Analytical Structure Characterization and Stability Analysis for a General Class of Mamdani Fuzzy Controllers

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Abstract Stability of a fuzzy control system is closely related to the analytical structure of the fuzzy controller, which is determined by its components such as input and output fuzzy sets and fuzzy rules. We first characterize the mathematical input–output structure of fuzzy controllers and then utilize the structure characteristics to advance stability analysis. We study how the components of a general class of Mamdani fuzzy controllers dictate the controller's input–output relationship. The controllers can use input fuzzy sets of any types, arbitrary fuzzy rules, arbitrary inference methods, either Zadeh or the product fuzzy logic AND operator, singleton output fuzzy sets, and the centroid defuzzifier. We theoretically prove that regardless of the choices for the other components, if and only if Zadeh fuzzy AND operator and piecewise linear (e.g., trapezoidal or triangular) input fuzzy sets are used, the fuzzy controllers become a peculiar class of nonlinear controllers with the following interesting characteristics: (1) they are linear with respect to input variables; (2) their control gains dynamically change with the input variables; and (3) they become linear controllers with constant gains around the system equilibrium point. These properties make the fuzzy controllers suitable for analysis and design using conventional control theory. This necessary and sufficient condition becomes a sufficient condition if the product AND operator is employed instead. We name the fuzzy controllers of this peculiar class type-A fuzzy controllers. Taking advantage of this new structure knowledge, we have established a necessary and sufficient local stability condition for the type-A fuzzy control systems. It can be used not only for the stability determination, but also for

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practically designing a type-A fuzzy control system that is at least stable at the equilibrium point even when model of the controlled system is mathematically unknown. Three numerical examples are provided to demonstrate the utility of our new findings.

1 Introduction

Efforts have been made to rigorously derive and study analytical structure of fuzzy controllers. By analytical structure, we mean the mathematical relationship between the input and output of a fuzzy controller. Precise understanding of the structure is fundamentally important because it can enable one to analyze and design fuzzy control systems more effectively with the aid of conventional control theory [2, 7, 14]. The analytical structure is determined by a fuzzy controller's components including input fuzzy sets, output fuzzy sets, fuzzy rules, fuzzy inference, fuzzy logic operators, and defuzzifier. Different component choices obviously result in different analytical structures. The explicit structures of some fuzzy controllers have been derived [3, 11] (recently, the explicit structures of some type-2 fuzzy controllers have been investigated [1, 15]). They are related to classical controllers such as PID control [4, 10, 13] and sliding mode control [5]. No work, however, has been reported in the literature that characterizes analytical structures into different types with respect to component choices.

Our motivations for the current study are two folds—we first characterize the mathematical input–output structure of fuzzy controllers of a broad class by class and then utilize the structure characteristics and class to advance system stability analysis. Structurally speaking, such classification can produce structure information that is broader than what an individual analytical structure can because one class can cover many different fuzzy controllers with various nonlinear input–output structures. Subsequently, the structure classification can provide useful guidelines for the controller developer to choose appropriate types of the components (e.g., triangular fuzzy sets instead of Gaussian ones) at the early development stage, reducing time and effort on design and analysis in practice. Up to date, triangular and trapezoidal fuzzy sets have been most widely used for input variable fuzzification whereas other types (e.g., Gaussian and bell-shape fuzzy sets) are utilized to a lesser extent. As to fuzzy logic AND operators, Zadeh type and the product type are used far more often than any other types. As a matter of fact, the remaining types have hardly been used. Historically, these current preferred choices were determined mainly based on the results of a great deal of the trial-and-error effort, computer simulation study and the empirical development of successful fuzzy control applications. No analytically rigorous reasons in the context of conventional control theory have been given in the literature. In this paper, we attempt to provide a more rigorous answer and show that these popular choices are indeed theoretically sensible.

Another important benefit of classifying fuzzy controller structures is to make stability analysis more tractable and effective for a wide class of fuzzy controllers as one could concentrate on one class of fuzzy controllers a time instead of on all fuzzy controllers at once. Consequently, the stability results could be less conservative and more practically meaningful. In this paper, we will develop an analytical stability criterion that can be used to judge (local) stability of a broad class of fuzzy control systems regardless of the availability of the mathematical model of the system under control. The criterion can also be utilized to analytically design a stable fuzzy control systems not only when the system model is given, but also when it is unknown.

2 Configuration of a General Class of Mamdani Fuzzy Controllers

This class of Mamdani fuzzy controllers has m input variables, designated as $x_i(n), i = 1, 2, \ldots, m$, where n signifies sampling instance. $x_i(n)$ is computed using the current and/or historical output of a dynamic plant to be controlled (e.g., $y(n)$ and $y(n-1)$) as well as target output signal $S(n)$. This means the input space to be m-dimensional. $x_i(n)$ is multiplied by a scaling factor k_i, resulting in the scaled input variable $X_i(n)$. The universe of discourse for $X_i(n)$ is $[a_i, b_i]$. Assuming that each of the intervals is divided into M_i-1 subintervals, all the subintervals of the m intervals produce a total of

$$\Phi = \prod_{i=1}^{m} (M_i - 1)$$

m-dimensional "blocks" (i.e., divisions), each is designated as $\Theta_i, i = 1, \ldots, \Phi$.

