^{\mathrm{sf}}(\zeta) \wedge \mathrm{d}\log \mathcal{X}^{\mathrm{sf}}(\zeta)\rangle. \tag{13}$$

So the $\mathcal{X}^{\mathrm{sf}}_\gamma(\zeta)$ are "holomorphic Darboux coordinates" on $\mathcal{M}'$, determining the holomorphic symplectic structures for all $\zeta \in \mathbb{C}^\times$, and hence the hyperkähler metric g^{sf}. In short: *knowing the functions $\mathcal{X}^{\mathrm{sf}}_\gamma(\zeta)$ is equivalent to knowing the hyperkähler metric g^{sf}.*

A global way of thinking about this construction is to say that for each $\zeta \in \mathbb{C}^\times$ we *pull back* the structure of holomorphic Poisson manifold from T to $\mathcal{M}'$, using the section $\mathcal{X}^{\mathrm{sf}}(\zeta)$ of T.[2] After pullback the Poisson structure is actually nondegenerate, i.e. it arises from a holomorphic symplectic structure.

4 Instanton Corrections to $\mathcal{X}$

We explained above how the semiflat section $\mathcal{X}^{\mathrm{sf}}$ can be used to construct the holomorphic-symplectic forms $\varpi(\zeta)$ corresponding to the semiflat metric g^{sf}. We now want to construct a new, "quantum-corrected" section $\mathcal{X}$. In the next section we will use $\mathcal{X}$ to build a quantum-corrected metric g.

4.1 BPS Degeneracies and Riemann–Hilbert Problem

The key new ingredient determining the quantum corrections is:

Data 6: A function $\Omega : \Gamma \to \mathbb{Z}$.

For each local section γ of Γ this gives a locally defined integer-valued function $\Omega(\gamma)$ on $\mathcal{B}'$. I emphasize that Ω is *not* required to be continuous: indeed Condition 7 below will imply that it is generally not continuous, but jumps in a specific way (governed by the Kontsevich–Soibelman wall-crossing formula) at real-codimension-1 loci in $\mathcal{B}'$.

[1] Note that this computation uses Condition 2, the fact $\langle \mathrm{d}Z \wedge \mathrm{d}Z\rangle = 0$—if we did not impose this condition, then computing the right side of (13) would produce a term $\langle \mathrm{d}Z \wedge \mathrm{d}Z\rangle/\zeta^2$, which would not match the form of $\varpi^{\mathrm{sf}}(\zeta)$.

[2] Of course T is a local system of tori, not a single torus, so the last sentence does not strictly make sense; but locally we can view $\mathcal{X}^{\mathrm{sf}}(\zeta)$ as a map into the space of local flat sections of T, which *is* a single holomorphic Poisson torus.

Ω should obey a simple parity-invariance condition:

Condition 5: $\Omega(\gamma; u) = \Omega(-\gamma; u)$.

We can now formulate the key ingredient in our construction, a certain Riemann–Hilbert problem. We need a little notation. Any $\gamma \in \Gamma_u$ gives a birational Poisson automorphism $\mathscr{K}_\gamma$ of T_u, defined by

$$\mathscr{K}_\gamma^* X_{\gamma'} = X_{\gamma'}(1 - X_\gamma)^{\langle \gamma, \gamma' \rangle}. \tag{14}$$

$\mathscr{K}_\gamma$ and $\mathscr{K}_{\gamma'}$ commute if and only if $\langle \gamma, \gamma' \rangle = 0$. Define a ray associated to each $\gamma \in \Gamma_u$,

$$\ell_\gamma(u) := Z_\gamma(u)\mathbb{R}_-. \tag{15}$$

Then to each ray ℓ running from the origin to infinity in the ζ-plane, associate a certain birational Poisson automorphism of T_u (first written down in [2]),

$$S_\ell(u) := \prod_{\gamma : \ell_\gamma(u) = \ell} \mathscr{K}_\gamma^{\Omega(\gamma; u)}. \tag{16}$$

We call the ℓ for which $S_\ell(u) \neq 1$ "BPS rays." Finally, we define an antiholomorphic involution ρ of T_u by

$$\rho^* X_\gamma = \overline{X}_{-\gamma}. \tag{17}$$

Now we can formulate the Riemann–Hilbert problem. Fix $u \in \mathscr{B}'$. We seek a map

$$\mathscr{X} : \mathscr{M}_u \times \mathbb{C}^\times \to T_u \tag{18}$$

with the following properties:

1. $\mathscr{X}$ depends piecewise-holomorphically on $\zeta \in \mathbb{C}^\times$, with discontinuities only at the rays $\ell_\gamma(u)$ for $\gamma \in \Gamma_u$ with $\Omega(\gamma; u) \neq 0$.
2. The limits $\mathscr{X}^\pm$ of $\mathscr{X}$ as ζ approaches any ray ℓ from both sides exist and are related by

$$\mathscr{X}^+ = S_\ell^{-1} \circ \mathscr{X}^-. \tag{19}$$

3. $\mathscr{X}$ obeys the reality condition

$$\mathscr{X}(-1/\bar{\zeta}) = \rho^* \mathscr{X}(\zeta). \tag{20}$$

4. For any γ, $\lim_{\zeta \to 0} \mathscr{X}_\gamma(\zeta) / \mathscr{X}_\gamma^{\rm sf}(\zeta)$ exists and is real.

We expect that the $\mathcal{X}$ with these properties should be unique if it exists, by analogy with what is known for similar Riemann–Hilbert problems appearing in [11, 12].

4.2 Solving the Riemann–Hilbert Problem

To find a solution of these conditions we contemplate the integral equation

$$\mathcal{X}_\gamma(x,\zeta) = \mathcal{X}_\gamma^{\mathrm{sf}}(x,\zeta)\exp\left[-\frac{1}{4\pi \mathrm{i}}\sum_{\gamma'}\Omega(\gamma';u)\langle\gamma,\gamma'\rangle\int_{\ell_{\gamma'}(u)}\frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\right.$$
$$\left.\times\ \log(1-\mathcal{X}_{\gamma'}(x,\zeta'))\right]. \tag{21}$$

For any fixed $x \in \mathcal{M}'$, (21) is a functional equation for the functions $\mathcal{X}_\gamma(x,\cdot) : \mathbb{C}^\times \to \mathbb{C}^\times$. We claim that if we find a collection of functions $\mathcal{X}_\gamma$ obeying this equation, they are a solution of our Riemann–Hilbert problem (in other words they obey the 4 conditions set out in the last section).

A natural way to try to produce a solution of (21) is by iteration, beginning with $\mathcal{X} = \mathcal{X}^{\mathrm{sf}}$. In [1] we sketch a proof that this iteration indeed converges for large enough R, to the *unique* solution of (21), under "reasonable" growth conditions on the $\Omega(\gamma;u)$ (stated more precisely in [1]):

Condition 6: $\Omega(\gamma;u)$ does not grow too quickly as a function of γ for fixed u.

Let us make a few remarks about this:

- This approach to the Riemann–Hilbert problem was inspired by the treatment of a similar problem in [11, 12]. At least morally speaking, ours is an infinite-dimensional version of the one discussed there, with the group $GL(K,\mathbb{R})$ replaced by the group of symplectomorphisms of the torus T.
- Our arguments are not strong enough to give *uniform* convergence of the iteration as we vary u, since $\Omega(\gamma;u)$ and $Z_\gamma(u)$ depend on u; in particular, the correct notion of "large enough R" may depend on u. Roughly speaking, the speed of the convergence is set by the largest $e^{-2\pi R|Z_\gamma(u)|}$ for which $\Omega(\gamma;u)\neq 0$.
- We did not give a complete proof that the $\mathcal{X}_\gamma$ obey our asymptotic Condition 4; we expect though that it should be possible to prove it directly, at least for large enough values of the parameter R, along similar lines to what was discussed

in [11,12]. Essentially the idea is that for large R the integrals in (21) have a finite limit as $\zeta \to 0$: this is easy to check directly if we replace $\mathcal{X}$ by $\mathcal{X}^{\mathrm{sf}}$, and we expect that this property should be preserved by the iteration.

- The $\mathcal{X}_\gamma$ are "quantum-corrected" versions of the original functions $\mathcal{X}_\gamma^{\mathrm{sf}}$. As with the $\mathcal{X}_\gamma^{\mathrm{sf}}$, the $\mathcal{X}_\gamma$ can be thought of as $\mathcal{X}_\gamma = \mathcal{X}^* X_\gamma$ for some section $\mathcal{X}$ of the complex torus bundle T.

4.2.1 Sums Over Trees

We also give a formula for a solution $\mathcal{X}_\gamma$ of (21) as a sum over certain iterated integrals, as follows. (It is not clear at the moment whether this sum actually *converges* or gives instead only an asymptotic series.)

We first introduce $\mathbb{Q}$-valued invariants related to the $\Omega(\gamma)$ by a "multi-cover formula" [2],

$$c(\gamma) = \sum_{n=1}^{\infty} \frac{\Omega(\gamma/n)}{n^2}. \tag{22}$$

(Here we take $\Omega(\gamma/n) = 0$ by definition whenever n does not divide γ.) We consider rooted trees, with edges labeled by pairs (i,j) (where i is the node closer to the root), and each node decorated by some $\gamma_i \in \Gamma$. Let $\mathcal{T}$ denote such a tree. Define a weight attached to $\mathcal{T}$ by

$$c(\mathcal{T}) = \frac{1}{|\mathrm{Aut}(\mathcal{T})|} \prod_{i \in \mathrm{Nodes}(\mathcal{T})} c(\gamma_i) \prod_{(i,j) \in \mathrm{Edges}(\mathcal{T})} \langle \gamma_i, \gamma_j \rangle. \tag{23}$$

Let $\gamma_{\mathcal{T}}$ denote the decoration at the root node of $\mathcal{T}$. We define a function $\mathcal{G}_{\mathcal{T}}(x,\zeta)$ on (a patch of) $\mathcal{M}$ inductively as follows: deleting the root node from $\mathcal{T}$ leaves behind a set of trees $\mathcal{T}_a$, and

$$\mathcal{G}_{\mathcal{T}}(x,\zeta) = \frac{1}{4\pi \mathrm{i}} \int_{\ell_{\gamma_{\mathcal{T}}}} \frac{\mathrm{d}\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \mathcal{X}^{\mathrm{sf}}_{\gamma_{\mathcal{T}}}(x,\zeta') \prod_a \mathcal{G}_{\mathcal{T}_a}(x,\zeta'). \tag{24}$$

Then a formal solution of (21) can be given as

$$\mathcal{X}_\gamma(x,\zeta) = \mathcal{X}_\gamma^{\mathrm{sf}}(x,\zeta) \exp\left[\sum_{\mathcal{T}} \langle \gamma, \gamma_{\mathcal{T}} \rangle c(\mathcal{T}) \mathcal{G}_{\mathcal{T}}(x,\zeta)\right]. \tag{25}$$

4.3 Wall-Crossing Formula

Define the "locus of marginal stability" by

$$W = \{u : \exists\gamma_1, \gamma_2 \text{ with } \Omega(\gamma_1; u) \neq 0, \Omega(\gamma_2; u) \neq 0, Z_{\gamma_1}(u)/Z_{\gamma_2}(u) \in \mathbb{R}_+\} \subset \mathscr{B}'. \tag{26}$$

This W is a union of countably many components ("walls") each of which has real codimension 1 in $\mathscr{B}'$. For our construction to work, the integers $\Omega(\gamma; u)$ must *jump* as u crosses any of these walls. More precisely, they must jump in accordance with the celebrated wall-crossing formula of Kontsevich and Soibelman [2]. We now describe this formula, essentially following [2], with a few slight adaptations to our context.

Let V be a strictly convex cone in $\mathbb{C}$ with apex at the origin. Then for any $u \notin W$, define

$$A_V(u) = \prod_{\gamma : Z_\gamma(u) \in V} \mathscr{K}_\gamma^{\Omega(\gamma;u)} = \prod_{\ell \subset V} S_\ell(u), \tag{27}$$

where the product is taken in order of increasing $\arg Z_\gamma(u)$. $A_V(u)$ is a birational Poisson automorphism of T_u.[3] Knowing $A_V(u)$ is sufficient to determine the $\Omega(\gamma; u)$ for γ with $Z_\gamma(u) \in V$; thus we can think of $A_V(u)$ as a sophisticated kind of generating function.

Define a *V–good path* to be a path $p \subset \mathscr{B}'$ along which there is no point u with $Z_\gamma(u) \in \partial V$ and $\Omega(\gamma; u) \neq 0$. (So as we travel along a V–good path, no BPS rays enter or exit V.)

Condition 7: If u and u' are the endpoints of a V–good path p, then $A_V(u)$ and $A_V(u')$ are related by parallel transport in T along p.

Condition 7 is essentially the wall-crossing formula of Kontsevich and Soibelman [2]. It is strong enough to determine all $\Omega(\gamma; u)$, if we have Data 1–4 and also know the $\Omega(\gamma; u_0)$ for some fixed u_0. In fact, at first sight it might seem to imply simply that $\Omega(\gamma; u)$ are locally constant functions of u on $\mathscr{B}'$. This is almost right: what it actually implies is that $\Omega(\gamma; u)$ are locally constant functions of u on $\mathscr{B}' \setminus W$. The point is that when u hits W the order of the factors in the product (27) is changed; as a result, for A_V to remain constant, the individual factors must in general also change. In other words, the $\Omega(\gamma; u)$ must jump.

[3] This statement needs a little amplification since the product in (27) may be infinite. One should more precisely think of $A_V(u)$ as living in a certain prounipotent completion of the group generated by $\{\mathscr{K}_\gamma\}_{\gamma : Z_\gamma(u) \in V}$ as explained in [2].

Condition 7 determines precisely how the $\Omega(\gamma; u)$ jump when u crosses some component of W. It is in this sense that it is a wall-crossing formula.

4.4 Absence of Unwanted Jumps in $\mathcal{X}$

Under this condition, let us revisit the solution $\mathcal{X}$ of the Riemann–Hilbert problem, and now vary the point $u \in \mathcal{B}$ as well as $\zeta \in \mathbb{C}^\times$. We have already noted that for any fixed u, $\mathcal{X}$ is discontinuous along the BPS rays. Letting u vary this becomes the statement that $\mathcal{X}$ is discontinuous along the locus

$$L = \left\{(u, \zeta) : \exists \gamma \in \Gamma_u \text{ with } Z_\gamma(u)/\zeta \in \mathbb{R}_- \text{ and } \Omega(\gamma; u) \neq 0\right\} \subset \mathcal{B}' \times \mathbb{C}^\times. \quad (28)$$

If Condition 7 is not obeyed, it is straightforward to show that these cannot be the only discontinuities of $\mathcal{X}$: there must be additional jumps when u meets the walls of marginal stability $W \subset \mathcal{B}'$. Such additional jumps would be a problem for our construction of the corrected hyperkähler metric below.

On the other hand, if Condition 7 *is* obeyed, then we claimed in [1] that $\mathcal{X}$ is actually continuous. This statement would follow directly from uniqueness of the solution of our Riemann–Hilbert problem, since Condition 7 says that the two Riemann–Hilbert problems we obtain by approaching the wall W from two sides are actually the same.

5 Corrected Metric

5.1 Construction

Having defined the section $\mathcal{X}(\zeta)$ of T, we are ready to describe the corrected hyperkähler metric g. The idea is similar to one we used above in our description of g^{sf}. Namely, for each $\zeta \in \mathbb{C}^\times$, we use $\mathcal{X}(\zeta)$ to pull back a holomorphic symplectic structure $\varpi^{(\zeta)}$ from T to $\mathcal{M}'$. As we have noted, $\mathcal{X}(\zeta)$ is not continuous; it has jumps along the locus $\pi^{-1}(L) \subset \mathcal{M}' \times \mathbb{C}^\times$, given by (19). Fortunately this jump is by composition with a Poisson morphism of T, and thus does not affect $\varpi^{(\zeta)}$. So $\varpi^{(\zeta)}$ is continuous, and depends holomorphically on $\zeta \in \mathbb{C}^\times$. In order to define an honest holomorphic symplectic structure, $\varpi^{(\zeta)}$ should also be nondegenerate. One expects this to be true at least for large enough R, since it is true for $\mathcal{X}^{\mathrm{sf}}$ and $\mathcal{X}$ differs from $\mathcal{X}^{\mathrm{sf}}$ only by corrections that are exponentially suppressed at large R.

Now our key claim is that

$\varpi^{(\zeta)}$ is of the form (8), where $(\omega_\pm, \omega_3)$ are symplectic forms defining a hyperkähler structure on $\mathcal{M}'$.

This is our construction of the new hyperkähler metric g on $\mathcal{M}'$.

5.2 Twistor Space

Let us make a few comments about how the claim above is motivated. One obvious necessary condition is $\varpi^{(-1/\bar{\zeta})} = \overline{\varpi^{(\zeta)}}$. This follows from the reality condition (20) (property 3 of the Riemann–Hilbert problem). We also need to see that $\varpi^{(\zeta)}$ has only a simple pole at $\zeta = 0$ (hence also at $\zeta = \infty$.) This follows from our asymptotic condition on $\mathscr{X}$ (property 4 of the Riemann–Hilbert problem). So $\varpi^{(\zeta)}$ indeed determines a complex 2-form ω_+ and a real 2-form ω_3. Of course this is still not enough to guarantee that these 2-forms fit together into an hyperkähler structure on $\mathscr{M}'$. The most delicate point is to show that indeed they do.

For this we use the "twistor space" construction [13, 14]. We consider the space $\mathscr{Z} = \mathscr{M} \times \mathbb{CP}^1$. The 2-form ϖ equips $\mathscr{Z}$ with a complex structure for which the projection to $\mathbb{CP}^1$ is holomorphic, and a fiberwise holomorphic symplectic form (globally twisted by $\mathscr{O}(2)$), obeying an appropriate reality condition. Moreover $\mathscr{Z}$ has a family of distinguished holomorphic sections labeled by points $x \in \mathscr{M}'$, given by the tautological-looking formula $s_x(\zeta) = (x, \zeta)$. In this situation, the twistor space construction promises us a hyperkähler metric on $\mathscr{M}'$, provided that the normal bundle $N(s_x)$ to each such section is a direct sum of copies of $\mathscr{O}(1)$. This condition on the normal bundle is the most delicate part of the story; we argue in [1] that it is a consequence of the asymptotic conditions obeyed by the section $\mathscr{X}$ as $\zeta \to 0$.

5.3 Improvement of Singularities

So far we have described how to construct a "quantum-corrected hyperkähler metric" g on $\mathscr{M}'$. The reader may be wondering why we have bothered to do so much work. After all, we already had a perfectly good hyperkähler metric g^{sf} on $\mathscr{M}'$.

However, g^{sf} has one important deficiency (in all but the most trivial examples): it is incomplete. The reason for this incompleteness is the fact that g^{sf} is defined only on $\mathscr{M}'$, which has smooth torus fibers over all points of $\mathscr{B}' \subset \mathscr{B}$, but does not include fibers over points of the "singular locus" $D \subset \mathscr{B}$. Typically one can complete $\mathscr{M}'$ topologically to a natural $\mathscr{M}$, with a projection $\pi : \mathscr{M} \to \mathscr{B}$, such that the fiber over a point of D is some kind of degenerate torus. One might then try to extend g^{sf} to a metric on the whole $\mathscr{M}$. This however appears to be impossible.

One answer to the question "why is g better than g^{sf}?" is that, if $\varOmega$ is chosen appropriately, we expect that g does admit an extension to a metric on $\mathscr{M}$, which in many cases will be complete. So morally the statement is that the quantum corrections "improve" the behavior of the metric near the singular locus D. We will discuss an example in the next section.

6 Ooguri-Vafa Metric

In [1] we discussed a model example of this phenomenon of improvement of singularities. Fix some constant $\Lambda \in \mathbb{C}$ (which enters the story in a trivial way: it is safe to fix $\Lambda = 1$ if you prefer.) We choose our data as follows:

Data 1: $\mathscr{B}$ is the disc $\{|u| < |\Lambda|\}$.
Data 2: The discriminant locus is $D = \{u = 0\} \subset \mathscr{B}$. So $\mathscr{B}'$ is the punctured disc.
Data 3: $\Gamma = \Gamma_{\rm g}$ is a local system of rank-2 lattices over $\mathscr{B}'$. With respect to a local basis of sections (γ_m, γ_e), with $\langle \gamma_m, \gamma_e \rangle = 1$, the monodromy around the puncture $u = 0$ is $\gamma_e \to \gamma_e, \gamma_m \to \gamma_m + \gamma_e$. $\Gamma_{\rm f}$ is trivial.
Data 4: With respect to the same local basis of sections, $Z_{\gamma_e}(u) = u$, $Z_{\gamma_m}(u) = \frac{1}{2\pi i}(u \log \frac{u}{\Lambda} - u)$. Note that analytically continuing around $u = 0$ we get $Z_{\gamma_m} \to Z_{\gamma_m} + Z_{\gamma_e}$, consistent with the monodromy of Γ; in other words Z is really globally defined.
Data 5: Since $\Gamma_{\rm f}$ is trivial, $\theta_{\rm f}$ is trivial.
Data 6: For all u, we have $\Omega(\gamma; u) = \begin{cases} 1 \text{ for } \gamma \in \{\gamma_e, -\gamma_e\}, \\ 0 \text{ otherwise.} \end{cases}$

In this case our construction can be carried out very explicitly (for any value of the parameter R): the integral equation (21) becomes simply an integral *formula*, or said otherwise, the iterative procedure of finding a solution actually terminates after a single step. So in this case we know the functions $\mathscr{X}_\gamma$ exactly. Applying our construction then yields an hyperkähler metric g on a torus fibration $\mathscr{M}' \to \mathscr{B}'$, which can be written down explicitly (it involves Bessel functions, but nothing worse). This is worked out in detail in [1].

Moreover, g admits an explicit smooth extension to a fibration $\mathscr{M} \to \mathscr{B}$, where $\mathscr{M} \setminus \mathscr{M}'$ consists of the fiber over $u = 0$, a nodal torus. This extended g coincides with the well-known "Ooguri-Vafa metric," first written down in [15]. So in this case our construction is a new picture of the hyperkähler structure on this known space.

One important drawback of this example is that it is only *local*—it is incomplete thanks to the boundary of the disc $\mathscr{B}$, and (as far as I know) has no suitable extension beyond this boundary. This drawback is eliminated in more interesting examples. On the other hand this example is extremely simple and computable, thanks to the fact that the γ for which $\Omega(\gamma; u) \neq 0$ generate an isotropic lattice for $\langle , \rangle$. Sadly, this virtue is also eliminated in more interesting examples.

7 More General Singular Loci

In more interesting examples we cannot so easily study the behavior of the metric on $\mathscr{M}'$ near the singular loci on $\mathscr{B}$. Nevertheless, we expect that the Ooguri–Vafa metric just discussed gives a kind of local model for what happens generally near the most generic kind of singular locus. Namely, consider some component $D_0 \subset D$, where

- $Z_{\gamma_0}(u) \to 0$ for some specific γ_0,
- $\Omega(\gamma_0; u) = 1$ for all u in a neighborhood of D_0,
- γ_0 is primitive (i.e. there exists some γ' with $\langle \gamma_0, \gamma' \rangle = 1$),
- the monodromy of Γ around D_0 is of "Picard–Lefshetz type", i.e.

$$\gamma \to \gamma + \langle \gamma, \gamma_0 \rangle \gamma_0. \tag{29}$$

The most essential difference between this situation and the Ooguri-Vafa metric we just discussed is that we no longer require that $\Omega(\gamma; u) = 0$ for all $\gamma \neq \pm\gamma_0$. Still, near D_0 and for large enough R, the quantum corrections coming from the charge γ_0, with $\Omega(\gamma_0; u) = 1$, should dominate all others, and so g should become similar to the Ooguri–Vafa metric. In particular, at least for large enough R, g should admit a smooth extension over D_0. This remains to be rigorously understood. I emphasize that it depends crucially on the condition $\Omega(\gamma_0; u) = 1$; otherwise we would have no reason (either mathematical or physical) to expect such a smooth extension of g to exist.

All of the above admits an extension to the case where γ_0 is not primitive, but rather is k times a primitive vector. In this case, instead of being smooth, we expect that the completed $(\mathcal{M}, g)$ has some mild singularities: there should be k orbifold singularities of type A_{k-1} lying over D_0. This is still a significant improvement over the behavior of g^{sf}.

The behavior of g near higher-codimension strata on D is more mysterious and should be very interesting. At the moment it is not clear (at least to me) how to use our construction to get really new information about it.

8 Pentagon

The next simplest example is already much more nontrivial. We fix a constant $\Lambda \in \mathbb{C}^\times$ (which enters the story in a trivial way: it is safe to fix $\Lambda = 1$ if you prefer.)

Data 1: $\mathcal{B}$ is the complex plane, coordinatized by u.

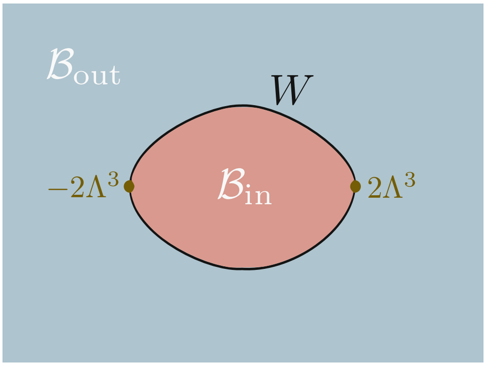
Data 2: The discriminant locus is $D = \{u = \pm 2\Lambda^3\} \subset \mathcal{B}$. So $\mathcal{B}'$ is the twice-punctured plane.

Data 3: Introduce a family of complex curves

$$\Sigma_u = \{y^2 = z^3 - 3\Lambda^2 z + u\} \subset \mathbb{C}^2. \tag{30}$$

For $u \in \mathcal{B}'$, Σ_u is a noncompact smooth genus 1 curve. Define $\Gamma_u = H_1(\Sigma_u, \mathbb{Z})$. Γ_u is a rank 2 lattice, the fiber of a local system Γ over $\mathcal{B}'$. It is equipped with the intersection pairing $\langle , \rangle$. Γ_{f} is trivial.

Fig. 1 The space $\mathscr{B}$ in the example of Sect. 8, divided into two chambers by a wall



Data 4: Introduce the 1-form $\lambda = y\,\mathrm{d}z$. λ is a holomorphic 1-form on Σ_u, which would be meromorphic if extended to the compactification of Σ_u (it has a pole of order 6 at the point at infinity, with zero residue). Then for $\gamma \in \Gamma_u$,

$$Z(\gamma) = \frac{1}{\pi} \oint_\gamma \lambda. \tag{31}$$

Data 5: Since Γ_{f} is trivial, θ_{f} is trivial.

Data 6: $\mathscr{B}$ is divided into two domains $\mathscr{B}_{\mathrm{in}}$ and $\mathscr{B}_{\mathrm{out}}$ (also sometimes called "strong coupling" and "weak coupling" respectively) by the locus

$$W = \{u : Z(\Gamma_u) \text{ is contained in a line in } \mathbb{C}\} \subset \mathscr{B}. \tag{32}$$

See Fig. 1. Since $\mathscr{B}_{\mathrm{in}}$ is simply connected we may trivialize Γ over $\mathscr{B}_{\mathrm{in}}$ by primitive cycles γ_1, γ_2 which collapse at the two points of D. We choose them so that $\langle \gamma_1, \gamma_2 \rangle = 1$. The set $\{\gamma_1, \gamma_2\}$ does not extend to a global trivialization of Γ, since it is not invariant under monodromy. However, the set $\{\gamma_1, \gamma_2, \gamma_1 + \gamma_2, -\gamma_1, -\gamma_2, -\gamma_1 - \gamma_2\}$ is invariant under the monodromy around infinity. Therefore the following definition of Ω makes global sense:

$$\text{For } u \in \mathscr{B}_{\mathrm{in}},\ \Omega(\gamma; u) = \begin{cases} 1 \text{ for } \gamma \in \{\gamma_1, -\gamma_1, \gamma_2, -\gamma_2\}, \\ 0 \text{ otherwise.} \end{cases} \tag{33}$$

$$\text{For } u \in \mathscr{B}_{\mathrm{out}},\ \Omega(\gamma; u) = \begin{cases} 1 \text{ for } \gamma \in \{\gamma_1, -\gamma_1, \gamma_2, -\gamma_2, \gamma_1 + \gamma_2, -\gamma_1 - \gamma_2\}, \\ 0 \text{ otherwise.} \end{cases} \tag{34}$$

All of our conditions on the data are more or less trivial to check. The most interesting one is the wall-crossing formula (Condition 7). Here the question is: choosing $u_{\mathrm{in,out}}$ to be two nearby points on opposite sides of W, and choosing V to be a narrow sector which contains the rays $\ell_{\gamma_1}(u)$ and $\ell_{\gamma_2}(u)$ both for $u = u_{\text{in}}$ and for $u = u_{\text{out}}$, do we have

$$A_V(u_{\text{in}}) = \mathcal{K}_{\gamma_1}\mathcal{K}_{\gamma_2} \stackrel{?}{=} \mathcal{K}_{\gamma_2}\mathcal{K}_{\gamma_1+\gamma_2}\mathcal{K}_{\gamma_1} = A_V(u_{\text{out}}). \tag{35}$$

This identity is indeed true: it is the "pentagon identity" given in [2]. This identity can easily be checked by hand. We remark in passing that (as also noted in [2]) this identity is also closely related to the five-term identity of the dilogarithm function and its quantum counterpart (see e.g. [16, 17].)

