Chapter 5
Determinism, Chaos and Reductionism

Everything that is necessary is also easy. You just have to accept it.

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5.1 General Remarks on Determinism

The term *determinism* has often been used in fields other than physics, such as psychology and sociology, causing some bewilderment. For instance, Popper (1992) feared a strictly deterministic world as a nightmare, because it would have meant that our universe is like a big robot, in which we merely play the role of small cogwheels or, at best, of sub-automata.[1] To avoid such misunderstandings, and because, at times, determinism has been improperly associated with reductionism,[2] we are going to briefly review the correct notion of determinism, which is used in physics. For a brilliant and exhaustive discussion on this subject, we refer the reader to Kojeve (1990).

One can readily acknowledge that a completely indeterminate world, whose phenomena obey no rules, would present totally uncorrelated facts and sequences of events, and we would have no chance of ever understanding it. For this reason, words such as *disorder* and *chaos* sound rather pejorative or disruptive to the ideal natural order usually associated with the idea of the "cosmos" since the beginning of Greek philosophy (Thuillier 1991). The historical development of science could be seen as the struggle against disorder, in an attempt to find regularity in phenomena that appear to be irregularly changing. This struggle reached its apogee with the mecha-

[1] This parallels with the totalitarian views is expressed in a paradigmatic way by two classical books of the period (Bauman 2000) by Orwell (1949) and Huxley (1932).

[2] For instance, Popper (1992) argued that a determinist must be a reductionist, although a reductionist is not necessarily a determinist; while others identify reductionism with determinism.

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nistic determinism shaped in the Enlightenment era, by the belief that the world is a sort of cosmic clock and thus completely predictable.

Our opinion is that the most interesting question is not the determinism of the laws of physics— something that, as we shall see, seems destined to remain unanswered, even neglecting the problems posed by quantum mechanics —but rather the discovery of apparently simple deterministic systems that behave in very irregular fashions. Furthermore, the theory of deterministic chaos has shown that complex behaviours are not the exclusive prerogative of systems of many interacting components, but may also be found in the dynamics of simple systems with few degrees of freedom. Because of this, scientists eventually abandoned the idea that the observed complexity of the world necessarily arises from the cooperation of many elementary building blocks.

But how is the theory of deterministic chaos relevant, in a book on the reduction of theories and the role of singular limits? First of all, recall the important message implied by Lorenz's work on simplified deterministic models of the atmosphere's dynamics (Lorenz 1963): the postulated "elementary building block" is often not elementary at all, and the effort to understand a given system by analyzing the equations governing its parts or constituents may fail. Then, observe that deterministic and stochastic descriptions are deeply different ontologically, but strong similarities can nevertheless be found between certain behaviours of deterministic chaotic systems and processes governed by stochastic laws. Such similarities are relevant on the practical level, e.g. when modeling complex systems. On the other hand, chaos presents both ontic and epistemic questions[3] which may generate confusion about the real conceptual relevance of chaos. We shall see that chaos allows us to unambiguosly introduce probabilistic concepts in a deterministic world. Such a possibility is not merely the consequence of our limited knoweledge of the state of the system of interest. Indeed, in order to account for this limited knowledge, one usually relies on a coarse-grained description, which requires a probabilistic approach, and finds that many important features of the dynamics do not depend on the scale ε of the graining, if it is fine enough. At the same time, many results for the $\varepsilon \to 0$ limit do not apply to the cases with $\varepsilon = 0$. Therefore, the probabilistic description of chaotic systems reveals one more instance of singular limits.

In the following, we would like to clarify some aspects of deterministic chaos which, in our opinion, are often misunderstood, leading to scientifically, as well as philosophically, questionable and confused claims. We begin by considering the relationship between determinism and predictability. Then, we shall consider the role of chaos in the statistical description of complex phenomena, for which statistical mechanics providing an important setting.

[3] We shall see how determinism refers to ontic descriptions, while predictability (and in some sense chaos) has an epistemic nature.

5.1.1 Determinism and Predictability

Among the aspects of mechanicism that have continued to influence scientific thought up to the present day, the impact of Laplace's statement reported in Chap. 3 is remarkable. This statement is a milestone of scientific thought, whose legacy has survived despite the advent of the quantum description of physical phenomena. Unfortunately, it has also often been misunderstood, in its technical and conceptual content cf. (Gleick 2008).

We believe that a fair interpretation of Laplace's "mathematical intelligence" was likely due to his desire to stress the importance of prediction in science, as it appears from a famous anecdote, probably apocryphal but frequently cited. We report here the version given by Cohen and Stewart (1994). After seeing Laplace's masterpiece, *Méchanique Céleste*, Napoleon addressed Laplace saying:

> *[t]hey tell me you have written this large book on the system of the universe, and have never even mentioned its Creator*. Laplace answered: *Sire, I have no need of this assumption*. To that Napoleon replied: *Ah! That is a beautiful assumption, it explains many things*, and Laplace: *This hypothesis, Sire, explains everything, but does not permit the prediction of anything. As a scholar, I must provide you with works permitting predictions.*

The Laplace's ideas (sometimes distorted) in the nineteenth century originated a widely accepted view of the science based on three elements (Kojeve 1990):

(a) *Determinism*: the metaphysical assumption of a deterministic causal structure of nature.[4] In mathematical terms, Laplace assumed that every phenomenon is described by a vector $\mathbf{X}$ (the system state) that evolves according to a deterministic law: i.e. if the state $\mathbf{X}(0)$ at the initial time, $t = 0$, is knonwn, then the state at every later instant $\mathbf{X}(t)$, $t > 0$, is uniquely determined. In modern terms, determinism is assured, in a world governed by Newtonian mechanics, by Cauchy's theorem on the existence and uniqueness of the solution of ordinary differential equations.[5]

(b) *Exact predictability*: the practical possibility of making predictions through mathematical laws. This is a delicate point, because it requires an explicitly computable rule for the evolution of $\mathbf{X}(t)$, once the initial state $\mathbf{X}(0)$ is known with arbitrary accuracy.

(c) *Mechanistic reductionism*: the possibility of explaining (at least in principle) any phenomenon from the motion of its elementary constituents, thought to interact through suitable forces.