M_i input fuzzy sets are defined over $[a_i, b_i]$. Like most fuzzy controllers in the literature, each subinterval has two fuzzy sets defined over it. The j-th fuzzy set for $X_i(n)$ is designated as $\tilde{A}_{i,j}$ whose membership function is denoted $\mu_{\tilde{A}_{i,j}}(x_i)$. The fuzzy sets can be any types. Of particular interest to this study is the piecewise linear fuzzy sets. Two examples of such fuzzy sets are illustrated in Fig. 1 (these examples are hypothetical and similar fuzzy sets are rarely used by fuzzy control in practice. We show this type of more dramatic examples to ensure a broad coverage). Note that the trapezoidal and triangular types, the most widely-used types among all the fuzzy set types available, are the special cases of the piecewise linear type. If $\mu_{\tilde{A}_{i,j}}(x_i)$ is of the trapezoidal type, it can be represented by

$$\mu_{\tilde{A}_{i,j}}(x_i) = \beta_{ij} \cdot x_i(n) + \lambda_{ij}$$

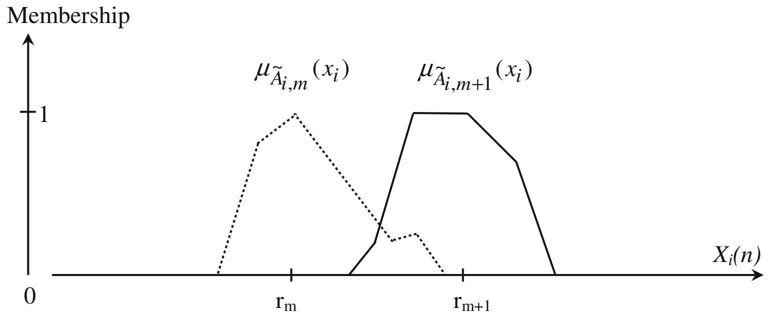

Fig. 1 Illustrative definition of piecewise linear fuzzy sets. Shown are two hypothetical ones. The widely-used *triangular* and *trapezoidal* fuzzy sets are their special cases

where λ_{ij} and β_{ij} are constants. β_{ij} is a constant taking different values in the different segments of $[a_i, b_i]$. When $x_i(n) \in [a_i, a_i^{'}], \beta_{ij} \geq 0$ and when $x_i(n) \in [b_i^{'}, b_i], \beta_{ij} \leq 0$, where $a_i^{'} < b_i^{'}$. When $x_i(n) \in [a_i^{'}, b_i^{'}], \mu_{\tilde{A}_{i,j}}(x_i) = 1$. Obviously, the trapezoidal type becomes the triangular type if $a_i^{'} = b_i^{'}$.

The fuzzy controllers use a total of $\Omega = \prod_{i=1}^{m} M_i$ fuzzy rules, each of which is in the following format:

$$\text{IF } X_1(n) \text{ is } \tilde{A}_{1,I_1} \text{ AND}\ldots\text{AND } X_m(n) \text{ is } \tilde{A}_{m,I_m} \text{ THEN } u(n) \text{ is } \tilde{V}_k$$

where $\tilde{V}_k$ is an output fuzzy set of the singleton type for the output variable $u(n)$ whose universe of discourse is $[U_L, U_H]$. That is, $\tilde{V}_k$ is nonzero only at one location in the interval and the nonzero value is designated as V_k. The fuzzy AND operator can be Zadeh type (i.e., the minimum operator) or the product type, but not both at the same time for the rule base. As for reasoning, any fuzzy inference method may be used in the rules. It will produce the same inference outcome because the output fuzzy sets are of the singleton type. The popular centroid defuzzifier is employed to combine the inference outcomes of the individual rules:

$$u(n) = \frac{\sum_{h=1}^{\Omega} \mu_h(\mathbf{x}) \cdot V_h}{\sum_{h=1}^{\Omega} \mu_h(\mathbf{x})} \tag{1}$$

Here, $\mathbf{x} = [x_1(n) \cdots x_m(n)]$ is the input vector and $\mu_h(\mathbf{x})$ is the resulting membership of executing all the fuzzy logic AND operations in the h-th rule whereas V_h signifies the nonzero value of the singleton output fuzzy set in the rule.

3 Structure Characterization and Local Stability Determination

3.1 Structure Characterization

In control theory, a general nonlinear controller is described by $u(n) = f(\mathbf{x})$ and a controller is linear if f is linear, that is

$$u(n) = \zeta_1 x_1(n) + \cdots + \zeta_m x_m(n) + \zeta_0 \tag{2}$$

where ζ_i, $i = 1, \ldots, m$, is a constant gain and ζ_0 is a constant control offset term. We use this formalism as a base to classify the fuzzy controllers into two types.

