

Fuzzy Resolution with Similarity-Based Reasoning

Banibrata Mondal and Swapan Raha

Abstract Resolution is an useful tool for mechanical theorem proving. Resolution models the refutation proof procedure, which is mostly used in constructing a ‘proof’ of a ‘theorem’. In this chapter, an attempt is made to derive a fuzzy resolvent from imprecise information expressed as standard rule using similarity based inverse approximate reasoning methodology. For complex clauses, we investigate similarity based ordinary approximate reasoning to derive a fuzzy resolvent. The proposal is well-illustrated with artificial examples and a real life problem.

1 Introduction

Man had for long wanted to find a general decision procedure to prove theorems. Since 1930, consistent efforts were made by Herbrand and others in this respect. It was in 1965 that Robinson [1] made a major breakthrough. He introduced a

This research has been partially supported by the UGC SAP (DRS) Phase-II project under the Department of Mathematics, Visva-Bharati and UGC major research project no. 36-293/2008 (SR);26.03.09.

B. Mondal (✉) · S. Raha

Department of Mathematics, Visva Bharati, Santiniketan, West Bengal, India
e-mail: mbanibrata@gmail.com

S. Raha

e-mail: swapan.raha@visva-bharati.ac.in

machine-oriented logic based on resolution principle to check the (un)satisfiability of a set of formulae represented in some standard form.

Mechanical theorem proving techniques were first applied to deductive question answering systems and then to problem solving, program synthesis, program analysis and many others. In automated theorem proving, resolution acts as a rule of inference leading to a refutation based theorem-proving technique.

Now, given some formulae modelling vague knowledge, can we define an automated procedure which decides whether a given conclusion is logically entailed by the knowledge? A first step towards formalization of automated deduction in fuzzy logic was taken by Lee and Chang [2]. Lee's works [2, 3] were continued and implemented by other researchers. Lee's fuzzy formulae are syntactically defined as classical first-order formulae, but they differ semantically as the formulae have a truth value in $[0,1]$. Mukaidono [4, 5] generalized Lee's [3] result. Dubois and Prade [6] established fuzzy resolution principle in the case of uncertain propositions. In Kim et al. [7], antonym-based fuzzy hyper-resolution was introduced and its completeness was proved. Fontana and Formato [8] introduced, a fuzzy resolution rule based on an extended most general unifier provided by the extended unification algorithm. Sessa [9] proposed a methodology to manage uncertain and imprecise information in the frame of the declarative paradigm of logic programming considering, a similarity relation between function and predicate symbols in the language of a logic program. Habiballa and Novak [10] presented refutational theorem proving system for fuzzy description logic based on the general resolution rule. Raha and Ray [11] investigated a generalized resolution principle that handles the inexact situation effectively and is applicable for both well-defined and undefined propositions. They associated a truth value to every proposition and used Zadeh's concept of approximate reasoning based on possibility theory to model a deductive process. In this research, we extend our knowledge to present, a generalised resolution principle that deals with fuzzy propositions and uses the technique of similarity based reasoning. Similarity is inherent in approximate reasoning, but none used it in fuzzy resolution. Indeed, human reasoning is often performed on the basis of analogy and/or similarity between entities. We, therefore, accomplish the task of choosing an almost complementary pair in fuzzy resolution on the basis of similarity/dissimilarity measure of fuzzy sets instead of contradictory degree of the pair. The advantage of our method is that, it reduces significant and important drawback of compositional rule of inference and executes effective resolution to show its flexibility for automated reasoning. In this article, we have avoided the generic problem in generalised modus ponens (GMP) by using similarity-based inverse approximate reasoning method [12] in fuzzy resolution. However, for complex set of clauses the method is not suitable. In this case, we investigate similarity-based ordinary approximate reasoning [13] to deduce fuzzy resolution.

2 Preliminaries

We start by summarizing a few basic concepts of fuzzy logic connected with fuzzy resolution.

Definition 1 A resolvent of two clauses C_1, C_2 containing the complementary literals p and $\neg p$ respectively, is defined as

$$res(C_1, C_2) = \{C_1 - \{p\}\} \cup \{C_2 - \{\neg p\}\},$$

\cup is understood as the disjunction of the literals present in them.

It is also a logical consequence of C_1 and C_2 .

Definition 2 A resolution deduction of a clause C from a set S of clauses is a finite sequence of clauses $C_1, C_2, \dots, C_n = C$ such that, each C_i is either a member of S or is a resolvent of two clauses taken from S . From the resolution principle in propositional logic we deduce that, if S is true under some truth valuation v , then $v(C_i) = TRUE$ for all i , and in particular, $v(C) = TRUE$ [14].

A simple resolution scheme is:

$$\begin{array}{l} \text{premise1 : } a \vee b \\ \text{premise2 : } \neg b \\ \hline \text{conclusion : } a \end{array}$$

In first order logic, resolution condenses the traditional syllogism of logical inference down to single rule. To recast the logical inference using the resolution technique, the formulae are represented in conjunctive normal form.

Definition 3 C_1 and C_2 being two clauses in fuzzy logic, $R(C_1, C_2)$ being a classical resolvent of C_1 and C_2 , and A being the literal C_1 and C_2 are resolved upon (i.e., $C_1 \in A$ and $C_2 \in \sim A$ or conversely), then fuzzy resolvent $R(C_1, C_2)_{cd(A)}$ of C_1 and C_2 , where $cd(A)$ is the contradictory degree of keyword A or the confidence associated with the resolvent, is defined by

$$R(C_1, C_2)_{cd(A)} \triangleq R(C_1, C_2) \vee (A \wedge \sim A)$$

An important theorem which was proved in [4, 5] is given in the following [15]:

Theorem 1 A set S of fuzzy clauses is unsatisfiable if and only if, there is a deduction of empty clause with its confidence of resolvent $cd \neq 0$ from S .