This example has the virtue that for every u only finitely many $\Omega(\gamma; u)$ are nonvanishing. This may lead to some technical simplifications (although we emphasize that there should be no essential difference between this case and the case where there are infinitely many nonvanishing $\Omega(\gamma; u)$, so long as the $\Omega(\gamma; u)$ grow slowly enough with γ).

As we commented in the previous section, we expect that the metric g on $\mathcal{M}'$ in fact extends to a *complete* metric on a space $\mathcal{M}$, obtained from $\mathcal{M}'$ by adding nodal torus fibers over the two points of D, and that the metric around either of these nodal fibers looks like the Ooguri-Vafa metric.

We believe that this complete metric actually has another name: it is the metric on a certain moduli space of rank-2 Higgs bundles on $\mathbb{CP}^1$ with an irregular singularity at ∞. This point of view is discussed at some length in [7]. Also, for any $\zeta \in \mathbb{C}^\times$, the complex manifold $(\mathcal{M}, J(\zeta))$ is isomorphic to a partial compactification $\mathcal{M}_{0,5}^{\text{cyc}}$ of $\mathcal{M}_{0,5}$, consisting of 5-tuples of points $(z_1, \ldots, z_5)$ on $\mathbb{CP}^1$ where $z_i \neq z_{i+1}$ (with i taken mod 5). Our description of this space is then closely related to the discussion in [18].

9 Hitchin Systems

Finally I briefly describe a more geometric family of examples, considered in [7].

Fix a compact complex smooth curve $\bar{C}$. Fix $n > 0$ marked points $z_i \in \bar{C}$, and let $C = \bar{C} \setminus \{z_1, \ldots, z_n\}$. Also fix parameters $m_i \in \mathbb{C}$ and $m_i^{(3)} \in \mathbb{R}/2\pi\mathbb{Z}$ associated to the marked points. Assume the m_i and $m_i^{(3)}$ generic (in particular, the m_i should be linearly independent over $\mathbb{Q}$.)

Data 1: $\mathcal{B}$ is the space of meromorphic quadratic differentials ϕ_2 on $\bar{C}$ with double poles at each z_i, of residue m_i^2. (So $\mathcal{B}$ is a complex affine space.) To stay consistent with our previous notation we will use either u or ϕ_2 to denote a point of $\mathcal{B}$.

Data 2: $D \subset \mathcal{B}$ is the locus of ϕ_2 which have at least one non-simple zero. So $\mathcal{B}'$ is the locus of ϕ_2 having only simple zeroes.

Data 3: Let T^*C be the holomorphic cotangent bundle to C. For any fixed $u \in \mathcal{B}$, consider the noncompact complex curve

$$\Sigma_u = \{(z \in C, \lambda \in T_z^*C) : \lambda^2 = \phi_2(z)\} \subset T^*C. \tag{36}$$

For $u \in \mathscr{B}'$, Σ_u is smooth. The obvious projection $\pi : \Sigma_u \to C$ is a double covering, branched over the zeroes of ϕ_2. Σ_u has a natural compactification $\bar{\Sigma}_u$ with a projection $\bar{\pi} : \bar{\Sigma}_u \to \bar{C}$.
Σ_u is equipped with the involution $\lambda \mapsto -\lambda$. Define Γ_u to be the subgroup of $H_1(\Sigma_u, \mathbb{Z})$ odd under this involution. Γ_u is the fiber of a local system Γ over $\mathscr{B}'$. It is equipped with the intersection pairing $\langle , \rangle$. $\Gamma_{\mathrm{f}} \subset \Gamma$ is the radical of the pairing $\langle , \rangle$, which has rank n. This radical does not undergo any monodromy as we vary u, so we can think of Γ_{f} as a single fixed lattice rather than a local system. Finally, $\Gamma_{\mathrm{g}} = \Gamma / \Gamma_{\mathrm{f}}$.

Data 4: By slight abuse of notation let λ denote the Liouville (tautological) 1-form on T^*C.
Then for $\gamma \in \Gamma_u$, define

$$Z(\gamma) = \frac{1}{\pi} \oint_\gamma \lambda. \tag{37}$$

Data 5: The 1-form λ restricted to Σ_u extends meromorphically to $\bar{\Sigma}_u$, with simple poles at the two preimages of z_i; let $z_i^\pm \in \bar{\Sigma}_u$ denote the preimage at which λ has residue $\pm m_i$. The lattice Γ_{f} has one generator $\gamma_{i,\mathrm{f}}$ for each puncture z_i, given by the sum of a counterclockwise loop around z_i^+ and a clockwise loop around z_i^-. We define θ_{f} by

$$\theta_{\mathrm{f}}(\gamma_{i,\mathrm{f}}) = m_i^{(3)}. \tag{38}$$

Data 6: The invariants $\Omega(\gamma; u)$ are defined in terms of the quadratic differential ϕ_2, as follows.
For any $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$, define a *$\vartheta$-trajectory* of ϕ_2 to be a real curve $c \subset C$ such that, for any real tangent vector v to c, $\phi_2(v \otimes v) \in e^{2i\vartheta}\mathbb{R}_+$; call a ϑ-trajectory *maximal* if it is not properly contained in any other ϑ-trajectory. The maximal ϑ-trajectories make up a singular foliation of C, with three-pronged singularities at the zeroes of ϕ_2.
Define the *mass* of a maximal ϑ-trajectory c to be $\int_c |\sqrt{\phi_2}|$. A generic maximal ϑ-trajectory has infinite mass; we are interested in the exceptional trajectories which have finite mass. Let a *finite ϑ-trajectory* be a maximal ϑ-trajectory with finite mass, and a *finite trajectory* be a pair (c, ϑ) where c is a finite ϑ-trajectory. Finite trajectories come in two types:

- *Saddle connections*: these are finite trajectories c which "run from one zero of ϕ_2 to another," i.e., their boundary $\bar{c} \setminus c$ consists of two points (which are then necessarily zeroes of ϕ_2).
- *Closed loops*: these are finite trajectories c with the topology of S^1. When such a trajectory occurs it sits in a 1-parameter family of such trajectories, sweeping out an open annulus on C.

Given a finite trajectory (c, ϑ), define its *lift* $\ell(c, \vartheta)$ to be the closure of $\pi^{-1}(c)$ on Σ. $\ell(c, \vartheta)$ has no boundary; it is a single loop if (c, ϑ) is a saddle connection (it is enlightening to draw a picture to see why), and the disjoint union of two loops if (c, ϑ) is a closed loop. The 1-form $e^{-i\vartheta}\lambda$ is real and nonvanishing on $\ell(c, \vartheta)$; hence it induces an orientation on $\ell(c, \vartheta)$. Note that if (c, ϑ) is a finite trajectory then $(c, \vartheta + \pi)$ is as well, and $\ell(c, \vartheta)$ differs from $\ell(c, \vartheta + \pi)$ only by orientation reversal. By construction, $\ell(c, \vartheta)$ is invariant under the combination of the deck transformation $\lambda \mapsto -\lambda$ and orientation reversal.

For any $\gamma \in \Gamma_u$, let $SC(\gamma; u)$ be the set of all saddle connections (c, ϑ) with $[\ell(c, \vartheta)] = \gamma$, and let $CL(\gamma; u)$ be the set of all isotopy classes of closed loops (c, ϑ) with $[\ell(c, \vartheta)] = \gamma$. Now finally we can define

$$\Omega(\gamma; u) = \#SC(\gamma; u) \ - 2\#CL(\gamma; u). \tag{39}$$

(The strange-looking coefficients $+1$ and -2 here are really necessary—otherwise the wall-crossing formula (Condition 7) would not be satisfied!)

These data satisfy all of our Conditions 1–7. The most difficult to see are the last two. Condition 6 follows from known results on quadratic differentials [19, 20] which say $\Omega(\gamma; u)$ grows at most quadratically as a function of the coefficients of γ. The wall-crossing formula (Condition 7) follows from a sort of inversion of the logic we have followed up to this point: namely, below we will give a direct description of the complex spaces $(\mathcal{M}, J(\zeta))$ and the functions $\mathcal{X}_\gamma(x, \zeta)$ thereon which solve the Riemann–Hilbert problem and are continuous except at the BPS rays. The existence of such functions $\mathcal{X}_\gamma(x, \zeta)$ then implies the wall-crossing formula (following the discussion of Sect. 4.4).

In [7] we argued that the hyperkähler space $\mathcal{M}$ in this example is a space of solutions of Hitchin equations on $\bar{C}$, with gauge group $PSU(2)$, and with ramification at the marked points z_i (with semisimple residues). This is a much-studied space, considered in particular in [21–24]. In particular, it is known that the complex spaces $(\mathcal{M}, J(\zeta))$ are moduli spaces of $PSL(2, \mathbb{C})$ connections on C, with fixed eigenvalues of monodromy around z_i, given by $\mu_\pm = \exp(\pm 2\pi \mathrm{i}(\zeta^{-1} m_i - m_i^{(3)} - \zeta \bar{m}_i))$.

The $\mathcal{X}_\gamma(x, \zeta)$ in this example are essentially functions considered earlier by Fock–Goncharov in [25], themselves complexifications of the "shear coordinates" familiar in Teichmüller theory. The main issue in identifying the Fock–Goncharov coordinates with our $\mathcal{X}_\gamma(x, \zeta)$ is to prove that the Fock–Goncharov coordinates have the correct asymptotic behavior as $\zeta \to 0, \infty$. This is accomplished by applying the WKB approximation to a family of flat connections on C of the form $\nabla(\zeta) = \varphi/\zeta + D + \bar{\varphi}\zeta$.

There is a generalization of this story to encompass quadratic differentials with poles of order greater than 2, also considered in [7]. This generalization in particular includes the "pentagon" example of Sect. 8; it corresponds to considering quadratic differentials $\varphi_2 = (z^3 - 3\Lambda^2 z + u)\mathrm{d}z^2$ on $\mathbb{CP}^1$, with order-7 poles at $z = \infty$.

Finally, we have extended many aspects of this story to the case of Hitchin equations with higher rank gauge group $PSU(K)$ [26]. In this case the coordinate functions $\mathcal{X}_\gamma$ involve more general coordinate systems than those which were described explicitly by Fock–Goncharov in [25]; conjecturally the $\mathcal{X}_\gamma$ exhaust the set of cluster coordinate systems.

10 DT Invariants

Finally let us briefly consider another viewpoint on this story, which is really where it began. The physical perspective on our construction makes clear that it should be closely related to the theory of generalized Donaldson–Thomas invariants (henceforth just "DT invariants.") In this section I briefly sketch that relation and a few examples.

10.1 The Dictionary

In the theory of DT invariants, one begins with a triangulated category $\mathcal{D}$ and constructs the space Stab($\mathcal{D}$) of *Bridgeland stability conditions* on $\mathcal{D}$ [27]. Under some further conditions[4] on $\mathcal{D}$, one then expects to be able to construct DT invariants depending on a point of Stab($\mathcal{D}$) [2, 3], whose dependence on the point of Stab($\mathcal{D}$) is governed by the wall-crossing formula. In what follows I assume some familiarity with this story and formulate the expected dictionary between the hyperkähler data in our construction and the theory of DT invariants. Many aspects of this dictionary are also described in Sect. 2.7 of [2].

We need a few technical preliminaries to "harmonize" the two sides first:

- On the hyperkähler data side: suppose given an example of our Data 1–6 obeying our Conditions 1–7. Fix a basepoint $u_0 \in \mathcal{B}'$. Let $\tilde{\mathcal{B}}'$ denote the universal cover of $\mathcal{B}'$. Over this cover we may globally trivialize the local system Γ, thus identifying all of its fibers with Γ_{u_0}. The fiberwise homomorphism $Z : \Gamma \to \mathbb{C}$ can thus be thought of as a family of homomorphisms from the fixed lattice Γ_{u_0} to $\mathbb{C}$, depending on a point $\tilde{u} \in \widetilde{\mathcal{B}'}$,

$$Z(\tilde{u}) : \Gamma_{u_0} \to \mathbb{C}. \tag{40}$$

- On the DT theory side: suppose given an appropriate category $\mathcal{D}$. Stab($\mathcal{D}$) is a complex Poisson manifold, carrying a natural "forgetful" map to Hom($K(\mathcal{D}), \mathbb{C}$)

[4] Which I am unfortunately not competent to summarize.

which is a local Poisson isomorphism [27]. We will consider a single connected component $\mathrm{Stab}^0(\mathscr{D}) \subset \mathrm{Stab}(\mathscr{D})$.

We then have the following expected dictionary:

DT theory	Hyperkähler data
$K(\mathscr{D})$	Γ_{u_0}
Euler pairing	$\langle \cdot, \cdot \rangle$
Stability functions $Z : K(\mathscr{D}) \to \mathbb{C}$	$Z(\tilde{u}) : \Gamma_{u_0} \to \mathbb{C}$
DT invariants of $\mathscr{D}$	$c(\gamma) \in \mathbb{Q}$ from (22)
A quotient of a Lagrangian $L \subset \mathrm{Stab}^0(\mathscr{D})$	$\mathscr{B}'$
???	$\mathscr{B}$
???	θ_{f}

This dictionary has one especially awkward feature: starting from the category $\mathscr{D}$ it is not at all clear how to choose the complex Lagrangian submanifold L. Because of this problem, at the moment we do not really have a recipe which begins with $\mathscr{D}$ alone and constructs a corresponding hyperkähler space. In particular examples which we do understand, L always has some nice geometric meaning (see the next section). It would be very interesting to understand how to get L in a purely categorical way.

10.2 Examples

For many examples of our construction of hyperkähler metrics (probably in all the examples that come from an underlying supersymmetric quantum field theory, which includes all of the examples discussed so far in this note), we expect that there is some triangulated category $\mathscr{D}$, fitting into the above dictionary. Let us now describe a few examples:

- Let $\mathscr{D}$ be the category of finite-dimensional modules over the Ginzburg dg algebra of the A_2 quiver. In recent work of Sutherland [28], one connected component $\mathrm{Stab}^0(\mathscr{D}) \subset \mathrm{Stab}(\mathscr{D})$ is identified with the universal cover of the total space of a particular $\mathbb{C}^\times$ bundle over the moduli space $\mathscr{M}_{1,1}$ of elliptic curves. This result fits well into the above dictionary: indeed we claim that the hyperkähler data corresponding to the category $\mathscr{D}$ is that of the "pentagon" example of Sect. 8 above. The elliptic curves appearing in Sutherland's picture are the curves Σ_u of Sect. 8.
- Upcoming work of Bridgeland and Smith [29] is also relevant to this dictionary.
  Begin with a real compact 2-manifold C, with $n > 1$ marked points. From the combinatorics of ideal triangulations of the curve C, one can build an associated quiver $Q(C)$, using a superpotential function first written down by

Labardini–Fragoso [30].[5] Let $\mathscr{D}(C)$ be the derived category of finite-dimensional modules over the Ginzburg dg algebra of $Q(C)$. Bridgeland and Smith show (roughly—for the precise statement see [29]) that there is a component $\mathrm{Stab}^0(\mathscr{D}(C)) \subset \mathrm{Stab}(\mathscr{D}(C))$, such that a point of $\mathrm{Stab}^0(\mathscr{D}(C))$ corresponds to a choice of complex structure on C and a meromorphic quadratic differential thereon, with double poles at the marked points. Among other things, this provides a family of nontrivial examples of categories $\mathscr{D}$ where one has a geometric interpretation for at least a component of $\mathrm{Stab}(\mathscr{D})$.

This result fits in well with the dictionary proposed above: it is consistent with the idea that the categories $\mathscr{D}(C)$ correspond to the hyperkähler data described in Sect. 9. Moreover, revisiting Sect. 9 we see that the mysterious Lagrangian subspace $L \subset \mathrm{Stab}(\mathscr{D}(C))$ appearing in the dictionary has a nice meaning here: it corresponds to fixing a particular complex structure on C and a choice of residues at the marked points on C.

Bridgeland and Smith also consider a generalization corresponding to allowing meromorphic quadratic differentials with higher-order poles. This generalization in particular gives another proof of Sutherland's results from [28] (by considering quadratic differentials on $\mathbb{CP}^1$ with a single pole of order 7).

- More ambitiously, at least on physical grounds we expect that given a complete non-compact Calabi–Yau threefold X, both sides of this dictionary should exist. Roughly speaking, $\mathscr{D} = \mathscr{D}(X)$ should be an appropriate version of the Fukaya category of X; $\mathscr{B}$ should be the moduli space of complex structures in X; Γ should be $H_3(X, \mathbb{Z})$; Z should be the period map; $c(\gamma)$ should be DT invariants counting special Lagrangian 3-cycles in X. The Lagrangian submanifold L is the period domain of X; the fact that it is Lagrangian is essentially Griffiths transversality. Finally, the hyperkähler space $\mathscr{M}$ built by our construction is some version of the family of intermediate Jacobians of X (I say "some version" because we are dealing with non-compact X).

  The examples studied by Bridgeland and Smith, i.e. the examples of Sect. 9 above, also fall into this class. The Calabi–Yau threefold $X(C)$ in this case is a conic bundle over the curve C, appropriately modified at the marked points; the Fukaya category $\mathscr{D}(X(C))$ is equivalent to the category $\mathscr{D}(C)$ mentioned above. This equivalence will also be explained in upcoming work of Bridgeland and Smith.

  (Incidentally, following through our various claims about the hyperkähler space $\mathscr{M}$ in these examples, we see that on the one hand $\mathscr{M}$ should be a version of the family of intermediate Jacobians of $X(C)$, while on the other hand $\mathscr{M}$ should be the $PSU(2)$ Hitchin system on C, with ramification at the marked points. So our claims would imply that these two integrable systems the same. This equivalence is not really novel: a version of it without the marked points was described in [32], a fact which gives us some additional confidence in our whole picture.)

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[5] This quiver and superpotential also appeared in the physics literature [31].

For general X, it is not clear a priori that the DT invariants will grow slowly enough to satisfy our Condition 6. Nevertheless, on physical grounds we would expect the hyperkähler manifold $\mathscr{M}$ to exist for general X.[6] Thus we expect that *either* the DT invariants do in fact grow slowly enough for us to prove that the Riemann–Hilbert problem has a solution, *or* they grow more quickly but have some hidden extra structure that allows the Riemann–Hilbert problem to have a solution anyway.

- Finally let me describe a *non*-example. It is natural to ask: what if we let $\mathscr{D}$ be the Fukaya category of a *compact* Calabi–Yau threefold X—will there be corresponding hyperkähler data then? It seems that the answer is "yes"—we can define the data by the same recipe as we use for non-compact X—but these data would not satisfy precisely our Conditions 1–7. In particular, Condition 4 (positive definiteness) will certainly be violated. However, this violation is of a rather controlled sort; there is just one negative direction. So, were this the only difficulty, the expected consequence would be that the space $\mathscr{M}$ we obtain is not hyperkähler but pseudo-hyperkähler, with one negative direction. ($\mathscr{M}$ in this case is the family of intermediate Jacobians of X, fibered over the moduli space of polarized complex structures on X. These intermediate Jacobians are quotients of $H^{3,0} \oplus H^{2,1}(X)$, and the negative direction is coming from $H^{3,0}(X)$; it is related to the fact that when equipped with its "Griffiths" complex structure, the intermediate Jacobian is not principally polarized.) However, there is also a second, more serious difficulty: the invariants $\Omega(\gamma)$ counting special Lagrangian 3-cycles in X are expected to grow very quickly as functions of γ (roughly $\Omega(\gamma) \sim \exp c\|\gamma\|^2$), badly violating our Condition 6. As a result it is far from clear whether our construction of hyperkähler metrics should be directly applicable to this situation.

  This difficulty is in some sense anticipated in the physics literature. Indeed, physics does *not* predict directly that there is an hyperkähler manifold associated to a compact Calabi–Yau threefold X. Rather it predicts the existence of a *quaternionic-Kähler* manifold. As in the hyperkähler case, it should be possible to construct the desired quaternionic-Kähler structure by beginning with a simple "semi-flat" metric g^{sf} and modifying it by quantum corrections.[7] The semi-flat metric in this case was first described by Ferrara and Sabharwal in [33], and was recently discussed by Hitchin in [34]. The description of the quantum corrections has been studied intensely in physics, with various interesting partial results. In particular, some of the quantum corrections are expected to be precise analogues of the ones we have described in the hyperkähler case, indeed related by a "quaternionic-Kähler/hyperkähler correspondence" [35, 36]. However, one also expects new quantum corrections in the quaternionic-Kähler case which do not

6The idea is that $\mathscr{M}$ is the moduli space of the IIB string theory formulated on the 10-manifold $X \times S^1 \times \mathbb{R}^{2,1}$.

7In this case "semi-flat" means that g^{sf} is locally invariant under a Heisenberg group of isometries, replacing the torus group that appeared in the hyperkähler case.

have an hyperkähler analogue. As far as I know, there are no examples yet of X where all quantum corrections have been fully described.

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Mirror Duality of Landau–Ginzburg Models via Discrete Legendre Transforms

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Abstract We recall the semi-flat Strominger–Yau–Zaslow (SYZ) picture of mirror symmetry and discuss the transition from the Legendre transform to a discrete Legendre transform in the large complex structure limit. We recall the reconstruction problem of the singular Calabi–Yau fibres associated to a tropical manifold and review its solution in the toric setting. We discuss the monomial-divisor correspondence for discrete Legendre duals and use this to give a mirror duality for Landau Ginzburg models motivated from the SYZ perspective and Floer theory. We mention its application for the construction of mirror symmetry partners for varieties of general type and discuss the straightening of the boundary of a tropical manifold corresponding to a smoothing of the divisor in the complement of a special Lagrangian fibration.

1 Strominger–Yau–Zaslow Fibrations and the Mirror of $(\mathbb{C}^*)^n$

We give a summary of the semi-flat picture of mirror symmetry following [9, §6–8] and discuss the example of an algebraic torus. Further references for the material are [4, 10–12, 20, 30, 31] and most recently [22]. Hitchin [28] first noticed the importance of the Legendre transform in this context while a Legendre transform already appeared in [2, 27] in a closely related context without the awareness of mirror symmetry and special Lagrangians.

Mirror symmetry has become intrinsic to the Calabi–Yau geometry by the work of Strominger–Yau–Zaslow [37] (short: SYZ), suggesting to explain the mirror

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duality of two Calabi–Yau manifolds X, $\check{X}$ as a duality of torus fibrations. There are supposed to be C^∞-maps

$$f : X \to B, \quad \check{f} : \check{X} \to B$$

with fibres homeomorphic to $(S^1)^n$ for $n = \dim_{\mathbb{C}} X = \dim_{\mathbb{R}} B$, in fact if $f^{-1}(b) = V/\Lambda$ for a real vector space V with lattice $\Lambda \cong \mathbb{Z}^n$ then $\check{f}^{-1}(b) = V^*/\Lambda^*$ where $V^* = \mathrm{Hom}(V, \mathbb{R})$, $\Lambda^* = \mathrm{Hom}(\Lambda, \mathbb{Z})$. Moreover, in the strong form of Strominger–Yau–Zaslow, the fibres of f and $\check{f}$ are required to be special Lagrangian, so by definition the restriction to the fibres of the symplectic form ω and the imaginary part of a fixed holomorphic volume form Ω vanish respectively. The base B carries the structure of a real affine manifold in two ways as follows. The transitions between coordinate charts of B are going to be elements of $\mathrm{GL}_n(\mathbb{Z}) \ltimes \mathbb{R}^n$ respectively.

One affine structure is determined by the complex structure of X and alternatively also by the symplectic structure of $\check{X}$. The other affine structure is determined by the symplectic structure of X and alternatively also by the complex structure of $\check{X}$. Let ν denote the vector field on $f^{-1}(b)$ given as a lift of a tangent vector $\bar{\nu}$ at a point $b \in B$ then the contraction of ω (respectively $\mathrm{im}\,\Omega$) by ν yields a one-form (respectively $(n-1)$-form) on $f^{-1}(b)$. That these are independent of the lift chosen follows from $f^{-1}(b)$ being special Lagrangian. McLean showed ([9, §6.1]) that these two forms on $f^{-1}(b)$ are both closed if and only if the infinitesimal deformation $\bar{\nu}$ of $f^{-1}(b)$ preserves the special Lagrangian property (which is true for a special Lagrangian fibration). Moreover, these two forms can be shown to be Hodge-star dual on $f^{-1}(b)$, so first order Lagrangian deformations correspond to harmonic one-forms on the Lagrangian. McLean proves that the moduli space of special Lagrangians is unobstructed [32, Thm. 3–4]. One deduces from this that B is locally the moduli space of the fibres of f as well as $\check{f}$. The just constructed maps descend to isomorphisms on cohomology

$$\begin{aligned} \mathcal{T}_{B,b} &\cong_\omega H^1(f^{-1}(b), \mathbb{R}), \\ \mathcal{T}_{B,b} &\cong_{\mathrm{im}\,\Omega} H^{n-1}(f^{-1}(b), \mathbb{R}), \end{aligned} \tag{1}$$

which give the tangent bundle two usually different flat connections. To distinguish the two, we denote the manifold B with the flat structure coming from f and ω by $\check{B}$ whereas the manifold with flat structure derived from f and $\mathrm{im}\,\Omega$ keeps the name B. For either of these, we call a set of coordinates $\{y_j\}$ *affine* if ∂_{y_j} are flat with respect to the respective flat structure. We also obtain a local systems of integral tangent vectors $\Lambda_B \subset \mathcal{T}_B$ isomorphic to the integral cohomology $H^{n-1}(f^{-1}(b), \mathbb{Z}) \subset H^{n-1}(f^{-1}(b), \mathbb{R})$ and similarly a system $\Lambda_{\check{B}} \subset \mathcal{T}_{\check{B}}$. A set of coordinates ∂_{y_j} on B (resp. $\check{B}$) is called *integral affine* if $\partial_{y_j} \in \Lambda_B$ (resp. in $\partial_{y_j} \in \Lambda_{\check{B}}$) and they form a basis over $\mathbb{Z}$. Thus, B and $\check{B}$ are real affine manifolds with coordinate transitions in $\mathrm{GL}_n(\mathbb{Z}) \ltimes \mathbb{R}^n$.

We assume that the torus bundle f is oriented and obtain from the second equation in (1), $\mathscr{T}_{B,b} \cong (H^1(f^{-1}(b),\mathbb{R}))^* = H_1(f^{-1}(b),\mathbb{R})$. Under this isomorphism, Λ_B becomes $H_1(f^{-1}(b),\mathbb{Z})$, so we have $X \cong \mathscr{T}_B/\Lambda_B$ as topological manifolds. Alternatively, we may also use $\mathscr{T}^*_{\check{B}} \cong H_1(f^{-1}(b),\mathbb{R})$ by means of the first equation in (1) to reconstruct X. We summarize

$$\mathscr{T}^*_{\check{B}}/\Lambda^*_{\check{B}} \underset{\omega}{\cong} X \underset{\mathrm{im}\,\Omega}{\cong} \mathscr{T}_B/\Lambda_B. \tag{2}$$

We can play the same game with $\check{f} : \check{X} \to B$ in place of $f : X \to B$ and the definition of SYZ mirror duality for $X, \check{X}$ is the statement that this is supposed to yield identical affine manifolds B, $\check{B}$ with swapped roles, i.e. the flat structure on B derives from the symplectic structure on $\check{X}$ and the flat structure on $\check{B}$ from the holomorphic structure on $\check{X}$, see Fig. 1.

