Image Thresholding by Grouping Functions: Application to MRI Images

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Abstract In this work we present a thresholding algorithm for greyscale images. Our proposal is the use of grouping functions to find the best threshold. These functions are able to measure the belongingness of a grey intensity to the background or to the object of the image, so the best threshold is the one associated with the highest grouping value.

1 Introduction

One of the most used techniques in image segmentation is thresholding or segmentation by greylevels [1], [2], [3]. In thresholding, the different objects of the image are characterized just by the intensity of each pixel. This technique consists in finding a threshold t such that the pixels whose intensities are lower or equal to t belong to the background of the image while the intensities that are greater than t belong to the object, or vice versa [4]. The advantages of this kind of procedures

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with respect to other segmentation algorithms are the simplicity and low computational cost. This is why thresholding is commonly used as a first step of more complex segmentation algorithms.

In this work we present a new thresholding algorithm, generalizing previous fuzzy approaches [4], [5]. Our proposal is based on the construction, for every possible grey intensity, of two fuzzy sets (Q_{B_t} and Q_{O_t}) representing the belongingness of every greylevel to the background $\left(\mu_{Q_{B_t}}(q)\right)$ and to the object $\left(\mu_{Q_{O_t}}(q)\right)$ of the image respectively. The objective is to find the threshold for wich the belongingness of every grey intensity to the object or to the background is maximum $\left(\mu_{Q_{B_t}} = 1 \text{ or } \mu_{Q_{O_t}} = 1\right)$, so we are completely sure that those pixels belong to the background or to the object of the image.

To solve the aim of the work, we propose the use of grouping functions, as a bivariate aggregation function that gets the maximum value if and only if one of the arguments is 1. In this work we study the axiomatization of these functions, propose some construction methods and relate grouping functions to overlap functions [6].

We also show an illustrative example for a medical imaging application, where we have to segment some magnetic resonance images. The purpose is to separate the gray matter from the white matter of a brain, which is a very helpful process to evaluate some diseases like Alzheimer or schizophrenia.

The rest of the contribution is organized in the following way. We start recalling some preliminary concepts in Sect. 2. In Sect. 3 we study grouping functions, their relations with overlap functions and some construction methods. In Sect. 4 we present our image thresholding algorithm, and in Sect. 5 we show an illustrative example. We finish with some conclusions in Sect. 6.

2 Preliminaries

A strict negation [7] is a continuous and strictly decreasing function $N : [0,1]^2 \rightarrow [0,1]$ such that $N(0) = 1$ and $N(1) = 0$. A strong negation is a strict negation that is also involutive, it means, $N(N(x)) = x$ for all $x \in [0,1]$.

A triangular norm (t-norm) is a symmetric and associative bivariate aggregation function $T : [0,1]^2 \rightarrow [0,1]$ such that $T(x,1) = x$ for all $x \in [0,1]$. Some examples of t-norms are minimum function $T_M(x,y) = \min(x,y)$ or product function $T_P(x,y) = x \cdot y$. A triangular conorm (t-conorm) is a symmetric and associative bivariate aggregation function $S : [0,1]^2 \rightarrow [0,1]$ such that $S(x,0) = x$ for all $x \in [0,1]$. Some examples of t-conorms are the maximum function $S_M(x,y) = \max(x,y)$ or the probabilistic sum function $S_P(x,y) = x + y - x \cdot y$ [8–10].

In this work we use restricted equivalence functions (REF) to build the fuzzy sets associated with an image [11], [4].

Definition 1 A function $REF : [0,1]^2 \to [0,1]$ is called restricted equivalence function if it satisfies the following conditions:

(1) $REF(x,y) = REF(y,x)$ for all $x, y \in [0,1]$;
(2) $REF(x,y) = 1$ if and only if $x = y$;
(3) $REF(x,y) = 0$ if and only if $x = 1$ and $y = 0$ or $x = 0$ and $y = 1$;
(4) $REF(x,y) = REF(N(x),N(y))$ for all $x, y \in [0,1]$, being N a strong negation;
(5) if $x \leq y \leq z$ then $REF(x,y) \geq REF(x,z)$ and $REF(y,z) \geq REF(x,z)$, for all $x, y, z \in [0,1]$.