Fuzzy Implication:

Typically, a fuzzy rule ‘If X is A then Y is B ’ (A and B are fuzzy sets) is expressed as $I(a, b)$, where I is a fuzzy implication and a and b are membership grades of A and B respectively. From an algebraic point of view, some implication operators basically identified in [16] are classified with four families $I_i^T, i \in \{1, 2, 3, 4\}$, where T and T^* denote the specific t-norm and correlated t-conorm respectively

Table 1 T-Norms and T-Conorms

T/T *	T(a, b)	T*(a, b)
M	$\min(a, b)$	$\max(a, b)$
P	$a.b$	$a + b - a.b$
B	$\max(0, a + b - 1)$	$\min(1, a + b)$

Table 2 Expression of fuzzy implication operators

I_i^T	M	P	B
I_1^T	$\max(\min(a, b), 1 - a)$	$1 - a + a^2.b$	$\max(1 - a, b)$
I_2^T	$\max(1 - a, b)$	$1 - a + a.b$	$\min(1 - a + b, 1)$
I_3^T	1 if $a \leq b$ b otherwise	1 if $a \leq b$ b/a otherwise	$\min(1 - a + b, 1)$
I_4^T	1 if $a \leq b$ $1 - a$ otherwise	1 if $a \leq b$ $\frac{1-a}{1-b}$ otherwise	$\min(1 - a + b, 1)$

(see, Table 1) to implement the implication operators. Here, we use only two families I_2 and I_3 of implication operators. Family I_2 , often named S-implication, derives from classical logic form $a \rightarrow b = \neg a \vee b$. Family I_3 , known as R-implication, reflects a partial ordering on propositions. With reference to the t-norms and t-conorms in Table 1, the explicit expressions of fuzzy implication operators I_i^T are presented in Table 2.

Similarity Index:

The notion of similarity plays a fundamental role in theories of knowledge and behaviour and has been dealt with extensively in psychology and philosophy. If we study the behaviour pattern of children we find that, children have a natural sense in recognizing regularities in the world and to mimic the behavior of competent members of their community. Children thus make decisions on similarity matching. The similarity between two objects suggests, the degree to which properties of one may be inferred from those of the other. The measure of similarity provided, depends mostly on the perceptions of different observers. Emphasis should also be given to different members of the sets, so that no one member can influence the ultimate result. Many measures of similarity have been proposed in the existing literature [17, 18]. A careful analysis of the different similarity measures reveals, that it is impossible to single out one particular similarity measure that works well for all purposes.

Suppose U be an arbitrary finite set, and $\mathcal{F}(U)$ be the collection of all fuzzy subsets of U . For $A, B \in \mathcal{F}(U)$, a similarity index between the pair $\{A, B\}$ is denoted as $S(A, B; U)$ or simply $S(A, B)$ which can also be considered as a function $S : \mathcal{F}(U) \times \mathcal{F}(U) \rightarrow [0, 1]$. In order to provide a definition for similarity index, a number of factors must be considered. We expect a similarity measure $S(A, B)$ to satisfy the following axioms :

- P1.** $S(B, A) = S(A, B)$, $S(\text{not } A, \text{not } B) = S(A, B)$, $\text{not } A$ being some negation of A .
- P2.** $0 \leq S(A, B) \leq 1$.

P3. $S(A, B) = 1$ if and only if $A = B$.

P4. For two fuzzy sets A and B , simultaneously *not null*, if $S(A, B) = 0$ then $\min(\mu_A(u), \mu_B(u)) = 0$ for all $u \in U$, i.e., $A \cap B = \Phi$.

P5. If either $A \subseteq B \subseteq C$ or $A \supseteq B \supseteq C$ then $S(A, C) \leq \min\{S(A, B), S(B, C)\}$.

We now consider a definition of measure of similarity which has been proposed in [13, 19].

Definition 4 *Similarity Indices* Let A and B be two fuzzy sets defined over the same universe of discourse U . The similarity index $S(A, B)$ of pair $\{A, B\}$ is defined by

$$S(A, B) = 1 - \left(\frac{\sum_u |\mu_A(u) - \mu_B(u)|^q}{n} \right)^{\frac{1}{q}},$$

where n is the cardinality of the universe of discourse and $q \geq 1$ is the family parameter.

Measure of dissimilarity is another measure of comparison of objects in literature. Many authors like Meunier et al. [17] have defined measure of dissimilarity in different way. However, we use the dissimilarity measure in the context of similarity measure and consider measure of dissimilarity of two fuzzy subsets A and B defined over $\mathcal{F}(U)$, denoted by $D(A, B)$ as $D(A, B) = 1 - S(A, B)$. Moreover, we assume $D(A, B) = S(\text{not } A, B) = S(A, \text{not } B)$. Through out the chapter, we use this concept of dissimilarity.

In the next section, principle of fuzzy resolution is discussed based on the method of inverse approximate reasoning [12].

3 Fuzzy Resolution Based on Inverse Approximate Reasoning

Instead of complementary literal in the set of clauses in fuzzy logic, we introduce here a concept of similar/dissimilar literal. Let us consider two clauses $C_1 = P \vee C_1'$ and $C_2 = P' \vee C_2'$. Resolvent of C_1 and C_2 denoted by $\text{res}(C_1, C_2) = C_1' \vee C_2'$ if and only if similarity between P and P' is less than Q or equivalently, dissimilarity between P and P' is greater than $1 - Q$, Q being pre-defined threshold.

The argument form of simple fuzzy resolution is as follows.

$$\frac{A \vee B \quad \text{not } B}{A}$$

The scheme for Generalised Fuzzy Resolution is given in Table 3.

Table 3 Generalised fuzzy resolution

Rule: X is A or	Y is B
Fact:	Y is B'
Conclusion: X is A'	

Table 4 Inverse approximate reasoning

Rule(p): if X is A then	Y is B
Fact(q):	Y is B^*
Conclusion: X is A^*	

In this case, we can say that the Disjunctive Syllogism holds if B' is close to $notB$, A' is close to A .