The work of Gross and Siebert on mirror symmetry by means of toric degenerations, starting out with [23], was motivated by reverse engineering X and $\check{X}$ from B. The real difficulty arises when $X, \check{X}$ are intended to be compact since then $f, \check{f}$ need to have singular fibres and the affine structures need to have singularities as well. We will not deal with singularities before Sect. 6 but we adopt the point of view of reconstructing X and $\check{X}$ from B. In what we discussed so far, at least topologically by (2), the reconstruction of $X, \check{X}$ is straightforward once we know Λ_B and $\Lambda_{\check{B}}$. In fact, this is a datum we need to fix in addition to B and $\check{B}$. This topological picture can be enhanced as follows. Given the real affine manifold B, we have

(A) a canonical symplectic structure on $\check{X} := \mathscr{T}^*_B/\Lambda^*_B$ locally given by $\omega = \sum_j d\bar{x}_j \wedge dy_j$ where y_j are affine coordinates of B and $\bar{x}_j = \partial_{y_j}$,
(B) a canonical complex structure on $X := \mathscr{T}_B/\Lambda_B$ locally given by complex coordinates $z_j = x_j + iy_j$ where y_j are integral affine coordinates of B, $x_j = dy_j$ and $i = \sqrt{-1}$. The holomorphic volume form is $\Omega = dz_1 \wedge \cdots \wedge dz_n$. We set $w_j = e^{2\pi i z_j}$.

Note that integrality of the coordinates only matters in (B). To obtain the complementary parts, i.e., the symplectic structure on X and complex structure on $\check{X}$, one uses the structure of a Hessian metric g on B. We obtain Kähler manifold X and $\check{X}$ by applying (A), (B) on the respective dual side using g to identify the tangent and cotangent bundle of B. More explicitly, g is locally given as $g_{ij} = \partial_{y_i}\partial_{y_j}K$ for some smooth strictly convex function $K : B \to \mathbb{R}$. Mirror duality appears in this setup in the disguise of the Legendre transform, see [9, Prop. 6.4]:

Definition 1. Given a real affine manifold B with Hessian metric g, the *Legendre transform* is the real affine manifold $\check{B}$ which is homeomorphic to B with coordinates given by $\check{y}_j := \partial_{y_j}K$ (where y_j are local affine coordinates on B and K is a local potential defining g) and dual potential $\check{K} : \check{B} \to \mathbb{R}$,

$$\check{K}(\check{y}_1, \ldots, \check{y}_n) = \sum_j \check{y}_j y_j - K(y_1, \ldots, y_n).$$

Note that also the integral structure dualizes: dual integral affine coordinates are those that are the Legendre dual of integral affine coordinates.

The symplectic structure on X and the complex structure on $\check{X}$ is given directly by

$$\begin{aligned} \omega &= 2i\partial\bar{\partial}(K \circ f) = \tfrac{i}{2}\sum g_{jk} dz_j \wedge d\bar{z}_k, \\ \bar{z}_j &= \bar{x}_j + i\partial_{y_j} K, \end{aligned} \tag{3}$$

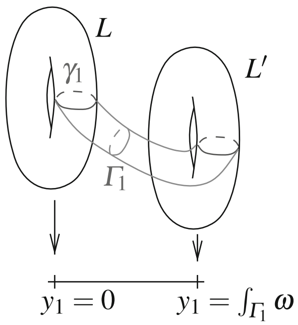
see [20, Prop 3.2], [9, Prop. 6.15].

The manifold X (resp $\check{X}$) is Ricci-flat (i.e., $\omega^n = c\Omega \wedge \bar{\Omega}$ for some $c \in \mathbb{C}$) if and only if $\det(\partial_{y_i}\partial_{y_j} K) = \det(g)$ is constant as follows from (3).

We discuss the following integrated version of the two affine structures which was pointed out to the author by Denis Auroux. It gives a hint at why mirror symmetry would exchanges periods and Gromov–Witten-invariants. Moreover, it leads towards Landau–Ginzburg potentials. Let X be a Calabi–Yau with Kähler form ω and non-vanishing holomorphic volume form Ω. The affine manifold B is the moduli space of special Lagrangian tori in X, i.e., the moduli of manifolds L homeomorphic to $(S^1)^n$ with $\omega|_L = 0$ and $\operatorname{im}\Omega|_L = 0$ (more generally one allows for a phase $\theta \in \mathbb{R}$, i.e., $\operatorname{im}(e^{i\theta}\Omega)|_L = 0$). Moreover, $\check{X}$ is given as the moduli space of pairs (L, ∇) where L is special Lagrangian and ∇ is a flat $U(1)$-connection of the trivial bundle with fibre $\mathbb{C}$ on L. The information of ∇ is equivalent to a map of groups $H_1(L, \mathbb{Z}) \to U(1)$. The local integral affine coordinates on the base are then given as

$$\begin{aligned} y_i &= \textstyle\int_{\Gamma_i} \omega, \\ \check{y}_i &= \textstyle\int_{\Gamma_i^*} \operatorname{im}\Omega \end{aligned} \tag{4}$$

where $\Gamma_i \in H_2(X, L \cup L')$ are cylinders traced out by a basis $\{\gamma_i\}$ of $H_1(L, \mathbb{Z})$ as we move L to L' and $\Gamma_i^* \in H_n(X, L \cup L')$ are traced out by a basis $\{\gamma_i^*\}$ of $H_{n-1}(L, \mathbb{Z})$ as we move L to L'.



Example 1 (The Mirror Dual of $(\mathbb{C}^*)^n$*).* The simplest example is $X = (\mathbb{C}^*)^n$. Its complex structure is indeed given as in (B) if we identify $B = \mathbb{R}^n$, $\mathscr{T}_B = \mathbb{R}^n \times \mathbb{R}^n$,

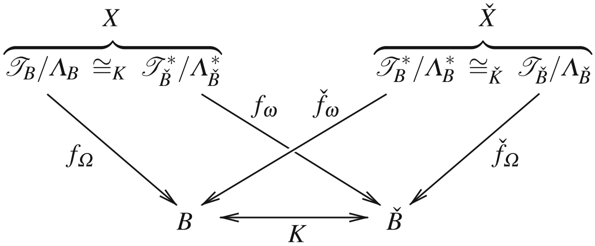


Fig. 1 $f_\Omega, \check{f}_\Omega$ are logarithm maps, $f_\omega, \check{f}_\omega$ are moment maps

$\Lambda = \mathbb{Z}^n$ where the latter is naturally contained in the second factor of $\mathscr{T}_B$. On the universal covers of $(\mathbb{C}^*)^n$ and $\mathscr{T}_B/\Lambda$ we set $z_j = x_j + iy_j$ where z_j are standard coordinates on $\mathbb{C}^n$, y_j are standard coordinates on B, $x_j = dy_j$ and $w_j = e^{2\pi i z_j}$ are standard coordinates on $(\mathbb{C}^*)^n$. We thus obtain

$$f : (\mathbb{C}^*)^n \to B, \qquad (w_1, \ldots, w_n) \mapsto \frac{-1}{2\pi}(\log|w_1|, \ldots, \log|w_n|) = (y_1, \ldots, y_n).$$

The holomorphic volume from is given by (B) as follows, we additionally pick the following symplectic form

$$\Omega = \tfrac{1}{(2\pi i)^n} \operatorname{dlog} w_1 \wedge \cdots \wedge \operatorname{dlog} w_n = dz_1 \wedge \cdots \wedge dz_n,$$
$$\omega = \tfrac{-1}{(2\pi)^2} \textstyle\sum_j \operatorname{dlog} r_j \wedge d\theta_j \quad = \textstyle\sum_j dx_j \wedge dy_j$$

where $w_j = r_j e^{i\theta_j}$. This choice turns f into a special Lagrangian fibration with $y_j = \check{y}_j$ as follows directly from (4). It determines $K = \frac{1}{2}\sum y_j^2$ up to a constant and g is the standard metric on B. We conclude from $y_j = \check{y}_j$ and $\mathscr{T}_B \cong \mathscr{T}^*_B$, that

the SYZ mirror dual of $((\mathbb{C}^*)^n, \Omega, \omega)$ *is* $((\mathbb{C}^*)^n, \Omega, \omega)$.

The setup in this example is very special in the sense that the two sets of affine coordinates on B coincide. It is easy to check that indeed $\omega = 2i\partial\bar{\partial}(K \circ f)$. More generally, the situation can be diagrammed as in Fig. 1. As verified in [4], Prop. 4.2, $\check{f}_\omega$ coincides with the moment map associated to ω and the natural fibrewise $(S^1)^n$-action on $\check{X} = \mathscr{T}^*_B/\Lambda^*$ and similarly for f_ω. Moreover, as in the above example, f_Ω is expressible as the map $\frac{-1}{2\pi}\log|\cdot|$ componentwise in the complex coordinates w_j on $X = \mathscr{T}_B/\Lambda$.

Example 2 (Further Mirror Duals of $(\mathbb{C}^*)^n$*).* While there aren't any interesting alternative algebraic choices for Ω in the previous example, there is a variety of choices for ω: for each equivariant embedding

$$\varphi : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^{m+1}/\mathbb{C}^*, \qquad (w_1, \ldots, w_n) \mapsto \left(\textstyle\prod_{k=1}^n w_k^{a_{0k}} : \cdots : \prod_{k=1}^n w_k^{a_{mk}}\right)$$

we can take $\omega = \varphi^*\omega_{\mathrm{FS}}$ where ω_{FS} is the Fubini–Study form on $\mathbb{P}^m = (\mathbb{C}^{m+1} \setminus \{0\})/\mathbb{C}^*$ (normalized by $\int_{\mathbb{P}^1} \omega = 1$), i.e.,

$$\pi^*\omega_{\mathrm{FS}} = \frac{i}{2\pi}\partial\bar{\partial}\log\|z\|^2 \tag{5}$$

for $\pi : \mathbb{C}^{m+1} \setminus \{0\} \to \mathbb{P}^m$ the natural projection. We want to compute f_ω. Let $S^{2m+1} = \{z \,|\, \|z\| = 1\}$ denote the unit sphere in $\mathbb{C}^{m+1}$. A straightforward computation shows that

$$\left(\partial\bar{\partial}\log\|z\|^2\right)\Big|_{S^{2m+1}} = \left(\sum_j dz_j \wedge d\bar{z}_j\right)\Big|_{S^{2m+1}}. \tag{6}$$

We represent $S^1 = \{e^{2\pi i\theta} | \theta \in \mathbb{R}\}$, so $\mathrm{Lie}(S^1)^* = \mathbb{R}\frac{1}{2\pi}\partial_\theta^*$ ($2\pi\partial_\theta$ is an integral coordinate). In this basis, a moment map for the Hamiltonian diagonal action of S^1 on $\mathbb{C}^{m+1}$ with respect to the symplectic form $\frac{i}{2\pi}\sum_j dz_j \wedge d\bar{z}_j$ is

$$z \mapsto 1 - \|z\|^2$$

(by setting the constant to 1), cf. [15, §2.3]. In particular, by (5), (6), ω_{FS} is the symplectic reduction of the form $\frac{i}{2\pi}\sum_j dz_j \wedge d\bar{z}_j$ on $\mathbb{C}^{m+1}$.

In order to obtain the desired moment map for ω, one may proceed as in [15, §6.6] as follows. The $(S^1)^n$ action induced by φ on $\mathbb{C}^{m+1}$ has moment map

$$(w_0, \dots, w_m) \mapsto -\sum_{j=0}^{m} |w_j|^2 a_j$$

with respect to $\frac{i}{2\pi}\sum_j dw_j \wedge d\bar{w}_j$ and a Lie algebra basis as above, cf. [15, Exc. 9]. The diagonal S^1 action commutes with the $(S^1)^n$ action and one can take successive symplectic reductions. One deduces that the moment map of the natural $(S^1)^n$ action on $(\mathbb{C}^*)^n$ with respect to ω is

$$f_\omega : (\mathbb{C}^*)^n \to \mathbb{R}^n, \qquad w \mapsto -\frac{\sum_{j=0}^m |\varphi_j(w)|^2 a_j}{\sum_{j=0}^m |\varphi_j(w)|^2}, \tag{7}$$

see [15, §6.6], cf. [17, §4.2]. In particular, if we are given a projective toric variety $\mathbb{P}_\Delta$ containing $(\mathbb{C}^*)^n$ as a dense orbit and given by a lattice polytope $\Delta \subset \mathbb{R}^n$, we may choose the a_j as the set of vertices of Δ which turns φ into the restriction of the rational map $\mathbb{P}_\Delta \to \mathbb{P}^m$ induced by linear system of $\mathcal{O}_{\mathbb{P}_\Delta}(1)$ with the basis of characters $\{z^{a_j} | a_j \text{ is a vertex of } \Delta\}$. We denote the resulting map by φ_Δ and ω_Δ denotes the symplectic form obtained from the φ_Δ by pulling back ω_{FS} as above. We have $\mathrm{im}\, f_\omega = -\mathrm{Int}(\Delta)$ by [17, §4.2] which is bounded unlike in Example 1. Since the complex manifold underlying the mirror is $\mathcal{T}_{-\mathrm{Int}(\Delta)}/\Lambda$, we have

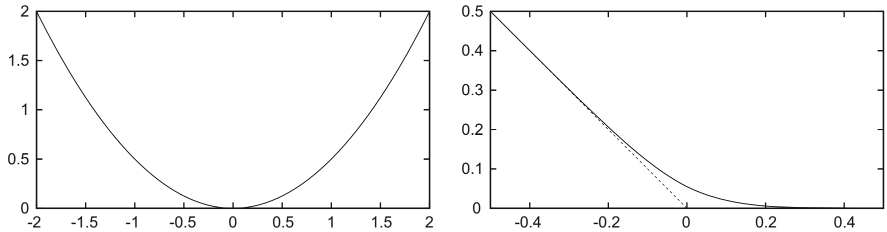


Fig. 2 $K(y) = \frac{1}{2}y^2$ and $K(y) = \frac{1}{4\pi}\log(1 + e^{-4\pi y}) = -\int_{-\infty}^{y} \frac{e^{-4\pi u}}{1+e^{-4\pi u}}du$

the mirror dual of $((\mathbb{C}^*)^n, \Omega, \omega_\Delta)$ *is a poly-annulus with cross-section* $\exp(2\pi\,\mathrm{Int}(\Delta))$,

see also [4, Prop. 4.2]. We obtain the potential K relating ω_Δ and Ω most easily by comparing (3) and (5), i.e., solving

$$2i\,\partial\bar\partial(K \circ f_\Omega) = \frac{i}{2\pi}\partial\bar\partial \log \sum_{j=0}^{m} |\varphi_j|^2$$

for K which yields

$$K(y_1, \ldots, y_n) = \frac{1}{4\pi}\log\Big(\sum_{j=0}^{m} \varphi_j(e^{-2\pi y_1}, \ldots, e^{-2\pi y_n})^2\Big).$$

Alternatively, we could solve the system $\check{y}_i = \partial_{y_i}K$ where $y_i, \check{y}_i$ are as in (4). We know $y_i = (f_\omega)_i$ from (7) and $\check{y}_i = \frac{-1}{2\pi}\log|w_i|$ from Example 1. Checking back the above K, we find that indeed

$$\begin{aligned}\partial_{y_i}K(y_1, \ldots, y_n) &= \frac{\sum_{j=0}^{m} \partial_{y_i}(\varphi_j(e^{-2\pi y_1}, \ldots, e^{-2\pi y_n})^2)}{4\pi\sum_{j=0}^{m}\varphi_j(e^{-2\pi y_1}, \ldots, e^{-2\pi y_n})^2} \\ &= -\frac{\sum_{j=0}^{m}(e^{-4\pi\sum_{k=0}^{m} a_{jk}y_k})a_{ji}}{\sum_{j=0}^{m}\varphi_j(e^{-2\pi y_1}, \ldots, e^{-2\pi y_n})^2} = f_\omega(e^{-2\pi y_1}, \ldots, e^{-2\pi y_n})_i.\end{aligned}$$

The boundedness of im f_ω is reflected in the asymptotic behaviour of the potential towards infinity, see e.g., Fig. 2 on the right.

Figure 2 shows the potentials for the construction of the mirror of $\mathbb{C}^*$ in Examples 1 and 2 respectively. In the latter case ω is obtained via the map $\varphi : \mathbb{C}^* \to$

$(\mathbb{C}^*)^2, \varphi = \{1\} \times \mathrm{id}_{\mathbb{C}^*}$, i.e., $a_{01} = 0, a_{11} = 1$. This corresponds to taking $\Delta = [0, 1]$ which is also the closure of all tangent slopes to K. Let us dwell on this for a moment and motivate the next section. Consider the sequence of symplectic forms on $\mathbb{C}^*$ given by $\omega_{r\Delta}$ for $r \in \mathbb{N}$. This corresponds to taking the sequence of embeddings φ^r_Δ inducing a sequence of potentials $K_r(y) = \frac{1}{4\pi}\log(1 + e^{-4\pi ry}) = K(ry)$ whose normalization has the limit

$$\lim_{r\to\infty} \frac{1}{r} K(ry) = \begin{cases} -y & \text{for } y \leq 0 \\ 0 & \text{for } y \geq 0. \end{cases} \tag{8}$$

Thus, looking at Fig. 2 on the right, the sequence of potentials approaches the piecewise linear function indicated by the positive real axis and the dotted line. Using this piecewise linear function, one can give a discrete version of the Legendre transform as we do in the following section.

2 Large Volume and Large Complex Structure Limit

Theoretical physicists studied Calabi–Yau manifolds in order to construct conformal field theories. To obtain such a theory from the more general concept of a quantum field theory (also via a Calabi–Yau manifold), a certain function needs to vanish (the β-function, see [9], §3.2.6.2) which can be enforced by taking a *large volume limit*. Since mirror symmetry is really about conformal field theories (at least by its origin), taking certain limits is an important step for its understanding. There are two related types of limits we are supposed to take, namely referring to (4),

$$\begin{array}{lll} \int_{\Gamma_i} \omega & \to \infty & \textit{large volume limit,} \\ \int_{\Gamma_i^*} \operatorname{im} \Omega & \to \infty & \textit{large complex structure limit.} \end{array} \tag{9}$$

Both of these limits amount to rescaling the affine base manifold B. Note that these interchange under mirror symmetry: a large volume limit on X turns into a large complex structure limit on $\check{X}$ and vice versa.

We intend to take both limits simultaneously. One needs to be a bit careful about how this works with the right choice of a potential: let us first rescale the coordinate y in (B) by r and see how this changes everything. All data become r-dependant which we indicate by making r an index. We set $y_{r,j} = ry_j$ and have

$$z_{r,j} = rz_j, \quad \partial_{y_{r,j}} = \frac{1}{r}\partial_{y_j}, \quad \Lambda_r = \frac{1}{r}\Lambda \quad \text{and} \quad \Omega_r = r^n\Omega.$$

The potential is as before determined by ω and this in turn is determined by the condition that the integral over a path scales by r: a priori, there are different ways to obtain an r-dependant potential:

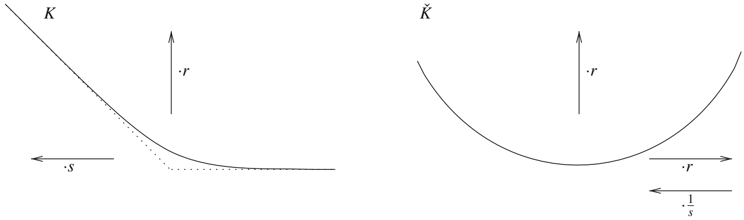


Fig. 3 $K(y) = \frac{1}{4\pi}\log(1 + e^{-4\pi y})$ and $\check{K}(y) = \frac{1}{4\pi}((y+1)\log(y+1) - y\log(-y))$

1. The first option is to just take the pullback of K via $y_{r,j} = ry_j$. This is $K'_r(y_r) := K(\frac{1}{r}y_r) = K(y)$. In terms of dual coordinates, this leads to $\check{y}'_j(y_r) = \partial_{y_{r,j}} K'_r(y_r) = \frac{1}{r}\partial_{y_j} K(y) = \frac{1}{r}\check{y}_j(\frac{1}{r}y_r)$. This is not what we want because it means that while enlarging the y-coordinates, we shrink the $\check{y}$-coordinates.
2. The next option is pulling back the dual coordinates via $y_{r,j} = ry_j$, i.e., set $\check{y}'_{r,j} = \check{y}_j(\frac{1}{r}y)$. With the previous calculation, it is easy to see that this corresponds to taking for the new potential the scaled pullback $K^0_r(y_r) := rK'_r(y_r)$.
3. Finally, in order to actually take the large volume limit simultaneously as the large complex structure limit, we need to scale the dual coordinates as well, i.e., $\check{y}_{r,j} = r\check{y}_j$. This is realized by rescaling the pullback potential even more by taking $K_r(y_r) := r^2 K'_r(y_r)$.

There are two types of limits that typically occur: metric limits and algebraic limits, for a discussion, see [9, 7.3.6.]. In some sense, these are represented by the two potentials shown in Fig. 2. Note that if we choose $K(y) = y^2/2$ then $K_r(y_r) = y_r^2/2$, so this potential remains invariant under taking the simultaneous limit. The effect is that the base B of f_ω and f_Ω becomes longer and longer as one approaches the limit. Rescaling the metric to normalize the diameter yields B itself as a limit the Calabi–Yaus. For an elliptic curve with potential $y_r^2/2$, the metric limit is thus a circle, cf. [20], Conj. 5.4. We are interested in algebraic limits and for such, the non-self-dual second potential in Fig. 2 is more relevant. Figure 3 illustrates how the scaling of the potential (here by factor r) and the base coordinate (here by factor s) influences the Legendre dual and dual potential. The diagram really only shows part of all rescaling options where the remaining ones come from applying the given ones on the dual side. In fact, in view of Fig. 3, the result of scaling by r on either side results in scaling both potentials by r^2 and both coordinates by r as we did in (3). above. There is still the degree of freedom of scaling by s which has a reciprocal effect on the dual. This explains why the limit we gave in (8) appears to be turned into the limit $r \to 0$ now. In truth, it was a limit with respect to the parameter s. The important point is that in algebraic examples, there is a non-trivial rescaling by s but it is non-homogenous along the base, i.e., in some regions it looks like a contraction, in others like an expansion. We will see this in the algebraic degeneration of an elliptic curve as well as in the mirror duality of $\mathbb{P}^1$. It is really this rescaling which yields a discretization of the Legendre transform. Before we give an example, we relate r to the algebraic coordinate: on $\mathscr{T}_B/\frac{1}{r}\Lambda$, we consider the two potentials derived from K_r, K_r^0 but modified by some inhomogeneously rescaling by some s. These potentials lead to symplectic forms ω, ω_0 via the first equation in (3) and we have $\omega = r\omega_0$. Reparametrizing $|t| = e^{-2\pi r}$ for t a coordinate on the unit disk, we get

$$\omega_t = \frac{\log|t|}{-2\pi}\,\omega_0.$$

While ω_t is going to infinity as $t \to 0$, we will find that ω_0 is bounded.

Example 3 (Elliptic Curve). The elliptic curve has been considered from an SYZ perspective many times before. We mostly follow [20], §6, see also [9], §8.4.1: We fix $n \in \mathbb{N}$ and consider the affine manifold $B = \mathbb{R}/n\mathbb{Z}$ with y being the standard coordinate on $\mathbb{R}$ and obtain the elliptic curve $X_r = \mathscr{T}_B/\frac{1}{r}\Lambda$ with periods 1 and irn. The family parameter r can be complexified: either ad hoc by using the complex coordinate t on the unit disc as before and then $X_t = \mathscr{T}_B/\frac{\log(t^n)}{2\pi i}\Lambda$ (note that we abuse notation here, we use the identification $\mathscr{T}_B = \mathbb{C}/n\mathbb{Z}$ via (B)) or more conceptually by invoking the B-field as in [9], §6.2.3. The limit for $t \to 0$ can be filled by a cycle of $\mathbb{P}^1$s of length n. This turns the total space of the family into a maximally unipotent degeneration.[1] Siebert had the idea to use log geometry to view the singular special fibre $\mathscr{X}_0$ as a (log) smooth Calabi–Yau. Indeed, let us compute the logarithmic cotangent sheaf on X_0, i.e., the restriction of the relative logarithmic cotangent sheaf $K_{X_0} = \Omega^1_{X_0}(\log X_0) := \Omega^1_{\mathscr{X}/O}(\log \mathscr{X}_0)|_{\mathscr{X}_0}$ with $O =$ unit disk. For each irreducible component $\mathbb{P}^1$ of $\mathscr{X}_0$, we have $K_{\mathscr{X}_0}|_{\mathbb{P}^1} = \Omega^1_{\mathbb{P}^1}(\log(\{0\}\cup\{\infty\})) \cong \mathscr{O}_{\mathbb{P}^1}$ and locally at an intersection point the pair $(\mathscr{X}, \mathscr{X}_0)$ is $(\operatorname{Spec}\mathbb{C}[u,v], V(t))$ with $t = uv$ and thus

$$\Omega^1_{\mathbb{C}^2/\mathbb{C}_t}(\log V(t)) = (\mathscr{O}_{\mathbb{C}^2}\frac{du}{u} \oplus \mathscr{O}_{\mathbb{C}^2}\frac{dv}{v})/\mathscr{O}_{\mathbb{C}^2}(\frac{du}{u} + \frac{dv}{v}) \cong \mathscr{O}_{\mathbb{C}^2}.$$

We deduce $K_{\mathscr{X}_0} \cong \mathscr{O}_{\mathscr{X}_0}$, so $\mathscr{X}_0$ is a log elliptic curve. To obtain a nowhere vanishing global section Ω of $K_{\mathscr{X}_0}$ we can just extend the local section $\frac{du}{u}$ in a standard chart of one of the components. There is a (degenerate) Strominger–Yau–Zaslow fibration $\mathscr{X}_0 \to B$ given as the compactification of the special Lagrangian fibration (with respect to Ω and ω_0) on the dense subset of $\mathscr{X}_0$ whose intersection with each $\mathbb{P}^1$ is $\mathbb{C}^* = \mathbb{P}^1\setminus(\{0\}\cup\{\infty\})$.

Let us discuss the potential K and Kähler form. We already mentioned that $K = y^2/2$ is not a useful choice here. In fact, Gross realized [20, §6], that if we take an open cover of X_0 in $\mathscr{X}$, the intersection of the nearby fibre with a neighbourhood of a node approaches $\mathscr{T}_{(0,1)}/\Lambda_r$ for $r \to \infty$ whereas away from the nodes it approaches

[1] This means $\mathscr{X}$ is flat over the base such that $X = \mathscr{X}_{t_0}$ for some $t_0 \neq 0$ and $T \in \operatorname{End}(H^\bullet(X,\mathbb{Q}))$, the monodromy operator around the special fibre at $t = 0$, satisfies $(T - \mathrm{id})^{n+1} = 0$ and $(T - \mathrm{id})^{n+1} \neq 0$ with $n = \dim X$.

$\mathscr{T}_{[0,0]}/\Lambda_r$, so all the mass in the complex geometry goes to the nodes. Conversely, all the mass in the symplectic geometry should leave any small neighbourhood of any node. This of course depends on the choice of potential which we make as follows.

In general, we want to have a relatively ample line bundle $\mathscr{L}$ on $\mathscr{X}$ and sections $s_0, \ldots, s_m$ which are in bijection with the zero-dimensional strata (which are the nodes of X_0 in this example) $v_0, \ldots, v_m$ in X_0 whose vanishing locus is contained in X_0 and such that s_j vanishes along precisely those components of X_0 that do not contain v_j. In analogy to Example 2, we then define the family of two-forms $\omega_r = \frac{i}{2\pi}\partial\bar{\partial}\log\sum_j |s_j|^{2r}$ on $\mathscr{X}$ that is fibrewise a symplectic form. Let ω denote the two-form on $\mathscr{X} \setminus X_0$ that restricts to $\omega_{\frac{\log|t|}{-2\pi}}$ on the fibre X_t. Its normalization is $\omega_0 = \frac{-2\pi}{\log|t|}\omega$.