3 Grouping Functions

Definition 2 A function $G_G : [0,1]^2 \to [0,1]$ is a grouping function if it satisfies the following conditions:

$(G_G1) G_G(x,y) = G_G(y,x)$ for all $x, y \in [0,1]$;
$(G_G2) G_G(x,y) = 0$ if and only if $x = y = 0$;
$(G_G3) G_G(x,y) = 1$ if and only if $x = 1$ or $y = 1$;
$(G_G4) G_G$ is non-decreasing;
$(G_G5) G_G$ is continuous.

Observe that a grouping function is a particular case of binary aggregation.

We can find a relation between grouping functions and overlap functions, defined in [6]. We use this relation to present several construction methods of grouping functions.

3.1 Overlap Functions

Definition 3 A function $G_O : [0,1]^2 \to [0,1]$ is an overlap function if it satisfies the following properties:

(G_O1) G_O is symmetric.
$(G_O2) G_O(x,y) = 0$ if and only if $xy = 0$.
$(G_O3) G_O(x,y) = 1$ if and only if $xy = 1$.
$(G_O4) G_O$ is non-decreasing.
$(G_O5) G_O$ is continuous.

Theorem 1 *Let G_O be an overlap function and let* N *be a strict negation. Then.*

$$G_G(x,y) = N(G_O(N(x),N(y))) \tag{1}$$

is a grouping function. Reciprocally, we have that

$$G_O(x,y) = N(G_G(N(x),N(y))) \tag{2}$$

is an overlap function.

Proof (G_G1), (G_G4) and (G_G5) are direct. $(G_G2) G_G(x,y) = 0 = N(G_O(N(x), N(y)))$ if and only if $G_O(N(x),N(y)) = 1$ if and only if $N(x) = N(y) = 1$ if and only if $x = y = 0$. $(G_G3), G_G(x,y) = 1 = N(G_O(N(x);N(y)))$ if and only if $G_O(N(x), N(y)) = 0$ if and only if $N(x) = 0$ or $N(y) = 0$ if and only if $x = 1$ or $y = 1$.

Based on the relation between overlap functions and t-norms, in this work we proof that any associative grouping function is also a t-conorm. However, the reciprocal of this theorem does not hold, as it is clear if we consider any non-continuous t-conorm.

Theorem 2 *Let G_G be an associative grouping function. Then G_G is a t-conorm.*

Proof We just need to proof that 0 is the neutral element of G_G. Because of the continuity of G_G and $G_G(0,1) = 1$ and $G_G(0,0) = 0$, we can say that for any $x \in]0,1[$ there exists a $y \in]0,1[$ such that $x = G_G(y,0)$. Then $G_G(x,0) = G_G(G_G(y,0),0) = G_G(y,G_G(0,0)) = G_G(y,0) = x$ and in a similar way $G_G(0,x) = x$.

Example 1 *An associative grouping function and therefore a t-conorm is the maximum function.*

$$G_G(x,y) = \max(x,y) \tag{3}$$

Theorem 3 Let $G_G1,\ldots,G_Gm$ be m grouping functions and let $w_1,\ldots,w_m$ be m non-negative weights such that $\sum_{i=1}^m w_i = 1$. Then the convex sum $G = \sum_{i=1}^m w_i G_G i$ is a grouping function.

Proof Direct.

3.2 Construction of Grouping Functions

In the next theorem we study a construction method of grouping functions from two functions f and h that satisfy certain properties.

Theorem 4 The function $G_G : [0,1]^2 \to [0,1]$ is a grouping function if and only if

$$G_G(x,y) = \frac{f(x,y)}{f(x,y) + h(x,y)} \tag{4}$$

for $f, h : [0,1]^2 \rightarrow [0,1]$ such that

(1) f and h are symmetric;
(2) f is non-decreasing and h is non-increasing;
(3) $f(x,y) = 0$ if and only if $x = y = 0$;
(4) $h(x,y) = 0$ if and only if $x = 1$ or $y = 1$;
(5) f y h are continuous functions.