The scheme in inverse approximate reasoning is as given in the following Table 4. Here, fuzzy sets A and A^* are defined over the universe of discourse $U = \{u_1, u_2, \dots, u_m\}$ and fuzzy sets B and B^* are defined over the universe of discourse $V = \{v_1, v_2, \dots, v_n\}$.

We, now, transform the disjunction form of rule into fuzzy implication or fuzzy relation and apply the method of inverse approximate reasoning to generate the required resolvent.

We proposed two algorithms SIAR and INAR for inverse approximate reasoning based on similarity given in [12]. We apply these methods to obtain a resolvent in fuzzy resolution.

In classical logic, we have

$$a \rightarrow b \equiv \neg a \vee b, \forall a, b \in \{0, 1\}. \tag{1}$$

Let us extend this classical logic equivalence to fuzzy logic, by interpreting the disjunction and negation as a fuzzy union (t-conorm) and a fuzzy complement, respectively. Fuzzy implication thus obtained is usually referred to in the literature as S-implication.

We now consider the classical logic tautology which is obtained from (1).

$$a \vee b \equiv \neg a \rightarrow b, \forall a, b \in \{0, 1\}. \tag{2}$$

Extending the classical equivalence (2) into fuzzy logic, we find that the fuzzy union is transformed to fuzzy implication. In fuzzy resolution, we deal with rule of type ' X is A or Y is B '. Like classical logic, we may transform the rule into ' X is $notA$ then Y is B ' under fuzzy logic. Hence, the equivalent scheme of Table 3 that conforms fuzzy resolution is considered in Table 5. It is noted that A^* is similar to A , whenever B^* is similar to $notB$.

We have demonstrated in [12] that, if the given data is sufficiently dissimilar to the consequent part of a given rule then we conclude that the resulting fuzzy set is sufficiently dissimilar to the antecedent part of the rule. Applying this method, in the scheme given in Table 5, we get the required resolvent, which establishes the fuzzy resolution principle. The algorithm is as follows:

Table 5 Equivalent scheme conforms fuzzy resolution

Rule: If X is <i>not</i> A then	Y is B
Fact:	X is B^*
Conclusion: X is A^*	

ALGORITHM—FRIAR:

- Step 1. Translate the rule into fuzzy implication as $\mu_R(u, v) = I(\mu_{notA}(u), \mu_B(v))$, where I is an implication operator;
- Step 2. Take cylindrical extension of B^* in V on $U \times V$, say R' , defined by $R' = \sum_{U \times V} \mu_{B^*}(v)/(u, v)$;
- Step 3. Compute $R^* = R \cap R'$, where \cap denotes any fuzzy conjunction operator;
- Step 4. Obtain $A^* = projR^*$ on U defined by

$$projR^* \text{ on } U = \sum_u \text{Sup}_v \mu_R(u, v)/u.$$

Mathematically, we get

$$\begin{aligned} \mu_{A^*}(u) &= proj_{v \in V} R^*(u, v) \\ &= \sup_{v \in V} T(\mu_R(u, v), \mu_{R'}(u, v)) \\ &= \sup_{v \in V} T(\mu_R(u, v), \mu_{B^*}(v)) \end{aligned} \tag{3}$$

where T is a t-norm used to describe fuzzy conjunction operator.

It is expected that, for the observation ‘ Y is *not* B ’ and the given premise ‘ X is A or Y is B ’ we can conclude ‘ X is A ’ by fuzzy resolution. However, for the the observation ‘ Y is B ’ no conclusion can be drawn. We establish the above criteria by the following theorem.

Theorem 2 Let $B^* = not B$ be normal and R be interpreted by any S-implication satisfying (3). Then $A^* \supseteq A$ for any t-norm T . (consistency)

Corollary Let $B^* = not B$ be normal and R be interpreted by any S-implication satisfying (3). Then $A^* = A$ for Lukasiewicz t-norm T .

Example 1 Consider the premises

$$\begin{aligned} p : & X \text{ is } LARGE \quad \text{or} \quad Y \text{ is } SMALL \\ q : & \quad \quad \quad \quad \quad \quad \quad Y \text{ is not } SMALL \end{aligned}$$

in which X and Y are defined over the universes $U = \{u_1, \dots, u_4\}$ and $V = \{v_1, \dots, v_4\}$ respectively and fuzzy sets labelled by *LARGE*, *SMALL* and *not SMALL* are defined by

$$A \triangleq LARGE = 0/u_1 + 0.45/u_2 + 0.95/u_3 + 1/u_4 \tag{4}$$

Table 6 A* for Reichenbach S-implication

$T/\mu A^*$	u_1	u_2	u_3	u_4	A^*	$S(A, A^*)$
M	0.35	0.53	0.95	1.0	$\supset A$	0.820
P	0.23	0.45	0.95	1.0	$\supset A$	0.886
B	0.0	0.45	0.95	1.0	A	1.0

Table 7 A* for Kleene-Dienes S-implication

$T/\mu A^*$	u_1	u_2	u_3	u_4	A^*	$S(A, A^*)$
M	0.35	0.45	0.95	1.0	$\supset A$	0.825
P	0.23	0.45	0.95	1.0	$\supset A$	0.886
B	0.0	0.45	0.95	1.0	A	1.0

Table 8 A* for Lukasiewicz S-implication

$T/\mu A^*$	u_1	u_2	u_3	u_4	A^*	$S(A, A^*)$
M	0.35	0.60	0.95	1.0	$\supset A$	0.809
P	0.23	0.51	0.95	1.0	$\supset A$	0.882
B	0.0	0.45	0.95	1.0	A	1.0

$$B \triangleq \text{SMALL} = 1/v_1 + 0.65/v_2 + 0.15/v_3 + 0.0/v_4 \tag{5}$$

$$B^* \triangleq \text{not SMALL} = 0/v_1 + 0.35/v_2 + 0.85/v_3 + 1/v_4 \tag{6}$$

The similarity between fuzzy sets B and B^* is 0.0, i.e., fuzzy set B^* in observation is dissimilar to fuzzy set B in the disjunctive form of rule.

