An Introduction to Exterior Differential Systems

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1 Introduction

In a rather precise sense the study of exterior differential systems is equivalent to the study of partial differential equations in the language of differential forms. Although the change of language from partial derivatives to differential forms may appear quite surprising nowadays, the concept developed in a time, when differential forms offered the most convenient way to do calculations in differential geometry without reference to local coordinates. Orthodox differential geometry has caught up in the meantime and this initial advantage has been lost to a large extent. Nevertheless exterior differential systems are still an interesting topic to study today, because they unify language, method and results for several different kinds of partial differential equations.

Studying partial differential equations in the unified framework of exterior differential systems allows us to take advantage of the beautiful theory of Cartan–Kähler about analytical solutions to analytical exterior differential systems, which is the central topic of these notes. In essence the theory of Cartan–Kähler replaces actual solutions to a given exterior differential system by formal power series solutions, an idea already used successfully in the predecessor of the Cartan–Kähler theory, the theorem of Cauchy–Kovalevskaya. Calculating the terms of a formal power series solution term by term reduces a complicated partial differential equation effectively to the problem of solving an inhomogeneous linear equation at each order of differentiation. Although this reduction to linear algebra is very appealing, a rather unpleasant problem arises in this approach: Inhomogeneous linear equations need not have a solution in general. An exterior differential system

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is called formally integrable, if all the inhomogeneous linear equations encountered at different orders of differentiation in calculating a formal power series solution allow solutions.

A partial answer to the problem of verifying formal integrability is given by the Spencer cohomology $H^{\bullet,\circ}(\mathscr{A})$ associated to an exterior differential system. Spencer cohomology tells us that we can always solve the inhomogeneous linear equation at differentiation order k provided the Spencer cohomology space $H^{k,2}(\mathscr{A}) = \{0\}$ vanishes, on the other hand we may still be able to solve the inhomogeneous linear equation at differentiation order k in case $H^{k,2}(\mathscr{A}) \neq \{0\}$. Despite being only a partial answer Spencer cohomology is a very useful tool in practice, because it is usually much easier to calculate the Spencer cohomology of an exterior differential system than to work with formal power series solutions directly. Moreover the algebraic roots of Spencer cohomology in commutative algebra ensure that only a finite number of problematic differentiation orders k exist with $H^{k,2}(\mathscr{A}) \neq \{0\}$.

Among the several excellent text books on the exterior differential systems let us point out the book [1], which can be seen as an authoritative reference on the topic. In writing these introductory notes I wanted to complement [1] with its numerous examples with a concise exposition of the theory of exterior differential systems and its relationship to Spencer cohomology. Moreover I wanted to discuss some of the points in more detail, which are treated rather superficially in the existing literature, say, for example, the distinction between the reduced symbol comodule $\mathscr{A}$ and the symbol comodule $\mathscr{R}$ and the precise definition of the Cartan character of an exterior differential system. In this way I hope that even the reader well acquainted with the Cartan–Kähler theory of exterior differential systems will find these introductory notes worth reading, the more so a reader looking for a panoramic view on the formal theory of partial differential equations.

Grosso modo these notes on exterior differential systems are structured into three essentially independent parts. In Sect. 2 we will construct the contact systems on three different kinds of jet bundles based on the notion of a canonical contact form. Sections 3 and 4 are dedicated to a detailed study of Spencer cohomology: Sect. 3 focuses on its general algebraic properties, whereas Sect. 4 links Spencer cohomology to three classical statements about partial differential equations. Last but not least the theory of Cartan–Kähler is the topic of Sect. 5, in which we will discuss the general setup of exterior differential system and sketch a proof of the Theorem of Cartan–Kähler about the analytical solutions to partial differential equations with analytical coefficients.

2 Jets and Contact Systems

In essence jets and jet bundles are introduced to geometrize differential operators and/or partial differential equations, splitting their study into algebraic and analytical problems. The resulting hybrid approach is the leitmotif of the formal theory of partial differential equations, which becomes the theory of exterior differential systems when formulated in the language of differential forms and exterior calculus. In this initial section we discuss the geometry of jets and jets bundle focusing on the construction of the contact systems on three different kinds of jets, the jets of smooth maps, the jets of sections of fiber bundles and the jets of submanifolds. In Sects. 3 and 4 we will discuss the algebraic aspects of exterior differential systems, before these two strands of the formal theory of partial differential equations are united into the theory of Cartan–Kähler in Sect. 5.

In order to begin our study of jets let us introduce an equivalence relation on the set of smooth maps from $\mathbb{R}^m$ to $\mathbb{R}^n$ by declaring two maps $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ and $g : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ to be in contact $f \sim_{k,x} g$ in a point $x \in \mathbb{R}^m$ to order $k \in \mathbb{N}_0$, if and only if there exists a constant $C > 0$ such that the difference $f - g$ is bounded by the estimate

$$| f(\xi) - g(\xi) | \;\leq\; C\, | \xi - x |^{k+1}$$

for all ξ in a compact neighborhood of x. Apparently this definition depends on the choice of norms $| \cdot |$ on $\mathbb{R}^m$ and $\mathbb{R}^n$ and compact neighborhoods, different choices of norms or neighborhood however only affect the constant, not the existence of the estimate itself. The equivalence class of a given function $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ under contact $\sim_{k,x}$ to the order k is called the kth order jet of f in x written $\mathrm{jet}^k_x f$. According to Taylor's Theorem smooth maps f and g are in contact in x to the order k, if and only if all their partial derivatives

$$\frac{\partial^{|A|} f}{\partial x^A}(x) \;=\; \frac{\partial^{|A|} g}{\partial x^A}(x) \qquad |A| \;\leq\; k \tag{1}$$

up to order k agree in x. In this case there exists a unique polynomial ψ of degree at most k on $\mathbb{R}^m$ with values in $\mathbb{R}^n$, which is in contact to both f and g to the order k in x. Thus the set $\mathrm{Jet}^k_x(\mathbb{R}^m, \mathbb{R}^n)$ of all kth order jets in x of smooth maps from $\mathbb{R}^m$ to $\mathbb{R}^n$ is in bijection

$$\mathrm{Sym}^{\leq k}\mathbb{R}^{m*} \otimes \mathbb{R}^n \;\stackrel{\cong}{\longrightarrow}\; \mathrm{Jet}^k_x(\mathbb{R}^m, \mathbb{R}^n), \qquad \psi \;\longmapsto\; \mathrm{jet}^k_x\, \psi(\cdot - x) \tag{2}$$

with the vector space $\mathrm{Sym}^{\leq k}\mathbb{R}^{m*} \otimes \mathbb{R}^n$ of polynomials of degree less than or equal to k on $\mathbb{R}^m$ with values in $\mathbb{R}^n$. The proper reason for including the translation $\mathrm{jet}^k_x\, \psi(\cdot - x)$ instead of the seemingly simpler $\mathrm{jet}^k_x\, \psi(\cdot)$ is that in this way the jet projections

$$\mathrm{pr} : \quad \mathrm{Jet}^k(\mathbb{R}^m, \mathbb{R}^n) \longrightarrow \mathrm{Jet}^{\tilde{k}}(\mathbb{R}^m, \mathbb{R}^n), \qquad \mathrm{jet}^k_x f \;\longmapsto\; \mathrm{jet}^{\tilde{k}}_x f$$

defined for all $k \geq \tilde{k} \geq 0$ become just the standard projections for polynomials

$$\mathrm{pr} : \quad \mathbb{R}^m \times \mathrm{Sym}^{\leq k}\mathbb{R}^{m*} \otimes \mathbb{R}^n \longrightarrow \mathbb{R}^m \times \mathrm{Sym}^{\leq \tilde{k}}\mathbb{R}^{m*} \otimes \mathbb{R}^n,$$

$$(x, \psi) \longmapsto (x, \operatorname{pr} \psi)$$

forgetting all homogeneous components of ψ of degree larger than $\tilde{k}$. Although the jet projections pr are defined in this way for all $k \geq \tilde{k} \geq 0$, only the two special cases $\tilde{k} = k-1$ and $\tilde{k} = 0$ are of any practical importance. Singling out the latter jet projection in notation, which becomes the evaluation map under the identification $\mathrm{Jet}^0(\mathbb{R}^m, \mathbb{R}^n) = \mathbb{R}^m \times \mathbb{R}^n$

$$\mathrm{ev}: \quad \mathrm{Jet}^k(\mathbb{R}^m, \mathbb{R}^n) \longrightarrow \mathbb{R}^m \times \mathbb{R}^n, \qquad \mathrm{jet}^k_x f \ \longmapsto \ (x, f(x))$$

we may say that the notation pr virtually always refers to the jet projection with $\tilde{k} = k - 1$.

In order to "globalize" the current definition of jets to smooth maps between manifolds it is very convenient to observe that the second definition (1) of kth order contact together with the general chain rule for iterated partial derivatives of compositions provide us with a well-defined jet composition map on the fibered product of two jet spaces

$$\begin{aligned} \mathrm{Jet}^k(\mathbb{R}^m, \mathbb{R}^n) \times_{\mathbb{R}^m} \mathrm{Jet}^k(\mathbb{R}^l, \mathbb{R}^m) &\longrightarrow \mathrm{Jet}^k(\mathbb{R}^l, \mathbb{R}^n), \\ (\mathrm{jet}^k_y f, \mathrm{jet}^k_x g) &\longmapsto \mathrm{jet}^k_x(f \circ g) \end{aligned}$$

provided the source y of $\mathrm{jet}^k_y f$ agrees $y = g(x)$ with the target $g(x)$ of $\mathrm{jet}^k_x g$, for this reason the composition map is only defined on the fibered product $\times_{\mathbb{R}^m}$. Defining the kth order jet of a smooth map $f : M \longrightarrow N$ between manifolds M and N with respect to local coordinates x and y about $p \in M$ and $f(p) \in N$ simply as $\mathrm{jet}^k_{x(p)}(y \circ f \circ x^{-1})$ we obtain

$$\mathrm{jet}^k_{\tilde{x}(p)}(\tilde{y} \circ f \circ \tilde{x}^{-1}) = \mathrm{jet}^k_{y(f(p))}(\tilde{y} \circ y^{-1}) \circ \mathrm{jet}^k_{x(p)}(y \circ f \circ x^{-1}) \circ \mathrm{jet}^k_{\tilde{x}(p)}(x \circ \tilde{x}^{-1})$$

for every other choice $\tilde{x}$ and $\tilde{y}$ of the local coordinates involved. With the jet composition map being well-defined we conclude that for two smooth maps $f : M \longrightarrow N$ and $g : M \longrightarrow N$ satisfying $f(p) = g(p)$ the validity of an equality of jets of the form

$$\mathrm{jet}^k_{x(p)}(y \circ f \circ x^{-1}) \ = \ \mathrm{jet}^k_{x(p)}(y \circ g \circ x^{-1})$$

is independent of the local coordinates x and y of M and N employed in its formulation:

Definition 2.1 (Jets of Smooth Maps). Two smooth maps $f : M \longrightarrow N$ and $g : M \longrightarrow N$ between manifolds M and N are said to be in contact $f \sim_{k,p} g$ in a point $p \in M$ to the order $k \in \mathbb{N}_0$ provided $f(p) = g(p)$ and

$$\mathrm{jet}^k_{x(p)}(y \circ f \circ x^{-1}) \ = \ \mathrm{jet}^k_{x(p)}(y \circ g \circ x^{-1})$$

for some and hence every local coordinates x of M and y of N about p and $f(p) = g(p)$ respectively. The set of all equivalence classes of smooth maps from M to N in contact

$$\mathrm{Jet}^k(\, M,\, N\,) \;:=\; \{\; \mathrm{jet}^k_p f \;\;|\;\; f : M \longrightarrow N \text{ smooth } \}$$

to order k is a fiber bundle over M with projection $\pi : \mathrm{Jet}^k(M, N) \longrightarrow M$, $\mathrm{jet}^k_p f \longmapsto p$, and over $M \times N$ under the evaluation $\mathrm{ev} : \mathrm{Jet}^k(M, N) \longrightarrow M \times N$, $\mathrm{jet}^k_p f \longmapsto (p, f(p))$.

It should be hardly surprising to see that the preceding definition reduces the contact relation for smooth maps between manifolds via a choice of local coordinates to the contact relation for smooth maps between Euclidean spaces, after all the definition of smoothness of maps between manifolds employs local coordinates in exactly the same way. This very observation implies directly that the jet composition map extends to:

$$\begin{aligned} \mathrm{Jet}^k(M, N) \times_M \mathrm{Jet}^k(L, M) &\longrightarrow \mathrm{Jet}^k(L, N), \\ (\mathrm{jet}^k_{g(p)} f, \mathrm{jet}^k_p g) &\longmapsto \mathrm{jet}^k_p(f \circ g) \end{aligned} \tag{3}$$

Using this generalized jet composition map we may define a second kind of jet bundles, the jet bundles of sections of fiber bundles over a manifold M. Consider therefore a smooth fiber bundle $\mathscr{F}M$ over a manifold M with projection $\pi : \mathscr{F}M \longrightarrow M$. The jet composition map induces a well-defined map from the space of jets of smooth maps $M \longrightarrow \mathscr{F}M$ to

$$\mathrm{Jet}^k(\, M,\, \mathscr{F}M\,) \;\longrightarrow\; \mathrm{Jet}^k(\, M,\, M\,), \qquad \mathrm{jet}^k_p f \;\longmapsto\; \mathrm{jet}^k_{f(p)}\pi \circ \mathrm{jet}^k_p f$$

which sends the kth order jet of a section $f \in \Gamma(\mathscr{F}M)$ to $\mathrm{jet}^k_p(\pi \circ f) = \mathrm{jet}^k_p \mathrm{id}_M$. In turn we define the bundle of kth order jets of sections of $\mathscr{F}M$ as the following submanifold

$$\mathrm{Jet}^k \mathscr{F}M := \{\mathrm{jet}^k_p f \;\;|\;\; \mathrm{jet}^k_{f(p)}\pi \circ \mathrm{jet}^k_p f \;=\mathrm{jet}^k_p\, \mathrm{id}_M \;\} \subset \mathrm{Jet}^k(\, M,\, \mathscr{F}M\,)$$

of $\mathrm{Jet}^k(M, \mathscr{F}M)$ with the induced bundle projection $\pi : \mathrm{Jet}^k \mathscr{F}M \longrightarrow M$, $\mathrm{jet}^k_p f \longmapsto p$. The jet projections $\mathrm{pr} : \mathrm{Jet}^k(M, \mathscr{F}M) \longrightarrow \mathrm{Jet}^{\tilde{k}}(M, \mathscr{F}M)$ clearly restrict to jet projections

$$\mathrm{pr} : \quad \mathrm{Jet}^k \mathscr{F}M \;\longrightarrow\; \mathrm{Jet}^{\tilde{k}} \mathscr{F}M, \qquad \mathrm{jet}^k_p f \;\longmapsto\; \mathrm{jet}^{\tilde{k}}_p f$$

for all $k \geq \tilde{k} \geq 0$ with the special case $\mathrm{ev} : \mathrm{Jet}^k \mathscr{F}M \longrightarrow \mathscr{F}M$, $\mathrm{jet}^k_p f \longmapsto f(p)$, singled out in notation to prevent ambiguities as discussed before. The jet projections pr and ev turn the jet bundles $\mathrm{Jet}^k\mathscr{F}M$, $k \geq 0$, associated to $\mathscr{F}M$ into a tower of fiber bundles over M

$$\ldots \xrightarrow{\mathrm{pr}} \mathrm{Jet}^3\mathscr{F}M \xrightarrow{\mathrm{pr}} \mathrm{Jet}^2\mathscr{F}M \xrightarrow{\mathrm{pr}} \mathrm{Jet}^1\mathscr{F}M \xrightarrow{\mathrm{ev}} \mathscr{F}M \xrightarrow{\pi} M \qquad (4)$$

in the sense that every manifold in this tower is a smooth fiber bundle over every manifold further down under the appropriate projection. It should be noted that the two types of jet bundles we have defined so far are very closely related, in fact we could have based all our considerations on the notion of jet bundles of sections. In this approach the jet bundle of smooth maps $M \longrightarrow N$ becomes the bundle $\mathrm{Jet}^k(\,M,\,N\,) := \mathrm{Jet}^k(\,N \times M\,)$ of jets of sections of the trivial N-bundle $N \times M$ over M, clearly a section $n : M \longrightarrow N \times M$ is essentially the same thing as the smooth map $\pi_N \circ n : M \longrightarrow N$.

The geometry of the tower of jets bundles (4) associated to a fiber bundle $\mathscr{F}M$ is governed by the structural property that all the fibers of the jet projection $\mathrm{pr} : \mathrm{Jet}^k\mathscr{F}M \longrightarrow \mathrm{Jet}^{k-1}\mathscr{F}M$ are naturally affine spaces, more precisely the fiber $\mathrm{pr}^{-1}(\mathrm{jet}^{k-1}_p f\,) \subset \mathrm{Jet}^k\mathscr{F}M$ over $\mathrm{jet}^{k-1}_p f$ is an affine space modelled on the vector space $\mathrm{Sym}^k T^*_p M \otimes \mathrm{Vert}_{f(p)}\mathscr{F}M$ for all $k \geq 1$. This additional affine structure is of the utmost importance for the formal theory of partial differential equations, because it reduces non-linear partial differential equations effectively to problems concerning affine linear maps, traditionally called symbol maps, which are significantly easier to deal with. In particular the resulting symbolic calculus allows us to climb up the tower (4) recursively one step at a time like we will do in the proof of the Theorem of Cartan–Kähler in Sect. 5.

Ironically enough the construction of the canonical affine structure on the fibers of the jet projection pr invariably involves a non-canonical choice, nevertheless this ambiguity can be reduced significantly by using the concept of anchored coordinate charts. A coordinate chart of a smooth manifold M anchored in a point $p \in M$ is a smooth map $\Phi^M : T_pM \longrightarrow M$ defined at least in some open neighborhood of $0 \in T_pM$ such that $\Phi^M(\,0\,) = p$ and such that the differential of Φ^M in the point $0 \in T_pM$ agrees with the identity of T_pM:

$$\Phi^M_{*,p} : \quad T_pM \;\cong\; T_0(\,T_pM\,) \longrightarrow T_pM, \qquad X \longmapsto \frac{d}{dt}\bigg|_0 \Phi^M(\,tX\,) \;\stackrel{!}{=}\; X \qquad (5)$$

Evidently the concept of coordinate charts of a manifold M anchored in a point $p \in M$ reflects the basic properties of the standard exponential maps studied in affine differential geometry. For every smooth fiber bundle $\pi : \mathscr{F}M \longrightarrow M$ over a manifold M and for every given $f_0 \in \mathscr{F}_{p_0}M$ in the fiber over a point $p_0 \in M$ we can easily find anchored coordinate charts $\Phi^M : T_{p_0}M \longrightarrow M$ and $\Phi^{\mathscr{F}} : T_{f_0}\mathscr{F}M \longrightarrow \mathscr{F}M$ anchored in p_0 and f_0 such that

$$\begin{array}{ccc} T_{f_0}\mathscr{F}M & \xrightarrow{\Phi^{\mathscr{F}}} & \mathscr{F}M \\ {\scriptstyle \pi_{*,\,f_0}}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ T_{p_0}M & \xrightarrow{\Phi^{M}} & M \end{array} \tag{6}$$

commutes wherever defined. Using such a pair of anchored coordinate charts we define

$$\mathrm{jet}^k_{p_0} f \;+\; \Delta f \;\;:=\;\; \mathrm{jet}^k_{p_0}\Big(p \;\longmapsto\; \Phi^{\mathscr{F}}\Big[\; \Phi^{\mathscr{F}^{-1}}(\, f(p)\,) \;+\; \Delta f\,(\,\Phi^{M^{-1}}(\,p\,)\,)\,\Big]\Big) \tag{7}$$

for all $\mathrm{jet}^k_{p_0} f \in \mathrm{Jet}^k_{p_0}\mathscr{F}M$ evaluating to $f(p_0) = f_0$ and all $\Delta f \in \mathrm{Sym}^k T^*_{p_0}M \otimes \mathrm{Vert}_{f_0}\mathscr{F}M$ considered as homogeneous polynomials of degree k on $T_{p_0}M$ with values in the subspace $\mathrm{Vert}_{f_0}\mathscr{F}M \subset T_{f_0}\mathscr{F}M$. The commutativity of the diagram (6) ensures that the expression

$$M \;\longrightarrow\; \mathscr{F}M, \qquad p \;\longmapsto\; \Phi^{\mathscr{F}}\Big[\; \Phi^{\mathscr{F}^{-1}}(\, f(p)\,) \;+\; \Delta f\,(\,\Phi^{M^{-1}}(\,p\,)\,)\,\Big]$$

results in a locally defined section of $\mathscr{F}M$. Needless to say this section depends on the pair of anchored coordinate charts $\Phi^{\mathscr{F}}$ and Φ^{M} used in its definition. Different choices for $\Phi^{\mathscr{F}}$ and Φ^{M} however will always lead to local sections in contact in the point $p_0 \in M$ up to order k, because Δf considered as a homogeneous polynomial of degree k on $T_{p_0}M$ has all its partial derivatives of order less than k vanishing in $0 \in T_{p_0}M$.

Taking partial derivatives the first time converts a composition like $\Phi^{\mathscr{F}} \circ \Delta f \circ \Phi^{M^{-1}}$ into a sum of products of partial derivatives of $\Phi^{\mathscr{F}}$, Δf and Φ^{M}, all subsequent partial derivatives are then calculated using the Leibniz rule for products. Hence all partial derivatives of the composition $\Phi^{\mathscr{F}} \circ \Delta f \circ \Phi^{M^{-1}}$ of order less than k in p_0 vanish and the only the non-zero contributions to partial derivatives of order k arise from choosing the critical factor Δf in *all* subsequent applications of the Leibniz rule. The net result is a sum of products of partial derivatives of Δf of order k in $0 \in T_{p_0}M$ with only first order partial derivatives of $\Phi^{\mathscr{F}}$ and Φ^{M} in $0 \in T_{f_0}\mathscr{F}M$ and $0 \in T_{p_0}M$. Exactly these first order derivatives however are fixed by the characteristic property (5) of anchored coordinate charts!

Certainly a lot of work needs to be done to make the argument sketched in the preceding paragraph precise, nevertheless we skip this problem for the time being and conclude that the addition (7) does not dependent on the choice of the pair of anchored coordinate charts $\Phi^{\mathscr{F}}$ and Φ^{M} used in its definition. Moreover the addition satisfies the axioms of a group action for the additive group underlying the vector space $\mathrm{Sym}^k T^*_{p_0}M \otimes \mathrm{Vert}_{f_0}\mathscr{F}M$:

$$\mathrm{jet}^k_{p_0} f + 0 \;=\; \mathrm{jet}^k_{p_0} f \qquad (\,\mathrm{jet}^k_{p_0} f + \Delta f\,) + \Delta\tilde{f} \;=\; \mathrm{jet}^k_{p_0} f + (\,\Delta f + \Delta\tilde{f}\,)$$

Both verifications necessary are essentially trivial, but involve rather bombastic formulas better omitted. Summarizing all our considerations on this topic we have constructed for all $k \geq 1$ a canonical vector group bundle action on $\mathrm{Jet}^k \mathscr{F}M$ fibered over $\mathscr{F}M$

$$+ : \quad \mathrm{Jet}^k \mathscr{F}M \ \times_{\mathscr{F}M} \ (\,\mathrm{Sym}^k T^*M \otimes \mathrm{Vert}\,\mathscr{F}M\,) \ \longrightarrow \ \mathrm{Jet}^k \mathscr{F}M$$

in the sense that the fiber of the vector bundle over $f_0 \in \mathscr{F}M$ acts on subset of jets evaluating to f_0. In the standard jet coordinates on $\mathrm{Jet}^k \mathscr{F}M$ introduced later on it is relatively easy to verify that the addition $+$ is a natural affine space structure on the fibers of the jet projection $\mathrm{pr} : \mathrm{Jet}^k \mathscr{F}M \longrightarrow \mathrm{Jet}^{k-1} \mathscr{F}M$ in the sense that it acts simply transitively on each fiber.

Without doubt the most important use of jets is to provide us with a concise definition of the intuitive notions of (non-linear) differential operators and partial differential equations. In particular the geometrization of partial differential equations brought about by jets can be used to reduce all possible kinds of partial differential equations to a single standard normal form, namely an exterior differential system. Before discussing this point let us point out very briefly that the jet bundles $\mathrm{Jet}^k FM$ of a vector bundle FM over a manifold M are naturally vector bundles again under the obvious choice of scalar multiplication and addition

$$\lambda\,\mathrm{jet}^k_p f \ := \ \mathrm{jet}^k_p(\,\lambda f\,) \qquad\qquad \mathrm{jet}^k_p f_1 \ + \ \mathrm{jet}^k_p f_2 \ := \ \mathrm{jet}^k_p(\,f_1 + f_2\,)$$

moreover the jet bundles $\mathrm{Jet}^k FM$ are also bundles of free modules over the algebra bundle $\mathrm{Jet}^k \mathbb{R}M$ of smooth k-jets of functions. Last but not least the vector bundle structure on $\mathrm{Jet}^k FM$ can be used for an alternative construction of the affine space structure on the fibers of the jet projection $\mathrm{pr} : \mathrm{Jet}^k FM \longrightarrow \mathrm{Jet}^{k-1} FM$. In this alternative construction the addition (7) is mediated by a canonical inclusion of vector bundles

$$\iota : \quad \mathrm{Sym}^k T^*M \otimes FM \ \longrightarrow \ \mathrm{Jet}^k FM, \qquad \psi_p \otimes f_p \ \longrightarrow \ \mathrm{jet}^k_p(\,\psi\, f\,) \tag{8}$$

so that $\mathrm{jet}^k_p f \, + \, \Delta f$ can be interpreted simply as the sum of $\mathrm{jet}^k_p f$ and $\iota(\,\Delta f\,)$ in the vector space $\mathrm{Jet}^k_p FM$, note that $\mathrm{Vert}_{f_0} FM \cong F_{p_0} M$ are canonically isomorphic for a vector bundle.

Definition 2.2 (Non-linear Partial Differential Equations). A smooth non-linear differential operator of order $k \geq 0$ from sections of a fiber bundle $\mathscr{F}M$ over M to sections of another fiber bundle $\mathscr{E}M$ is a map $D : \Gamma(\,\mathscr{F}M\,) \longrightarrow \Gamma(\,\mathscr{E}M\,)$ between the sets of locally defined sections such that the value $(\,Df\,)(\,p\,) \in \mathscr{E}_p M$ of the image of $f \in \Gamma(\,\mathscr{F}M\,)$ in a point $p \in M$ depends only on $\mathrm{jet}^k_p f \in \mathrm{Jet}^k_p \mathscr{F}M$. In particular D induces a well-defined smooth map of fiber bundles over M called the total symbol of D:

$$\sigma^{\mathrm{total}}_D : \quad \mathrm{Jet}^k \mathscr{F}M \ \longrightarrow \ \mathscr{E}M, \qquad \mathrm{jet}^k_p f \ \longmapsto \ (\,D\,f\,)(\,p\,)$$

A non-linear partial differential equation for local sections $f \in \Gamma(\mathscr{F}M)$ is an equation of the form $Df = *$ with a distinguished global section $* \in \Gamma(\mathscr{E}M)$ of the target bundle.

This notion of non-linear partial differential equations may not be the most general one, nevertheless it is sufficiently ample to illustrate the use of jets and comprises the important subclass of linear partial differential equations. Naturally enough a differential operator D of order $k \geq 0$ is called a linear differential operator provided both the source and target bundle FM and EM are vector bundles over M and $D : \Gamma(FM) \longrightarrow \Gamma(EM)$ is an $\mathbb{R}$-linear map between the vector spaces of sections, equivalently we may ask for its total symbol $\sigma_D^{\text{total}} : \text{Jet}^k FM \longrightarrow EM$ to be a homomorphism of vector bundles over M. A linear partial differential equation for sections $f \in \Gamma(FM)$ is in turn an equation of the form $D\,f = 0$ with a linear operator and the distinguished zero section $0 \in \Gamma(EM)$. Given now a partial differential equation $Df = *$ of order $k \geq 1$ we may "solve" the equation algebraically

$$\text{Eq}^k_p M \;:=\; \{\ \text{jet}^k_p f \in \text{Jet}^k_p \mathscr{F}M \ \mid\ \sigma_D^{\text{total}}(\,\text{jet}^k_p f\,) \;=\; *_p\ \} \;\subset\; \text{Jet}^k_p \mathscr{F}M$$

in terms of jets in every point $p \in M$. Evidently a local section $f \in \Gamma(\mathscr{F}M)$ is a solution to the partial differential equation $Df = *$, if and only if its image under the jet operator

$$\text{jet}^k :\quad \Gamma(\mathscr{F}M) \longrightarrow \Gamma(\text{Jet}^k \mathscr{F}M), \qquad f \longmapsto \Big(\, p \longmapsto \text{jet}^k_p f \,\Big) \tag{9}$$

takes values $\text{jet}^k_p f \in \text{Eq}^k_p M$ in every point $p \in M$. This observation motivates the minimal regularity assumption imposed in the formal theory of partial differential equations, namely we require that the family $\{\text{Eq}^k_p M\}_{p \in M}$ of subsets of $\text{Jet}^k \mathscr{F}M$ assembles into a subbundle $\text{Eq}^k M \subset \text{Jet}^k \mathscr{F}M$. A partial differential equation failing to satisfy this minimal regularity assumption is outside the scope of the formal theory and has to be treated differently.

Solving a partial differential equation algebraically in every point $p \in M$ introduces the concept of formal solutions into the picture, sections of the fiber bundle $\text{Eq}^k M \subset \text{Jet}^k \mathscr{F}M$. Only those formal solutions $f^k \in \Gamma(\text{Eq}^k M)$ though, which are in the image of the jet operator, correspond to actual solutions $f \in \Gamma(\mathscr{F}M)$. Hence it makes sense to distinguish sections in the image of the jet operator (9) from arbitrary sections of $\text{Eq}^k M$ and call them holonomic sections. In consequence the original partial differential equation $Df = *$ has been reformulated into the problem to find all holonomic sections of $\text{Eq}^k M$. In passing we remark that the question, whether every section of $\text{Eq}^k M$ is fiberwise homotopic to a holonomic section or not, has sparked intensive research on Gromov's h-principle [4].

Interestingly the holonomic sections $\text{jet}^k f \in \Gamma(\text{Jet}^k \mathscr{F}M)$ of a jet bundle are exactly those sections of $\text{Jet}^k \mathscr{F}M$, which satisfy a particular first order partial differential constraint, the contact constraint. In the setup of exterior differential systems this contact constraint is formulated in terms of a canonical 1-form specific to jet bundles, the canonical contact form. Restricting this canonical contact form to the subbundle $\mathrm{Eq}^k M \subset \mathrm{Jet}^k \mathscr{F} M$ of algebraic pointwise solutions of a partial differential equation $Df = *$ induces an exterior differential system on the manifold $\mathrm{Eq}^k M$, whose solutions correspond bijectively to local solutions $f \in \Gamma(\mathscr{F} M)$ of the original partial differential equation $Df = *$. In this way every partial differential equation satisfying the minimal regularity assumption is transformed into an equivalent exterior differential system.

In order to construct the canonical contact form $\gamma^{\mathrm{contact}}$ on the jet bundle $\mathrm{Jet}^k \mathscr{F} M$ of sections of a fiber bundle $\mathscr{F} M$ we remark that every smooth curve $c : \mathbb{R} \longrightarrow \mathrm{Jet}^k \mathscr{F} M$ in the total space of a jet bundle can be written in the form $c(t) = \mathrm{jet}^k_{p_t} f_t$ with smooth curves $t \longmapsto p_t$ in the base M and a curve $t \longrightarrow f_t$ in $\Gamma(\mathscr{F} M)$. Anticipating a Leibniz rule for such combined curves we can decompose every vector tangent to $\mathrm{Jet}^k \mathscr{F} M$ into two parts:

$$\frac{d}{dt}\bigg|_0 \mathrm{jet}^k_{p_t} f_t \;=\; \frac{d}{dt}\bigg|_0 \mathrm{jet}^k_{p_t} f_0 \;+\; \frac{d}{dt}\bigg|_0 \mathrm{jet}^k_{p_0} f_t \tag{10}$$

Although this formula is entirely correct the decomposition on the right depends on the specific representation of the given tangent vector on the left as a combination of a curve $t \longmapsto p_t$ in the base and a curve $t \longmapsto f_t$ in the local sections of $\mathscr{F} M$. Essentially the problem is that the first summand picks up partial derivatives of order $k + 1$ of f_0 in form of the partial derivatives of $\mathrm{jet}^k f_0$ in the direction of $\frac{d}{dt}\big|_0 p_t$. This problem is easily overcome using the jet projection pr and so we can define the contact form $\gamma^{\mathrm{contact}}$ on $\mathrm{Jet}^k \mathscr{F} M$ via:

$$\gamma^{\mathrm{contact}}\Big(\frac{d}{dt}\bigg|_0 \mathrm{jet}^k_{p_t} f_t \Big) \;:=\; \frac{d}{dt}\bigg|_0 \mathrm{jet}^{k-1}_{p_0} f_t \;\in\; \mathrm{Vert}_{\mathrm{jet}^{k-1}_{p_0} f_0} \mathrm{Jet}^{k-1} \mathscr{F} M$$

En nuce the contact form tells us, whether or not we are forced to change the local section f_0 in order to reproduce a given vector tangent to $\mathrm{Jet}^k \mathscr{F} M$. Thus every holonomic section $\mathrm{jet}^k f : M \longrightarrow \mathrm{Jet}^k \mathscr{F} M,\ p \longmapsto \mathrm{jet}^k_p f$, with $f \in \Gamma(\mathscr{F} M)$ pulls back the contact form to:

$$\begin{aligned} \Big((\mathrm{jet}^k f)^* \gamma^{\mathrm{contact}} \Big) \Big(\frac{d}{dt}\bigg|_0 p_t \Big) &:= \gamma^{\mathrm{contact}} \Big(\frac{d}{dt}\bigg|_0 \mathrm{jet}^k_{p_t} f \Big) \\ &= \frac{d}{dt}\bigg|_0 \mathrm{jet}^{k-1}_{p_0} f = 0 \end{aligned} \tag{11}$$

In order to simplify this characterization of holonomic sections $\mathrm{jet}^k f \in \Gamma(\mathrm{Jet}^k \mathscr{F} M)$ of the jet bundle $\mathrm{Jet}^k \mathscr{F} M$ it is convenient to replace the contact form $\gamma^{\mathrm{contact}}$, which is a 1-form on $\mathrm{Jet}^k \mathscr{F} M$ with values in the slightly unwieldy vector bundle $\mathrm{pr}^*(\mathrm{Vert}\,\mathrm{Jet}^{k-1} \mathscr{F} M)$, by its scalar components aka local sections of the contact subbundle of $T^*\mathrm{Jet}^k \mathscr{F} M$ defined by:

$$\mathrm{Contact\,Jet}^k \mathscr{F} M$$
$$:= \mathbf{im}\Big(\mathrm{pr}^*(\,\mathrm{Vert}^*\mathrm{Jet}^{k-1}\mathscr{F} M\,) \longrightarrow T^*\mathrm{Jet}^k\mathscr{F} M, \;\; \eta \longmapsto \langle\, \eta, \, \gamma^{\mathrm{contact}}\,\rangle \Big)$$

Every contact form $\gamma \in \Gamma(\,\mathrm{Contact\,Jet}^k \mathscr{F} M\,)$ is actually horizontal for the projection pr to $\mathrm{Jet}^{k-1}\mathscr{F} M$, because every pr-vertical tangent vector is π-vertical as well and thus has a presentation $\frac{d}{dt}\big|_0 \mathrm{jet}^k_p f_t$, in which the base point $p \in M$ does not vary. A fortiori we get:

$$\gamma^{\mathrm{contact}}\Big(\frac{d}{dt}\Big|_0 \mathrm{jet}^k_p f_t \Big) \;=\; \frac{d}{dt}\Big|_0 \mathrm{jet}^{k-1}_p f_t \;=\; \mathrm{pr}_*\Big(\frac{d}{dt}\Big|_0 \mathrm{jet}^k_p f_t \Big) \;=\; 0$$

The horizontality of contact forms allows us to define the contact system on $\mathrm{Jet}^k \mathscr{F} M$ as the following sequence of vector subbundles of the cotangent bundle of the jet bundle $\mathrm{Jet}^k \mathscr{F} M$

$$\mathrm{Contact\ Jet}^k \mathscr{F} M \;\subseteq\; \mathrm{Horizontal\ Jet}^k \mathscr{F} M \;\subseteq\; T^*\mathrm{Jet}^k \mathscr{F} M \tag{12}$$

where $\mathrm{Horizontal\,Jet}^k \mathscr{F} M$ denotes the subbundle of horizontal 1-forms with respect to pr.