In our example, this limit is the potential given by (8) on each $\mathbb{P}^1$ component of X_0 (up to the addition of an affine function). Indeed, only two s_j are non-vanishing on this $\mathbb{P}^1$ and they give the potential on the right of Fig. 2. For concreteness, let us refine the example by considering the family of Fermat elliptic curves in $\mathbb{P}^2$ given by $z_0z_1z_2 + t(z_0^3 + z_1^3 + z_2^3) = 0$, then $\mathscr{X}$ is the blow-up of $\mathbb{P}^2$ in the base locus of the family, $X_0 = \{z_0z_1z_2 = 0\}$, $B = \mathbb{R}/3\mathbb{Z}$, $\mathscr{L}$ can be chosen as $\mathscr{O}(1)$ and $s_j = z_j$. The upshot is: the potential on B becomes a piecewise affine function with non-linearity at the three integral points of $B/3\mathbb{Z}$ corresponding to the equators of the components of X_0.

The example led us to the consideration of a piecewise affine potential in the limit. We deal with a version of the Legendre transform for such potentials in the next section. Moreover, so far we have been dealing only with the situation where all fibres of the SYZ maps are smooth. Talking about compact manifolds with vanishing first Chern class, this restricts one to the study of complex tori, e.g., the elliptic curve just studied. For Hyperkähler manifolds or Calabi–Yau manifolds in the strong sense,[2] one has to allow singular torus fibres. The critical loci of these fibres play an important role in the theory. There is another way of obtaining interesting geometry, namely by allowing a boundary for the affine manifold over which the SYZ fibration takes lower-dimensional tori as fibres, a typical situation for the compactifications of moment maps from $(\mathbb{C}^*)^n$.

3 Algebraic Limits and the Discrete Legendre Transform of a Tropical Manifold

We have already seen in an example that an algebraic large complex structure limit with simultaneous large volume limit leads to a discretization of the Legendre transform. A general definition of this has been given in [24] for an affine manifold

[2]This means $h^\bullet(X, \mathscr{O}_X) = h^\bullet(S^n, \mathbb{Q})$ for $n = \dim X$.

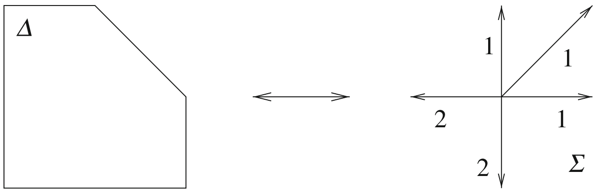


Fig. 4 A polytope is dual to a fan with piecewise linear function. The piecewise linear function is given up to addition of a linear function by its slope changes along the rays as given in the diagram

with (a certain type of) singularities that behave well with regard to the piecewise affine potential. A discrete Legendre transform on a vector space had been known before, see [3, §14]. We are going to give a natural extension to manifolds with polyhedral boundary. The simplest example of a discrete Legendre transform is the correspondence

$$\Delta \leftrightarrow (\Sigma, \varphi)$$

of a polytope with a fan and piecewise linear convex function, well-known in toric geometry, see [17], §3.4 as well as Fig. 4. The underlying manifolds are the polytope Δ and a real vector space respectively. Note that φ now plays the role of the strictly convex function $\check{K}$, but we need to weaken the assumption on $K, \check{K}$ from strictly convex as in the smooth case to just convex.[3]

The definition of a piecewise linear function φ associated to a polytope Δ can be given as

$$\varphi(n) = \max\{\langle n, m\rangle \mid m \in \Delta\} \tag{10}$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing of a vector space with its dual space. If we take $K \equiv 0$ for the piecewise linear function on the polytope, this coincides with the previous definition since one can check (e.g. [3, §14]) that

$$\check{K}(\check{y}) = \max_y \{\sum_i \check{y}_i y_i - K(y)\}.$$

Note that [17] uses $\varphi(n) = -\inf\{\langle n, m\rangle \mid m \in \Delta\}$. We should make a remark on sign conventions here that also explains the minus sign in (7). Since our discussion is governed by the Legendre transform and this associates to a point the tangent at a convex function over the point, positive directions should get mapped to positive directions under this transform unlike in [17] where concave functions are used.

[3]Confusingly in the discrete world (e.g. [17]), for a piecewise affine function on a polyhedral complex the notion *strictly convex* is used for the property where the maximal cells coincide with non-extendable domains of linearity of the function. This is actually the type of function we want.

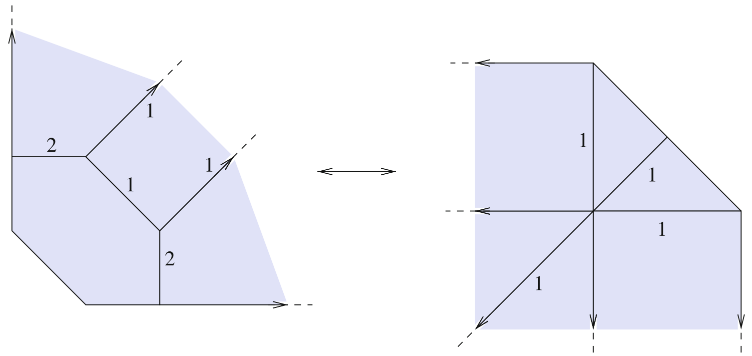


Fig. 5 An example of a discrete Legendre transform

The general construction of a discrete Legendre transform is obtained from patching this example in both directions: assume we have an integral affine manifold B, i.e., a real affine manifold with an atlas whose transition functions are in $\mathbb{Z}^n \rtimes \mathrm{GL}_n(\mathbb{Z})$. Moreover, we assume to have a polyhedral decomposition $\mathscr{P}$ of B, i.e., $\mathscr{P}$ is a set of lattice polytopes each of which comes with an immersion in B, the set $\mathscr{P}$ covers B, is closed under intersection in B and two polytopes in $\mathscr{P}$ coincide if there image in B does. We also need a polarization φ which is a section of $\mathrm{PAC}(B,\mathbb{R})/\mathrm{Aff}(B,\mathbb{R})$, the sheaf of piecewise affine convex functions on B (piecewise with respect to $\mathscr{P}$) with rational slopes modulo the sheaf of affine functions on B (both with rational slopes). We require that the non-extendable domains of linearity of φ coincide with the maximal cells in $\mathscr{P}$. We also require the boundary of B to be locally convex, more precisely, near each point in ∂B, the pair $(B, \partial B)$ looks like an open subset of a lattice polytope with its boundary. Such a triple $(B, \mathscr{P}, \varphi)$ is called a *tropical manifold*. The *discrete Legendre transform* (DLT) associates another tropical affine manifold to $(B, \mathscr{P}, \varphi)$ and is a duality:

$$(B, \mathscr{P}, \varphi) \longleftrightarrow (\check{B}, \check{\mathscr{P}}, \check{\varphi}).$$

The dual is constructed as follows: The neighbourhood of each vertex v in $\mathscr{P}$ can be identified with a neighbourhood of the origin of a fan Σ_v and φ restricts to a piecewise linear convex function on its support. Thus from the duality in Fig. 4, we obtain a lattice polytope $\check{v}$. On the other hand, for each maximal cell σ in $\mathscr{P}$, again by the duality in Fig. 4, we obtain a fan $\check{\Sigma}_\sigma$ with a piecewise linear function $\check{\varphi}_\sigma$. Finally, $\check{B}$ is given by gluing all these polytopes and fans according to their adjacency, see Fig. 5 for an example.

Example 4 (Duality of Cones as a DLT). Note that the duality of cones is a special case of a DLT: Let $\sigma \subset \mathbb{R}^n$ be a rationally generated polyhedral cone containing no non-trivial linear subspace and

$$\check{\sigma} = \{n \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}) \mid n(m) \geq 0 \text{ for all } m \in \sigma\}.$$

Taking trivial piecewise linear functions and for the polyhedral decompositions the set of faces respectively gives a discrete Legendre transform

$$\sigma \longleftrightarrow -\check{\sigma}.$$

Note that this is more general that the polytope-to-fan duality (e.g., Fig. 4) because given a polytope Δ, we may take σ to be

$$\operatorname{Cone}(-\Delta) = \{(rm, r) \mid m \in -\Delta, r \in \mathbb{R}_{\geq 0}\} \subset \mathbb{R}^n \times \mathbb{R},$$

the cone over $-\Delta$. Then (Σ, φ), the DLT of Δ, is obtained from the dual cone $\check{\sigma} \subseteq \operatorname{Hom}(\mathbb{R}^n \oplus \mathbb{R}, \mathbb{R})$ as follows: Σ is the projection of the proper faces of $\check{\sigma}$ under the restriction

$$\operatorname{Hom}(\mathbb{R}^n \oplus \mathbb{R}, \mathbb{R}) \twoheadrightarrow \operatorname{Hom}(\mathbb{R}^n, \mathbb{R})$$

and the graph of φ is the section of this projection given by $\partial\check{\sigma}$.

We will come back to this example later.

It is quite remarkable that the construction of the discrete Legendre transform even works if the tropical manifold has singularities as long as the local monodromy around the singularities respects the polyhedral decomposition, see [24]. The discriminant loci in B and $\check{B}$ are then homeomorphic. Singularities are an important feature of the story. If one wants to study compact Calabi–Yau manifolds, the base B of the SYZ fibration needs to be a homology sphere, see [24, Prop. 2.37]. Therefore, the fibration needs to have singular fibres which are reflected in the base as singular locus of the affine structure of codimension two. The singularities only affect the affine structure, the underlying topological space will still be a topological manifold (with boundary). The local monodromy on the tangent bundle $\mathscr{T}_B$ around a branch of the discriminant coincides with the monodromy in the cohomology of a nearby smooth torus fibre for case (A) and the homology for (B). See [19] for a systematic account on how to obtain a DLT from reflexive polytopes and nef partitions. For further examples on affine manifolds with singularities, see [12, 29, 34, 39].

Before closing this section, we would like to introduce natural refinements of the cell decompositions $(B, \mathscr{P})$ and $(\check{B}, \check{\mathscr{P}})$ that give topologically a common refinement on the interiors of B and $\check{B}$. This is given by the *barycentric subdivision* (cf. [24, Def. 1.25] for the compact case and [38, Def. 3.2] for an alternative definition in the non-compact case with the draw-down that doesn't seem natural in the context of SYZ fibrations). The definition we give requires that each unbounded cell $\tau \in \mathscr{P}$ has the property that the convex hull of its vertices $\operatorname{conv}(\tau^{[0]})$ is a face of τ. This is satisfied in Fig. 5.

We define a triangulation $\mathscr{P}^{\text{bar}}$ of B, which introduces one new vertex in each relative interior of a compact cell $\tau \in \mathscr{P}$. This vertex is the *barycenter* of the cell and is defined as the average of the cell's vertices $v_\sigma^{\text{bar}} = \frac{1}{\#\sigma^{[0]}} \sum_{v \in \sigma^{[0]}} v$ where $\sigma^{[0]}$ denotes the set of vertices of σ. We may use the same definition to associate a

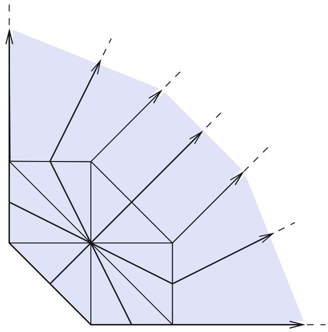


Fig. 6 The barycentric subdivision of the left hand side in Fig. 5 indicating by bold lines how to obtain the cell decomposition given by its right hand side

barycenter to an unbounded cell, so for τ unbounded we have $v_\tau^{\mathrm{bar}} = v_{\mathrm{conv}\tau^{[0]}}^{\mathrm{bar}}$. We then set

$$\mathscr{P}^{\mathrm{bar}} = \{\mathrm{conv}\{v_{\tau_0}^{\mathrm{bar}}, \dots, v_{\tau_k}^{\mathrm{bar}}\} \mid \tau_0 \subsetneq \cdots \subsetneq \tau_k, \tau_i \in \mathscr{P}, k \geq 0\} \cup \mathscr{P}_{\mathrm{unbounded}}^{\mathrm{bar}}$$

where conv means taking the convex hull and $\mathscr{P}_{\mathrm{unbounded}}^{\mathrm{bar}}$ will be empty if each cell in B is bounded. It is defined as

$$\mathscr{P}_{\mathrm{unbounded}}^{\mathrm{bar}} = \{\mathrm{conv}\{v_{\tau_0}^{\mathrm{bar}}, \dots, v_{\tau_k}^{\mathrm{bar}}\} + \sum_{i=1}^{k} \mathbb{R}_{\geq 0}\rho_{\tau_i} \mid \tau_0 \subsetneq \dots \subsetneq \tau_k, \tau_i \in \mathscr{P}, \tau_i \text{ is unbounded}\}$$

where ρ_{τ_i} is the sum of all primitive integral generators of the rays in τ_i, so $\sum_{i=1}^{k} \mathbb{R}_{\geq 0}\rho_{\tau_i}$ is a cone generated by such rays and its sum with $\mathrm{conv}\{v_{\tau_0}^{\mathrm{bar}}, \dots, v_{\tau_k}^{\mathrm{bar}}\}$ should be read as a Minkowski sum (i.e., pointwise sum).

Note that indeed $\mathscr{P}^{\mathrm{bar}}$ is a refinement of the polyhedral decomposition $\mathscr{P}$ of B and, after removing the boundary respectively, topologically also of $\check{\mathscr{P}}$ of $\check{B}$, namely by respectively merging all cells in $\mathscr{P}^{\mathrm{bar}}$ which contain a vertex that is in $\mathscr{P}$ but not in $\mathscr{P}^{\mathrm{bar}}$. See Fig. 6 for an example.

4 The Degenerate Calabi–Yau Fibre and the Reconstruction Problem

As in the case of the elliptic curve, the tropical manifold $(B, \mathscr{P}, \varphi)$ encodes a degenerate fibre as follows [24]: each cell $\sigma \in \mathscr{P}$ gives a projective toric variety

$$\mathbb{P}_\sigma = \mathrm{Proj}\, \mathbb{C}[\mathrm{Cone}(\sigma) \cap (\mathbb{Z}^n \oplus \mathbb{Z})]$$

where $\mathrm{Cone}(\sigma)$ was defined in Example 4. This is functorial for inclusions of cells: $\tau \subseteq \sigma \Rightarrow \mathbb{P}_\tau \subseteq \mathbb{P}_\sigma$, so we may form the limit

$$\check{X}_0(B, \mathscr{P}, \varphi) := \varinjlim_{\sigma \in \mathscr{P}} \mathbb{P}_\sigma$$

which is called the degenerate Calabi–Yau in the *cone picture*. This should be thought of as a degeneration of (A) in Sect. 1. Dually, concerning a degeneration of (B), for each $\sigma \in \mathscr{P}$, we may consider the *fan along* σ by which we mean the following. Let U_σ be a sufficiently small neighbourhood of the relative interior of σ. The image of $\{\tau \in \mathscr{P} \mid \sigma \subset \tau\}$ under the projection $U_\sigma \twoheadrightarrow U_\sigma/\sigma$ (where two points are identified if their difference is parallel to σ) gives a neighbourhood of the origin of a fan Σ_σ in $\mathbb{R}^{n-\dim\sigma}$ unique up to isomorphism. Let X_{Σ_σ} denote the corresponding toric variety. This construction is contravariantly functorial for inclusions: $\tau \subseteq \sigma \Rightarrow X_{\Sigma_\sigma} \subseteq X_{\Sigma_\tau}$, so we may form

$$X_0(B, \mathscr{P}, \varphi) := \varprojlim_{\sigma \in \mathscr{P}} X_{\Sigma_\sigma}$$

which we call the degenerate Calabi–Yau in the *fan picture*. It is not hard to see that in fact

$$X_0(B, \mathscr{P}, \varphi) = \check{X}_0(\check{B}, \check{\mathscr{P}}, \check{\varphi})$$

which should be compared to Fig. 1: indeed, there is a continuous map

$$f_\omega : \check{X}_0(B, \mathscr{P}, \varphi) \to B$$

by taking a direct limit over all moment maps $f_{\omega_\sigma} : \mathbb{P}_\sigma \to \sigma$ for each $\sigma \in \mathscr{P}$, see Example 2 for the definition of ω_σ. This does not coincide with the limit map f_{ω_0} discussed in Example 3 but it is f_{ω_1} restricted to the central fibre. The meaning of f_ω could be understood as this: suppose we have a nearby fibre X_t, then we can use symplectic parallel transport to get a retraction map $\check{X}_t \to \check{X}_0$ and we can compose this with f_ω to get a Lagrangian fibration $\check{X}_t \to B$. It is currently not clear how to turn this into a special Lagrangian fibration. We have a diagram:

$$\begin{array}{ccc}
\check{X}_0(B, \mathscr{P}, \varphi) & & X_0(\check{B}, \check{\mathscr{P}}, \check{\varphi}) \\
\downarrow{\scriptstyle \check{f}_\omega} & & \downarrow{\scriptstyle f_\omega} \\
(B, \mathscr{P}, \varphi) & \xleftrightarrow{\text{DLT}} & (\check{B}, \check{\mathscr{P}}, \check{\varphi})
\end{array}$$

We have called $X_0(B, \mathscr{P}, \varphi)$ and $\check{X}_0(B, \mathscr{P}, \varphi)$ *Calabi–Yau*. This is justified if its canonical bundle is trivial. These spaces have a log structure would be entirely encoded in $\mathscr{P}$ for the first and in φ for the second if were no singularities in the affine structure. The singularities however contribute non-discrete moduli of the log structure encoded in so-called *slab functions*, see [26]. We will not go into defining log structures, but recall that in the case of the elliptic curve we constructed a sheaf of log differential forms which was trivial. This generalizes as long as the transition functions of B can be chosen in $\mathbb{Z}^n \rtimes \mathrm{SL}_n(\mathbb{Z})$, i.e., B is orientable.

The log differential forms restricted to each component $\mathbb{P}_\sigma$ of $\check{X}_0(B, \mathscr{P}, \varphi)$ are just $\Omega^k_{\mathbb{P}_\sigma}(\log D_\sigma)$, the differential forms with logarithmic poles along D_σ where D_σ is the complement of the dense torus in $\mathbb{P}_\sigma$. These sheaves glue to a sheaf $\Omega^k := \Omega^k_{\check{X}_0(B,\mathscr{P},\varphi)^\dagger/\operatorname{Spec}\mathbb{C}^\dagger}$, though the gluing is non-trivial whenever singularities appear (the dagger indicating the presence of a log structure), see [25, 35, §3.2]. If B is orientable, Ω^n is trivial and a section gives a global holomorphic volume form with logarithmic poles.

The *reconstruction problem* is the question of whether one can reconstruct a smooth (or at most orbifold) Calabi–Yau X_t from its degeneration X_0; more precisely, whether we can lift X_0 from a space over a point to a flat family $\mathscr{X}$ over the unit disk whose non-zero fibres have at most orbifold singularities. In general, so in presence of singularities, this is a very difficult problem towards which Gross and Siebert accomplished a major break-through in [26] by proving a canonical liftability to $\operatorname{Spec}\mathbb{C}[\![t]\!]$ assuming that the local monodromy of the affine singularities of B cannot be factored (locally rigid). The parametrization of the disk is also important and Gross and Siebert obtain the one trivializing the Gauss–Manin connection (flat coordinates). Their proof is constructive and involves *wall-crossings*. We will come back to this in a later section.

We now treat an easy case: Assume that B is a lattice polyhedron in $\mathbb{R}^n$ and $\mathscr{P}$ a subdivision of it given by a piecewise linear function φ. So in particular, we have no singularities. It is not hard to see that the dual $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ also has the property that it globally embeds in a vector space (the dual space). The DLT here can be worked out as follows. Let $\Delta(B, \mathscr{P}, \varphi)$ be the polyhedron in $\mathbb{R}^n \oplus \mathbb{R}$ given as

$$\Delta_{(B,\mathscr{P},\varphi)} = \{(m, r) \in \mathbb{R}^n \oplus \mathbb{R} | \varphi(m) \geq r\}$$

and let

$$\Sigma_{(B,\mathscr{P},\varphi)} = \{0\} \cup \{\overline{\operatorname{Cone}(\tau)} \mid \tau \in \mathscr{P}\}$$

be the fan in $\mathbb{R}^n \oplus \mathbb{R}$ where $\overline{\operatorname{Cone}(\tau)}$ denotes the closure of $\operatorname{Cone}(\tau)$ in $\mathbb{R}^n \oplus \mathbb{R}$. We define the piecewise linear function $\varphi_{(B,\mathscr{P},\varphi)}(m, r) = r\varphi(m)$. We have a DLT

$$\Delta_{(B,\mathscr{P},\varphi)} \leftrightarrow (\Sigma_{(\check{B},\check{\mathscr{P}},\check{\varphi})}, \varphi_{(\check{B},\check{\mathscr{P}},\check{\varphi})})$$

which is really just the classical toric story as in Fig. 4. The original DLT $(B, \mathscr{P}, \varphi) \leftrightarrow (\check{B}, \check{\mathscr{P}}, \check{\varphi})$ is contained in this as a "sub-DLT" by intersecting with $\mathbb{R}^n \times \{1\}$. Moreover, this picture solves the reconstruction problem: The fan $\Sigma_{(B,\mathscr{P},\varphi)}$ maps to the fan of $\mathbb{A}^1$ by the projection to the second factor $\mathbb{R}^n \oplus \mathbb{R} \twoheadrightarrow \mathbb{R}$, so we have a map of toric varieties

$$f : \mathscr{X}(B, \mathscr{P}, \varphi) := X_{\Sigma_{(B,\mathscr{P},\varphi)}} \to \operatorname{Spec}\mathbb{C}[t]$$

such that $f^{-1}(0) = X_0(B, \mathscr{P}, \varphi)$ and $f^{-1}(t)$ is irreducible for $t \neq 0$. In fact $f^{-1}(t)$ is isomorphic to the toric variety given by the *asymptotic fan* of $(B, \mathscr{P}, \varphi)$

which is just the sub-fan of $\Sigma_{(B,\mathscr{P},\varphi)}$ contained in $\mathbb{R}^n \times \{0\}$. So if this gives a smooth toric variety, a general fibre of f is smooth. This is the total space description for the fan picture.

There is a dual version, the cone picture $\check{\mathscr{X}}(\check{B},\check{\mathscr{P}},\check{\varphi})$ of the total space satisfying

$$\check{\mathscr{X}}(\check{B},\check{\mathscr{P}},\check{\varphi}) = \mathscr{X}(B,\mathscr{P},\varphi).$$

Gross and Siebert use this cone picture description to prove the more general reconstruction (non-embedded situation). Those familiar with toric geometry will know that we have

$$\check{\mathscr{X}}(B,\mathscr{P},\varphi) = \operatorname{Proj} \mathbb{C}[\overline{\mathrm{Cone}(\Delta_{(B,\mathscr{P},\varphi)})} \cap \mathbb{Z}^{n+2}].$$

Let us recall how this works by gluing charts: To each vertex v of $\Delta_{(B,\mathscr{P},\varphi)}$ we associate the ring $R_v = \mathbb{C}[\mathbb{R}_{\geq 0}(\Delta_{(B,\mathscr{P},\varphi)} - v) \cap (\mathbb{Z}^n \oplus \mathbb{Z})]$ which is naturally a $\mathbb{C}[t]$-algebra by mapping t to the monomial given by the unique generator of the second summand in $\mathbb{Z}^n \oplus \mathbb{Z}$ (indeed, it is contained in $\Delta_{(B,\mathscr{P},\varphi)} - v$). The affine varieties Spec R_v will give an open cover of $\mathscr{X}$. The intersection of two such, Spec R_v and Spec R_w, is empty if no cell in $\mathscr{P}$ contains both v and w and otherwise for τ being the minimal cell containing both, we may localize the rings R_v and R_w by inverting all elements that are sums of monomials with exponents contained in $\mathbb{R}_{\geq 0}(\tau - v)$ (respectively $\mathbb{R}_{\geq 0}(\tau - w)$). Denoting the resulting rings $R_{v,\tau}$ and $R_{w,\tau}$, we have a natural isomorphism $R_{v,\tau} \to R_{w,\tau}$ induced by

$$\mathbb{R}_{\geq 0}(\Delta_{(B,\mathscr{P},\varphi)} - v) + \mathbb{R}(\tau - v) = \mathbb{R}_{\geq 0}(\Delta_{(B,\mathscr{P},\varphi)} - w) + \mathbb{R}(\tau - w).$$

All these isomorphisms are compatible and glue to give $\check{\mathscr{X}}(B,\mathscr{P},\varphi)$ and a map $\check{f} : \check{\mathscr{X}}(B,\mathscr{P},\varphi) \to \mathbb{C}$ such that $\check{f}^{-1}(0) = \check{X}_0(B,\mathscr{P},\varphi)$. To see the latter, note that we identify

$$\operatorname{Proj} \mathbb{C}[\mathrm{Cone}(\sigma) \cap (\mathbb{Z}^n \oplus \mathbb{Z})] \cong \operatorname{Proj} \mathbb{C}[\mathrm{Cone}(\varphi(\sigma)) \cap ((\mathbb{Z}^n \oplus \mathbb{Z}) \oplus \mathbb{Z})]. \quad (11)$$

5 Compactifying Divisors and the Landau–Ginzburg Potential

We have already dealt with the situation where B has a boundary when we discussed discrete Legendre transforms. We now want to match it with the discussion of SYZ fibrations from Sect. 1. For this, let us consider the mirror dual of $\mathbb{P}^1$. We have already treated the mirror dual of $\mathbb{C}^*$ with respect to its Fubini–Study-metric coming from the embedding in $\mathbb{P}^1$, we have

$$\mathbb{R} \xleftarrow{f_\Omega} \mathbb{C}^* \xrightarrow{f_\omega} (0,1).$$

The map f_ω naturally extends to $\mathbb{P}^1 \to [0,1]$. We may think of the compactifying divisor $D = \{0\} \cup \{\infty\}$ as adding (partially) contracted SYZ fibres. In fact, we contract the 1-cycle which we used to define our base coordinate via the first integral in (4). Phrased differently, the holomorphic cylinders[4] which we used to define the base coordinate y on $(0,1)$ becomes a holomorphic disk. By the maximum principle, there are no holomorphic disks in $\mathbb{C}^*$, but they do appear as we compactify to $\mathbb{P}^1$. It is insightful to interpret the presence of holomorphic disks from the point of view of Floer theory, see [5] for a detailed account. We already mentioned that the mirror $\check{X}$ of $X = \mathbb{C}^*$ can be considered as the moduli space of pairs (L, ∇) where L special Lagrangian tori with a $U(1)$-connection ∇ on $L \times \mathbb{C}$. Fukaya–Oh–Ohta–Ono [16] give an obstruction for the intersection Floer homology complex to be a complex. If we are interested in the Floer homology $HF^\bullet(\mathscr{L}, \mathscr{L})$ of $\mathscr{L} = (L, \nabla)$ with itself, the obstruction is

$$m_0(\mathscr{L}) = \sum_{\substack{\beta \in \pi_2(X,L) \\ \mu(\beta)=2}} n_\beta(\mathscr{L}) z_\beta(\mathscr{L}) \tag{12}$$

where $n_\beta(\mathscr{L})$ is the (virtual) number of holomorphic disks of homotopy class β which contain a pre-determined general marked point in L, $\mu(\beta)$ denotes the Maslov index of β and

$$z_\beta(\mathscr{L}) = \exp(-\int_\beta \omega)\mathrm{hol}_\nabla(\partial\beta) \in \mathbb{C}^* \tag{13}$$

for $\mathrm{hol}_\nabla(\partial\beta)$ the holonomy of ∇ along $\partial\beta$. The important observation is that $z_\beta(\mathscr{L})$ gives a holomorphic function on $\check{X}$. Just note its similarity with the holomorphic coordinate

$$w_j = \exp(2\pi i(x_j + i \int_{\Gamma_j} \omega))$$

on $\check{X}$ given in Sect. 1. By [4], Lemma 3.1, the condition $\mu(\beta) = 2$ is equivalent to $\beta.D = 1$ where D is the compactifying divisor and the dot denotes the algebraic intersection number. We learn that a partial compactification of X yields a holomorphic function m_0 on $\check{X}$ (assuming that (12) has finitely many summands or converges). Motivated by physics, this function is called a *Landau–Ginzburg-potential* (LG potential) and denoted W. The pair $(\check{X}, W)$ is called a *Landau–Ginzburg model* (LG model). In fact more generally, an LG model will simply be a variety with a flat holomorphic function to $\mathbb{C}$ as well as a restriction of such to an open subset in the analytic topology. Coming back to the example of $\mathbb{P}^1$, the two-point compactification of $\check{X}$, we obtain a LG potential on

[4]It is possible to choose them holomorphic, in fact there is a natural choice.