Definition 1 A fuzzy controller of the general class is defined as *local type-A fuzzy controller* in an m-dimensional region of input space if its input–output relation satisfies

$$u(n) = c_1(\mathbf{x})x_1(n) + \cdots + c_m(\mathbf{x})x_m(n) + c_0(\mathbf{x}) \tag{3}$$

$c_i(\mathbf{x})$, gain for $x_i(n)$, must be either a constant or a fractional expression whose numerator is constant. All the terms, from $x_1(n)$ to $x_m(n)$, must be present. $c_0(\mathbf{x})$ must be either a constant (including 0) or a fractional expression whose numerator does not contain $a_i x_i(n), i = 1, \ldots, m$, or their linear combination (a_i is constant).

Definition 2 A fuzzy controller is defined as *local type-B fuzzy controller* in the region if it is not of local type-A.

Definition 3 If a fuzzy controller is of local type-A in every region of input space, it is defined as a global type-A fuzzy controller. Otherwise, it is defined as a global type-B fuzzy controller.

A type-A fuzzy controller, global or local, possesses interesting properties—it is linear with respect to its input variables and nonlinear in terms of the gains, $c_i(\mathbf{x})$, which vary with $\mathbf{x}$. On the other hand, a type-B fuzzy controller cannot be linear with respect to input variables.

We are now ready to formally characterize the structure of the fuzzy controllers.

Theorem 1 A *necessary and sufficient* condition for a Mamdani fuzzy controller of the general class that uses Zadeh AND operator to be of global type-A is that all of its input fuzzy sets are piecewise linear.

Proof We first prove the necessity of the condition—if at least one of the input fuzzy sets involving a particular m-dimensional region of the input space is not piecewise linear, the fuzzy controller cannot be of global type-A.

Without loss of generality, assume that $\mu_{\tilde{A}_{i,j}}(x_i)$, a non-piecewise linear function, is the membership function of the j-th fuzzy set for $x_i(n)$. To derive the input–output analytical relationship for the fuzzy controllers, the input space must be divided into many m-dimensional regions in such a way that in each of the regions one and only one of the membership functions in a fuzzy rule will always be the smallest [13]. As a result, that membership function will be the resulting membership function for that rule after Zadeh AND operation is carried out. The total number of such regions depends on m and the shape of the input fuzzy sets. Inevitably, $\mu_{\tilde{A}_{i,j}}(x_i)$ will be the smallest membership function in at least one region (or this fuzzy set would not be useful and hence should be removed used—cannot be true). After this step of considering the individual rules, the obtained regions must be put together (i.e., superimposed) so that all the rules can be considered at the same time. This process will create the final m-dimensional regions. Let us assume that there are a total of Φ regions, in each of which, up to 2^m fuzzy rules will be involved (here we assume, without loss of generality, that the input fuzzy set for $x_i(n)$ intersects only with its adjacent (two) fuzzy sets and intersects only once with each of them). Let Ψ_i be the number of the rules involved in the i-th final region and suppose that in the region the input fuzzy sets for $x_1(n)$ is the smallest for α_1 rules, $x_2(n)$ for α_2 rules,…, and $x_m(n)$ for α_m rules, where $\sum_{p=1}^{m} \alpha_p = \Psi_i$ (some α_p may be 0). For the i-th final region, we get the input–output relation:

$$u(n) = \frac{\sum_{j=1}^{m} \sum_{k=1}^{\alpha_j} \mu_{h_k}(x_j) \cdot V_{h_k}}{\sum_{j=1}^{m} \sum_{k=1}^{\alpha_j} \mu_{h_k}(x_j)} \tag{4}$$

where $\mu_{h_k}(x_j)$ represents the resulting smallest membership function and $V_{h_k}$is the singleton output fuzzy set involved in the fuzzy rule. Because V_{h_k} represents the rule consequent and all the rule consequents should not have the same singleton fuzzy sets, thus the above equation should not/cannot be equal to V_{h_k} for all the final regions. Hence, if at least one input fuzzy set, say $\mu_{\tilde{A}_{i,j}}(x_i)$, is not linear or piecewise linear in one of the final regions, $u(n)$ in that region will not be able to be written in the form of (3) because $x_i(n)$ cannot be factored out of $\mu_{\tilde{A}_{i,j}}(x_i)$. That is to say that the fuzzy controller cannot be a local type-A controller in that region and therefore the fuzzy controller cannot be a global type-A controller.