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[4] The idea of causality explicitly enters Laplace's as well as our reasoning. Indeed, the strict notion of "causality" leads to considerable difficulties from epistemological and ontological points of view. This does not concern us. In its evolution, classical mechanics has developed a principle of legal determinism, in which the notions of cause and effect are not explicitly invoked. The idea was anticipated by Kant, who stated that the *geschiet* (the effect) presupposes an antecedent (*worauf*) from which it follows according to a rule. The adjective "causal" is still used in the same sense.

[5] We stress the importance of identifying the state vector $\mathbf{X}$ which fully describes the phenomenon under investigation. For instance, in classical mechanics, $\mathbf{X}$ is given by the positions and velocities of particles. This is a fundamental step which took a long time to be understood. For instance, in Aristotelian physics only the positions were considered.

Together, items (a), (b) and (c) summarise Laplace's view, which can be called mechanistic determinism. The followers of mechanistic determinism are reductionists and expect a scientific theory to describe reality in mathematical terms. Given the equations ruling the temporal evolution of a system, and given its initial conditions, the knowledge of the system at any future time can then be obtained. It is important to stress that Newtonian mechanics, which was founded on such premises, was not restricted to a small class of phenomena; it was believed capable, *in principle*, of yielding predictions in all conceivable phenomena: from the orbit of the moon to the motion of falling apples. This fundamental idea is the very essence of mechanistic (or causal) determinism.

Unfortunately, except for extremely simple cases such as the motion of two gravitationally interacting bodies, or the periodic behaviour of pendulums, the time evolution of a system is typically hard to determine explicitly. However, in principle, the equations of motion can be solved, with more or less complicated mathematical calculations. Indeed, generations of astronomers have computed with incredible patience and perseverance the orbits of planets and asteroids, from the equations of classical mechanics. Their successes were numerous, beginning with the derivation of Kepler's laws from the principles of mechanics and the universal law of gravitation, which was given by Newton himself. After obtaining strong agreement between the theoretical calculations and the observations, this approach was systematically confirmed in astronomy. One of its sensational successes was the discovery of the planet Neptune in the nineteenth century. A series of observations indicated a significant deviation of the motion of Uranus from the positions predicted by Newtonian mechanics. Assuming that this discrepancy was not due to a shortcoming of the Newtonian theory, but the presence of an unknown planet, the laws of dynamics and of gravitation led astronomers to calculate the position of this hypothetical planet which, sure enough, was observed by telescope a short while later.

In less rhetorical terms, the essence of Laplace's famous statement is that the laws of classical physics are perfectly deterministic: if the state of a system at a given time is known exactly, its subsequent evolution is uniquely determined. The calculations that took years in the past, when astronomers could only rely on pen and paper, are today performed very rapidly on computers, which determine with great precision the motion of celestial bodies and artificial satellites. The successful use of modern computers in the exploration of space can be seen as another confirmation of Laplace's idea. At least, indeed, popular writers have taken it to be this way.

In 1867, after 20 years of pen-and-paper work, the French astronomer Delaunay completed the calculation of the position of the moon as a function of time, with an accuracy never reached before. In 1970, Deprit, Henrard and Rom checked that calculation with one of the earliest computer algebra systems. The verification took twenty hours and found only three minor mistakes. It is interesting that computers first allowed people to find Delaunay's minor errors, while today the roles are reversed: the analytical calculation of the great astronomer is used to check the accuracy of the new computer algebra systems[1] (Pavelle et al. 1981).

Despite its undisputed success, however, the mechanistic deterministic approach appears to contradict everyday evidence, where there is no way of predicting many events. Consider the evolution of the weather, falling leaves, or a stone rolling down a slope. How can we reconcile the fundamental assumptions made by Laplace with the apparent irregularity of most phenomena? The simplest way is to think that irregular phenomena appear so only because they require the solution of a very large number of equations, which may also be very complicated. In such cases, which are actually very frequent, it is not possible to solve the problem by pen and paper, and one may think that a sufficiently powerful computer could provide the answer with the desired accuracy.

It is therefore necessary to distinguish the questions concerning the deterministic nature of the laws of physics from those posed by the possibility of making predictions. This is essential to avoid confusion. For example, unlike the majority of physicists and mathematicians, by *deterministic system* Popper (1992) means a system governed by a deterministic evolution law, whose evolution can be in principle predicted with arbitrary accuracy.

Determinism amounts to the metaphysical doctrine that same events always follow from same antecedents. But, as Maxwell had already pointed out in 1873, it is impossible to confirm this fact, because nobody has ever experienced the same situation twice:

> It is a metaphysical doctrine that from the same antecedents follow the same consequences. No one can gainsay this. But it is not of much use in a world like this, in which the same antecedents never again concur, and nothing ever happens twice... The physical axiom which has a somewhat similar aspect is "that from like antecedents follow like consequences". But here we have passed .... from absolute accuracy to a more or less rough approximation.[6]

In these few lines, Maxwell touches on issues which will be later investigated, and anticipates their solution. The issues are:

1. the impossibility of proving (or refuting) the deterministic character of the laws of nature;
2. the practical impossibility of making long-term predictions for a class of phenomena, referred to here as chaotic, despite their deterministic nature.

About 30 years after Maxwell, Duhem (1991), making a remark on a result obtained by Hadamard, concerning a case of what is currently called deterministic chaos, reached the same conclusion. Very similarly to Maxwell, he noted that mathematical deductions are not useful to physicists if they merely state that a proposition, rigorously true, implies the exact truth of another. To be useful to physicists, the mathematical argument must also prove that the second proposition approximately holds if the first is only approximately verified. More formally, Duhem stressed the importance of the fact that solutions of differential equations enjoy a continuous dependence on initial and boundary data, if they have to be of practical interest e.g. in physics.

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[6] From the conference *Does the progress of Physical Science tend to give advantage to opinion of Necessity* (or Determinism) over that of the Contigency of Events and the Freedom of the Will?, cf. Campbell and Garnett (1882) Chap. XIV.

After the development of quantum mechanics, many think that discussing the deterministic nature of the laws of physics is too academic an exercise to deserve serious consideration. For instance, in a speech motivated by the heated controversy on chaos and determinism between philosophers and scientists, Kampen (1991) bluntly said that the problem does not exist, as it is possible to show that:

> the ontological determinism à la Laplace can neither be proved nor disproved on the basis of observations.[7]

While we fully agree with this statement, we think that the dichotomy concerning determinism and chaos deserves attention well beyond mere scholarly discussions. For instance, there are uncountably many situations lacking a solid mathematical model, such as those common in biology, in which our question has practical implications.

