Nominal Sets over Algebraic Atoms*

Joanna Ochremiak

Institute of Informatics, University of Warsaw, Warsaw, Poland
ochremiak@mimuw.edu.pl

Abstract. Nominal sets, introduced to Computer Science by Gabbay and Pitts, are useful for modeling computation on data structures built of atoms that can only be compared for equality. In certain contexts it is useful to consider atoms equipped with some nontrivial structure that can be tested in computation. Here, we study nominal sets over atoms equipped with both relational and algebraic structure. Our main result is a representation theorem for orbit-finite nominal sets over such atoms, a generalization of a previously known result for atoms equipped with relational structure only.

1 Introduction

Nominal sets [Pit13] are sets whose elements depend on *atoms* – elements of a fixed countably infinite set $\mathbb{A}$. Examples include:

- the set $\mathbb{A}$ itself,
- the set $\mathbb{A}^n$ of n-tuples of atoms,
- the set $\mathbb{A}^{(n)}$ of n-tuples of distinct atoms,
- the set $\mathbb{A}^*$ of finite words over $\mathbb{A}$,
- the set of graphs edge-labeled with atoms, etc.

Any such set is acted upon by permutations of the atoms in a natural way, by renaming all atoms that appear in it. We require the result of applying a permutation of atoms to each element of a nominal set to be determined by a finite set of atoms, called a *support* of this element. Sets $\mathbb{A}$, $\mathbb{A}^n$ and $\mathbb{A}^{(n)}$ are nominal, since each tuple of atoms is supported by the finite set of atoms that appear in it. Another example of a nominal set is $\mathbb{A}^*$, where a word is supported by the set of its letters. The set of all cofinite subsets of atoms is also nominal: one of the supports of a cofinite set is simply its complement.

Nominal sets were introduced in 1922 by Fraenkel as an alternative model of set theory. In this context they were further studied by Mostowski, which is why they are sometimes called Fraenkel-Mostowski sets. Rediscovered for the computer science community in the 90s by Gabbay and Pitts [GP02], nominal sets gained a lot of interest in semantics. In this application area atoms, whose

* This work was supported by the Polish National Science Centre (NCN) grant 2012/07/B/ST6/01497.

P. Höfner et al. (Eds.): RAMiCS 2014, LNCS 8428, pp. 429–445, 2014.

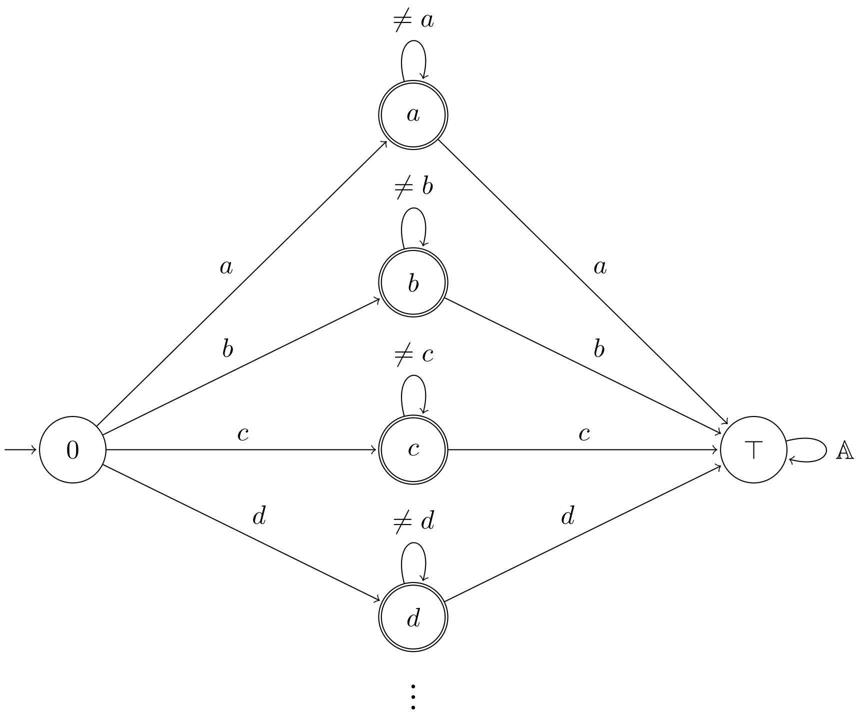
only structure is equality, are used to describe variable names in programs or logical formulas. Permutations of atoms correspond to renaming of variables.