Proof We have to take into account that $f(x,y) + h(x,y) \neq 0$ for all $(x,y) \in [0,1]^2$. Then the necessity is straightaway taking $f(x,y) = G_G(x,y)$ and $h(x,y) = 1 - G_G(x,y)$. (Sufficiency) (G_G1) (G_G2) (G_G3) and (G_G5) are direct. (G_G4) If $x_1 \leq x_2$ then $f(x_1, y) \leq f(x_2, y)$ and $h(x_2,y) \leq h(x_1,y)$. So we have that $f(x_1,y)h(x_2,y) \leq f(x_2,y)h(x_1,y)$. Multiplying both sides of the equality we have $f(x_1,y)f(x_2,y) + f(x_1,y)h(x_2,y) \leq f(x_1,y)f(x_2,y) + f(x_2,y)h(x_1,y)$. We can rewrite $G_G(x_1,y) = \frac{f(x_1,y)}{f(x_1,y)+h(x_1,y)} \leq \frac{f(x_2,y)}{f(x_2,y)+h(x_2,y)} = G_G(x_2,y)$.

Example 2 If we take $f(x,y) = \max(x,y)$ and $h(x,y) = \sqrt{(1-x)(1-y)}$ we have

$$G_G(x,y) = \frac{\max(x,y)}{\max(x,y) + \sqrt{(1-x)(1-y)}} \tag{5}$$

Example 3 If we take $f(x,y) = \max(x,y)$ and $h(x,y) = (1-x)(1-y)$ we have

$$G_G(x,y) = \frac{\max(x,y)}{\max(x,y) + (1-x)(1-y)} \tag{6}$$

Example 4 If we take $f(x,y) = 1 - \sqrt{(1-x)(1-y)}$ and $h(x,y) = \min((1-x),(1-y))$ we have

$$G_G(x,y) = \frac{1 - \sqrt{(1-x)(1-y)}}{1 - \sqrt{(1-x)(1-y)} + \min((1-x),(1-y))} \tag{7}$$

A deeper study on some properties of grouping functions can be found in [12].

4 Thresholding Algorithm Based on Grouping Functions

In this work we propose the use of grouping functions as the metric to calculate the optimal threshold of an image. To do so, we construct, for every greylevel, a fuzzy set associated with the background and a fuzzy set associated with the object of the image. Applying a convex combination of several grouping functions to these sets, we choose the suitable threshold for each image. The scheme of our proposal is shown in Algorithm 1.

4.1 Construction of Fuzzy Sets Associated with the Image

In thresholding problems with only one threshold, we suppose that the image is divided into two areas, so we can separate one object from the background. Based on the study presented in [4], in this work we construct two fuzzy sets (Q_{B_t} associated with the background and Q_{O_t} associated with the object) from restricted equivalence functions, bearing in mind the following reasoning: the more similar is a greylevel (q) to the average of the background intensities (analogously to the object intensities) the higher the membership value of that intensity to the fuzzy set associated with the background (object) is.

Algorithm 1 Thresholding algorithm

1: **for** $t = \{0, 1, \ldots, L-1\}$ (For every greylevel) **do**
2: Construct a fuzzy set associated with the background of the image (Q_{B_t}).
3: Construct a fuzzy set associated with the object of the image (Q_{O_t}).
4: **for** $q = \{0, 1, \ldots, L-1\}$ (For every greylevel) **do**
5: Calculate several grouping functions for $Q_{B_t}(q)$ and $Q_{O_t}(q)$.
6: Calculate the convex combination of previous functions obtaining a new grouping function $G_{G_{comb}}(q)$.
7: **end for**
8: Calculate the weighted sum of previous grouping

$$\sum_{q=0}^{L-1} G_{G_{comb}}(\mu_{B_t}(q), \mu_{O_t}(q)) \cdot h(q)$$

where $h(q)$ is the number of pixels whose intensity is q.
9: **end for**
10: Take as best threshold $t*$ the one associated with the maximum sum of grouping:

$$t* = \arg\max_t \sum_{q=0}^{L-1} G_{G_{comb}}(\mu_{B_t}(q), \mu_{O_t}(q)) \cdot h(q)$$

For a fixed greylevel t, we start by calculating the average value of the intensities belonging to the background ($m_B(t)$) and to the object ($m_O(t)$) using the following expressions:

$$m_B(t) = \frac{\sum_{q=0}^{t} q \cdot h(q)}{\sum_{q=0}^{t} h(q)} \quad m_O(t) = \frac{\sum_{q=t+1}^{L-1} q \cdot h(q)}{\sum_{q=t+1}^{L-1} h(q)}$$