Popper (1992) was an avowed non-determinist, in the sense that he did not accept what he called "scientific determinism": the doctrine according to which the world can be rationally predicted, to any desired degree of accuracy, if a sufficiently precise description of past events, along with all the laws of nature is available. But it is worth remarking that the Popperian definition of "determinism" is different from the one commonly used in physics, because it includes an arbitrarily precise predictability, not required in physics. Apart from questions of terminology, which can be clarified, Popper has made a very important contribution to the issues related to determinism and predictability, since he has convincingly shown that a possible determinism of the laws of nature would not suffice to produce a forecast from "inside". In other words, assuming that Laplace's infinitely capable Intelligence is part of our world, it should predict itself: but no Intelligence can predict all the results of its own forecasts. Nevertheless, a prediction by an external agent remains possible, requiring that the Laplacian Intelligence be placed outside the world, hence requiring that it does not affect the evolution of the world. The discovery of deterministic chaos gave new impulse to these questions.

We have thus argued that determinism and predictability constitute two quite distinct issues, and the former does not imply the latter.

Roughly speaking, determinism can be traced back to a vision of the nature of causality and can be cast in mathematical terms, by saying that the laws of nature are expressed by ordinary differential equations. It is fair to say that most macroscopic phenomena can be described in this way, as confirmed, for instance, by the impressive successes of astronomy in the past and by technological realisations today. However, as noted by Maxwell, the objectively ontological determinism of the the laws of nature cannot be proven; but one might find it convenient to use deterministic descriptions. Moreover, even at a macroscopic level, many phenomena are chaotic

[7] In brief, van Kampen's argument is the following. Suppose the existence of a world A which is not deterministic and consider a second world B obtained from the first using the following deterministic rule: every event in B is the copy of an event occurred one million years earlier in A. Therefore, all the observers in B and their prototypes live the same experiences despite the different natures of the two worlds.

and, in some sense, appear to be “random”. The meaning of these terms will be clarified shortly. On the other hand, the microscopic phenomena described by quantum mechanics, fall directly within a probabilistic framework. They appear ontologically and epistemologically non-deterministic.

Concerning predictability, the presence of “chaos” in phenomena governed by deterministic laws and the logical aporia proposed by Popper shows that predictability is far from trivial. Two main issues arise: are deterministic phenomena always predictable? And what does prediction mean?

5.2 An Excursus on Chaos

Ironically, in spite of the success of Newtonian mechanics in the discovery of Neptune, the first clear example of what today we call chaos was found in celestial mechanics, the science of regular and predictable phenomena *par excellence*. This is the case of the long standing three-body problem: the motion of three gravitationally interacting bodies, such as the moon, Earth and sun, which was a nightmare for many great early mathematicians as Newton, Euler and Lagrange. In spite of its deterministic nature, Poincaré (1982) found that the evolution of the three-body system can be chaotic, meaning that small perturbations in the initial state, such as a slight change in the initial position of one of the three objects, may lead to dramatic differences in the later states of the system. As a vivid example of sensitivity to initial conditions, we mention the effect of a very distant single electron on massive bodies (Berry 1978). An electron at the limit of the observable universe (a distance of $O(10^{10})$ light years) will lead in just a few collisions to a complete breakdown of the predictability of systems of billiard balls.

There is a widespread vulgate, see e.g. Gleick (2008), which claims that the line of scientific research opened by Poincaré remained basically neglected until 1963, when meteorologist Lorenz rediscovered deterministic chaos while studying the evolution of a simple model of the atmosphere. Therefore, it is often claimed that the new paradigm of deterministic chaos originated in the sixties. This is not true; mathematicians never forgot Poincaré’s legacy, although it was not so well known to physicists, (Aubin and Dalmedico 2002).

Here, we briefly recall the essential characteristics of a deterministic chaotic system[8]:

(i) the evolution is given by a deterministic rule, for example, by a set of differential equations;
(ii) solutions sensitively depend on the initial conditions: i.e. two initially almost identical states $\mathbf{X}(0)$ and $\mathbf{X}'(0)$, characterised by a very small initial displacement $|\mathbf{X}(0) - \mathbf{X}'(0)| = \delta_0$, separate at an exponential rate:

$$|\mathbf{X}(t) - \mathbf{X}'(t)| \sim \delta_0 e^{\lambda t}, \tag{5.1}$$

[8] We consider systems whose phase space is bounded.

where λ is positive and is called the Lyapunov exponent[9];

(iii) the evolution of the state $\mathbf{X}(t)$ is not periodic and appears quite irregular, similar in many respects to that of random systems.

Let us start from point (iii) and its relevance to the issue of reductionism. In the deterministic mechanistic approach, the undeniable irregularity of many natural phenomena is thought to be only "apparent". For instance, it is seen as due to a very large number of causes, which are individually thought to be simple. An example of this interpretation of irregular phenomena, which we might call the philosophy of the "simple elementary brick", is afforded by Landau's theory of the onset of turbulence (Landau 1944). This theory states that the very complicated behaviour of a turbulent fluid arises from the superposition of many periodic oscillations, whose individual behaviour is simple by definition. This influential philosophical point of view was, however, refuted by the discovery made by Lorenz (1963), one of the pioneers of the modern theory of chaos. While investigating a minimal model for the dynamics of the atmosphere, he unequivocally realised that the erratic behaviour, typical of turbulent fluids, is not necessarily due to a large number of variables, since it can be found in quite simple and low dimensional dynamics, as a consequence of deterministic chaos. This led to the important conclusion that the elementary bricks are not always "simple". Within this new vision, Ruelle and Takens (1971) showed some years later that the onset of turbulence was not due to a superposition of simple oscillations.

The sensitive dependence on the initial conditions drastically limits the potential to make predictions: if the initial state is known with a certain uncertainty δ_0, the evolution of the system can be accurately predicted with precision Δ only up to a time that depends on the Lyapunov exponent. This quantity is inherent in the system and does not depend on our ability to determine the initial state; hence, recalling 5.1, the time within which the error on the prediction does not exceed the desired tolerance is given by:

$$T_p \sim \frac{1}{\lambda} \ln \frac{\Delta}{\delta_0}. \tag{5.2}$$

Deterministic systems, which are often fairly good models for macroscopic phenomena, can display a behaviour which is chaotic. Their sensitivity to initial conditions introduces an error in predictions which grows exponentially in time. As the exponent is an intrinsic characteristic of the system, predictions remain meaningful only within a time given by 5.2. It is well evident, therefore, that a deterministic nature does not imply the possibility of an arbitrarily accurate prediction.