Now let us prove the condition to be sufficient. When all the input fuzzy sets are linear or piecewise linear, the smallest membership function in every final region will always be a linear function of $x_i(n)$ (note that a piecewise linear membership function can always be decomposed into a series of linear membership functions). Suppose that the linear fuzzy sets are

$$\mu_{h_k}(x_j) = \beta_{h_k} x_j(n) + \lambda_{h_k}.$$

As a result,

$$u(n)=\frac{\sum_{j=1}^{m}\sum_{k=1}^{\alpha_j}\mu_{h_k}(x_j)\cdot V_{h_k}}{\sum_{j=1}^{m}\sum_{k=1}^{\alpha_j}\mu_{h_k}(x_j)}=\sum_{j=1}^{m}\frac{\sum_{k=1}^{\alpha_j}\beta_{h_k}\cdot V_{h_k}}{\sum_{j=1}^{m}\sum_{k=1}^{\alpha_j}\mu_{h_k}(x_j)}x_j(n)+\frac{\sum_{j=1}^{m}\sum_{k=1}^{\alpha_j}\lambda_{h_k}\cdot V_{h_k}}{\sum_{j=1}^{m}\sum_{k=1}^{\alpha_j}\mu_{h_k}(x_j)}$$
$$=c_1(\mathbf{x})x_1(n)+\cdots+c_m(\mathbf{x})x_m(n)+c_0(\mathbf{x})$$

where

$$c_j(\mathbf{x})=\frac{\sum_{k=1}^{\alpha_j}\beta_{h_k}\cdot V_{h_k}}{\sum_{j=1}^{m}\sum_{k=1}^{\alpha_j}\mu_{h_k}(x_j)},\quad j=1,2,\ldots,m \text{ and } c_0(\mathbf{x})=\frac{\sum_{j=1}^{m}\sum_{k=1}^{\alpha_j}\lambda_{h_k}\cdot V_{h_k}}{\sum_{j=1}^{m}\sum_{k=1}^{\alpha_j}\mu_{h_k}(x_j)}$$

Unless the fuzzy sets are such chosen that $\sum_{j=1}^{m}\sum_{k=1}^{\alpha_j}\mu_{h_k}(x_j)$ is a constant, the value of $c_j(\mathbf{x})$ changes with $x_j(n)$ and hence is variable gain for $x_j(n)$. According to Definition 3, the fuzzy controller is a global type-A fuzzy controller. ■

We can rigorously show that Theorem 1 also holds for the fuzzy controllers that use the product AND operator when the popular input fuzzy sets in the literature are utilized. They include triangular, trapezoidal, Gaussian, Bell, Generalized Bell, Sigmoid, and so on. This kind of result, however, is not comprehensive enough. We know that the necessity portion of the theorem will not hold if the input fuzzy sets are allowed to be any nonlinear functions. Subsequently, we decided to present the following sufficient condition. This result can be extended to become a necessary and sufficient condition if some (mild) mathematical constraint is applied to the fuzzy sets. The constraint will most likely not affect the practicality of the fuzzy controllers, if at all.

In the next section, we will conduct local stability analysis. For that purpose, we only need to know the properties of the relevant part of the fuzzy controllers—the local fuzzy controller covering the system equilibrium point. The following results are obvious in light of Theorem 1 and Definitions 1 to 3.

Corollary 1 A necessary and sufficient condition for a Mamdani fuzzy controller of the general class that uses Zadeh AND operator to be of local type-A around the system equilibrium point is that all the input fuzzy sets covering the point are linear.

Note that the input fuzzy sets have to be linear, not piecewise linear, because the latter is not sensible when we consider only a point.

Corollary 2 A Mamdani fuzzy controller of the general class that uses Zadeh AND operator is a global type-B fuzzy controller if at least one input fuzzy set is not piecewise linear.

Theorem 2 A *sufficient* condition for a Mamdani fuzzy controller of the general class that uses the product AND operator to be of global type-A is that all of its input fuzzy sets are piecewise linear.

Proof Due to the product fuzzy AND operator, in each m-dimensional "block", the combined membership function for the h-th rule can be represented as

$$\mu_h(\mathbf{x}) = \mu_{\tilde{A}_{1,J_1}}(x_1) \times \cdots \times \mu_{\tilde{A}_{m,J_m}}(x_m)$$

and hence

$$\begin{aligned} u(n) &= \frac{\sum_{h=1}^{\Omega} \mu_h(\mathbf{x}) \cdot V_h}{\sum_{h=1}^{\Omega} \mu_h(\mathbf{x})} = \sum_{h=1}^{\Omega} \frac{V_h}{\sum_{h=1}^{\Omega} \mu_h(\mathbf{x})} \mu_h(\mathbf{x}) \\ &= \sum_{h=1}^{\Omega} \theta_h(\mathbf{x}) \times \mu_h(\mathbf{x}) = \sum_{h=1}^{\Omega} \theta_h(\mathbf{x}) \times \mu_{\tilde{A}_{1,J_1}}(x_1) \times \cdots \times \mu_{\tilde{A}_{m,J_m}}(x_m) \end{aligned} \tag{5}$$