Let REF be a restricted equivalence function, we construct the fuzzy sets Q_{B_t} and Q_{O_t} with the following membership functions, for every greylevel $q = 0, 1, \ldots, L-1$:

$$\mu_{Q_{B_t}}(q) = REF\left(\frac{q}{L-1}, \frac{m_B(t)}{L-1}\right) \tag{8}$$

$$\mu_{Q_{O_t}}(q) = REF\left(\frac{q}{L-1}, \frac{m_O(t)}{L-1}\right) \tag{9}$$

We can proof that, with this construction method and due to property (2) of Definition 1, a greylevel has maximum membership degree to the background (object) fuzzy set only if its intensity is the same as the average intensities of the background (object) of the image.

- $\mu_{Q_{B_t}}(q) = 1$ if and only if $q = m_B(t)$.
- $\mu_{Q_{O_t}}(q) = 1$ if and only if $q = m_O(t)$.

4.2 *Grouping Calculus*

To calculate the grouping value associated with each possible threshold t, we use n different grouping functions $(G_G1, G_G2, \ldots, G_Gn)$. A grouping function takes two arguments and calculates the group level between both of them. In this case, we compute the grouping, for every greylevel, between the membership degree to the fuzzy set associated with the background and to the fuzzy set associated with the object.

Using the result obtained in Theorem 3, we combine the n grouping functions previously calculated. In this way, we obtain a new grouping function that, experimentally, it outperforms the result obtained by the worst grouping expression selected. This step helps us to solve the problem of choosing a grouping expression not suitable for a specific problem, what finishes in wrong results.

Once we have one sole value for the grouping of every greylevel, we calculate the sum. This is the value for the grouping associated with the threshold t.

4.3 Selection of the Maximum Grouping

Every possible threshold $t = \{0, 1, \ldots, L-1\}$ has a grouping value associated with it, calculated as the sum of the grouping function in several points. To get the best threshold, we choose the one associated with the highest grouping value. We choose the maximum value because of grouping functions properties. It means, the sum is maximum if $G_G(\mu_{B_t}(q), \mu_{O_t}(q)) = 1$ for all $q = \{0, 1, \ldots, L-1\}$. By property ($G_G2$) of Definition 2 this is achieved in two cases:

- $\mu_{B_t}(q) = 1$, so $q = m_B(t)$. In this case we are completely sure that the pixels whose intensity is q belong to the background of the image, because this intensity is exactly the average intensity of all the pixels of the background.
- $\mu_{O_t}(q) = 1$, so $q = m_O(t)$. In this case we are completely sure that the pixels whose intensity is q belong to the object of the image, because this intensity is exactly the average intensity of all the pixels of the object.

In this sense, by choosing the highest grouping value we are selecting the threshold for which all the pixels whose intensity is lower than the threshold are very closed to the average of background (object) intensities and all the pixels whose intensity is greater than the threshold are very closed to the average intensity of the object (background).

5 Illustrative Example

In this section we show the performance of the proposed algorithm over 10 T1-weighted magnetic resonance images (see Fig. 1). These images are provided by the Center for Morphometric Analysis at Massachusetts General Hospital (available at http://www.cma.mgh.harvard.edu/ibsr/). The aim of the segmentation of this kind of images is to separate each of the pixels inside the brain into one of the following two types: grey matter and white matter. This segmentation can be viewed as part of a volumetric analysis of the brain regions, which is very useful to evaluate the evolution of diseases such as Alzheimer, epilepsy or schizophrenia [13, 14]. To measure the quality of the segmented results, we compare them with an ideal handmade segmentation provided at the same webpage (see Fig. 2). This comparison is measured by the percentage of well classified pixels.