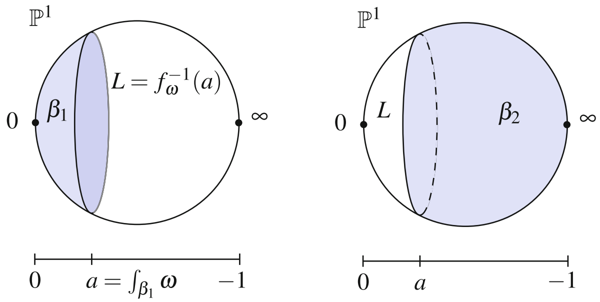


Fig. 7 The two holomorphic disks giving the LG potential of the mirror of $\mathbb{P}^1$

$$X = \{w \in \mathbb{C}^* \mid e^{2\pi 0} < |w| < e^{2\pi 1}\}$$

given by $W = e^{2\pi}(w + \frac{1}{w})$, see Fig. 7. We could have gotten rid of the factor $e^{2\pi}$ had we rescaled m_0. This generalizes to smooth toric Fano varieties, see [5], Prop. 2.5:

Proposition 1. *Given a smooth projective toric Fano variety $\mathbb{P}_\Delta$, the LG potential on its mirror is given by*

$$W = \sum_{\tau \subset \Delta \textit{ is a facet}} e^{-2\pi\alpha_\tau} z^{n_\tau}$$

where $n_\tau \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$ is the primitive integer inward normal vector to τ, such that τ is given by intersecting the affine hyperplane $n_\tau + \alpha_\tau = 0$ with Δ. Moreover, z^{n_τ} is the character associated to n_τ for the torus containing the poly-annulus which is the mirror of the dense $(\mathbb{C}^*)^n$ in $\mathbb{P}_\Delta$.

Note that we may also study non-compact Fanos, e.g., by embedding $\mathbb{C}^n$ in $\mathbb{P}^n$, we have that the mirror dual of $\mathbb{C}^n$ is the LG model

$$\{(w_1, \ldots, w_n) \in (\mathbb{C}^*)^n \mid 1 < |w_i| < e^{2\pi}\} \xrightarrow{w_1 + \ldots + w_n} \mathbb{C}.$$

Let us now consider the large volume limit of this picture. By taking $\lim_{r\to\infty} r\omega$, we enlarge the mirror poly-annulus until it becomes all of $(\mathbb{C}^*)^n$. The potential will also move to infinity, but can be normalized similarly as we normalized the symplectic form previously, see [4], §4.2. Under normalization, it remains the same and we have in the large volume limit

the mirror dual of $((\mathbb{C})^n, \Omega, \omega_{\mathbb{P}^n})$ *is the LG model* $(\mathbb{C}^*)^n \xrightarrow{w_1 + \ldots + w_n} \mathbb{C}.$

Let us now see how we find the LG potential in the context of the discrete Legendre transform, i.e., in the degeneration limit.

Example 5 (LG Potential on the Mirror of $\mathbb{P}^1$*).* We consider the example of $\mathbb{P}^1$ again, which is given by the cone picture

$$\mathbb{P}^1 = \check{X}_0([-1,0],\{\{-1\},\{0\},[-1,0]\},0)$$

and the DLT of $B = [-1,0]$ is the fan of $\mathbb{P}^1$ with piecewise linear function φ whose slope changes by 1 at the origin. We have for the mirror degenerate Calabi–Yau the fan picture

$$X_0 := X_0([-1,0],\{\{-1\},\{0\},[-1,0]\},0) = \mathbb{A}^1 \sqcup_{\{0\}} \mathbb{A}^1 = V(uv) \subseteq \mathbb{A}^2$$

We take the potential $W_0 = u + v$, i.e., the standard coordinate on each $\mathbb{A}^1$. The reconstruction of X_0 is given by $\mathscr{X} = \mathbb{A}^2 \to \mathbb{A}^1, (u,v) \mapsto uv$. Let us view the same from the perspective of the cone picture. We denote the DLT of B by $(\check{B},\check{\mathscr{P}},\check{\varphi})$ and may assume $\check{\varphi}(0) = 0$. We have the monoid algebra

$$\mathscr{X} = \operatorname{Spec}\mathbb{C}[P], \qquad P = \Delta_{(\check{B},\check{\mathscr{P}},\check{\varphi})} \cap (\mathbb{Z}\oplus\mathbb{Z}).$$

Let e_1, e_2 be generators for the two summands of $\mathbb{Z}\oplus\mathbb{Z}$ respectively. We set $w = z^{e_1}$ and $t = z^{e_2}$. Since $e_2 \in P$, $\mathbb{C}[P]$ is a $\mathbb{C}[t]$-algebra giving the map $\operatorname{Spec}\mathbb{C}[P] \to \operatorname{Spec}\mathbb{C}[t] = \mathbb{A}^1$. The generators of the P are $e_1 + \varphi(e_1)e_2$ and $-e_1 + \varphi(-e_1)e_2$, so the generators of $\operatorname{Spec}\mathbb{C}[P]$ are $wt^{\varphi(e_1)}$ and $w^{-1}t^{\varphi(-e_1)}$. Denoting these by u, v, we have $\mathbb{C}[P] = \mathbb{C}[u,v]$. We claim that the sum $u + v$ is the reconstruction of the LG potential:

$$W = wt^{\varphi(e_1)} + w^{-1}t^{\varphi(-e_1)}.$$

Indeed this restricts to W_0 on X_0. Inserting the φ as given by (10) from $\Delta = [-1,0]$, we get for $t \neq 0$

$$W = w + w^{-1}t$$

for the potential on $X_t = V(xy - t) \cong \mathbb{C}^*$. Taking $t = 1$ reproduces the mirror of $\mathbb{P}^1$ constructed before Proposition 1 up to a factor of $e^{2\pi}$ and up to the restriction to an annulus.

What we did for $\mathbb{P}^1$ here generalizes directly to the case of a general $(B,\mathscr{P},\varphi)$, see [13]. The potential W_0 on a component $\mathbb{P}_\sigma$ of $\check{X}_0(B,\mathscr{P},\varphi)$ is 0 if σ is compact. Otherwise, let $\operatorname{rays}(\sigma)$ denote the set of equivalence classes of (unbounded) extremal rays of σ up to translation. The potential on $\mathbb{P}_\sigma$ is given by

$$W_0|_{\mathbb{P}_\sigma} = \sum_{(n_0+\mathbb{R}_{\geq 0}n)\,\in\,\operatorname{rays}(\sigma)} z^n \tag{14}$$

where $(n_0 + \mathbb{R}_{\geq 0}n)$ denotes a representative of an element in $\operatorname{rays}(\sigma)$ for which we require n to be a primitive integral vector. Clearly z^n doesn't depend on the choice of representative. These local potentials glue to a LG potential W_0 on $\check{X}_0(B, \mathscr{P}, \varphi)$. See [13] for a solution of the reconstruction problem for this potential. We again restrict ourselves to the easy case: Let us assume that $(B, \mathscr{P}, \varphi)$ is embedded in $\mathbb{R}^n$. Recall from the end of Sect. 4 the local description of the total space $\check{\mathscr{X}} = \check{\mathscr{X}}(B, \mathscr{P}, \varphi)$ of the smoothing of $\check{X}_0(B, \mathscr{P}, \varphi)$. For each vertex $v \in \mathscr{P}$, we have an affine chart Spec R_v of $\check{\mathscr{X}}$. The reconstructed potential $W : \check{\mathscr{X}} \to \mathbb{C}$ is given in each R_v by the sum

$$W = \sum_{(n_0+\mathbb{R}_{\geq 0}n)\,\in\,\bigcup\{\mathrm{rays}(\sigma)|\sigma\in\mathscr{P}\}} z^n t^{\varphi(n+n_0)-\varphi(n_0)}. \tag{15}$$

It can be shown that $\varphi(n+n_0)-\varphi(n_0)$ is an invariant of the equivalence class of $n_0+\mathbb{R}_{\geq 0}n$. Note that this indeed restricts to W_0 on X_0 making use of the identification (11). It is also in line with the above example for the mirror of $\mathbb{P}^1$ where the sum was $u + v$ and we had $\varphi(n_0) = \varphi(0) = 0$. More generally in the presence of singularities of the affine structure, one needs to sum over all *broken lines* which we have implicitly done here, too. Broken lines are an analogue of holomorphic disks in tropical geometry. See [13, 21] for more details.

6 Mirror Duality for Landau–Ginzburg Models

We are now in the position to study a duality of Landau Ginzburg models. We understood in the first section that the mirror dual of $(\mathbb{C}^*)^n$ is again $(\mathbb{C}^*)^n$ or some analytic open subset thereof depending on the choice of symplectic form. We understood in the previous section that partial compactifications on one side lead to a LG potential on the other side. LG models are well-known to be the mirror duals of projective Fano varieties, some of which are compactifications of $(\mathbb{C}^*)^n$, some others (possibly all) can be degenerated torically such that the mirror is also obtained from the given discrete Legendre transform construction. However, in principle, there is nothing stopping us from looking at partial compactifications of $(\mathbb{C}^*)^n$ on both sides as in Fig. 5, e.g., the reader will meanwhile hopefully agree with the slogan

the mirror dual of $\mathbb{C}^n \xrightarrow{w_1+\ldots+w_n} \mathbb{C}$ *is* $\mathbb{C}^n \xrightarrow{w_1+\ldots+w_n} \mathbb{C}.$

The discrete Legendre transform underlying this slogan is the duality of very simple cones, namely

$$\mathbb{R}^n_{\geq 0} \stackrel{\mathrm{DLT}}{\longleftrightarrow} -\mathbb{R}^n_{\geq 0},$$

more precisely, one $\mathbb{R}^n_{\geq 0}$ sits in the dual space of the vector space containing the other. While the DLT provides a very general framework for the construction of very sophisticated Landau–Ginzburg models (e.g., with singularities in the affine structure), we give here a simple and yet very useful subset of the wide range of DLT duals.

Let us fix a free abelian group $M \cong \mathbb{Z}^n$, $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$, $N = \mathrm{Hom}_\mathbb{Z}(M, \mathbb{Z})$, $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$. Consider a strictly[5] convex rational polyhedral cone $\sigma \subseteq M_\mathbb{R}$ with $\dim \sigma = \dim M_\mathbb{R}$, and let $\check{\sigma} \subseteq N_\mathbb{R}$ be the dual cone,

$$\check{\sigma} := \{n \in N_\mathbb{R} \mid \langle n, m\rangle \geq 0 \text{ for all } m \in \sigma\}.$$

We already explained in Example 4 that the duality $\sigma \leftrightarrow -\check{\sigma}$ constitutes a DLT. For the simplicity of the exposition, we remove the minus sign from $\check{\sigma}$ in the following and call $\sigma \leftrightarrow \check{\sigma}$ and related constructions a DLT. Note that in this notation, the previous slogan results from starting with the cone $\sigma = \mathbb{R}_{\geq 0}e_1 \oplus \ldots \oplus \mathbb{R}_{\geq 0}e_n$ where $e_1, \ldots, e_n$ is a basis of M. Note that if $e_1, \ldots, e_n$ were only a basis of $M \otimes_\mathbb{Z} \mathbb{Q}$ but not of M, we would already be studying an interesting duality of quotient singularities (in fact this relates to the Berglund–Hübsch construction [6]), cf. [7]. Let us remain in the smooth world. So since the corresponding toric varieties

$$\check{X}_0(\check{\sigma}) = X_0(\sigma) = X_\sigma = \mathrm{Spec}\, \mathbb{C}[\check{\sigma} \cap N]$$
$$\check{X}_0(\sigma) = X_0(\check{\sigma}) = X_{\check{\sigma}} = \mathrm{Spec}\, \mathbb{C}[\sigma \cap M]$$

are usually singular, we choose toric desingularizations by choosing fans Σ and $\check{\Sigma}$ which are refinements of σ and $\check{\sigma}$ respectively, with Σ and $\check{\Sigma}$ consisting only of standard cones, i.e., cones generated by part of a basis for M or N.

We now obtain smooth toric varieties X_Σ and $X_{\check{\Sigma}}$. However, the resolution has broken the DLT property: Σ is not the DLT of $\check{\Sigma}$ in general. This can be fixed as follows. We may assume that we have chosen resolutions given by a piecewise linear functions φ, $\check{\varphi}$ respectively. Then there are polytopes $P \subseteq M_\mathbb{R}$, $\check{P} \subseteq N_\mathbb{R}$ such that we have DLTs

$$\begin{aligned} (\sigma, \Sigma, \varphi) &\leftrightarrow \check{P} \\ (\check{\sigma}, \check{\Sigma}, \check{\varphi}) &\leftrightarrow P. \end{aligned} \tag{16}$$

Moreover, these have the property that

$$\overline{\mathrm{Cone}(P)} \cap M_\mathbb{R} = \sigma$$
$$\overline{\mathrm{Cone}(\check{P})} \cap N_\mathbb{R} = \check{\sigma},$$

[5] This means it doesn't contain a non-trivial linear subspace.

where the overline means taking the closure and the cones are contained in $M_{\mathbb{R}} \oplus \mathbb{R}$ (resp. $N_{\mathbb{R}} \oplus \mathbb{R}$) so that intersection with $M_{\mathbb{R}}$ (resp. $N_{\mathbb{R}}$) makes sense. Note that we have the fan pictures $X_{\Sigma} = X_0(\sigma, \Sigma, \varphi)$, $X_{\check{\Sigma}} = X_0(\check{\sigma}, \check{\Sigma}, \check{\varphi})$. By the construction in the previous section, we obtain reconstructed potentials $\check{W} : \check{\mathscr{X}}(\sigma, \Sigma, \varphi) \to \mathbb{C}$, $W : \mathscr{X}(\check{\sigma}, \check{\Sigma}, \check{\varphi}) \to \mathbb{C}$, which make sense to write down as elements

$$\check{W} = \sum_{\substack{\mathbb{R}_{\geq 0} n \text{ is a ray in } \check{\Sigma} \\ n \in N \text{ is primitive}}} z^n t^{\check{\varphi}(n)} \in \mathbb{C}[\overline{\mathrm{Cone}(\check{\sigma})} \cap (N \oplus \mathbb{Z})] = \mathbb{C}[\check{\sigma} \cap N] \otimes_{\mathbb{C}} \mathbb{C}[t]$$
$$W = \sum_{\substack{\mathbb{R}_{\geq 0} m \text{ is a ray in } \Sigma \\ m \in M \text{ is primitive}}} z^m t^{\varphi(m)} \in \mathbb{C}[\overline{\mathrm{Cone}(\sigma)} \cap (M \oplus \mathbb{Z})] = \mathbb{C}[\sigma \cap M] \otimes_{\mathbb{C}} \mathbb{C}[t].$$

So the potentials pull back from $X_\sigma \times \mathbb{A}^1_t$, $X_{\check{\sigma}} \times \mathbb{A}^1_t$ respectively. For a fixed t, we have diagrams

(17)

One needs to take a close look to observe that this duality is actually “balanced” in the following sense. One might wonder what happens if one chooses a different resolution Σ_{new} instead of Σ. Then X_Σ becomes $X_{\Sigma_{\text{new}}}$ but the potential W remains “the same” (being the pullback of the same potential on X_σ). However, while on the dual side $X_{\check{\Sigma}}$ remains the same space, its potential $\check{W}$ changes to $\check{W}_{\text{new}}$ because it is a sum over all rays in Σ_{new}. So it is not possible to change only one side by adding exceptional divisors. Of course, the geometry of $X_{\Sigma_{\text{new}}}$ might be very different from that of X_Σ, e.g., one of them might have a trivial canonical bundle while the other has a more positive one. To ensure that the geometry of X_Σ doesn’t differ considerably from that of X_σ, we would want $X_\Sigma \to X_\sigma$ to be a crepant resolution. Such does not always exist in the category of smooth schemes, however it does exist in general in the category of orbifolds which should be the slightly more general framework to be used here.

While the balancing argument just given is a weak one to rectify mirror symmetry, we should actually argue by the discrete Legendre transform. There are four DLTs in place three of which we have seen already, see Fig. 8. It has been shown in [18] that there exists a (non-unique) DLT pair $(B, \mathscr{P}, \varphi) \leftrightarrow (\check{B}, \check{\mathscr{P}}, \check{\varphi})$ which “dominates” the two DLTs given in (16). Most importantly, the potentials constructed for $\mathscr{X}(B, \mathscr{P}, \varphi)$, $\mathscr{X}(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ via the previous section agree with $W, \check{W}$ respectively in the following sense: the space $\mathscr{X}(B, \mathscr{P}, \varphi)$ relates to $\mathscr{X}(\sigma, \mathscr{P}, \varphi)$ by a deformation, i.e., there is a flat family with general fibre isomorphic to $\mathscr{X}(\sigma, \mathscr{P}, \varphi)$ and special fibre given by $\mathscr{X}(B, \mathscr{P}, \varphi)$. Moreover this family is birational to the trivial family with fibre $\mathscr{X}(\sigma, \mathscr{P}, \varphi)$ and the potential on $\mathscr{X}(B, \mathscr{P}, \varphi)$ is the pullback of the potential W from the trivial family.

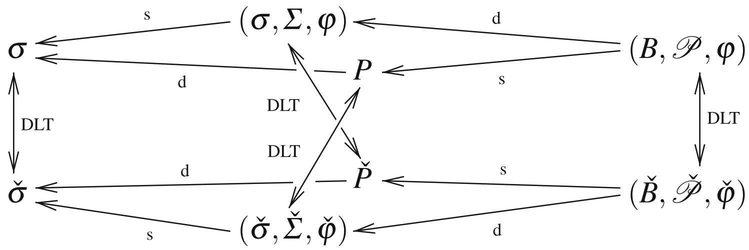


Fig. 8 Tropical manifolds and their relationships: DLT marks a discrete Legendre transform (up to sign convention), s marks a subdivision, d marks a deformation/degeneration

The mirror duality of Landau–Ginzburg models given in (17) has been used in [18] to construct mirror duals for varieties which are not necessarily Fano or Calabi–Yau, e.g., for varieties of general type. A notion of mirror symmetry for such varieties didn't exist before the cited work had been started, so this relatively simple construction for duals is already quite powerful, cf. [1, 8, 14]. Note also that the famous mirror construction of Batyrev–Borisov is reproducible from this duality, see [18]. Note that the potentials in loc.cit. had been permitted to have more general coefficients, i.e.,

$$W = \sum_{\substack{\mathbb{R}_{\geq 0} m \text{ is a ray in } \Sigma \\ m \in M \text{ is primitive}}} c_m z^m t^{\varphi(m)}$$

for some (general) $c_m \in \mathbb{C}$ and similarly for $\check{W}$ (independently of the coefficients of W). This can be argued to make sense by changing the (complexified) symplectic form on either side, recall from (13) that the monomials are integrals of the symplectic form.

There is yet one flaw in the picture: The potential which we give in (15) is the "naive potential". It agrees with the Floer theoretic one in the Fano case by Proposition 1, however X_Σ, $X_{\check{\Sigma}}$ are rarely Fano. More generally, there will be non-rigid rational curves in X_Σ or $X_{\check{\Sigma}}$ and these cause *disk bubbling* and non-geometric virtual counts of holomorphic disks (see [5]). Such give rise to (possible infinitely many) additional Maslov index two holomorphic disks and thus terms in the potential. To keep this under control, the authors of [13] required the boundary of B and $\check{B}$ to be smooth. In fact they suggested to smooth the boundary by trading "corners" in B (or $\check{B}$) for singularities of the affine structure of B (or $\check{B}$), see Fig. 9.

The advantage is that the tropical potential (the generalization of (15) to affine manifolds with singularities) for a smooth boundary of B (or $\check{B}$) seems to agree with the Floer theoretic one. The additional terms arise from holomorphic disks attaching to the singularities in the SYZ fibration and these can be accounted for tropically. We shall study this for an example in the next section. Let us record here the main result of [18] which supports the mirror duality (17) from a cohomological point of

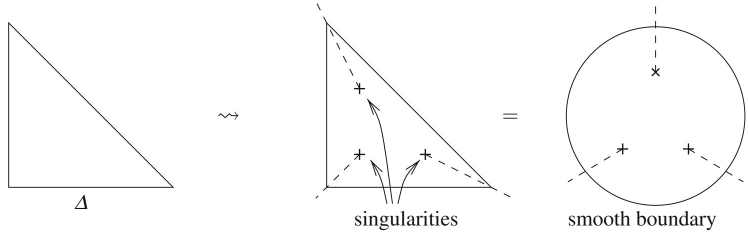


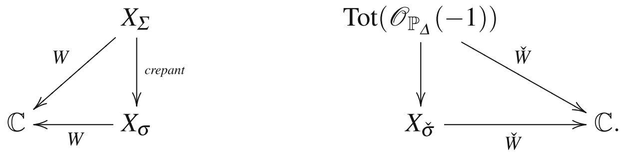
Fig. 9 The cone picture of $\mathbb{P}^2$ and how to trade corners for singularities

view. For this, the general setup of (17) is restricted to the situation where σ has the special shape of a Gorenstein cone, i.e., there is a lattice polytope Δ such that

$$\sigma = \mathrm{Cone}(\Delta).$$

For this to make sense, we need to write $M_{\mathbb{R}}$ as $(M_0 \oplus \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ where $M_0 \cong \mathbb{Z}^{n-1}$ and $\Delta \subset M_0 \otimes_{\mathbb{Z}} \mathbb{R}$. Note that the existence of a toric crepant resolution $X_\Sigma \to X_\sigma$ is equivalent with the existence of a triangulation $\mathscr{P}$ of Δ into simplices for which the edges emanating from a vertex in each form a basis of M_0. The authors of [18] prove the following:

Theorem 1. *Assume that Δ has at least one interior lattice point, $\mathbb{P}_\Delta = \mathrm{Proj}\,\mathbb{C}[\sigma \cap M]$ is smooth and that there is a projective crepant toric resolution $X_\Sigma \to X_\sigma$ factoring through the blowup of the origin $X_\Sigma \to \mathrm{Bl}_0 X_\sigma \to X_\sigma$ then the blow-up of the origin $X_{\check\Sigma} = \mathrm{Bl}_0 X_{\check\sigma} \to X_{\check\sigma}$ is a toric resolution. The diagram (17) specializes to*



where $\mathrm{Tot}(\mathscr{L}) = \mathrm{Spec}(\mathrm{Sym}(\mathscr{L}^{-1}))$ *denotes the total space of a line bundle. The critical locus of $\check W$ is a hypersurface $S \subset \mathbb{P}_\Delta$ which is smooth if the coefficients of $\check W$ were chosen general. The Kodaira dimension of S is*

$$\kappa(S) = \min\{\dim \Delta', n-2\}$$

where Δ' is the convex hull of the lattice points in the interior of Δ. We have that

$$W^{-1}(0) = D_{v_1} \cup \ldots \cup D_{v_r} \cup \tilde W_0$$

is normal crossings, $\tilde{W}_0$ *is the strict transform of the zero fibre of* $W : X_\sigma \to \mathbb{C}$ *and* D_{v_i} *are toric exceptional divisors of* $X_\Sigma \to X_\sigma$ *projecting to the origin. They are indexed by the lattice points in the interior of* Δ. *The critical set near the origin* $\check{S} = \operatorname{Sing} W^{-1}(0)$ *supports the sheaf of vanishing cycles* $\mathscr{F}_{\check{S}} = (\phi_{W,0}\mathbb{C})[1]$ *which carries the structure of a cohomological mixed Hodge complex. Denoting*

$$h^{p,q}(\check{S}, \mathscr{F}_{\check{S}}) = \dim \operatorname{Gr}_p^F \mathbb{H}^{p+q}(\check{S}, \mathscr{F}_{\check{S}}),$$

we have

$$h^{p,q}(S) = h^{d-p,q}(\check{S}, \mathscr{F}_{\check{S}})$$

where $d = \dim S = n - 2$.

7 Moving the Compactifying Divisor and Corrected Potentials

We already mentioned the concept of trading corners for singularities, see Fig. 9. Geometrically this means the following: Recall that we started our discussion with the mirror duality of $(\mathbb{C}^*)^n$ and continued by partially compactifying it to a toric variety X_Σ using a toric divisor $D = X_\Sigma \backslash (\mathbb{C}^*)^n$. The special Lagrangian fibration (SYZ fibration) is still entirely given on $(\mathbb{C}^*)^n$ with parts of the torus fibres contracting towards D. There are moduli of the pair $(X_{\check{\Sigma}}, D)$ by moving D in its equivalence class, in particular D becomes non-toric by doing so. It is not known whether $X_{\check{\Sigma}} \backslash D$ for such a non-toric D still supports a special Lagrangian fibration (using for Ω a section of $\Omega^n_{X_{\check{\Sigma}}}(\log D)$). This is already unknown for the complement of a smooth cubic in $\mathbb{P}^2$. Nonetheless, we already have a good expectation of what the affine base of such a special Lagrangian fibration should look like. In the case of $\mathbb{P}^2$, we depicted it on the right of Fig. 9. See [33] for a treatment of the case of a partial smoothing of the hyperplanes in $\mathbb{P}^2$, see also [11]. What happens to the mirror as we smooth D? We have a natural bijection between the components of D and the terms in the potential of the mirror $\check{W} : X_{\check{\Sigma}} \to \mathbb{C}$, so by smoothing D, we expect only one monomial to contribute to the mirror potential near D. On the other hand, the special Lagrangian fibration on $X_{\check{\Sigma}} \backslash D$—should such exist—or at least the affine model for its base acquires singularities there are additional disks attaching to these singularities and to D. It can be checked in simple Fano examples that the monomials in the potential remain the same (up to changing coefficients) when smoothing the toric boundary divisor. Summing over rays in (14) is replaced by summing over *broken lines* in the presence of singularities [13], see Fig. 11.