where

$$\theta_h(\mathbf{x}) = \frac{V_h}{\sum_{h=1}^{\Omega} \mu_h(\mathbf{x})}.$$

Note that due to the use of the product AND operation, $\mu_h(\mathbf{x})$ here differs from the case above that uses Zadeh AND operator. Assume that in the entire input space, all the input fuzzy sets are linear or piecewise linear. Then in any Θ_i, carrying out all the multiplication operations in (5) and simplifying the resultant expression gives

$$\begin{aligned} u(n) &= \sum_{i=1}^{m} a_i(\mathbf{x}) \cdot x_i(n) + \sum_{i=1}^{m} \sum_{j>i}^{m} a_{ij}(\mathbf{x}) \cdot x_i(n) \cdot x_j(n) + \cdots + a_{1\ldots m}(\mathbf{x}) x_1(n) \cdots x_m(n) + \delta(\mathbf{x}) \\ &= \sum_{i=1}^{m} a_i(\mathbf{x}) \cdot x_i(n) + \Gamma(\mathbf{x}) \end{aligned}$$

where

$$\Gamma(\mathbf{x}) = \sum_{i=1}^{m} \sum_{j>i}^{m} a_{ij}(\mathbf{x}) \cdot x_i(n) \cdot x_j(n) + \cdots + a_{1\ldots m}(\mathbf{x}) x_1(n) \cdots x_m(n) + \delta(\mathbf{x})$$

The denominators of $a_i(\mathbf{x})$ and $\Gamma(\mathbf{x})$ are the same—$\sum_{h=1}^{\Omega} \mu_h(\mathbf{x})$. The numerators of $a_i(\mathbf{x})$ are constants determined by the slops and y-intercepts of the fuzzy sets. The numerator of $\Gamma(\mathbf{x})$ contains the cross-products of $x_1(n), \ldots, x_m(n)$ as well as a constant term, all determined by the slopes and y-intercepts of the fuzzy sets. Note that none of the terms in the numerator can be expressed as $a_i x_i(n), i = 1, \ldots, m$, or their linear combination (a_i is constant). By Definition 1, the fuzzy controller is a global type-A fuzzy controller. ■

For the same reason as for obtaining Corollary 1, we state Corollary 3 for Theorem 2.

Corollary 3 A sufficient condition for a Mamdani fuzzy controller of the general class that uses the product AND operator to be of local type-A around the system equilibrium point is that all the input fuzzy sets covering the point are linear.

Note that a Mamdani fuzzy controller of the general class that uses the product AND operator may (very likely) be a global type-B fuzzy controller if at least one input fuzzy set is not piecewise linear.

In a sense, type-A fuzzy controllers are most similar to the linear controller as far as the controller structure is concerned. These fuzzy controllers can be treated as nonlinear controllers with variable gains. The variable gains are mathematically complicated in general. Deriving their explicit expressions is possible for some configurations [12] but impossible for others. The variable gains can empower the fuzzy controllers to outperform the linear controller (e.g., PID control), especially when the system under control is nonlinear or with time delay. Our previous study of Mamdani as well as TS fuzzy controllers with variable gains has shown this point [12, 13]. This may provide one theoretical justification/explanation for the dominant use of triangular/trapezoidal input fuzzy sets and Zadeh or product fuzzy AND operator in the current practice of fuzzy control. There are many other choices available, but most of them have hardly been used. Such strong preferences have existed in the literature for many years with little theoretical support. The prior justifications are based on perceived component simplicity and empirical observation of good control performance attributed to these particular preferred selections.

3.2 Local Stability Analysis

We now turn our attention to local stability determination of the fuzzy controllers regulating nonlinear dynamic systems. Without loss of generality, assume that when a system to be controlled is at the equilibrium point of our interest (i.e., $y(n) = 0$), $\mathbf{x} = \mathbf{0}$. We want to study two related issues: (1) conditions for the fuzzy control system to be stable at least in the area around the equilibrium point, and (2) design of the fuzzy control system that will be at least stable in the area.

At $\mathbf{x} = \mathbf{0}$, a local fuzzy controller satisfying Corollary 1 or 3 becomes a local linear controller:

$$u(n) = c_1(\mathbf{0})x_1(n) + \cdots + c_m(\mathbf{0})x_m(n) + c_0(\mathbf{0}) \tag{6}$$

where $c_0(\mathbf{0}), \ldots, c_m(\mathbf{0})$ are the values of $c_i(\mathbf{x})$ and are the constant gains. The term of $c_0(\mathbf{0})$ does not affect system stability; it will be dropped from the stability study below. If both the system to be controlled and the local fuzzy controller are linearizable at the equilibrium point, then the system stability at that point can be decided by applying Lyapunov's linearization method [6] to the linearized local fuzzy controller (i.e. (2)) and the linearized system. Thus, we obtain the following result.

Theorem 3 Suppose that a fuzzy controller of the general class is used to control a nonlinear system that is linearizable at the equilibrium point. If the fuzzy controller is a local type-A controller at the equilibrium point that is also linearizable, the fuzzy control system is locally stable (or unstable) at the equilibrium point if and only if the linearized control system is strictly stable (or unstable) at the equilibrium point.