The preceding calculations offer a good insight into the geometry of the contact system, nevertheless some readers will certainly prefer a more down to earth approach vindicating our findings explicitly in local coordinates on jet bundles. For the time being we will restrict to the jet bundles $\mathrm{Jet}^k(\,M, \,\mathscr{F}\,)$ of smooth maps from a manifold M to a manifold $\mathscr{F}$, in any case the difference between $\mathrm{Jet}^k(\,M, \,\mathscr{F}\,)$ and $\mathrm{Jet}^k \mathscr{F} M$ virtually disappears in local coordinates for a fiber bundle $\mathscr{F} M$ over M with model fiber $\mathscr{F}$. Choosing local coordinates (x, U) on M and (f, V) on $\mathscr{F}$ we may then define local coordinates on the subset

$$(\,\mathrm{Jet}^k(\,M, \,\mathscr{F}\,)\,)_{(x,U),\,(f,V)} \;:=\; \{\, \mathrm{jet}^k_p f \;\;|\;\; p \in U \text{ and } f(p) \in V \,\}$$

of kth order jets of maps $f : M \longrightarrow \mathscr{F}$ with source in U and target in V by setting

$$x^\alpha(\,\mathrm{jet}^k_p f\,) \;:=\; x^\alpha(\,p\,) \qquad\qquad f^\lambda_A(\,\mathrm{jet}^k_p f\,) \;:=\; \frac{\partial^{|A|} f^\lambda}{\partial x^A}(\,x^1(p), \ldots, x^m(p)\,)$$

for all $\alpha = 1, \ldots, m$, $\lambda = 1, \ldots, n$ and all multi-indices A on $\{1, \ldots, m\}$ of order $|\,A\,| \leq k$. The standard jet coordinates constructed in this way from smooth atlases for both M and $\mathscr{F}$ define a smooth atlas for $\mathrm{Jet}^k(\,M, \,\mathscr{F}\,)$ turning it into a smooth manifold of dimension:

$$\dim\ \mathrm{Jet}^k(\,M, \,\mathscr{F}\,) \;=\; m + n \binom{m+k}{m}$$

Standard jet coordinates are adapted to the projections $\mathrm{pr} : \mathrm{Jet}^k(M, \mathscr{F}) \longrightarrow \mathrm{Jet}^{\tilde{k}}(M, \mathscr{F})$ for all $k \geq \tilde{k} \geq 0$ in the sense that pr simply forgets all the coordinate functions f^λ_A with $|A| > \tilde{k}$, this observation proves explicitly that (4) really is the stipulated tower of smooth fiber bundles over M. Moreover standard jet coordinates on $\mathrm{Jet}^k(M, \mathscr{F})$ allow us to decompose every tangent vector $\frac{d}{dt}\big|_0 \mathrm{jet}^k_{p_t} f_t$ in the way predicted by Leibniz's rule

$$
\begin{aligned}
&\frac{d}{dt}\bigg|_0 \mathrm{jet}^k_{p_t} f_t \\
&= \sum_{\alpha=1}^m \Big[\frac{d}{dt}\bigg|_0 x^\alpha(\mathrm{jet}^k_{p_t} f_t) \Big] \frac{\partial}{\partial x^\alpha} + \sum_{\substack{|A|\leq k\\ \lambda}} \Big[\frac{d}{dt}\bigg|_0 f^\lambda_A(\mathrm{jet}^k_{p_t} f_t) \Big] \frac{\partial}{\partial f^\lambda_A} \\
&= \sum_{\alpha=1}^m \delta x^\alpha \Big[\frac{\partial}{\partial x^\alpha} + \sum_{\substack{|A|\leq k\\ \lambda}} \Big(\frac{\partial}{\partial x^\alpha} \frac{\partial^{|A|} f^\lambda_0}{\partial x^A} \Big)(x(p_0)) \frac{\partial}{\partial f^\lambda_A} \Big] + \sum_{\substack{|A|\leq k\\ \lambda}} \delta f^\lambda_A \frac{\partial}{\partial f^\lambda_A}
\end{aligned}
$$

where $\delta x^\alpha := \frac{d}{dt}\big|_0 x^\alpha(p_t)$ and $\delta f^\lambda_A := \frac{d}{dt}\big|_0 \frac{\partial^A f^\lambda_t}{\partial x^A}(x(p_0))$. Evaluating $\frac{d}{dt}\big|_0 \frac{\partial^{|A|} f^\lambda_t}{\partial x^A}(x(p_t))$ for a multi-index A of highest order $|A| = k$ we pick up derivatives of f^λ_0 of order $k+1$ as anticipated above, albeit only in the coefficients of the basis vector $\frac{\partial}{\partial f^\lambda_A}$ associated to A. For all multi-indices A of order $|A| < k$ on the other hand the value of the partial derivative $(\frac{\partial}{\partial x^\alpha} \frac{\partial^{|A|+1} f^\lambda_0}{\partial x^{A+\alpha}})(p_0)$ equals $f^\lambda_{A+\alpha}(\mathrm{jet}^k_{p_0} f_0)$ by construction and so we obtain eventually:

$$
\begin{aligned}
\mathrm{pr}_*\Big(\frac{d}{dt}\bigg|_0 \mathrm{jet}^k_{p_t} f_t \Big) &= \sum_{\alpha=1}^m \delta x^\alpha \Big(\frac{\partial}{\partial x^\alpha} + \sum_{\substack{|A|<k\\ \lambda}} f^\lambda_{A+\alpha}(\mathrm{jet}^k_{p_0} f_0) \frac{\partial}{\partial f^\lambda_A} \Big) \\
&\quad + \sum_{\substack{|A|<k\\ \lambda}} \delta f^\lambda_A \frac{\partial}{\partial f^\lambda_A}
\end{aligned}
$$

Evidently the first part in this decomposition comes from the variation $\delta p := \frac{d}{dt}\big|_0 p_t$ of the point $p_0 \in M$, while the second part is caused by the variation $\delta f := \frac{d}{dt}\big|_0 \mathrm{jet}^k_{p_0} f_t$ of the kth order jet of the smooth map $f_0 : M \longrightarrow \mathscr{F}$. In this decomposition the canonical contact form $\gamma^{\mathrm{contact}}$ on $\mathrm{Jet}^k(M, \mathscr{F})$ is simply the projection to the second part, so we conclude:

$$
\gamma^{\mathrm{contact}}\Big(\frac{\partial}{\partial x^\alpha} \Big) = - \sum_{\substack{|A|<k\\ \lambda}} f^\lambda_{A+\alpha} \frac{\partial}{\partial f^\lambda_A} \qquad \gamma^{\mathrm{contact}}\Big(\frac{\partial}{\partial f^\lambda_A} \Big) = + \, \delta_{|A|<k} \frac{\partial}{\partial f^\lambda_A}
$$

More succinctly this explicit version of the canonical contact form γ^{contact} reads:

$$\gamma^{\text{contact}} = \sum_{\substack{|A|<k \\ \lambda}} \underbrace{\left(df^\lambda_A - \sum_{\alpha=1}^m f^\lambda_{A+\alpha}\, dx^\alpha \right)}_{=:\ \gamma^\lambda_A} \otimes \frac{\partial}{\partial f^\lambda_A} = \sum_{\substack{|A|<k \\ \lambda}} \gamma^\lambda_A \otimes \frac{\partial}{\partial f^\lambda_A}$$

In standard jet coordinates the contact system on $\mathrm{Jet}^k(M, \mathscr{F})$ can thus be written

$$\begin{aligned} \text{Contact } \mathrm{Jet}^k(M, \mathscr{F}) &:= \operatorname{span}\{\ \gamma^\lambda_A \mid \text{for all } \lambda,\ |A| < k\ \} \\ \text{Horizontal } \mathrm{Jet}^k(M, \mathscr{F}) &:= \operatorname{span}\{\ dx^\alpha,\ df^\lambda_A \mid \text{for all } \alpha,\ \lambda,\ |A| < k\ \} \end{aligned} \tag{13}$$

because $\text{Contact}\,\mathrm{Jet}^k(M, \mathscr{F})$ and $\text{Horizontal}\,\mathrm{Jet}^k(M, \mathscr{F})$ are respectively the subbundles of scalar components of γ^{contact} and of horizontal forms with respect to pr. For the calculations to come it is important to observe that the exterior derivative of the scalar contact form γ^λ_A

$$\begin{aligned} d\gamma^\lambda_A &= -\sum_{\alpha=1}^m df^\lambda_{A+\alpha} \wedge dx^\alpha \\ &= -\sum_{\alpha=1}^m \left(\gamma^\lambda_{A+\alpha} + \sum_{\tilde\alpha=1}^m f^\lambda_{A+\alpha+\tilde\alpha}\, dx^{\tilde\alpha} \right) \wedge dx^\alpha \stackrel{!}{=} -\sum_{\alpha=1}^m \gamma^\lambda_{A+\alpha} \wedge dx^\alpha \end{aligned}$$

with a multi-index A of order $|A| < k-1$ lies in the ideal generated by all the scalar contact forms taken together. This is no longer true for multi-indices A of highest order $|A| = k-1$, but at least $d\gamma^\lambda_A = -\sum_\alpha df^\lambda_{A+\alpha} \wedge dx^\alpha$ is an element of the ideal generated by horizontal forms. In other words the contact system satisfies the characteristic compatibility condition:

$$d: \quad \Gamma(\text{Contact}\,\mathrm{Jet}^k(M,\mathscr{F})) \longrightarrow \Gamma(\text{Horizontal}\,\mathrm{Jet}^k(M,\mathscr{F}) \wedge T^*\mathrm{Jet}^k(M,\mathscr{F})) \tag{14}$$

An illustrative example for the contact system, whose axiomatization has become a topic of research by itself under the keyword contact manifolds, is the first order jet bundle $\mathrm{Jet}^1\mathbb{R}M$ of the trivial real line bundle $\mathbb{R}M := \mathbb{R} \times M$ over M, whose sections correspond to smooth functions $f : M \longrightarrow \mathbb{R}$. Rather atypically this bundle splits into the Cartesian product

$$\mathrm{Jet}^1\mathbb{R}M \xrightarrow{\cong} \mathbb{R} \times T^*M, \qquad \mathrm{jet}^1_p f \longmapsto (\ f(p),\ d_p f\ ) \tag{15}$$

which identifies the cotangent bundle T^*M of the manifold M with the pointed jet bundle $T^*M :=^* \mathrm{Jet}^1\mathbb{R}M$ of first order jets of functions evaluating to zero $\mathrm{ev}(\mathrm{jet}^1_p f) = (0,\ p)$ in $\mathrm{Jet}^0\mathbb{R}M = \mathbb{R} \times M$. From this point of view it is natural to define the higher order cotangent bundle as the bundle of pointed kth order jets of functions, compare for example [8]:

$$T^{*k}M = {}^*\mathrm{Jet}^k\mathbb{R}M := \{\ \mathrm{jet}^k_p f \mid f : M \longrightarrow \mathbb{R} \text{ smooth and } f(p) = 0\ \}$$

More important for our present purpose is that the vertical tangent bundle of $\mathrm{Jet}^0\mathbb{R}M$ is canonically the trivial line bundle $\mathrm{Vert}\,(\mathbb{R}\times M\,) = (\,T\,\mathbb{R}\,)\times M$ over $\mathbb{R}\times M$ due to $T\,\mathbb{R}\cong\mathbb{R}\times\mathbb{R}$, the contact form thus becomes a scalar valued differential form on $\mathrm{Jet}^1\mathbb{R}M$:

$$\gamma^{\mathrm{contact}}\Big(\,\frac{d}{dt}\Big|_0 \mathrm{jet}^1_{p_t} f_t\,\Big) \;=\; \frac{d}{dt}\Big|_0 \mathrm{jet}^0_{p_0} f_t \;\hat{=}\; \frac{d}{dt}\Big|_0 f_t(\,p_0\,)$$

Comparing this expression with the differential of the tautological function on $\mathrm{Jet}^1\mathbb{R}M$, which is just the projection $f^{\mathrm{taut}}(\mathrm{jet}^1_p f\,) := f(\,p\,)$ to the first factor in decomposition (15)

$$df^{\mathrm{taut}}\Big(\,\frac{d}{dt}\Big|_0 \mathrm{jet}^1_{p_t} f_t\,\Big) \;=\; \frac{d}{dt}\Big|_0 f_t(\,p_t\,) \;=\; \frac{d}{dt}\Big|_0 f_t(\,p_0\,) \;+\; d_{p_0} f_0\Big(\,\frac{d}{dt}\Big|_0 p_t\,\Big)$$

we conclude that the contact form comprises $\gamma^{\mathrm{contact}} = df^{\mathrm{taut}} - \mathrm{pr}^*_{T^*M}\theta$ both the differential of the tautological function $f^{\mathrm{taut}} \in C^\infty(\mathrm{Jet}^1\mathbb{R}M\,)$ and the tautological 1-form θ on T^*M. Correspondingly we get in standard jet coordinates $(\,x^1,\ldots,x^m,\ f,\ f_1,\ldots,f_m\,)$ on $\mathrm{Jet}^1\mathbb{R}M$ the classical expression for contact forms in Darboux coordinates for contact manifolds:

$$\gamma^{\mathrm{contact}} \;=\; df \;-\; \sum_{\mu=1}^{m} f_\mu\, dx^\mu$$

Besides higher order cotangent bundles we can also define higher order tangent bundles:

Definition 2.3 (Higher Order Tangent Bundles). Recalling the definition of the tangent bundle TM of a manifold M as the set $\mathrm{Jet}^1_0(\mathbb{R},\ M\,)$ of equivalence classes of smooth curves $c:\ \mathbb{R}\longrightarrow M$ under the relation of first order contact in $0\in\mathbb{R}$ we define the kth order tangent bundle as the set of equivalence classes of curves

$$T^kM \;:=\; \mathrm{Jet}^k_0(\,\mathbb{R},\ M\,)$$

under kth order contact in 0 with projection $\mathrm{Jet}^k_0(\mathbb{R},\ M\,)\longrightarrow M$, $\mathrm{jet}^k_0 c =: \frac{d^{\leq k}}{dt^{\leq k}}\Big|_0 c \longmapsto c(0)$.

In difference to the classical tangent bundle $TM = T^1M$ the higher order tangent bundles T^kM with $k>1$ do not carry a natural vector bundle structure. The proper way to think of this problem is to consider the canonical embedding $\Phi:\ T^kM\longrightarrow \mathrm{Hom}\,(\,^*\mathrm{Jet}^k\mathbb{R}M,\ \mathbb{R}^kM\,)$ of T^kM into the bundle $\mathrm{Hom}\,(\,^*\mathrm{Jet}^k\mathbb{R}M,\ \mathbb{R}^kM\,)$ of linear maps from the kth order cotangent bundle $T^{*k}M =^* \mathrm{Jet}^k\mathbb{R}M$ to the trivial vector bundle $\mathbb{R}^kM$ with fiber $\mathbb{R}^k$ defined by:

$$\Phi\Big[\,\frac{d^{\leq k}}{dt^{\leq k}}\Big|_0 c\,\Big](\,\mathrm{jet}^k_{c(0)} f\,) \;:=\; \Big(\,\frac{d^1}{dt^1}\Big|_0 (f\circ c),\ \frac{d^2}{dt^2}\Big|_0 (f\circ c),\ldots,\ \frac{d^k}{dt^k}\Big|_0 (f\circ c)\Big)$$

In the special case $k = 1$ the canonical embedding Φ is of course a version of the canonical pairing $TM \times_M T^*M \longrightarrow \mathbb{R}M$ between the tangent and cotangent bundles TM and T^*M, as such it induces an isomorphism of fiber bundles. In general however the embedding Φ looks in suitable coordinates on T^kM and $\mathrm{Hom}(\,^*\mathrm{Jet}^k\mathbb{R}M,\, \mathbb{R}^kM\,)$ like the polynomial map

$$(\,t_1, \ldots, t_k\,) \longmapsto (\,t_1,\, t_1^2 + t_2,\, t_1^3 + 2\,t_1\,t_2 + t_3,\, \ldots\,)$$

from the set $(T_pM)^k$ of k-tuples of vectors in T_pM to the vector space $(\,\mathrm{Sym}^{1\leq k}T_pM\,)^k$ of k-tuples of polynomials of degree at most k without constant term in T_pM. Since not every quadratic polynomial is the square of a linear polynomial, the embedding Φ is not surjective for any $k > 1$, nor does it induce a vector space structure on T^kM.

Whereas the jets of smooth maps and jets of local sections are very similar and virtually indistinguishable in local coordinates the third kind of jets we want to discuss in this section are slightly different in nature, namely jets of submanifolds. The jet bundles of submanifolds or generalized Graßmannians are introduced to deal with geometrically motivated partial differential equations, which actually ask for a submanifold solution, not a smooth map or local section, consider for example the partial differential equations describing minimal or totally geodesic submanifolds. Generalized Graßmannians can be used as well to describe multivalued solutions to standard partial differential equations as submanifolds of a jet bundle as discussed for example in [6].

In order to define the contact equivalence relation between submanifolds of a given manifold M we recall that the higher order tangent bundles of a manifold M are defined as the set $T_p^kM := \mathrm{Jet}_0^k(\,\mathbb{R},\, M\,)$ of equivalence classes $\frac{d^{\leq k}}{dt^{\leq k}}\Big|_0 c$ of curves $c : \mathbb{R} \longrightarrow M$ under contact to order $k \geq 0$ in the point $0 \in \mathbb{R}$. Thinking of the higher order tangent bundle T_p^kN of a submanifold $N \subset M$ in a point $p \in N$ as a subset of the higher order tangent bundle of M

$$T_p^kN := \left\{ \left.\frac{d^{\leq k}}{dt^{\leq k}}\right|_0 c \;\;\Big|\;\; c : \mathbb{R} \longrightarrow N \;\subset\; M \text{ smooth curve with } c(\,0\,) = p \right\}$$
$$\subset T_p^kM$$

we may say that two submanifolds N and $\tilde{N}$ are in contact up to order $k \geq 0$ in a common point $p \in N \cap \tilde{N}$ provided $T_p^kN = T_p^k\tilde{N} \subset T_p^kM$, equivalently for every curve $c : \mathbb{R} \longrightarrow N$ with $c(0) = p$ there exists a curve $\tilde{c} : \mathbb{R} \longrightarrow \tilde{N}$ in contact to c to order k and vice versa:

Definition 2.4 (Jets of Submanifolds and Graßmannians). Two submanifolds of a manifold M are said to be in contact $N \sim_{k,p} \tilde{N}$ to order $k \geq 0$ in a common point $p \in N \cap \tilde{N}$, if their kth order tangent spaces in p agree $T_p^kN = T_p^k\tilde{N}$ considered as subsets of T_p^kM. The equivalence class of a submanifold N under contact to order

$k \geq 0$ in a point $p \in N$ is called the kth order $\mathrm{jet}^k_p N$ of N in p, the set of all kth order jets of submanifolds of dimension n defines the generalized Graßmannian:

$$\mathrm{Gr}^k_n M \quad := \quad \{ \ \mathrm{jet}^k_p N \ \mid \ N \text{ an } n\text{–dimensional submanifold of } M \text{ and } p \in N \ \}$$

Two submanifolds N and $\tilde{N}$ sharing a point $p \in N \cap \tilde{N}$ are in contact to order 0 in p irrespective of their dimensions, because $T^0_p M$ is just the manifold M. The 0th order Graßmannian $\mathrm{Gr}^0_n M = M$ is thus not particularly interesting. The first order Graßmannian $\mathrm{Gr}^1_n M$ on the other hand agrees with the fiber bundle of all linear subspaces $\mathrm{Gr}_n(\,TM\,)$ of dimension n of the tangent bundle. In consequence two submanifolds N and $\tilde{N}$ of different dimensions $n \neq \tilde{n}$ are never in contact to first and thus never in contact to positive order $k > 0$ due to the existence of the by now familiar tower of fiber bundles over M

$$\ldots \xrightarrow{\ \mathrm{pr}\ } \mathrm{Gr}^3_n M \xrightarrow{\ \mathrm{pr}\ } \mathrm{Gr}^2_n M \xrightarrow{\ \mathrm{pr}\ } \mathrm{Gr}^1_n M \xrightarrow{\ \pi\ } \mathrm{Gr}^0_n M \quad = \quad M \tag{16}$$

under the jet projections $\mathrm{pr}: \ \mathrm{Gr}^k_n M \longrightarrow \mathrm{Gr}^{\tilde{k}}_n M$, $\mathrm{jet}^k_p N \longmapsto \mathrm{jet}^{\tilde{k}}_p N$. Unlike the towers of jet bundles we have discussed before there is no meaningful evaluation $\mathrm{ev}: \ \mathrm{Gr}^1_n M \longrightarrow \mathrm{Gr}^0_n M$ defined in this tower other than the fiber bundle projection π.

This minor difference between jet bundles and generalized Graßmannians is reflected faithfully in local standard coordinates on $\mathrm{Gr}^k_n M$. In fact for every $k > 0$ and every choice of local coordinates $(x^1, \ldots, x^m)$ on an open subset $U \subset M$ we may consider the subset

$$\begin{aligned} &(\,\mathrm{Gr}^k_n M\,)_{(x,U)} \\ &\quad := \{\, \mathrm{jet}^k_p N \ \mid \ p \in U \text{ and } \ d_p x^1\big|_{T_pN}\,, \ \ldots, \ d_p x^n\big|_{T_pN} \ \text{ linearly independent}\,\} \end{aligned}$$

of the generalized Graßmannian $\mathrm{Gr}^k_n M$ consisting of the kth order jets of n-dimensional submanifolds $\mathrm{jet}^k_p N$ such that the first n coordinate functions $x^1, \ldots, x^n$ restrict to local coordinates $x^1\big|_N, \ldots, x^n|_N$ on N in a neighborhood $U_N \subset N \cap U$ of p. Upon restriction to N the other $(m-n)$ coordinate functions $x^{n+1}, \ldots, x^m$ thus become smooth functions of $x^1\big|_N, \ldots, x^n|_N$ turning the submanifold N into the graph of the smooth map

$$\begin{aligned} (\,x^{n+1}_N, \ldots, x^m_N\,): \quad & \mathbb{R}^n \longrightarrow \mathbb{R}^{m-n}, \\ (\,x^1(q), \ldots, x^n(q)\,) & \longmapsto (\,x^{n+1}(q), \ldots, x^m(q)\,) \end{aligned}$$

defined on $(\,x^1\big|_N, \ldots, x^n|_N)(\,U_N\,)$ by:

$$(\,x^{n+1}_N, \ldots, x^m_N\,) \quad := \quad (\,x^{n+1}, \ldots, x^m\,) \circ (\,x^1\big|_N, \ldots, x^n|_N\,)^{-1}$$

In this local description of submanifolds of M the difference between two n-dimensional submanifolds N and $\tilde{N}$ becomes the difference between the associated tuples $(x_N^{n+1}, \ldots, x_N^m)$ and $(x_{\tilde{N}}^{n+1}, \ldots, x_{\tilde{N}}^m)$ of functions of $(x^1, \ldots, x^n)$. Clearly two n-dimensional submanifolds N and $\tilde{N}$ are in contact in a common point $p \in N \cap \tilde{N} \cap U$ to order $k \geq 0$, if and only if

$$\frac{\partial^{|A|} x_N^\beta}{\partial x^A}(\, x^1(p), \ldots, x^n(p)\,) \;=\; \frac{\partial^{|A|} x_{\tilde{N}}^\beta}{\partial x^A}(\, x^1(p), \ldots, x^n(p)\,)$$

for all $\beta = n{+}1, \ldots, m$ and all multi-indices A on $\{1, \ldots, n\}$ of order $|A| \leq k$. With this observation in mind we define the standard jet coordinates on $(\,\mathrm{Gr}_n^k M\,)_{(x,U)}$ by setting

$$x^\alpha(\,\mathrm{jet}_p^k N\,) \;:=\; x^\alpha(\,p\,) \qquad\qquad x_A^\beta(\,\mathrm{jet}_p^k N\,) \;:=\; \frac{\partial^{|A|} x_N^\beta}{\partial x^A}(\, x^1(p), \ldots, x^n(p)\,)$$

for $\alpha = 1, \ldots, n$, for $\beta = n+1, \ldots, m$ and A a multi-index on $\{1, \ldots, n\}$ of order $|A| \leq k$. Clearly the domains $(\,\mathrm{Gr}_n^k M\,)_{(x,U)} \subset \mathrm{Gr}_n^k M$ of these standard jet coordinates associated to local coordinates (x, U) on M cover $\mathrm{Gr}_n^k M$ making it a smooth manifold of dimension

$$\dim\ \mathrm{Gr}_n^k M = n + (\,m - n\,) \binom{n+k}{n}$$

say $\dim \mathrm{Gr}_n^0 M = m$ and $\dim \mathrm{Gr}_n^1 M = m + (m-n)\,n$ as expected. Moreover these standard jet coordinates are well adapted to the jet projections $\mathrm{pr} : \mathrm{Gr}_n^k M \longrightarrow \mathrm{Gr}_n^{\tilde{k}} M$ and the projection $\pi : \mathrm{Gr}_n^k M \longrightarrow M$ to the base manifold M proving explicitly that the tower (16) of projections specifies a tower of smooth fiber bundles over M.

Among the subtle differences between the jet bundles of smooth maps or sections and the generalized Graßmannians $\mathrm{Gr}_n^k M$ the definition of the canonical contact form is certainly the most significant. In fact we may not simply copy the definition of the canonical contact form $\gamma^{\mathrm{contact}}$ we have used before, because a tangent vector to $\mathrm{Gr}_n^k M$ written in the form

$$\left.\frac{d}{dt}\right|_0 \mathrm{jet}_{p_t}^k N_t \;\in\; T_{\mathrm{jet}_{p_0}^k N_0} \mathrm{Gr}_n^k M$$

implicitly requires $p_t \in N_t$ for all t to be well-defined, so neither the expression $\frac{d}{dt}\big|_0 \mathrm{jet}_{p_0}^k N_t$ nor its counterpart $\frac{d}{dt}\big|_0 \mathrm{jet}_{p_t}^k N_0$ make any sense. For all $k \geq 1$ however we may lift the canonical inclusion $\iota_N : N \longrightarrow M$ of an n-dimensional submanifold $N \subset M$ to the Graßmannian $\mathrm{jet}^{k-1}\iota_N : N \longrightarrow \mathrm{Gr}_n^{k-1} M$, $p \longmapsto \mathrm{jet}_p^{k-1} N$, in such a way that $\pi \circ \mathrm{jet}^{k-1}\iota_N = \iota_N$. The differential of the lifted inclusion $\mathrm{jet}^{k-1}\iota_N$ of the submanifold N in a point $p \in N$

$$(\mathrm{jet}_p^{k-1}\iota_N)_{*,p} : \quad T_pN \longmapsto T_{\mathrm{jet}_p^{k-1}N}\,\mathrm{Gr}_n^{k-1}M, \quad \frac{d}{dt}\Big|_0 p_t \longmapsto \frac{d}{dt}\Big|_0 \mathrm{jet}_{p_t}^{k-1}N$$

is thus an embedding, whose image in the tangent space $T_{\mathrm{jet}_p^{k-1}N}\mathrm{Gr}_n^{k-1}M$ turns out to depend only on the kth order jet of the submanifold $\mathrm{jet}_p^kN \in \mathrm{Gr}_n^kM$. In consequence we can define the canonical contact form on Gr_n^kM simply by the projection to the corresponding quotient:

$$\gamma^{\mathrm{contact}}\Big(\frac{d}{dt}\Big|_0 \mathrm{jet}_{p_t}^kN_t\Big) := \frac{d}{dt}\Big|_0 \mathrm{jet}_{p_t}^{k-1}N_t + \mathbf{im}\,(\mathrm{jet}_{p_0}^{k-1}\iota_{N_0})_{*,p_0}$$

Although significantly different in definition this contact form serves the same purpose as before, namely it tells us, whether we are forced to vary the submanifold in order to reproduce a given vector tangent to Gr_n^kM. In fact $\mathbf{im}\,(\mathrm{jet}_{p_0}^{k-1}\iota_{N_0})_{*,p_0}$ is precisely the subspace of tangent vectors, which can be realized without a variation of the submanifold N_0!

One advantage of the preceding definition of the canonical contact form on Gr_n^kM is that it is evidently horizontal for the jet projection $\mathrm{pr} : \mathrm{Gr}_n^kM \longrightarrow \mathrm{Gr}_n^{k-1}M$, because a tangent vector $\frac{d}{dt}\big|_0 \mathrm{jet}_{p_t}^kN_t$ vertical under pr satisfies $\frac{d}{dt}\big|_0 \mathrm{jet}_{p_t}^{k-1}N_t = 0$ by definition and thus vanishes under $\gamma^{\mathrm{contact}}$. Due to this horizontality we can extend the canonical contact form $\gamma^{\mathrm{contact}}$ to the contact system on the generalized Graßmann bundle Gr_n^kM of order $k \geq 1$

$$\mathrm{Contact}\,\mathrm{Gr}_n^kM \subseteq \mathrm{Horizontal}\,\mathrm{Gr}_n^kM \subseteq T^*\mathrm{Gr}_n^kM \qquad (17)$$

where $\mathrm{Horizontal}\,\mathrm{Gr}_n^kM$ denotes the subbundle of horizontal forms with respect to pr and $\mathrm{Contact}\,\mathrm{Gr}_n^kM$ the subbundle of scalar components of the canonical contact form $\gamma^{\mathrm{contact}}$:

$$\mathrm{Contact}_{\mathrm{jet}_p^kN}\,\mathrm{Gr}_n^kM$$
$$:= \mathbf{im}\Big(\mathrm{Ann}\,\mathbf{im}\,(\mathrm{jet}^{k-1}\iota_N)_{*,p} \longrightarrow T^*_{\mathrm{jet}_p^kN}\mathrm{Gr}_n^kM, \ \eta \longmapsto \langle\,\eta,\,\gamma^{\mathrm{contact}}\,\rangle\Big)$$

In order to find an explicit description of the canonical contact form in standard jet coordinates (x^α, x_A^β) on Gr_n^kM let us consider a submanifold $N \subset M$ of dimension n with canonical inclusion $\iota_N : N \longrightarrow M$ written locally as a graph of the smooth map $(x_N^{n+1}, \ldots, x_N^m)$:

$$\iota_N : \quad (x^1,\ldots,x^n) \longmapsto (x^1,\ldots,x^n;\, x_N^{n+1}(x^1,\ldots,x^n),\ldots,x_N^m(x^1,\ldots,x^n))$$

The lift of the inclusion to the Graßmannian $N \longrightarrow \mathrm{Gr}_n^{k-1}M$, $p \longmapsto \mathrm{jet}_p^{k-1}N$, is given by

$$\mathrm{jet}^{k-1}\iota_N : \quad (x^1,\ldots,x^n) \longmapsto (x^1,\ldots,x^n;\, \Big\{\frac{\partial^{|A|}x_N^\beta}{\partial x^A}(x^1,\ldots,x^n)\Big\}_{|A|<k,\,\beta})$$

hence its differential $(\mathrm{jet}^{k-1}\iota_N)_{*,p} : T_pN \longrightarrow T_{\mathrm{jet}^{k-1}_p N}\mathrm{Gr}^{k-1}_n M$ in a point $p \in N$ satisfies:

$$(\mathrm{jet}^{k-1}\iota_N)_{*,p} : \frac{\partial}{\partial x^\alpha} \longmapsto \frac{\partial}{\partial x^\alpha} + \sum_{\substack{|A|<k \\ \beta}} \Big(\frac{\partial}{\partial x^\alpha} \frac{\partial^{|A|} x^\beta_N}{\partial x^A} \Big)(x^1(p), \ldots, x^n(p)) \frac{\partial}{\partial x^\beta_A}$$

On the other hand the definition of the standard jet coordinates (x^α, x^β_A) on $\mathrm{Gr}^k_n M$ becomes

$$x^\beta_{A+\alpha}(\mathrm{jet}^k_p N) = \Big(\frac{\partial}{\partial x^\alpha} \frac{\partial^{|A|} x^\beta_N}{\partial x^A} \Big)(x^1(p), \ldots, x^n(p))$$

for all multi-indices of order $|A| < k$ less than k, in consequence the image of the differential $(\mathrm{jet}^{k-1}\iota_N)_{*,p}$ depends only on the coordinates (x^α, x^β_A) of the point $\mathrm{jet}^k_p N$ in the generalized Graßmannian $\mathrm{Gr}^k_n M$ as claimed. Specifically we obtain the following congruences

$$\gamma^{\mathrm{contact}}\Big(\frac{\partial}{\partial x^\alpha} \Big) \equiv - \sum_{\substack{|A|<k \\ \beta}} x^\beta_{A+\alpha} \frac{\partial}{\partial x^\beta_A}$$

$$\gamma^{\mathrm{contact}}\Big(\frac{\partial}{\partial x^\beta_A} \Big) \equiv + \delta_{|A|<k} \frac{\partial}{\partial x^\beta_A}$$

modulo the would be image of the differential $(\mathrm{jet}^{k-1}\iota_N)_{*,p}$ defined as the subspace:

$$\begin{aligned}\Sigma(x^\alpha, x^\beta_A) &:= \text{“}\mathbf{im}\,(\mathrm{jet}^{k-1}\iota_N)_{*,p}\text{”} \\ &= \mathrm{span}\Big\{ \frac{\partial}{\partial x^\alpha} + \sum_{\substack{|A|<k \\ \beta}} x^\beta_{A+\alpha} \frac{\partial}{\partial x^\beta_A} \;\Big|\; \alpha = 1, \ldots, n \Big\}\end{aligned}$$

It is comforting to know that the contact form $\gamma^{\mathrm{contact}}$ thus looks virtually the same as before

$$\gamma^{\mathrm{contact}} = \sum_{\substack{|A|<k \\ \beta}} \underbrace{\Big(dx^\beta_A - \sum_{\alpha=1}^{n} x^\beta_{A+\alpha}\, dx^\alpha \Big)}_{=:\ \gamma^\beta_A} \otimes \Big(\frac{\partial}{\partial x^\beta_A} + \Sigma(x^\alpha, x^\beta_A) \Big)$$

in standard jets coordinates on $\mathrm{Gr}^k_n M$, in particular the contact system has the familiar form:

$$\begin{array}{rcll} \text{Contact } \mathrm{Gr}^k_n M & := \text{span} \{ & \gamma^\beta_A & \mid \text{ for all } \beta,\ |A| < k \ \} \\ \text{Horizontal } \mathrm{Gr}^k_n M & := \text{span} \{ & dx^\alpha,\ dx^\beta_A & \mid \text{ for all } \alpha,\ \beta,\ |A| < k \ \} \end{array}$$

With the construction of the canonical contact system we have established an almost complete analogy between the generalized Graßmannian $\mathrm{Gr}^k_n M$ and the jet bundles of maps or sections. What we are still lacking though is an analogue of the addition (7), which turns the fiber of the jet projection $\mathrm{pr} : \mathrm{Jet}^k \mathscr{F} M \longrightarrow \mathrm{Jet}^{k-1}\mathscr{F} M$ over a point $\mathrm{jet}^{k-1}_p f$ into an affine space modelled on the vector space $\mathrm{Sym}^k T^*_p M \otimes \mathrm{Vert}_{f(p)} \mathscr{F} M$ for all $k \geq 1$. Much to our chagrin the fiber of the jet projection $\pi : \mathrm{Gr}^1_n M \longrightarrow M$ is *not* an affine space, rather we may identify it via $\mathrm{jet}^1_p N \longmapsto T_p N$ with the compact Graßmannian $\mathrm{Gr}_n(T_p M)$. Despite this disappointment we observe that the vertical tangent space of $\mathrm{Gr}^1_n M$ in a point $\mathrm{jet}^1_p N = T_p N$

$$\mathrm{Vert}_{\mathrm{jet}^1_p N} \mathrm{Gr}^1_n M \quad = \quad T_{T_p N} \mathrm{Gr}_n(T_p M) \quad \cong \quad \mathrm{Hom}(T_p N,\ T_p M / T_p N) \qquad (18)$$

can be written in a form $\mathrm{Hom}(T_p N,\ T_p M / T_p N) = \mathrm{Sym}^1 T^*_p N \otimes (T_p M / T_p N)$ reminiscent of the vector space acting on the first order jet bundle $\mathrm{Jet}^1 \mathscr{F} M$. Somewhat more precisely the identification (18) of the tangent space of the Graßmannian $\mathrm{Gr}_n(T_p M)$ associates to a homomorphism $A \in \mathrm{Hom}(T_p N,\ T_p M / T_p N)$ the following tangent vector in the point $T_p N$

$$\frac{d}{dt}\Big|_0 \ \mathbf{im}\Big(\mathrm{id} + tA^{\mathrm{lift}} : \ T_p N \longrightarrow T_p M, \quad X \longmapsto X + tA^{\mathrm{lift}} X \Big)$$
$$\in T_{T_p N} \mathrm{Gr}_n(T_p M)$$

where $A^{\mathrm{lift}} : T_p N \longrightarrow T_p M$ is a linear lift of A. Of course the curve $t \longmapsto \mathbf{im}(\mathrm{id} + tA^{\mathrm{lift}})$ of n-dimensional subspaces of $T_p M$ defined for t sufficiently small depends on the lift A^{lift} chosen, nevertheless the tangent vector to this curve in $t = 0$ only depends on A.

En nuce the principal idea of the identification (18) of the tangent spaces of the Graßmannian $\mathrm{Gr}_n(T_p M)$ is to replace a subspace $T_p N \subset T_p M$ by its inclusion $T_p N \longrightarrow T_p M$, being an application the latter is more easy to deform. In the context of jets of submanifolds we do not loose information in replacing similarly a submanifold $N \subset M$ by its canonical inclusion $\iota_N : N \longrightarrow M$, because $\mathrm{jet}^k_p \iota_N$ determines $\mathrm{jet}^k_p N$ completely, to wit the inclusion $T^k_p N \longrightarrow T^k_p M$ used to define $\mathrm{jet}^k_p N$ is just the jet composition (3) with $\mathrm{jet}^k_p \iota_N$. In the same vein the addition (7) of jets of smooth maps becomes an addition of jets of submanifolds

$$\mathrm{jet}^k_p N + \Delta N := \mathrm{jet}^k_p \,\mathbf{im}\Big(N \longrightarrow M,\ q \longmapsto \Phi^M \Big[\Phi^{M^{-1}}(q) + \Delta N(\Phi^{N^{-1}}(q)) \Big] \Big) \qquad (19)$$

with homogeneous polynomials $\Delta N \in \mathrm{Sym}^k T_p^* N \otimes T_p M$ of degree k on $T_p N$ with values in $T_p M$. Although this addition is well-defined for all $k \geq 1$ independent of the choice of the anchored coordinate charts Φ^N and Φ^M for N and M, a peculiar problem arises in the case $k = 1$ singled out in our discussion above: The image of the deformed smooth map is not even locally a submanifold of dimension n, because we modify the linear inclusion $T_p N \subset T_p M$ by linear terms. Evidently this problem disappears for jet orders $k \geq 2$ and the equality $\mathrm{jet}_p^{k-1} \, \mathbf{im} \, \iota_N = \mathrm{jet}_p^{k-1} N$ ensures that our addition acts on the fibers of the projection $\mathrm{Gr}_n^k M \longrightarrow \mathrm{Gr}_n^{k-1} M$ in the sense that $\mathrm{jet}_p^k N + \Delta N$ still lies over $\mathrm{jet}_p^{k-1} N$.