The singularities emanate walls (indicated dashed in Fig. 11) into the affine manifold which ought to contain the image of Maslov index zero holomorphic disks under the SYZ map f_Ω should such exist. These can be attached to the

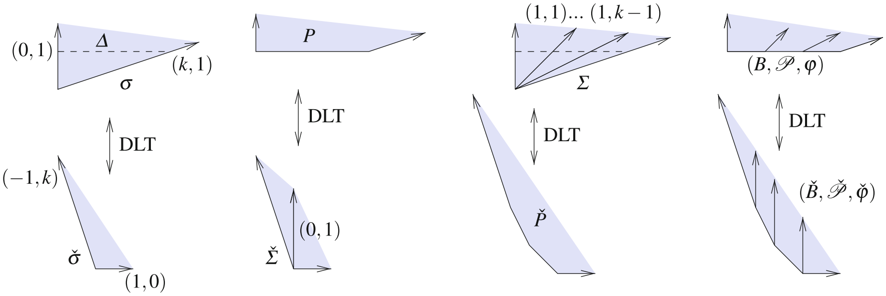


Fig. 10 The four DLTs for $\mathrm{Tot}(\mathscr{O}_{\mathbb{P}^1}(-k))$

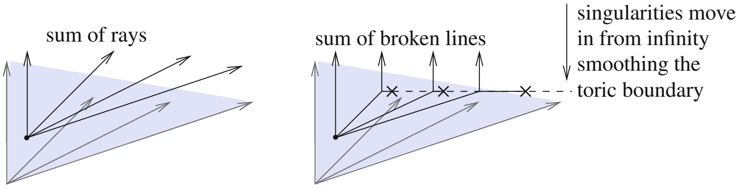


Fig. 11 Fan pictures for the minimal crepant resolution of the singularity $uv - z^3 = 0$ with and without smoothing of the toric boundary divisor. Interpreted dually, these are cone pictures for a degeneration of the singularity C^2/ζ_3 where ζ_3 is a primitive third root of unity acting diagonally. The monomials in the LG potential on this singularity remain the same when smoothing the toric boundary divisor of the mirror dual: summing over rays becomes summing over broken lines

holomorphic disk touching D and give rise to further terms in the potential. As long as D itself does not contribute such walls, the tropical potential obtained in this way by counting broken lines is expected to be the correct potential meaning that it agrees with the one given in (12). Moreover the smoothing of D makes W proper as has been argued in [13]. The process of pulling in the corners is very ad hoc and hasn't been systematized yet. This will be treated in [36]. In non-Fano cases, where Proposition 1 possibly fails, the right count of holomorphic disks seems more accessible when the boundary divisor has been smoothed by means of counting broken lines. We close this article by studying the corner-pull-in-process in an example:

Example 6 (Corrected Potential for $\mathrm{Tot}(\mathscr{O}_{\mathbb{P}^1}(-k))$ *and Its Mirror).* Let $\sigma = \mathrm{Cone}(\Delta)$ with Δ an interval of length k and Σ be the unique subdivision giving a crepant resolution of X_σ. Let $\check{\sigma}$ be the dual cone of σ and $\check{\Sigma}$ be the fan of $\mathrm{Tot}(\mathscr{O}_{\mathbb{P}^1}(-k))$ which resolves $X_{\check{\sigma}}$. See Fig. 10 for a how the diagram in Fig. 8 visualizes for this setup. We start from the DLT pair $(B, \mathscr{P}, \varphi) \leftrightarrow (\check{B}, \check{\mathscr{P}}, \check{\varphi})$ and straighten out the boundary in these each at a time. See this process in Fig. 12. Even though we started with very simple cones, we eventually obtain a fairly interesting DLT pair whose singularities will feature scattering. The upshot of this example is that the corrections that come to the potentials don't impact the critical locus of the potential. This can be deduced from the positions of the invariant directions of the

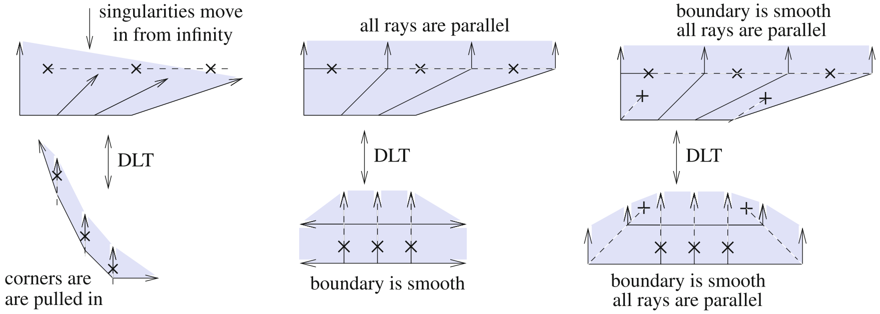


Fig. 12 Flattening the boundary

singularities towards the direction of the boundary divisor in the respective cone pictures. The critical loci together with the sheaf of vanishing cycles were the main objects of study in [18].

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Mirror Symmetry in Dimension 1 and Fourier–Mukai Transforms

Nicolò Sibilla

Abstract In this paper we will describe an approach to mirror symmetry for appropriate one-dimensional DM stacks of arithmetic genus $g \leq 1$, called *tcnc* curves, which was developed by the author with Treumann and Zaslow in Sibilla et al. (Ribbon Graphs and Mirror Symmetry I, arXiv:1103.2462). This involves introducing a conjectural sheaf-theoretic model for the Fukaya category of punctured Riemann surfaces. As an application, we will investigate derived equivalences of *tcnc* curves, and generalize classic results of Mukai on dual abelian varieties (Mukai, Nagoya Math. J. **81**, 153–175, 1981).

1 Introduction

As originally formulated by Kontsevich [12], Homological Mirror Symmetry (from now on, HMS) relates the derived category of coherent sheaves on a Calabi–Yau variety X, $D^b(Coh(X))$, and the Fukaya category of a symplectic manifold $\hat{X}$, by stating that if X and $\hat{X}$ are mirror partners, then $D^b(Coh(X)) \cong Fuk(\hat{X})$. Since its proposal, much work has been done towards establishing Kontsevich's conjecture in important classes of examples, see [23, 25, 29], and references therein.

One of the main obstacles for tackling Kontsevich's conjecture is gaining a sufficient understanding of the Fukaya category.[1] Starting in 2009, in various talks, Kontsevich has argued [13] that the Fukaya category of a Stein manifold should have good local-to-global properties, and therefore conjecturally could be recovered as

[1]For foundational material on the Fukaya category, the reader should consult [9, 24].

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the global sections of a suitable sheaf of dg categories.[2] This is in keeping with previous work of Nadler and Zaslow who, in [18, 21], establish an equivalence between the Fukaya category of exact Lagrangians in a cotangent bundle T^*X, and the dg category of (complexes of cohomologically) constructible sheaves[3] over X, $Sh(X)$.

Following Kontsevich's insight, in [32], joint with Treumann and Zaslow, we equip the Lagrangian skeleton of a punctured Riemann surface Σ with a sheaf of dg categories, called CPM$(-)$,[4] such that its local behavior is dictated by Nadler and Zaslow's work on cotangent bundles, while its global sections are conjecturally quasi-equivalent to the Fukaya category of exact Lagrangians in Σ, $Fuk(\Sigma)$.[5] Further, in [32], using this model as a stand in for the Fukaya category, we prove a version of HMS in dimension 1 which pairs suitable stacky, degenerate elliptic curves, called *tcnc* curves (see Sect. 3.1), and punctured symplectic tori.

In this paper we review the results contained in [32], by focusing on motivations and examples, and keeping the presentation of the arguments as explicit and concrete as possible. Then, we will apply this framework to investigate derived equivalences of tcnc curves. A more detailed outline of the paper is given below.

In Sect. 2, after reviewing the necessary background, we define CPM$(-)$ as a sheaf of dg categories on a suitable Grothendieck site of decorated ribbon graphs, and open inclusions. The applications to mirror symmetry are explained in Sect. 3. Given a tcnc curve C, we explain how to construct a ribbon graph $D_{\hat{C}}$, which arises as the skeleton of a punctured symplectic torus $\hat{C}$, and we prove that there is an equivalence $\mathcal{P}\mathrm{erf}(C) \cong \mathrm{CPM}(\hat{C})$. Granting the conjectural equivalence $\mathrm{CPM}(D_{\hat{C}}) \cong Fuk(\hat{C})$, we obtain an HMS statement relating C and $\hat{C}$.

The HMS statement proved in [32] can be used to explore the algebraic geometry of tcnc curves.[6] In Sect. 4 we prove that, up to derived equivalence, tcnc curves are classified by the sum of the orders of the isotropy groups at the nodes. This generalizes work of Mukai on derived auto-equivalences of smooth elliptic curves [17], and of Burban and Kreussler who considered the case of the nodal $\mathbb{P}^1$ [6]. From the stand-point of mirror symmetry, this result corresponds to the simple fact that the Fukaya category of a punctured Riemann surface depends exclusively on genus, and number of punctures.

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[2]For relevant work in this direction, see also [19, 20, 26, 27].

[3]From now on, we will refer to objects in $Sh(X)$ simply as 'constructible sheaves.' See [11] for a comprehensive introduction to the subject.

[4]CPM stands for 'constructible plumbing model,' as this framework can be more generally applied to investigate the Fukaya category of a plumbing of cotangent bundles, for which see also [1].

[5]Lagrangian branes in Fuk(Σ) are further required to be, in a suitable sense, 'adapted' to the skeleton, and thus in particular compact when Γ_Σ is. Also, when referring to the 'Fukaya category,' we shall mean the *split closure* of the category of *twisted complexes* over the Fukaya category, see [24].

[6]See also [30, 31]. In [30] we define an action of the mapping class group of a torus with n punctures on $\mathcal{P}\mathrm{erf}(X_n)$, where X_n is a cycle of n projective lines.

2 A Model for the Fukaya Category of Punctured Riemann Surfaces

In this section we review the construction of CPM(−). We will follow quite closely the exposition of [32], but we shall gloss over many technical aspects of the theory, for which we refer the reader to the original paper. Section 2.1 contains a brief overview of definitions and results from microlocal sheaf theory which will be needed later, and a preliminary, 'local,' definition of CPM(−). Section 2.2 discusses a useful dictionary between category of sheaves, and categories of quiver representations. In Sect. 2.3 we introduce the notion of *chordal* ribbon graph, and give the full definition of CPM(−), as a sheaf of dg categories over the Grothendieck site of chordal ribbon graphs.

Before proceeding, it might be useful to clarify what we mean by sheaf of dg categories. Recall that, after Tabuada [33], the category of small dg categories, dg$\mathcal{C}$at, can be equipped with a model structure. A sheaf on a site $\mathcal{C}$ with values in a model category $\mathcal{D}$ is a pre-sheaf F, such that, whenever $S = \{U_i\}$ is a covering sieve for $U \in \mathcal{C}$, the diagram

$$F(U) \to [\Pi_i F(U_i) \rightrightarrows \Pi_{i,j} F(U_{ij}) \overset{\to}{\underset{\to}{\to}} \dots]$$

is a *homotopy* limit in $\mathcal{D}$. The sheaf property can be verified in practice quite easily, using the following description of equalizers in dg$\mathcal{C}$at.

Lemma 1. *Let* $C \overset{F}{\underset{G}{\rightrightarrows}} C'$ *be a diagram in* $dg\mathcal{C}at$*, and denote* $\mathcal{E}$ *the dg category having*

- *as objects, pairs* (C, u)*, where* $C \in \mathcal{C}$*, and* $u : F(C) \to G(C)$ *is a degree zero, closed morphism, which becomes invertible in the homotopy category,*
- *as morphisms, pairs* $(f, H) \in hom^k(C, C') \oplus hom^{k-1}(F(C), G(C'))$*, with differential given by* $d(f, H) = (df, dH - (u'F(f) - G(f)u))$*. The composition is obvious.*

Then $\mathcal{E}$*, endowed with the natural forgetful functor* $\mathcal{E} \to \mathcal{C}$*, is a homotopy equalizer for* F *and* G*.*

Proof. Lemma 1 depends on the availability of an explicit construction of the *path object* $P(\mathcal{C}')$ for $\mathcal{C}'$, which can be found in Lemma 4.1 of [34]. This allows us to compute the homotopy equalizer in the usual way, by taking appropriate fibrant replacements. We leave the details to the reader.

2.1 Microlocal Sheaf Theory in Dimension 1

Let X be a manifold, and let $Sh(X)$ be the category of constructible sheaves over X. In [11], Kashiwara and Schapira explain how to attach to a constructible sheaf

$\mathcal{F} \in Sh(X)$ a *conical* (i.e. invariant under fiberwise dilation) Lagrangian subset of T^*X, called *singular support*, and denoted $SS(\mathcal{F})$. Informally, $SS(\mathcal{F})$ is an invariant encoding the co-directions along which $\mathcal{F}$ does not 'propagate.' Rather than giving the general definition, for which we refer the reader to Sect. 5.1 of [11], we will describe the singular support in the simpler set up which will be needed in the following.

Assume that X is a one-dimensional manifold equipped with affine structure. Let x be a point of X, and let f be an affine $\mathbb{R}$-valued function on X around x. For $\epsilon > 0$ sufficiently small let A be the sublevel set $\{y \in X \mid f(y) < f(x) + \epsilon\}$ and let B be the sublevel set $\{y \in X \mid f(y) < f(x) - \epsilon\}$. We define a functor $\mu_{x,f}$: $Sh(X) \to \mathbb{C}$-mod to be the cone on the natural map $\Gamma(A; F|_A) \to \Gamma(B; F|_B)$.

Since every constructible sheaf F is locally constant in a deleted neighborhood of x, this functor does not depend on ϵ as long as it is sufficiently small. Clearly $\mu_{x,f}$ depends only on x and df_x. When $(x, \xi) \in T^*X$ we let $\mu_{x,\xi}$ denote the functor associated to the point x and the affine function whose derivative at x is ξ.

Definition 1. For each $F \in Sh(X)$ we define $SS(F) \subset T^*X$, the *singular support* of F, to be the closure of the set of all $(x, \xi) \in T^*X$ such that $\mu_{x,\xi} F \neq 0$.

Note that, as $\mu_{x,\xi} = \mu_{x,t\cdot\xi}$ when $t > 0$, the set $SS(F)$ is conical. In fact, if $(x, \xi) \in SS(F)$ and $t \in \mathbb{R}_{>0}$, then $(x, t \cdot \xi) \in SS(F)$. Further, $SS(F)$ is one-dimensional and therefore a Lagrangian subset of T^*X with its usual symplectic form.

Definition 2. Suppose $\Lambda \subset T^*X$ is a conical Lagrangian. Define $Sh(X, \Lambda) \subset Sh(X)$ to be the full triangulated subcategory of sheaves with $SS(F) \subset \Lambda$.

Example 1. Let $\Lambda = X \cup T^*_{s_1}X \cup \cdots \cup T^*_{s_n}X$ be the union of the zero section and the cotangent spaces at finitely many points $\{s_1, \ldots, s_n\}$. Then $Sh(X, \Lambda)$ is the category of sheaves that are locally constant away from $\{s_1, \ldots, s_n\}$.

If F is a sheaf in $Sh(X, \Lambda)$, and $\xi \neq 0$, $\mu_{x,\xi}(F) \in \mathbb{C}$-mod should be thought of as the (microlocal) 'stalk' of F over $(x, \xi) \in \Lambda \setminus X$. This suggests that sheaves with singular support in Λ have a local nature over Λ, as well as over X. The locality of $Sh(X, \Lambda)$ over X can be encoded in the claim that the assignment

$$U \subset^{open} X \mapsto Sh_\Lambda(U) := Sh(U, T^*U \cap \Lambda),$$

defines a sheaf of dg categories over X. In an analogous fashion, in Definition 3 we will introduce a sheaf of dg categories, denoted CPM(−), which, in an appropriate sense, is an extension of $Sh_\Lambda(-)$ to Λ. In particular, we will have CPM$(\Lambda) \cong Sh(X, \Lambda)$.

Let $U \subset T^*X$ be an open subset, and let $\mathcal{P}(X, U)$ be the Verdier quotient of $Sh(X)$ by the thick subcategory of all sheaves F with $SS(F) \cap U = \varnothing$ (see [11], Sect. 6.1). Consider the full subcategory of $\mathcal{P}(X, U)$ spanned by sheaves F with singular support in Λ, and denote it $\mathcal{P}_\Lambda(X, U)$. Both $\mathcal{P}(X, -)$, and $\mathcal{P}_\Lambda(X, -)$, naturally define pre-sheaves of dg categories on T^*X. We can therefore consider the sheafification of $\mathcal{P}_\Lambda(X, -)$ over T^*X, which we denote $MSh_\Lambda(-)$.[7]

Definition 3. Define CPM$(-)$ to be the sheaf of dg categories over Λ obtained by pulling back $MSh(-)$ along i, CPM$(-) \cong i^*MSh(-)$.

2.2 Microlocal Sheaves and Quiver Representations

Assume that X is a one-dimensional manifold and $\Lambda \hookrightarrow T^*X$ is a conical Lagrangian subset. The category $Sh(X, \Lambda)$, and the sheaf CPM$(-)$ over Λ, can be very explicitly described in terms of quiver representations.

Let us call the connected components of $\Lambda - X$ the *spokes* of Λ. They are divided into two groups depending on which component of $T^*X - X$ they fall into. Using an orientation of X we may label these groups "upward" and "downward." The conic Lagrangian Λ determines a partition P_Λ of X into subintervals (which may be open, half-open, or closed) and points. Let us describe this partition in case $X = \mathbb{R}$, the general case is similar. Each spoke of Λ is incident with a point $x \in \mathbb{R}$, which we may order $x_1 < \ldots < x_k$. We put $\{x_i\} \in P_\Lambda$ if x_i is incident with both an upward and a downward spoke. We put an interval I from x_i to x_{i+1} in $\mathcal{P}_\Lambda$ whose boundary conditions are determined by the following rules

- If x_i is incident with an upward spoke but not incident with a downward spoke, then x_i is included in I. Otherwise x_i is not included in I.
- If x_{i+1} is incident with a downward spoke but not incident with an upward spoke, then x_{i+1} is included in I. Otherwise x_{i+1} is not included in I.

We put $(-\infty, x_1)$ in P_Λ if x_1 is incident with an upward spoke and $(-\infty, x_1]$ in P_Λ if x_1 is incident with a downward spoke, and similarly we put (x_k, ∞) (resp. $[x_k, \infty)$) in P_Λ if x_k is incident with a downward (resp. upward) spoke.

Define a quiver (that is, directed graph) Q_Λ whose vertices are the elements of P_Λ and with and edge joining I to J (in that orientation) if the closure of J has nonempty intersection with I. If there are n spokes then this is a quiver of type A_{n+1} (i.e. shaped like the Dynkin diagram A_{n+1}) whose edges are in natural bijection with the spokes of Λ: an upward spoke corresponds to a left-pointing arrow and a downward spoke to a right-pointing arrow.

Theorem 1. *There is a natural equivalence of dg categories*

$$Sh(M; \Lambda) \cong Rep(Q_\Lambda)$$

If (x, ξ) belongs to a spoke of Λ corresponding to an arrow f of Q_Λ, then under this equivalence the functor $\mu_{x,\xi}$ intertwines with the functor $\mathsf{C}one(f)$.

[7]Note that, if $\pi : T^*X \to X$ is the natural projection, then $\pi_* MSh(-) \cong Sh_\Lambda(-)$.

Example 2. 1. Let $\perp\!\!\!+\!\!\top \subset T^*\mathbb{R}$ be the union of the zero section, the fiber at 0, an upward spoke at some $x_- < 0$ and a downward spoke at some $x_+ > 0$. Then

$$Sh(\mathbb{R}, \perp\!\!\!+\!\!\top) \cong Rep(\bullet \leftarrow \bullet \leftarrow \bullet \rightarrow \bullet \rightarrow \bullet).$$

2. Let $\Lambda = S^1 \cup T^*_{x_0} S^1 \hookrightarrow T^* S^1$ be the union of the zero section, and the cotangent fiber at some $x_0 \in S^1$. Then

$$Sh(S^1, \Lambda) \cong Rep(\bullet \rightrightarrows \bullet).$$

Proposition 1. *Let M be a one-dimensional manifold and let Λ_1 and Λ_2 be conical Lagrangians in T^*M. Suppose that in each connected component U of $T^*M - M$, $\Lambda_1 \cap U$ and $\Lambda_2 \cap U$ have an equal number of components (i.e. Λ_1 and Λ_2 have an equal number of spokes in each group). Then $Sh(M; \Lambda_1) \cong Sh(M; \Lambda_2)$.*

Proof. For a general quiver Q, if a is an arrow let $s(a)$ and $t(a)$ denote the source and target of a, respectively. A vertex v of Q is called a sink (resp. source) if all the arrows incident to it have $t(a) = v$ (resp. $s(a) = v$). If x is a sink or a source, then Bernstein–Gelfand–Ponomarev [4] define a new quiver $S_x Q$ obtained by reversing the orientation of all the arrows in Q incident to x.

Let Q be a quiver, and let $x \in Q$ be a sink or a source. It follows from [4] that there is an equivalence of dg categories $Rep(Q) \cong Rep(S_x Q)$. Thus, if Q_1 and Q_2 are quivers with same underlying undirected graph, then $Rep(Q_1) \cong Rep(Q_2)$. Proposition 1 follows by applying this theorem to quivers of the form Q_Λ.

We conclude this section, by showing how Theorem 1 yields a very explicit description of the sheaf CPM$(-)$. For concreteness, we focus on the example $X = S^1$ and $\Lambda = S^1 \cup T^*_{x_0} S^1$, the general case is similar. Denote R^+ and R^- respectively the up-ward and down-ward spoke of Λ. We shall describe the sections of CPM$(-)$ on *contractible* open subsets $U \subset \Lambda$, and the assignment defining, on objects, the restriction functors

$$Res_U : \mathrm{CPM}(\Lambda) = Sh(X, \Lambda) \cong Rep(\bullet \rightrightarrows \bullet) \rightarrow \mathrm{CPM}(U).$$

The definition on morphisms will be obvious. This is sufficient to reconstruct CPM$(-)$.

Assume that $M = V_1 \overset{f}{\underset{g}{\rightrightarrows}}$ is an object in $Rep(\bullet \rightrightarrows \bullet)$. Then

- if $U \subset S^1$, CPM$(U) \cong \mathbb{C}$-mod, and $Res_U(M) = V_2$,
- if $U \subset R^+$, CPM$(U) \cong \mathbb{C}$-mod, and $Res_U(M) = \mathrm{C}one(f)$,
- if $U \subset R^-$, CPM$(U) \cong \mathbb{C}$-mod, and $Res_U(M) = \mathrm{C}one(g)$,
- if $x_0 \in U$, CPM$(U) \cong Rep(\bullet \leftarrow \bullet \rightarrow \bullet)$, and $Res_U(M) = V_2 \overset{f}{\leftarrow} V_1 \overset{g}{\rightarrow} V_2$.

2.3 *Chordal Ribbon Graphs and* CPM

Recall that a cyclic order $\mathcal{R}$ on a set S is a *ternary* relation on S, which allows us to speak unambiguously about ordered triples, and satisfies the obvious properties enjoyed by a set of points arranged on a circle. We shall define a graph to be a pair (D, V_D), where D is a one-dimensional CW-complex, and V_D is the set of 0-cells, called vertices.

Definition 4. Let (D, V_D) be a graph in which every vertex has degree $\geq$ 2. A *ribbon structure* on (D, V_D) is a collection $\{\mathcal{R}_v\}_{v\in V_D}$ where $\mathcal{R}_v$ is a cyclic order on the set of half-edges incident with v. We call a graph equipped with a ribbon structure a ribbon graph.

Definition 5. A *chordal ribbon graph* is a pair (D, Z) where

- D is a ribbon graph.
- Z is a closed, bivalent subgraph containing each vertex of X.
- Let $\mathcal{R}_v$ denote the ternary relation defining the cyclic order on the set of half-edges incident with v. If e and f are the two half-edges of Z incident with v, then there is at most one half-edge g so that $(e, g, f) \in \mathcal{R}_v$ and at most one half-edge h so that $(f, h, e) \in \mathcal{R}_v$.

In particular, the last condition requires that each vertex of a chordal ribbon graph has degree at most 4. We refer to Z as the *zero section* of the chordal ribbon graph.

Let Chord denote the category whose objects are chordal ribbon graphs, and where $\mathrm{Hom}((C, W), (D, Z))$ is given by the set of open immersions $j : C \hookrightarrow D$ with $j(W) \subset Z$ and preserving the cyclic orders at each vertex. We endow Chord with a Grothendieck topology in the evident way.

The simplest examples of chordal ribbon graph, called *fishbones*, are pairs of the form $(\Lambda, X \cap \Lambda)$, where X is a one-dimensional manifold, and $\Lambda \subset T^*X$ is a conical Lagrangian subset. Section 2.1 gives a recipe for constructing a sheaf CPM(−) on any fishbone $(\Lambda, X \cap \Lambda)$ (see Definition 3). As the full subcategory of fishbones is a basis for the Grothendieck topology on Chord, we can make the following definition.

Definition 6. Denote CPM : Chord $\to$ dg$\mathcal{C}$at the sheaf of dg categories on Chord whose restriction to the sub-category of fishbones recovers Definition 3. We call $\mathrm{CPM}(D, Z)$ the *constructible plumbing model* of the chordal ribbon graph (D, Z).

Chordal structure and restriction on valency are just convenient technical assumptions which could be removed as CPM(−) is expected not to depend on them. More precisely, up to quasi-equivalence, the constructible plumbing model of the chordal ribbon graph (D, Z) should be a function solely of the 'deformation class,' appropriately defined, of the ribbon graph D. A sketch of the full theory, which will take as input (suitably graded) ribbon graphs of any valency, is discussed in the last part of [32]. Work is in progress to fill in the remaining details.[8]

Setting technical complications aside, let's assume for the moment that CPM$(-)$ can be evaluated on a general ribbon graph. Then the expected relationship with the Fukaya category can be formulated as in Conjecture 1 below. Recall that ribbon graphs label cells in the moduli space of punctured Riemann surface (see e.g. [10, 22]). Further, if Σ lies in the cell labeled by Γ_Σ, there is an embedding $\Gamma_\Sigma \hookrightarrow \Sigma$, and a nicely behaved retraction of Σ onto Γ_Σ. In the language of Stein geometry, Γ_Σ is the *skeleton* of Σ.

Conjecture 1. Let Σ be a punctured Riemann surface with skeleton Γ_Σ, then CPM(Γ_Σ) is quasi-equivalent to $Fuk(\Sigma)$.

3 Homological Mirror Symmetry for tcnc Curves

In this section we will prove the main theorem of [32], which establishes a version of homological mirror symmetry for a class of nodal, stacky, curves of genus $g \leq 1$, introduced in Sect. 3.1 below. The proof of HMS will be discussed in Sect. 3.2, and will make use of the model for the Fukaya category supplied by the sheaf CPM$(-)$.

3.1 tcnc Curves

Let $\mathbb{P}^1(a_1, a_2)$ be a projective line, with stacky points at 0, and ∞, and isotropy groups isomorphic, respectively, to $\mathbb{Z}_{a_1}$, and $\mathbb{Z}_{a_2}$.[9] We call $\mathbb{P}^1(a_1, a_2)$ a *Beilinson–Bondal* (or, BB) curve.

Definition 7. A *tcnc curve* C is a connected, reduced DM stack of dimension 1, with nodal singularities, such that its normalization $\tilde{C} \stackrel{\pi}{\to} C$ is a disjoint union of n BB curves $P_1, \dots, P_n$. Further, if $Z \hookrightarrow C$ is the singular set, we require that $\pi^{-1}(Z)$ interesects each P_i in at most two points.[10]

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[8] In fact, any ribbon graph is deformation equivalent, in the above sense, to a ribbon graph admitting chordal structure. This gives us a concrete, although not 'functorial,' way of computing the global sections of CPM$(-)$ on a general ribbon graph up to quasi-equivalence.

[9] Note that our conventions differ from the ones commonly found in the literature. Weighted projective lines, which are denoted $\mathbb{P}^1(a_1, a_2)$, are usually defined as quotients of $\mathbb{C}^2 - \{0\}$ by $\mathbb{C}^*$ acting with weights a_1, a_2. According to the latter definition, if $gcd(a_1, a_2) \neq 1$, $\mathbb{P}^1(a_1, a_2)$ has non-trivial generic isotropy group. However, the two definitions agree if $gcd(a_1, a_2) = 1$.

[10] The scheme-theoretic notions employed in the definition, such as 'normalization,' can be easily adapted to DM stacks. We leave it to the reader to fill in the details.

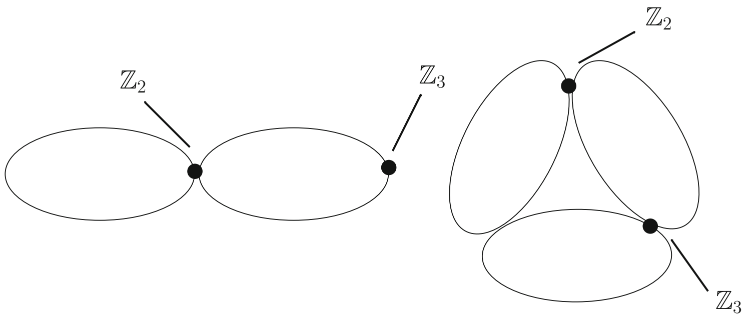


Fig. 1 Above is a picture of the tcnc curves considered in Example 3. The *labels* indicate the isotropy subgroups at the stacky points

It follows from the definition, that the coarse moduli space of a tcnc curve must have arithmetic genus $g \leq 1$, and thus be equal to a cycle of rational curves (i.e., a Galois cover of a nodal $\mathbb{P}^1$), if $g = 1$, and to a chain of rational curves if $g = 0$.

A tcnc curve C is uniquely determined by its genus, together with a tuple of positive integers, which we shall call the W-vector, and which specifies the orders of the isotropy groups at points 0 and ∞, on the different irreducible components of C. We will not give a formal definition of the W-vector, as it easier to see how this works in an example.