Again, by INAR, we study the shape of the resolvent A^* for data given in (4), (5) and (6) with different S-implications and different t-norms, which is described in the Tables 6, 7 and 8, where M, p and B indicate minimum, product and bounded product respectively for t-norms.

The result shows that the dissimilarity between B^* and B assures the similarity between A^* and A when the reasoning mechanism is handled using inverse approximate reasoning. Thus the proposition ‘given a disjunction and the negation of one of the disjuncts, the other may be inferred’ is established in fuzzy logic.

Example 2 Now, we consider the scheme and data of *Example 1* except B^* . Consider $B^* = 0.0/v_1 + 0.1225/v_2 + 0.7225/v_3 + 1.0/v_4$ in (6). We shall observe the results for the given premise ‘ p ’ and data in (4) and (5).

In this case, $S(B, B^*) = 0.1304$, i.e., fuzzy sets B and B^* are dissimilar.

Let us execute the reasoning mechanism by INAR. The results are shown in Table 9, 10 and 11 respectively for different implications and t-norms.

Table 9 A* for Reichenbach S-implication

$T/\mu A^*$	u_1	u_2	u_3	u_4	A^*	$S(A, A^*)$
M	0.15	0.53	0.95	1.0	$\supset A$	0.9144
P	0.11	0.45	0.95	1.0	$\supset A$	0.9458
B	0.0	0.45	0.95	1.0	A	1.0

Table 10 A* for Kleene-Dienes S-implication

$T/\mu A^*$	u_1	u_2	u_3	u_4	A^*	$S(A, A^*)$
M	0.15	0.45	0.95	1.0	$\supset A$	0.9250
P	0.11	0.45	0.95	1.0	$\supset A$	0.9458
B	0.0	0.45	0.95	1.0	A	1.0

Table 11 A* for Lukasiewicz S-implication

$T/\mu A^*$	u_1	u_2	u_3	u_4	A^*	$S(A, A^*)$
M	0.15	0.60	0.95	1.0	$\supset A$	0.8939
P	0.11	0.45	0.95	1.0	$\supset A$	0.9458
B	0.0	0.45	0.95	1.0	A	1.0

That is, if B^* is not exactly match with $notB$ but these are dissimilar, the fuzzy resolvent can be obtained through inverse approximate reasoning method. The technique is new.

Theorem 3 Let $B^* = B$ be normal and R be interpreted by any implication satisfying (3). Then $A^* = UNKNOWN$ for any t-norm T .

We prove the theorem for Reichenbach S-implication and $T = min$ only, but the above theorem can be proved for any other implications and any other t-norms in the similar way.

Example 3 In *Example 1*, if we take $B^* = B \triangleq SMALL = 1.0/v_1 + 0.65/v_2 + 0.15/v_3 + 0.0/v_4$ in (6) then either by SIAR or by INAR we get $A^* = 1.0/u_1 + 1.0/u_2 + 1.0/u_3 + 1.0/u_4 = UNKNOWN$, for all of the cases.

Theorem 4 Let $B^* = not B$ be normal and R be interpreted by Rescher-Gaines R-implication satisfying (3). Then $A^* \subseteq A$ for any t-norm T .

Example 4 For the data given in *Example 1*, applying INAR for Rescher-Gaines R-implication combined with any t-norm T , we get the fuzzy resolvent A^* as

$A^* = 0.0/u_1 + 0.35/u_2 + 0.85/u_3 + 1.0/u_4$ which is a subset of A and $S(A, A^*) \approx 0.929$. It establishes *Theorem 4*.

We, now, apply another method SIAR [12] to obtain fuzzy resolvent for the scheme given in Table 3. Let us consider another classical logic equivalence

$$a \vee b \equiv b \vee a \equiv \neg b \rightarrow a \tag{7}$$

The classical logic equivalence (7) can be extended in fuzzy logic with implication and negation function. Then we transform the rule in Table 3 into its equivalent form ‘ p^1 : If Y is *not* B then X is A ’ over the domain of $[0, I]_V \times U$. A fuzzy rule may be defined by means of a conjunction for defining a fuzzy Cartesian product rather than in terms of a multivalued logic implication [12, 20].

Therefore, the rule in p^1 is transformed into fuzzy relation R as

$$\mu_R(v, u) = T(1 - \mu_B(v), \mu_A(u)), \tag{8}$$

where T is a t-norm describing a fuzzy conjunction.

Now, we can apply our method SIAR described in [12]. The algorithm is as follows :

ALGORITHM—FRSIAR:

- Step 1. Translate given premise p^1 and compute $R(\text{not } B, A)$ by (8);
- Step 2. Compute similarity measure $S(\text{not } B, B^*)$ using some suitable definition;
- Step 3. Modify $R(\text{not } B, A)$ with $S(\text{not } B, B^*)$ to obtain the modified conditional relation $R(\text{not } B, A|B^*)$ using *scheme* in [12];
- Step 4. Use *sup-projection* operation on $R(\text{not } B, A|B^*)$ to obtain A^* as

$$\mu_{A^*}(u) = \sup_v \mu_{R(\text{not } B, A|B^*)}(v, u). \tag{9}$$

We now illustrate the method applied here by some suitable examples.

Example 5 Let us consider the data of *Example 1*. For completely dissimilar B^* with B and for different t-norms T , the shapes of fuzzy resolvent A^* in U are studied here, when we apply SIAR. In each case, it turns out exactly the fuzzy set A which corresponds ‘*LARGE*’.

Example 6 Let us consider the data in *Example 1* where B^* is not completely dissimilar with B , but dissimilarity exceeds certain threshold. Applying SIAR we observe the shapes of A^* and compare it with given A for different t-norms. Since $S(A, A^*) \approx 0.92$, i.e., A^* is almost similar to A , it establishes fuzzy resolution in reasoning.