Proof The conclusion can be obtained using Lyapunov's linearization method (see [10] for similar proof). The method assumes the nonlinear control system to be continuously differentiable at the equilibrium point. In essence, it states that if the linearized system is strictly stable (or unstable) at the equilibrium point, then the equilibrium point is locally stable (or unstable) for the original nonlinear system. ■

The linearizability test must be met before the stability condition can be used because it is the precondition for the theorem. Our previous study indicates that the use of Zadeh fuzzy logic AND operator results in more than one control structure to cover $\mathbf{x} = \mathbf{0}$, which can sometimes fail the test. This is usually not true for the fuzzy controllers using the product AND operator because in most cases there is only one control structure for the entire area around $\mathbf{x} = \mathbf{0}$ [12]. A test failure only means inapplicability of the theorem; it does not imply system instability.

Theorem 3 offers some practically important advantages. First, it is a necessary and sufficient condition. Unlike sufficient conditions or necessary conditions, it is not conservative and is the "tightest" possible stability condition. Second, only explicit structure of the local fuzzy controller covering the equilibrium point is required. As long as all the input fuzzy sets covering the equilibrium point are linear, the theorem is usable. Third, the theorem can be used not only when the system model is available, but also when it is unavailable but is known linearizable at the equilibrium point. (Most physical systems are linearizable.) In the latter case, one can devise a linear controller and use it to control the system. If the resulting control system is observed to be locally stable (unstable), then the same system controlled by a linearized fuzzy controller whose gains at the equilibrium point equal to the gains of the linear controller will be locally stable (unstable). This design approach can be attractive as in practice, physical systems are often too complex and/or costly to be precisely modeled.

4 Numerical Examples

Since Theorems 1 and 2 and their corollaries are straightforward to apply, no numerical examples are needed. We now use three examples to illustrate the utility of Theorem 3 and its above-mentioned advantages in system analysis and design.

Example 1 (Local stability determination when system model is known) Assume that a continuous-time system to be controlled is

$$y''(t) + 20y'(t) + 5\sin(y(t)) = 4u(t)$$

and the sampling period for the system is 0.1. Suppose that a type-A fuzzy controller has been designed. It uses the product fuzzy AND operator and its input variables are $x_1(n) = S(n) - y(n)$ and $x_2(n) = y(n-1) - y(n)$, where $S(n)$ is the reference input signal. The controller's output variable is $\Delta u(n) = u(n) - u(n-1)$. There can be numerous fuzzy sets for the scaled input variables $X_1(n)$ and $X_2(n)$; but to determine the local stability one only needs to know whether the fuzzy sets covering the area around $\mathbf{x} = \mathbf{0}$ are linear and, if so, what their mathematical expressions are. Supposedly they are

$$\mu_{\tilde{A}_{1,1}}(x_1) = -0.95x_1(n) + 0.4, \quad \mu_{\tilde{A}_{1,2}}(x_1) = x_1(n) + 0.5,$$

for $x_1(n) \in [-0.5, 0.5]$ and

$$\mu_{\tilde{A}_{2,1}}(x_2) = -1.1x_2(n) + 0.5, \quad \mu_{\tilde{A}_{2,2}}(x_2) = 0.9x_2(n) + 0.2$$

for $x_2(n) \in [-0.5, 0.5]$.

The four fuzzy rules resulted from the four combinations of these fuzzy sets are

IF $X_1(n)$ is $\tilde{A}_{1,1}$ AND $X_2(n)$ is $\tilde{A}_{2,1}$ THEN $\Delta u(n)$ is $\tilde{V}_1$
IF $X_1(n)$ is $\tilde{A}_{1,1}$ AND $X_2(n)$ is $\tilde{A}_{2,2}$ THEN $\Delta u(n)$ is $\tilde{V}_2$
IF $X_1(n)$ is $\tilde{A}_{1,2}$ AND $X_2(n)$ is $\tilde{A}_{2,1}$ THEN $\Delta u(n)$ is $\tilde{V}_3$
IF $X_1(n)$ is $\tilde{A}_{1,2}$ AND $X_2(n)$ is $\tilde{A}_{2,2}$ THEN $\Delta u(n)$ is $\tilde{V}_4$

where $\tilde{V}_1, \tilde{V}_2, \tilde{V}_3$, and $\tilde{V}_4$ are singleton fuzzy sets whose $\Delta u(n)$ values corresponding to the nonzero memberships of these sets are at 1, 0.71, −1.083, and −1, respectively.

The question is: Is this fuzzy control system stable at $\mathbf{x} = \mathbf{0}$?