Unluckily however the vector space $\mathrm{Sym}^k T_p^* N \otimes T_p M$ is too large to provide us with a simply transitive group action on the fibers of the projection in analogy to the addition (7) on jets of maps or sections. In order to understand this problem let us have another look at the identification (18) of the tangent spaces of Graßmannian $\mathrm{Gr}_n(T_p M)$. The construction of an explicit curve in $\mathrm{Gr}_n(T_p M)$ representing the tangent vector associated to a linear map $A : T_p N \longrightarrow T_p M / T_p N$ required us to lift A to $A^{\mathrm{lift}} : T_p N \longrightarrow T_p M$. The representing curve depended on this lift, but not the tangent vector itself. Changing the homogeneous polynomial $\Delta N \in \mathrm{Sym}^k T_p^* N \otimes T_p M$ used in the addition (19) by a homogeneous polynomial of degree k on $T_p N$ with values in $T_p N$ similarly changes the image submanifold

$$\mathbf{im}\Big(N \longrightarrow M, \quad q \longmapsto \Phi^M \Big[\, {\Phi^M}^{-1}(q) \; + \; \Delta N(\, {\Phi^N}^{-1}(q) \,) \, \Big] \Big)$$

but not its equivalence class under contact of submanifolds to order k in p. For example we may always choose the anchored coordinate chart Φ^M in such a way that $\Phi^M(T_p N) \subset N$ holds true. For such a choice and arbitrary $\Delta N \in \mathrm{Sym}^k T_p^* N \otimes T_p N$ the smooth map

$$\varphi : \; N \longrightarrow N, \qquad q \longmapsto \Phi^M \Big[\, {\Phi^M}^{-1}(q) \; + \; \Delta N(\, {\Phi^N}^{-1}(q) \,) \, \Big]$$

is actually a local diffeomorphism (sic!) of N due to $\mathrm{jet}_p^{k-1} \varphi = \mathrm{jet}_p^{k-1} \mathrm{id}_N$ and $k \geq 2$ so that $\mathrm{jet}_p^k N = \mathrm{jet}_p^k N + \Delta N$. Modifying this argument slightly to make it work for changes of $\Delta N \in \mathrm{Sym}^k T_p^* N \otimes T_p M$ by a homogeneous polynomial of degree k with values in $T_p N$ we conclude that the addition (19) descends to a well-defined addition of jets of submanifolds

$$\begin{aligned} &\mathrm{jet}_p^k N + \Delta N \\ &\quad := \mathrm{jet}_p^k \, \mathbf{im}\Big(N \longrightarrow M, \, q \longmapsto \Phi^M \Big[\, {\Phi^M}^{-1}(q) + (\Delta N)^{\mathrm{lift}} ({\Phi^N}^{-1}(q)) \Big] \Big) \end{aligned} \tag{20}$$

with $\Delta N \in \mathrm{Sym}^k T_p^* N \otimes (T_p M / T_p N)$ lifted arbitrarily to $(\Delta N)^{\mathrm{lift}} \in \mathrm{Sym}^k T_p^* N \otimes T_p M$. Although it seems difficult to verify the axioms of a group action for the addition $+$ directly due to the ambiguities in choosing Φ^N and Φ^M as well as the lift $(\Delta N)^{\text{lift}}$, this problem disappears in the local standard jet coordinates $(\,x^\alpha,\ x^\beta_A\,)$ on $\mathrm{Gr}^k_n M$. As an additional bonus this local coordinate presentation makes it rather obvious that $\mathrm{Sym}^k T^*_p N \otimes (T_p M / T_p N)$ acts simply transitive on the fibers of the projection $\mathrm{Gr}^k_n M \longrightarrow \mathrm{Gr}^{k-1}_n M$.

For the purpose of writing the addition (20) as a smooth group bundle action on the Graßmannian $\mathrm{Gr}^k_n M$ of jets of submanifolds we recall that the tautological vector bundle on $\mathrm{Gr}^1_n M = \mathrm{Gr}_n(\,TM\,)$ is defined as the subbundle of the pull back $\pi^* TM$ of the tangent bundle of M via the projection $\pi : \mathrm{Gr}^1_n M \longrightarrow M$, whose fiber in $\mathrm{jet}^1_p N \in \mathrm{Gr}^1_n M$ reads:

$$\mathrm{Taut}_{\mathrm{jet}^1_p N}\,\mathrm{Gr}^1_n M \quad := \quad T_p N$$

Implicitly we have used the tautological vector bundle already in the identification (18)

$$\mathrm{Vert}\,\mathrm{Gr}^1_n M \quad = \quad \mathrm{Taut}^*\mathrm{Gr}^1_n M \otimes \Big(\,\pi^* TM /_{\mathrm{Taut}\,\mathrm{Gr}^1_n M}\,\Big)$$

of the vertical tangent bundle of $\mathrm{Gr}^1_n M$, in a similar vein the tautological vector bundle appears in the definition of the canonical contact form γ^{contact} on $\mathrm{Gr}^1_n M$ as the composition

$$T\,\mathrm{Gr}^1_n M \ \stackrel{\pi_*}{\longrightarrow}\ \pi^* TM \ \stackrel{\mathrm{pr}}{\longrightarrow}\ \pi^* TM /_{\mathrm{Taut}\,\mathrm{Gr}^1_n N}$$

of the differential of $\pi : \mathrm{Gr}^1_n M \longrightarrow M$ with the projection to $\pi^* TM / \mathrm{Taut}\,\mathrm{Gr}^1_n M$. The tautological vector bundle pulls back from $\mathrm{Gr}^1_n M$ to a vector bundle on $\mathrm{Gr}^k_n M$, in turn this pull back bundle allows us to write the addition (20) as a smooth group bundle action

$$+ : \quad \mathrm{Gr}^k_n M \ \times_{\mathrm{Gr}^1_n M}\ \mathrm{Sym}^k \mathrm{Taut}^*\mathrm{Gr}^1_n M \otimes \Big(\,\pi^* TM /_{\mathrm{Taut}\,\mathrm{Gr}^1_n M}\,\Big) \longrightarrow \mathrm{Gr}^k_n M$$

defined on $\mathrm{Gr}^k_n M$ for all $k \geq 2$, which preserves the fibers of the projection to $\mathrm{Gr}^{k-1}_n M$. With the construction of this group bundle action we have established a complete analogy between the three types of jets discussed in this section: Jets of maps, jets of sections of fiber bundles and jets of submanifolds. In particular the contact systems associated to these three types of jets allow us to treat partial differential equations for maps, for sections and for submanifolds in the unified language of exterior differential systems.

3 Comodules and Spencer Cohomology

A comodule over a symmetric coalgebra can be seen as the algebraic analogue of a jet bundle in differential geometry, in a rather precise sense this analogy dualizes the better known analogy between differential operators and modules over polynomial algebras. In the formal theory of partial differential equations the latter concept is usually studied under the key word D-modules, which is essentially a proper subtheory of commutative algebra. From our point of view however it is the former notion of a comodule, which fits nicely into the theory of exterior differential systems, because the notion can be seen as a straightforward axiomatization of the commutativity of partial derivatives.

In this section and Sect. 4 we will study the algebraic properties of comodules over symmetric coalgebras in depth starting from their axiomatic definition in terms of partial derivatives, introducing the important subclass of tableau comodules on the way and ending with a detailed discussion of the three most important theorems about tableau comodules from the point of view of partial differential equations. Needless to say all the ideas, properties and theorems discussed in this context are essentially dual to ideas, properties and theorems of commutative algebra. A good complementary reading to these notes would thus be [2]. Nevertheless we hope that the reader will find our reformulation of commutative algebra in terms of comodules helpful for explicit applications in differential geometry:

Definition 3.1 (Comodules over $\mathrm{Sym}\, T^*$**).** A comodule over the symmetric coalgebra $\mathrm{Sym}\, T^*$ is a graded vector space $\mathscr{A}^\bullet$ together with a bilinear map $T \times \mathscr{A}^\bullet \longrightarrow \mathscr{A}^{\bullet-1}$, $(t, a) \longmapsto \frac{\partial a}{\partial t}$, called the directional derivative such that the endomorphism $\frac{\partial}{\partial t} : \mathscr{A}^\bullet \longrightarrow \mathscr{A}^{\bullet-1}$, $a \longmapsto \frac{\partial a}{\partial t}$, of $\mathscr{A}^\bullet$ with a fixed direction $t \in T$ is homogeneous of degree -1 and the endomorphisms $\frac{\partial}{\partial t_1}$ and $\frac{\partial}{\partial t_2}$ commute for all $t_1, t_2 \in T$:

$$\frac{\partial}{\partial t_1} \circ \frac{\partial}{\partial t_2} \;=\; \frac{\partial}{\partial t_2} \circ \frac{\partial}{\partial t_1}$$

In consequence we may iterate the axiomatic directional derivatives of a comodule $\mathscr{A}$ in order to obtain well-defined homogeneous endomorphisms like $\frac{\partial^2}{\partial t_1 \partial t_2}$: $\mathscr{A}^\bullet \longrightarrow \mathscr{A}^{\bullet-2}$ etc.

Although intimidating in nomenclature the notion of a comodule is nothing but an axiomatization of a very familiar concept, that of the directional derivatives of functions on the vector space T. The example motivating this axiomatization is the vector space $\mathrm{Sym}^\bullet T^* \otimes V$ of polynomials on T with values in a vector space V graded by homogeneity together with

$$T \times \mathrm{Sym}^\bullet T^* \otimes V \longmapsto \mathrm{Sym}^{\bullet-1} T^* \otimes V, \qquad (t, \psi) \longmapsto \frac{\partial \psi}{\partial t}$$

which associates to a polynomial ψ and a direction $t \in T$ the directional derivative:

$$\frac{\partial \psi}{\partial t}(\, p\,) \;:=\; \frac{d}{d\varepsilon}\Big|_0 \psi(\, p + \varepsilon\, t\,)$$

In this interpretation of comodules as an axiomatization of directional derivatives it is natural to define the Spencer coboundary operator on alternating forms with values in a comodule

$$B: \quad \mathscr{A}^{\bullet} \otimes \Lambda^{\circ} T^* \longrightarrow \mathscr{A}^{\bullet-1} \otimes \Lambda^{\circ+1} T^*, \qquad \omega \longmapsto B\,\omega$$

in analogy to the de Rham coboundary operator on differential forms by setting

$$(\, B\,\omega\,)(\, t_0, \ldots, t_r\,) \;:=\; \sum_{\mu=0}^{r} (-1)^{\mu} \frac{\partial}{\partial t_{\mu}}\, \omega(\, t_0, \ldots, \widehat{t_{\mu}}, \ldots, t_r\,)$$

for an alternating r-form $\omega \in \mathscr{A}^k \otimes \Lambda^r T^*$ with values in $\mathscr{A}^k$. Evidently $B\,\omega$ is then an $(r+1)$-form on T with values in $\mathscr{A}^{k-1}$, in this sense the Spencer coboundary operator B is bihomogeneous of bidegree $(-1, +1)$. The axiomatic commutation of directional derivatives ensures that the Spencer operator B satisfies the coboundary condition, in other words

$$\begin{aligned} & B^2\omega(\, t_0, \ldots, t_{r+1}\,) \\ &= \sum_{0 \le \mu < \nu \le r+1} (-1)^{\mu+\nu} \Big(+ \frac{\partial^2}{\partial t_{\mu} \partial t_{\nu}}\, \omega(\, t_0, \ldots, \widehat{t_{\mu}}, \ldots, \widehat{t_{\nu}}, \ldots, t_{r+1}\,) \\ &\quad - \frac{\partial^2}{\partial t_{\nu} \partial t_{\mu}}\, \omega(\, t_0, \ldots, \widehat{t_{\mu}}, \ldots, \widehat{t_{\nu}}, \ldots, t_{r+1}\,) \Big) \end{aligned}$$

vanishes irrespective of ω. In turn B defines a bigraded cohomology theory for comodules:

Definition 3.2 (Spencer Cohomology of a Comodule). The Spencer cohomology of a comodule $\mathscr{A}$ over the symmetric coalgebra $\mathrm{Sym}\, T^*$ of a vector space T is the bigraded cohomology $H^{\bullet,\circ}(\, \mathscr{A}\,)$ associated to the bigraded Spencer complex

$$\ldots \stackrel{B}{\longrightarrow} \mathscr{A}^{\bullet+1} \otimes \Lambda^{\circ-1} T^* \stackrel{B}{\longrightarrow} \mathscr{A}^{\bullet} \otimes \Lambda^{\circ} T^* \stackrel{B}{\longrightarrow} \mathscr{A}^{\bullet-1} \otimes \Lambda^{\circ+1} T^* \stackrel{B}{\longrightarrow} \ldots$$

of alternating, multilinear forms on T with values in $\mathscr{A}$:

$$H^{\bullet,\circ}(\, \mathscr{A}\,) \;:=\; \frac{\mathbf{ker}(\, B: \;\; \mathscr{A}^{\bullet} \otimes \Lambda^{\circ} T^* \longrightarrow \mathscr{A}^{\bullet-1} \otimes \Lambda^{\circ+1} T^*\,)}{\mathbf{im}(\, B: \;\; \mathscr{A}^{\bullet+1} \otimes \Lambda^{\circ-1} T^* \longrightarrow \mathscr{A}^{\bullet} \otimes \Lambda^{\circ} T^*\,)}$$

In order to get some idea about Spencer cohomology theory let us calculate it for some examples. Every graded vector space $\mathscr{A}^\bullet$ can be made a comodule $\mathscr{A}^\bullet_{\text{trivial}}$ by declaring all its directional derivatives to vanish $\frac{\partial a}{\partial t} := 0$ for all $a \in \mathscr{A}^k$ and all $t \in T$. The Spencer cohomology of such a comodule aptly called trivial is certainly given by:

$$H^{\bullet,\circ}(\,\mathscr{A}_{\text{trivial}}\,) \quad = \quad \mathscr{A}^\bullet \otimes \Lambda^\circ T^*$$

Somewhat more interesting are the free comodules $\text{Sym}^\bullet T^* \otimes V$ of polynomials on T with values in a vector space V introduced before. The Spencer operator associated to such a free comodule $B : (\text{Sym}^\bullet T^* \otimes V) \otimes \Lambda^\circ T^* \longrightarrow (\text{Sym}^{\bullet-1} T^* \otimes V) \otimes \Lambda^{\circ+1} T^*$ can be written as a sum

$$B \quad = \quad \sum_{\mu=1}^n \frac{\partial}{\partial t_\mu} \otimes \text{id}_V \otimes dt_\mu \wedge$$

over a dual pair of bases $t_1, \ldots, t_n$ and $dt_1, \ldots, dt_n$ of T and T^*. In order to calculate the cohomology of the Spencer complex we introduce the operator of integration along rays through the origin $B^* : \text{Sym}^\bullet T^* \otimes V \otimes \Lambda^\circ T^* \longrightarrow \text{Sym}^{\bullet+1} T^* \otimes V \otimes \Lambda^{\circ-1} T^*$ as the sum:

$$B^* \quad := \quad \sum_{\mu=1}^n dt_\mu \cdot \otimes \text{id}_V \otimes t_\mu \lrcorner$$

After some more or less straightforward calculations we find that the formal Laplace operator

$$\Delta \quad := \{\, B,\ B^* \,\} \quad = \quad B \circ B^* \ + \ B^* \circ B$$

is diagonalizable on $(\text{Sym}^k T^* \otimes V) \otimes \Lambda^r T^*$ with eigenvalue $k + r$. In consequence every closed Spencer cochain $\psi \in (\text{Sym}^k T^* \otimes V) \otimes \Lambda^r T^*$ of bidegree (k, r) satisfying $k + r > 0$ is exact

$$\psi = \frac{1}{k+r}\,\Delta\,\psi = \frac{1}{k+r}\Big(\, B\,(\,B^*\psi\,) \ + \ B^*\,(\,B\psi\,)\Big) = B\Big(\,\frac{1}{k+r}\,B^*\,\psi\,\Big)$$

by $B\psi = 0$. Hence the Spencer cohomology of a free comodule $\text{Sym}^\bullet T^* \otimes V$ is concentrated

$$H^{0,0}(\,\text{Sym}\,T^* \otimes V\,) \quad = \quad V$$

in comodule and form degrees 0. The preceding calculation of the Spencer cohomology of free comodules is an elementary version of Hodge theory and by no means restricted to this special case. Considered as a method to calculate the cohomology of a given coboundary operator B it relies on making a suitable guess for the operator B^* such that the formal Laplace operator $\Delta := \{ B, B^* \}$ is diagonalizable. The original complex then decomposes into a direct sum of "eigensubcomplexes" under Δ, because Δ and B commute $[\Delta, B] = 0$, however all these eigensubcomplexes are exact except for the kernel subcomplex!

The limited stock of examples discussed so far can be augmented by simple modifications of the underlying graded vector spaces. For example the shift in grading by an integer $d \in \mathbb{Z}$

$$(\mathscr{A}^{+d})^\bullet \quad := \quad \mathscr{A}^{\bullet+d}$$

certainly results in the shift in grading $H^{\bullet,\circ}(\mathscr{A}^{+d}) = H^{\bullet+d,\circ}(\mathscr{A})$ in Spencer cohomology. A theoretically important variation of the shift is the twist of a comodule $\mathscr{A}$ defined by

$$\mathscr{A}^\bullet(d) \quad := \quad \mathscr{A}^{\bullet+d}$$

for $\bullet \geq 0$ with $\mathscr{A}^\bullet(d) := \{0\}$ for all $\bullet < 0$, here the directional derivatives of $\mathscr{A}^\bullet(d)$ equal the directional derivatives of $\mathscr{A}$ in positive degrees $\bullet > 0$ only. In consequence the Spencer cohomology $H^{\bullet,\circ}(\mathscr{A}(d))$ vanishes in all comodule degrees $\bullet < 0$ and equals

$$H^{0,\circ}(\mathscr{A}(d)) \quad = \quad (\mathscr{A}^d \otimes \Lambda^\circ T^*)\Big/_{B(\mathscr{A}^{d+1} \otimes \Lambda^{\circ-1} T^*)} \tag{21}$$

in comodule degree $\bullet = 0$, while $H^{\bullet,\circ}(\mathscr{A}(d)) = H^{\bullet+d,\circ}(\mathscr{A})$ as before in degrees $\bullet > 0$. Another interesting variation of the shift is the idea of a free comodule $\mathscr{A}^\bullet = \mathrm{Sym}^\bullet T^* \otimes V^\bullet$ generated by a graded vector space $V^\bullet$, which is essentially a direct sum of shifted free comodules with associated Spencer cohomology $V^\bullet$ concentrated in form degree $\circ = 0$:

$$\mathrm{Sym}^\bullet T^* \otimes V^\bullet \quad = \quad \bigoplus_{k\in\mathbb{Z}} \mathrm{Sym}^{\bullet-k} T^* \otimes V^k$$

Coming back to the general theory we observe that the Spencer operator B commutes with the extended directional derivatives $\frac{\partial}{\partial t} \otimes \mathrm{id}$ on the graded vector space $\mathscr{A}^\bullet \otimes \Lambda^r T^*$ of Spencer cochains of fixed form degree $\circ = r$. In turn the Spencer complex becomes a complex of comodules, the directional derivatives induced on the Spencer cohomology however are all trivial due to the formal version of Cartan's Homotopy Formula

$$\{ B, (\mathrm{id} \otimes t \lrcorner) \} \quad := \quad B \circ (\mathrm{id} \otimes t \lrcorner) + (\mathrm{id} \otimes t \lrcorner) \circ B \quad = \quad \frac{\partial}{\partial t} \otimes \mathrm{id} \tag{22}$$

which implies for every cohomology class $[\,\omega\,] \in H^{\bullet,\circ}(\,\mathscr{A}\,)$ with $B\,\omega = 0$ and all $t \in T$:

$$\frac{\partial}{\partial t}[\,\omega\,] \;:=\; [\,(\frac{\partial}{\partial t} \otimes \mathrm{id})\,\omega\,] \;=\; [\,B\,(\mathrm{id} \otimes t \lrcorner\,)\,\omega \;+\; (\mathrm{id} \otimes t \lrcorner\,)\,B\omega\,] \;=\; 0$$

Although the induced comodule structure on Spencer cohomology is thus trivial, the Spencer cohomology of a comodule $H^{\bullet,\circ}(\,\mathscr{A}\,)$ carries an interesting algebraic structure, namely the right multiplication of Spencer cochains with elements of $\Lambda^\circ T^*$ commutes with the Spencer coboundary operator B and thus descends to a natural graded right $\Lambda^\circ T^*$-module structure. To see this point more clearly we expand the Spencer coboundary operator into the sum

$$B \;=\; \sum_{\mu=1}^{n} \frac{\partial}{\partial t_\mu} \otimes dt_\mu \wedge$$

over a dual pair of bases $t_1, \ldots, t_n$ and $dt_1, \ldots, dt_n$ of T and T^* respectively and conclude that right multiplication with $\omega \in \Lambda^\circ T^*$ commutes with left multiplication by dt_μ due to associativity. In the literature the additional module structure on $H^{\bullet,\circ}(\mathscr{A})$ is hardly ever mentioned. Nevertheless it is not only practical in explicit calculations, it is important for the theory as well: In quite precise a sense we can reconstruct a comodule $\mathscr{A}$ from its Spencer cohomology $H^{\bullet,\circ}(\,\mathscr{A}\,)$ considered as a graded right module over $\Lambda^\circ T^*$.

A pleasant aspect of the very general and abstract Definition 3.1 of comodules we have adopted in these notes is that it very easy to introduce the complementary concept of homomorphisms of comodules. In general a homomorphism of degree $d \in \mathbb{Z}$ from a comodule $\mathscr{A}$ to a comodule $\mathscr{B}$ is a homogeneous linear map $\Phi : \mathscr{A}^\bullet \longrightarrow \mathscr{B}^{\bullet+d}$ between the underlying graded vector spaces, which intertwines the directional derivatives

$$\Phi(\,\frac{\partial}{\partial t}\bigg|_{\mathscr{A}}\,a\,) \;=\; \frac{\partial}{\partial t}\bigg|_{\mathscr{B}}\,(\,\Phi\,a\,)$$

for all $t \in T$. The set of all comodule homomorphisms $\Phi : \mathscr{A}^\bullet \longrightarrow \mathscr{B}^{\bullet+d}$ of fixed degree $d \in \mathbb{Z}$ is evidently a vector space $\mathrm{Hom}^{\,d}_{\mathrm{Sym}\,T^*}(\,\mathscr{A},\,\mathscr{B}\,)$, in consequence we can talk about the abelian category of comodules over the symmetric coalgebra $\mathrm{Sym}\,T^*$ by defining the vector space of morphisms $\mathscr{A} \longrightarrow \mathscr{B}$ in this category as the direct sum of all these vector spaces:

$$\mathrm{Hom}^{\,\bullet}_{\mathrm{Sym}\,T^*}(\,\mathscr{A},\,\mathscr{B}\,) \;:=\; \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}^{\,d}_{\mathrm{Sym}\,T^*}(\,\mathscr{A},\,\mathscr{B}\,)$$

This rather complicated definition of morphisms has the advantage of making the following functor from the category of comodules to the category of graded vector spaces representable:

Definition 3.3 (Finitely Generated and Bounded Comodules). The space of generators of a comodule $\mathscr{A}$ over the symmetric coalgebra $\operatorname{Sym} T^*$ of a vector space T is the graded vector space of elements of $\mathscr{A}^\bullet$ constant under all partial derivatives:

$$\operatorname{Gen}^\bullet \mathscr{A} := \bigoplus_{k \in \mathbb{Z}} \operatorname{Gen}^k \mathscr{A} \quad \operatorname{Gen}^k \mathscr{A} := \left\{ a \in \mathscr{A}^k \;\middle|\; \frac{\partial a}{\partial t} = 0 \text{ for all } t \in T \right\}$$

A comodule $\mathscr{A}$ is called finitely generated and bounded below in case $\operatorname{Gen} \mathscr{A}$ is a finite-dimensional vector space and $\mathscr{A}^k = \{0\}$ vanishes for all sufficiently small $k \ll 0$.

In passing we observe that the homogeneous subspaces of a finitely generated comodule $\mathscr{A}$ bounded below are finite-dimensional $\dim \mathscr{A}^k < \infty$ for all $k \in \mathbb{Z}$ due to a straightforward induction based on $\mathscr{A}^k = \{0\}$ for $k \ll 0$ and an induction step using the exact sequence:

$$0 \longrightarrow \operatorname{Gen}^\bullet \mathscr{A} \stackrel{\subset}{\longrightarrow} \mathscr{A}^\bullet \stackrel{B}{\longrightarrow} \mathscr{A}^{\bullet-1} \otimes T^*$$

In order to understand the significance of generators let us consider the real numbers as a trivial comodule $\mathbb{R}^\bullet$ concentrated in degree 0 with all directional derivatives necessarily vanishing. The image of $1 \in \mathbb{R}$ under a homomorphism $\Phi : \mathbb{R}^\bullet \longrightarrow \mathscr{A}^{\bullet+k}$ of comodules homogeneous of degree $k \in \mathbb{Z}$ is then a generator $\Phi(1) \in \operatorname{Gen}^k \mathscr{A}$ of $\mathscr{A}$ of degree k due to

$$\frac{\partial}{\partial t}\,\Phi(\,1\,) \;=\; \Phi(\,\frac{\partial 1}{\partial t}\,) \;=\; 0$$

and vice versa every $a \in \operatorname{Gen}^k \mathscr{A}$ defines the homomorphism $\Phi_a : \mathbb{R}^\bullet \longrightarrow \mathscr{A}^{\bullet+k}$, $x \longmapsto xa$. In other words the functor $\operatorname{Gen}^\bullet$ to the category of graded vector spaces is represented by $\mathbb{R}$:

$$\operatorname{Hom}^\bullet_{\operatorname{Sym} T^*}(\,\mathbb{R},\, \mathscr{A}\,) \stackrel{\cong}{\longrightarrow} \operatorname{Gen}^\bullet \mathscr{A}, \qquad \Phi \longmapsto \Phi(\,1\,)$$

Using a suitable projective resolution of the representing comodule $\mathbb{R}$ it is then easy to prove:

$$H^{\bullet,\circ}(\,\mathscr{A}\,) \;\cong\; \operatorname{Ext}^{\circ,\bullet}_{\operatorname{Sym} T^*}(\,\mathbb{R},\, \mathscr{A}\,)$$

In consequence the Spencer cohomology calculates the derived functor $\operatorname{Ext}^{\circ,\bullet}_{\operatorname{Sym} T^*}(\mathbb{R}, \cdot)$ associated to the functor $\operatorname{Gen}^\bullet = \operatorname{Hom}^\bullet_{\operatorname{Sym} T^*}(\,\mathbb{R}, \cdot)$ from comodules to graded vector spaces!

Remark 3.4 (Interpretation of Spencer Cohomology). In general it seems to be difficult to say directly, what exactly a non-zero Spencer cohomology class tells us about the underlying comodule. Direct interpretations are available however for the Spencer cohomology of a comodule $\mathscr{A}$ over $\mathrm{Sym}\,T^*$ in form degrees 0 and $n := \dim\, T$, namely $H^{\bullet,0}(\mathscr{A}) = \mathrm{Gen}^{\bullet}\mathscr{A}$ is true for $\circ = 0$ by our preceding discussion, whereas

$$H^{\bullet,0}(\,\mathscr{A}\,) = \mathrm{Gen}^{\bullet}\mathscr{A} \qquad H^{\bullet,n}(\,\mathscr{A}\,) \cong \mathscr{A}^{\bullet}\Big/_{\mathrm{span}\{\ \frac{\partial a}{\partial t}\ \mid\ a \in \mathscr{A}^{\bullet+1} \text{ and } t \in T\ \}}$$

is satisfied in form degree $\circ = n$ by a straightforward and not too complicated calculation.

Lemma 3.5 (Finiteness of Spencer Cohomology). *Consider a finitely generated comodule $\mathscr{A}$ bounded below. Every subcomodule $\mathscr{B} \subset \mathscr{A}$ and every quotient comodule $\mathscr{A}/\mathscr{B}$ of $\mathscr{A}$ are likewise finitely generated and bounded below. In particular the Spencer cohomology $H^{\bullet,\circ}(\mathscr{A})$ of $\mathscr{A}$ is a finite dimensional vector space:*

$$\dim\ H^{\bullet,\circ}(\,\mathscr{A}\,)\ <\ \infty$$

Needless to say the hard part in the proof of this lemma is the assertion that a quotient $\mathscr{A}/\mathscr{B}$ of a finitely generated comodule $\mathscr{A}$ bounded below by a subcomodule $\mathscr{B}$ is finitely generated, all other assertions of the lemma are trivial or direct consequences of this finiteness. For example the rather surprising conclusion about the Spencer cohomology of a finitely generated comodule $\mathscr{A}$ bounded below simply observes that the Spencer complex

$$\ldots \stackrel{B}{\longrightarrow} \mathscr{A}^{\bullet+1} \otimes \Lambda^{\circ-1}T^* \stackrel{B}{\longrightarrow} \mathscr{A}^{\bullet} \otimes \Lambda^{\circ}T^* \stackrel{B}{\longrightarrow} \mathscr{A}^{\bullet-1} \otimes \Lambda^{\circ+1}T^* \stackrel{B}{\longrightarrow} \ldots$$

associated to $\mathscr{A}$ is a complex of finitely generated comodules $\mathscr{A} \otimes \Lambda^{\circ}T^*$ bounded below with generators $\mathrm{Gen}^{\bullet}(\mathscr{A} \otimes \Lambda^{\circ}T^*) = (\mathrm{Gen}^{\bullet}\mathscr{A}) \otimes \Lambda^{\circ}T^*$. Assuming finite generation of quotients the subquotient comodule $H^{\circ}(\mathscr{A})$ of the finitely generated comodule $\mathscr{A} \otimes \Lambda^{\circ}T^*$ bounded below is itself finitely generated and bounded below, on the other hand we have seen that $H^{\circ}(\mathscr{A})$ is a trivial comodule in the sense that all its directional derivatives vanish. In consequence

$$H^{\bullet,\circ}(\,\mathscr{A}\,)\ =\ \mathrm{Gen}^{\bullet}\ H^{\circ}(\,\mathscr{A}\,)$$

is finite-dimensional as claimed. All in all Lemma 3.5 reduces very easily to the non-trivial statement that quotients of finitely generated comodule bounded below are finitely generated.

In order to give at least a sketch of the principal argument leading to Lemma 3.5 let us consider a quotient $\mathscr{A}/\mathscr{B}$ of a finitely generated comodule $\mathscr{A}$ bounded below. For sufficiently large $d \gg 0$ the spaces of generators $\mathrm{Gen}^{k+d}\mathscr{A} = \{0\}$ vanish for all $k \geq 0$ due to the finite generation of $\mathscr{A}$. Hence for all $k \geq 0$ the following composition of injective maps

$$\mathscr{A}^{k+d} \longrightarrow T^* \otimes \mathscr{A}^{k+d-1} \longrightarrow T^* \otimes T^* \otimes \mathscr{A}^{k+d-2}$$
$$\longrightarrow \quad \ldots \longrightarrow \underbrace{T^* \otimes \ldots \otimes T^*}_{k} \otimes \mathscr{A}^d$$

is injective itself for all $k \geq 0$ and factorizes by coassociativity over the embedding

$$\mathscr{A}^{\bullet}(\, d\,) \stackrel{\Delta}{\longrightarrow} \mathrm{Sym}^{\bullet} T^* \otimes \mathscr{A}^d \tag{23}$$

by means of the comultiplication Δ (sic!), which is defined for $a \in \mathscr{A}^{k+d}$ as the sum

$$\Delta\, a \quad := \quad \frac{1}{k!} \sum_{\mu_1,\ldots,\mu_k=1}^{n} dt_{\mu_1} \cdot \ldots \cdot dt_{\mu_k} \otimes \frac{\partial^k a}{\partial t_{\mu_1} \ldots \partial t_{\mu_k}} \tag{24}$$

over a dual pair $t_1, \ldots, t_n$ and $dt_1, \ldots, dt_n$ of bases. In consequence the twisted quotient comodule $(\mathscr{A}/\mathscr{B})(\, d\,)$ embeds via Δ into a quotient of the free comodule generated by $\mathscr{A}^d$:

$$(\mathscr{A}/_{\mathscr{B}})^{\bullet}(\, d\,) \;\cong\; \mathscr{A}^{\bullet}(\, d\,)/_{\mathscr{B}^{\bullet}(\, d\,)} \;\longrightarrow\; \mathrm{Sym}^{\bullet} T^* \otimes \mathscr{A}^d/_{\Delta(\,\mathscr{B}^{\bullet}(\, d\,))}$$

All generators of $\mathscr{A}/\mathscr{B}$ of degree at least d are thus generators of a quotient of the free comodule $\mathrm{Sym}\, T^* \otimes \mathscr{A}^d$ as well. An upper bound for the dimension of the space of generators of quotients of free comodules however can be calculated quite effectively using the fundamental ideas underlying the construction of Gröbner bases. On the other hand the quotient comodule $\mathscr{A}/\mathscr{B}$ has finite dimensional homogeneous subspaces and thus only a finite dimensional space of generators of degrees less than d, hence we end up with a finite dimensional space $\mathrm{Gen}\,(\mathscr{A}/\mathscr{B})$ of generators of arbitrary degree.

In general the direct calculation of the Spencer cohomology of a comodule can get quite involved. A convenient alternative, at least for a comodule $\mathscr{A}$ with a large symmetry group, is to construct an initial free resolution of length $r \geq 0$ for $\mathscr{A}$ first, this is an exact sequence

$$0 \longrightarrow \mathscr{A}^{\bullet+d_0} \stackrel{\Phi_0}{\longrightarrow} \mathrm{Sym}^{\bullet} T^* \otimes V_0 \stackrel{\Phi_1}{\longrightarrow} \mathrm{Sym}^{\bullet-d_1} T^* \otimes V_1$$
$$\stackrel{\Phi_2}{\longrightarrow} \ldots \stackrel{\Phi_r}{\longrightarrow} \mathrm{Sym}^{\bullet-d_1-\ldots-d_r} T^* \otimes V_r$$

with suitable comodule homomorphisms $\Phi_0, \ldots, \Phi_r$ of degrees $-d_0, \ldots, -d_r$ respectively. The difference here to an actual free resolution of the comodule $\mathscr{A}$ is that we do not ask for Φ_r to be surjective. Comodules allowing an initial free resolution of some length $r \geq 0$ are rather special of course, to the very least they are isomorphic via Φ_0 to subcomodules of free comodules. In practice however it is often easy to guess an initial free resolution and apply the following lemma to obtain information about the Spencer cohomology:

Lemma 3.6 (Initial Free Resolutions and Spencer Cohomology). *Consider a comodule $\mathscr{A}^\bullet$, which allows an initial free resolution of length $r \geq 0$ of the form*

$$0 \longrightarrow \mathscr{A}^{\bullet+d_0} \stackrel{\Phi_0}{\longrightarrow} \mathrm{Sym}^\bullet T^* \otimes V_0 \stackrel{\Phi_1}{\longrightarrow} \mathrm{Sym}^{\bullet-d_1} T^* \otimes V_1$$
$$\stackrel{\Phi_2}{\longrightarrow} \ldots \stackrel{\Phi_r}{\longrightarrow} \mathrm{Sym}^{\bullet-d_1-\ldots-d_r} T^* \otimes V_r$$

with comodule homomorphisms $\Phi_0, \ldots, \Phi_r$ of degrees $-d_0, \ldots, -d_r$ respectively. Independent of whether the last comodule homomorphism Φ_r is surjective or not the only non-vanishing Spencer cohomology spaces of $\mathscr{A}$ of form degree $\circ \leq r$ at most equal to r are:

$$H^{d_0,0}(\mathscr{A}) \cong V_0 \quad H^{d_0+d_1-1,1}(\mathscr{A}) \cong V_1 \quad \ldots \quad H^{d_0+\ldots+d_r-r,r}(\mathscr{A}) \cong V_r$$

It is a pity that the only conceptual proof of this lemma I know of requires some knowledge of spectral sequences, which is quite formidable a concept from homological algebra for an introductory text like this one on exterior differential systems. In essence however spectral sequences are just a highly efficient tool to facilitate certain types of diagram chases. The spectral sequence accelerated diagram chases proving the Lemma of Five, the Lemma of Nine and the Snake Lemma for example are almost trivial. Perhaps our use of spectral sequences in this section motivates the reader unacquainted with the concept to study spectral sequences from this point of view to accelerate her or his future diagram chases.

Using spectral sequences the proof of Lemma 3.6 proceeds along the following line of argument. In a first step we extend the given initial free resolution of length $r \geq 0$ to the right by the projection onto the cokernel of Φ_r in order to obtain an exact sequence:

$$0 \longrightarrow \mathscr{A}^{\bullet+d_0} \stackrel{\Phi_0}{\longrightarrow} \mathrm{Sym}^\bullet T^* \otimes V_0 \stackrel{\Phi_1}{\longrightarrow} \mathrm{Sym}^{\bullet-d_1} T^* \otimes V_1$$
$$\stackrel{\Phi_2}{\longrightarrow} \ldots \stackrel{\Phi_r}{\longrightarrow} \mathrm{Sym}^{\bullet-d_1-\ldots-d_r} T^* \otimes V_r \stackrel{\mathrm{pr}}{\longrightarrow} \mathscr{C}^{\bullet-d_1-\ldots-d_r} \longrightarrow 0$$

Thinking of this exact sequence of comodules as a complex with trivial homology and taking Spencer cochains we obtain a double complex with columns given by the Spencer complexes of the comodules involved, while the rows are all copies of the original exact sequence tensored with $\Lambda^\circ T^*$. Of course we would prefer to have the two coboundary operators in this double complex anticommuting instead of commuting, the difference however plays a negligible role in the construction of the two spectral sequences associated to a double complex.