In the above methods, we applied INAR or SIAR when the disjunctive knowledge can be transformed into fuzzy implication. However, it may not always be the case. Moreover, when the expert knowledge is in complex form of disjunction it is difficult to apply INAR or SIAR. So, we extend our method in such a way that can deal with complex premises.

Table 12 Generalised fuzzy resolution-extended form

p:	X is A or	Y is B
q:	Y is B' or	Z is C
r:	X is A' or	Z is C'

4 Fuzzy Resolution with Complex Clauses

In this section, we extend the scheme given in Table 3. Let X, Y and Z be three linguistic variables that take values from the domain U, V and W respectively. We consider the derivation of an inexact conclusion ‘ r ’ from two typical knowledge (premises) ‘ p ’ and ‘ q ’ according to the scheme given in Table 12, where A ’s, B ’s and C ’s are approximations of possibly inexact concepts by fuzzy sets over U, V and W respectively. In 1993, Raha and Ray [11] applied Zadeh’s [21] concept of approximate reasoning with the application of possibility theory to model a deductive process ‘Generalised Disjunctive Syllogism’. They used projection principle and conjunction principle to deduce fuzzy resolvent. Here, we investigate another method which is described in the following algorithm.

ALGORITHM—FRCEP:

Step 1. Translate the premise p into fuzzy relation

$$R_1 \subseteq \mathcal{F}(U \times V) \text{ as } \mu_{R_1}(u, v) = \min(\mu_A(u), 1 - \mu_B(v));$$

Step 2. Translate the premise q into fuzzy relation

$$R_2 \subseteq \mathcal{F}(V \times W) \text{ as } \mu_{R_2}(v, w) = \min(\mu_{B'}(v), \mu_C(w));$$

Step 3. Take cylindrical extension of R_1 in $U \times V$ on $U \times W$, say R_1' , defined by

$$R_1' = \sum_{U \times V \times W} \mu_{R_1}(u, v)/(u, v, w);$$

Step 4. Take cylindrical extension of R_2 in $V \times W$ on $U \times V \times W$, say R_2' , defined by

$$R_2' = \sum_{U \times V \times W} \mu_{R_2}(v, w)/(u, v, w);$$

Step 5. Construct $R' = R_1' \cap R_2'$, where \cap denotes any fuzzy conjunction operator;

Step 6. Compute $S(\text{not}B, B')$ and, say, s ;

Step 7. Modify R' with s by a Scheme in [12] and, say, R^* ;

Step 8. Obtain $R = \text{proj}R^*$ on $U \times W$ defined by

$$\text{proj}R^* \text{ on } U \times W = \sum_{U \times W} \sup_v \mu_{R^*}(u, v, w)/(u, w);$$

Step 9. Obtain A' and C' by projecting R separately on U and W such that

$$A' = proj_U R = \sum_U \sup_w \mu_R(u, w)/u \text{ and}$$

$$C' = proj_W R = \sum_W \sup_u \mu_R(u, w)/w$$

Symbolically, the fuzzy resolvent R is obtained by, for $u \in U, v \in V$ and $w \in W, \mu_R(u, w)$

$$\begin{aligned} &= \sup_v \mu_{R^*}(u, v, w) \\ &= \sup_v \{s \rightarrow \mu_{R'}(u, v, w)\}, \\ &= s \rightarrow \sup_v \mu_{R'}(u, v, w) \\ &= s \rightarrow \sup_v T(\mu_{R_1'}(u, v, w), \mu_{R_2'}(u, v, w)) \\ &= s \rightarrow \sup_v T(\mu_{R_1}(u, v), \mu_{R_2}(u, v)) \\ &= s \rightarrow \sup_v T(\min(\mu_A(u), 1 - \mu_B(v)), \min(\mu_{B'}(v), \mu_C(w))) \\ &= s \rightarrow T\left(\min\left(\mu_A(u), 1 - \inf_v \mu_{B'}(v)\right), \min\left(\sup_v \mu_{B'}(v), \mu_C(w)\right)\right) \\ &= s \rightarrow T(\mu_A(u), \mu_C(w)), \text{ iff } 1 - \inf_v \mu_{B'}(v) = \sup_v \mu_{B'}(v) = 1. \end{aligned}$$

This derivation can be achieved if there is a $v_0 \in V$ such that $\mu_B(v_0) = 0$ and $\mu_{B'}(v_0) = 1$ which is possible if the fuzzy sets B and B' are dissimilar, i.e., *not* B and B are similar for any implication \rightarrow in derivation. We observe two criteria here.

Criterion 1: Taking $x \rightarrow y = \min(1, y/x)$, we get

$$\mu_R(u, w) = \min(1, T(\mu_A(u), \mu_C(w))/s);$$

Criterion 2: Taking $x \rightarrow y = 1 - x + xy$, we get

$$\mu_R(u, w) = 1 - s + T(\mu_A(u), \mu_C(w)).s.$$

From the two criteria above, we observe that when $s = S(notB, B') = 0$, i.e., when B and B' are completely similar, $R = U \times W = UNKNOWN$. Therefore, fuzzy resolvent could be anything. However, if s is close to unity, i.e., if B and B' are almost dissimilar we have R is close to $\sum_{U \times W} T(\mu_A(u), \mu_C(w))/(u, w)$ which, after re-translation, gives ‘If X is A or Z is C ’. Again, we observe that a small change in B' produces a small change in fuzzy resolvent—which ensures our method is reasonable one.