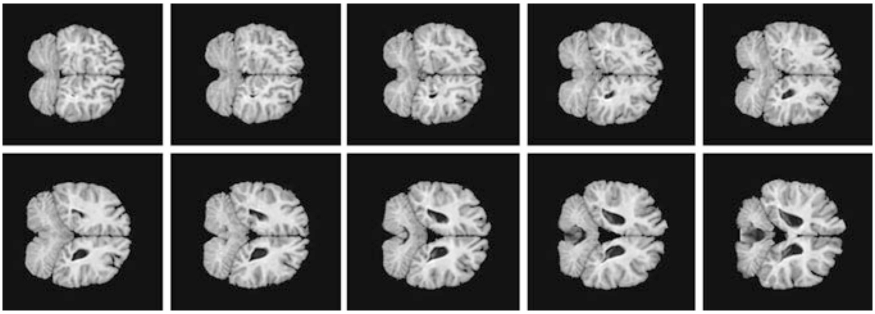

Fig. 1 Original images

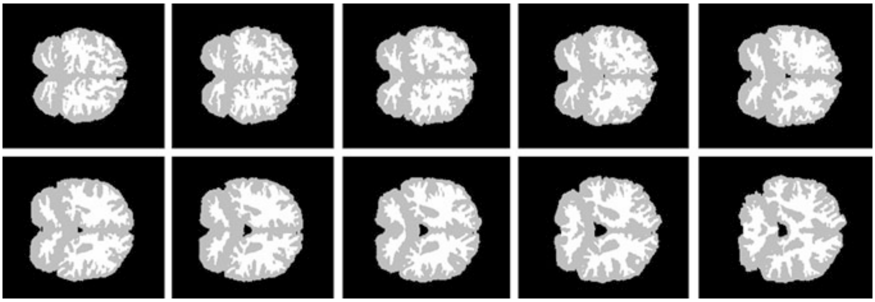

Fig. 2 Ideal handmade segmentations

In this example we use four grouping functions for the step 5 of the algorithm:

- $G_G1(x, y) = \max(x, y)$
- $G_G2(x, y) = \frac{\max(x,y)}{\max(x,y)+\sqrt{(1-x)(1-y)}}$
- $G_G3(x, y) = 1 - \sqrt{(1-x)(1-y)}$
- $G_G4(x, y) = x + y - xy$

To calculate the convex combination of grouping functions we use the same weight for each of them. In this way, each of the four weights is 0.25. If we a priori know that some grouping functions are more suitable than others for a specific image, then it is recommended to use different values for the weights, using greater values for these suitable functions (for example, using weighted means).

In Fig. 3 we see the segmentations obtained by our method for every image.

Next we study the different segmentations obtained by each one of the four proposed grouping functions in relation to the final result obtained by the consensus of all of them. To do so, in Figs. 4, 5 and 6 we show different results for three images. In the first row we see the segmentation obtained by each one of the four grouping functions. In the second row we show in white the pixels well

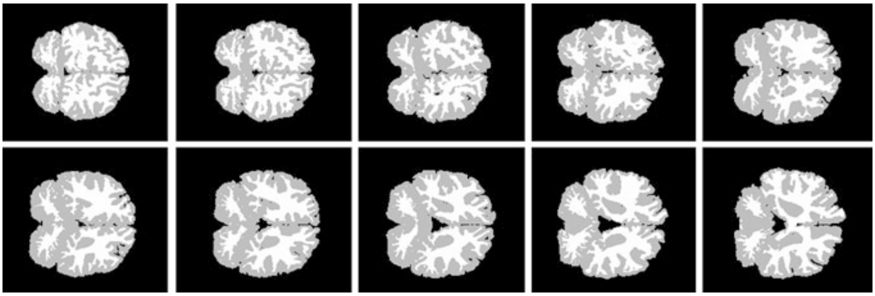

Fig. 3 Obtained segmentations by our algorithm

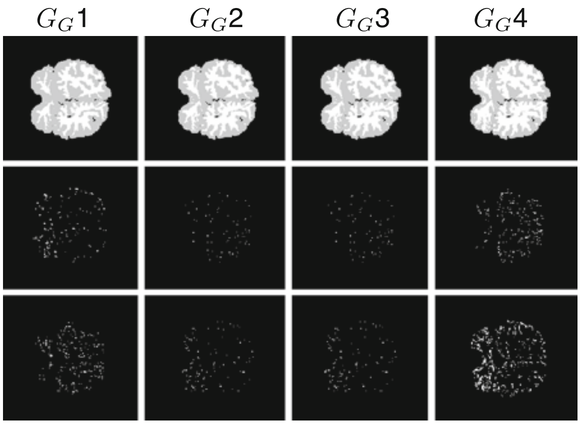

Fig. 4 Segmentations obtained by each one of the four grouping functions (*first row*). In the *second row* we show in *white* the pixels well classified by every individual grouping function that are wrong classified in the convex combination. In the *third row* we show the pixels wrong classified by every individual grouping function that are well classified in the convex combination

classified by every individual grouping function that are wrong classified in the convex combination. Finally, in the third row we show in white the pixels wrong classified by every individual grouping function that are well classified in the convex combination.