By assumption the initial free resolution extended by the projection to the cokernel comodule $\mathscr{C}$ of Φ_r is exact everywhere, hence the rows first spectral sequence associated to our double complex collapses at its E^1-term, simply because it equals $\{0\}$ everywhere, in consequence the columns first spectral sequence necessarily converges to $\{0\}$ as well. On calculating its E^1-term however we obtain the Spencer cohomology of $\mathscr{A}$ in the first column, the Spencer cohomology of $\mathscr{C}$ in the last column with the vector spaces $V_0, \ldots, V_r$ in between in the first row representing the Spencer cohomology of the free comodules forming the initial free resolution of $\mathscr{A}$. A spectral sequence with such an E^1-term has only one chance left to converge to $\{ 0 \}$, namely the higher order coboundary operators must induce isomorphisms

$$H^{\bullet + d_0 - s,\, s}(\, \mathscr{A}\,) \stackrel{\cong}{\longrightarrow} [\, V_s\,]_{\bullet = d_1 + \ldots + d_s}$$

of graded vector spaces for all $s = 0, \ldots, r$ as well as for all $s > r$ isomorphisms:

$$H^{\bullet + d_0 - r - 1,\, s}(\, \mathscr{A}\,) \stackrel{\cong}{\longrightarrow} H^{\bullet - d_1 - \ldots - d_r,\, s - r - 1}(\, \mathscr{C}\,)$$

Apropos spectral sequences by far the most useful spectral sequence in the theory of comodules is not the spectral sequence discussed above, but the spectral sequence arising from a peculiar double Spencer complex. In general the graded tensor product $\mathrm{Sym}^\bullet T^* \otimes \mathscr{A}^\bullet$ of the free comodule $\mathrm{Sym}^\bullet T^*$ with a comodule $\mathscr{A}^\bullet$ can be turned into a comodule in two different ways with different directional derivatives. Namely it can be considered as a free comodule $\mathrm{Sym}^\bullet T^* \otimes \mathscr{A}^\bullet_{\mathrm{trivial}}$ generated by the graded vector space $\mathscr{A}^\bullet_{\mathrm{trivial}}$ underlying $\mathscr{A}$ with directional derivatives $\frac{\partial}{\partial t} \otimes \mathrm{id}_{\mathscr{A}}$ or it can be considered as a tensor product $\mathrm{Sym}^\bullet T^* \otimes \mathscr{A}^\bullet$ of comodules with directional derivatives dictated by the usual Leibniz rule:

$$\Big(\frac{\partial}{\partial t} \Big)^{\otimes} \;:=\; \frac{\partial}{\partial t} \otimes \mathrm{id}_{\mathscr{A}} \;+\; \mathrm{id}_{\mathrm{Sym}\, T^*} \otimes \frac{\partial}{\partial t}$$

For a comodule $\mathscr{A}$ bounded below the resulting two comodules are actually isomorphic via

$$\exp\ P : \quad \mathrm{Sym}^\bullet T^* \otimes \mathscr{A}^\bullet \stackrel{\cong}{\longrightarrow} \mathrm{Sym}^\bullet T^* \otimes \mathscr{A}^\bullet_{\mathrm{trivial}}, \qquad \psi \longmapsto \sum_{r \geq 0} \frac{1}{r!} P^r \psi$$

where $P : \mathrm{Sym}^\bullet T^* \otimes \mathscr{A}^\bullet \longrightarrow \mathrm{Sym}^{\bullet+1} T^* \otimes \mathscr{A}^{\bullet - 1}$ is defined as the sum over a dual pair

$$P \;:=\; \sum_{\mu=1}^{n} dt_\mu \cdot \otimes \frac{\partial}{\partial t_\mu}$$

of bases $t_1, \ldots, t_n$ and $dt_1, \ldots, dt_n$ for T and T^* respectively. In fact P is at least locally nilpotent for a comodule $\mathscr{A}$ bounded below so that its exponential $\exp\ P$ is well-defined, moreover the commutator $[\, \frac{\partial}{\partial t} \otimes \mathrm{id}_{\mathscr{A}},\ P\,] = \mathrm{id}_{\mathrm{Sym}\, T^*} \otimes \frac{\partial}{\partial t}$ commutes with P and the identity

$$\big(\frac{\partial}{\partial t}\otimes \mathrm{id}_{\mathscr{A}}\big)\circ \exp\ P = \exp\ P\circ\big(\frac{\partial}{\partial t}\otimes\mathrm{id}_{\mathscr{A}}\big) + \big[\,\big(\frac{\partial}{\partial t}\otimes\mathrm{id}_{\mathscr{A}}\big),\ \exp\ P\,\big]$$
$$= \exp\ P\circ\Big(\frac{\partial}{\partial t}\otimes\mathrm{id}_{\mathscr{A}} + \mathrm{id}_{\mathrm{Sym}\,T^*}\otimes\frac{\partial}{\partial t}\Big)$$

shows that $\exp\ P$ is a homomorphism of comodules with inverse $\exp(-P)$. In a sense the resulting isomorphism $\mathrm{Sym}^\bullet T^*\otimes\mathscr{A}^\bullet\cong\mathrm{Sym}^\bullet T^*\otimes\mathscr{A}^\bullet_{\mathrm{trivial}}$ of comodules tells us that a general comodule $\mathscr{A}$ bounded below is not too different from a free comodule. A convenient method to make this structural statement about comodules bounded below precise is to consider the two spectral sequences associated to the double Spencer complex

$$\begin{array}{ccccc}
 & \Big\downarrow{\scriptstyle B} & & \Big\downarrow{\scriptstyle B} & \\
\xrightarrow{\ b\ } & \mathrm{Sym}^{\bullet+1}T^*\otimes\mathscr{A}^{\blacklozenge+1}\otimes\Lambda^{\circ}T^* & \xrightarrow{\ b\ } & \mathrm{Sym}^{\bullet}T^*\otimes\mathscr{A}^{\blacklozenge+1}\otimes\Lambda^{\circ+1}T^* & \xrightarrow{\ b\ } \\
 & \Big\downarrow{\scriptstyle B} & & \Big\downarrow{\scriptstyle B} & \\
\xrightarrow{\ b\ } & \mathrm{Sym}^{\bullet+1}T^*\otimes\mathscr{A}^{\blacklozenge}\otimes\Lambda^{\circ+1}T^* & \xrightarrow{\ b\ } & \mathrm{Sym}^{\bullet}T^*\otimes\mathscr{A}^{\blacklozenge}\otimes\Lambda^{\circ+2}T^* & \xrightarrow{\ b\ } \\
 & \Big\downarrow{\scriptstyle B} & & \Big\downarrow{\scriptstyle B} &
\end{array} \tag{25}$$

where B and b are the anticommuting Spencer operators for $\mathscr{A}$ and $\mathrm{Sym}\,T^*$ respectively with the other factor merely serving as additional coefficients. The b-first spectral sequence collapses at its E^1-term, simply because it is concentrated in forms degree $\circ = 0$

$$\delta_{\bullet=0=\circ}\ \mathscr{A}^{\blacklozenge}$$

and so it is impossible that any of the higher coboundary operators are non-trivial. Things are quite different for the B-first spectral sequence however, which turns into an efficient algorithm to reconstruct a comodule $\mathscr{A}$ from its Spencer cohomology:

Lemma 3.7 (Standard Spectral Sequence of a Comodule). *Every finitely generated comodule $\mathscr{A}$ bounded below carries a canonical complete filtration*

$$\mathscr{A}^\bullet\ \supseteq\ \ldots\ \supseteq\ (F^{-1}\mathscr{A})^\bullet\ \supseteq\ (F^0\mathscr{A})^\bullet\ \supseteq\ (F^{+1}\mathscr{A})^\bullet\ \supseteq\ \ldots\ \supseteq\ \{0\}$$

by the subcomodules $F^k\mathscr{A}$ generated in degrees greater than or equal to $k\in\mathbb{Z}$ in the sense:

$$(F^k\mathscr{A})^\bullet\ :=\ \mathbf{ker}\Big(\mathscr{A}^\bullet\ \xrightarrow{\ \Delta\ }\ \mathrm{Sym}^{\bullet-k+1}T^*\otimes\mathscr{A}^{k-1}\Big)$$

Whereas the b-first spectral sequence associated to the double Spencer complex (25) collapses at its E^1-term, the E^1-term of the B-first spectral sequence reflects the Spencer cohomology

$$\mathrm{Sym}^\bullet T^*\otimes H^{\blacklozenge,\circ}(\mathscr{A})\ \Longrightarrow\ \delta_{\circ=0}\,(F^{\blacklozenge}\mathscr{A}/_{F^{\blacklozenge+1}\mathscr{A}})^{\bullet+\blacklozenge}$$

of $\mathscr{A}$ *and the spectral sequence converges to the successive quotients of the filtration subcomodules* $F^{\blacklozenge}\mathscr{A}$*. In addition the coboundary operator* B_1 *for the* E^1*-term is completely determined by the right* $\Lambda^{\circ}T^*$*-module structure on the Spencer cohomology* $H^{\blacklozenge,\circ}(\mathscr{A})$.

Perhaps the most striking application of the standard spectral sequence with a very practical appeal is the following explicit formula for the dimensions of the homogeneous subspaces of a finitely generated comodule bounded below, which reflects the equality of the E^1-Euler characteristics of the two spectral sequences associated to the double Spencer complex (25):

Corollary 3.8 (Poincaré Function of a Comodule). *The dimensions of the homogeneous subspaces* $\mathscr{A}^k$, $k \in \mathbb{Z}$, *of a finitely generated comodule* $\mathscr{A}$ *bounded below can be calculated from the Betti numbers* $\dim\ H^{\bullet,\circ}(\mathscr{A})$ *of its Spencer cohomology and the dimension* $n := \dim\ T$ *of the vector space* T *by means of the formula:*

$$\dim\ \mathscr{A}^k \quad = \sum_{\substack{r=0,\ldots,n\\ d\in\mathbb{Z}\\ d+r\leq k}} (-1)^r \binom{k-d-r+n-1}{n-1} \dim\ H^{d,r}(\mathscr{A})$$

In particular $\dim\ \mathscr{A}^k$ *equals the value of a polynomial in* k *of degree at most* $n-1$ *for all* $k > d_{\max}$*, where* $d_{\max} \in \mathbb{Z}$ *is chosen so that* $H^{d,r}(\mathscr{A}) = \{0\}$ *for all* $d > d_{\max}$ *and all* r.

Proof. The two spectral sequences associated to the double Spencer complex (25) arise from the two anticommuting Spencer coboundary operators B and b, which are trihomogeneous of tridegrees $(0, -1, +1)$ and $(-1, 0, +1)$ respectively with respect to the trigrading on $\mathrm{Sym}^{\bullet}T^* \otimes \mathscr{A}^{\blacklozenge} \otimes \Lambda^{\circ}T^*$. In particular both B and b preserve the total grading so that the both spectral sequences actually decompose into the direct sum of spectral sequences

$$\mathrm{Sym}^{\bullet}T^* \otimes \mathscr{A}^{\blacklozenge} \otimes \Lambda^{\circ}T^* \quad = \bigoplus_{k\in\mathbb{Z}} \Big(\mathrm{Sym}^{\bullet}T^* \otimes \mathscr{A}^{\blacklozenge} \otimes \Lambda^{\circ}T^*\Big)_{\bullet+\blacklozenge+\circ=k}$$

parametrized by the total degree $k \in \mathbb{Z}$. The total degree k part of the b-first spectral sequence collapses as before at its E^1-term $\delta_{\bullet=0=\circ}\mathscr{A}^k$ of Euler characteristic $\dim\ \mathscr{A}^k$. According to Lemma 3.7 the total degree k-part of the E^1-term of the B-first spectral sequence reads $\mathrm{Sym}^{k-\blacklozenge-\circ}T^* \otimes H^{\blacklozenge,\circ}(\mathscr{A})$, its Euler characteristic is thus finite and given by

$$\sum_{\substack{r=0,\ldots,n\\ d\in\mathbb{Z}}} (-1)^r \dim\ \mathrm{Sym}^{k-d-r}T^* \otimes H^{d,r}(\mathscr{A})$$

$$= \sum_{\substack{r=0,\ldots,n\\ d\in\mathbb{Z}\\ d+r\leq k}} (-1)^r \binom{k-d-r+n-1}{n-1} \dim\ H^{d,r}(\mathscr{A})$$

because $H(\mathscr{A})$ is a finite-dimensional vector for the finitely generated comodule $\mathscr{A}$ bounded below. The Euler characteristic of every complex on the other hand equals the Euler characteristic of its cohomology, in turn the Euler characteristic is constant all along a spectral sequence, which is in essence a sequence of coboundary operators each defined on the *cohomology* of the previous operator. With the E^∞-terms of the two spectral sequences arising from the double Spencer complex (25) being isomorphic the stipulated formula for $\dim \mathscr{A}^k$ simply reflects the equality of the two different E^1-Euler characteristics.

The Spencer cohomology of the finitely generated comodule $\mathscr{A}$ bounded below is a finite dimensional vector space according to Lemma 3.5, hence we may certainly choose $d_{\max} \in \mathbb{Z}$ so that $H^{d,r}(\mathscr{A}) = \{0\}$ for all $d > d_{\max}$ and all $r = 0, \ldots, n$. For all degrees $k > d_{\max}$ the original summation calculating $\dim \mathscr{A}^k$ can be simplified to read

$$\dim \mathscr{A}^k \quad = \sum_{\substack{r=0,\ldots,n \\ d\in\mathbb{Z}}} (-1)^r \begin{pmatrix} k-d-r+n-1 \\ n-1 \end{pmatrix} \dim H^{d,r}(\mathscr{A}) \qquad (26)$$

because all summands with $d + r > k$ vanish automatically. In fact either $d > d_{\max}$ or $d \leq d_{\max} < k$, in the first case $\dim H^{d,r}(\mathscr{A}) = 0$, whereas $\binom{k-d-r+n-1}{n-1} = 0$ in the second case due to $n-1 > k-d-r+n-1 \geq 0$. The simplified summation (26) however defines a polynomial of degree at most $n-1$ in k equal to $\dim \mathscr{A}^k$ for $k > d_{\max}$. □

Another direct application of the standard spectral sequence leads to a kind of converse to Lemma 3.6. Consider a comodule $\mathscr{A}$ bounded below satisfying the additional condition that its only non-vanishing Spencer cohomology in form degrees $\circ = 0, \ldots, r$ is concentrated in

$$V_0 := H^{d_0,0}(\mathscr{A}) \quad V_1 := H^{d_0+d_1-1,1}(\mathscr{A}) \quad \ldots \quad V_r := H^{d_0+\ldots+d_r-r,r}(\mathscr{A})$$

for suitable integers $d_0, \ldots, d_r \in \mathbb{Z}$. The integers $d_1, \ldots, d_r \geq 1$ are then actually positive except for d_0 and the comodule $\mathscr{A}$ has an initial free resolution of length $r \geq 0$ by free comodules linked by comodule homomorphisms $\Phi_0, \ldots, \Phi_r$ of degrees $-d_0, \ldots, -d_r$

$$\begin{aligned} 0 \longrightarrow \mathscr{A}^{\bullet+d_0} \xrightarrow{\Phi_0} \mathrm{Sym}^\bullet T^* \otimes V_0 \xrightarrow{\Phi_1} \mathrm{Sym}^{\bullet-d_1} T^* \otimes V_1 \\ \xrightarrow{\Phi_2} \ldots \xrightarrow{\Phi_r} \mathrm{Sym}^{\bullet-d_1-\ldots-d_r} T^* \otimes V_r \end{aligned}$$

which are determined by the higher order coboundary operators of the standard spectral sequence of Lemma 3.7. In particular the comodule homomorphism Φ_0 identifies $\mathscr{A}^\bullet$ with a subcomodule of the shifted free comodule $\mathrm{Sym}^{\bullet-d_0} T^* \otimes V_0$ determined by the tableau:

$$\mathscr{A}^{d_0+d_1} \;\cong\; \mathbf{ker}\Big(\; \Phi_1 : \;\; \mathrm{Sym}^{d_1} T^* \otimes V_0 \;\longrightarrow\; V_1 \;\Big)$$

In the following section we will study the structure of such tableau comodules in more detail.

4 Algebraic Properties of Tableau Comodules

In the algebraic analysis of exterior differential systems the comodules of interest are usually tableau comodules, comodules which arise as the kernels of homogeneous homomorphisms between free comodules. Tableau comodules and the partial differential equations they represent are classified, albeit rather superficially, into underdetermined, determined and overdetermined tableau comodules depending on the ranks of the free comodules involved in their definition. Underdetermined partial differential equations can usually be studied successfully with methods from functional analysis, while integrability constraints will likely thwart such an approach for a given overdetermined partial differential equation.

Perhaps the most interesting case of this superficial classification of partial differential equations is the limiting case of both realms: The Euler–Lagrange equations associated to a variational principle and the elliptic differential equations studied in global analysis are always determined partial differential equations. Mathematical physics for example favors determined partial differential equations according to the following metaprinciple: Reasonable field equations should allow for a unique solution for arbitrarily given Cauchy data. In this section we will discuss the three classical statements about under- and overdetermined partial differential equations from the point of view of their associated tableau comodules:

- Formal Integrability of underdetermined differential equations.
- Complex Characterization of finite type differential equations.
- Cartan's Test for Involutivity of first order tableau comodules.

In order to begin our study of tableau comodules let us have a closer look at a non-trivial homogeneous homomorphism $\Phi : \mathrm{Sym}^{\bullet} T^* \otimes V \longrightarrow \mathrm{Sym}^{\bullet - d} T^* \otimes E$ between free comodules. As a homomorphism of comodules Φ maps the space V of generators of the domain to generators of the codomain $\mathrm{Sym}\, T^* \otimes E$ including 0 so that $d \in \mathbb{N}_0$ is necessarily non-negative. Moreover it is easily seen that Φ is completely determined by its restriction ϕ to the subspace $\mathrm{Sym}^d T^* \otimes V \subset \mathrm{Sym}^{\bullet} T^* \otimes V$ of elements of degree d. Conversely every linear map $\phi : \mathrm{Sym}^d T^* \otimes V \longrightarrow E$ extends in a unique way to a homomorphism of comodules

$$\Phi : \quad \mathrm{Sym}^{\bullet} T^* \otimes V \;\longrightarrow\; \mathrm{Sym}^{\bullet - d} T^* \otimes E$$

of degree $-d$, which can be written in terms of directional derivatives as an iterated sum

$$\Phi(\,\psi \otimes v\,) \;:=\; \sum_{\mu_1,\ldots,\mu_d=1}^{n} \frac{\partial^d \psi}{\partial t_{\mu_1} \ldots \partial t_{\mu_d}} \otimes \phi\Big(\,\frac{1}{d!}\,dt_{\mu_1} \cdot \ldots \cdot dt_{\mu_d} \otimes v\,\Big) \tag{27}$$

over a basis $t_1, \ldots, t_n$ of T and its dual basis $dt_1, \ldots, dt_n$ of T^*. In the spirit of partial differential equations we may interpret the original linear map $\phi\,:\,\mathrm{Sym}^d T^* \otimes V \longrightarrow E$ as a linear differential operator $D_\phi\,:\,C^\infty(\,T,\,V\,) \longrightarrow C^\infty(\,T,\,E\,)$ of order d defined by:

$$(\,D_\phi \psi\,)(\,p\,) \;:=\; \phi\Big(\,\sum_{\mu_1,\ldots,\mu_d=1}^{n} \frac{1}{d!}\,dt_{\mu_1} \cdot \ldots \cdot dt_{\mu_d} \otimes \frac{\partial^d\,\psi}{\partial t_{\mu_1} \ldots \partial t_{\mu_d}}(\,p\,)\,\Big) \tag{28}$$

The associated partial differential equation $D_\phi \psi = 0$ can be written as a system of $\dim\,E$ scalar differential equations in the $\dim\,V$ unknown scalar components of $\psi \in C^\infty(\,T,\,V\,)$, for this reason the equation is called underdetermined, determined or overdetermined respectively, if there are less, an equal number of or more equations than unknown functions:

$$\begin{array}{ll} \textbf{underdetermined:} & \dim\,E \leq \dim\,V \\ \textbf{determined:} & \dim\,E = \dim\,V \\ \textbf{overdetermined:} & \dim\,E \geq \dim\,V \end{array} \tag{29}$$

The homomorphism $\Phi\,:\,\mathrm{Sym}^\bullet T^* \otimes V \longrightarrow \mathrm{Sym}^{\bullet-k} T^* \otimes E$ of free comodules associated to ϕ is nothing else but the restriction of the operator D_ϕ to the subspace $\mathrm{Sym}\,T^* \otimes V \subset C^\infty(\,T,\,V\,)$ of polynomials on T with values in V. In particular its kernel comodule agrees with the space of polynomial solutions $\psi \in \mathrm{Sym}\,T^* \otimes V$ to the partial differential equation $D_\phi \psi = 0$:

Definition 4.1 (Tableaux and Comodules). A tableau of order $d \geq 1$ is by definition a subspace $\mathscr{A}^d \subset \mathrm{Sym}^d T^* \otimes V$ of the vector space $\mathrm{Sym}^d T^* \otimes V$ of homogeneous polynomials of degree d on T with values in V. The tableau comodule $\mathscr{A}^\bullet \subset \mathrm{Sym}^\bullet T^* \otimes V$ associated to a tableau $\mathscr{A}^d$ is the kernel of the homomorphism

$$0 \longrightarrow \mathscr{A}^\bullet \stackrel{\subset}{\longrightarrow} \mathrm{Sym}^\bullet T^* \otimes V \stackrel{\Phi}{\longrightarrow} \mathrm{Sym}^{\bullet-d} T^* \otimes E$$

of free comodules induced by some linear map $\phi\,:\,\mathrm{Sym}^d T^* \otimes V \longrightarrow E$ with kernel $\mathscr{A}^d$. A tableau comodule $\mathscr{A}$ is called underdetermined, determined or overdetermined provided:

$$\mathrm{codim}\,\mathscr{A}^d \;\leq\; \dim\,V \qquad \mathrm{codim}\,\mathscr{A}^d \;=\; \dim\,V \qquad \mathrm{codim}\,\mathscr{A}^d \;\geq\; \dim\,V$$

Of course one possible choice for the linear map ϕ in the definition is simply the canonical projection pr : $\mathrm{Sym}^d T^* \otimes V \longrightarrow \mathrm{Sym}^d T^* \otimes V/\mathscr{A}^d$, other choices however are convenient to avoid the typographical monster $\mathrm{Sym}^d T^* \otimes V/\mathscr{A}^d$. Whatever the preferred choice the tableau comodule $\mathscr{A}$ does only depend on $\mathscr{A}^d = \mathbf{ker}\,\phi$, for this reason its homogeneous subspaces $\mathscr{A}^{d+1}, \mathscr{A}^{d+2}, \ldots$ are sometimes called the first and the second prolongation of $\mathscr{A}^d$ etc. Of little concern is the equality $\mathscr{A}^k = \mathrm{Sym}^k T^* \otimes V$ for $k < d$, because in general we are interested in the behavior of $\mathscr{A}^k$ at large degrees $k \gg 0$. According to Lemma 3.6 the non-vanishing Spencer cohomology of a tableau comodule in form degrees $\circ = 0, 1$ reads:

$$H^{0,0}(\,\mathscr{A}\,) \;=\; V \qquad\qquad H^{d-1,1}(\,\mathscr{A}\,) \;=\; \mathrm{Sym}^d T^* \otimes V/_{\mathscr{A}^d} \tag{30}$$

In order to reduce the complexity it seems like a good idea to replace the linear map ϕ from the complicated and high-dimensional vector space $\mathrm{Sym}^d T^* \otimes V$ with its localizations

$$\phi_\xi : \quad V \longrightarrow E, \qquad v \longmapsto \phi(\,\tfrac{1}{d!}\xi^d \otimes v\,)$$

at covectors $\xi \in T^*$. In this way we are interpreting the linear map $\phi : \mathrm{Sym}^d T^* \otimes V \longrightarrow E$ via the vector space isomorphism $\mathrm{Hom}\,(\,\mathrm{Sym}^d T^* \otimes V,\ E\,) \cong \mathrm{Sym}^d T \otimes \mathrm{Hom}\,(\,V,\ E\,)$ as a homogeneous polynomial of degree d on T^* (sic!) with values in $\mathrm{Hom}\,(\,V,\ E\,)$. Motivated by this interpretation of ϕ we define the characteristic (projective) variety of a tableau $\mathscr{A}^d$ by

$$\mathscr{X}(\,\mathscr{A}^d\,) \;:=\; \{\,[\,\xi\,] \in \mathbb{P}T^* \mid \phi_\xi : V \longrightarrow E \text{ is not surjective }\} \tag{31}$$

this is $\xi \in T^*$ is a characteristic covector, if and only if ϕ_ξ *fails* to be surjective. Evidently the tableau $\mathscr{A}^d$ has to be underdetermined to allow some non-characteristic covector:

Theorem 4.2 (Formal Integrability of Underdetermined Equations). *Consider a linear map $\phi : \mathrm{Sym}^d T^* \otimes V \longrightarrow E$ possessing at least one non-characteristic covector $\xi \in T^*$ in the sense that the localization $\phi_\xi : V \longrightarrow E,\ v \longmapsto \phi(\frac{1}{d!}\xi^d \otimes v)$, of ϕ at ξ is surjective. The homomorphism of free comodules defining the tableau comodule $\mathscr{A}$ associated to the tableau $\mathscr{A}^d := \mathbf{ker}\,\phi$ is surjective, too, with associated short exact sequence:*

$$0 \longrightarrow \mathscr{A}^\bullet \stackrel{\subset}{\longrightarrow} \mathrm{Sym}^\bullet T^* \otimes V \stackrel{\Phi}{\longrightarrow} \mathrm{Sym}^{\bullet-d} T^* \otimes E \longrightarrow 0$$

According to Lemma 3.6 the only non-vanishing Spencer cohomology spaces of $\mathscr{A}$ are:

$$H^{0,0}(\,\mathscr{A}\,) \;=\; V \qquad\qquad H^{d-1,1}(\,\mathscr{A}\,) \;=\; E$$

By far the most important conclusion of this theorem is that underdetermined partial differential equations have no Spencer cohomology of form degree $\circ = 2$, in consequence there are no obstructions at all to the recursive procedure discussed in Sect. 5 to construct infinite order formal power series solutions for arbitrarily specified Cauchy data. In other words Theorem 4.2 is exactly the reason, why the term Spencer cohomology is never even mentioned in text books studying partial differential equations in the language of Functional Analysis: Banach and Sobolev spaces etc.

Despite its importance the proof of Theorem 4.2 is rather straightforward. Fixing a non-characteristic covector $\xi \in T^*$ with surjective localization $\phi_\xi : V \longrightarrow E,\, v \longmapsto \phi(\frac{1}{d!}\xi^d \otimes v)$, we try to construct a preimage of a vector $\frac{1}{k!}\alpha^k \otimes e \in \mathrm{Sym}^k T^* \otimes E$ under the comodule homomorphism $\Phi : \mathrm{Sym}^\bullet T^* \otimes V \longrightarrow \mathrm{Sym}^{\bullet - d} T^* \otimes E$ extending ϕ by making an ansatz

$$\sum_{\mu=0}^{k} \tfrac{1}{(k-\mu)!}\,\alpha^{k-\mu}\,\tfrac{1}{(d+\mu)!}\,\xi^{d+\mu} \otimes v_\mu \quad \in \quad \mathrm{Sym}^{k+d} T^* \otimes V$$

with as yet unknown parameter vectors $v_0, \ldots, v_k \in V$. Inserting this ansatz into the definition (27) of the comodule homomorphism Φ we get after some auxiliary calculations:

$$\begin{aligned}
&\Phi\Big(\sum_{\mu=0}^{k} \tfrac{1}{(k-\mu)!}\,\alpha^{k-\mu}\,\tfrac{1}{(d+\mu)!}\,\xi^{d+\mu} \otimes v_\mu \Big) \\
&= \sum_{s=0}^{k} \tfrac{1}{(k-s)!}\,\alpha^{k-s}\,\tfrac{1}{s!}\,\xi^{s} \otimes \phi\Big[\sum_{\mu=0\vee(s-d)}^{s} \tfrac{1}{(s-\mu)!}\,\alpha^{s-\mu}\,\tfrac{1}{(d+\mu-s)!}\,\xi^{d+\mu-s} \otimes v_\mu \Big] \\
&= \sum_{s=0}^{k} \tfrac{1}{(k-s)!}\,\alpha^{k-s}\,\tfrac{1}{s!}\,\xi^{s} \\
&\qquad \otimes \Big(\phi_\xi v_s \;+\; \phi\Big[\sum_{\mu=0\vee(s-d)}^{s-1} \tfrac{1}{(s-\mu)!}\,\alpha^{s-\mu}\,\tfrac{1}{(d+\mu-s)!}\,\xi^{d+\mu-s} \otimes v_\mu \Big] \Big)
\end{aligned}$$

Due to the surjectivity of the localization $\phi_\xi : V \longrightarrow E$ at the non-characteristic covector ξ we may thus choose the parameters $v_0, \ldots, v_k \in V$ of our ansatz recursively to satisfy

$$\begin{aligned}
\phi_\xi v_0 &= e \\
\phi_\xi v_1 &= -\phi\Big[\tfrac{1}{1!}\,\alpha^1\,\tfrac{1}{(d-1)!}\,\xi^{d-1} \otimes v_0 \Big] \\
\phi_\xi v_2 &= -\phi\Big[\tfrac{1}{2!}\,\alpha^2\,\tfrac{1}{(d-2)!}\,\xi^{d-2} \otimes v_0 \;+\; \tfrac{1}{1!}\,\alpha^1\,\tfrac{1}{(d-1)!}\,\xi^{d-1} \otimes v_1 \Big]
\end{aligned}$$

etc. in order to obtain a preimage of $\frac{1}{k!}\alpha^k \otimes e \in \mathrm{Sym}^k T^* \otimes E$ under Φ. In this argument $\alpha \in T^*$ and $e \in E$ as well as $k \in \mathbb{N}_0$ were all arbitrary so that Φ is surjective

$$\mathbf{im}\,\Phi \quad \supset \quad \mathrm{span}\{\, \tfrac{1}{k!}\,\alpha^k \otimes e \;\mid\; \alpha \in T^*,\ e \in E,\ k \in \mathbb{N}_0 \,\} \quad = \quad \mathrm{Sym}\,T^* \otimes E$$

because the polarization formula says that the vectors $\frac{1}{k!}\alpha^k \otimes e$ span $\mathrm{Sym}^k T^* \otimes E$. Unluckily the other two classical statements about tableau comodules discussed in this section are more difficult, in particular the following characterization of partial differential equations of finite type as complex elliptic differential equations requires confidence in multilinear algebra:

Theorem 4.3 (Complex Elliptic Partial Differential Equations). *The homomorphism Φ of free comodules extending a given linear map $\phi : \mathrm{Sym}^d T^* \otimes V \longrightarrow E$ with kernel tableau $\mathscr{A}^d := \mathbf{ker}\,\phi$ has a finite-dimensional kernel comodule $\mathscr{A}$*

$$0 \longrightarrow \mathscr{A}^\bullet \stackrel{\subset}{\longrightarrow} \mathrm{Sym}^\bullet T^* \otimes V \stackrel{\Phi}{\longrightarrow} \mathrm{Sym}^{\bullet - d} T^* \otimes E$$

if and only if only the complex localizations of ϕ at complex valued linear forms $\xi_{\mathbb{C}} \in T^* \otimes_{\mathbb{R}} \mathbb{C}$:

$$\phi_{\xi_{\mathbb{C}}} :\; V \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow E \otimes_{\mathbb{R}} \mathbb{C}, \qquad v_{\mathbb{C}} \longmapsto \phi(\, \tfrac{1}{d!}\,\xi_{\mathbb{C}}^d \otimes v_{\mathbb{C}}\,)$$

are injective for every non-zero complex-valued linear form $\xi_{\mathbb{C}} \in T^* \otimes_{\mathbb{R}} \mathbb{C} \setminus \{0\}$.

Essentially this theorem is a consequence of Hilbert's Nullstellensatz in algebraic geometry, although the necessary reformulation (37) can hardly be called obvious. Nevertheless it is well worth the effort to try to understand the main idea of this reformulation, because it provides us with a peculiar kind of upper bound on the growth of a tableau comodule $\mathscr{A}$ in terms of the homogeneous ideal $I^\bullet \subset \mathrm{Sym}^\bullet T$ defining the characteristic variety:

$$\mathscr{X}(\,\mathscr{A}^d\,) \;:=\; \{\, [\,\xi\,] \in \mathbb{P}T^* \mid \phi_\xi : V \longrightarrow E \text{ is not injective } \} \tag{32}$$

Needless to say this definition is different to our previous definition (31) of the characteristic variety, although we may reconcile both definitions by asking for covectors $\xi \in T^*$ such that ϕ_ξ fails to be of the maximal possible rank $\min\{\dim V, \dim E\}$. In other words the redefinition (32) is specific to the study of overdetermined partial differential equations.

Let us begin our discussion of Theorem 4.3 with a small side remark about the rank of a linear map $\phi : V \longrightarrow E$. The canonical isomorphism $\mathrm{Hom}(V, E) \cong V^* \otimes E$ allows us to think of ϕ as an element of the algebra $\Lambda V^* \otimes \Lambda E$ with the (untwisted) tensor product multiplication. Pairing powers of ϕ in this algebra with elements of the dual space we get

$$\langle\, \tfrac{1}{r!}\,\phi^r,\, (\,v_1 \wedge \ldots \wedge v_r\,) \otimes \eta \,\rangle \;=\; \eta(\,\phi\, v_1, \ldots, \phi\, v_r\,) \tag{33}$$

for all $r \geq 1$ and all $v_1, \ldots, v_r \in V$, $\eta \in \Lambda^r E^*$, in particular $\frac{1}{r!}\phi^r = 0$ is equivalent to ϕ being of rank less than r. Similarly we may interpret the linear map $\phi : \mathrm{Sym}^d T^* \otimes V \longrightarrow E$ defining the tableau $\mathscr{A}^d$ as a homogeneous polynomial of degree d on T^* with values in $\mathrm{Hom}(V, E)$ or as an element $\phi \in \mathrm{Sym}^d T \otimes V^* \otimes E$ of the algebra $\mathrm{Sym}\, T \otimes \Lambda V^* \otimes \Lambda E$. In direct generalization of Eq. (33) the powers of ϕ in this algebra satisfy

$$\langle \tfrac{1}{r!}\phi^r, \tfrac{1}{(rd)!}\xi^{rd} \otimes (v_1 \wedge \ldots \wedge v_r) \otimes \eta \rangle = \eta(\phi_\xi v_1, \ldots, \phi_\xi v_r) \tag{34}$$

when paired with elements of the dual space $\mathrm{Sym}^{rd} T^* \otimes \Lambda^r V \otimes \Lambda^r E^*$ with arbitrary $\xi \in T^*$, $v_1, \ldots, v_r \in V$ and $\eta \in \Lambda^r E^*$. Equation (34) implicitly characterizes the covectors $\xi \in T^*$, for which the localization $\phi_\xi : V \longrightarrow E$ fails to have rank at least r, in terms of the power $\frac{1}{r!}\phi^r \in \mathrm{Sym}^{rd} T \otimes \Lambda^r V^* \otimes \Lambda^r E$. In order to make this characterization somewhat more explicit let us consider the following two rearrangements of the factors in (34)

$$\begin{aligned} \iota_r^\phi &: \quad \Lambda^r V \otimes \Lambda^r E^* \longrightarrow \mathrm{Sym}^{rd} T \\ \mu_r^\phi &: \Lambda^{r-1} V \otimes \Lambda^r E^* \longrightarrow \mathrm{Sym}^{rd} T \otimes V^* \end{aligned}$$

characterized as linear maps by:

$$\begin{aligned} \eta(\phi_\xi v_1, \ldots, \phi_\xi v_r) &=: \langle \iota_r^\phi(v_1 \wedge v_2 \wedge \ldots \wedge v_r \otimes \eta), \tfrac{1}{(rd)!}\xi^{rd} \rangle \\ &=: \langle \mu_r^\phi(v_2 \wedge \ldots \wedge v_r \otimes \eta), \tfrac{1}{(rd)!}\xi^{rd} \otimes v_1 \rangle \end{aligned} \tag{35}$$

The localization $\phi_\xi : V \longrightarrow E$ of $\phi : \mathrm{Sym}^d T^* \otimes V \longrightarrow E$ at a covector $\xi \in T^*$ thus has rank less than r, if and only if ξ is a common zero of all polynomials in $\mathbf{im}\, \iota_r^\phi \subset \mathrm{Sym}^{rd} T$ and thus a common zero of all polynomials in the homogeneous ideal generated by $\mathbf{im}\, \iota_r^\phi$:

$$I_r^{\phi\,\bullet} := \langle \mathbf{im}\, \iota_r^\phi \rangle \subset \mathrm{Sym}^\bullet T$$

In consequence the homogeneous ideal $I_r^{\phi\,\bullet}$, $r \geq 1$, defines the projective algebraic variety

$$\begin{aligned} \mathscr{Z}_r(\mathscr{A}^d) &:= \{ [\xi] \in \mathbb{P}T^* \mid \psi(\xi) = 0 \text{ for all polynomials } \psi \in I_r^\phi \} \\ &= \{ [\xi] \in \mathbb{P}T^* \mid \phi_\xi : V \longrightarrow E \text{ has rank less than } r \} \end{aligned}$$

associated to the linear map $\phi : \mathrm{Sym}^d T^* \otimes V \longrightarrow E$ or its tableau comodule $\mathscr{A}$, which is called the rth systolic variety in [7]. In particular the characteristic variety (32) is defined by the homogeneous ideal $I_N^{\phi\,\bullet}$ corresponding to $N := \dim V$.

In the same vein we may consider the graded submodule $M_r^{\phi\,\bullet} := \langle \mathbf{im}\, \mu_r^\phi \rangle \subset \mathrm{Sym}^\bullet T \otimes V^*$ generated by the image of $\mu_r^\phi : \Lambda^{r-1} V \otimes \Lambda^r E^* \longrightarrow \mathrm{Sym}^{rd} T \otimes V^*$.

Although the submodules $M_r^{\phi\,\bullet} \subset \mathrm{Sym}^\bullet T \otimes V^*$ seem to have no direct geometric interpretation in terms of the characteristic variety, they possess an interesting algebraic property in that they are upper bounds for the tableau comodule $\mathscr{A}$ associated to the tableau $\mathscr{A}^d = \mathbf{ker}\,\phi$. To see this point clearly let us rewrite the definition of $\mu_r^\phi : \Lambda^{r-1}V \otimes \Lambda^r E^* \longrightarrow \mathrm{Sym}^{rd}T \otimes V^*$ in the form

$$\begin{aligned}
&\langle\, \mu_r^\phi(\, v_2 \wedge \ldots \wedge v_r \otimes \eta\,),\ \tfrac{1}{(rd)!}\,\xi^{rd} \otimes v\,\rangle \\
&\quad := \eta(\,\phi(\,\tfrac{1}{r!}\,\xi^r \otimes v\,),\ \phi^{v_2}(\,\tfrac{1}{r!}\,\xi^r\,), \ldots, \phi^{v_r}(\,\tfrac{1}{r!}\,\xi^r\,)\,) \\
&\quad = \langle\, \eta,\ (\,\phi \wedge \phi^{v_2} \wedge \ldots \wedge \phi^{v_r}\,)(\,\Delta[\,\tfrac{1}{(rd)!}\,\xi^{rd} \otimes v\,]\,)\,\rangle
\end{aligned}$$

where $\phi^v : \mathrm{Sym}^d T^* \longrightarrow E,\ \frac{1}{d!}\xi^d \longmapsto \phi(\,\frac{1}{d!}\xi \otimes v\,)$, denotes the localization of ϕ at some $v \in V$ and $\Delta : \mathrm{Sym}^{rd}T \otimes V^* \longrightarrow (\,\mathrm{Sym}^d T \otimes V^*\,) \otimes \mathrm{Sym}^d T \otimes \ldots \otimes \mathrm{Sym}^d T$ the comultiplication:

$$\Delta[\,\tfrac{1}{(rd)!}\,\xi^{rd} \otimes v\,] \;:=\; (\,\tfrac{1}{d!}\,\xi^d \otimes v\,) \otimes \underbrace{(\,\tfrac{1}{d!}\,\xi^d\,) \otimes \ldots \otimes (\,\tfrac{1}{d!}\,\xi^d\,)}_{r-1 \text{ times}}$$

The decisive observation linking the tableau comodule $\mathscr{A}$ to the submodules M_r^ϕ, $r \geq 1$, and eventually to the ideals I_r^ϕ, $r \geq 1$, is that the comultiplication Δ restricts to a map

$$\Delta : \quad \mathscr{A}^{rd} \longrightarrow \mathscr{A}^d \otimes \mathrm{Sym}^d T^* \otimes \ldots \otimes \mathrm{Sym}^d T^*$$

simply because $\mathscr{A}$ is after all a comodule over the symmetric coalgebra $\mathrm{Sym}\,T^*$. Hence

$$\langle\, \mu_r^\phi(\, v_2 \wedge \ldots \wedge v_r \otimes \eta\,),\ a\,\rangle \;=\; \langle\, \eta,\ (\,\phi \wedge \phi^{v_2} \wedge \ldots \wedge \phi^{v_r}\,)(\,\Delta\, a\,)\,\rangle \;=\; 0$$

vanishes for all $a \in \mathscr{A}^{rd}$ and all $v_2, \ldots, v_r \in V$, $\eta \in \Lambda^r E^*$ due to the consequence $(\phi \wedge \phi^{v_2} \wedge \ldots \wedge \phi^{v_r})(\,\Delta\, a\,) = 0$ of the equality $\mathscr{A}^d = \mathbf{ker}\,\phi$. In turn the canonical pairing between $\mathrm{Sym}^{rd}T \otimes V^*$ and $\mathrm{Sym}^{rd}T^* \otimes V$ vanishes $\langle\, m,\ a\,\rangle = 0$ on all pairs $a \in \mathscr{A}^{rd}$ and $m \in \mathbf{im}\,\mu_r^\phi = (M_r^\phi)^{rd}$, and this mutual annihilation property extends immediately

$$\langle\, m,\ a\,\rangle \;=\; 0 \tag{36}$$

to all $m \in M_r^{\phi\,\bullet}$, $r \geq 1$, and all $a \in \mathscr{A}^\bullet$, because the submodule $M_r^{\phi\,\bullet} \subset \mathrm{Sym}^\bullet T \otimes V^*$ generated by $\mathbf{im}\,\mu_r^\phi$ is spanned by elements of the form $\frac{1}{s!}\,t^s \cdot m$ with $t \in T$, $s \in \mathbb{N}_0$ and $m \in \mathbf{im}\,\mu_r^\phi$, however all these elements satisfy $\langle\, \frac{1}{s!}\,t^s \cdot m, a\,\rangle = \langle\, m,\ \frac{1}{s!}\,\frac{\partial^s}{\partial t^s}\,a\,\rangle = 0$:

Corollary 4.9 (Upper Bound for Tableau Comodules). *Consider the tableau comodule $\mathscr{A}$ associated to a tableau $\mathscr{A}^d \subset \mathrm{Sym}^d T^* \otimes V$ of order $d \geq 1$ and* *a linear map* $\phi : \mathrm{Sym}^d T^* \otimes V \longrightarrow E$ *realizing* $\mathscr{A}^d$ *in the sense* $\mathbf{ker}\,\phi = \mathscr{A}^d$. *The powers of the linear map* $\phi \in \mathrm{Sym}^d T \otimes V^* \otimes E$ *in the algebra* $\mathrm{Sym}\, T \otimes \Lambda\, V^* \otimes \Lambda\, E$ *give rise to a sequence of linear maps* $\mu^\phi_r : \Lambda^{r-1} V \otimes \Lambda^r E^* \longrightarrow \mathrm{Sym}^{rd} T \otimes V^*$, $r \geq 1$, *with the property*

$$\mathscr{A} \quad \subset \quad \mathrm{Ann}\, M^\phi_r \quad := \quad \{\ a \in \mathrm{Sym}\, T^* \otimes V \ \mid \ \langle\, m,\, a\,\rangle = 0 \textit{ for all } m \in M^\phi_r\ \}$$

where $M^\phi_r := \langle\, \mathbf{im}\,\mu^\phi_r\,\rangle$ *denotes the* $\mathrm{Sym}^\bullet T$*-submodule of* $\mathrm{Sym}^\bullet T \otimes V^*$ *generated by* $\mathbf{im}\,\mu^\phi_r$.