In parallel, nominal sets were studied in automata theory [Pis99], under the name of *named sets with symmetries*[1], and used to model computation over infinite alphabets that can only be accessed in a limited way.

An example of such a model that predates nominal sets are Francez-Kaminski register automata [KF94] that, over the alphabet of atoms $\mathbb{A}$, recognize languages such as "the first letter does not appear any more":

$$L = \{a_1 \dots a_n \;:\; a_1 \neq a_i \text{ for all } i > 1\}.$$

To this end, after reading the first letter the automaton stores it in its register. Then it reads the rest of the input word and rejects if any letter equals the one in the register. The automaton has one register and three states: $0, 1, \top$, where 0 is initial and $\top$ is rejecting. Alternatively, in the framework of nominal sets, this may be modelled as an automaton with an infinite state space $\{0, \top\} \cup \mathbb{A}$ and the transition relation defined by the graph:



In [BKL11] and [BKL] Bojańczyk, Klin and Lasota showed that automata over infinite alphabets whose letters are built of atoms that can only be tested

[1] The equivalence between named sets and nominal sets was proven in [FS06] and [GMM06].

for equality, are essentially automata in the category of nominal sets. As a continuation of this line of research, Turing machines that operate over such alphabets were studied in [BKLT13].

The key notion in the above constructions is *orbit-finiteness* – a more relaxed notion of finiteness provided by nominal sets. A nominal set is considered orbit-finite if it has finitely many elements, up to permutations of atoms. The set $\mathbb{A}$ of atoms is orbit-finite: in fact, it has only one orbit. This single orbit can be represented by any atom a, because a can be mapped to every other atom by a suitable permutation. Another example of an orbit-finite set is the set of configurations of any register automaton. The automaton described above has infinitely many configurations. However, there are only three of them up to permutations: the initial state 0 with an empty register, state 1 with an atom stored in the register and the rejecting state $\top$ with an empty register.

Atoms turn out to be a good framework to speak of data that can be accessed only in a limited way. Nominal sets, as defined in [Pit13], intuitively correspond to data with no structure except for equality. To model a device with more access to its alphabet one may use atoms with additional structure. An example here are atoms with total order. A typical language recognized by a nominal automaton over such atoms is the language of all monotonic words:

$$L = \{a_1 \dots a_n \;:\; a_i < a_j \text{ for all } i < j\}.$$

In [BKL] atoms are modelled as countable relational structures. In this setting the definition of a nominal set remains essentially the same. The only change is that we consider only those permutations of atoms that preserve and reflect the relational structure, i.e., we talk about *automorphisms* of atoms. A choice of such automorphisms is called an *atom symmetry*.

Since interesting orbit-finite nominal sets are usually infinite (for example, the transition relation of the automaton above), to manipulate them effectively we need to represent them in a finite way. In [BKL] Bojańczyk et al. provide such a concrete, finite representation of orbit-finite nominal sets for atoms that are *homogeneous* relational structures over finite vocabularies (the corresponding atom symmetries are called *Fraïssé symmetries*). Each element of a nominal set is represented as a finite substructure of atoms modulo some group of local automorphisms. There are two technical assumptions needed for the theorem to hold: existence of least supports and so-called *fungibility* (meaning roughly that one can always find an automorphism that fixes a concrete substructure of atoms without fixing other atoms).

In some contexts, a relational structure of atoms is not enough. In [BL12] Bojańczyk and Lasota use the theory of nominal sets to obtain a machine-independent characterization of the languages recognized by deterministic timed automata. To do so they introduce atoms with a total order and a function symbol $+1$ and they relate deterministic timed automata to automata over these *timed atoms*. An example of a language recognized by such a nominal automaton is the set of all monotonic words where the distance between any two consecutive letters is smaller than 1:

$$L = \{a_1 \dots a_n \ : \ a_{i-1} + 1 > a_i > a_{i-1} \text{ for all } i > 1\}.$$

One could easily think of other types of potentially useful functional dependencies on atoms, such as composing two atoms to get another atom. It is therefore natural to ask if the representation theorem can be generalized to cover atoms with algebraic as well as relational structure. This paper gives a positive answer to this question.