As we can see, depending on the image we segment, some grouping functions are more suitable for thresholding purposes than others. This fact is confirmed with the number of pixels well and wrong classified with respect to the convex combination. In Table 1 we show the thresholds obtained for every image and the percentage of well classified pixels. The first column shows the results obtained by the convex combination of grouping functions, and columns 2–5 show the results obtained by each one of the functions.

As we can experimentally see, the consensus of grouping functions always provides a middle threshold value among the ones obtained by every grouping function and so forth, in most cases the percentage of well classified pixels is intermediate too. In this way, we know that our algorithm does not always get the best possible result with grouping functions. However, the threshold got by the

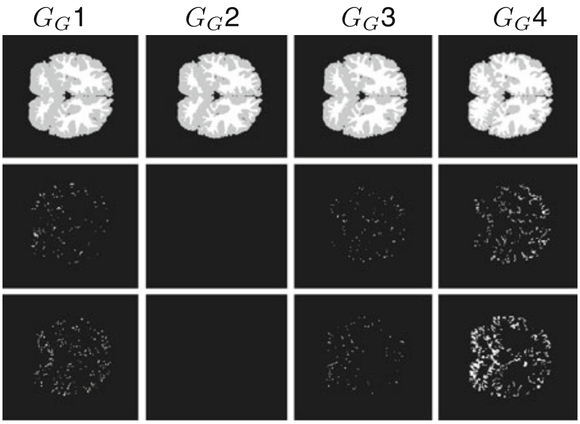

Fig. 5 Segmentations obtained by each one of the four grouping functions (*first row*). In the *second row* we show in *white* the pixels well classified by every individual grouping function that are wrong classified in the convex combination. In the *third row* we show the pixels wrong classified by every individual grouping function that are well classified in the convex combination

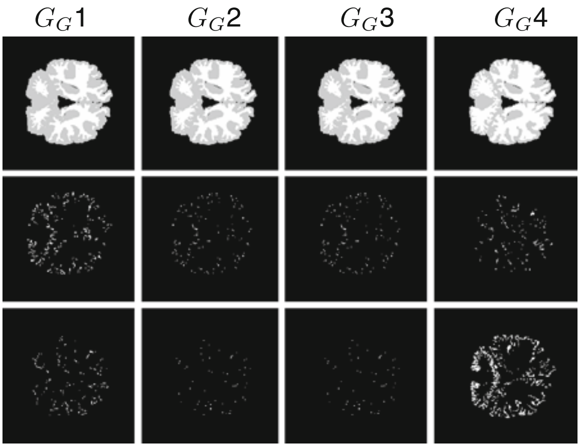

Fig. 6 Segmentations obtained by each one of the four grouping functions (*first row*). In the *second row* we show in *white* the pixels well classified by every individual grouping function that are wrong classified in the convex combination. In the *third row* we show the pixels wrong classified by every individual grouping function that are well classified in the convex combination

agreement tends to be better or equal than the worst of the thresholds got by every of the grouping functions. As we have said, this fact solves the problem of choosing a grouping function suitable for every one of the images.

Finally, we compare our thresholding method based on grouping functions with Otsu's algorithm [3], as it is one of the most used thresholding methods. In Table 2 we show the obtained threshold by both algorithms as well as the percentage of well classified pixels. We can see that our method gets the best results for all images in the experiment, so we can say it improves Otsu's thresholding method for this set of images.