Solution Due to the use of the linear fuzzy sets, the local fuzzy controller covering the region around $\mathbf{x} = \mathbf{0}$ satisfies Corollary 3 and hence is a local type-A fuzzy controller. For the stability determination, we need to have the explicit structure of this local fuzzy controller, which can be derived by plugging the four fuzzy sets and the four fuzzy rules into (1). The result is

$$\Delta u(n) = c_1(\mathbf{x})x_1(n) + c_2(\mathbf{x})x_2(n) + c_0(\mathbf{x})$$

where

$$c_1(\mathbf{x}) = \frac{1.23578}{0.035x_1(n) - 0.18x_2(n) - 0.01x_1(n)x_2(n) + 0.63},$$
$$c_2(\mathbf{x}) = \frac{0.109605}{0.035x_1(n) - 0.18x_2(n) - 0.01x_1(n)x_2(n) + 0.63},$$
$$c_0(\mathbf{x}) = -\frac{0.00914327}{0.035x_1(n) - 0.18x_2(n) - 0.01x_1(n)x_2(n) + 0.63}.$$

The controlled system is obviously linearizable at $\mathbf{x} = \mathbf{0}$ and the linearized system is

$$y''(t) + 20y'(t) + 5y(t) = 4u(t).$$

With the sampling period 0.1, the discrete-time pulse transfer function is

$$H(z) = \frac{Y(z)}{U(z)} = \frac{0.01131\ \mathrm{z} + 0.005913}{\mathrm{z}^2 - 1.114\ \mathrm{z} + 0.1353}.$$

The local fuzzy controller is obviously linearizable and the resultant controller is

$$\Delta u(n) = 1.96155x_1(n) + 0.173977x_2(n) - 0.0145131. \tag{7}$$

For the stability determination, one only needs to consider $\Delta u(n) = 1.96155e(n) + 0.173977r(n)$. Because

$$\begin{aligned} u(n) &= u(n-1) + \Delta u(n) = u(n-1) + 1.96155x_1(n) + 0.173977x_2(n) \\ &= u(n-1) + 2.135527x_1(n) - 0.173977x_1(n-1), \end{aligned}$$

the transfer function of the linearized local fuzzy controller is

$$C(z) = \frac{U(z)}{X_1(z)} = \frac{2.135527z - 0.173977}{z - 1}.$$

The closed-loop control system at $\mathbf{x} = \mathbf{0}$ is

$$\frac{H(z)C(z)}{1 + H(z)C(z)} = \frac{0.024159\ (\mathrm{z} + 0.5226)\ (\mathrm{z} - 0.08146)}{(\mathrm{z} - 0.1376)\ (\mathrm{z}^2 - 1.952\mathrm{z} + 0.9913)}.$$

The poles are $z = 0.1376$, and $z = 0.9760 \pm 0.1965\mathrm{i}$, all of which are inside the unit circle. Therefore, the linearized control system is stable at $\mathbf{x} = \mathbf{0}$ stable, so is the local fuzzy control system.

In next example, we show how to determine the local stability even when system model is unknown.

Example 2 (Local stability determination when system model is unknown) Suppose that the system model in Example 1 is unavailable but is known to be linearizable at $\mathbf{x} = \mathbf{0}$. Also, assume that we have devised a linear PI controller $\Delta u(n) = 1.96155x_1(n) + 0.173977x_2(n)$ and found it to be able to control the system stably at $\mathbf{x} = \mathbf{0}$. Note that this PI controller is exactly the same as the local fuzzy controller (4) without the constant offset term. With these modifications, can the local stability of the fuzzy control system in Example 1 be determined ?

Solution According to Example 1, the new fuzzy controller system is linearizable. Thus Theorem 3 is applicable. Since the PI control system containing $\Delta u(n) = 1.96155x_1(n) + 0.173977x_2(n)$, which is the linearized local fuzzy controller, is known to be locally stable, the fuzzy control system is logically locally stable too.

Example 3 (Design of at least locally stable fuzzy control system when system model is unknown) Suppose that the $\Delta u(n)$values for $\tilde{V}_1, \tilde{V}_2, \tilde{V}_3$, and $\tilde{V}_4$ are not given in the above example, but all the other conditions remain the same. How should one design their values so that the resulting fuzzy control system is stable at least at $\mathbf{x} = \mathbf{0}$?

Solution It can be derived that the local fuzzy controller is $\Delta u(n) = c_1(\mathbf{x})x_1(n) + c_2(\mathbf{x})x_2(n) + c_0(\mathbf{x})$ where

$$c_1(\mathbf{x}) = \frac{0.2V_1 + 0.5V_2 - 0.19V_3 - 0.475V_4}{0.035x_1(n) - 0.18x_2(n) - 0.01x_1(n)x_2(n) + 0.63},$$
$$c_2(\mathbf{x}) = \frac{0.45V_1 - 0.55V_2 + 0.36V_3 - 0.44V_4}{0.035x_1(n) - 0.18x_2(n) - 0.01x_1(n)x_2(n) + 0.63},$$
$$c_0(\mathbf{x}) = \frac{0.1V_1 + 0.25V_2 + 0.08V_3 + 0.2V_4 + x_1^2(n)(0.9V_1 - 1.1V_2 - 0.855V_3 + 1.045V_4)}{0.035x_1(n) - 0.18x_2(n) - 0.01x_1(n)x_2(n) + 0.63}.$$