The proof of the representation theorem for atoms with both relational and function symbols follows the same pattern as the proof for relational structures given in [BKL]. There are, however, some subtleties, since instead of finite supports one has to consider *finitely generated supports* (which can be infinite) and, as a result, the notion of fungibility becomes less clear.

The structure of this paper is as follows. In Section 2 we define atom symmetries and introduce the category of G-sets. In Section 3, following [BKL11,BKL], we focus on the theory of nominal sets for Fraïssé symmetries, introduce the category of nominal sets, and explain the notion of the least finitely generated support. In Section 4 we define the property of fungibility and finally prove the representation theorem for fungible Fraïssé symmetries that admit least finitely generated supports.

2 Atom Symmetries

A (right) *group action* of a group G on a set X is a binary operator $\cdot \colon X \times G \to X$ that satisfies following conditions:

$$\text{for all } x \in X \quad x \cdot e = x, \;\text{ where } e \text{ is the neutral element of } G,$$

$$\text{for all } x \in X \text{ and } \pi, \sigma \in G \quad x \cdot (\pi\sigma) = (x \cdot \pi) \cdot \sigma.$$

The set X equipped with such an action is called a *G-set*.

Example 2.1. For a set X let $\mathrm{Sym}(X)$ denote the symmetric group on X, i.e., the group of all bijections of X. Take any subgroup G of the symmetric group $\mathrm{Sym}(X)$. There is a natural action of the group G on the set X defined by $x \cdot \pi = \pi(x)$.

Definition 2.2. An *atom symmetry* $(\mathbb{A}, G)$ is a set $\mathbb{A}$ of *atoms*, together with a subgroup $G \leq \mathrm{Sym}(\mathbb{A})$ of the symmetric group on $\mathbb{A}$.

Example 2.3. Examples of atom symmetries include:

- the *equality symmetry*, where $\mathbb{A}$ is a countably infinite set, say the natural numbers, and $G = \mathrm{Sym}(\mathbb{A})$ contains all bijections of $\mathbb{A}$,
- the *total order symmetry*, where $\mathbb{A} = \mathbb{Q}$ is the set of rational numbers, and G is the group of all monotone permutations,

– the *timed symmetry*, where $\mathbb{A} = \mathbb{Q}$ is the set of rational numbers, and G is the group of all permutations of rational numbers that preserve the order relation $\leq$ and the successor function $x \mapsto x + 1$[2].

For any element x of a G-set X the set

$$x \cdot G = \{x \cdot \pi \mid \pi \in G\} \subseteq X$$

is called the *orbit* of x. Orbits form a partition of X. The set X is called *orbit-finite* if the partition has finitely many parts. Each of the orbits can be perceived as a separate G-set. Therefore we can treat any G-set X as a disjoint union of its orbits.

Example 2.4. For any atom symmetry $(\mathbb{A}, G)$ the action of G on $\mathbb{A}$ extends pointwise to an action of G on the set of tuples $\mathbb{A}^n$. In the equality symmetry, the set $\mathbb{A}^2$ has two orbits:

$$\{(a,a) \mid a \in \mathbb{A}\} \qquad\qquad \{(a,b) \mid a \neq b \in \mathbb{A}\}.$$

In the timed symmetry, the set $\mathbb{A}^2$ is not orbit-finite. Notice that for any $a \in \mathbb{Q}$ each of the elements $(a, a+1)$, $(a, a+2), \ldots$ is in a different orbit.

Let X be a G-set. A subset $Y \subseteq X$ is *equivariant* if $Y \cdot \pi = Y$ for every $\pi \in G$, i.e., it is preserved under group action. Considering a pointwise action of a group G on the Cartesian product $X \times Y$ of two G-sets X, Y we can define an *equivariant relation* $R \subseteq X \times Y$. In the special case when the relation is a function $f \colon X \to Y$ we obtain a following definition of an *equivariant function*

$$f(x \cdot \pi) = f(x) \cdot \pi \text{ for any } x \in X,\ \pi \in G.$$

The identity function on any G-set is equivariant, and the composition of two equivariant functions is again equivariant, therefore for any group G, G-sets and equivariant functions form a category, called G-**Set**.