Let us consider a scheme given in Table 13 where variables $X_i (i = 1, 2, \dots, m)$ and the respective fuzzy subsets $A_i (i = 1, 2, \dots, m)$ are defined on universe $U_i (i = 1, 2, \dots, m)$ respectively; variables $Y_j (j = 1, 2, \dots, n)$ and the respective fuzzy subsets $B_j (j = 1, 2, \dots, n)$ are defined on universe $V_j (j = 1, 2, \dots, n)$ respectively. (A_k, B_l) is almost dissimilar over the same universe $U_k (= V_l)$ with the degree of confidence of keyword A_k is cd

Table 13 Generalised fuzzy resolution-another extension

$C_1: X_1 \text{ is } A_1 \text{ or } X_2 \text{ is } A_2 \text{ or } \dots \text{ or } X_m \text{ is } A_m;$
$C_2: Y_1 \text{ is } B_1 \text{ or } Y_2 \text{ is } B_2 \text{ or } \dots \text{ or } Y_n \text{ is } B_n;$
$R(C_1, C_2): X_1 \text{ is } A_1' \text{ or } \dots \text{ or } X_m \text{ is } A_m'$ or $Y_1 \text{ is } B_1' \text{ or } \dots \text{ or } Y_n \text{ is } B_n'$

$(A_k) = 1 - S(A_k, B_l)$ and the corresponding variables X_k, Y_l , defined over $U_k (= V_l)$ respectively, assign same linguistic variable.

The algorithm is as follows:

ALGORITHM—FRAE:

- Step 1. Check for pair of literals $(A_i, B_j), \forall i, j$ from clauses C_1 and C_2 whether these are defined over same universe and are assigned by same linguistic variable; Otherwise, resolution is not possible;
- Step 2. If dissimilarity $D(A_k, B_l)$ is high, i.e., $D(A_k, B_l) > 1 - Q$, Q is pre-defined threshold then go to the next step and say, A_k is keyword; Otherwise, there is no fuzzy resolvent;
- Step 3. Modify B_j either by $B_j' = \min(1, B_j/D(A_k, B_l))$ or by $B_j' = 1 - (1 - B_j) \cdot S(A_k, B_l)$;
- Step 4. Fuzzy resolvent is $R(C_1, C_2) = A_1' \vee \dots \vee A_m' \vee B_1' \vee \dots \vee B_n'$ and $R(C_1, C_2)_{cd} = cd(A_k)$ which is measured as $cd(A_k) = D(A_k, B_l)$;
- Step 5. Repeat the process until empty clause, with the confidence $cd \neq 0$, is derived for more than two clauses.

Hence, we prove the (un)satisfiability of a theorem by the deduction of empty clause from a set of fuzzy clauses.

Let us consider an example to illustrate the method. Suppose variables that range over finite sets or can be approximated by variables ranging over such sets.

Example 7 Let us consider the premises

- $p : X \text{ is LARGE} \quad \text{or } Y \text{ is SMALL or } Z \text{ is LARGE};$
- $q : X \text{ is not LARGE};$
- $r : Y \text{ is not SMALL};$

in which X, Y and Z are defined over the universes $U = \{u_1, \dots, u_4\}, V = \{v_1, \dots, v_4\}$ and $W = \{w_1, w_2, w_3, w_4\}$ respectively. Fuzzy sets labelled by LARGE, SMALL and not SMALL defined over the universes U, V and W respectively are given in *Example 1* and fuzzy set LARGE defined over W is given by

$$C \triangleq \text{LARGE} = 0/w_1 + 0.20/w_2 + 0.75/w_3 + 1/w_4.$$

$$A' \triangleq \text{not LARGE} = 1.0/u_1 + 0.55/u_2 + 0.05/u_3 + 0.0/u_4.$$

We execute the following steps with the given data, using ALGORITHM-FRCEP.

Execution:

Step 1. Compute the fuzzy relation $R_I = A \vee B \vee C$ by

$$\mu_{R_I}(u, v, w) = \min(1 - \mu_A(u), 1 - \mu_B(v), \mu_C(w));$$

Step 2. Extend A' in U cylindrically on $U \times V \times W$ as

$$R_2 = \sum_{U \times V \times W} \mu_{A'}(u)/(u, v, w);$$

Step 3. Compute $R' = R_I \cap R_2$ by $\mu_{R'}(u, v, w) = T(\mu_{R_I}(u, v, w), \mu_{R_2}(u, v, w))$, taking fuzzy conjunction \cap as t-norm T ;

Step 4. Modify R' with $S(1 - \mu_A(u), \mu_{A'}(u)) = s_1$ to get R^* as $\mu_{R^*}(u, v, w) = 1 - (1 - \mu_{R'}(u, v, w)) \cdot s_1$;

Step 5. Project R^* on $V \times W$ such that

$$R = \sum_{V \times W} \sup_u \mu_{R^*}(u, v, w) = \begin{pmatrix} 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.20 & 0.45 & 0.45 \\ 0.00 & 0.20 & 0.75 & 0.85 \\ 0.00 & 0.20 & 0.75 & 1.00 \end{pmatrix}$$

Step 6. Extend B' in V cylindrically on $V \times W$ as $R_3 = \sum_{V \times W} \mu_{B'}(v, w)$

Step 7. Compute $R'' = R \cap R_3$ by $\mu_{R''}(v, w) = T(\mu_R(v, w), \mu_{R_3}(v, w))$, taking fuzzy conjunction \cap as t-norm T ;

Step 8. Modify R'' with $S(1 - \mu_B(v), \mu_{B'}(v)) = s_2$ to get R^{**} as $\mu_{R^{**}}(v, w) = 1 - (1 - \mu_{R''}(v, w)) \cdot s_2$;

Step 9. Project R^{**} on W such that

$$C' = \sum_w \sup_v \mu_{R^{**}}(v, w) = 0/w_1 + 0.20/w_2 + 0.75/w_3 + 1/w_4,$$

which is completely similar to C , i.e., $S(C', C) = 1$.

Even if A' and B' in the respective premises p and q , are not completely dissimilar to A and B respectively, but the dissimilarity measures attain values greater than certain predefined threshold then we can get a fuzzy resolvent C' which is almost similar to C , using FRCEP.