Furthermore, this major result holds even for simple low-dimensional systems, which leads to a second major conclusion: the reductionistic idea that complex systems can be analysed as an agglomerate of simple elements is incorrect. In general, complex systems cannot be *reduced* to a sum of simple elementary constituents.

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[9] Equation (5.1) holds for infinitesimal distance. Because the phase space is bounded, the distance between the two trajectories cannot grow forever and reaches its maximum in a finite time.

5.3 Chaos and Complexity

We have seen that chaos has major consequences for predictability. However, noting that T_p, Eq. (5.2), could be made arbitrarily large by reducing δ_0, though at great costs, because of the slow divergence of the logarithm, it might seem that the problem is only of practical order and not intrinsic to chaotic evolutions. In other words, the limitations on predictability may appear simply epistemological and not ontological, which would imply that the transformation of a deterministic mechanistic problem into a probabilistic one has to be blamed only on our technical inability to sufficiently reduce the error on the initial conditions.[10] This a point of crucial importance.

We shall give some evidence of the impossibility of circumventing this problem, simply by asserting that a deterministic system is, in principle, predictable, on the grounds that the desired accuracy at any given (finite) time t merely requires sufficiently accurate knowledge of the initial conditions with the necessary (finite) precision. Let us consider a deceptively simple dynamical system:

$$x(t+1) = 2x(t) \ \ mod\, 1. \tag{5.3}$$

This system is chaotic and its Lyapunov exponent is $\lambda = \ln(2)$. This means that a small error in the initial conditions doubles at every step. Suppose that $x(0)$ is a known real number in the interval [0, 1], it can be expressed by an infinite sequence of 0 and 1, because it can be written as

$$x(0) = \frac{a_1}{2} + \frac{a_2}{4} + \cdots + \frac{a_n}{2^n} + \cdots$$

where every a_n takes either the value 0 or the value 1. It is also interesting to note that the above binary notation allows us to determine the time evolution by means of a very simple rule: at every step, move the "binary point" of the binary expansion of $x(0)$ by one position to the right and eliminate the integer part. For example, take

$$x(0) = 0.110010100101100100101001011 10\ldots \tag{5.4a}$$

Then

$$x(1) = 0.10010100101100100101001011 10\ldots \tag{5.4b}$$

$$x(2) = 0.0010100101100100101001011 10\ldots \tag{5.4c}$$

$$x(3) = 0.010100101100100101001011 10\ldots \tag{5.4d}$$

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[10] It is worth stressing how dramatically chaos affects our predictions. Because of the logarithm in 5.2, increasing the predictability time T_p by a factor 5 increases the required precision of the initial conditions by five orders of magnitude, e.g. from metre-order precision to micrometre-order precision. For all relevant phenomena this is and will forever remain impossible to be achieved. This is why our local weather forecast are restricted to 5–7 days predictions (roughly speaking the time given by the Lyapunov exponent) and one cannot hope to greatly improve on that by making more accurate measurements of the initial conditions.

and so on. In terms of the sequence $\{a_1, a_2, \ldots, a_n, \ldots\}$, it becomes quite clear how crucially the temporal evolution depends on the initial condition.

Let us now make a brief digression on the notion of "complexity" of a binary sequence. Generally speaking, different types of sequences are possible, for example consider the following ones:

$$11111111111111\ldots \tag{5.4}$$

$$10101010101010\ldots \tag{5.5}$$

$$00101000110100\ldots \tag{5.6}$$

One would presumably state that sequences (5.4) and (5.5) appear to be "ordered", whereas sequence (5.6) seems "complex". Why should one classify the sequences in this way?

In the case of (5.4) and (5.5) the knowledge of the first n values $a_1, a_2, a_3, \ldots, a_n$ appears to be sufficient to predict the following values $a_{n+1}, a_{n+2}, \ldots$. This is not true for sequence (5.6), which seems to be generated by a stochastic, rather than a deterministic, rule. In this case, one could think that the sequence of 0 and 1 is generated tossing a coin, and writing 1 for heads and 0 for tails. One way to formalise this intuitive concept of "complex" behaviour is to associate it with the lack of a constructive rule; then the cases of (5.4) and (5.5) are not complex because they can be generated by means of very simple rules. On a computer, for instance, (5.4) can be generated through a single statement:

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WRITE 1 N TIMES
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and similarly for (5.5):

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WRITE 10 N/2 TIMES.
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By contrast, (5.6) seems to require a program of the kind:

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WRITE 0  WRITE 0  WRITE 1  WRITE 0  WRITE 1  ...
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A precise mathematical formalisation of the complexity of a sequence has been proposed independently in 1965 by Kolmogorov, Chaitin and Solomonoff (Li and Vitanyi 1992). Given the sequence $a_1, a_2, a_3, \ldots, a_n$, among all possible programs which generate this sequence, one considers that with the smallest number of instructions. Denoting by $K^{(N)}$ the number of these instructions, the algorithmic complexity of the sequence is defined by

$$K = \lim_{N\to\infty} \frac{K^{(N)}}{N}. \tag{5.7}$$

Therefore, if there is a simple rule, which can be expressed by a few instructions, the complexity vanishes. If there is no explicit rule, which is not just the complete list of 0 and 1, the complexity is maximal, that is 1. Intermediate values of K between 0 and 1 correspond to situations with no obvious rules, but such that part of the information necessary to do a given step is contained in the previous steps.

To give an intuitive idea of the concept of complexity, let us consider a situation related to the transmission of messages (Chaitin 1990): A friend on Mars needs the tables of logarithms.[11] It is easy to send him the tables in binary language; this method is safe but would naturally be very expensive. It is cheaper to send the instructions necessary to implement the algorithm which computes logarithms.

However, if the friend is not interested in mathematics, but rather in football or the lottery, and wants to be informed of the results of football matches or the lottery draw, there is no way of compressing the information in terms of an algorithm whose repeated use produces the relevant information for the different events; the only option is the transmission of the entire information.

To sum up: the cost of the transmission of the information contained in the algorithm of logarithms is independent of the number of logarithms one wishes to compute. On the contrary, the cost of the transmission of football or lottery results increases linearly with the number of events. One might think that the difference is that there are precise mathematical rules for logarithms, but not for football matches and lottery drawings, which are then classified as random events.