The preceding lemma is certainly interesting for all $r \geq 1$, nevertheless it has an additional twist for r equal to the dimension $N := \dim\, V$ of V in that the inclusion $M^\phi_r \subset I^\phi_r \otimes V^*$ becomes an actual equality $M^\phi_N = I^\phi_N \otimes V^*$ for this r. Choosing a dual pair of bases $v_1, \ldots, v_N$ and $dv_1, \ldots, dv_N$ for the vector spaces V and V^* we may in fact reformulate the identity $\langle\, \mu^\phi_r(\,\tilde{v}_2 \wedge \ldots \wedge \tilde{v}_r \otimes \eta\,),\ \cdot\ \otimes \tilde{v}_1\,\rangle = \langle\, \iota^\phi_r(\,\tilde{v}_1 \wedge \tilde{v}_2 \wedge \ldots \wedge \tilde{v}_r \otimes \eta\,),\ \cdot\ \rangle$ derived from the definition (35) of ι^ϕ_r and μ^ϕ_r into an expansion valid for all $\tilde{v}_2, \ldots, \tilde{v}_r \in V$ and all $\eta \in \Lambda^r E^*$:

$$\mu^\phi_r(\,\tilde{v}_2 \wedge \ldots \wedge \tilde{v}_r \otimes \eta\,) \quad = \quad \sum_{\lambda=1}^{N} \iota^\phi_r(\,v_\lambda \wedge \tilde{v}_2 \wedge \ldots \wedge \tilde{v}_r \otimes \eta\,) \otimes dv_\lambda$$

This expansion tells us $\mathbf{im}\,\mu^\phi_r \subset \mathbf{im}\,\iota^\phi_r \otimes V^*$ and so $M^\phi_r \subset I^\phi_r \otimes V^*$ for all $r \geq 1$. The penultimate exterior power $\Lambda^{N-1} V$ of V however is spanned by the multivectors obtained by removing a factor v_s from $v_1 \wedge \ldots \wedge v_N \in \Lambda^N V$, the preceding equation thus becomes

$$\mu^\phi_r(\,v_1 \wedge \ldots \wedge \widehat{v_s} \wedge \ldots \wedge v_N \otimes \eta\,) \quad = \quad (-1)^{s-1}\, \iota^\phi_r(\,v_1 \wedge \ldots \wedge v_N \otimes \eta\,) \otimes dv_s$$

for these multivectors and all $\eta \in \Lambda^N E^*$, $s = 1, \ldots, N$ so that $\mathbf{im}\,\mu^\phi_N = \mathbf{im}\,\iota^\phi_N \otimes V^*$ and in turn $M^\phi_N = I^\phi_N \otimes V^*$. Combined with Corollary 4.9 this insight establishes the direct link

$$\mathscr{A}^\bullet \quad \subset \quad \mathrm{Ann}(\,I^{\phi\,\bullet}_N \otimes V^*\,) \tag{37}$$

between the tableau comodule $\mathscr{A}$ associated to a tableau $\mathscr{A}^d$ and the homogeneous ideal $I^{\phi\,\bullet}_N$ defining its characteristic variety $\mathscr{X}(\,\mathscr{A}^d\,)$. Before using this direct link in the proof of Theorem 4.3 we want to state an alternative version of (37) in terms of differential operators:

Corollary 4.5 (Scalar Differential Constraints). *Consider a linear map* $\phi : \mathrm{Sym}^d T^* \otimes V \longrightarrow E$ *realizing a tableau* $\mathscr{A}^d \subset \mathrm{Sym}^d T^* \otimes V$ *of order* $d \geq 1$ *in the sense* $\mathscr{A}^d = \mathbf{ker}\,\phi$. *Equation (28) associates to* ϕ *a linear differential operator* $D_\phi : C^\infty(\,T,\, V\,) \longrightarrow C^\infty(\,T,\, E\,)$ *of order* d, *in complete analogy every homogeneous element* $D \in I^k_N \subset \mathrm{Sym}^k T$ *in the ideal* I^ϕ_N *defining the* *characteristic variety* $\mathscr{X}(\mathscr{A}^d)$ *can be interpreted as a scalar differential operator* $D : C^\infty(T) \longrightarrow C^\infty(T)$ *of order* k*. For every solution* $\psi \in C^\infty(T, V)$ *of the differential equation* $D_\phi\psi = 0$ *it holds then true that:*

$$(D \otimes \mathrm{id}_V)\, \psi = 0$$

Solutions $\psi \in C^\infty(T, V)$ of the differential equation $D_\phi\psi = 0$ are characterized by the fact that the homogeneous pieces of their Taylor series $\mathrm{taylor}_p\psi \in \overline{\mathrm{Sym}}\, T^* \otimes V$ taken in an arbitrary point $p \in T$ are elements of $\mathscr{A}^k$ for all $k \geq 0$. On the other hand the value of the scalar differential operator associated to $D \in I^k_N$ on $\psi \in C^\infty(T, V)$ is given by a sum

$$\Big((D \otimes \mathrm{id}_V)\, \psi \Big)(p) = \sum_{\lambda=1}^{N} \Big(D\langle\, dv_\lambda,\, \psi\,\rangle \Big)(p)\, v_\lambda = \sum_{\lambda=1}^{N} \langle\, D \otimes dv_\lambda,\, \mathrm{taylor}^k_p \psi\,\rangle\, v_\lambda$$

over a dual pair $v_1, \ldots, v_N$ and $dv_1, \ldots, dv_N$ of bases for V and V^*. Equation (37) thus tells us that the right hand side vanishes for a solution ψ of the equation $D_\phi\psi = 0$.

Proof of Theorem 4.3. According to Hilbert's Nullstellensatz from algebraic geometry every homogeneous polynomial $\psi \in \mathbb{C}[\,x^1, \ldots, x^n\,]$ of positive degree vanishing on all points of a projective variety $\mathscr{X}_\mathbb{C}$, which is the vanishing variety of some homogeneous ideal I

$$\mathscr{X}_\mathbb{C} \;=\; \{\; [\,\xi_\mathbb{C}\,] \in \mathbb{P}\mathbb{C}^n \;|\; p(\,\xi_\mathbb{C}\,) = 0 \text{ for all } p \in I \;\}$$

lies in the radical $\sqrt{I} \subset \mathbb{C}[\,x^1, \ldots, x^n\,]$ of I in the sense $\psi^e \in I$ for sufficiently large exponent $e \in \mathbb{N}$. Among the well-known consequences of this theorem is that the radical of a homogeneous ideal I with empty vanishing variety $\mathscr{X}_\mathbb{C} = \emptyset$ equals the "irrelevant" ideal $\sqrt{I} = \mathbb{C}^+[\,x^1, \ldots, x^n\,]$ consisting of all polynomials of positive degree [5]. Exactly this particular consequence of Hilbert's Nullstellensatz is what Theorem 4.3 is all about.

Unluckily we have been working over the real numbers as of now and not over the algebraically closed field $\mathbb{C}$ required by Hilbert's Nullstellensatz. Multilinear algebra however behaves nicely under complexification inasmuch as we have canonical identifications

$$(\,\mathrm{Sym}\, T^*\,) \otimes_\mathbb{R} \mathbb{C} \;=\; \mathrm{Sym}\,(\,T^* \otimes_\mathbb{R} \mathbb{C}\,) \qquad\qquad (\,\Lambda\, V^*\,) \otimes_\mathbb{R} \mathbb{C} \;=\; \Lambda\,(\,V^* \otimes_\mathbb{R} \mathbb{C}\,)$$

of symmetric and exterior powers as well as tensor products etc. Of course we could go about and repeat all our calculations and constructions for the complexified linear map

$$\phi \otimes_\mathbb{R} \mathrm{id} : \quad \mathrm{Sym}^d(\,T^* \otimes_\mathbb{R} \mathbb{C}\,) \otimes_\mathbb{C} (\,V \otimes_\mathbb{R} \mathbb{C}\,) = (\,\mathrm{Sym}^d T^* \otimes V\,) \otimes_\mathbb{R} \mathbb{C} \;\longrightarrow\; E \otimes_\mathbb{R} \mathbb{C}$$

with complex localizations $\phi_{\xi_\mathbb{C}} : V \otimes_\mathbb{R} \mathbb{C} \longrightarrow E \otimes_\mathbb{R} \mathbb{C}$, $v_\mathbb{C} \longmapsto (\phi \otimes_\mathbb{R} \mathrm{id})(\frac{1}{d!}\xi_\mathbb{C}^d \otimes v_\mathbb{C})$, etc., however the upshot of all these calculations is that the complex characteristic variety

$$\mathscr{X}_\mathbb{C}(\mathscr{A}^d)$$
$$:= \{[\xi_\mathbb{C}] \in \mathbb{P}(T^* \otimes_\mathbb{R} \mathbb{C}) \mid \phi_{\xi_\mathbb{C}} : (V \otimes_\mathbb{R} \mathbb{C}) \longrightarrow (E \otimes_\mathbb{R} \mathbb{C}) \text{ is not injective}\}$$

can be defined by the complexified ideal $I_N^{\phi\,\bullet} \otimes_\mathbb{R} \mathbb{C} \subset \mathrm{Sym}^\bullet(T \otimes_\mathbb{R} \mathbb{C})$, while the kernel of the extension $(\Phi \otimes_\mathbb{R} \mathrm{id}) : (\mathrm{Sym}^\bullet T^* \otimes V) \otimes_\mathbb{R} \mathbb{C} \longrightarrow (\mathrm{Sym}^{\bullet-d} T^* \otimes E) \otimes_\mathbb{R} \mathbb{C}$ of the complexified linear map $\phi \otimes_\mathbb{R} \mathrm{id}$ to comodules over the symmetric coalgebra $\mathrm{Sym}\,(T^* \otimes_\mathbb{R} \mathbb{C})$ is nothing else but the complexification $\mathscr{A}^\bullet \otimes_\mathbb{R} \mathbb{C}$ of the tableau comodule associated to ϕ.

Let us suppose now that the localization $\phi_{\xi_\mathbb{C}} : V \otimes_\mathbb{R} \mathbb{C} \longrightarrow E \otimes_\mathbb{R} \mathbb{C}$ at some non-zero complex valued form $\xi_\mathbb{C} \in T^* \otimes_\mathbb{R} \mathbb{C}$ is not injective and let $v_\mathbb{C} \in V \otimes_\mathbb{R} \mathbb{C}$ be a non-zero vector in its kernel. With these choices made the product vector $\frac{1}{k!}\xi_\mathbb{C}^k \otimes v_\mathbb{C} \neq 0$ is for all $k \geq 0$ a non-zero element of the kernel $\mathscr{A}^k \otimes_\mathbb{R} \mathbb{C}$ of the comodule extension $\Phi \otimes_\mathbb{R} \mathrm{id}$ of $\phi \otimes_\mathbb{R} \mathrm{id}$

$$(\Phi \otimes_\mathbb{R} \mathrm{id})(\tfrac{1}{k!}\xi_\mathbb{C}^k \otimes v_\mathbb{C}) \;=\; \tfrac{1}{(k-d)!}\xi_\mathbb{C}^{k-d} \otimes (\phi \otimes_\mathbb{R} \mathrm{id})(\tfrac{1}{d!}\xi_\mathbb{C}^d \otimes v_\mathbb{C}) \;=\; 0$$

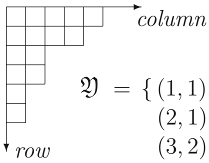
so that $\dim \mathscr{A}^k \geq 1$ for all $k \geq 0$ leading to the comodule $\mathscr{A}$ of infinite dimension. Conversely suppose that all localizations $\phi_{\xi_\mathbb{C}} : V \otimes_\mathbb{R} \mathbb{C} \longrightarrow E \otimes_\mathbb{R} \mathbb{C}$ at non-zero complex valued forms $\xi_\mathbb{C} \in T^* \otimes_\mathbb{R} \mathbb{C}$ are injective. The vectors $t_1, \ldots, t_n$ of a basis for T considered as homogeneous polynomials of degree 1 on $T^* \otimes_\mathbb{R} \mathbb{C}$ trivially vanish on the complex characteristic variety $\mathscr{X}_\mathbb{C}(\mathscr{A}^d) = \emptyset$ as it is empty, hence Hilbert's Nullstellensatz guarantees the existence of exponents $e_1, \ldots, e_n \in \mathbb{N}$ such that the powers $t_1^{e_1}, t_2^{e_2}, \ldots, t_n^{e_n} \in I_N^\phi$ of the basis vectors are real elements of the homogeneous ideal $I_N^\phi \otimes_\mathbb{R} \mathbb{C}$ describing the complex projective variety $\mathscr{X}_\mathbb{C}(\mathscr{A}^d)$. In turn the drawers principle asserts that every monomial $t^{k_1} \ldots t^{k_n}$ in the basis vectors $t_1, \ldots, t_n$ of total degree $k_1 + \ldots + k_n > e_1 + \ldots + e_n - n$ is an element of the ideal I_N^ϕ, because at least one of the basis vectors t_μ occurs with an exponent $k_\mu \geq e_\mu$. Since the monomials in basis vectors span the symmetric powers we conclude

$$I_N^{\phi\,\bullet} \;=\; \mathrm{Sym}^\bullet T$$

and so $\mathscr{A}^\bullet = \{0\}$ for all $\bullet > e_1 + \ldots + e_n - n$ due to $\mathscr{A}^\bullet \subset \mathrm{Ann}(I_N^{\phi\,\bullet} \otimes V^*) = \{0\}$ according to the direct link (37) between $\mathscr{A}$ and I_N^ϕ. With all homogeneous subspaces of $\mathscr{A}^\bullet \subset \mathrm{Sym}^\bullet T^* \otimes V$ being finite dimensional we conclude $\dim \mathscr{A} < \infty$. □

In the second part of this section we want to discuss the main ideas and their ramifications related to Cartan's Involutivity Test for first order tableaux $\mathscr{A}^1 \subset T^* \otimes V$. In contrast to tableaux of higher order tableaux of first order $d = 1$ possess a very interesting discrete invariant, the so-called Cartan character, under the natural action of $\mathbf{GL}\,T \times \mathbf{GL}\,V$ on the subspaces of $T^* \otimes V = \mathrm{Hom}\,(T,\ V)$. The nomenclature adopted by Cartan and his collaborators with respect to tableaux alludes directly to the fact that this Cartan character, although usually written as a decreasing sequence of non-negative integers, is actually a Young diagram, a very interesting combinatorial structure with strong ties to the representation theory of the general linear groups: A Young diagram with additional "filling" is traditionally called a Young tableau (sic!) in representation theory.

A Young diagram is by definition a finite set $\mathfrak{Y} \subset \mathbb{N}^2$ of tuples of natural numbers with the property that for every tuple $(r, c) \in \mathfrak{Y}$ all tuples $(\tilde{r}, \tilde{c}) \in \mathbb{N}^2$ of natural numbers satisfying both inequalities $\tilde{r} \leq r$ and $\tilde{c} \leq c$ are elements $(\tilde{r}, \tilde{c}) \in \mathfrak{Y}$, too. In the parlance of partially ordered sets and lattices we may equivalently define a Young diagram as a finite lower subset $\mathfrak{Y} \subset \mathbb{N}^2$ with respect to the componentwise partial order $\geq$ on $\mathbb{N}^2$. It is much more appropriate though to think of a Young diagram as a picture of little squares neatly aligned in rows and columns in an arrangement similar to matrices:



$$\mathfrak{Y} = \{\, (1,1),\ (1,2),\ (1,3),\ (1,4),\ (1,5),\ (2,1),\ (2,2),\ (2,3),\ (2,4),\ (3,1),\ (3,2),\ (4,1),\ (4,2),\ (5,1),\ (6,1) \,\}$$

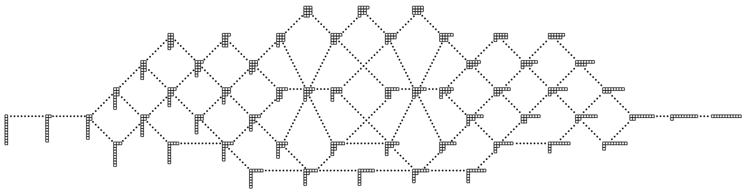
Due to this interpretation the elements of a Young diagram are called its boxes, the number $\sharp\,\mathfrak{Y}$ of boxes of a Young diagram $\mathfrak{Y}$ is called its order. Say the Young diagram depicted above has order 15 with boxes arranged in columns of lengths $6 \geq 4 \geq 2 \geq 2 \geq 1$ and rows of lengths $5 \geq 4 \geq 2 \geq 2 \geq 1 \geq 1$. Similarly every Young diagram $\mathfrak{Y}$ is completely determined by the lengths $c_1 \geq c_2 \geq c_3 \geq \ldots$ of its columns or the lengths $r_1 \geq r_2 \geq r_3 \geq \ldots$ of its rows. The image of a Young diagram $\mathfrak{Y} \subset \mathbb{N}^2$ under the reflection along the main diagonal $(r, c) \longmapsto (c, r)$ interchanging rows and columns is again a Young diagram of the same order called the diagram $\mathfrak{Y}^*$ conjugated to $\mathfrak{Y}$. Moreover the finite set

$$\mathbf{YD}(\,D\,) \;:=\; \{\, \mathfrak{Y} \subset \mathbb{N}^2 \mid \mathfrak{Y} \text{ is a Young diagram of order } \sharp\,\mathfrak{Y} = D \,\}$$

of all Young diagrams of fixed order $D \in \mathbb{N}_0$ comes along with a partial order $\geq$ defined by:

$$\mathfrak{Y} \geq \tilde{\mathfrak{Y}} \qquad \Leftrightarrow \qquad \sum_{\mu=1}^{s} c_\mu \geq \sum_{\mu=1}^{s} \tilde{c}_\mu \quad \text{for all } s \geq 1 \tag{38}$$

In other words $\mathfrak{Y} \geq \tilde{\mathfrak{Y}}$, if and only if $\mathfrak{Y}$ has at least as many boxes in the first column as $\tilde{\mathfrak{Y}}$, at least as many boxes in the first two columns together as $\tilde{\mathfrak{Y}}$ and so on. Under this partial order the set $\mathbf{YD}(\,D\,)$ of Young diagrams of order $D \geq 0$ is actually a self dual lattice with antimonotone involution $* : \mathbf{YD}(\,D\,) \longrightarrow \mathbf{YD}(\,D\,),\ \mathfrak{Y} \longmapsto \mathfrak{Y}^*$. Lacking a pretext we will not discuss these beautiful examples of self-dual lattices in more detail, because only the partial order $\geq$ enters into the definition of the Cartan character of a first order tableau $\mathscr{A}^1 \subset \mathrm{Hom}(T, V)$. Perhaps the reader will enjoy studying the following Hasse diagram of $\mathbf{YD}(11)$ though, in which the Young diagrams are ordered descendingly from left to right:



The similarity between Young diagrams and matrices mentioned before provides the fundamental link between Young diagrams and tableaux. Recall from your first semesters at university that the choice of bases $t_1, \ldots, t_n$ for T and $v_1, \ldots, v_N$ for V turns the vector space $\mathrm{Hom}(T, V)$ into the vector space of all $N \times n$-matrices via the following linear map

$$\mathrm{mat}: \quad \mathrm{Hom}(T, V) \xrightarrow{\cong} \mathrm{Mat}_{N\times n}\mathbb{R}, \qquad A \longmapsto \begin{pmatrix} dv_1(At_1) & \ldots & dv_1(At_n) \\ \vdots & & \vdots \\ dv_N(At_1) & \ldots & dv_N(At_n) \end{pmatrix}$$

where $dv_1, \ldots, dv_N \in V^*$ is the basis dual to $v_1, \ldots, v_N$. Modulo the choice of bases for T and V every tableau $\mathscr{A}^1 \subset \mathrm{Hom}(T, V)$ may thus be thought of as a subspace of matrices, in turn the coefficient of the image matrix in row r and column c becomes the linear functional:

$$\mathrm{mat}_{rc}: \quad \mathscr{A}^1 \longrightarrow \mathbb{R}, \qquad A \longmapsto \mathrm{mat}_{rc}(A) := dv_r(At_c)$$

It should be noted that the matrix coefficients $\mathrm{mat}_{rc} \in \mathscr{A}^{1*}$ span the space $\mathscr{A}^{1*}$ of linear functionals on $\mathscr{A}^1$ since $\mathrm{mat}: \mathscr{A}^1 \longrightarrow \mathrm{Mat}_{N\times n}\mathbb{R}$ is injective. Hence we may choose a basis $\{\mathrm{mat}_{rc}\}_{(r,c)\in\mathfrak{Y}}$ of $\mathscr{A}^{1*}$ consisting entirely of the matrix coefficients indexed by a suitable subset $\mathfrak{Y} \subset \{1, \ldots, N\} \times \{1, \ldots, n\}$. The remaining matrix coefficients are then *fixed* linear combinations of the matrix coefficients in $\mathfrak{Y}$.

It may be somewhat surprising, but the preceding rather esoteric discussion about the linear independence of matrix coefficients captures exactly what we do automatically, whenever we specify a subspace of matrices. Consider for example the following subspace

$$\left\{ \begin{pmatrix} a & b \\ 0 & 2a-b \end{pmatrix} \;\middle|\; a, b \in \mathbb{R} \right\} \quad = \quad \left\{ \begin{pmatrix} \frac{1}{2}b + \frac{1}{2}d & b \\ 0 & d \end{pmatrix} \;\middle|\; b, d \in \mathbb{R} \right\}$$

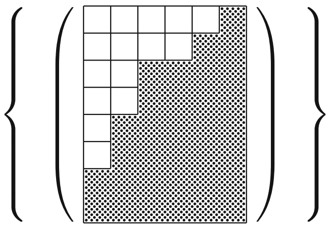
of $\mathrm{Mat}_{2\times 2}\mathbb{R}$. Although defining the same subspace the left and right hand side definitions differ significantly in form, implicitly we have chosen the basis $\{\, a,\ b\,\}$ of matrix coefficients corresponding to $\{\,(1,1),\ (1,2)\,\}$ on the left and the basis $\{\, b,\ d\,\}$ corresponding to $\{\,(1,2),\ (2,2)\,\}$ on the right hand side. The subset $\{\,(1,1),\ (2,2)\,\}$ of matrix coefficients corresponds to a third alternative definition of the same subspace of 2×2-matrices, whereas subsets containing $(2,1)$ certainly do not correspond to bases of matrix coefficients:

Definition 4.6 (Young Diagrams Presenting a Tableau). A Young diagram $\mathfrak{Y}$ of order D is said to present a first order tableau $\mathscr{A}^1 \subset \mathrm{Hom}\,(\,T,\ V\,)$ of dimension $D := \dim \mathscr{A}^1$, if there exist some bases $t_1, \ldots, t_n$ and $v_1, \ldots, v_N$ of T and V respectively such that the associated matrix coefficients $\mathrm{mat}_{rc} \in \mathscr{A}^{1*}$ indexed by $(r,c) \in \mathfrak{Y}$

$$\mathrm{mat}_{rc}: \quad \mathscr{A}^1 \longrightarrow \mathbb{R}, \qquad A \longmapsto dv_r(\, At_c\,)$$

are a basis of $\mathscr{A}^{1*}$. Schematically we may then write $\mathrm{mat}(\,\mathscr{A}^1\,) \subset \mathrm{Mat}_{N\times n}\mathbb{R}$ in the form

$$\mathrm{mat}(\,\mathscr{A}^1\,) \quad =$$



where the coefficients in $\mathfrak{Y}$ can be assigned arbitrary values, the fixed linear combinations of these values calculating the other coefficients characterize the subspace $\mathrm{mat}(\,\mathscr{A}^1\,)$.

In saying that a Young diagram $\mathfrak{Y}$ of order D presents a tableau $\mathscr{A}^1 \subset \mathrm{Hom}\,(\,T,\ V\,)$ of dimension $D = \dim \mathscr{A}^1$ we deliberately draw attention away from the bases of T and V realizing this presentation. In this way the set of Young diagrams presenting a given tableau $\mathscr{A}^1$ of dimension D becomes an invariant of the tableau under the natural action $\mathbf{GL}\,T \times \mathbf{GL}\,V$ on the Graßmannian of D-dimensional subspaces of $\mathrm{Hom}\,(\,T,\ V\,)$. It should not pass by unnoticed that this invariant with values in the subsets of $\mathbf{YD}(\,D\,)$ has a compelling interpretation in terms of the Plücker embedding $\mathrm{Gr}_D\,\mathrm{Hom}\,(\,T,\ V\,) \longrightarrow \mathbb{P}(\,\Lambda^D\mathrm{Hom}\,(\,T,\ V\,)\,)$. According to the representation theory of general linear groups [3] the domain of the Plücker embedding decomposes under $\mathbf{GL}\,T \times \mathbf{GL}\,V$ into a direct sum of irreducible subrepresentations

$$\Lambda^D\mathrm{Hom}\,(\,T,\ V\,) \quad = \bigoplus_{\substack{\mathfrak{Y}\subset\mathbf{YD}(\,D\,)\\ \sharp\,\mathrm{rows}\leq N\\ \sharp\,\mathrm{columns}\leq n}} \mathrm{Schur}^{\mathfrak{Y}^*}T^* \otimes \mathrm{Schur}^{\mathfrak{Y}}V$$

parametrized by all Young diagrams of order D with at most $n := \dim T$ columns and $N := \dim V$ rows. The subset of Young diagrams $\mathfrak{Y}$ presenting a tableau $\mathscr{A}^1$ agrees with the subset of the irreducible subrepresentations in this decomposition, to which the Plücker line in $\Lambda^D \mathrm{Hom}\,(T,\ V)$ associated to $\mathscr{A}^1$ projects non-trivially.

Leaving the construction of invariants aside there is another good reason not to spend too much importance on the bases of T and V used to write a given tableau $\mathscr{A}^1 \subset \mathrm{Hom}\,(T,\ V)$ as a subspace of matrices of size $N \times n$ in $\mathfrak{Y}$-schematical form: This characteristic property does not pertain to the bases themselves, but actually to their associated flags. Recall at this point that a complete flag on T is an increasing sequence $F_\bullet$ of subspaces

$$\{0\} \;=:\; F_0 \;\subsetneq\; F_1 \;\subsetneq\; \ldots \;\subsetneq\; F_{n-1} \;\subsetneq\; F_n \;:=\; T$$

satisfying $\dim F_s = s$ for all $s = 1, \ldots, n$. A basis $t_1, \ldots, t_n$ of T is called adapted to a complete flag $F_\bullet$ provided $t_s \in F_s \setminus F_{s-1}$ for all $s = 1, \ldots, n$. Evidently every basis $t_1, \ldots, t_n$ of T is adapted to exactly one complete flag defined by $F_s := \mathrm{span}\{\, t_1, \ldots, t_s \,\}$ for all s, on the other hand there are certainly many different bases adapted to a given flag. Nevertheless two bases $t_1, \ldots, t_n$ and $t'_1, \ldots, t'_n$ adapted to the same complete flag $F_\bullet$ on T are necessarily related by an invertible lower triangular matrix $B \in \mathrm{Mat}_{n\times n}\mathbb{R}$ via:

$$t'_c \;=\; \sum_{s=1}^{c} B_{cs}\, t_s$$

For the matrix coefficients $\mathrm{mat}'_{rc} \in \mathscr{A}^{1*}$ associated to the basis $t'_1, \ldots, t'_n$ this becomes

$$\mathrm{mat}'_{rc}(\,A\,) \;=\; dv_r\Big(\,A(\,\sum_{s=1}^{c} B_{cs} t_s\,)\Big) \;=\; \sum_{s=1}^{c} B_{cs}\ \mathrm{mat}_{rs}(\,A\,)$$

so that the matrix coefficients $\mathrm{mat}'_{rc} \in \mathscr{A}^{1*}$ for $(r, c) \in \mathfrak{Y}$ are invertible linear combinations of the matrix coefficients $\mathrm{mat}_{rs} \in \mathscr{A}^{1*}$ with $(r, s) \in \mathfrak{Y}$. A very similar argument applies to changing the basis of V, while keeping the associated complete flag on V unchanged. This dependence on flags is precisely the reason, why we are *not* interested in arbitrary subsets of matrix coefficients, but in subsets specified by Young diagrams.

Another way to understand the relationship between complete flags on T and Young diagram presentations of a given first order tableau $\mathscr{A}^1$ is to study its restrictions to subspaces $F \subset T$. The linear restriction map $\mathrm{res}_F : \mathrm{Hom}\,(T,\ V) \longrightarrow \mathrm{Hom}\,(F,\ V)$, $A \longmapsto A|_F$, associated to a subspace $F \subset T$ gives rise in fact to a short exact sequence of tableaux

$$0 \longrightarrow \mathscr{A}^1_F \stackrel{\subset}{\longrightarrow} \mathscr{A}^1 \stackrel{\mathrm{res}_F}{\longrightarrow} \mathrm{res}_F \mathscr{A}^1 \longrightarrow 0 \qquad (39)$$

where $\mathrm{res}_F\mathscr{A}^1 \subset \mathrm{Hom}(F, V)$ is the image of $\mathscr{A}^1 \subset \mathrm{Hom}(T, V)$ under res_F and $\mathscr{A}^1_F \subset \mathscr{A}^1$ is the subspace of all $A \in \mathscr{A}^1$ satisfying $At = 0$ for all $t \in F$. In general the dimensions of the two derived tableaux $\mathscr{A}^1_F$ and $\mathrm{res}_F\mathscr{A}^1$ depend delicately on the chosen subspace $F \subset T$, hence it makes sense to call a subspace $F \subset T$ of dimension s a regular subspace provided:

$$\dim\ \mathrm{res}_F\mathscr{A}^1 \quad = \quad \max\{\ \dim\ \mathrm{res}_{\hat{F}}\mathscr{A}^1 \ \mid\ \hat{F} \subset T \text{ is a subspace of dimension } s\ \}$$

In the opposite case F is a singular subspace with respect to the tableau $\mathscr{A}^1 \subset \mathrm{Hom}(T, V)$ in the sense that the dimension of the intersection $\mathscr{A}^1_F = \mathscr{A}^1 \cap \mathrm{Hom}(T/F, V)$ is larger than it needs to be. After a little bit of multilinear algebra of the kind we used to establish Eq. (36) above the latter characterization of singular subspaces turns into an explicit space of polynomials on the Graßmannian $\mathrm{Gr}_s T$ considered as an algebraic variety such that a subspace $F \subset T$ of dimension s is singular with respect to $\mathscr{A}^1$, if and only if $F \in \mathrm{Gr}_s T$ is a common zero of all these polynomials. The complementary subset of regular subspaces of T in dimension s is thus a non-empty Zariski dense subset:

$$\mathrm{Gr}^{\mathrm{reg}}_s T := \{\ F \mid\ F \subset T \text{ is an } \mathscr{A}^1\text{–regular subspace of dimension } s\ \} \subseteq \mathrm{Gr}_s T$$

Coming back to complete flags we conclude that the set of all complete flags $F_\bullet$ on T, which feature a regular subspace $F_s \in \mathrm{Gr}^{\mathrm{reg}}_s T$ in a given dimension s, is a Zariski dense subset of the algebraic variety $\mathrm{Flag}\, T$ of all complete flags on T. Finite intersections of Zariski dense subsets however are still Zariski dense, in consequence the subset of regular flags on T

$$\mathrm{Flag}^{\mathrm{reg}} T \quad := \quad \{\ F_\bullet \in \mathrm{Flag}\, T \mid\ \text{every } F_s \text{ is } \mathscr{A}^1\text{–regular in its dimension } s\ \}$$

is a Zariski dense subset of the algebraic variety $\mathrm{Flag}\, T$ of all complete flags on T, in particular $\mathrm{Flag}^{\mathrm{reg}} T$ is a non-empty, dense subset of $\mathrm{Flag}\, T$ with respect to the manifold topology as well. The existence of regular complete flags for arbitrary first order tableaux allows us to define the Cartan character $\mathfrak{Y}^{\mathscr{A}}$ of a tableau $\mathscr{A}^1 \subset \mathrm{Hom}(T, V)$ in the following way:

Lemma 4.7 (Cartan Character of First Order Tableaux). *Associated to every first order tableau $\mathscr{A}^1 \subset T^* \otimes V$ of dimension $D := \dim \mathscr{A}^1$ is the set of all Young diagrams of order D presenting $\mathscr{A}^1$. With respect to the partial order $\geq$ this subset of $\mathbf{YD}(D)$ has a unique maximal element called the Cartan character $\mathfrak{Y}^{\mathscr{A}}$ of the tableau $\mathscr{A}^1$, its column lengths $c^{\mathscr{A}}_1 \geq c^{\mathscr{A}}_2 \geq \ldots \geq c^{\mathscr{A}}_n \geq 0$ satisfy for all $s = 1, \ldots, n$:*

$$c^{\mathscr{A}}_1 + c^{\mathscr{A}}_2 + \ldots + c^{\mathscr{A}}_s \quad := \quad \max\{\ \dim\ \mathrm{res}_F\mathscr{A}^1 \ \mid\ F \in \mathrm{Gr}_s T\ \}$$

Proof. For the purpose of proof let us assume that the column lengths of the Cartan character $\mathfrak{Y}^{\mathscr{A}}$ in spe of a fixed first order tableau $\mathscr{A}^1 \subset \mathrm{Hom}(T, V)$

of dimension $D \geq 0$ are defined simply as a sequence of non-negative numbers $c_1^{\mathscr{A}}, c_2^{\mathscr{A}}, \ldots, c_n^{\mathscr{A}} \geq 0$ via:

$$c_1^{\mathscr{A}} + c_2^{\mathscr{A}} + \ldots + c_s^{\mathscr{A}} \quad := \quad \max\{ \dim \operatorname{res}_F \mathscr{A}^1 \mid F \in \operatorname{Gr}_s T \} \tag{40}$$

For every given Young diagram $\mathfrak{Y} \in \mathbf{YD}(D)$ presenting $\mathscr{A}^1$ we may choose bases $t_1, \ldots, t_n$ and $v_1, \ldots, v_N$ of T and V respectively such that the matrix coefficients $A \longmapsto dv_r(At_c)$ indexed by $(r, c) \in \mathfrak{Y}$ are a basis of $\mathscr{A}^{1*}$. The specific matrix coefficients $\operatorname{mat}_{rc} \in \mathscr{A}^{1*}$ indexed by boxes $(r, c) \in \mathfrak{Y}$ in the first $s = 1, \ldots, n$ columns with $c \leq s$ actually come from the restriction of $\mathscr{A}^1$ to the subspace $F_s \in \operatorname{Gr}_s T$ spanned by $\{t_1, \ldots, t_s\}$ in the sense $\operatorname{mat}_{rc}(A) := dv_r([A|_{F_s}]t_c)$. Hence the image of the adjoint $(\operatorname{res}_{F_s}\mathscr{A}^1)^* \longrightarrow \mathscr{A}^{1*}$ of the restriction $\mathscr{A}^1 \longrightarrow \operatorname{res}_{F_s}\mathscr{A}^1$, $A \longmapsto A|_{F_s}$, to F_s contains the $c_1 + \ldots + c_s$ linearly independent matrix coefficients indexed by boxes $(r, c) \in \mathfrak{Y}$ in the first s columns and so:

$$\begin{aligned} c_1 + c_2 + \ldots + c_s &\leq \dim \operatorname{res}_{F_s}\mathscr{A}^1 \\ &\leq \max\{ \dim \operatorname{res}_F \mathscr{A}^1 \mid F \in \operatorname{Gr}_s T \} \\ &= c_1^{\mathscr{A}} + c_2^{\mathscr{A}} + \ldots + c_s^{\mathscr{A}} \end{aligned}$$

Since this inequality is true for all $s = 1, \ldots, n$, we conclude that $\mathfrak{Y} \leq \mathfrak{Y}^{\mathscr{A}}$ provided we can show that the non-negative numbers $c_1^{\mathscr{A}}, \ldots, c_n^{\mathscr{A}} \geq 0$ are actually the column lengths of a Young diagram $\mathfrak{Y}^{\mathscr{A}}$ presenting the tableau $\mathscr{A}^1$.