Suppose,

$$A' = 1.0/\mu_1 + 0.3025/\mu_2 + 0.0025/\mu_3 + 0.0/\mu_4;$$

$$B' = 0.0/v_1 + 0.1225/v_2 + 0.7225/v_3 + 1.0/v_4.$$

$S(notA, A') = 0.873992$; $S(notB, B') = 0.869604$ are computed.

Then, it yields $C' = 0.24/w_1 + 0.39/w_2 + 0.81/w_3 + 1.0/w_4$ with $S(C, C') = 0.843441$.

Therefore, it is possible to get a fuzzy resolvent from a set of fuzzy clauses if there is a pair of dissimilar literals contained in the respective clauses.

5 Example in Real Life

The statements in real life are mostly fabricated by vague, imprecise, uncertain and incomplete information which can be dealt with using fuzzy logic. In daily life, we have to often decide whether one statement follows from some other statements. Fuzzy resolution can be a way to deal with such phenomena. Let us consider the following example.

Example 8 Suppose, the stock price is low if the prime interest rate is high. Suppose, also, people are not happy when stock price is low. Assume that the prime interest rate is high. Can we conclude that people are not happy? Again, if we assume that the interest rate is very high, how can we conclude about the happiness of people?

For this problem, we construct the rule base in Table 14.

(i) **For Observation 1:** The above scheme may be transformed into the equivalent scheme given in Table 15. in which *prime interest rate*, *stock price* and *happiness* are defined over the respective universes $U = [0, 20]\%$, $V = \text{Rs. } [10,000, 22,000]$ and $W = [0,100]$. Fuzzy sets defined over the respective universes U, V and W are given in Table 16.

The membership grade of fuzzy complement is defined by $1 - a$ here, where a is the membership grade of a fuzzy set.

The similarity between complement of fuzzy set *LOW* in p and fuzzy set *not LOW* in q (denoted by s_1) is 1.0. Therefore, in two clauses, fuzzy sets are dissimilar. Hence, we can resolve upon these dissimilar pair and apply ALGORITHM—FRCEP to get the resolvent. Here,

$$R_1 = \sum_{U \times V} \min(\mu_{not\ HIGH}(u), 1 - \mu_{LOW}(v)) / (u, v)$$

$$R_2 = \sum_{V \times W} \min(\mu_{not\ LOW}(u), \mu_{not\ HAPPY}(w)) / (v, w)$$

We construct $R' = R_1' \cap R_2'$, where \cap denotes any fuzzy conjunction operator (here, $\cap = \min$) and

$$R_1' = \sum_{U \times V \times W} \mu_{R_1}(u, v) / (u, v, w),$$

$$R_2' = \sum_{U \times V \times W} \mu_{R_2}(v, w) / (u, v, w).$$

We modify R' with dissimilarity $s_1 = 1.0$ as we get $\mu_{R^*} = (u, v, w) = 1 - (1 - \mu_{R'}) \cdot s_1$ and we get

$$R = projR^* \text{ on } U \times W$$

Let us extend the fuzzy sets ‘HIGH’ cylindrically in the premise r on $U \times W$ as

$$R_3 = \sum_{U \times W} \mu_{HIGH} / (u, w);$$

Table 14 Rule base and observations

Rule 1:	IF the prime interest rate is HIGH THEN the stock price is LOW;
Rule 1:	IF the stock price is LOW THEN people are not HAPPY;
Observation 1:	The prime interest rate is HIGH;
Observation 2:	The prime interest rate is very HIGH;

Table 15 Equivalent scheme for observation 1

p:	The prime interest rate is not HIGH OR the stock price is LOW;
q:	The stock price is not LoW OR the people are not HAPPY;
r:	The prime interest rate is HIGH;

Table 16 Fuzzy sets in scheme

HIGH	=	0.0/0 + 0.0/2 + 0.1/4 + 0.3/6 + 0.5/8 +0.6/10 + 0.7/12 + 0.9/14 + 1.0/16 (C U)
LOW	=	1.0/10, 000 + 0.8/12,000 + 0.75/14, 000 +0.5/16,000 + 0.3/18, 000 + 0.1/20,000 +0.0/22,000 (C V)
HAPPY	=	0.0/0 + 0.25/25 + 0.5/50 + 0.75/75 +1.0/100 (C W)

and compute $R'' = R \cap R_3$ by

$$\mu_{R''}(u, w) = T(\mu_R(u, w), \mu_{R_3}(u, w)),$$

taking fuzzy conjunction \cap as t-norm T (here, $\cap = \min$);

Next, we calculate the dissimilarity measure of the fuzzy set *not HIGH* in p and the fuzzy set *HIGH* in r (denoted by s_2) and R'' is induced by the dissimilarity measure $s_2 = 1$ to get R^{**} as

$$\mu_{R^{**}}(u, w) = 1 - (1 - \mu_{R''}(u, w)) \cdot s_2;$$

At last, we project R^{**} on W such that

$$C' = \sum_{w'} \sup_v \mu_{R^{**}}(u, w) = 0.50/0 + 0.50/25 + 0.50/50 + 0.25/75 + 0.0/100$$

and $S(C', \textit{not HAPPY}) = 0.75$, which is rather similar to *not HAPPY*.

(ii) **For Observation 2:** In this case, the observation is changed into

r : The prime interest rate is *very HIGH*

to the scheme given in Table 15.

Describing the membership grades of the fuzzy set ‘*very HIGH*’ in q as a^2 , where a is the membership grades of the fuzzy set ‘*HIGH*’ defined over U , defining other parameters in the same way and executing in a similar way, we can derive a fuzzy set

$$C' = 0.46/0 + 0.46/25 + 0.46/50 + 0.37/75 + 0.16/100,$$

the similarity of which is 0.71 with the fuzzy set *not HAPPY* in the premise q , whenever dissimilarity between ‘*not HIGH*’ $\subset U$ in premise p and ‘*HIGH*’ $\subset U$ in observation r is 0.84.