Let us now analyse the problem of transmission, with accuracy Δ, of a sequence $x(t), 0 < t < T$, generated by the rule (5.3). At first glance, the problem seems similar to sending the tables of logarithms, and we could opt for transmitting $x(0)$ and the rule (5.3), which costs a number of bits independent of T. The friend on Mars would then be left with the task of generating the sequence $x(1), x(2), \ldots, x(T)$. However, we must also choose the number of bits to which $x(0)$ should be specified. From (5.2), the accuracy Δ at time T requires accuracy $\delta_0 \sim 2^{-T}\Delta$ for $x(0)$, hence that the number of bits specifying $x(0)$ grows with T. Again, we have to tackle the problem of the complexity of a sequence of symbols, $\{a_0, a_1, \ldots\}$. The fact is that there are "simple" initial conditions, of the type (5.4) or (5.5), which can be specified by a number of instructions independent of the length of the sequence, but there are complex sequences as well.

The determination of the algorithmic complexity of a sequence is impossible, as implied by Gödel's incompleteness theorem. Notwithstanding this impossibility, a result of Martin-Löf (1966) shows that "almost all" binary sequences, which express the real numbers in [0, 1], are complex. Therefore, the major conclusion is that the details of the time evolution are well hidden in the initial condition and that, in general, is complex.

The immediate and striking consequence of these facts is that determining with arbitrary precision of the initial conditions is hopeless. Hence, long-term predictions with the desired accuracy are impossible in principle, despite seeming quite

[11] In the pre-computer age, numerical computations relied on tabulated numbers for, e.g. logarithmic and trigonometric functions.

reasonable for such a simple time evolution. Insisting nonetheless on following such a path, one faces an infinite regression[12] and ineluctably runs into an impossibility which is not merely practical (Li and Vitanyi 1992).

So far, we have observed that deterministic systems, even with just a few degrees of freedom, may exhibit chaos (that is, a sensitive dependence on initial conditions). This fact strongly impacts on the possibility of making accurate predictions beyond a certain predictability time T_p. In turn, most initial conditions are complex, hence the predictability of chaotic systems is intrinsically limited in time. One may then state that predictable deterministic systems and chaotic unpredictable ones are related by a singular limit. Indeed, as mentioned earlier, the conclusions one may draw for vanishing and for arbitrarily small, but finite errors, are completely different. This singularity highlights the relevance of chaos to reductionism. It also shows that elementary constituents of a given object may indeed have very complex behaviour themselves. Moreover, this singularity clarifies how, in many situations, stochastic macroscopic properties emerge from chaos. We will return to this point at the end of the chapter.

5.4 Chaos and Probability

Because of their irregular behaviour, deterministic chaotic systems share many features with stochastic processes. In particular, the unpredictability of a chaotic system calls for statistical or probabilistic approaches, analogously to the case of stochastic processes. For instance, trying to predict the motion of a fluid particle in a turbulent flow is meaningless, while it is possible and appropriate to predict its statistical features such as its average velocity, kinetic energy, etc. This fact is very interesting from a practical point of view, but it is rather subtle and can lead to confusion. An important characterisation of the dynamics, on a coarse-grained scale, is given the Kolmogorov–Sinai (K–S) entropy, defined as follows.[13]

Let $\mathscr{A} = \{A_1, \ldots, A_N\}$ be a finite partition of the phase space, made up of the N disjoint sets A_i, and consider the sequence of points

$$\{\mathbf{x}(0),\ \mathbf{x}(1),\ \mathbf{x}(2),\ \ldots, \mathbf{x}(n), \ldots\} \tag{5.9}$$

which constitutes the trajectory with initial condition $\mathbf{x}(0)$. This trajectory can be associated with the symbol sequence

$$\{\sigma(\mathbf{x}(0)),\ \sigma(\mathbf{x}(1)),\ \sigma(\mathbf{x}(2)), \ldots, \sigma(\mathbf{x}(n)), \ldots\} = \{i_0, i_1, i_2, \ldots, i_n, \ldots\} \tag{5.10}$$

[12] In philosophical language the classical trilemma of Agrippa: if we are asked to prove how we know something, we can provide a proof or an argument. Nonetheless, a proof of the proof can be then asked and so on, leading to an infinite process which never ends.

[13] For the sake of simplicity, we restrict ourselves to the case of discrete-time dynamical systems, but continuous systems may be treated analogously.

where $i_n \in \{1, 2, \ldots, N\}$ and $\sigma(\mathbf{x}(k)) = i_k$ if $\mathbf{x}(k) \in A_{i_k}$. The coarse-grained properties of chaotic trajectories can be therefore studied through the discrete time sequence (5.10). Let $C_m = (i_1 i_2 \ldots i_m)$ be a "word" of length m and probability $P(C_m)$. The quantity

$$H_m(\mathscr{A}) = -\sum_{C_m} P(C_m) \ln P(C_m) \tag{5.11}$$

is called the block entropy of the m-sequences.[14] In the limit of infinitely long sequences, the asymptotic entropy increment

$$h_S(\mathscr{A}) = \lim_{m\to\infty} (H_{m+1} - H_m)$$

is called the Shannon entropy, and depends on the partition $\mathscr{A}$. Taking the largest value over all possible partitions we obtain the so-called Kolmogorov–Sinai entropy:

$$h_{KS} = \sup_{\mathscr{A}} h_S(\mathscr{A}).$$

A more tractable and intuitive definition of h_{KS} starts from the partition $\mathscr{A}_\varepsilon$ made of a grid of hypercubes of sides of length ε, and takes the following limit:

$$h_{KS} = \lim_{\varepsilon\to 0} h(\mathscr{A}_\varepsilon).$$

Although h_{KS} and K are conceptually very different characterisations of a symbol sequence,[15] their numerical values are simply related:

$$h_{KS} = \lim_{N\to\infty} \frac{\langle K^{(N)} \rangle}{N \ln 2}, \tag{5.12}$$

where $\langle . \rangle$ denotes an average over all sequences of length N. This leads to the following maxim:

Complex = Incompressible = Unpredictable.

[14] Shannon (1948) showed that, once the probabilities $P(C_m)$ are known, the entropy (5.11) measures, under natural conditions, the *surprise* or information carried by $\{C_m\}$.