For this purpose let us fix a regular flag $F_\bullet \in \operatorname{Flag}^{\operatorname{reg}} T$ for the tableau $\mathscr{A}^1$ and an adapted basis $t_1, \ldots, t_n$ for T satisfying $F_s = \operatorname{span}\{t_1, \ldots, t_s\}$ for all $s = 1, \ldots, n$. Evidently the kernel $\mathscr{A}^1_{F_s} \subset \mathscr{A}^1$ of the restriction to F_s consists of those $A \in \mathscr{A}^1 \subset \operatorname{Hom}(T, V)$ satisfying $A\,t_\mu = 0$ for all $1 \leq \mu \leq s$, this simple observation gives rise to the short exact sequences

$$0 \longrightarrow \mathscr{A}^1_{F_s} \xrightarrow{\subset} \mathscr{A}^1_{F_{s-1}} \xrightarrow{\frac{\partial}{\partial t_s}} \mathscr{A}^1_{F_{s-1}}\, t_s \longrightarrow 0 \tag{41}$$

for all $s = 1, \ldots, n$, where $\frac{\partial}{\partial t_s} A := A\,t_s$ and $\mathscr{A}^1_{F_0} := \mathscr{A}^1$ in case of doubt. In consequence

$$\begin{aligned} \dim \operatorname{res}_{F_s}\mathscr{A}^1 &= \dim \mathscr{A}^1 - \dim \mathscr{A}^1_{F_s} = c_1^{\mathscr{A}} + \ldots + c_s^{\mathscr{A}} \\ \dim \mathscr{A}^1_{F_{s-1}}\, t_s &= \dim \mathscr{A}^1_{F_{s-1}} - \dim \mathscr{A}^1_{F_s} = c_s^{\mathscr{A}} \end{aligned}$$

where the first equation simply reflects the regularity of the chosen flag $F_\bullet \in \operatorname{Flag}^{\operatorname{reg}} T$ with respect to the tableau $\mathscr{A}^1$ and the second the short exact sequence (41).

The crucial observation to be made at this point is that the sequence of subspaces $\mathscr{A}^1_{F_{s-1}} t_s$, $s = 1, \ldots, n$, of dimensions $c_s^{\mathscr{A}}$ is actually a monotonely decreasing filtration

$$V \supseteq \mathscr{A}^1_{F_0} t_1 \supseteq \mathscr{A}^1_{F_1} t_2 \supseteq \mathscr{A}^1_{F_2} t_3 \supseteq \ldots \supseteq \mathscr{A}^1_{F_{n-1}} t_n \supseteq \{0\} \tag{42}$$

so that the non-negative integers $c_1^{\mathscr{A}}, \ldots, c_n^{\mathscr{A}} \geq 0$ we have been using up to now are monotonely decreasing $c_1^{\mathscr{A}} \geq c_2^{\mathscr{A}} \geq \ldots \geq c_n^{\mathscr{A}} \geq 0$ as appropriate for the column lengths of a Young diagram $\mathfrak{Y}^{\mathscr{A}}$ of order $c_1^{\mathscr{A}} + \ldots + c_n^{\mathscr{A}} = \dim \mathscr{A}^1$. By the our choice of a regular flag $F_\bullet \in \mathrm{Flag}^{\mathrm{reg}} T$ all the subspaces $F_s \in \mathrm{Gr}_s^{\mathrm{reg}} T$ are regular in their dimension s with respect to the tableau $\mathscr{A}^1$. With regularity being a Zariski open condition we conclude that for an arbitrary deformation vector $t \in T$ the deformation $F_s^\varepsilon := F_{s-1} \oplus \mathbb{R}(t_s + \varepsilon t)$ with ε sufficiently close to 0 is still a regular subspace $F_s^\varepsilon \in \mathrm{Gr}_s^{\mathrm{reg}} T$. Comparing the short exact sequence (41) for F_s with the short exact sequence constructed similarly for F_s^ε

$$0 \longrightarrow \mathscr{A}^1_{F_s^\varepsilon} \xrightarrow{\subset} \mathscr{A}^1_{F_{s-1}} \longrightarrow \mathscr{A}^1_{F_{s-1}}(t_s + \varepsilon t) \longrightarrow 0$$

we observe that the regularity of F_s^ε is equivalent to $\dim \mathscr{A}^1_{F_{s-1}}(t_s + \varepsilon t) = \dim \mathscr{A}^1_{F_{s-1}} t_s$, hence sufficiently close to 0 the curve $\varepsilon \longmapsto \mathscr{A}^1_{F_{s-1}}(t_s + \varepsilon t)$ is a smooth curve in the Graßmannian of subspaces of V of dimension $c_s^{\mathscr{A}}$. In particular the trivial inclusion of subspaces

$$\mathscr{A}^1_{F_s} t = \mathscr{A}^1_{F_s}(t_s + \varepsilon t) \subseteq \mathscr{A}^1_{F_{s-1}}(t_s + \varepsilon t)$$

valid for all $\varepsilon \neq 0$ continues to hold true for $\varepsilon = 0$ by the way the topology is defined on the Graßmannians. In consequence $\mathscr{A}^1_{F_{s-1}} t_s \supseteq \mathscr{A}^1_{F_s} t$ for all $s = 1, \ldots, n$ and an arbitrary deformation vector $t \in T$, in particular $\mathscr{A}^1_{F_{s-1}} t_s \supseteq \mathscr{A}^1_{F_s} t_{s+1}$ in filtration (42).

Last but not least we complement the chosen basis $t_1, \ldots, t_n$ for T adapted to the regular flag $F_\bullet \in \mathrm{Flag}^{\mathrm{reg}} T$ by a basis $v_1, \ldots, v_N$ of V adapted to the decreasing filtration (42) in the sense that for all $s = 1, \ldots, n$ the filtration subspace $\mathscr{A}^1_{F_{s-1}} t_s$ is spanned by the first $c_s^{\mathscr{A}}$ basis vectors $v_1, \ldots, v_{c_s^{\mathscr{A}}}$. In order to show that the special matrix coefficients $\mathrm{mat}_{rc} \in \mathscr{A}^{1*}$ indexed by boxes $(r, c) \in \mathfrak{Y}^{\mathscr{A}}$ with respect to these bases are actually a basis of $\mathscr{A}^{1*}$ it is sufficient to verify that they generate $\mathscr{A}^{1*}$, in other words we need to prove that every $A \in \mathscr{A}^1$ satisfying $\mathrm{mat}_{rc}(A) = 0$ for all $(r, c) \in \mathfrak{Y}^{\mathscr{A}}$ necessarily vanishes $A = 0$.

Due to $A \in \mathscr{A}^1 = \mathscr{A}^1_{F_0}$ the vector $At_1 \in \mathscr{A}^1_{F_0} t_1$ is a linear combination of the first $c_1^{\mathscr{A}}$ basis vectors $v_1, \ldots, v_{c_1^{\mathscr{A}}} \in V$, hence the assumption $\mathrm{mat}_{r1}(A) = 0$ for all $(r, 1) \in \mathfrak{Y}^{\mathscr{A}}$ implies $At_1 = 0$ or equivalently $A \in \mathscr{A}^1_{F_1}$. Iterating this argument we find that $At_2 \in \mathscr{A}^1_{F_1} t_2$ is a linear combination of the first $c_2^{\mathscr{A}}$ basis vectors of V and conclude $At_2 = 0$ or $A \in \mathscr{A}^1_{F_2}$ as before from the assumption $\mathrm{mat}_{r2}(A) = 0$ for all $(r, 2) \in \mathfrak{Y}^{\mathscr{A}}$. Continuing in this way we eventually arrive at the conclusion $A \in \mathscr{A}^1_{F_n} = \{0\}$ or equivalently $A = 0$. Summing up this argument we conclude that the matrix coefficients $\mathrm{mat}_{rc} \in \mathfrak{Y}^{\mathscr{A}}$ indexed by the boxes $(r, c) \in \mathfrak{Y}^{\mathscr{A}}$ are a basis of $\mathscr{A}^{1*}$ so that $\mathfrak{Y}^{\mathscr{A}}$ presents the tableau $\mathscr{A}^1$. □

Theorem 4.8 (Cartan's Involutivity Test for Tableaux). *Every Young diagram $\mathfrak{Y}$ presenting a first order tableau $\mathscr{A}^1 \subset \mathrm{Hom}(T, V)$ with associated tableau comodule $\mathscr{A}$ over the symmetric coalgebra $\mathrm{Sym}\, T^*$ of a vector space T of dimension $n := \dim T$ provides us with an a priori estimate on the dimension of the first prolongation $\mathscr{A}^2 \subset \mathrm{Sym}^2 T^* \otimes V$ of the tableau in terms of its column lengths $c_1 \geq c_2 \geq \ldots \geq c_n \geq 0$*

$$\dim \mathscr{A}^2 \;\leq\; c_1 + 2c_2 + 3c_3 + 4c_4 + \ldots + n\,c_n$$

which may only be sharp for the Cartan character $\mathfrak{Y}^{\mathscr{A}}$ of $\mathscr{A}^1$. If this estimate is in fact sharp for the Cartan character, then the Spencer cohomology of the comodule $\mathscr{A}$ is concentrated in comodule degree zero with $H^{\bullet,\circ}(\mathscr{A}) = \{0\}$ for $\bullet \neq 0$. Moreover the dimensions of $\mathscr{A}^k$ and $H^{0,r}(\mathscr{A})$ can be calculated for all $k, r > 0$ from the column lengths of $\mathfrak{Y}^{\mathscr{A}}$ via:

$$\dim \mathscr{A}^k = +\binom{k-1}{0} c_1^{\mathscr{A}} + \binom{k}{1} c_2^{\mathscr{A}} + \ldots + \binom{k+n-2}{n-1} c_n^{\mathscr{A}}$$

$$\dim H^{0,r}(\mathscr{A}) = \binom{n}{r} \dim V - \binom{n-1}{r-1} c_1^{\mathscr{A}} - \binom{n-2}{r-1} c_2^{\mathscr{A}} + \ldots - \binom{0}{r-1} c_n^{\mathscr{A}}$$

Last but not least the comodule $\mathscr{A}$ has a canonical resolution by free comodules of the form:

$$0 \longrightarrow \mathscr{A}^\bullet \overset{\subset}{\longrightarrow} \mathrm{Sym}^\bullet T^* \otimes H^{0,0}(\mathscr{A}) \longrightarrow \mathrm{Sym}^{\bullet-1} T^* \otimes H^{0,1}(\mathscr{A})$$
$$\longrightarrow \mathrm{Sym}^{\bullet-2} T^* \otimes H^{0,2}(\mathscr{A}) \longrightarrow \ldots \longrightarrow \mathrm{Sym}^{\bullet-n} T^* \otimes H^{0,n}(\mathscr{A}) \longrightarrow 0$$

Without doubt Cartan's Involutivity Test is the most beautiful gem in the theory of exterior differential systems, although or perhaps because it is in essence a theorem of commutative algebra. Involutivity of a tableau is a notion actually defined by the theorem: A first order tableau $\mathscr{A}^1 \subset \mathrm{Hom}(T, V)$ is called an involutive tableau provided it passes Cartan's Test positively with $\dim \mathscr{A}^2 = c_1^{\mathscr{A}} + 2\,c_2^{\mathscr{A}} + \ldots + n\,c_n^{\mathscr{A}}$, in consequence the associated Spencer cohomology is concentrated in comodule degree $\bullet = 0$ with $H^{\bullet,\circ}(\mathscr{A}) = \{0\}$ for all $\bullet \neq 0$. The converse of is statement is true as well, although we will not prove this fact: A first order tableau $\mathscr{A}^1$, whose Spencer cohomology is concentrated in comodule degree zero, necessarily passes Cartan's Test $\dim \mathscr{A}^2 = c_1^{\mathscr{A}} + 2\,c_2^{\mathscr{A}} + \ldots + n\,c_n^{\mathscr{A}}$ positively. The proof presented below of the direct implication of Theorem 4.8 relies heavily on the following technical lemma:

Lemma 4.9 (Technical Lemma for Cartan's Involutivity Test). *Consider for a given comodule $\mathscr{A}$ over the symmetric coalgebra $\mathrm{Sym}\, T^*$ of a vector space T the subcomodule $\mathscr{A}^\bullet_{\mathbb{R}t} \subset \mathscr{A}^\bullet$ of elements constant in the direction of a fixed vector $t \in T$:*

$$\mathscr{A}^{\bullet}_{\mathbb{R}t} \;:=\; \mathbf{ker}\Big(\;\frac{\partial}{\partial t}\;:\;\mathscr{A}^{\bullet}\;\longrightarrow\;\mathscr{A}^{\bullet-1},\;\;a\;\longmapsto\;\frac{\partial a}{\partial t}\;\Big)$$

In case the directional derivative $\frac{\partial}{\partial t} : \mathscr{A}^{k+1} \longrightarrow \mathscr{A}^{k}$ in the direction t is surjective for some $k \in \mathbb{Z}$ the inclusion $\mathscr{A}^{\bullet}_{\mathbb{R}t} \longrightarrow \mathscr{A}^{\bullet}$ induces a surjection on the level of Spencer cohomology:

$$H^{k,\circ}(\,\mathscr{A}_{\mathbb{R}t}\,)\;\longrightarrow\;H^{k,\circ}(\,\mathscr{A}\,),\qquad[\,\omega\,]\;\longmapsto\;[\,\omega\,]$$

If in addition to $\frac{\partial}{\partial t} : \mathscr{A}^{k+1} \longrightarrow \mathscr{A}^{k}$ being surjective the following Spencer cohomology spaces

$$H^{k+1,0}(\,\mathscr{A}\,)\;=\;0\qquad H^{k+1,1}(\,\mathscr{A}\,)\;=\;0\qquad H^{k,2}(\,\mathscr{A}_{\mathbb{R}t}\,)\;=\;0$$

of $\mathscr{A}$ and $\mathscr{A}_{\mathbb{R}t}$ vanish, then the directional derivative $\frac{\partial}{\partial t} : \mathscr{A}^{k+2} \longrightarrow \mathscr{A}^{k+1}$ is surjective again.

The first statement is an almost trivial consequence of Cartan's Homotopy Formula. Starting with an arbitrary representative $\omega \in \mathscr{A}^{k} \otimes \Lambda^{r}T^{*}$ of a cohomology class $[\,\omega\,] \in H^{k,r}(\,\mathscr{A}\,)$ we use the surjectivity of the directional derivative $\frac{\partial}{\partial t} : \mathscr{A}^{k+1} \longrightarrow \mathscr{A}^{k}$ and the algebraic analogue of Cartan's Homotopy Formula (22) to find a cochain $\omega^{\mathrm{pre}} \in \mathscr{A}^{k+1} \otimes \Lambda^{r}T^{*}$ satisfying:

$$\begin{aligned}\omega = (\,\frac{\partial}{\partial t}\otimes\mathrm{id}\,)\,\omega^{\mathrm{pre}} \;&=\; \{\,B,\;(\,\mathrm{id}\otimes t\lrcorner\,)\,\}\,\omega^{\mathrm{pre}}\\ &:= (\,\mathrm{id}\otimes t\lrcorner\,)\,B\,\omega^{\mathrm{pre}}\;+\;B\,(\,\mathrm{id}\otimes t\lrcorner\,)\,\omega^{\mathrm{pre}}\end{aligned}$$

In consequence $(\,\mathrm{id} \otimes t\lrcorner\,)\,B\,\omega^{\mathrm{pre}} \equiv \omega$ modulo $\mathbf{im}\,B$ is still closed and represents the same cohomology class $[\,(\,\mathrm{id} \otimes t\lrcorner\,)\,B\,\omega^{\mathrm{pre}}\,] = [\,\omega\,] \in H^{k,r}(\,\mathscr{A}\,)$, however its directional derivative

$$\begin{aligned}(\,\frac{\partial}{\partial t}\otimes\mathrm{id}\,)\Big(\,(\,\mathrm{id}\otimes t\lrcorner\,)\,B\,\omega^{\mathrm{pre}}\,\Big) &= (\,\mathrm{id}\otimes t\lrcorner\,)\,B\,\Big(\,(\,\frac{\partial}{\partial t}\otimes\mathrm{id}\,)\,\omega^{\mathrm{pre}}\,\Big)\\ &= (\,\mathrm{id}\otimes t\lrcorner\,)\,B\,\omega\;\;=\;\;0\end{aligned}$$

in the direction of t vanishes, recall that ω is assumed to represent a cohomology class and so it is necessarily closed $B\,\omega = 0$. The modified representative $(\,\mathrm{id} \otimes t\lrcorner\,)\,B\,\omega^{\mathrm{pre}} \in \mathscr{A}^{k}_{\mathbb{R}t} \otimes \Lambda^{r}T^{*}$ is thus constant in the direction of t and provides us with the looked for preimage of $[\,\omega\,]$ under the map $H^{k,r}(\,\mathscr{A}_{\mathbb{R}t}\,) \longrightarrow H^{k,r}(\,\mathscr{A}\,)$ induced by the inclusion $\mathscr{A}^{\bullet}_{\mathbb{R}t} \longrightarrow \mathscr{A}^{\bullet}$. Turning to the proof of the second statement we study the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & & & 0 \\
 & & & & & & & & \big\downarrow \\
 & & & & \mathscr{A}^{k+1}_{\mathbb{R}t} \otimes T^* & \xrightarrow{B} & \mathscr{A}^{k}_{\mathbb{R}t} \otimes \Lambda^2 T^* & \xrightarrow{B} & \mathscr{A}^{k-1}_{\mathbb{R}t} \otimes \Lambda^3 T^* \\
 & & & & \big\downarrow \subset & & \big\downarrow \subset & & \big\downarrow \subset \\
 & & \mathscr{A}^{k+2} & \xrightarrow{B} & \mathscr{A}^{k+1} \otimes T^* & \xrightarrow{B} & \mathscr{A}^{k} \otimes \Lambda^2 T^* & \xrightarrow{B} & \mathscr{A}^{k-1} \otimes \Lambda^3 T^* \\
 & & \big\downarrow \frac{\partial}{\partial t} & & \big\downarrow \frac{\partial}{\partial t} \otimes \mathrm{id} & & \big\downarrow \frac{\partial}{\partial t} \otimes \mathrm{id} & & \\
0 & \longrightarrow & \mathscr{A}^{k+1} & \xrightarrow{B} & \mathscr{A}^{k} \otimes T^* & \xrightarrow{B} & \mathscr{A}^{k-1} \otimes \Lambda^2 T^* & & \\
 & & & & \big\downarrow & & & & \\
 & & & & 0 & & & &
\end{array} \tag{43}$$

whose rows and columns are all complexes. By assumption the row complexes are exact on the diagonal $\mathscr{A}^{k+1}$, $\mathscr{A}^{k+1} \otimes T^*$ and $\mathscr{A}^{k}_{\mathbb{R}t} \otimes \Lambda^2 T^*$, while the column complexes are exact on the parallel diagonal $\mathscr{A}^{k} \otimes T^*$, $\mathscr{A}^{k} \otimes \Lambda^2 T^*$ and $\mathscr{A}^{k-1}_{\mathbb{R}t} \otimes \Lambda^3 T^*$ due to the definition of the comodule $\mathscr{A}_{\mathbb{R}t}$ and our assumption that the directional derivative $\frac{\partial}{\partial t} : \mathscr{A}^{k+1} \longrightarrow \mathscr{A}^{k}$ is surjective. A very delightful diagram chase all over this diagram proves that the directional derivative $\frac{\partial}{\partial t} : \mathscr{A}^{k+2} \longrightarrow \mathscr{A}^{k+1}$ on the left is surjective as well.

Proof of Theorem 4.8: Once and for all let us fix a first order tableau $\mathscr{A}^1 \subset \mathrm{Hom}\,(T, V)$ of dimension $D \geq 0$ on vector spaces T and V of dimensions n and N respectively. On the set of Young diagrams of order D with at most n columns we define the weighted sum $\|\cdot\| : \mathbf{YD}_n(D) \longrightarrow \mathbb{N}_0$ of the column lengths $c_1 \geq c_2 \geq \ldots \geq c_n \geq 0$ of the argument by:

$$\begin{aligned}
\|\mathfrak{Y}\| &:= c_1 + 2c_2 + \ldots + n\,c_n \\
&= (n+1)\,D - (c_1) - (c_1 + c_2) - \ldots - (c_1 + c_2 + \ldots + c_n)
\end{aligned}$$

Using the second expression for the weighted sum $\|\cdot\|$ and the definition (38) of the partial order $\geq$ on the lattice $\mathbf{YD}(D)$ we conclude directly that $\mathfrak{Y} \geq \tilde{\mathfrak{Y}}$ implies $\|\mathfrak{Y}\| \leq \|\tilde{\mathfrak{Y}}\|$ with equality, if and only if $\mathfrak{Y} = \tilde{\mathfrak{Y}}$. Every Young diagram presenting $\mathscr{A}^1$ has at most n columns of course and the maximality of the Cartan character $\mathfrak{Y}^{\mathscr{A}}$ among the diagrams presenting $\mathscr{A}^1$ implies that $\|\cdot\|$ attains its minimum in and only in the Cartan character:

$$\|\mathfrak{Y}^{\mathscr{A}}\| \;=\; \min\{\ \|\mathfrak{Y}\|\ \mid\ \mathfrak{Y} \in \mathbf{YD}_n(D) \text{ presents the tableau } \mathscr{A}^1\ \} \tag{44}$$

In the following paragraphs we will provide two different arguments to establish the *a priori* estimate $\dim \mathscr{A}^2 \leq \|\mathfrak{Y}\|$ for the dimension of the first prolongation $\mathscr{A}^2$ in terms of a Young diagram $\mathfrak{Y}$ presenting $\mathscr{A}^1$. Whereas the first argument is rather explicit and is intended to provide the reader with a meaningful interpretation (46) for the weighted sum $\|\cdot\|$, the second argument is more elegant and provides us with the a useful description of the equality case $\dim \mathscr{A}^2 = \| \mathfrak{Y} \|$, on which the inductive proof of the main statement is based.

Consider an arbitrary Young diagram $\mathfrak{Y}$ presenting the tableau $\mathscr{A}^1$ and corresponding bases $t_1, \ldots, t_n$ and $v_1, \ldots, v_N$ for T and V respectively so that the matrix coefficients $\mathrm{mat}_{rc}(\,A\,) := dv_r(At_c)$ indexed by the boxes $(r, c) \in \mathfrak{Y}$ are a basis of $\mathscr{A}^{1*}$. Recall that the directional derivative in the tableau comodule $\mathscr{A}^\bullet$ associated to $\mathscr{A}^1$ agrees with insertion $\frac{\partial}{\partial t} : \mathscr{A}^1 \longrightarrow V,\ A \longmapsto At$, in comodule degree $\bullet = 1$, this observation allows us to generalize the matrix coefficients $\mathrm{mat}_{rc} \in \mathscr{A}^{1*}$ to matrix coefficients defined on $\mathscr{A}^2$ by:

$$\mathrm{mat}_{r;\,\bar{c}c} : \quad \mathscr{A}^2 \longrightarrow \mathbb{R}, \qquad a \longmapsto dv_r\Big(\, \frac{\partial^2}{\partial t_{\bar{c}}\, \partial t_c}\, a \,\Big)$$

It is well known that these generalized matrix coefficients $\mathrm{mat}_{r;\bar{c}c}$ with their symmetry $\mathrm{mat}_{r;\,\bar{c}c} = \mathrm{mat}_{r;\,c\bar{c}}$ taken into account are a basis of the vector space dual to $\mathrm{Sym}^2 T^* \otimes V$, hence they certainly generate $\mathscr{A}^{2*}$ due to $\mathscr{A}^2 \subset \mathrm{Sym}^2 T^* \otimes V$. On the other hand we know that the matrix coefficients $\mathrm{mat}_{\bar{r}\bar{c}} \in \mathscr{A}^{1*}$ indexed by $(\bar{r}, \bar{c}) \notin \mathfrak{Y}$ are fixed linear combinations

$$\mathrm{mat}_{\bar{r}\bar{c}} \;=\; \sum_{(r,c) \in \mathfrak{Y}} C^{rc}_{\bar{r}\bar{c}}\ \mathrm{mat}_{rc} \tag{45}$$

of the matrix coefficients indexed by $(r, c) \in \mathfrak{Y}$, where the constants $C^{rc}_{\bar{r}\bar{c}} \in \mathbb{R}$ are characteristic for the tableau $\mathscr{A}^1$, the trivial identity $\mathrm{mat}_{\bar{r};\,s\bar{c}}(\,a\,) = \mathrm{mat}_{\bar{r}\bar{c}}(\,\frac{\partial a}{\partial t_s}\,)$ thus implies $\mathrm{mat}_{\bar{r};\,s\bar{c}} = \sum_{(r,c)\in\mathfrak{Y}} C^{rc}_{\bar{r}\bar{c}}\ \mathrm{mat}_{r;\,sc}$ for every $s = 1, \ldots, n$ as well. In the light of the symmetry $\mathrm{mat}_{r;\,c\bar{c}} = \mathrm{mat}_{r;\,\bar{c}c}$ we conclude that the generalized matrix coefficients $\mathrm{mat}_{r;\,\bar{c}c}$ indexed by triples $(r; \bar{c}, c)$ satisfying $c \geq \bar{c} \geq 1$ and $(r, c) \in \mathfrak{Y}$ already generate all of $\mathscr{A}^{2*}$, hence:

$$\dim \mathscr{A}^2 \;\leq\; \| \mathfrak{Y} \| \;=\; \sharp\{\, (r;\, \bar{c},\, c\,) \;|\; r \geq 1,\ c \geq \bar{c} \geq 1 \text{ and } (\,r,\, c\,) \in \mathfrak{Y} \;\} \tag{46}$$

Somewhat more elegantly the *a priori* estimate $\dim \mathscr{A}^2 \leq \| \mathfrak{Y} \|$ can be established using the complete flag $F_\bullet$ associated to the chosen basis $t_1, \ldots, t_n \in T$ with $F_s := \mathrm{span}\{\, t_1, \ldots, t_s \,\}$. Associated to this complete flag is a descending filtration of the tableau $\mathscr{A}^1$ by subtableaux

$$\mathscr{A}^1 \;=\; \mathscr{A}^1_{F_0} \;\supseteq\; \mathscr{A}^1_{F_1} \;\supseteq\; \mathscr{A}^1_{F_2} \;\supseteq\; \ldots \;\supseteq\; \mathscr{A}^1_{F_{n-1}} \;\supseteq\; \mathscr{A}^1_{F_n} \;=\; \{\,0\,\}$$

which in turn define tableau comodules $\mathscr{A}^\bullet_{F_s}$ for all $s = 1, \ldots, n$. The most important observation to be made at this point is that these tableau comodules are interrelated by

$$\mathscr{A}^{\bullet}_{F_s} \;=\; \mathbf{ker}\Big(\;\frac{\partial}{\partial t_s}\;:\;\mathscr{A}^{\bullet}_{F_{s-1}}\;\longrightarrow\;\mathscr{A}^{\bullet-1}_{F_{s-1}},\;\;a\;\longmapsto\;\frac{\partial a}{\partial t_s}\;\Big)\;=:\;(\,\mathscr{A}^{\bullet}_{F_{s-1}}\,)_{\mathbb{R}\,t_s} \tag{47}$$

valid for all $s = 1, \ldots, n$. In fact the tableau comodule $\mathscr{A}^{\bullet}_{F_{s-1}} \subset \mathrm{Sym}^{\bullet}T^* \otimes V$ is more or less by definition the subspace of polynomials ψ on T with values in V, whose differential $B_p\psi : T \longrightarrow V,\, t \longrightarrow \frac{\partial\psi}{\partial t}(\,p\,)$, is an element of the subspace $\mathscr{A}^{1}_{F_{s-1}} \subset \mathrm{Hom}\,(\,T,\; V\,)$ in every point $p \in T$. Clearly such a polynomial is constant in the direction of t_s, if and only if its differential in every point kills the vector t_s and thus lies in the subspace $\mathscr{A}^1_{F_s}$. The resulting equality (47) between the tableau comodule $\mathscr{A}^{\bullet}_{F_s}$ and the kernel of $\frac{\partial}{\partial t_s}$ in the tableau comodule $\mathscr{A}^{\bullet}_{F_{s-1}}$ for all $s = 1, \ldots, n$ for all $s = 1, \ldots, n$ implies the exactness of the sequences:

$$0 \;\longrightarrow\; \mathscr{A}^2_{F_s} \;\stackrel{\subset}{\longrightarrow}\; \mathscr{A}^2_{F_{s-1}} \;\stackrel{\frac{\partial}{\partial t_s}}{\longrightarrow}\; \mathscr{A}^1_{F_{s-1}} \tag{48}$$

Combining this exactness with the estimate $\dim\,\mathscr{A}^1_{F_s} = D - \dim\,\mathrm{res}_{F_s}\mathscr{A}^1 \leq c_{s+1} + \ldots + c_n$ established in the proof of Lemma 4.7 to obtain the *a priori* estimate as a telescope sum

$$\dim\,\mathscr{A}^2 \;=\; \sum_{s=1}^{n}\Big(\,\dim\,\mathscr{A}^2_{F_{s-1}} \;-\; \dim\,\mathscr{A}^2_{F_s}\,\Big) \;\leq\; \sum_{s=1}^{n}\dim\,\mathscr{A}^1_{F_{s-1}} \;\leq\; \|\,\mathfrak{Y}\,\|$$

telescoping to $\dim\,\mathscr{A}^2_{F_0} - \dim\,\mathscr{A}^2_{F_n} = \dim\,\mathscr{A}^2$ according to $\mathscr{A}^1_{F_0} = \mathscr{A}^1$ and $\mathscr{A}^1_{F_n} = \{\,0\,\}$.

For the second part of the proof let us assume that the *a priori* estimate is actually sharp $\dim\,\mathscr{A}^2 = \|\,\mathfrak{Y}^{\mathscr{A}}\,\|$ for the Cartan character $\mathfrak{Y}^{\mathscr{A}}$. Under this assumption all inequalities in the preceding argument must be equalities, in particular $\dim\,\mathscr{A}^1_{F_s} = c^{\mathscr{A}}_{s+1} + \ldots + c^{\mathscr{A}}_n$ holds true for all $s = 1, \ldots, n$ and all the exact sequences (48) are surjective on the right and thus short exact. This simple observation provides the basis in $k = 1$ for an inductive argument to the end that we have for all $k \geq 1$ and all $s = 1, \ldots, n$ short exact sequences:

$$0 \;\longrightarrow\; \mathscr{A}^{k+1}_{F_s} \;\stackrel{\subset}{\longrightarrow}\; \mathscr{A}^{k+1}_{F_{s-1}} \;\stackrel{\frac{\partial}{\partial t_s}}{\longrightarrow}\; \mathscr{A}^{k}_{F_{s-1}} \;\longrightarrow\; 0 \tag{49}$$

In the induction step from k to $k+1$ we apply the first statement of the technical Lemma 4.9 to all tableau comodules $\mathscr{A}^{\bullet}_{F_s} = (\mathscr{A}^{\bullet}_{F_{s-1}})_{\mathbb{R}t_s}$ in turn to obtain a chain of surjections

$$H^{k,\circ}(\,\mathscr{A}_{F_n}\,) \;\longrightarrow\; H^{k,\circ}(\,\mathscr{A}_{F_{n-1}}\,) \;\longrightarrow\; \ldots \;\longrightarrow\; H^{k,\circ}(\,\mathscr{A}_{F_1}\,) \;\longrightarrow\; H^{k,\circ}(\,\mathscr{A}_{F_0}\,)$$

in Spencer cohomology, in which the first term $H^{k,\circ}(\,\mathscr{A}_{F_n}\,) = \{\,0\,\}$ vanishes due to $k > 0$, after all the tableau comodule $\mathscr{A}^{\bullet}_{F_n} = \delta_{\bullet=0}\,V$ is concentrated in degree zero. In consequence

$$H^{k+1,0}(\mathscr{A}_{F_s}) = \{0\} \qquad H^{k+1,1}(\mathscr{A}_{F_s}) = \{0\} \qquad H^{k,2}(\mathscr{A}_{F_s}) = \{0\}$$

vanish for all $s = 0, \ldots, n$, in fact we just have finished proving the third assertion, whereas the first two are true for every first order tableau comodule according to Eq. (30) and the assumption $k > 0$. All requirements of the second statement of Lemma 4.9 are thus met, and we conclude that the short exact sequences one degree higher up are exact

$$0 \longrightarrow \mathscr{A}^{k+2}_{F_s} \xrightarrow{\subset} \mathscr{A}^{k+2}_{F_{s-1}} \xrightarrow{\frac{\partial}{\partial t_s}} \mathscr{A}^{k+1}_{F_{s-1}} \longrightarrow 0$$

on the right for all $s = 1, \ldots, n$ completing thus the induction. More or less as a by-product we have proved that the Spencer cohomology $H^{\bullet,\circ}(\mathscr{A}_{F_s}) = \{0\}$ vanishes in comodule degrees $\bullet \neq 0$ for all comodules $\mathscr{A}_{F_s}$, $s = 0, \ldots, n$ and thus for $\mathscr{A} = \mathscr{A}_{F_0}$ as well. With such a Spencer cohomology the E^1-term of the standard spectral sequence of Lemma 3.7

$$\mathrm{Sym}^\bullet T^* \otimes H^{\blacklozenge,\circ}(\mathscr{A}) \implies \delta_{\blacklozenge=0=\circ}\,\mathscr{A}^\bullet$$

is concentrated in degrees $\blacklozenge = 0$. Hence this spectral sequence has only one chance left to converge to the original comodule $\mathscr{A}$: The coboundary operators leading to the E^2-term have to link up to form a resolution of $\mathscr{A}$ by the free comodules with basis $H^{0,\circ}(\mathscr{A})$, in particular the standard spectral sequence collapses at its E^2-term equal to $\mathscr{A}$.

The short exact sequences (49) valid for all $k \geq 1$ and $s = 1, \ldots, n$ allow us to calculate the dimension of the homogeneous subspaces $\mathscr{A}^k_{F_s}$ of the comodules $\mathscr{A}_{F_s}$ as telescope sums

$$\dim \mathscr{A}^k_{F_s} = \sum_{\mu \geq 1} \Big(\dim \mathscr{A}^k_{F_{s+\mu-1}} - \dim \mathscr{A}^k_{F_{s+\mu}} \Big) = \sum_{\mu \geq 1} \dim \mathscr{A}^{k-1}_{F_{s+\mu-1}}$$

for all $k \geq 2$. Straightforward induction on $k \geq 1$ using this equation in the induction step and the equality $\dim \mathscr{A}^1_{F_s} = c^{\mathscr{A}}_{s+1} + \ldots + c^{\mathscr{A}}_n$ established as a direct consequence of the assumption $\dim \mathscr{A}^2 = \| \mathfrak{Y}^{\mathscr{A}} \|$ as induction base thus proves the proves the explicit formula

$$\dim \mathscr{A}^k_{F_s} = \sum_{\mu=1}^{n-s} \binom{k-2+\mu}{\mu-1} c^{\mathscr{A}}_{s+\mu}$$

for all $k \geq 1$ and $s = 0, \ldots, n$, which becomes the stipulated formula for $\dim \mathscr{A}^k$ in the special case $s = 0$. Eventually the Betti numbers $\dim H^{0,r}(\mathscr{A})$ of the comodule $\mathscr{A}$ can be calculated by binomial inversion from the following identity, which is obtained by equating the preceding formula for $\dim \mathscr{A}^k$ with the formula from Corollary 3.8:

$$\sum_{s=1}^{n} \binom{k-2+s}{s-1} c_s^{\mathscr{A}} \quad = \quad \sum_{r=0,\ldots,n} (-1)^r \binom{k-r+n-1}{n-1} \dim \; H^{0,\,r}(\,\mathscr{A}\,)$$

□

Perhaps the reader may have wondered, why we took the time to prove the *a priori* estimate $\dim \mathscr{A}^2 \leq \|\mathfrak{Y}\|$ for the dimension of the first prolongation of a tableau $\mathscr{A}^1 \subset \mathrm{Hom}\,(\,T,\;V\,)$ in terms of a Young diagram $\mathfrak{Y}$ presenting $\mathscr{A}^1$. The point is that the argument using matrix coefficients generalizes to all degrees $k \geq 1$ in the formulation that the matrix coefficients

$$\mathrm{mat}_{r;\,c^1\ldots c^k}(\,a\,) \quad := \quad dv_r\Big(\,\frac{\partial^k}{\partial t_{c^1}\ldots\partial t_{c^k}}\,a\,\Big)$$

indexed by tuples $(\,r;\;c^1,\ldots,c^k\,)$ satisfying $c^k \geq \ldots \geq c^1 \geq 1$ and $(r, c^k) \in \mathfrak{Y}$ generate $\mathscr{A}^{k*}$. For the Cartan character $\mathfrak{Y} = \mathfrak{Y}^{\mathscr{A}}$ of an involutive tableau $\mathscr{A}^1 \subset \mathrm{Hom}\,(\,T,\;V\,)$ however the formulas for $\dim \mathscr{A}^k$ in terms of the column lengths of $\mathfrak{Y}^{\mathscr{A}}$ imply

$$\dim\;\mathscr{A}^k \;=\; \sharp\{\,(\,r;\;c^1,\ldots,c^k\,)\mid r \geq 1,\;c^k \geq \ldots \geq c^1 \geq 1 \text{ and } (\,r,\;c^k\,) \in \mathfrak{Y}^{\mathscr{A}}\;\}$$

so that these special matrix coefficient are a basis of $\mathscr{A}^{k*}$. In a sense this statement can be seen as an interpolation formula, because it implies that for every choice of real constants $a_{r;\,c^1\ldots c^k} \in \mathbb{R}$ for every tuple with $r \geq 1$ and $c^k \geq \ldots \geq c^1 \geq 1$ as well as $(r, c^k) \in \mathfrak{Y}$ there exists a unique element $a \in \mathscr{A}^k$ of the comodule satisfying:

$$\mathrm{mat}_{r;\,c^1\ldots c^k}(\,a\,) \quad = \quad a_{r;\,c^1\ldots c^k}$$

In other words there exists an essentially algorithmic way to calculate all elements of the tableau comodule $\mathscr{A}$ corresponding to an involutive tableau $\mathscr{A}^1 \subset \mathrm{Hom}\,(\,T,\;V\,)$ in terms of its structure constants $C^{rc}_{\tilde{r}\tilde{c}}$ of Eq. (45), although it seems difficult to write an actual computer program to implement this algorithm. In light of this interpolation property of involutive tableaux, it is very interesting to know that every finitely generated comodule $\mathscr{A}$ bounded below becomes eventually an involutive tableau comodule:

Theorem 4.10 (Twisted Comodules and Involutivity). *Consider a finitely generated comodule $\mathscr{A}$ bounded below and let $d_{\max} < \infty$ be the maximal comodule degree $\bullet$ realized by a non-trivial Spencer cohomology space $H^{\bullet,\circ}(\,\mathscr{A}\,) \neq \{\,0\,\}$ in its finite dimensional Spencer cohomology, this is $H^{d,r}(\,\mathscr{A}\,) = \{\,0\,\}$ for all $d > d_{\max}$ and all r. For all $d \geq d_{\max}$ the twist $\mathscr{A}^\bullet(\,d\,) \subset \mathrm{Sym}^\bullet T^* \otimes \mathscr{A}^d$ associated to $\mathscr{A}$ is an involutive tableau comodule associated to the "prolonged" tableau $B(\,\mathscr{A}^{d+1}\,) \subset \mathrm{Hom}\,(\,T,\;\mathscr{A}^d\,)$ with:*

$$B(\mathscr{A}^{d+1}) \;:=\; \{\, B\,a : T \longrightarrow \mathscr{A}^d,\;\; t \longmapsto \tfrac{\partial a}{\partial t} \;\mid\; a \in \mathscr{A}^{d+1} \,\}$$

Its Cartan character $\mathfrak{Y}^{\mathscr{A}(d)}$ *has columns of length* $c_1^{\mathscr{A}(d)} \geq c_2^{\mathscr{A}(d)} \geq \ldots \geq c_n^{\mathscr{A}(d)} \geq 0$ *given by:*

$$c_s^{\mathscr{A}(d)} \;=\; \sum_{\substack{r=0,\ldots,n \\ l\in\mathbb{Z}}} (-1)^r \binom{n+d-s-l-r}{n-s} \dim\, H^{l,r}(\mathscr{A})$$

where $\binom{x}{m}$ *denotes the binomial polynomial* $\frac{1}{m!}\,x\,(x-1)\cdots(x-m+1)$ *for all* $m \in \mathbb{N}_0$. *In passing we note the identity* $\dim\,\mathscr{A}^d = c_1^{\mathscr{A}(d)} + \dim\, H^{d,n}(\mathscr{A})$ *valid for all* $d \geq d_{\max}$.