Hence, in either case, we can derive a conclusion like ‘people are not so happy’ for the given observations. Therefore, our resolution method is effective in deriving a conclusion from some given statements in real life.

6 Conclusion

This chapter presented a resolution principle for fuzzy formulae based on similarity and approximate reasoning methodology. Similarity is inherent in approximate reasoning and resolution deduction can be used as a rule of inference to generate new clause from a given set of clauses. The essential idea of resolution of two clauses is to search for a literal in a clausal formula that is almost complementary to a literal in the other form. The clause formed by the disjunction of the remaining literals and subsequent removal of the pair of almost complementary literals is a logical consequence. If we put the resolvent in the set of clauses its behaviour (satisfiability) never changes. It can be applied directly to any set S of clausal formulae (not necessarily to ground clauses) to test the (un)satisfiability of S . To test the unsatisfiability it checks whether S contains the empty clause, (as a resolution deduction). This could be a powerful technique in constructing a proof of a theorem using refutation procedure. Examples cited in the chapter attempted to demonstrate how resolution can be effectively used to construct a proof of a theorem or to make a decision in real life. Inverse approximate reasoning may be applied to model different goal-directed search techniques. We apply inverse approximate reasoning method to avoid the inherent problem of GMP. Instead of testing complementary literals we use dissimilarity concept of fuzzy literals.

References

1. Robinson, J.A.: A machine oriented logic based on the resolution principle. J.ACM **12**(1), 23–41 (1965)
2. Lee, R., Chang, C.L.: Some properties of fuzzy logic. Inf. Control **19**(1), 417–431 (1971)
3. Lee, R.: Fuzzy logic and the resolution principle. J. ACM **19**(1), 109–119 (1972)
4. Shen, Z., Ding, L., Mukaidono, M.: Fuzzy resolution principle. In: Proceedings of the Eighteenth International Symposium on Multivalued Logic, Spain, May 24–26, 1988

5. Mukaidono, M.: Fuzzy inference of resolution style. In: Yager, R.R. (ed.) *Fuzzy Set and Possibility Theory*, pp. 224–231. Pergamon Press, New York (1982)
6. Dubois, D., Prade, H.: Necessity measure and the resolution principle. *IEEE Trans. Syst. Man Cybern.* **17**(3), 474–478 (1987)
7. Kim, C., Kim, D.S., Park, J.: A new fuzzy resolution principle based on the antonym. *Fuzzy Sets Syst.* **113**(2), 299–307 (2000)
8. Fontana, F.A., Formato, F.: A similarity-based resolution principle. *Int. J. Intell. Syst.* **17**(9), 853–872 (2002)
9. Sessa, M.I.: Approximate reasoning by similarity-based sld resolution. *Theor. Comput. Sci.* **275**, 389–426 (2002)
10. Habiballa, H., Novak, V.: Fuzzy general resolution. In: *Proceedings of International Conference Aplimat 2002*, pp. 199–206. Slovak Technical University, Bratislava (2002)
11. Raha, S., Ray, K.S.: Approximate reasoning based on generalised disjunctive syllogism. *Fuzzy Sets Syst.* **61**(2), 143151 (1994)
12. Mondal, B., Raha, S.: Similarity based inverse approximate reasoning. *IEEE Trans. Fuzzy Syst.* **19**(6), 10581071 (2011)
13. Mondal, B., Mazumdar, D., Raha, S.: Similarity in approximate reasoning. *Int. J. Comput. Cognit.* **4**(3), 46–56 (2006)
14. Kelly, J.J.: *The Essence of Logic*. Prentice-Hall, New Delhi (1997)
15. Zadeh, L.A.: The concept of linguistic variable and its application to approximate reasoning: Part I and part II. *Inf. Sci.* **8**, 301–357 (1975)
16. Lazzarini, B., Marcelloni, F.: Some considerations on input and output partitions to produce meaningful conclusions in fuzzy inference. *Fuzzy Sets Syst.* **113**(2), 221–235 (2000)
17. Bouchon-Meunier, B., Rifqi, M., Bothorel, S.: Towards general measures of comparison of objects. *Fuzzy Sets Syst.* **84**(2), 143–153 (1996)
18. Zwick, R., Carlstein, E., Budescu, D.V.: Measures of similarity among fuzzy concepts: a comparative analysis. *Int. J. Approx. Reason.* **1**(2), 221–242 (1987)
19. Raha, S., Pal, N.R., Ray, K.S.: Similarity based approximate reasoning: methodology and application. *IEEE Trans. Syst. Man Cybern. Part A: Syst. Hum.* **32**(4), 541547 (2002)
20. Ughetto, L., Dubois, D., Prade, H.: Implicative and conjunctive fuzzy rule -a tool for reasoning from knowledge and examples. In: *Proceedings of AAAI/IAAI'1999*, Menlo Paru, CA, USA, pp. 214–219 (1999)
21. Zadeh, L.A.: A theory of approximate reasoning. In: Hayes, J.E., Michie, D., Mikulich, L.I. (eds.) *Machine Intelligence*, vol. 9, pp. 149–194. Elsevier, New York (1979)

Authors' Biography

Banibrata Mondal received B.Sc. and M.Sc. degrees in Mathematics from Visva-Bharati, Santiniketan, West Bengal, India. He is currently working toward the Ph.D. degree with the Department of Mathematics, Visva Bharati. His current research interest includes formulation of inverse approximate reasoning based on similarity.

Swapan Raha received B.Sc. and M.Sc. in Mathematics and M.Tech. (Computer Science) from Calcutta University, Kolkata, India. He received the Ph.D. degree from Indian Statistical Institute, Kolkata. He is currently a Professor with the Department of Mathematics, Visva Bharati, Santiniketan, West Bengal, India. His research interests include fuzzy logic and approximate reasoning, fuzzy automata and fuzzy control.