Thus,

$$c_1(\mathbf{0}) = 0.3175V_1 + 0.7937V_2 - 0.3019V_3 - 0.7540V_4,$$
$$c_2(\mathbf{0}) = 0.7143V_1 - 0.8730V_2 + 0.5714V_3 - 0.6984V_4,$$
$$c_0(\mathbf{0}) = 0.1587V_1 + 0.3968V_2 + 0.1270V_3 + 0.3175V_4.$$

We know from Example 2 that the linear PI controller $\Delta u(n) = 1.96155x_1(n) + 0.173977x_2(n)$, which is the same as the local fuzzy controller (4) without the constant term, can control the system stably at least at $\mathbf{x} = \mathbf{0}$. Thus, the fuzzy control system of interest will be locally stable too if the values of the design parameters satisfy the following simultaneous equations:

$$\begin{cases} 0.1587V_1 + 0.3968V_2 + 0.1270V_3 + 0.3175V_4 = 0 \\ 0.3175V_1 + 0.7937V_2 - 0.3019V_3 - 0.7540V_4 = 1.96155 \\ 0.7143V_1 - 0.8730V_2 + 0.5714V_3 - 0.6984V = 0.173977. \end{cases}$$

The number of solution set is infinite, and every set achieves the local stability. One set, for instance, is: $V_1 = 1, V_2 = -0.40, V_3 = -1.25$, and $V_4 = -0.5$.

5 Conclusion

We have achieved two objectives: (1) to establish the conditions for a subset of a general class of Mamdani fuzzy controllers to be a specific type of nonlinear controllers described in (3), and (2) to utilize these conditions and establish a tight local stability criterion for analyzing or designing the fuzzy control systems even when the controlled system model is mathematically unavailable. This type of controllers has some desirable characteristics suitable for analysis and design using conventional control theory.

Based on our results, we recommend that the trapezoidal and triangular fuzzy sets, the only two widely-used piecewise linear types, be used for input fuzzy sets as the first choice. As we have demonstrated theoretically and through examples, the benefits of doing so include (1) clearer connection between the fuzzy controllers and conventional control (2) easier (local) stability analysis, and (3) more practically meaningful system design.

References

1. Du, X., Ying, H.: Derivation and analysis of the analytical structures of the interval type-2 fuzzy PI and PD controllers. IEEE Trans. Fuzzy Syst. **8**, 802–814 (2010)
2. Farinwata, S.S., Vachtsevanos, G.: Stability analysis of the fuzzy logic controller designed by the phase portrait assignment algorithm. In: Proceedings of the 2nd IEEE International conference Fuzzy Systems, San Francisco, pp. 1377–1382 (1993)
3. Langari, R.: A nonlinear formulation of a class of fuzzy linguistic control algorithms. In: Proceedings of the 1992 American Control Conference, Chicago, IL (1992)
4. Lewis, F.L., Liu, K.: Towards a paradigm for fuzzy logic control. Automatica **32**, 167–181 (1996)
5. Li, H.X., Gatland, H.B.: A new methodology for designing a fuzzy logic controller. IEEE Trans. Syst. Man Cybern. **25**, 505–512 (1995)
6. Slotine, J.-J.E., Li, W.: Applied Nonlinear Control. Prentice Hall, Englewood Cliffs (1991)
7. Wang, L.X.: Adaptive Fuzzy Systems and Control. PTR Prentice Hall, Upper Saddle River (1994)
8. Wang, H.O., Tanaka, K., Griffin, M.F.: An approach to fuzzy control of nonlinear systems: stability and design issue. IEEE Trans. Fuzzy Syst. **4**, 14–23 (1996)
9. Yager, R.R., Filev, D.P.: Essentials of Fuzzy Modeling and Control. Wiley, New York (1994)

10. Ying, H.: The simplest fuzzy controllers using different inference methods are different nonlinear proportional-integral controllers with variable gains. Automatica **29**, 1579–1589 (1993)
11. Ying, H.: Practical design of nonlinear fuzzy controllers with stability analysis for regulating processes with unknown mathematical models. Automatica **30**, 1185–1195 (1994)
12. Ying, H.: Fuzzy Control and Modeling: Analytical Foundations and Applications. IEEE Press, New York (2000)
13. Ying, H., Siler, W., Buckley, J.J.: Fuzzy control theory: a nonlinear case. Automatica **26**, 513–520 (1990)
14. Zeng, X.J., Singh, M.G.: Decomposition property of fuzzy systems and its applications. IEEE Trans. Fuzzy Syst. **4**, 167–181 (1996)
15. Zhou, H., Ying, Hao: A method for deriving the analytical structure of a broad class of typical interval type-2 Mamdani fuzzy controllers. IEEE Trans. Fuzzy Syst. **21**, 447–458 (2013)