The Prolongation Theorem is actually a recompilation of all the properties we have discussed in the last two sections, for this reason we will not go into the details of its proof. Perhaps the strangest conclusion of Theorem 4.10 is that the Betti numbers of every finitely generated comodule $\mathscr{A}$ bounded below satisfy the following *a priori* inequalities for all $s = 1, \ldots, n$

$$\sum_{\substack{r=0,\ldots,n \\ l\in\mathbb{Z}}} (-1)^r \binom{n+d_{\max}-s-l-r-1}{n-s} \dim\, H^{l,r}(\mathscr{A}) \;\geq\; 0$$

which reflect the standard column length inequalities for the Cartan character $\mathfrak{Y}^{\mathscr{A}(d_{\max})}$.

5 Cartan–Kähler Theory

In essence the notion of an exterior differential system studied in this section can be seen as an axiomatization of the contact systems on the jet bundles of maps or section and the similar contact system on the generalized Graßmannians constructed in Sect. 2. Exaggerating somewhat we may say that exterior differential systems axiomatize the very concept of partial differential equations itself. En nuce the Cartan–Kähler theory of exterior differential systems is based on the simple idea to replace the submanifold solutions passing through a point by their infinite order Taylor series in this point, an idea already present in the beautiful theorem of Cauchy–Kovalevskaya for underdetermined partial differential equations. The purpose of this section is to sketch a proof of the formal version of the theorem of Cartan–Kähler, which generalizes the theorem of Cauchy–Kovalevskaya to other partial differential equations, while linking the topic to the Spencer cohomology of comodules discussed in Sects. 3 and 4.

Certainly the most striking feature common to both the contact system (12) on the bundle of jets of maps or sections and the contact system (17) on the generalized Graßmannians is the existence of a filtration of the cotangent bundle of the total space M by subbundles

$$0 \subseteq CM \subseteq HM \subseteq T^*M$$

such that the characteristic compatibility condition $d\,\Gamma(CM) \subset \Gamma(HM \wedge T^*M)$ holds true:

Definition 5.1 (Exterior Differential Systems). An exterior differential system on a manifold M is a filtration of the cotangent bundle T^*M by subbundles CM and HM called the bundles of contact and horizontal forms respectively

$$0 \subseteq CM \subseteq HM \subseteq T^*M$$

such that the exterior derivative of every contact form $\gamma \in \Gamma(CM)$ is a section of the ideal bundle $d\gamma \in \Gamma(HM \wedge T^*M)$ generated by HM. The annihilator subbundles

$$C^\perp M := \operatorname{Ann} CM = \{ X_p \in TM \mid \gamma(X_p) = 0 \text{ for all } \gamma \in C_pM \} \subset TM$$

and $H^\perp M := \operatorname{Ann} HM$ defining the reciprocal filtration of the tangent bundle of M

$$TM \supseteq C^\perp M \supseteq H^\perp M \supseteq \{0\}$$

are called the vector bundles of admissible and vertical vectors on M respectively.

The reader may well wonder how such a definition may be used to treat partial differential equations in the language of differential forms, this question however is as futile as asking for the proper meaning of an answer 42 without knowing the question exactly. In other words the preceding definition is pretty useless without being accompanied by the complementary notion of a solution to a given exterior differential system $CM \subseteq HM \subseteq T^*M$:

Definition 5.2 (Solutions to Exterior Differential Systems). A solution to an exterior differential system $CM \subset HM$ on a manifold M is a submanifold $N \subset M$ of dimension $n := \dim HM - \dim CM$ such that every vector tangent to N is both admissible $T_pN \subset C_p^\perp M$ and non-vertical $T_pN \cap H_p^\perp M = \{0\}$. In every point $p \in N$ the tangent space T_pN is thus a linear complement to the vertical in the admissible vectors:

$$C_p^\perp M = T_pN \oplus H_p^\perp M$$

Whatever else exterior differential systems and their solutions may be good for, their *raison d'être* is to unify different types of partial differential into a common framework formulated in the language of differential forms. For this reason let us postpone the development of the general theory for the moment in order to verify that the solutions to the contact systems discussed in Sect. 2 faithfully represent our intuitive understanding of what a solution to a partial differential equation should be. For convenience we will only consider the contact system on the generalized Graßmannian $\mathrm{Gr}^k_n M$, the reader is invited to repeat this analysis with the contact systems on $\mathrm{Jet}^k(N, M)$ and/or $\mathrm{Jet}^k \mathscr{F} M$.

Recall to begin with that the standard jet coordinates on the generalized Graßmannian $\mathrm{Gr}^k_n M$ associated to local coordinates $(x^1, \ldots, x^m)$ on M take the form (x^α, x^β_A) with indices $\alpha = 1, \ldots, n$ and $\beta = n+1, \ldots, m$ as well as multi-indices A on $\{1, \ldots, n\}$ of order $|A| \le k$. Moreover the scalar components of the canonical contact form $\gamma^{\mathrm{contact}}$ on $\mathrm{Gr}^k_n M$ are indexed by $\beta = n+1, \ldots, m$ and multi-indices A of order $|A| < k$ and read:

$$\gamma^\beta_A \;:=\; dx^\beta_A \;-\; \sum_{\alpha=1}^{n} x^\beta_{A+\alpha}\, dx^\alpha$$

Augmented by horizontal forms the contact system (17) on $\mathrm{Gr}^k_n M$ can thus be written:

$$\begin{array}{rcl} C(\mathrm{Gr}^k_n M) := \mathrm{span}\,\{ & \gamma^\beta_A & \mid \text{ for all } \beta,\ |A| < k\ \} \\ H(\mathrm{Gr}^k_n M) := \mathrm{span}\,\{ & dx^\alpha,\ dx^\beta_A & \mid \text{ for all } \alpha,\ \beta,\ |A| < k\ \} \end{array}$$

In particular the annihilator subbundles of the reciprocal filtration of $T\mathrm{Gr}^k_n M$ are given by

$$\begin{array}{rcl} H^\perp(\mathrm{Gr}^k_n M) := \mathrm{span}\,\{ & \frac{\partial}{\partial x^\beta_A} & \mid \text{ for all } \beta,\ |A| = k\ \} \\ C^\perp(\mathrm{Gr}^k_n M) := \mathrm{span}\,\{ & \frac{\partial}{\partial x^\beta_A},\ \frac{d}{dx^\alpha} & \mid \text{ for all } \alpha,\ \beta,\ |A| = k\ \} \end{array}$$

where the total derivatives $\frac{d}{dx^\alpha}$ associated to the jet coordinates (x^α, x^β_A) are defined by:

$$\frac{d}{dx^\alpha} \;:=\; \frac{\partial}{\partial x^\alpha} \;+\; \sum_{\substack{|A|<k \\ \beta}} x^\beta_{A+\alpha}\, \frac{\partial}{\partial x^\beta_A}$$

With a view on the calculations to come we remark that in this special exterior differential system the dual quotient bundles $H(\mathrm{Gr}^k_n M)/C(\mathrm{Gr}^k_n M)$ and $C^\perp(\mathrm{Gr}^k_n M)/H^\perp(\mathrm{Gr}^k_n M)$ are spanned by the dual classes represented by $dx^1, \ldots, dx^n$ and $\frac{d}{dx^1}, \ldots, \frac{d}{dx^n}$. Every linear complement to the vertical in the admissible vectors in a point $p \in \mathrm{Gr}^k_n M$ is of the form

$$\operatorname{span}\left\{ \frac{d}{dx^\alpha}\Big|_p + \sum_{\substack{|A|=k\\ \beta}} x^\beta_{A,\alpha} \frac{\partial}{\partial x^\beta_A}\Big|_p \right\} \subset C^\perp_p(\operatorname{Gr}^k_n M) \tag{50}$$

with suitably chosen constants $x^\beta_{A,\alpha} \in \mathbb{R}$ defined for all α, β and multi-indices A of order $|A| = k$. According to this description of all linear complements possible in $p \in \operatorname{Gr}^k_n M$ the differentials $d_p x^1, \ldots, d_p x^n$ of the coordinate functions $x^1, \ldots, x^n$ stay linearly independent upon restriction to the tangent space $T_p N$ of a solution submanifold $N \subset \operatorname{Gr}^k_n M$ passing through p, hence N can be written at least locally as the graph of a smooth map

$$(x^1, \ldots, x^n) \longmapsto (x^1, \ldots, x^n,\ x^\beta_A(x^1, \ldots, x^n))$$

with parameter functions $x^\beta_A(x^1, \ldots, x^n)$ to be specified for all β and all multi-indices A of order $|A| \leq k$. In terms of these parameter functions the tangent space $T_p N$ can be written

$$T_p N = \operatorname{span}\left\{ \frac{\partial}{\partial x^\alpha}\Big|_p + \sum_{\substack{|A|\leq k\\ \beta}} \frac{\partial x^\beta_A}{\partial x^\alpha}(x^1, \ldots, x^n) \frac{\partial}{\partial x^\beta_A}\Big|_p \right\}$$

and comparing coefficients with the general form (50) of linear complements to the vertical in the admissible vectors we obtain the following constraints on the functions $x^\beta_A(x^1, \ldots, x^n)$

$$x^\beta_{A+\alpha}(x^1, \ldots, x^n) = \frac{\partial x^\beta_A}{\partial x^\alpha}(x^1, \ldots, x^n) \qquad\qquad x^\beta_{A,\alpha} = \frac{\partial x^\beta_A}{\partial x^\alpha}(x^1, \ldots, x^n)$$

for all α, β and all multi-indices A of order $|A| < k$ respectively $|A| = k$. By a straightforward induction all solutions to these constraints are completely determined by the parameter functions $x^\beta(x^1, \ldots, x^n)$ corresponding to the empty multi-index via the expected formula:

$$x^\beta_A(x^1, \ldots, x^n) = \frac{\partial^{|A|} x^\beta}{\partial x^A}(x^1, \ldots, x^n) \qquad x^\beta_{A,\alpha} = \frac{\partial^{|A|+1} x^\beta}{\partial x^{A+\alpha}}(x^1, \ldots, x^n) \tag{51}$$

In consequence every solution submanifold $N \subset \operatorname{Gr}^k_n M$ to the contact system on the generalized Graßmannian $\operatorname{Gr}^k_n M$ is holonomic in the sense that there exists at least locally a submanifold $N_{\text{base}} \subset M$ of dimension n with the property $N = \{ \operatorname{jet}^k_{\pi(p)} N_{\text{base}} \mid p \in N \}$. Concluding our excursion to jet coordinates we recall that the exterior derivative of the scalar component γ^β_A of the contact form γ^{contact} indexed by a multi-index A of order $|A| < k - 1$

$$d\,\gamma^\beta_A \;=\; -\sum_\alpha\Big(\gamma^\beta_{A+\alpha}+\sum_{\tilde\alpha}x^\beta_{A+\alpha+\tilde\alpha}\,dx^{\tilde\alpha}\Big)\wedge dx^\alpha \;\stackrel{!}{=}\; -\sum_\alpha\gamma^\beta_{A+\alpha}\wedge dx^\alpha$$

lies in the ideal generated by the components of $\gamma^{\mathrm{contact}}$, because $\sum x^\beta_{A+\alpha+\tilde\alpha}\,dx^{\tilde\alpha}\wedge dx^\alpha = 0$ vanishes due to symmetry. For multi-indices A of order $|\,A\,| = k-1$ on the other hand the exterior derivative $d\gamma^\beta_A = -\sum dx^\beta_{A+\alpha}\wedge dx^\alpha$ of γ^β_A restricts to a non-trivial 2-form

$$(d\gamma^\beta_A)_p\Big(\frac{d}{dx^{\tilde\alpha}}\Big|_p+\sum_{\substack{|\,\tilde A\,|=k\\ \tilde\beta}}x^{\tilde\beta}_{\tilde A,\tilde\alpha}\frac{\partial}{\partial x^{\tilde\beta}_{\tilde A}}\Big|_p,\;\;\frac{d}{dx^{\hat\alpha}}\Big|_p+\sum_{\substack{|\,\hat A\,|=k\\ \hat\beta}}x^{\hat\beta}_{\hat A,\hat\alpha}\frac{\partial}{\partial x^{\hat\beta}_{\hat A}}\Big|_p\Big)$$
$$\stackrel{!}{=} x^\beta_{A+\tilde\alpha,\hat\alpha} - x^\beta_{A+\hat\alpha,\tilde\alpha} \tag{52}$$

on a general linear complement of the vertical in the admissible vectors in a point $p\in\mathrm{Gr}^k_nM$ written in the form (50) with suitably chosen constants $x^\beta_{A,\alpha}\in\mathbb{R}$.

A partial differential equation of order $k\geq 1$ for submanifolds of dimension n of a manifold M is in essence the same as the associated subset $\mathrm{Eq}^kM\subset\mathrm{Gr}^k_nM$ of *algebraic* solutions. In practice Eq^kM is usually a smooth subbundle of the fiber bundle $\pi:\mathrm{Gr}^k_nM\longrightarrow M$, although in principle it could arbitrarily complicated. Partial differential equations satisfying this regularity assumption can be transformed into an equivalent exterior differential system on the manifold Eq^kM simply by restricting the differential forms comprising the contact system on Gr^k_nM to the submanifold Eq^kM. Exterior differential systems of general type for example can be reduced to an exterior differential system in the sense of Definition 5.1, because they are invariably first order partial differential equations for submanifolds.

A peculiar consequence of the observation (52) is that the tangent space T_pN of a solution N to an exterior differential system $CM\subseteq HM\subseteq T^*M$ on a manifold M has to satisfy additional *quadratic* constraints besides being a linear complement to the vertical in the admissible vectors, namely the exterior derivative of every contact form $\gamma\in\Gamma(CM)$ needs to vanish $(d\gamma)_p|_{T_pN\times T_pN}=0$ when restricted to T_pN. More precisely $\gamma|_{T_pN}=0$ for every contact form $\gamma\in\Gamma(CM)$, because every vector tangent to N is admissible and hence in $C^\perp_pM$, in terms of the inclusion $\iota_N:N\longrightarrow M$ we may write this $\iota^*_N\gamma=0$ and obtain

$$\iota^*_N(d\gamma)\;=\;d\,(\iota^*_N\gamma)\;=\;0\qquad\Longrightarrow\qquad(d\gamma)_p\big|_{T_pN\times T_pN}\;=\;0 \tag{53}$$

using the naturality of the exterior derivative. Recall now that the linear complements to the vertical in the admissible vectors correspond directly to sections of the short exact sequence

$$0\longrightarrow H^\perp_pM\stackrel{\subset}{\longrightarrow}C^\perp_pM\stackrel{\mathrm{pr}}{\longrightarrow}C^\perp_pM\big/_{H^\perp_pM}\longrightarrow 0 \tag{54}$$

namely the *image* of a section $s : C_p^\perp M / H_p^\perp M \longrightarrow C_p^\perp M$ is a linear complement and every linear complement T equals the image of a unique section s_T : $C_p^\perp M / H_p^\perp M \longrightarrow C_p^\perp M$. Sections of a short exact sequence like (54) on the other hand form an affine space modelled on the vector space $(H_p M / C_p M) \otimes H_p^\perp M$ of linear maps $\Delta s : C_p^\perp M / H_p^\perp M \longrightarrow H_p^\perp M$. Only the images of those sections $s : C_p^\perp M / H_p^\perp M \longrightarrow C_p^\perp M$ qualify as candidates for the tangent space of a solution N passing through $p \in M$, which satisfy the quadratic constraint

$$(d\gamma)_p(\, sX, \, sY \,) \;=\; 0 \tag{55}$$

for every contact form $\gamma \in \Gamma(TM)$ and all $X, Y \in C_p^\perp M / H_p^\perp M$. In due course we will analyze this quadratic constraint in more detail, in particular a description of the set of all possible solutions $s : C_p^\perp M / H_p^\perp M \longrightarrow C_p^\perp M$ is given in Corollary 5.6.

Leaving the analytical description of exterior differential systems aside and turning to the associated algebraic theory of comodules we begin by casting the characteristic compatibility condition $d\,\Gamma(\,CM\,) \subset \Gamma(\,HM \wedge T^*M\,)$ between contact and horizontal forms into more manageable terms. Multilinear algebra tells us that ideal $H \wedge \Lambda^{\circ-1}T^* \subset \Lambda^\circ T^*$ in the exterior algebra of alternating forms on a vector space T generated by a subspace $H \subset T^*$ equals the ideal of alternating forms vanishing on all tuples of arguments in $H^\perp \subset T$:

$$H \wedge \Lambda^{\circ-1}T^* \;=\; \{\, \gamma \in \Lambda^\circ T^* \;\mid\; \gamma(\, V_1, \ldots, V_\circ \,) = 0 \;\text{ for all }\; V_1, \ldots, V_\circ \in H^\perp \,\}$$

More succinctly this statement reads $\Lambda^\circ H^\perp = (\, H \wedge \Lambda^{\circ-1}T^* \,)^\perp$ in terms of the duality between $\Lambda^\circ T$ and $\Lambda^\circ T^*$, in particular it can be seen as the supersymmetric analogue of the statement that a polynomial $\gamma \in \operatorname{Sym} T^*$ on T lies in the ideal generated by $H \subset T^*$, if and only if it vanishes identically on the subspace $H^\perp \subset T$. In consequence the characteristic compatibility condition imposed on an exterior differential system is equivalent to

$$d\gamma(\, V_1, \, V_2 \,) \;=\; 0 \tag{56}$$

for every contact form $\gamma \in \Gamma(\,CM\,)$ and all vertical vector fields $V_1, V_2 \in \Gamma(\,H^\perp M\,)$. Replacing one of the two vertical vector fields by an admissible vector field $A \in \Gamma(\,C^\perp M\,)$ we obtain an expression $\Sigma(\gamma, A, V) := d\gamma(A, V) \in C^\infty(M)$, which does only depend on the class represented by A in the sections of the quotient bundle $C^\perp M / H^\perp M$. Despite first appearance $\Sigma(\gamma, A, V)$ depends $C^\infty(M)$-linearly not only on the vector fields A and V, but on the contact form $\gamma \in \Gamma(\,CM\,)$ as well, because $\gamma(A) = 0 = \gamma(V)$ both vanish so that:

$$\Sigma(\, f\gamma, \, A, \, V \,) \;=\; (\, df \wedge \gamma \,)(\, A, \, V \,) \;+\; f\, d\gamma(\, A, \, V \,) \;=\; f\, \Sigma(\, \gamma, \, A, \, V \,)$$

for every smooth function $f \in C^\infty(M)$ and every contact form $\gamma \in \Gamma(\,CM\,)$:

Definition 5.3 (Symbol of an Exterior Differential System). The symbol of an exterior differential system $CM \subseteq HM \subseteq T^*M$ on a manifold M is the $C^\infty(M)$-trilinear map $\Sigma : \Gamma(CM) \times \Gamma(C^\perp M/H^\perp M) \times \Gamma(H^\perp M) \longrightarrow C^\infty(M)$ defined for a contact form $\gamma \in \Gamma(CM)$ and vector fields $A \in \Gamma(C^\perp M)$ and $V \in \Gamma(H^\perp M)$ by:

$$\Sigma(\gamma, A, V) := d\gamma(A, V)$$

Being $C^\infty(M)$-trilinear the symbol Σ can be thought of as a homomorphism of vector bundles in many different ways, the preferred interpretation for exterior differential system reads:

$$\Sigma_p : \quad H_p^\perp M \longrightarrow (H_pM/C_pM) \otimes C_p^*M,$$
$$V_p \longmapsto \Big(A_p \otimes \gamma_p \longmapsto (d\gamma)_p(A, V) \Big)$$

Even more important than the symbol of an exterior differential system $CM \subseteq HM \subseteq T^*M$ on a manifold M are the two $\mathrm{Sym}(H_pM/C_pM)$-comodules associated to Σ in a point $p \in M$:

Definition 5.4 (Symbol and Reduced Symbol Comodule). Consider an exterior differential system $CM \subseteq HM \subseteq TM$ on a manifold M. The reduced symbol comodule of this exterior differential system in a point $p \in M$ is the tableau comodule $\mathscr{A}_p^\bullet \subset \mathrm{Sym}^\bullet(H_pM/C_pM) \otimes C_p^*M$ associated to the image of Σ_p considered as a tableau:

$$\mathscr{A}_p^1 := \textbf{im}\,\Sigma_p \subset (H_pM/C_pM) \otimes C_p^*M$$

The symbol comodule $\mathscr{R}_p^\bullet$ in the point $p \in M$ is the kernel of the composition

$$\mathrm{Sym}^\bullet(H_pM/C_pM) \otimes H_p^\perp M$$
$$\stackrel{\mathrm{id}\otimes\Sigma_p}{\longrightarrow} \mathrm{Sym}^\bullet(H_pM/C_pM) \otimes (H_pM/C_pM) \otimes C_p^*M$$
$$\stackrel{B\otimes\mathrm{id}}{\longrightarrow} \mathrm{Sym}^{\bullet-1}(H_pM/C_pM) \otimes \Lambda^2(H_pM/C_pM) \otimes C_p^*M$$

of comodule homomorphisms involving the Spencer coboundary operator B of Sect. 3.

Interestingly the symbol comodule $\mathscr{R}_p$ of a general exterior differential system is never even mentioned in the otherwise authoritative reference [1] on exterior differential systems. The most important reason for this strange omission seems to be that the symbol comodule $\mathscr{R}_p$ and its reduced counterpart $\mathscr{A}_p$ are related by the very simple short exact sequence

$$0 \longrightarrow \mathrm{Sym}^\bullet(H_pM/C_pM) \otimes \textbf{ker}\,\Sigma_p \stackrel{\subset}{\longrightarrow} \mathscr{R}_p^\bullet \stackrel{\mathrm{id}\otimes\Sigma_p}{\longrightarrow} \mathscr{A}_p^\bullet(1) \longrightarrow 0 \quad (57)$$

of comodules and thus have very similar Spencer cohomologies. In addition Σ_p is injective for many interesting examples so that not only the Spencer cohomology of $\mathscr{R}_p$ and $\mathscr{A}_p$, but the comodules themselves are easily confounded. It is symbol comodule $\mathscr{R}_p$ though, which has the direct bearance on the solution space of an exterior differential system erroneously attributed to the reduced symbol comodule in [1]. In any case the family of subspaces $\mathbf{ker}\,\Sigma_p \subset H_p^{\perp}M \subset T_pM$ parametrized by $p \in M$ appears in [1] in the guise of the so-called special Cauchy characteristic vector fields $\Gamma(\,\mathbf{ker}\,\Sigma\,) \subset \Gamma(\,TM\,)$.

In order to justify the short exact sequence (57) linking the two symbol comodules associated to an exterior differential system we recall from Theorem 4.10 that the twist $\mathscr{A}_p(\,1\,)$ of the tableau comodule $\mathscr{A}_p$ is again a tableau comodule, in fact it is the tableau comodule arising from the tableau $\mathscr{A}_p^2 \subset (H_pM/C_pM) \otimes \mathscr{A}_p^1$. In turn this tableau can be written as the kernel of the Spencer coboundary operator $B \otimes \mathrm{id}$ in the exact sequence

$$0 \longrightarrow \mathscr{A}_p^2 \stackrel{\subset}{\longrightarrow} (H_pM/C_pM) \otimes \mathscr{A}_p^1 \stackrel{B\otimes\mathrm{id}}{\longrightarrow} \Lambda^2(H_pM/C_pM) \otimes C_p^*M$$

due to the generic property $H^{0,\,2}(\,\mathscr{A}_p\,) = \{\,0\,\} = H^{1,\,1}(\,\mathscr{A}_p\,)$ of tableau comodules established in Eq. (30). In consequence the twist $\mathscr{A}_p^{\bullet}(\,1\,)$ of the reduced symbol comodule $\mathscr{A}_p^{\bullet}$ can be written as the kernel of the following homomorphism of free comodules:

$$\begin{aligned}&\mathrm{Sym}^{\bullet}(\,H_pM/C_pM\,) \otimes \mathscr{A}_p^1\\ &\stackrel{B\otimes\mathrm{id}}{\longrightarrow}\ \mathrm{Sym}^{\bullet-1}(\,H_pM/C_pM\,) \otimes \Lambda^2(\,H_pM/C_pM\,) \otimes C_p^*M\end{aligned}$$

With $\mathscr{A}_p^1 := \mathbf{im}\,\Sigma_p$ the symbol comodule $\mathscr{R}_p^{\bullet}$ is thus by its very definition the preimage of the subcomodule $\mathscr{A}_p^{\bullet}(\,1\,) \subset \mathrm{Sym}^{\bullet}(H_pM/C_pM) \otimes (H_pM/C_pM) \otimes C_p^*M$ under the homomorphism $\mathrm{id} \otimes \Sigma_p\,:\, \mathrm{Sym}^{\bullet}(H_pM/C_pM) \otimes H_p^{\perp}M \longrightarrow \mathrm{Sym}^{\bullet}(H_pM/C_pM) \otimes (H_pM/C_pM) \otimes C_p^*M$ of free comodules induced by Σ_p so that the sequence (57) is short exact.

With the machinery of symbol and symbol comodules at our disposal let us now come back to the discussion of the quadratic constraint (55) characterizing the set of linear complements to the vertical in the admissible vectors, which are proper candidates for the tangent spaces T_pN of solutions N passing through $p \in M$. Modifying the section $s\,:\,C_p^{\perp}M/H_p^{\perp}M \longrightarrow C_p^{\perp}M$ of the short exact sequence (54) corresponding to an arbitrary linear complement by a linear map $\Delta s\,:\,C_p^{\perp}M/H_p^{\perp}M \longrightarrow H_p^{\perp}M$ we obtain for all vectors $X,\,Y \in C_p^{\perp}M/H_p^{\perp}M$

$$\begin{aligned}&(d\gamma)_p(\,(s+\Delta s)\,X,\,(s+\Delta s)\,Y\,)\\ &\quad= (d\gamma)_p(\,s\,X,\,s\,Y\,) \;+\; (d\gamma)_p(\,s\,X,\,(\Delta s)\,Y\,) \;-\; (d\gamma)_p(\,s\,Y,\,(\Delta s)\,X\,)\\ &\quad= (d\gamma)_p(\,s\,X,\,s\,Y\,) \;+\; \Sigma_p(\,\gamma_p,\;X,\;(\Delta s)\,Y\,) \;-\; \Sigma_p(\,\gamma_p,\;Y,\;(\Delta s)\,X\,)\end{aligned}$$

because $(d\gamma)_p$ vanishes on two vertical arguments due to the reformulation (56) of the axiomatic compatibility condition $d\,\Gamma(\,CM\,) \subset \Gamma(\,HM \wedge T^*M\,)$ between the contact and the horizontal forms. The modified section $s + \Delta s$ is a solution to the quadratic constraint (55), if and only if Δs satisfies the following inhomogeneous linear equation:

$$\Sigma_p(\,\gamma_p,\ X,\ (\Delta s)\,Y\,) \;-\; \Sigma_p(\,\gamma_p,\ Y,\ (\Delta s)\,X\,) \;=\; -(d\gamma)_p(\,s\,X,\ s\,Y\,) \qquad (58)$$

Our preferred interpretation $\Sigma_p : H_p^\perp M \longrightarrow (H_pM/C_pM) \otimes C_p^*M$ of the symbol Σ_p on the other hand allows us to write the trilinear form $(\,X,\ Y,\ \gamma_p\,) \longmapsto \Sigma_p(\,\gamma_p,\ Y,\ (\Delta s)X\,)$ as the image of Δs considered as an element of $(H_pM/C_pM) \otimes H_p^\perp M$ under the linear map:

$$\mathrm{id} \otimes \Sigma_p : \quad (\,H_pM/C_pM\,) \otimes H_p^\perp M \;\longrightarrow\; (\,H_pM/C_pM\,) \otimes \mathscr{A}_p^1$$

In addition the skew-symmetrization of this trilinear form on the left hand side of the linear equation (58) for Δs implements a special case of the Spencer coboundary operator

$$\begin{aligned}\langle\ B\Big[\,(\mathrm{id} \otimes \Sigma_p)(\,\Delta s\,)\,\Big](\,X,\ Y\,),\ \gamma_p\,\rangle \\ = \Sigma_p(\,\gamma_p,\ X,\ (\Delta s)\,Y\,) \;-\; \Sigma_p(\,\gamma_p,\ Y,\ (\Delta s)\,X\,)\end{aligned}$$

namely $B : (H_pM/C_pM) \otimes \mathscr{A}_p^1 \longrightarrow \Lambda^2(H_pM/C_pM) \otimes \mathscr{A}_p^0$ with $\mathscr{A}_p^0 := C_p^*M$ by definition, the most difficult problem here is to convince oneself of the correctness of the sign. Since every section of the short exact sequence (54) can be written in the form $s + \Delta s$ for an arbitrarily chosen base section $s : C_p^\perp M/H_p^\perp M \longrightarrow C_p^\perp M$ and a suitable modification Δs we conclude that the linear complements T to the vertical vectors $H_p^\perp M$ in the admissible vectors $C_p^\perp M$ satisfying the quadratic constraint (55) correspond via $s_T = s + \Delta s$ bijectively to the solutions $\Delta s \in (H_pM/C_pM) \otimes H_p^\perp M$ of the inhomogeneous linear equation

$$B\Big(\,(\mathrm{id} \otimes \Sigma_p)(\,\Delta s\,)\,\Big) \;=\; -\,\Theta_p(\,s\,) \qquad (59)$$

with right hand side given by $\langle\,\Theta_p(s)(\,X,\ Y\,),\ \gamma_p\,\rangle := (d\gamma)_p(\,sX,\ sY\,)$, compare the original equation (58). In turn this inhomogeneous linear equation gives rise to the concept of torsion:

Definition 5.5 (Torsion). Consider an exterior differential system $CM \subseteq HM \subseteq TM$ on a manifold M. The torsion of this exterior differential system in a point $p \in M$ is the Spencer cohomology class

$$\begin{aligned}[\,\Theta_p(\,s\,)\,] \;&\in\; H^{0,2}(\,\mathscr{A}_p\,) \\ &:= \Big(\,\Lambda^2(\,H_pM/C_pM\,) \otimes C_p^*M\,\Big)\Big/_{B\big(\,(H_pM/C_pM) \otimes \mathscr{A}_p^1\,\big)}\end{aligned}$$

represented by the 2-form $\Theta_p(s) \in \Lambda^2(H_pM/C_pM) \otimes C_p^*M$ with values in C_p^*M defined for an arbitrary section $s : C_p^\perp M/H_p^\perp M \longrightarrow C_p^\perp M$ of the short exact sequence (54) by:

$$\langle\ \Theta_p(s)(X, Y), \gamma_p\ \rangle \ := \ (d\gamma)_p(sX, sY)$$

A classical theorem of linear algebra asserts that the inhomogeneous linear equation (59) characterizing the candidates $\mathbf{im}(s + \Delta s) \subset C_p^\perp M$ for the tangent spaces of solution submanifolds $N \subset M$ passing through $p \in M$ has a solution $\Delta s \in (H_pM/C_pM) \otimes H_p^\perp M$, if and only if $-\Theta_p(s)$ lies in the image of the Spencer coboundary operator B, if and only if the torsion vanishes. After all $\mathrm{id} \otimes \Sigma_p : (H_pM/C_pM) \otimes H_p^\perp M \longrightarrow (H_pM/C_pM) \otimes \mathscr{A}_p^1$ is surjective by definition, hence the vanishing $[\Theta_p(s)] = 0$ of the torsion implies that every preimage $\Delta s \in (H_pM/C_pM) \otimes H_p^\perp M$ of an element of $(H_pM/C_pM) \otimes \mathscr{A}_p^1$ making $-\Theta_p(s)$ exact is a solution to the inhomogeneous equation (59). A very similar argument implies that the torsion is actually independent of the section used to define the representative $\Theta_p(s)$ due to the identity $\Theta_p(s + \Delta s) = \Theta_p(s) + B[(\mathrm{id} \otimes \Sigma_p)(\Delta s)]$:

Corollary 5.6 (Significance of Torsion). *No solution submanifold $N \subset M$ to an exterior differential system $CM \subseteq HM \subseteq T^*M$ on a manifold M passes through a point $p \in M$, unless the torsion $[\Theta_p(s)] \in H^{0,2}(\mathscr{A}_p)$ vanishes for one and hence every section $s : C_p^\perp M/H_p^\perp M \longrightarrow C_p^\perp M$ of the short exact sequence (54). In the latter case the linear complements T to the vertical in the admissible vectors satisfying the constraint $(d\gamma)_p(s_TX, s_TY) = 0$ for all contact forms $\gamma \in \Gamma(CM)$ and all $X, Y \in C_p^\perp M/H_p^\perp M$ form an affine space modelled on the vector space $\mathscr{R}_p^1$.*

Somewhat surprisingly it is the homogeneous subspace $\mathscr{R}_p^1$ of the symbol comodule $\mathscr{R}_p$, which parametrizes the possible candidates for the tangent spaces T_pN of solution submanifolds $N \subset M$ in the case of vanishing torsion $[\Theta_p(s)] = 0$ in the point $p \in M$, not a homogeneous subspace of the more prominent reduced symbol comodule $\mathscr{A}_p$. The reason for this is simple: By its very definition $\mathscr{R}_p^1$ equals the kernel of the linear map $B \circ (\mathrm{id} \otimes \Sigma_p)$ and thus acts naturally on the solutions to the inhomogeneous linear equation (59).

The generalization of Corollary 5.6 to higher orders of differentiation forms the cornerstone of the Cartan–Kähler theory of exterior differential systems. Similar to its historic precursor, the theorem of Cauchy–Kovalevskaya the Cartan–Kähler theory tries to reconstruct the solution submanifolds from their infinite order Taylor series in a given point, a notion made precise by the projective limit $\mathrm{Gr}_n^\infty M$ of the tower (16) of Graßmannians:

$$\ldots \xrightarrow{\mathrm{pr}} \mathrm{Gr}_n^3 M \xrightarrow{\mathrm{pr}} \mathrm{Gr}_n^2 M \xrightarrow{\mathrm{pr}} \mathrm{Gr}_n^1 M \xrightarrow{\pi} \mathrm{Gr}_n^0 M \ = \ M$$

In such a power series approach we are inevitably led to consider jet solutions of sorts:

Definition 5.7 (Jet Solutions and Semisolutions). A jet solution of order $k \geq 1$ to an exterior differential system $CM \subseteq HM \subseteq T^*M$ on a manifold M is a kth order jet of a submanifold $\mathrm{jet}^k_p N \in \mathrm{Gr}^k_n M$, whose tangent space in p is a linear complement $T_pN \subset C^\perp_p M$ to the subspace $H^\perp_p M$ of vertical vectors, such that

$$\mathrm{jet}^{k-1}_p(\, \iota^*_N\, \gamma\,) \;=\; 0 \;=\; \mathrm{jet}^{k-1}_p(\, \iota^*_N\, d\gamma\,)$$

for all contact forms $\gamma \in \Gamma(\,CM\,)$. Similarly a jet semisolution of order $k \geq 1$ is an element $\mathrm{jet}^k_p N \in \mathrm{Gr}^k_n M$ represented by a submanifold $N \subset M$ satisfying $C^\perp_p M = T_pN \oplus H^\perp_p M$ and $\mathrm{jet}^{k-1}_p(\,\iota^*_N \gamma\,) = 0$ for all $\gamma \in \Gamma(\,CM\,)$. Solutions and semisolutions assemble into the sets:

$$\mathrm{Eq}^k_p M := \{\; \mathrm{jet}^k_p N \;\mid\; \mathrm{jet}^k_p N \in \mathrm{Gr}^k_n M \text{ is a jet solution of order } k \;\}$$

$$\overline{\mathrm{Eq}}^k_p M := \{\; \mathrm{jet}^k_p N \;\mid\; \mathrm{jet}^k_p N \in \mathrm{Gr}^k_n M \text{ is a jet semisolution of order } k \;\}$$

In light of the identification $\mathrm{jet}^1_p N \leftrightarrow T_pN$ of the generalized Graßmannian $\mathrm{Gr}^1_n M$ with the Graßmann bundle $\mathrm{Gr}_n(\,TM\,)$ of n-dimensional subspaces of TM the preceding definition of jet solutions and jet semisolutions faithfully reflects our considerations above for order $k = 1$. Jet semisolutions of order $k = 1$ say are simply linear complements T_pN to the vertical in the admissible vectors, while jet solutions are linear complements satisfying the quadratic constraint $\mathrm{jet}^0_p(\,\iota^*_N d\gamma\,) = (d\gamma)_p|_{T_pN \times T_pN} = 0$. Hence $\overline{\mathrm{Eq}}^1_p M$ is always the affine space of sections of the short exact sequence (54), while $\mathrm{Eq}^1_p M$ is described by Corollary 5.6 as an affine space modelled on $\mathscr{R}^1_p$ in the case of vanishing torsion, otherwise it is empty. This classification of jet solutions and jet semisolutions of order $k = 1$ generalizes to the picture at higher orders of differentiation $k \geq 1$, which is best remembered as a tower

$$\begin{array}{ccccc}
\cdots & & \cdots & & \\
\downarrow & \swarrow & & & \\
\mathrm{Eq}^3 M & \xrightarrow{\;\subset\;} & \overline{\mathrm{Eq}}^3 M & \xrightarrow{\;\mathrm{tor}^2\;} & H^{2,2}(\,\mathscr{A}\,) \\
\mathrm{pr}\downarrow & \swarrow\mathrm{pr} & & & \\
\mathrm{Eq}^2 M & \xrightarrow{\;\subset\;} & \overline{\mathrm{Eq}}^2 M & \xrightarrow{\;\mathrm{tor}^1\;} & H^{1,2}(\,\mathscr{A}\,) \\
\mathrm{pr}\downarrow & \swarrow\mathrm{pr} & & & \\
\mathrm{Eq}^1 M & \xrightarrow{\;\subset\;} & \overline{\mathrm{Eq}}^1 M & \xrightarrow{\;\mathrm{tor}^0\;} & H^{0,2}(\,\mathscr{A}\,) \\
\mathrm{pr}\downarrow & \swarrow\mathrm{pr} & & & \\
M & & & &
\end{array} \tag{60}$$

we need to climb up one step at a time. The diagonal projections $\mathrm{pr} : \overline{\mathrm{Eq}}^{k+1}_p M \longrightarrow \mathrm{Eq}^k_p M$ are always surjective with fiber an affine space modelled on $\mathrm{Sym}^{k+1}(H_pM/C_pM) \otimes H^\perp_p M$, while the rows in this tower are "exact" in the following sense: There exists a jet solution $\mathrm{jet}^{k+1}_p N \in \mathrm{Eq}^{k+1}_p M$ over a jet solution

$\mathrm{jet}^k_p N \in \mathrm{Eq}^k_p M$, if and only if the higher torsion $\mathrm{tor}^k : \overline{\mathrm{Eq}}^{k+1}_p M \longrightarrow H^{k,2}(\mathscr{A}_p)$ vanishes on some and hence on every $\mathrm{jet}^{k+1}_p \overline{N} \in \overline{\mathrm{Eq}}^{k+1}_p M$ lying over $\mathrm{jet}^k_p N$. Interestingly all obstructions against formal integrability live in the Spencer cohomology $H^{\bullet,2}(\mathscr{A}_p)$ of the reduced symbol comodule of form degree $\circ = 2$.