[15] Consider the following two m-sequences, produced by tossing a fair coin:

$$01010101010\ldots010101$$
$$01001010110\ldots101001$$

One finds that the first sequence is compressible, while the second appears to be stochastic, in spite of the fact that both occur with probability 2^{-m}. This shows that algorithmic complexity, which characterises a single sequence, and information, which amounts to a probabilistic notion, are conceptually different.

which is valid for stochastic processes, e.g. Markov chains, as well. In the final section of this chapter, we will return to this similarity of chaotic deterministic systems and random sequences.

We conclude this section noting that initial conditions play a key role even in deterministic chaotic dynamics, just as they do in the problem of irreversibility. In Chap. 4, we saw that reversible large mechanical systems display an irreversible behaviour, for almost all nonequilibrium initial conditions. Analogously, in deterministic chaotic systems, unpredictable evolutions arise for almost all initial conditions, apparently in conflict with the deterministic nature of the dynamics. Hence, both chaotic systems and systems with many degree of freedoms enjoy a complex nature, requiring probabilistic approaches. Both are characterised by the transition to a complex probabilistic state through a singular limit, which is $\varepsilon \to 0$ for chaotic systems and $N \to \infty$ for systems with many degree of freedom.

5.5 Quarrels on Chaos and Determinism: Chaos and Probability Revisited

The discovery of chaos, in particular the impossibility of making long-term predictions for deterministic systems, has generated a debate about determinism, randomness and, more generally, complexity. The debate has often been heated (Amsterdamski et al. 1990). Here, it suffices to recall some of its most interesting aspects. In his long-lasting diatribe against Prigogine, the father of catastrophe theory, the mathematician Thom, argues in uncompromising terms that being attracted by the charm of randomness is the symptom *par excellence* of an anti-scientific attitude, since it largely proceeds from admiration to confusion. According to Thom, humanists could be forgiven for such an attitude, but not scientists, who should be accustomed to the rigour of scientific rationality. He insists with great determination that randomness is a negative concept, hollow, and devoid of any scientific interest, whereas determinism is an object of fascinating richness (Amsterdamski et al. 1990).

Although not in complete agreement with all positions taken by Thom, we share the concern that chaos could be used as some sort of anti-science passkey. Unlike what some may think, deterministic chaos, and its inability to make predictions, does not provide any proof of the weakness of "classical" or "standard" science, which would have been eventually forced to abandon determinism. Chaos provides no evidence of the inability of official science to deal with the complexity of the real world; nor does it highlight any difficulty that calls for an alternative science.

The impossibility for a deterministic science to make long-term, arbitrarily accurate predictions, is indeed a consequence of deterministic chaos, but by no means does it lead to the impossibility of any form of accurate prediction. In particular, according to Thom, any model of a real phenomenon must be deterministic, in order to teach us something.

Because of chaos, the role of probability in physics takes further meanings. In the nineteenth century, this role was acknowledged by Maxwell and Boltzmann in relation to thermodynamics. To understand the properties of a gas starting from the microscopic details of the dynamics of its molecules is not only hard but is also misleading; only a statistical description, which takes advantage of the huge number of particles involved, and describes the gas in terms of a few macroscopic variables such as pressure, temperature, etc. is appropriate. Indeed, in Chap. 3 we observed that a system containing a very large number of particles is described by particular emerging laws: the so-called statistical laws, which are due to the large number of particles constituting the body,[16] and which cannot in any way be derived from purely mechanical laws. Although the elementary constituents of a system with a large number of degrees of freedom obey the same laws of mechanics as those of a system with a small number of degrees of freedom, the large number implies *qualitatively* different new laws (Landau and Lifshitz 1980).

As a consequence of the large number of particles, the macroscopic level is characterised by a sort of "statistical determinism", as in transport equations: the Navier-Stokes equations for the velocity of a fluid, Fourier's law for the temperature, Fick's law for diffusion are all deterministic, and result from the fact that the statistical analysis is exact with such large numbers of interacting objects. An example of "statistical determinism" in our daily lives is the sweeping of dust with a broom. In order to collect the dust into a corner, one tries to push the dust grains with the broom horsehair. Of course, the single hair cannot push a grain. However, the broom has lots of hair, so sweeping many times eventually achieves the goal.

Regardless of the rivers of ink shed in philosophical controversies, perhaps the greatest technical significance of the discovery of deterministic chaos is that it reveals that the statistical approach is necessary also in systems with few degrees of freedom. A statistical approach is obviously necessary if the number of degrees of freedom is very large, but in the presence of deterministic chaos it is necessary, independent of the number of variables involved.

An example is given by the Lorentz gas, further idealised by the Sinai Billiard, in which a particle moves with constant speed bouncing over fixed circular obstacles (Dorfman 1999), see Fig. 5.1.

Because the dynamics is unstable, the motion looks very similar to Brownian motion. Indeed, with regard to diffusion in the long-time limit, it is possible to prove that the particle in the Sinai billiard enjoys exactly the same statistical properties as a Brownian particle. In the latter case the irregularity of the motion is due to the presence of many fluid molecules randomly colliding with it; the motion in the Sinai billiard has no randomness, but trajectories are chaotic, due to the instability. The overall effect, as far as diffusion is concerned, is the same.

As observed above about the statistical description of both thermodynamic and chaotic systems, the probabilistic approach could be considered as merely a method to cope with our limited ability to accurately control the systems of interest. In statistical

[16] To be rigorous, this is true for particles interacting through a potential, i.e. in all cases of physical interest.

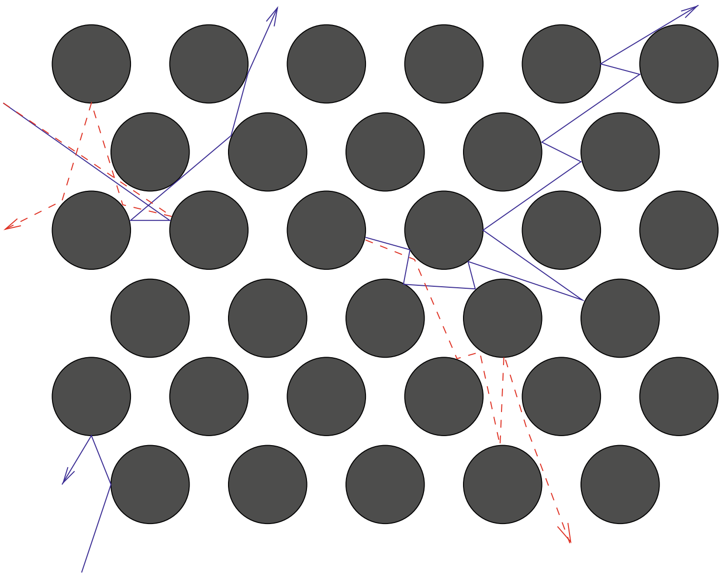

Fig. 5.1 Examples of trajectories of particles bouncing over fixed *circular* obstacles; note the divergence of initially close trajectories

mechanics, the difficulty is due to the large number of degrees of freedom, whereas in chaotic systems it arises from the sensitive dependence on initial conditions. By contrast, quantum mechanics is intrinsically stochastic; the position and momentum of the system cannot be determined with arbitrary precision, because a bound is imposed by Heisenberg's uncertainty principle. Hence, probabilities are unavoidable.