Example 3. Consider the weighted projective plane $\mathbb{P}^2(1,2,3) = [(\mathbb{C}^3 - \{0\})/\mathbb{C}^*]$, where $\mathbb{C}^*$ acts with weights $1, 2, 3$.

- Let $C \hookrightarrow \mathbb{P}^2(1,2,3)$ be the sub-stack defined by the equation $x_0x_1 = 0$. C is a tcnc curve of genus 0, and can be encoded in the W-vector $(1,2,3) \in \mathbb{N}^3$. Note that the reverse tuple $(3,2,1)$ is an equally valid W-vector for C.
- Let $C' \hookrightarrow \mathbb{P}^2(1,2,3)$ be defined by $x_0x_1x_2 = 0$. C' has genus 1, and is described by the W-vector $(1,2,3) \in \mathbb{N}^3$. As before, because of the evident symmetries of C', there are other viable choices of W-vector for C', such as for instance $(2,3,1)$.

Definition 8. Denote C^i_A the tcnc curve of genus $i \in \{0,1\}$, with W-vector $A \in \mathbb{N}^m_{>0}$.

Theorem 2 gives a description of the category of perfect complexes over a tcnc curve which will play a key role in our proof of homological mirror symmetry.

Theorem 2. *Let C be a tcnc curve with singular set Z,*[11] *and normalization $\pi : \tilde{C} \to C$. Let σ, τ be two non overlapping sections of $\pi^{-1}(Z) \to Z$, then the diagram*

[11] Note that Z is a disjoint union of classifying stacks of the form $[Spec(\mathbb{C})/\mu_{a_i}]$.

$$\mathcal{P}\mathrm{erf}(C) \xrightarrow{\pi^*} \mathcal{P}\mathrm{erf}(\tilde{C}) \underset{\tau^*}{\overset{\sigma^*}{\rightrightarrows}} \mathcal{P}\mathrm{erf}(Z)$$

is an equalizer in dg$\mathcal{C}$at*.*

We will not prove Theorem 2 in full generality. Instead, we show below, see Theorem 3, that the analogous statement holds for all nodal curves. This in particular implies Theorem 2 when C is an actual scheme, i.e. its W-vector is a tuple filled with 1-s. The general case of Theorem 2 follows easily from here, but we refer the reader to [32] for a complete proof.

Lemma 2. *Let C be a nodal curve, with normalization $\tilde{C} \xrightarrow{\pi} C$, then for every $\tilde{\mathcal{F}}$ vector bundle on $\tilde{C}$ and isomorphism $u : \sigma^*(\tilde{\mathcal{F}}) \to \tau^*(\tilde{\mathcal{F}})$, the assignment:*

$$U \subset^{open} C \mapsto \{s \in \tilde{\mathcal{F}}(\pi^{-1}(U)) | u(\sigma^*(s)) = \tau^*(s)\},$$

defines a vector bundle $\tilde{\mathcal{F}}^u$ on C such that $\pi^*(\mathcal{F}^u) \cong \tilde{\mathcal{F}}$. Conversely, if $\mathcal{F}$ is a vector bundle on C such that $\pi^*\mathcal{F} \cong \tilde{\mathcal{F}}$, then $\mathcal{F} \cong \tilde{\mathcal{F}}^u$ for some isomorphism $u : \sigma^*(\tilde{\mathcal{F}}) \to \tau^*(\tilde{\mathcal{F}})$.

Proof. See Proposition 4.4 in [14].

Theorem 3. *Let C be a nodal curve, with singular set Z, and normalization $\pi : \tilde{C} \to C$. Let $\sigma, \tau : Z \to C$ be two non-overlapping sections of $\pi^{-1}(Z) \to Z$, then the diagram*

$$\mathcal{P}\mathrm{erf}(C) \xrightarrow{\pi^*} \mathcal{P}\mathrm{erf}(\tilde{C}) \underset{\tau^*}{\overset{\sigma^*}{\rightrightarrows}} \mathcal{P}\mathrm{erf}(Z)$$

is an equalizer in dg$\mathcal{C}$at*.*

Proof. Recall that $\mathcal{P}\mathrm{erf}(-)$ satisfies Zariski descent, see Proposition 11 in [35]. As a consequence, it is sufficient to prove the claim for affine C, and we will rectrict to this case. Let E be the equalizer of the diagram

$$\mathcal{P}\mathrm{erf}(\tilde{C}) \underset{\tau^*}{\overset{\sigma^*}{\rightrightarrows}} \mathcal{P}\mathrm{erf}(Z)$$

constructed according to the prescriptions of Lemma 1. Recall that the objects of E are pairs $(\tilde{\mathcal{F}}, u)$, where $\tilde{\mathcal{F}}$ is an object of $D^b(Coh(\tilde{C}))$, and u is a degree zero, closed morphism $\sigma^*\tilde{\mathcal{F}} \xrightarrow{u} \tau^*\tilde{\mathcal{F}}$, which becomes invertible in the homotopy category. The morphisms of E are pairs $(f, H) \in hom^k(\tilde{\mathcal{F}}, \tilde{\mathcal{G}}) \oplus hom^{k-1}(\sigma^*\pi^*\tilde{\mathcal{F}}, \tau^*\pi^*\tilde{\mathcal{G}})$, and the differential is given by $d(f, H) = (df, dH - (u'\sigma^*(f) - \tau^*(f)u))$.

Fix a natural equivalence $\alpha : \sigma^*\pi^* \cong \tau^*\pi^*$. As $\mathcal{P}\mathrm{erf}(C)$ is generated by line bundles, and E is generated by objects of the form $(\tilde{\mathcal{F}}, u)$ with $\tilde{\mathcal{F}}$ a line bundle on $\tilde{C}$, it is sufficient to define a (quasi-)equivalence ψ between these two linear sub-categories. Define ψ as follows,

- if $\mathcal{F}$ is a line bundle on C, then $\psi(\mathcal{F}) = (\pi^*\mathcal{F}, \sigma^*\pi^*\mathcal{F} \xrightarrow{\alpha} \tau^*\pi^*\mathcal{F})$,
- if $\mathcal{F}, \mathcal{G}$ are line bundles on C, and $f \in hom^k(\mathcal{F}, \mathcal{G})$, then $\psi(f) = (\pi^* f, 0)$.

Consider a line bundle $\tilde{\mathcal{F}}$ over $\tilde{C}$. It follows from Lemma 2 that the set of isomorphism classes of line bundles $\mathcal{F}$ on C such that $\pi^*\mathcal{F} \cong \tilde{\mathcal{F}}$ carries a transitive action by $(\mathbb{C}^*)^{|Z|}$ (given by pointwise rescaling the 'compatibility' isomorphisms u, see Lemma 2). Further, the same is true for the set of isomorphism classes of objects of $(\tilde{\mathcal{G}}, v) \in E$, such that $(\tilde{\mathcal{G}}, v) \cong (\tilde{\mathcal{F}}, u)$ for some $u \in hom^0(\sigma^*\tilde{\mathcal{F}}, \tau^*\tilde{\mathcal{F}})$. Essential surjectivity follows from the fact that ψ defines a $(\mathbb{C}^*)^{|Z|}$-equivariant map between these two sets of isomorphism classes.

We shall prove next that ψ is quasi-fully faithful, i.e. that the map between hom-complexes defined by ψ induces an isomorphism in the homotopy category. Denote HoE the homotopy category of E. It is sufficient to show that for all line bundles $\mathcal{F}$ on C, and for all $i \in \mathbb{N}$,

$$\psi : Hom^i_C(\mathcal{O}_C, \mathcal{F})(= H^i_C(\mathcal{F})) \xrightarrow{\cong} Hom^i_{HoE}(\psi(\mathcal{O}_C), \psi(\mathcal{F})).$$

Note that, as C and $\tilde{C}$ are affine, cohomology vanishes in positive degree. It follows that $Hom^i_{HoE}(\psi(\mathcal{O}_C), \psi(\mathcal{F})) = 0$ for all $i > 0$.[12] Further, in degree-zero, the hom-space fits in the following short exact sequence

$$0 \to Hom^0_{HoE}(\psi(\mathcal{O}_C), \psi(\mathcal{F})) \to Hom^0_{\tilde{C}}(\mathcal{O}_{\tilde{C}}, \pi^*\mathcal{F}) \to Hom^0_Z(\sigma^*\mathcal{O}_{\tilde{C}}, \tau^*\pi^*\mathcal{F}) \to 0.$$

Thus, proving fully faithfulness boils down to showing exactness of

$$0 \to H^0_C(\mathcal{F}) \xrightarrow{\pi^*} H^0_{\tilde{C}}(\pi^*\mathcal{F}) \to H^0_Z(\tau^*\pi^*\mathcal{F}) \to 0. \tag{1}$$

Now, (1) is obtained by taking global sections of the sequence

$$0 \to \mathcal{F} \to \pi_*(\pi^*\mathcal{F}) \to \pi_*\tau_*\tau^*(\pi^*\mathcal{F}) \to 0,$$

which is exact (see the proof of Proposition 4.4 of [14]). Since C is affine, taking global section is an exact operation, and this concludes the proof of Theorem 3.

3.2 Wheels, Dualizable Ribbon Graphs, and HMS

A *wheel* is a conical Lagrangian Λ in T^*S^1 that contains the zero section. We can equip a wheel with canonical chordal structure, given by the pair $(Z = S^1, \Lambda)$.

[12] Note that $Hom^1_{HoE}(\psi(\mathcal{O}_C), \psi(\mathcal{F}))$ vanishes, since it is isomorphic to the quotient of $Hom^0_Z(\sigma^*\mathcal{O}_{\tilde{C}}, \tau^*\pi^*\mathcal{F}) \cong \mathbb{C}$ by the image of the differential, which is easily seen to be surjective.

Recall from Sect. 2.3 that a choice of orientation on S^1 yields a subdivision of the spokes of Λ into two groups, called respectively "upward" and "downward." We will denote a wheel with a_1 upward spokes, and a_2 downward spokes, Λ_{a_1,a_2}.

Theorem 4. *If $a_1, a_2 \in \mathbb{N}_{>0}$, there is an equivalence $\mathcal{P}\mathrm{erf}(\mathbb{P}^1(a_1, a_2)) \cong Sh(S^1, \Lambda_{a_1,a_2})$.*

Proof. Theorem 4 is due to Bondal [5], who first suggested this should be interpreted as an instance of mirror symmetry. Partially inspired by Bondal's insights, Fang, Liu, Treumann and Zalow develop an approach to HMS for (stacky) toric varieties [7, 8], which in particular implies this result, and is the starting point for the project pursued in [32]. Note that when $a_1 = a_2 = 1$, this recovers the classic result of Beilinson [3], according to which there is an equivalence $D^b(Coh(\mathbb{P}^1)) \cong Rep(\bullet \rightrightarrows \bullet)$. In fact, Theorem 1 gives an equivalence $Rep(\bullet \rightrightarrows \bullet) \cong Sh(S^1, \Lambda_{1,1})$.

Remark 1. Theorem 4 can be refined, by requiring that the equivalence intertwine appropriate 'stalk functors.' More precisely, take $i \in \{1, 2\}$, and let $j_i : [*/\mathbb{Z}_{a_i}] \to \mathbb{P}^1(a_1, a_2)$ be the inclusion. If χ is a character of $\mathbb{Z}_{a_i}$, we denote S^i_χ the following composition

$$S^i_\chi : \mathcal{P}\mathrm{erf}(\mathbb{P}^1(a_1, a_2)) \xrightarrow{j_i^*} \mathcal{P}\mathrm{erf}([*/\mathbb{Z}_{a_i}]) \xrightarrow{\chi} \mathbb{C}\text{-mod}.$$

We can label the upward (resp. downward) spokes of Λ_{a_1,a_2} with characters of $\mathbb{Z}_{a_2}$ (resp. $\mathbb{Z}_{a_1}$).[13] We denote R^i_χ the upward ($i = 2$), or downward ($i = 1$), spoke of Λ_{a_1,a_2} labeled by χ. Note that there is a restriction functor

$$Res^i_\chi : \mathrm{CPM}(\Lambda_{a_1,a_2}) \longrightarrow \mathrm{CPM}(R^i_\chi) \cong \mathbb{C}\text{-mod}.$$

The claim is that we can define $\Phi : \mathcal{P}\mathrm{erf}(\mathbb{P}^1(a_1, a_2)) \cong \mathrm{CPM}(\Lambda_{a_1,a_2})$ in such a way that we get commutative diagrams of dg categories

$$\begin{array}{ccc} \mathcal{P}\mathrm{erf}(\mathbb{P}^1(a_1, a_2)) & \xrightarrow{S^i_\chi} & \mathbb{C}\text{-mod} \\ \Big\downarrow{\scriptstyle \Phi} & & \Big\downarrow{\scriptstyle \cong} \\ \mathrm{CPM}(\Lambda_{a_1,a_2}) & \xrightarrow{Res^i_\chi} & \mathrm{CPM}(R^i_\chi). \end{array}$$

The chordal ribbon graphs which are most relevant in the context of mirror symmetry have special properties, and are called *dualizable*. Dualizable ribbon

[13] Both the set of characters and the set of up-/down-ward spokes come with natural cyclic orders (the spokes inherit it from the ribbon structure on Λ_{a_1,a_2}). The labelling cannot therefore be entirely arbitrary, as it must preserve this cyclic order, see [32].

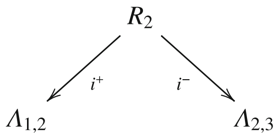
graphs are obtained by gluing together wheels along matching sets of in- and outgoing edges. We will limit ourselves to explain the geometry of dualizable ribbon graphs through concrete examples, while referring the reader to [32] for rigorous definitions. Also, we will mostly consider *trivalent* dualizable ribbon graphs, as this will somewhat simplify the exposition, and will not reduce generality in any serious way (in fact, any chordal ribbon graph is, in an appropriate sense, 'deformation equivalent' to a trivalent graph, cf. Footnote 8).

Let $a \in \mathbb{N}_{>0}$, and denote R_a the chordal ribbon graph given by a disjoint union of positive rays, with empty vertex set, and trivial chordal structure, $R_a = (\coprod 1 \leq i \leq a \mathbb{R}_{>0}, \varnothing)$. If Λ_{a_1,a_2} is a wheel, we can choose morphisms in Chord

$$R_{a_2} \xrightarrow{i^-} \Lambda_{a_1,a_2} \xleftarrow{i^+} R_{a_1},$$

mapping homeomorphically the components of R_{a_1}, and R_{a_2}, respectively onto the upward, and downward, spokes of Λ_{a_1,a_2}.

Example 4. 1. Let $A = (1,2,3) \in \mathbb{N}^3$, and denote $\Lambda_A^0 = (D_A^0, Z_A)$ the chordal ribbon graph obtained as the push-out of the following diagram in Chord,

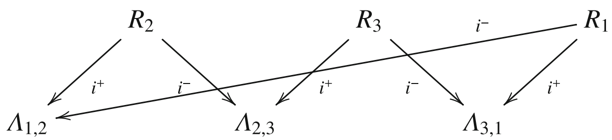


That is, D_A^0 is the push-out of the underlying one-dimensional CW-complexes, and is equipped with the unique chordal structure rendering the natural inclusions

$$\Lambda_{1,2} \hookrightarrow \Lambda_A^0 \hookleftarrow \Lambda_{2,3}$$

morphisms in Chord. Thus, Z_A is the disjoint union of two circles. Note that D_A^0 is the non-compact skeleton of a punctured curve of genus 0, endowed with appropriate Stein structure.

2. Let $A = (1,2,3) \in \mathbb{N}^3$, and let $\Lambda_A^1 = (D_A^1, Z_A)$ be the push-out of the following diagram in Chord



The ribbon graph D_A^1 is isomorphic to the skeleton of a Stein torus with 6 punctures.

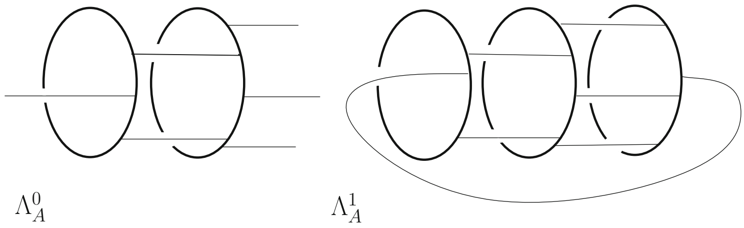


Fig. 2 The dualizable ribbon graphs considered in Example 4(1) and (2) are sketched above. We have signaled the chordal basis by drawing it with a *thicker line*

Dualizable ribbon graphs are constructed by joining together wheels as in the two examples above.[14] Recall, after Proposition 1, that the sections of CPM(−) on a wheel Λ_{a_i,a_j} depend exclusively on a_i and a_j. That is, we can assume that Λ_{a_i,a_j} has any convenient shape, so long as we do not change these two integers. As a consequence, given any dualizable ribbon graph, all the geometric information which is required to compute the sections of CPM(−) over it can be encoded in a discrete set of data.

Namely, disregarding finer geometric features which do not affect the sections of CPM(−), dualizable ribbon graphs are identified by their genus, which is equal to 0 or 1,[15] and by a tuple of positive integers recording the number of edges connecting the different connected components of the chordal basis, Z. This is entirely analogous to the case of tcnc curves, which was discussed in Sect. 3.1.

Let $i \in \{0, 1\}$, and let $A = (a_1, \ldots, a_m)$ be a tuple of positive integers, and denote Λ^i_A any dualizable ribbon graph whose geometry fits these numerical data, in the manner explained above.

Theorem 5 (HMS). *There is an equivalence of dg categories*

$$\mathcal{P}\mathrm{erf}(C^i_A) \cong \mathrm{CPM}(\Lambda^i_A).$$

Proof. There is a covering of Λ^i_A given by wheels $W_i = \Lambda_{a_i,a_{i+1}}$. Then, by the sheaf property of CPM we have an equalizer diagram

$$\mathrm{CPM}(C^0_A) \to \mathrm{CPM}(\coprod W_i) \rightrightarrows \mathrm{CPM}(\coprod W_i \cap W_{i+1}).$$

---

[14]It is important to point out that, as shown in Fig. 2, in a dualizable ribbon graph the strands joining together the components of the chordal basis cannot be (non-trivially) ‘braided.’ This can be translated in appropriate conditions of coherency on the maps $R_{a_i} \to \Lambda_i$. We refer the reader to [32] for further details.

[15]The genus of a ribbon graph D can be described geometrically as the genus of any surface in which D can be embedded, in a way compatible with the ribbon structure, as a deformation retract. Thus D^0_A in Example 4(1) has genus 0, while D^1_A in Example 4(2), has genus 1. For a formal, combinatorial definition of the genus of a ribbon graph, see [32].

The theorem then follows immediately from Theorem 4 (and Remark 1), and Theorem 2.

As discussed above, dualizable ribbon graphs Λ^i_A arise as skeleta of punctured curves of genus i with appropriate Stein structure. Granting Conjecture 1, Theorem 5 can therefore be interpreted as a HMS statement, relating punctured symplectic surfaces, and degenerate, nodal algebraic curves, having equal genus $i \in \{0, 1\}$. In particular, this confirms the well known mirror symmetry heuristics according to which the mirror of a symplectic torus with n punctures should be a cycle of n rational curves.[16]

4 Tcnc Curves and Fourier–Mukai Equivalences

The Fukaya category of a punctured Riemann surface Σ should depend solely on the symplectic geometry of Σ, which is encoded in its genus, and in its number of punctures.[17] In view of Conjecture 1, this suggests that if D and D' are (chordal) ribbon graphs arising as skeleta of a unique punctured surface Σ equipped with two different Stein structures, there should be an equivalence $\mathrm{CPM}(D) \cong \mathrm{CPM}(D')$.

In this section we sketch a proof that this is indeed the case for dualizable ribbon graphs, by introducing a simple graphical calculus which will enable us to construct this equivalence in a step-by-step fashion. A precise statement of our theorem is collected below. If $n \in \mathbb{N}$, we denote $\mathbf{1}(n) \in \mathbb{N}^n$ the tuple filled with 1-s. By slight abuse of notation, we shall also denote $(a, \mathbf{1}(n), b)$ a tuple of length $2 + n$, of the form $(a, 1, 1, \dots, 1, b)$.

Theorem 6. *If $A = (a_1, \dots, a_m)$ is a tuple of positive integers, there are equivalences*

1. $\mathrm{CPM}(\Lambda^0_A) \cong \mathrm{CPM}(\Lambda^0_{A'})$, *where* $A' = (a_1, \mathbf{1}(a_2 + \cdots + a_{m-1}), a_m)$,
2. $\mathrm{CPM}(\Lambda^1_A) \cong \mathrm{CPM}(\Lambda^1_{A'})$, *where* $A' = \mathbf{1}(a_1 + \cdots + a_n)$.

Our interest in this result depends on the fact that, using the dictionary provided by Theorem 5, it can be translated in a statement regarding derived equivalences of tcnc curves.

Corollary 1. *If $A = (a_1, \dots, a_m)$ is a tuple of positive integers, there are equivalences*

1. $\mathcal{P}\mathrm{erf}(C^0_A) \cong \mathcal{P}\mathrm{erf}(C^0_{A'})$, *where* $A' = (a_1, \mathbf{1}(a_2 + \cdots + a_{m-1}), a_m)$,
2. $\mathcal{P}\mathrm{erf}(C^1_A) \cong \mathcal{P}\mathrm{erf}(C^1_{A'})$, *where* $A' = \mathbf{1}(a_1 + \cdots + a_n)$.

---

[16] Kontsevich announced related results in [13]. HMS for the nodal $\mathbb{P}^1$ is also treated in [15].

[17] Note that this is true, without further specifications, only if we are considering the Fukaya category of *compact* Lagrangians in Σ.

Denote X_n a cycle of rational curves with n components. Corollary 1 implies in particular that there is an equivalence $\mathcal{P}\mathrm{erf}(X_n) \cong \mathcal{P}\mathrm{erf}([X_1/\mu_n])$, where μ_n is the group of n-th roots of unity, acting on X_1 in the obvious manner, and $[X_1/\mu_n]$ is the quotient stack. As we shall explain, this result can be interpreted as a generalization to the singular case of Mukai's classic work on derived equivalences of smooth elliptic curves (and, more generally, of principally polarised abelian varieties) [17]. Recall that Mukai shows that, if X and $X^\vee$ are dual abelian varieties, there is a non-trivial equivalence $D^b(Coh(X)) \cong D^b(Coh(X^\vee))$, which he defines via a pull–push formalism, by taking as kernel the universal bundle on the product $X \times X^\vee$.

As in the smooth case, the nodal projective line X_1 is isomorphic to its dual $X_1^\vee$, which is the moduli space of rank 1, degree 0, torsion-free sheaves over X_1 [6]. Further, one can show that X_n, which is the n-fold cover of X_1, parametrizes μ_n-*equivariant* sheaves on X_1 satisfying the properties just listed. In this perspective, we can interpret the covering map $X_n \to X_1$ as induced by 'forgetting the equivariant structure.' Thus, X_n is isomorphic to the moduli space of rank 1, degree 0, torsion-free sheaves over the quotient stack $[X_1/\mu_n]$ or, in other words, X_n is dual, in the sense discussed above, to $[X_1/\mu_n]$.

The existence of an equivalence $\mathcal{P}\mathrm{erf}(X_n) \cong \mathcal{P}\mathrm{erf}([X_1/\mu_n])$ therefore fits well with what we would expect based on the smooth case.[18] Note that the case $n = 1$ was also studied by Burban and Kreussler [6], who use the theory of spherical functors to define a non-trivial derived equivalence $D^b(Coh(X_1)) \cong D^b(Coh(X_1))$ having the required properties.[19]

4.1 Elementary Moves

In this section we introduce a set of operations, called *elementary moves*, which can be used to modify the geometry of chordal ribbon graphs while preserving the global sections of CPM(−). First, however, we spell out the behaviour of CPM(−) on some especially simple chordal ribbon graphs, which can be used as building blocks for all trivalent graphs in Chord.

Let E be a ribbon graph with empty vertex set, and underlying CW complex homeomorphic to $\mathbb{R}$. We can equip E with two distinct chordal structures (E, W), by setting either $W = E$, or $W = \varnothing$. In both cases, $\mathrm{CPM}(E, W) \cong \mathbb{C}$-mod. Thus, if (D, Z) is a chordal ribbon graph, and $\{e_i\}_{i\in I}$ is the set of edges of D, restriction to the edges yields *stalk* functors, indexed by I,

---

[18] Note that any such equivalence would extend to an equivalence of the full derived categories, see Theorem 1.2 in [2].

[19] In [30], extending results of [6], we defined an action of the mapping class group of a torus with n punctures on $D^b(Coh(X_n))$. The argument we shall describe below can be interpreted, roughly, as defining an action of an appropriate version of the mapping class *groupoid*. For a definition of spherical functor, see [28].

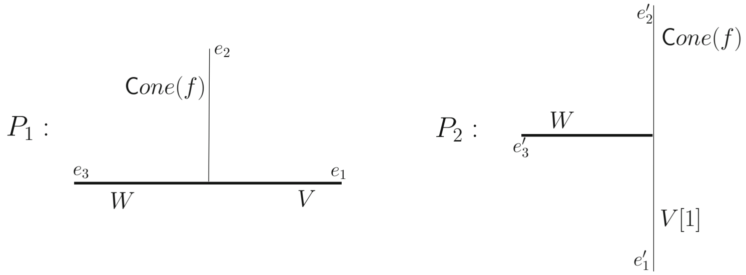


Fig. 3 Up to isomorphism, there are only two chordal structures on a pitchfork, which are represented above, and are denoted P_1 and P_2

$$Res_i : \mathrm{CPM}(D, Z) \to \mathbb{C}\text{-mod} \cong \mathrm{CPM}(e_i),$$

which generalize the 'microlocal stalks' discussed in Sect. 2.1. It is often convenient to indicate an object $L \in \mathrm{CPM}(D, Z)$ by assigning the collection of its stalks $Res_i(L)$, which can be visualized as labels attached to the edges e_i of D.

A *pitchfork* is a chordal ribbon graph $P = (D, Z)$, such that D is isomorphic to the union of the real line $\mathbb{R}$, and an upward spoke R^+. As shown by Fig. 3 above, there are only two possible choices of chordal basis, which yield inequivalent chordal ribbon graphs P_1, P_2. In either case, using Theorem 1, we can see that the global sections of CPM(−) are given by $Rep(\bullet \to \bullet)$. The edges of the graphs represented in Fig. 3 are decorated with labels corresponding to an object $L = (V \xrightarrow{f} W) \in Rep(\bullet \to \bullet)$. Thus, for instance, the picture indicates that the stalk of $L \in \mathrm{CPM}(P_1) = Rep(\bullet \to \bullet)$ on any point lying on the edge e_2, is isomorphic to $\mathsf{C}one(f)$.

The set of the *Elementary Moves*, or *EM*-s, which we shall use in the proof of Theorem 6, is given in the table below (Fig. 4). Note that the ribbon graphs considered in Fig. 4 are obtained by gluing together pitchforks along common edges, and thus we can easily compute the sections of CPM(−) over them using Lemma 1.

For each elementary move *EMi*, let (D_{l_i}, Z_{l_i}) be the graph appearing on the left of the '⇔' symbol, and (D_{r_i}, Z_{r_i}) the graph appearing on the right. *EM*-s preserve global sections of CPM(−), and there is a preferred isomorphism $\Phi_i : \mathrm{CPM}(D_{l_i}, Z_{l_i}) \cong \mathrm{CPM}(D_{r_i}, Z_{r_i})$. We have labelled the edges of (D_{l_i}, Z_{l_i}), and (D_{r_i}, Z_{r_i}), with the stalks of $L \in \mathrm{CPM}(D_{l_i}, Z_{l_i})$, and $\Phi_i(L) \in \mathrm{CPM}(D_{r_i}, Z_{r_i})$, respectively. This schematics gives enough information for reconstructing Φ_i entirely. In order to simplify notations, we have indicated the cone of a map $f : V \to W \in \mathbb{C}$-mod simply by $\mathsf{C}(f)$. Note that $EM1'$ can be obtained simply by iterating $EM1$. We have included it in Fig. 4 because in Sect. 4.2 it will be convenient to apply this transformation directly, without factoring it into simpler *EM*-s.