A more detailed study of the tower (60) has to wait a little bit until we have clarified the subtle interplay between the jets of submanifolds and the jets of differential forms, on which Definition 5.7 is based. Conceptually it is easier in this endeavor to consider the more general case of smooth maps $\varphi : N \longrightarrow M$ between manifolds N and M and specify to canonical inclusions $\iota_N : N \longrightarrow M$ of submanifolds later on. A smooth map $\varphi : N \longrightarrow M$ can be written in local coordinates $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^m)$ around a point $p \in N$ and its image $\varphi(p) \in M$ as m smooth functions of n variables, namely the pull backs of:

$$\varphi^* y^\mu \quad =: \quad y^\mu(x^1, \ldots, x^n) \qquad \Longrightarrow \qquad \varphi^* dy^\mu \;=\; \sum_{\alpha=1}^{n} \frac{\partial y^\mu}{\partial x^\alpha}(x^1, \ldots, x^n)\, dx^\alpha$$

In this local coordinate description the pull back of a general differential form reads:

$$\varphi^* \Big[\, f(y^1, \ldots, y^m)\, dy^{\mu_1} \wedge \ldots \wedge dy^{\mu_r} \,\Big]$$
$$= \sum_{\alpha_1,\ldots,\alpha_r=1}^{n} f(y^1(x), \ldots, y^m(x))\, \frac{\partial y^{\mu_1}}{\partial x^{\alpha_1}}(x) \ldots \frac{\partial y^{\mu_r}}{\partial x^{\alpha_r}}(x)\, dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_r}$$

In consequence the partial derivatives up to order k of the coefficients of the right hand side with respect to the monomial basis $dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_r}$ depend on the partial derivatives of the original coefficient $f(y^1, \ldots, y^m)$ up to order k and on the partial derivatives of the functions y^μ up to order $k+1$ only. In other words we have a well-defined linear map of vector spaces

$$[\![\, \mathrm{jet}^{k+1}_p \varphi \,]\!] : \quad \mathrm{Jet}^k_{\varphi(p)} \Lambda^\circ T^* M \longrightarrow \mathrm{Jet}^k_p \Lambda^\circ T^* N, \qquad \mathrm{jet}^k_{\varphi(p)} \omega \longmapsto \mathrm{jet}^k_p(\varphi^* \omega) \tag{61}$$

which only depends on $\mathrm{jet}^{k+1}_p \varphi \in \mathrm{Jet}^{k+1}_p(N, M)$. In a similar vein we recall that the exterior derivative d is a linear first order differential operator, its composition $\mathrm{jet}^{k-1} \circ d$ is thus a linear differential operator of order k, whose total symbol map in the sense of Definition 2.2 induces for every point $p \in N$ a linear map of the jet fiber vector spaces

$$d^{\mathrm{formal}} : \quad \mathrm{Jet}^k_p \Lambda^\circ T^* N \longrightarrow \mathrm{Jet}^{k-1}_p \Lambda^{\circ+1} T^* N, \qquad \mathrm{jet}^k_p \omega \longmapsto \mathrm{jet}^{k-1}_p(d\omega)$$

as well as its analogue $d^{\mathrm{formal}} : \mathrm{Jet}^k_{\varphi(p)} \Lambda^\circ T^* M \longrightarrow \mathrm{Jet}^{k-1}_{\varphi(p)} \Lambda^{\circ+1} T^* M$, such that the diagram

$$
\begin{array}{ccc}
\mathrm{Jet}^k_{\varphi(p)}\Lambda^\circ T^*M & \xrightarrow{[\![\,\mathrm{jet}^{k+1}_p\varphi\,]\!]} & \mathrm{Jet}^k_p\Lambda^\circ T^*N \\
{\scriptstyle d^{\mathrm{formal}}}\downarrow & & {\scriptstyle d^{\mathrm{formal}}}\downarrow \\
\mathrm{Jet}^{k-1}_{\varphi(p)}\Lambda^{\circ+1}T^*M & \xrightarrow{[\![\,\mathrm{jet}^k_p\varphi\,]\!]} & \mathrm{Jet}^{k-1}_p\Lambda^{\circ+1}T^*N
\end{array}
\tag{62}
$$

commutes due to the naturality $d(\varphi^*\omega) = \varphi^*(d\omega)$ of the exterior derivative. In passing we remark that the *principal* symbol of the differential operator $\mathrm{jet}^{k-1}\circ d$, which is by definition the restriction of its *total* symbol d^{formal} to the symbol subspace defined in (8)

$$
\mathrm{Sym}^k T_p^*N \otimes \Lambda^\circ T_p^*N \;=\; \mathbf{ker}\Big(\,\mathrm{pr}:\ \mathrm{Jet}^k_p\Lambda^\circ T^*N \;\longrightarrow\; \mathrm{Jet}^{k-1}_p\Lambda^\circ T^*N\,\Big) \tag{63}
$$

agrees with the Spencer coboundary operator B defined in Sect. 3. Of course this is the *conditio sine qua non* for the usefulness of Spencer cohomology in the study of exterior differential systems.

A rather surprising aspect of the commutative diagram (62) should not pass by unnoticed, the linear map $[\![\,\mathrm{jet}^k\varphi\,]\!]$ does only depend on the jet of φ of order k, whereas $[\![\,\mathrm{jet}^{k+1}\varphi\,]\!]$ invariably involves the partial derivatives of φ of order $k+1$. In order to resolve this apparent contradiction to the commutativity of (62) we observe that the only terms in the partial derivatives of the coefficients of $\varphi^*[\,f(y^1,\ldots,y^m)\,dy^{\mu_1}\wedge\ldots\wedge dy^{\mu_r}\,]$ of order up to k, which actually involve partial derivatives of the functions y^μ of order $k+1$, can be written:

$$
\begin{aligned}
& f(\varphi(p))\,\Big(\sum_{\alpha=1}^n \frac{\partial^{|A|+1}y^{\mu_1}}{\partial^{A+\alpha}x}\,dx^\alpha\Big)\wedge\varphi_p^*dy^{\mu_2}\wedge\ldots\wedge\varphi_p^*dy^{\mu_r} \\
&\quad +\; f(\varphi(p))\,\varphi_p^*dy^{\mu_1}\wedge\Big(\sum_{\alpha=1}^n \frac{\partial^{|A|+1}y^{\mu_2}}{\partial^{A+\alpha}x}\,dx^\alpha\Big)\wedge\ldots\wedge\varphi_p^*dy^{\mu_r} \;+\;\ldots
\end{aligned}
$$

Quite remarkably this expression looks like a derivation applied to $f(\varphi(p))\,dy^{\mu_1}\wedge\ldots\wedge dy^{\mu_r}$! The change in the pull back of jets of differential forms resulting from a modification of the highest order partial derivatives of φ by adding $\Delta\varphi\in\mathrm{Sym}^{k+1}T_p^*N\otimes T_{\varphi(p)}M$ thus reads

$$
[\![\,\mathrm{jet}^{k+1}_p\varphi+\Delta\,\varphi\,]\!](\,\mathrm{jet}^k_p\omega\,) = [\![\mathrm{jet}^{k+1}_p\varphi\,]\!](\,\mathrm{jet}^k_p\omega\,) + B\,\Big((\mathrm{id}\otimes\varphi_p^*)(\,\Delta\varphi\,\lrcorner\,\omega_{\varphi(p)}\,)\Big) \tag{64}
$$

where the additional term involves the Spencer coboundary operator B and the composition

$$
\Lambda^\circ T^*_{\varphi(p)}M \;\xrightarrow{\Delta\varphi\,\lrcorner}\; \mathrm{Sym}^{k+1}T_p^*N\otimes\Lambda^{\circ-1}T^*_{\varphi(p)}M \;\xrightarrow{\mathrm{id}\otimes\varphi_p^*}\; \mathrm{Sym}^{k+1}T_p^*N\otimes\Lambda^{\circ-1}T_p^*N
$$

applied to the value $\omega_{\varphi(p)} \in \Lambda^\circ T^*_{\varphi(p)} M$ of the differential form ω in $\varphi(p)$. Since the Spencer coboundary operator equals d^{formal} on the symbol subspace, the additional term in (64) lies in the kernel of d^{formal} and is thus irrelevant to the commutativity of (62).

Specializing the preceding observations to the inclusion maps $\iota_N : N \longrightarrow M$ of submanifolds of M we remark that $\text{jet}^k_p \iota_N \in \text{Jet}^k(N, M)$ and $\text{jet}^k_p N \in \text{Gr}^k_n M$ encode essentially the same object, the jet of a submanifold, hence we may write in a shorthand notation

$$[\![\text{jet}^k_p N]\!] : \quad \text{Jet}^{k-1}_p \Lambda^\circ T^* M \longrightarrow \text{Jet}^{k-1}_p \Lambda^\circ T^* N, \qquad \text{jet}^{k-1}_p \omega \longmapsto \text{jet}^{k-1}_p (\iota_N^* \omega)$$

for the restriction maps appearing prominently in Definition 5.7 of jet solutions and semisolutions. Feeling somewhat uneasy with the fact that the target vector space depends on the representative submanifold N we sooth our conscience by observing that for every two submanifolds N_1 and N_2 representing $\text{jet}^k_p N_1 = \text{jet}^k_p N_2$ there exists a distinguished class $\text{jet}^k_p \varphi \in \text{Jet}^k_p(N_1, N_2)$ represented by those diffeomorphisms $\varphi : N_1 \longrightarrow N_2$, which satisfy

$$\left.\frac{d^{\leq k}}{dt^{\leq k}}\right|_0 c \;=\; \left.\frac{d^{\leq k}}{dt^{\leq k}}\right|_0 (\varphi \circ c) \;\in\; T^k_p M$$

for every curve $c : \mathbb{R} \longrightarrow N_1$. For every diffeomorphism φ in this class $\text{jet}^k_p \iota_{N_1} = \text{jet}^k_p(\iota_{N_2} \circ \varphi)$ so that the two realizations of $[\![\text{jet}^k_p N]\!]$ are intertwined by the well-defined isomorphism:

$$[\![\text{jet}^k_p \varphi]\!] : \quad \text{Jet}^{k-1}_p \Lambda^\circ T^* N_2 \xrightarrow{\cong} \text{Jet}^{k-1}_p \Lambda^\circ T^* N_1, \quad \text{jet}^{k-1}_p \omega \longmapsto \text{jet}^{k-1}_p (\varphi^* \omega)$$

Although these comments may look rather pedantic, they are directly related to a delicate subtlety, which has bothered the author for quite a while. If we modify the highest order partial derivatives of $\text{jet}^{k+1}_p N \in \text{Gr}^{k+1}_n M$ by an element $\Delta N \in \text{Sym}^{k+1} T^*_p N \otimes (T_p M / T_p N)$ using the addition (20), then the modification formula (64) tells us

$$\begin{aligned}
&[\![\text{jet}^{k+1}_p N + \Delta N]\!](\text{jet}^k_p \omega) \\
&\quad = [\![\text{jet}^{k+1}_p N]\!](\text{jet}^k_p \omega) + B\Big((\text{id} \otimes \text{res}_{T^*_p N})((\Delta N)^{\text{lift}} \lrcorner\, \omega_p)\Big)
\end{aligned} \tag{65}$$

the additional term on the right however depends on the lift $(\Delta N)^{\text{lift}} \in \text{Sym}^{k+1} T^*_p N \otimes T_p M$ we need to chose in the addition (20). The resolution to this paradox is that a representative submanifold $\tilde{N}$ for $\text{jet}^{k+1}_p N + \Delta N$ is in contact with N to order k in p only, hence we no longer sport a distinguished vector space isomorphism $[\![\text{jet}^{k+1}_p \varphi]\!] : \text{Jet}^k_p \Lambda^\circ T^* \tilde{N} \xrightarrow{\cong} \text{Jet}^k_p \Lambda^\circ T^* N$. Cum grano salis the modification formula still makes sense: The ambiguity in choosing $(\Delta N)^{\text{lift}}$ given

ΔN is countered exactly by the ambiguity of lifting the distinguished class $\mathrm{jet}^k_p\varphi$ of diffeomorphisms $\varphi : N \longrightarrow \tilde{N}$ to a vector space isomorphism $[\![\, \mathrm{jet}^{k+1}_p\varphi \,]\!]$.

Let us now put the formulas derived above to the test and study the tower (60) of jet solutions and jet semisolutions in more detail. In a first step we observe that every jet semisolution $\mathrm{jet}^{k+1}_p N \in \overline{\mathrm{Eq}}^{k+1}_p M$ of order $k+1$ projects under $\mathrm{pr} : \mathrm{Gr}^{k+1}_n M \longrightarrow \mathrm{Gr}^k_n N$ to a jet solution $\mathrm{jet}^k_p N$. By assumption $\mathrm{jet}^k_p(\,\iota^*_N\gamma\,) = 0$ vanishes for every contact form $\gamma \in \Gamma(\,CM\,)$, thus

$$\mathrm{pr}[\,\mathrm{jet}^k_p(\,\iota^*_N\,d\gamma\,)\,] \;=\; \mathrm{jet}^{k-1}_p(\,\iota^*_N\,d\gamma\,) \;=\; d^{\,\mathrm{formal}}\Big(\mathrm{jet}^k_p(\,\iota^*_N\gamma\,)\Big) \;=\; 0$$

for all $\gamma \in \Gamma(\,CM\,)$ as claimed. In this way we have proved the first statement of the lemma:

Lemma 5.8 (Lifting Jet Solutions to Semisolutions). *Every jet semisolution* $\mathrm{jet}^{k+1}_p N$ *of order* $k+1$ *of an exterior differential system* $CM \subseteq HM$ *on a manifold* M *projects under* $\mathrm{pr} : \mathrm{Gr}^{k+1}_n M \longrightarrow \mathrm{Gr}^k_n M$ *to a jet solution* $\mathrm{jet}^k_p N$ *of order* k. *Conversely the set of all jet semisolutions lying over a given solution* $\mathrm{jet}^k_p N$ *of order* $k \geq 1$ *is a non-empty affine subspace of* $\mathrm{pr}^{-1}(\,\mathrm{jet}^k_p N\,)$ *modelled on the vector subspace:*

$$\mathrm{Sym}^{k+1}T^*_p N \otimes H^{\perp}_p M \;\;\subset\;\; \mathrm{Sym}^{k+1}T^*_p N \otimes (\,T_p M / T_p N\,)$$

Proof. Consider a submanifold $N \subset M$ of dimension n representing a given jet solution $\mathrm{jet}^k_p N$ of order $k \geq 1$. By definition N represents a jet semisolution of order $k+1$, if and only if the $\mathbb{R}$-linear map $\Theta : \Gamma(\,CM\,) \longrightarrow \mathrm{Jet}^k_p T^*N$, $\gamma \longmapsto \mathrm{jet}^k_p(\,\iota^*_N\gamma\,)$, is trivial. In light of our discussion above Θ depends on the representative submanifold N only through $\mathrm{jet}^{k+1}_p N$:

$$\Theta(\,\gamma\,) \;:=\; [\![\,\mathrm{jet}^{k+1}_p N\,]\!](\,\mathrm{jet}^k_p\gamma\,) \;:=\; \mathrm{jet}^k_p(\,\iota^*_N\gamma\,)$$

Since N represents a jet solution of order k, we find $\mathrm{pr}[\mathrm{jet}^k_p(\,\iota^*_N\gamma\,)] = \mathrm{jet}^{k-1}_p(\,\iota^*_N\gamma\,) = 0$, hence $\Gamma(\,CM\,)$ gets mapped under Θ into the kernel of the jet projection, the symbol subspace $\mathrm{Sym}^k T^*_p N \otimes T^*_p N$ of observation (63). In particular Θ is $C^\infty(M)$-linear with

$$\Theta(\,f\,\gamma\,) \;=\; \mathrm{jet}^k_p(\,\iota^*_N f \wedge \iota^*_N\gamma\,) \;=\; \mathrm{jet}^k_p(\,\iota^*_N f\,) \wedge \mathrm{jet}^k_p(\,\iota^*_N\gamma\,) \;=\; f(\,p\,)\,\mathrm{jet}^k_p(\,\iota^*_N\gamma\,)$$

for all $f \in C^\infty(M)$, where we use the natural algebra structure induced on the jet fiber of the algebra bundle $\Lambda^\circ T^*N$ in the second and $\mathrm{jet}^{k-1}_p(\,\iota^*_N\gamma\,) = 0$ in the third equality. Due to $C^\infty(M)$-linearity $\Theta(\,\gamma\,)$ depends on the value of γ in p only, moreover the composition

$$C_p M \xrightarrow{\Theta} \mathrm{Sym}^k T_p^* N \otimes T_p^* N \xrightarrow{B} \mathrm{Sym}^{k-1} T_p^* N \otimes \Lambda^2 T_p^* N$$

vanishes, because the Spencer coboundary B agrees with d^{formal} on the symbol subspace:

$$B[\,\Theta(\gamma_p)\,] = d^{\text{formal}}[\,\mathrm{jet}_p^k(\iota_N^*\gamma)\,] = \mathrm{jet}_p^{k-1}(d\,\iota_N^*\gamma) = \mathrm{jet}_p^{k-1}(\iota_N^* d\gamma) = 0$$

The calculation of the Spencer cohomology of free comodules on the other hand implies that

$$0 \longrightarrow \mathrm{Sym}^{k+1} T_p^* N \xrightarrow{B} \mathrm{Sym}^k T_p^* N \otimes T_p^* N \xrightarrow{B} \mathrm{Sym}^{k-1} T_p^* N \otimes \Lambda^2 T_p^* N$$

is exact for $k \geq 1$. In consequence there exists a unique $\Theta^{\text{pre}} \in \mathrm{Sym}^{k+1} T_p^* N \otimes C_p^* M$ with

$$B(\langle\, \Theta^{\text{pre}},\ \gamma_p \,\rangle) = \Theta(\gamma_p) = \mathrm{jet}_p^k(\iota_N^*\gamma)$$

What remains to do, now that the existence of Θ^{pre} is established, is to write the standard short exact sequence associated to the 3-step filtration $T_p N \subset C_p^\perp M \subset T_p M$ with a view

$$0 \longrightarrow H_p^\perp M \longrightarrow T_p M/_{T_p N} \longrightarrow C_p^* M \longrightarrow 0 \qquad (66)$$

on the canonical isomorphisms $T_p M / C_p^\perp M \cong C_p^* M$ and $H_p^\perp M \cong C_p^\perp M / T_p N$. The element $\Theta^{\text{pre}} \in \mathrm{Sym}^{k+1} T_p^* N \otimes C_p^* M$ can thus be lifted to $\Delta N \in \mathrm{Sym}^{k+1} T_p^* N \otimes (T_p M / T_p N)$, although not uniquely, and all lifts satisfy the decisive property $\Delta N \lrcorner \gamma_p = \langle\, \Theta^{\text{pre}}, \gamma_p \,\rangle$. Together with the modification formula (65) this property implies that the modification $\mathrm{jet}_p^{k+1} N - \Delta N \in \mathrm{Gr}_n^{k+1} M$ is a jet semisolution of order $k+1$ lying over $\mathrm{jet}_p^k N$ due to:

$$[\![\, \mathrm{jet}_p^{k+1} N - \Delta N \,]\!](\,\mathrm{jet}_p^k \gamma\,) = \mathrm{jet}_p^k(\iota_N^*\gamma) - B\,(\,\Delta N \lrcorner \gamma_p\,) = 0$$

Last but not least the exactness of the sequence (66) tells us that the difference of two lifts of Θ^{pre} corresponds exactly to an element of $\mathrm{Sym}^{k+1} T_p^* N \otimes H_p^\perp M$. □

Lemma 5.9 (Obstructions against Formal Integrability). *Consider a jet semisolution* $\mathrm{jet}_p^{k+1} N \in \overline{\mathrm{Eq}}_p^{k+1} M$ *of order* $k+1$ *to an exterior differential system on a manifold* M. *There exists a jet solution* $\mathrm{jet}_p^{k+1} \tilde{N} \in \mathrm{Eq}_p^{k+1} M$ *of order* $k+1$ *lifting the jet solution* $\mathrm{jet}_p^k N = \mathrm{jet}_p^k \tilde{N}$, *if and only if* $k+1$ *recursively defined obstructions*

$$B_{r+1}\Big[\,\Theta^{\text{pre}}(\,\mathrm{jet}_p^{k+1} N\,)\,\Big] \in \mathrm{Sym}^{k-r} T_p^* N \otimes H^{r,2}(\,\mathscr{A}_p\,)$$

vanish for all $r = 0, \ldots, k$. *In the latter case the set of all possible jet solutions* $\mathrm{jet}_p^{k+1} N$ *over the jet solution* $\mathrm{jet}_p^k N$ *form an affine space modelled on the vector space* $\mathscr{R}_p^{k+1}$.

In a very precise sense the statement of this lemma reflects the standard spectral sequence of Lemma 3.7, the operators B_{r+1} say are exactly the higher order coboundary operators of this spectral sequence. In particular the obstructions are defined strictly recursively in the sense that $B_{r+1}\,[\,\Theta^{\mathrm{pre}}(\mathrm{jet}_p^{k+1} N)\,]$ is only defined, if the preceding obstructions $B_1\,[\,\Theta^{\mathrm{pre}}(\,\mathrm{jet}_p^{k+1} N\,)\,], \ldots, B_r\,[\,\Theta^{\mathrm{pre}}(\,\mathrm{jet}^{k+1} N\,)\,] = 0$ all vanish. For this reason in particular it is rather difficult to calculate these integrability obstructions explicitly.

Proof. In order to begin we choose a representative submanifold $N \subset M$ for the given jet semisolution $\mathrm{jet}_p^{k+1} N \in \overline{\mathrm{Eq}}_p^{k+1} M$ of order $k \geq 0$ and consider the associated the $\mathbb{R}$-linear map $\Theta : \Gamma(\,CM\,) \longrightarrow \mathrm{Jet}_p^k\,\Lambda^2 T^*N,\ \gamma \longmapsto \mathrm{jet}_p^k(\,\iota_N^* d\gamma\,)$, whose triviality characterizes $\mathrm{jet}_p^{k+1} N$ as a jet solution of order $k+1$. We recall that $\mathrm{jet}_p^{k+1} N$ projects to a jet solution $\mathrm{jet}_p^k N$

$$\mathrm{pr}\,[\,\mathrm{jet}_p^k(\,\iota_N^* d\gamma\,)\,] \;=\; \mathrm{jet}_p^{k-1}(\,d\,\iota_N^*\gamma\,) \;=\; d^{\,\mathrm{formal}}(\,\mathrm{jet}_p^k(\,\iota_N^* d\gamma\,)\,) \;=\; 0$$

so that the image of Θ lies in the symbol subspace $\mathrm{Sym}^k T_p^* N \otimes \Lambda^2 T_p^* N \subset \mathrm{Jet}_p^k \Lambda^2 T^* N$ defined in (63). In consequence the $\mathbb{R}$-linear map Θ is actually $C^\infty(M)$-linear with

$$\begin{aligned}\Theta(\,f\,\gamma\,) &= \mathrm{jet}_p^k(\,\iota_N^*(\,df \wedge \gamma \;+\; f\,d\gamma\,)\,)\\ &= \mathrm{jet}_p^k(\,\iota_N^* df\,) \wedge \mathrm{jet}_p^k(\,\iota_N^*\gamma\,) \;+\; \mathrm{jet}_p^k(\,\iota_N^* f\,) \wedge \mathrm{jet}_p^k(\,\iota_N^* d\gamma\,)\\ &= f(\,p\,)\,\mathrm{jet}_p^k(\,\iota_N^* d\gamma\,)\end{aligned}$$

for all $f \in C^\infty(M)$, because $\mathrm{jet}_p^k(\,\iota_N^*\gamma\,) = 0$ as well as $\mathrm{jet}_p^{k-1}(\,\iota_N^* d\gamma\,) = 0$. We may thus think of Θ as a linear map $C_p M \longrightarrow \mathrm{Sym}^k T_p^* N \otimes \Lambda^2 T_p^* N$ with the additional property that

$$C_p M \;\overset{\Theta}{\longrightarrow}\; \mathrm{Sym}^k T_p^* N \otimes \Lambda^2 T_p^* N \;\overset{B}{\longrightarrow}\; \mathrm{Sym}^{k-1} T_p^* N \otimes \Lambda^3 T_p^* N$$

vanishes as a linear map, after all B agrees with $d^{\,\mathrm{formal}}$ on the symbol subspace (63) and so:

$$B[\,\Theta(\,\gamma_p\,)\,] \;=\; d^{\,\mathrm{formal}}[\,\mathrm{jet}_p^k(\,\iota_N^* d\gamma\,)\,] \;=\; \mathrm{jet}_p^{k-1}(\,\iota_N^*\,d^2\gamma\,) \;=\; 0$$

Up to this point we have followed the proof of Lemma 5.8 closely with only minute changes in the argument, but now we have to deviate from the path laid out above.

Although we may still choose a preimage $\Theta^{\mathrm{pre}} \in \mathrm{Sym}^{k+1}T_p^*N \otimes T_p^*N \otimes C_p^*M$ of Θ with the property

$$B(\langle\ \Theta^{\mathrm{pre}},\ \gamma_p\,\rangle) \;=\; \Theta(\,\gamma_p\,) \;=\; \mathrm{jet}_p^k(\,\iota_N^* d\gamma\,)$$

this preimage is no longer unique, because $B\ :\ \mathrm{Sym}^{k+1}T_p^*N \otimes T_p^*N \longrightarrow \mathrm{Sym}^k T_p^*N \otimes \Lambda^2 T_p^*N$ is no longer injective, to wit its kernel equals the image of $\mathrm{Sym}^{k+2}T_p^*N$ under B.

Keeping an eye on this non-uniqueness problem of the chosen preimage Θ^{pre} we observe that the restriction $\mathrm{res}_{T_pN}\ :\ H_pM \longrightarrow T_p^*N$ is surjective due to $T_pN \cap H_p^{\perp}M = \{\,0\,\}$ with kernel equal to C_pM by $T_pN \subset C_p^{\perp}M$, in other words it induces a canonical isomorphism

$$H_pM/C_pM \;\overset{\mathrm{res}_{T_pN}}{\longrightarrow}\; T_p^*N \tag{67}$$

equivalently $T_pN \subset C_p^{\perp}M$ is a complete set of representatives for the quotient $C_p^{\perp}M/H_p^{\perp}M$. This canonical isomorphism by restriction allows us to interpret the tableau $\mathscr{A}_p^1$ as a subspace of $T_p^*N \otimes C_p^*M \cong (H_pM/C_pM) \otimes C_p^*M$, in turn we will consider the class

$$[\,\Theta^{\mathrm{pre}}\,] \;\in\; \mathrm{Sym}^{k+1}T_p^*N \otimes \Big[\ T_p^*N \otimes C_p^*M/_{\mathscr{A}_p^1}\ \Big] \;=\; \mathrm{Sym}^{k+1}T_p^*N \otimes H^{0,1}(\,\mathscr{A}_p\,)$$

represented by Θ^{pre}. The vector space on the right is one the trihomogeneous subspaces of the E^1-term of the standard spectral sequence for the reduced symbol comodule $\mathscr{A}_p$

$$\mathrm{Sym}^{\bullet}T_p^*N \otimes H^{\blacklozenge,\circ}(\,\mathscr{A}_p\,) \;\Longrightarrow\; \delta_{\circ=0=\blacklozenge}\,\mathscr{A}_p^{\bullet}$$

constructed in Lemma 3.7. More precisely the total degree $k+2$ part of the E^1-term reads

$$\begin{array}{ccccc}
\mathrm{Sym}^{k+2}T_p^*N \otimes C_p^*M & \xrightarrow{B_1} & \mathrm{Sym}^{k+1}T_p^*N \otimes \Big[T_p^*N \otimes C_p^*M/_{\mathscr{A}_p^1}\Big] & \xrightarrow{B_1} & \mathrm{Sym}^{k}\,T_p^*N \otimes H^{0,2}(\mathscr{A}_p) \\
0 & & 0 & \searrow & \mathrm{Sym}^{k-1}T_p^*N \otimes H^{1,2}(\mathscr{A}_p) \\
\vdots & & \vdots & & \vdots \\
0 & & 0 & \searrow & \mathrm{Sym}^{1}\,T_p^*N \otimes H^{k-1,2}(\mathscr{A}_p) \\
0 & & 0 & \searrow & \mathrm{Sym}^{0}\,T_p^*N \otimes H^{k,2}(\mathscr{A}_p)
\end{array}$$

in form degree 0, 1 and 2, where B_1 is the coboundary operator for the E^1-term and the higher order coboundary operators $B_2, \ldots, B_{k+1}$ relevant for our argument have been indicated, although they are defined only on the kernel of all preceding coboundary operators. In consequence the standard spectral sequence results in $k+1$ recursively defined obstructions

$$B_{r+1}[\,\Theta^{\rm pre}\,] \;\in\; \mathrm{Sym}^{k-r}T_p^*N \otimes H^{r,2}(\,\mathscr{A}_p\,)$$

which are independent of the preimage $\Theta^{\rm pre}$ chosen for Θ, because the resulting ambiguity of the class $[\,\Theta^{\rm pre}\,]$ lies in the image of the left coboundary operator B_1.

Because the standard spectral sequence converges to $\{\,0\,\}$ in all positive form degrees, the class $[\,\Theta^{\rm pre}\,]$ lies in the image of the left B_1 coboundary operator, if and only if all the recursively defined obstructions $B_{r+1}[\,\Theta^{\rm pre}\,] = 0$ vanish for all $r = 0,\ldots,k$. Under this assumption we can modify our chosen preimage to a possibly different preimage of Θ with:

$$\Theta^{\rm pre} \;\in\; \mathrm{Sym}^{k+1}T_p^*N \otimes \mathscr{A}_p^1 \;\subset\; \mathrm{Sym}^{k+1}T_p^*N \otimes T_p^*N \otimes C_p^*M \tag{68}$$

Recall now that the set of jet semisolutions of order $k+1$ lying over jet_p^kN is an affine space modelled on $\mathrm{Sym}^{k+1}T_p^*N \otimes H_p^\perp M$ according to Lemma 5.8, where $H_p^\perp M$ serves as a set of representatives for the subspace $C_p^\perp M/T_pN \subset T_pM/T_pN$.

In case that we can chose a preimage $\Theta^{\rm pre}$ of Θ of the special form (68), equivalently in case that all recursively defined obstructions vanish, we can lift such a $\Theta^{\rm pre}$ to a preimage $\Delta N \in \mathrm{Sym}^{k+1}T_p^*N \otimes H_p^\perp M$ under the surjective symbol map $\mathrm{id}\otimes\Sigma_p$. Since the symbol map Σ_p is based on the idea of inserting vertical vectors $V \in H_p^\perp M$ into the exterior derivatives

$$\mathrm{res}_{T_pN}\Big(\,V \lrcorner\,(d\gamma)_p\,\Big) \;=\; -\mathrm{res}_{T_pN}\Big(\,\Sigma_p(\,\gamma_p,\,\cdot\,,\,V\,)\Big) \;=\; -\Sigma_p(\,\gamma_p,\,\cdot\,,\,V\,)$$

of contact forms, every ΔN chosen in this way satisfies the decisive equation:

$$(\mathrm{id}\otimes\mathrm{res}_{T_pN})(\,\Delta N \lrcorner\,(d\gamma)_p\,) \;=\; -\Sigma_p(\,\gamma_p,\,\cdot\,,\,\Delta N\,) \;=\; -\langle\,\Theta^{\rm pre},\;\gamma_p\,\rangle$$

Note that the restriction res_{T_pN} implements the canonical isomorphism $H_pM/C_pM \xrightarrow{\cong} T_p^*N$ only and can be dropped from notation. For all contact forms $\gamma \in \Gamma(\,CM\,)$ we thus find:

$$\begin{aligned} &[\![\,\mathrm{jet}_p^{k+1}N + \Delta N\,]\!](\,\mathrm{jet}_p^k(\,d\gamma\,)\,) \\ &= \mathrm{jet}_p^k(\,\iota_N^*\,d\gamma\,) \;+\; B\Big(\,(\mathrm{id}\otimes\mathrm{res}_{T_pN})(\,\Delta N \lrcorner\,(d\gamma)_p\,)\Big) \\ &= \mathrm{jet}_p^k(\,\iota_N^*\,d\gamma\,) \;-\; B\,\langle\,\Theta^{\rm pre},\;\gamma_p\,\rangle \;=\; 0 \end{aligned}$$

In consequence the modification $\mathrm{jet}_p^kN + \Delta N \in \mathrm{Eq}_p^{k+1}M$ is a jet solution of order $k+1$ lying over jet_p^kN. Being the kernel of $(B\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Sigma_p)$ in $\mathrm{Sym}^{k+1}T_p^*N \otimes$

$H_p^{\perp}M$ the homogeneous subspace $\mathscr{R}_p^{k+1}$ of the symbol comodule $\mathscr{R}_p$ parametrizes the possible choices for the difference element $\Delta N \in \mathrm{Sym}^{k+1}T_p^*N \otimes H_p^{\perp}M$. □

Unluckily Lemma 5.9 does not exclude the possibility of an infinite number of obstructions against formal integrability occurring at arbitrarily high orders of differentiation. Only the last of the recursively defined obstructions $B_{k+1}[\,\Theta^{\mathrm{pre}}(\,\mathrm{jet}_p^{k+1}N\,)\,]$ however appears to convey genuine information, the preceding obstructions are simply partial derivatives of the obstructions at lower order of differentiation. This intuitive idea gives rise to the conjecture:

Conjecture 5.10 (Vanishing Criterion for Obstructions). Consider an exterior differential system $CM \subseteq HM \subseteq T^*M$ on a manifold M. If the sets Eq^kM and $\mathrm{Eq}^{k-1}M$ of jet solutions of order k and $k-1$ form a smooth subtower

$$\begin{array}{ccccc} \mathrm{Eq}^kM & \xrightarrow{\ \mathrm{pr}\ } & \mathrm{Eq}^{k-1}M & \xrightarrow{\ \pi\ } & M \\ {\scriptstyle\subset}\downarrow & & {\scriptstyle\subset}\downarrow & & {\scriptstyle=}\downarrow \\ \mathrm{Gr}^k_nM & \xrightarrow{\ \mathrm{pr}\ } & \mathrm{Gr}^{k-1}_nM & \xrightarrow{\ \pi\ } & M \end{array}$$

of the tower (16) of Graßmannians in a neighborhood of a point $p \in M$, then all the recursively defined obstructions on $\overline{\mathrm{Eq}}_p^{k+1}M$ vanish except possibly the last, the higher torsion:

$$\mathrm{tor}^k : \quad \overline{\mathrm{Eq}}_p^{k+1}M \longrightarrow H^{k,2}(\,\mathscr{A}_p\,)$$

For the time being the author has been unable to prove this conjecture, nevertheless he is quite convinced of its validity. The point is that the conjecture is definitely true in an essentially dual formulation of the formal theory of partial differential equations, however this proof appears to require the use of so-called semiholonomic jets and is thus not easily translated into the language of exterior differential systems. Assuming the validity of this conjecture and climbing up the tower (60) one step at a time using Lemmas 5.8 and 5.9 alternatingly the reader will find no difficulties to prove the following version of the Theorem of Cartan–Kähler inductively starting with the fact that $\mathrm{Eq}^1M \longrightarrow M$ is a smooth subbundle of the tower of generalized Graßmannians in the case of vanishing torsion:

Theorem 5.11 (Formal Version of Cartan–Kähler). *Consider an exterior differential system $CM \subseteq HM \subseteq T^*M$ on a manifold M. In a given point $p \in M$ we choose $d_{\max}$ so that the Spencer cohomology of the reduced symbol comodule vanishes $H^{d,2}(\mathscr{A}_p) = \{0\}$ in form degree $\circ = 2$ for all $d > d_{\max}$. If the torsion maps*

$$\mathrm{tor}^k : \quad \overline{\mathrm{Eq}}_p^{k+1}M \longrightarrow H^{k,2}(\,\mathscr{A}_p\,)$$

all vanish for $k = 0, \ldots, d_{\max}$, *then* $\mathrm{Eq}_p^{k+1} M$ *is an affine fiber bundle over* $\mathrm{Eq}_p^k M$ *with fiber modelled on* $\mathscr{R}_p^{k+1}$ *for all* $k \geq 0$. *In particular there exist as many formal submanifold solutions in* $\mathrm{Gr}_n^\infty M$ *as predicted by the dimensions of the homogeneous subspaces of* $\mathscr{R}_p$.

In its original formulation the theorem of Cartan–Kähler treats involutive reduced symbol comodules only, for which we may choose $d_{\max} = 0$ according to Lemma 4.8, the theorem of Cauchy–Kovalevskaya for underdetermined partial can be seen as the case, where $d_{\max} = -1$ is already sufficient. In general a formal solution to a given exterior differential system need not correspond to a real submanifold solution, under the additional assumption that M is an analytical manifold and both CM and HM are analytical subbundles of the cotangent bundle T^*M however, every formal solution defines an actual submanifold solution within its radius of convergence. Under this analyticity assumption the theorem of Cartan–Kähler extends to the statement that there exist as many formal solutions with *positive* radius of convergence as predicted by the dimensions of the homogeneous subspaces of $\mathscr{R}_p$.

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