However, in light of our arguments, it seems fair to claim that the vexed question of whether the laws of physics are deterministic or probabilistic has, and will have, no definitive answer. On the sole basis of empirical observations, it does not appear possible to decide between these two contrasting arguments:

(i) Laws governing the universe are inherently random, and the determinism that is believed to be observed is in fact a result of the probabilistic nature implied by the large number of degrees of freedom;
(ii) the fundamental laws are deterministic, and seemingly random phenomena appear so due to deterministic chaos.

Basically these two positions can be viewed as a reformulation of the endless debate on quantum mechanics: thesis (i) expresses the inherent indeterminacy claimed by the Copenhagen school, whereas thesis (ii) illustrates the hidden determinism advocated by Einstein (Pais 2005).

5.6 Concluding Remarks

By way of conclusion, we would like to make a few remarks on the importance of chaos from a conceptual point of view and in the context of present-day research.

The most important findings are the following:

1. Deterministic systems, even with just a few degrees of freedom, may be sensitive to initial conditions, hence unpredictable except in the short term.
2. Chaotic systems are complex. Complexity can be rigorously defined in terms of algorithmic complexity, which is a notion of incompressibility hence of unpredictability. Moreover, *almost all* initial conditions of a generic deterministically chaotic system are complex, hence almost all trajectories are complex.
3. The elementary bricks of complex systems may have far from elementary behaviour, and be complex themselves.
4. A probabilistic description is needed both for chaotic systems and for systems with many degrees of freedom. In both cases, new *statistical* laws emerge from the underlying deterministic framework.
5. If a given phenomenon appears irregular or disordered, it is practically impossible to check whether this is due to chaos, to the presence of many interacting degrees of freedoms, or to some intrinsic randomness.
6. Analogously to the case of the singular limit of statistical mechanics, the singular nature of the chaotic limit allows neither practically nor conceptually the reduction of chaotic macroscopic phenomena to deterministic mechanistic laws. From a philosophical perspective, this is another case of strong emergence.

The discovery of an irregular chaotic behaviour in systems with few degrees of freedom and apparently innocent evolution laws seems to contradict the supporters of the "elementary brick" philosophy. A revealing example of the difficulties faced by this philosophy is given by Lorenz's celebrated model: if we reduce the hydrodynamic equations to elementary, or simple, structures we do not necessarily find simple behaviour, hence we do not necessarily increase our understanding of (for example) turbulence. This teaches us two general lessons, which are of practical importance:

(a) complex (unpredictable) behaviours are not necessarily produced by complicated structures, such as structures made of many components, but are common in simple and low dimensional dynamics;
(b) the methodological approach ["micro-reductionism" in the words of Smith (1998)], which seeks to understand and control dynamics by determining the equations ruling the interactions of its parts, can fail. We may say that: *knowing the Navier-Stokes equation does not solve the problem of understanding turbulence*.

It is a matter of fact that finding solutions, or merely approximate solutions, to the classical "initial value" problem (i.e. to differential equations once the initial state is given) is not a viable approach in many interesting situations characterised by complex (or complicated) behaviour. Even when detailed knowledge of the evolution laws is given, or presumed to be given, the presence of chaos and/or large numbers of degrees of freedom foils the initial value problem, because an unlimited amount of information on the initial state would be required.

Therefore, rather than considering the properties of specific trajectories originating from given initial states, one is forced to adopt a new strategy based on the statistical information carried by an ensemble of trajectories. This task is usually accomplished with the aid of computers, which have thus played a key role in developing the theory of dynamical systems and chaos. Indeed, the wealth of behaviour of nonlinear systems has been unveiled and systematically and quantitatively characterised thanks only to the fast computations and visualisations made possible by computers. Understanding the practical and conceptual problems posed by chaotic dynamics has led to a shift towards probabilistic or, at times, qualitative approaches, in science.

To better appreciate this recent shift in approach, consider the paradigmatic example of pre-chaos approaches to complex systems, constituted by von Neumann's belief that powerful computers and a clever use of numerical analysis would eventually lead to accurate forecasts, and even to the control, of weather and climate:

> The computer will enable us to divide the atmosphere at any moment into stable regions and unstable regions. Stable regions we can predict. Unstable regions we can control.[17]

The great scientist von Neumann was wrong, but he did not know the phenomenon of deterministic chaos.

Despite the exponentially fast growth of computing power, the forecasting ability of even the largest weather forecasting centres advances rather slowly (Yoden 2007). Modern weather forecasters have two goals: ever more accurate and detailed predictions, and advances in cognition and qualitative understanding. However, even the standard activity is carried out with perspectives different from von Neumann's. The intrinsic limitations on predictability, inherent in the chaotic nature of the atmosphere, require meteorologists to run series of forecasts, known as ensemble forecasts, each member of which starts from a slightly different initial condition, in order to produce data for a probabilistic concept of the forecasts. Is this surrendering before the tasks of prediction and detailed description of weather and climate? In fact, we simply believe that this change of perspective is dictated by the evidence that, in the field of complex systems, one may only investigate problems that are physically well-posed.

Because detailed predictions are impossible in chaotic systems, one wonders whether the study of oversimplified chaotic models of physical phenomena improves our understanding of the behaviour of real-world systems, or is irrelevant to that end. This raises, in turn, the general question of the relationship between scientific theories and the part of the real world they aim to describe, including, in particular, the role played in this relationship by mathematical models and numerical simulations.

Roughly speaking, we can identify two main categories of numerical simulation, although their boundaries are sometimes blurred:

[17] Cited in Dyson (2009).