Table 1 Thresholds and percentages got by every grouping function and the consensus of all of them

	Consensus		Grouping 1		Grouping 2		Grouping 3		Grouping 4	
	Threshold	%	Threshold	%	Threshold	%	Threshold	%	Threshold	%
Image 1	158	97.15	162	97.53	158	97.15	158	97.15	156	96.93
Image 2	161	97.25	166	97.75	161	97.25	161	97.25	157	96.73
Image 3	167	98.25	172	98.2	165	98.18	165	98.18	158	97.32
Image 4	166	97.4	175	97.53	166	97.4	166	97.4	159	96.64
Image 5	173	97.67	178	97.6	173	97.67	173	97.67	160	96.54
Image 6	173	97.7	177	97.67	173	97.7	173	97.7	160	96.61
Image 7	174	97.6	180	97.48	174	97.6	172	97.56	158	96.06
Image 8	173	97.94	177	97.83	173	97.94	173	97.94	156	96.15
Image 9	164	96.5	171	96.93	166	96.67	166	96.67	152	94.7
Image 10	165	96.2	165	96.2	165	96.2	165	96.2	150	94.46

Table 2 Thresholds and percentages got by our method and Otsu's one

	Consensus		Otsu	
	Threshold	%	Threshold	%
Image 1	158	97.15	155	96.82
Image 2	161	97.25	156	96.58
Image 3	167	98.25	159	97.45
Image 4	166	97.4	159	96.64
Image 5	173	97.67	162	96.8
Image 6	173	97.7	157	96.13
Image 7	174	97.6	154	95.44
Image 8	173	97.94	150	95.24
Image 9	164	96.5	137	92.35
Image 10	165	96.2	143	93.52

6 Conclusions

In this work we have presented a thresholding algorithm for greyscale images based on grouping functions. These functions, applied to our problem, measure the belongingness of a greylevel intensity to the background or to the object of the image. In this way, we choose the threshold associated with the highest grouping value to segment the image. One of the advantages of our proposal is avoiding the selection of a suitable grouping function for each image, by means of a convex combination of several of them.

References

1. Sahoo, P.K., Soltani, S., Wong, A.K.C., Chen, Y.C.: A survey of thresholding techniques. Comput. Vis. Graph. Image Process. **41**, 233–260 (1988)
2. Sezgin, M., Sankur, B.: Survey over image thresholding techniques and quantitative performance evaluation. J. Electron. Imaging **13**, 146–168 (2004)
3. Otsu, N.: A threshold selection method from gray level histograms. IEEE Trans. Syst. Man Cybern. **9**, 62–66 (1979)
4. Bustince, H., Barrenechea, E., Pagola, M.: Image thresholding using restricted equivalence functions and maximizing the measure of similarity. Fuzzy Sets Syst. **158**, 496–516 (2007)
5. Huang, L.K., Wang, M.J.J.: Image thresholding by minimizing the measure of fuzziness. Pattern Recogn. **28**, 41–51 (1995)
6. Bustince, H., Fernandez, J., Mesiar, R., Montero, J., Orduna, R.: Overlap functions. Nonlinear Anal. Theor. Meth. Appl. **78**, 1488–1499 (2010)
7. Trillas, E.: Sobre funciones de negación en la teoría de conjuntos difusos. Stochastica **III**(1), 47–59 (1979)
8. Alsina, C., Frank, M., Schweizer, B.: Associative Functions. Triangular Norms and Copulas. World Scientific, Singapore (2006)
9. Beliakov, G., Pradera, A., Calvo, T.: Aggregation functions: a guide for practitioners. Studies in Fuzziness and Soft Computing, vol. 221. Springer, Heidelberg (2007)

10. Klement, E.P., Mesiar, R., Pap, E.: Triangular norms, trends in logic. Studia Logica Library, vol. 8. Kluwer Academic Publishers, Dordrecht (2000)
11. Bustince, H., Barrenechea, E., Pagola, M.: Restricted equivalence functions. Fuzzy Sets Syst. **157**, 2333–2346 (2006)
12. Jurio, A., Bustince, H., Pagola, M., Pradera, A., Yager, R.R.: Some properties of overlap and grouping functions and their application to image thresholding. Fuzzy Sets Syst. **229**, 69–90 (2013)
13. Caldairou, B., Passat, N., Habas, P., Studholme, C., Rousseau, F.: A non-local fuzzy segmentation method: application to brain MRI. Pattern Recogn. **44**(9), 1916–1927 (2011)
14. Mayer, A., Greenspan, H.: An adaptive mean-shift framework for MRI brain segmentation. IEEE Trans. Med. Imaging **28**(8), 105–118 (2009)