Giving an explicit definition of the preferred equivalences Φ_i, based on the information contained in Fig. 4, is not hard and we omit the details. However, as an example of the kind of arguments involved, it might be useful to discuss briefly the case of $EM1$.

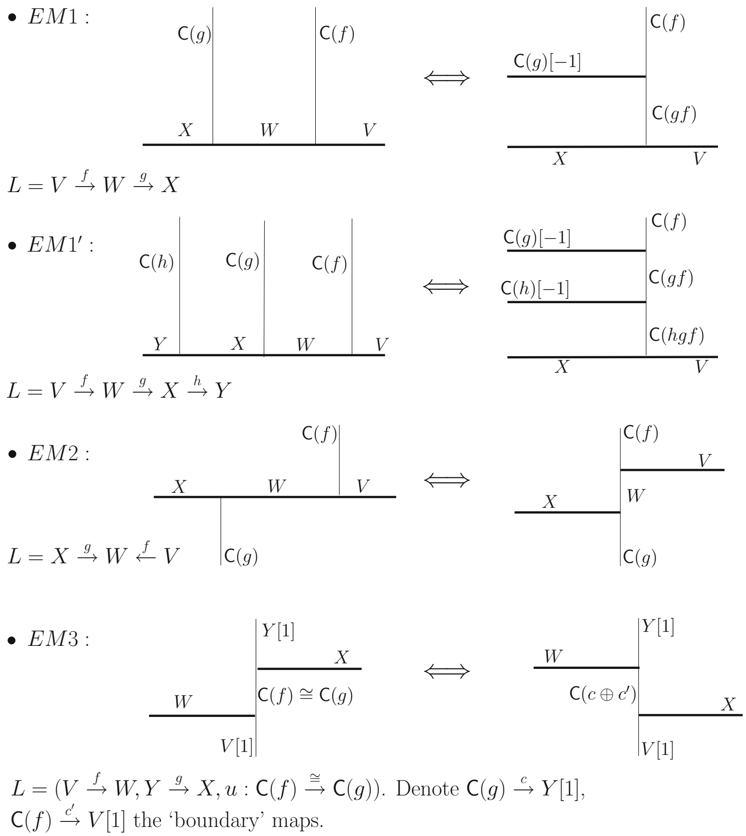


Fig. 4

Proof (The Definition of Φ_1). Following the notations of Fig. 3, (D_{r_1}, Z_{r_1}) can be constructed by gluing together edges e_2 of P_1, and e_3' of P_2. An object in $\mathrm{CPM}(D_{r_1}, Z_{r_1})$ is given therefore by a triple of the form $(A \xrightarrow{l} B, C \xrightarrow{m} D, u : \mathsf{C}(l) \xrightarrow{\cong} C[1])$. Now, take $L = (V \xrightarrow{f} W \xrightarrow{g} X)$ in $\mathrm{CPM}(D_{l_1}, Z_{l_1})$. Since $\mathbb{C}$-mod is dg triangulated, there exist a natural morphism $p : \mathsf{C}(gf) \to \mathsf{C}(g)$, and an isomorphism $\mathsf{C}(p) \cong \mathsf{C}(f)$. This follows from the very general fact that triangulated dg categories satisfy an appropriate enhancement of the *octahedral axiom*.[20] However, for completeness, we construct p explicitly.

[20] In the closely related context of stable $(\infty, 1)$-categories, a parallel statement is proved in [16], see Theorem 1.1.2.14: in fact, both statement and proof in [16] apply word for word to triangulated dg categories as well.

Consider the commutative diagram

$$\begin{array}{ccccc} V & \xrightarrow{f} & W & \longrightarrow & \mathsf{C}(f) \\ {\scriptstyle gf}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow \\ X & \xrightarrow{=} & X & \longrightarrow & 0 \end{array}$$

Computing the co-cones of the vertical arrows, and using the availability of functorial cones in triangulated dg categories,[21] we obtain maps $\mathsf{C}(gf)[-1] \xrightarrow{p} \mathsf{C}(g)[-1] \to \mathsf{C}(f)$. Both rows of the diagram are exact, that is, they are isomorphic to mapping cone sequences. It follows that $\mathsf{C}(gf)[-1] \xrightarrow{p} \mathsf{C}(g)[-1] \to \mathsf{C}(f)$ is exact as well. In particular, there is a natural isomorphism $\mathsf{C}(f) \cong \mathsf{C}(p)$.

We define Φ_1 on objects by setting

$$\Phi_1(L) = (V \xrightarrow{gf} X, \mathsf{C}(gf)[-1] \xrightarrow{p} \mathsf{C}(g)[-1], \mathsf{C}(gf) \xrightarrow{=} \mathsf{C}(gf)).$$

The definition on morphisms is obvious, as the assignment relies on the repeated application of the cone construction, and is therefore functorial. The fact that the microlocal stalks of $\Phi_1(L)$ match the indications of Fig. 4 follows from the natural identification $\mathsf{C}(f) \cong \mathsf{C}(p)$.

Next, we show that Φ_1 is a quasi-equivalence, by constructing a quasi-inverse Φ_1^{-1}. Let $M = (A \xrightarrow{l} B, C \xrightarrow{m} D, u : \mathsf{C}(l) \xrightarrow{\cong} C[1])$ be an object in $\mathrm{CPM}(D_{r_1}, Z_{r_1})$. Consider the map $c : C[1] \to A[1]$ obtained by composing u^{-1} with the boundary map $\mathsf{C}(m) \to A[1]$. We have a diagram

$$\begin{array}{ccccc} C & \xrightarrow{c[-1]} & A & \xrightarrow{l} & B \\ \downarrow{\scriptstyle m} & & \downarrow & & \downarrow{\scriptstyle n} \\ D & \longrightarrow & 0 & \longrightarrow & D[1] \end{array}$$

As the rows are exact, functoriality of cones yields a map n, indicated with a dashed arrow, which makes the diagram commute. Taking co-cones of the two rightmost vertical arrows gives maps $r : \mathsf{C}(n)[-1] \to B$, and $s : A \to \mathsf{C}(n)[-1]$. Note that, by construction, $s \circ r = m$. The quasi-inverse Φ_1^{-1} sends the object M to

$$\Phi_1^{-1}(M) := (A \xrightarrow{r} \mathsf{C}(n)[-1] \xrightarrow{s} B) \in \mathrm{CPM}(D_{r_1}, Z_{r_1}).$$

We leave it to the reader to check that Φ_1^{-1} is a quasi-inverse to Φ_1: this follows again from a simple application of the octahedral axiom.

[21] See Sect. 5.1 of [35] for further details.

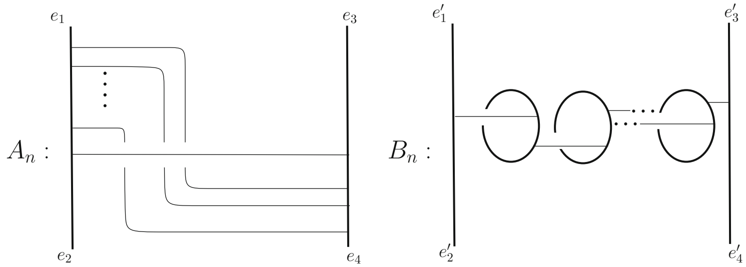


Fig. 5 The parameter $n \in \mathbb{N}_{>0}$ indicates the number of strands in A_n, and loops in B_n

4.2 The Proof of Theorem 6

Figure 5 represents two different kinds of chordal ribbon graphs, which are denoted A_n, and B_n, with $n \in \mathbb{N}_{>0}$. All tri-valent dualizable ribbon graphs can be assembled by gluing along their external edges a certain number of copies of graphs of type A and B. In order to prove Theorem 6, it is therefore enough to show that for every n there exists an equivalence $\Phi_n : \mathrm{CPM}(A_n) \cong \mathrm{CPM}(B_n)$, with the property that Φ_n preserves the stalks on the 4 external edges e_i, and e'_i.[22] In fact, starting with any dualizable ribbon graph, we can turn it into a dualizable graph having weight vector with entries all equal to 1 by successively replacing its subgraphs of type A with subgraphs of type B. The availability of the equivalences Φ_n insures that, while doing so, we are not affecting the sections of CPM(−) (up to isomorphism).

In defining the Φ_n-s, we have to break down the algorithm just described in yet smaller subroutines. For all $n \in \mathbb{N}_{>0}$, we identify suitable subgraphs of A_n isomorphic to the graphs appearing in Fig. 4, and we modify their geometry via the appropriate elementary move. This gives rise to a new graph, which we can manipulate in similar manner, until, after a finite number of steps, we achieve the geometry of B_n. This procedure involves keeping track of what happens to the stalks across *EM*-s, to make sure that our operations, which are local in nature, determine equivalences at the level of global sections of CPM(−). This can be easily done, using the information on stalks given by Fig. 4.

The proof of Theorem 6 can therefore be reduced to a simple graphical calculus, which is illustrated in Figs. 6 and 7 below, for the cases $n = 2$, and $n = 3$. At each step we apply an elementary move, which is explicitly indicated over the symbol '$\Leftrightarrow$.' It is important to notice that, in some of these steps, we are simultaneously applying the same elementary move to two distinct subgraphs. The strategy for

[22]It might be surprising that the strands of A_n are 'non-trivially braided.' The existence of the equivalence Φ_n depends, in fact, in a crucial way on the choice of this particular geometry. Note however that the chordal basis of a dualizable ribbon graph is a union of loops. Considered as edges of the larger graph, the strands in a subgraph of type A can therefore be un-braided, cf. also Footnote 14.

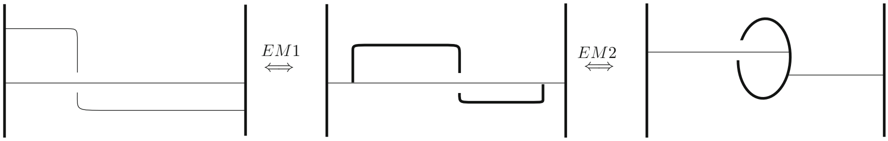


Fig. 6 The case $n = 2$

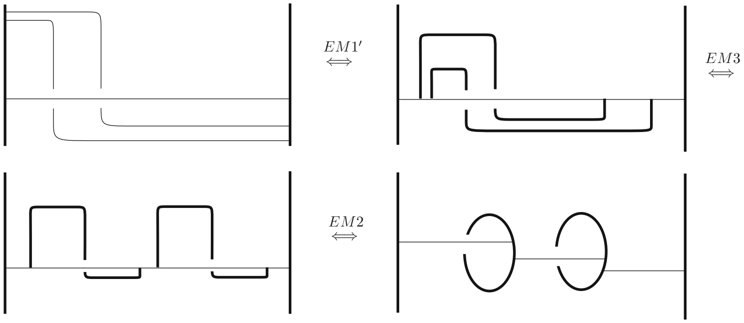


Fig. 7 The case $n = 3$

proving the statement in the general case can be easily extrapolated from here, and therefore we will not discuss it in any further detail.

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The Very Good Property for Moduli of Parabolic Bundles and the Additive Deligne–Simpson Problem

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Abstract In "Quantization of Hitchin's Integrable System and Hecke Eigensheaves", Beilinson and Drinfeld introduced the "very good" property for a smooth complex equidimensional stack. They prove that for a semisimple group G over $\mathbb{C}$, the moduli stack $\mathrm{Bun}_G(X)$ of G-bundles over a smooth complex projective curve X is "very good", as long as X has genus $g > 1$. In the case of the projective line, when $g = 0$, this is not the case. However, the result can sometimes be extended to the projective line and reductive group $G = \mathrm{GL}(n, \mathbb{C})$, by introducing additional parabolic structure at a collection of marked points and slightly modifying the definition of a "very good" stack. Using the modified definition, we provide a sufficient condition for the moduli stack of parabolic vector bundles over $\mathbb{P}^1$ to be very good, and use this property to study the space of solutions to the additive Deligne–Simpson problem.

1 The Very Good Property

In [1] Beilinson and Drinfeld introduce the notion of a "very good" stack. They require this property in order to avoid using derived categories in their study of D-modules on the moduli stack $\mathrm{Bun}_G(X)$ of G-bundles over X, where G is a semisimple algebraic group and X is a smooth complex projective curve.

A smooth complex equidimensional stack $\mathscr{Y}$ will be called *very good* if

$$\operatorname{codim}\{y \in \mathscr{Y} \mid \dim G_y = n\} > n, \text{for } n > 0,$$

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where G_y is the automorphism group of $y \in \mathscr{Y}$. Beilinson and Drinfeld demonstrate that $\mathrm{Bun}_G(X)$ is very good when X has genus $g > 1$. However, in the $g = 0$ case, when $X = \mathbb{P}^1$, this is no longer true.

To approach the very good property in the genus $g = 0$ case, we introduce additional parabolic structure at a finite collection of marked points. Furthermore, we will consider this case for the group $G = \mathrm{GL}(n, \mathbb{C})$.

Note that the reductive group $\mathrm{GL}(n, \mathbb{C})$ contains $\mathbb{C}^*$ as a central subgroup, so the automorphism group of any $\mathrm{GL}(n, \mathbb{C})$-bundle contains a one-dimensional subgroup that acts by dilation on the fibers. Therefore, regardless of the curve X, the moduli stack of parabolic bundles over X cannot be very good.

To remedy this, we will modify the definition of a very good stack. The stack $\mathscr{Y}$ will be called *almost very good* if

$$\operatorname{codim} \{y \in \mathscr{Y} | \dim \operatorname{Aut}(y) - m = n\} > n, \text{for } n > 0,$$

where m is the maximal nonnegative integer such that $\dim \operatorname{Aut}(y) \geq m$ for all $y \in \mathscr{Y}$.

It turns out that a sufficiently elaborate parabolic structure on a vector bundle is enough to make the corresponding moduli stack of parabolic bundles over $\mathbb{P}^1$ almost very good. This is equivalent to showing that the quotient of the moduli stack by the classifying stack of $\mathbb{C}^*$ is very good.

2 The Very Good Property for Moduli of Parabolic Bundles

Seshadri introduced the notion of parabolic structures on vector bundles in [17], furnishing them with a stability condition analogous to the usual one for vector bundles. Expanding upon this, Mehta and Seshadri proved the existence of a moduli space of semistable parabolic bundles on a smooth projective curve of genus $g \geq 2$ in [16]. Since we do not require semistability, we define parabolic bundles in a slightly different way from [16].

Parabolic bundles over an algebraic curve generalize vector bundles by defining additional structure in the fibers over specified points. Namely, let X be a smooth connected complex projective curve. In the future we restrict ourselves to the case when $X = \mathbb{P}^1$.

A *parabolic bundle* $\mathbf{E}$ over X consists of a vector bundle E over X, a collection of distinct points $(x_1, \ldots, x_k)$ on X, and a flag $E_{x_i} = E_{i0} \supseteq E_{i1} \supseteq \cdots \supseteq E_{iw_{i-1}} \supseteq E_{iw_i} = 0$ in the fiber over each such point x_i. If $D = (x_1, \ldots, x_k)$ and $w = (w_1, \ldots, w_k)$, we say that the parabolic bundle $\mathbf{E}$ has *weight type* (D, w). If $\alpha_0 = \operatorname{rk} E$ and $\alpha_{ij} = \dim E_{ij}$, for $1 \leq i \leq k$ and $1 \leq j \leq w_{i-1}$, we say that $\mathbf{E}$ has *dimension vector* $\alpha = (\alpha_0, \alpha_{ij})$.

The isomorphism classes of parabolic bundles over X of weight type (D, w) and dimension vector α form an algebraic stack. If one wishes to consider a moduli scheme instead, one can introduce stability and semistability for parabolic bundles.

One way of defining these notions is to introduce *parabolic degree*, using it to define *parabolic slope* (see [16]).

To do this, additional numbers called *weights* are assigned to each subspace in each flag. Since we do not limit ourselves to stable or semistable parabolic bundles, we do not require weights to be part of the definition. Parabolic bundles without weights are sometimes referred to as "quasi-parabolic" bundles.

In order to formulate our main result, we introduce two more definitions that will specify which dimension vector give rise to very good parabolic bundles. Let $I = \{0\} \cup \{(i,j) | 1 \le i \le k, 1 \le j \le w_{i-1}\}$. For a dimension vector $\alpha \in \mathbb{Z}^I$, we write: $\delta(\alpha) = -2\alpha_0 + \sum_i \alpha_{i1}$. We say that α is in the *fundamental region* if

$$\delta(\alpha) \ge 0$$
$$-2\alpha_{ij} + \alpha_{ij-1} + \alpha_{ij+1} \ge 0, \text{ for } 1 \le i \le k \text{ and } 1 \le j \le w_{i-1}$$

(note that we assume $\alpha_{i0} = \alpha_0$, for all i). We now introduce our main result.

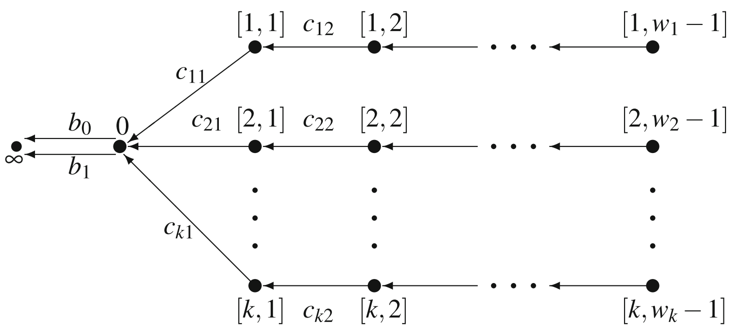
Theorem 1. *The moduli stack of parabolic bundles of weight type* (D, w) *and dimension vector* α *over* $\mathbb{P}^1$ *is "almost very good" if* α *is in the fundamental region and* $\delta(\alpha) > 0$.

The proof of this theorem will be given in another publication.

3 Parabolic Bundles and Squids

Crawley-Boevey's approach [3, 4] to the Deligne–Simpson problems involves relating them to representations of an algebra called a squid. We introduce Crawley-Boevey's terminology and explain how his work relates to the special case of Theorem 1 we will use to study solutions of the additive Deligne–Simpson problem.

Let (D, w) be as before. Consider the quiver pictured below.



A *squid* $S_{D,w}$ is the quotient of the path algebra of this quiver by the relations $(\lambda_{i0}b_0 + \lambda_{i1}b_1)c_{i1} = 0$, for $x_i = (\lambda_{i0} : \lambda_{i1}) \in \mathbb{P}^1$. We consider representations of $S_{D,w}$ to be quiver representations in the standard complex coordinate spaces that satisfy these relations. A squid representation is said to be *Kronecker-preinjective* if $\lambda_0 b_0 + \lambda_1 b_1$ is surjective as a linear map for all $(\lambda_0 : \lambda_1) \in \mathbb{P}^1$.

Let (d, α) be the vector of dimensions of the spaces in a representation of $S_{D,w}$. That is, d is the dimension of the space at the vertex ∞, while $\alpha_0 + d$ is the dimension of the space at vertex 0 and α_{ij} is the dimension of the space at the vertex $[i, j]$. We will consider the case when $d = 0$, and the Kronecker-preinjective squid representations reduce to just representations of the star-shaped quiver (which is the quiver pictured above without the vertex ∞ and the arrows adjacent to it).

Consider the symmetrization of the Euler–Ringel form, defined on the standard coordinate vectors ϵ_v, where $v \in I = \{0\} \cup \{(i, j)|1 \leq i \leq k, 1 \leq j \leq w_{i-1}\}$, as:

$$(\epsilon_{v_1}, \epsilon_{v_2}) = \begin{cases} 2 \text{ (if } v_1 = v_2) \\ -1 \text{ (if an edge joins } v_1 \text{ and } v_2) \\ 0 \text{ (otherwise)} \end{cases}$$

It defines a generalized Cartan matrix, so we may interpret a subset of the dimension vectors α as roots of a Kac–Moody Lie algebra. The corresponding Weyl group is generated by the reflections:

$$s_v(\alpha) = \alpha - (\alpha, \epsilon_v)\epsilon_v,$$

for all $v \in I$. A dimension vector $\alpha \in \mathbb{Z}^I_{\geq 0}$ is in the fundamental region if it has connected support and if $(\alpha, \epsilon_v) \leq 0$ for all $v \in I$. That is, none of the coordinates with respect to the basis consisting of the vectors ϵ_v are made smaller by reflections. This is consistent with the definition of the fundamental region given in the previous section and the definition given by Crawley-Boevey in [4].

Note that the fundamental region is part of $-C^\vee$, where $C^\vee$ is the dual to the fundamental chamber for the Weyl group action on the Kac–Moody algebra mentioned above (see [7] for details). Let $q(\alpha)$ be the quadratic form associated to the symmetric form above, and let $p(\alpha) = 1 - q(\alpha)$.

Also note that the vector α can be used to define a product of partial flag varieties

$$Fl(\alpha) = \prod_i Fl(\alpha_0, \alpha_{i1}, \ldots, \alpha_{iw_i}).$$

That is, α_0 is the dimension of the ambient space $\mathbb{C}^{\alpha_0}$, and for a fixed i, each α_{ij} is the dimension of the ith subspace in the flag. The group $\mathrm{PGL}(\alpha_0)$ acts diagonally on $Fl(\alpha)$, so it makes sense to discuss the very good property of the resulting quotient stack. Indeed, when the underlying vector bundle is trivial, Theorem 1 gives us:

Theorem 2. *The quotient stack $PGL(\alpha_0)\backslash Fl(\alpha)$ is very good, if α is in the fundamental region and $\delta(\alpha) > 0$.*

Theorem 2 may also be obtained from Crawley-Boevey's results in [2] (see e.g. Theorem 1.1), after noticing that $Fl(\alpha)$ is the quotient of the space of star-shaped quiver representations of dimension α with injective arrows by the group $H(\alpha) = \prod_{i,j} GL(\alpha_{ij})$, acting by conjugation on the arrows. In this case, the very good property is equivalent to Crawley-Boevey's inequality $p(\alpha) > \sum_i p(\beta_i)$, for any decomposition $\alpha = \sum_i \beta_i$ into the sum of positive roots. The condition that α is in the fundamental region and $\delta(\alpha) \geq 0$ implies this inequality.

4 Applications to the Additive Deligne–Simpson Problem

The original version of the Deligne–Simpson problem was suggested in a letter from Deligne to Simpson, who considered it in his paper [18]. It asks whether there exist k complex $n \times n$ matrices $A_1, \ldots, A_k$ from prescribed conjugacy classes $C_1, \ldots, C_k$ such that $A_1 \cdot A_2 \cdots A_k = \text{Id}$. We will consider this problem's additive analogue, studied by Kostov in [9–15], and Crawley-Boevey in [3], among others.

The additive Deligne–Simpson problem can be formulated in the following way: Given k conjugacy classes $C_1, \ldots, C_k$ of complex matrices in $\mathfrak{gl}_n$, do there exist $A_1 \in C_1, \ldots, A_k \in C_k$ such that

$$A_1 + \cdots + A_k = 0?$$

One can interpret this problem in terms of local systems on $\mathbb{P}^1$ with a collection of k marked points. That is, the additive Deligne–Simpson problem is equivalent to the problem of existence of logarithmic connections on a trivial bundle over $\mathbb{P}^1$ with residues in $C_1, \ldots, C_k$ at the marked points. The multiplicative problem may be obtained by instead considering the monodromy operators, corresponding to loops around each of the marked points, for logarithmic connections on (not necessarily trivial) vector bundles over $\mathbb{P}^1$.

There are several approaches to solving the Deligne–Simpson problem. In [8], Katz describes an algorithm for the existence of rigid local systems, which Kostov applies in [9–15] to determine when solutions to various cases of the Deligne–Simpson problems exist. The algorithm, called the *middle convolution algorithm*, proceeds by changing the rank of the local system by a number δ, called the *defect*, dependent on $C_1, \ldots, C_k$. Solutions exist for the original problem, as long as they exist for the altered rank. This continues while $\delta < 0$ or until one arrives at a situation when solutions cannot exist. When δ becomes nonnegative, a nontrivial existence theorem guarantees that there are solutions. In [3], Crawley-Boevey proposes another approach to the additive Deligne–Simpson problem by examining fibers of the moment map on the cotangent bundle to the space of representations of the star-shaped quiver.

We relate the additive Deligne–Simpson problem for semisimple conjugacy classes $C_1, \ldots, C_k$ to the very good property for $\text{PGL}(\alpha_0)\backslash Fl(\alpha)$. Indeed, given semisimple $C_1, \ldots, C_k$ one can write a dimension vector α, where $\alpha_0 = n$ is the size of the matrix in C_i and α_{ij} is the dimension of the direct sum of the first j

eigenspaces, ordered by dimensions from least to greatest. This defines a product of partial flag varieties $Fl(\alpha)$. As a consequence of Theorem 2, we have:

Theorem 3. *If the conjugacy classes C_i are semisimple, the corresponding quotient stack $PGL(\alpha_0)\backslash Fl(\alpha)$ is very good and the eigenvalues of all the C_i add up to* 0*, then the space of solutions of the additive Deligne–Simpson problem for $C_1, \dots, C_k$ is a nonempty, irreducible variety of dimension $2 \cdot \dim Fl(\alpha) - \alpha_0^2 + 1$.*

Applying Theorem 2 we obtain:

Corollary 1. *If the conjugacy classes C_i are semisimple, α is in the fundamental region and $\delta(\alpha) > 0$, then the space of solutions of the additive Deligne–Simpson problem is a nonempty, irreducible variety of dimension $2 \cdot \dim Fl(\alpha) - \alpha_0^2 + 1$.*

Remark 1. In the above corollary, $\delta(\alpha)$ is actually equal to the defect δ that appears in Katz's middle convolution algorithm.

Remark 2. In [3], the condition $p(\alpha) > \sum_i p(\beta_i)$, for any decomposition $\alpha = \sum_i \beta_i$ into the sum of positive roots (see [3]), is used in place of α being in the fundamental region and $\delta(\alpha) > 0$. While the former is both necessary and sufficient for the existence of solutions to the additive Deligne–Simpson problem, the latter is somewhat easier to check.

5 Further Discussion

We have seen that applying Theorem 1 in the case when the underlying vector bundle is trivial yields a result (Theorem 2) that can be used to study the space of solutions of the additive Deligne–Simpson problem. We hope that in the nontrivial bundle case, the very good property for moduli of parabolic bundles may be used to study solutions to the multiplicative Deligne–Simpson problem. Indeed, in the semisimple case, the product of conjugacy classes $C_1 \times \cdots \times C_k$ may be thought of as a twisted cotangent bundle over $Fl(\alpha)$. The very good property for $Fl(\alpha)$ implies that the moment map for the $\prod_{i=1}^{k} GL(n, \mathbb{C})$ action on $C_1 \times \cdots \times C_k$ (the moment map sends $A_1, \dots, A_k$ to $A_1 + \cdots + A_k$) contains 0 in its image. Furthermore, the very good property implies the fiber over 0 is an irreducible complete intersection.

In general, when the underlying vector bundles is not necessarily trivial, one can also consider twisted cotangent bundles over moduli of parabolic bundles. These twisted cotangent bundles relate to the Deligne–Simpson problems as they parametrize pairs $(\mathbf{E}, \nabla)$, consisting of a parabolic bundle $\mathbf{E}$ and a logarithmic connection ∇, defined on the underlying vector bundle, with residues at the marked points compatible with the parabolic structure. Therefore, we hope that the very good property may be used to obtain similar results for corresponding moment map and, passing to monodromy, for the space of solutions to the multiplicative Deligne–Simpson problem.

It would also be interesting to extend the result of Theorem 1 to other semisimple groups. By analogy with flag varieties, it is possible to define a parabolic structure on G-bundles, when G is not $GL(n, \mathbb{C})$, by specifying parabolic subgroups P_i at each marked point $x_i \in \mathbb{P}^1$. Although there is no correspondence with quiver representations for a general G, it may be possible to modify Beilinson and Drinfeld's original proof of the very good property for Bun_G. A key part of their argument consists of showing that the *global nilpotent cone* $\mathrm{Nilp}(G)$, the fiber over 0 in the Hitchin system, is Lagrangian. One can consider the parabolic analogue of the Hitchin system, which has its own global nilpotent cone. It has been proved to be Lagrangian in specific instances, such as for complete flags [5, 19] or rank 3 [6]. However, the author is unaware of a proof for the case of partial flags.

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