(i) Accurate numerical simulations which approximate the solution of equations representing, or thought to represent, a given phenomenon.
(ii) Numerical implementations of models which, retaining the basic features of a real system, are crude simplifications, or phenomenological caricatures of "realistic" models.

Class (i) includes, for example, standard direct numerical simulations of the Navier-Stokes equations, or the full N-body gravitational problem in celestial mechanics. This computational approach is the most obvious, and reflects the etymological origin of the term "computer": from the Latin computare "to count", "to sum up". The idea underlying this use of computers is that systems can be completely known and reproduced *in silico*, once the equations representing their properties are solved numerically.

Class (ii), instead, presupposes some kind of modelling activity. As an explicit connection between models and reality is not available or, more generally, is not even required, the results of numerical computations only concern the abstract mathematical structures of the model. As such, they can be considered as mere metaphors for of the original phenomenon. Typical examples are: Lorenz's model, which is a caricature of Boussinesq's equation; coupled map lattices that constitute a prototype for spatially extended systems, but are far from representing any of them; the Lotka-Volterra equations, that describe some basic mechanisms of competition between prey and predator species, whose real dynamics is unknown (Cencini et al. 2009).

One should also beware of the possible confusion between ontic and epistemic descriptions, when studying the problems of chaos.

Determinism simply means that: given the same initial state $\mathbf{X}(0)$, one always finds the same evolved state $\mathbf{X}(t)$, at any fixed later time $t > 0$. Therefore, determinism refers exclusively to ontic descriptions, and it does not deal with predictions. This has been clearly stressed by Atmanspacher, in a paper by the rather eloquent title *Determinism is ontic, determinability is epistemic*, (Atmanspacher 2002). This distinction between ontic and epistemic descriptions was obvious to Maxwell; after having noted the metaphysical nature of the problem of determinism in physics, he stated that:

> There are certain classes of phenomena... in which a small error in the data only introduces a small error in the result...There are other classes of phenomena which are more complicated, and in which cases of instability may occur.[18]

On the contrary, Popper (1992) confused determinism and prediction:

> Scientific determinism is the doctrine that the state of any closed physical system at any future instant can be predicted.

In the previous section, we considered arguments, e.g. by van Kampen, which deny that determinism may be decided on the basis of observations. This conclusion is also

[18] From the conference *Does the progress of Physical Science tend to give advantage to opinion of Necessity* (or Determinism) over that of the Contingency of Events and the Freedom of the Will?, see Campbell and Garnett (1882).

reached from detailed analyses of sequences of data produced by the time evolutions of interest. Computing the so-called ε-entropy and the Finite-Size Lyapunov Exponents, at different resolution scales ε, one cannot distinguish potentially underlying deterministic dynamics from stochastic ones. The analysis of temporal series can only be used, at best, to pragmatically classify the stochastic or chaotic character of the observed signal, within certain scales (Cencini et al. 2009).

At first, this could be disturbing: not even the most sophisticated time-series analysis that we could perform reveals the "*true nature*" of the system under investigation, the reason simply being the unavoidable finiteness of the resolution we can achieve. More sophisticated instruments, with the much higher resolution that can be envisaged for the future, will not change this fact, as their resolutions will nevertheless always be finite. On the other hand, one may be satisfied with a non-metaphysical point of view, in which the *true nature* of the object of investigation is not at stake. The advantage is that one may choose whatever model is more appropriate or convenient to describe the phenomenon of interest, especially considering the fact that, in practice, one observes and wishes to account for only a limited set of coarse-grained properties. These properties are typically equivalently obtained from a variety of different underlying dynamics.

Chaotic systems and, more precisely, those which are ergodic, naturally lead to probabilistic descriptions in the presence of deterministic dynamics. In particular, ergodic theory justifies the *frequentist* interpretation of probability, according to which the probability of a given event is defined by its relative frequency. Therefore, assuming ergodicity, it is possible to obtain an empirical notion of probability which is an objective property of the trajectory (von Plato 1994).

There is no universal agreement on this issue; for instance, Popper (2002) believed that probabilistic concepts are extraneous to a deterministic description of the world, while Einstein held the opposite view, as expressed in his letter to Popper:

> I do not believe that you are right in your thesis that it is impossible to derive statistical conclusions from a deterministic theory. Only think of classical statistical mechanics (gas theory, or the theory of Brownian movement).[19]

Naively, one might consider the statistical properties of chaotic systems to be illusory, because they only result from observational limitations. Apparently, such a conclusion is confirmed by the fact that important measures of the dynamical complexity, such as the Lyapunov exponent λ and the Kolmogorov–Sinai entropy h_{KS}, are defined via finite, albeit arbitrarily high, resolutions. For instance, in the computation of λ one considers two trajectories, which are initially very close $|\mathbf{X}'(0)-\mathbf{X}(0)| = \delta_0$ and diverge in time from each other. Similarly, h_{KS} is computed introducing a partition of the phase space, whose elementary cells have a finite size ε. However, in the small-ε limit, the value of h_{KS} asymptotically tends to a value that no longer depends on ε, as happens to λ in the small-δ_0 limit. Therefore, these measures of the chaotic properties of given dynamics can be considered intrinsic properties of the dynamics themselves: they do not depend on our observation ability, provided it is finite, i.e. provided ε and δ_0 do not vanish.

[19] The letter is reprinted in Popper (2002).

According to Primas (2002), measures of stability, such as the Lyapunov exponent, concern ontic descriptions, whereas measures of information content or information loss, such as the Kolmogorov–Sinai entropy, relate to epistemic descriptions. We agree as far as stability is concerned.

Regarding the epistemic character of h_{KS}, we observe that the Shannon entropy of a sequence of data, as well as the Kolmogorov–Sinai entropy, enjoy an epistemic status from a certain viewpoint, but not from another. The epistemic status arises from the fact that information theory deals with transmission and reception of data, which is necessarily finite. On the other hand, h_{KS} is definitely an objective quantity, which does not depend on our observational limitations, as demonstrated by the fact that it can be expressed in terms of Lyapunov exponents.[20] Therefore, the Kolmogorov–Sinai entropy can be considered as a concept which links deterministic and stochastic descriptions.

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[20] The so-called Pesin formula:

$$h_{KS} = \sum_{i:\lambda_i>0} \lambda_i,$$

expresses the Kolmogorov–Sinai entropy as the sum of the positive Lyapunov exponents, (Cencini et al. 2009). The first Lyapunov exponent λ_1 is the λ introduced in (5.1).

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