Definition 2.5. For any x in a G-set X, the group

$$G_x = \{\pi \in G \mid x \cdot \pi = x\} \leq G$$

is called the *stabilizer* of x.

Lemma 2.6. *If $G_x \leq G$ is the stabilizer of an element x of a G-set X then $G_{x\cdot\pi} = \pi^{-1} G_x \pi$ for each $\pi \in G$.*

Proof. Obviously $\pi^{-1} G_x \pi \subseteq G_{x\cdot\pi}$. On the other hand, $x \cdot (\pi\sigma\pi^{-1}) = x$ for any $\sigma \in G_{x\cdot\pi}$. Hence $\pi G_{x\cdot\pi} \pi^{-1} \subseteq G_x$, which means that $G_{x\cdot\pi} \subseteq \pi^{-1} G_x \pi$. As a result $G_{x\cdot\pi} = \pi^{-1} G_x \pi$, as required.

[2] The timed symmetry was originally defined in [BL12] for $\mathbb{A} = \mathbb{R}$. Considering the rational numbers instead of the reals makes little difference but is essential for our purposes. To fit the Fraïssé theory we need the set of atoms to be countable.

Proposition 2.7. *Let x be an element of a single-orbit G-set X. For any G-set Y equivariant functions from X to Y are in bijective correspondence with those elements $y \in Y$ for which $G_x \leq G_y$.*

Proof. Given an equivariant function $f\colon X \to Y$, let $y = f(x)$. If $\pi \in G_x$ then

$$y \cdot \pi = f(x) \cdot \pi = f(x \cdot \pi) = f(x) = y,$$

hence $G_x \leq G_y$. On the other hand, given $y \in Y$ such that $G_x \leq G_y$, define a function $f\colon X \to Y$ by $f(x \cdot \pi) = y \cdot \pi$. Function f is well-defined. Indeed, if $x \cdot \pi = x \cdot \sigma$ then $\pi\sigma^{-1} \in G_x \subseteq G_y$, hence $y \cdot \pi = y \cdot \sigma$.

It is easy to check that the two above constructions are mutually inverse.

3 Fraïssé Symmetries

In the following, we shall consider atom symmetries that arise as automorphism groups of algebraic structures. Such symmetries behave particularly well if those structures arise as so-called *Fraïssé limits*, which we introduce in this sections.

3.1 Fraïssé Limits

An *algebraic signature* is a set of relation and function names together with (finite) arities. We will consider structures over a fixed finite algebraic signature. For two structures $\mathfrak{A}$ and $\mathfrak{B}$, an *embedding* $f\colon \mathfrak{A} \to \mathfrak{B}$ is an injective function from the carrier of $\mathfrak{A}$ to the carrier of $\mathfrak{B}$ that preserves and reflects all relations and functions in the signature.

Definition 3.1. A class $\mathcal{K}$ of finitely generated structures over some fixed algebraic signature is called a *Fraïssé class* if it:

- is closed under isomorphisms as well as finitely generated substructures and has countably many members up to isomorphism,
- has *joint embedding property*: if $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}$ then there is a structure $\mathfrak{C}$ in $\mathcal{K}$ such that both $\mathfrak{A}$ and $\mathfrak{B}$ are embeddable in $\mathfrak{C}$,
- has *amalgamation*: if $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \mathcal{K}$ and $f_{\mathfrak{B}}\colon \mathfrak{A} \to \mathfrak{B}$, $f_{\mathfrak{C}}\colon \mathfrak{A} \to \mathfrak{C}$ are embeddings then there is a structure $\mathfrak{D}$ in $\mathcal{K}$ together with two embeddings $g_{\mathfrak{B}}\colon \mathfrak{B} \to \mathfrak{D}$ and $g_{\mathfrak{C}}\colon \mathfrak{C} \to \mathfrak{D}$ such that $g_{\mathfrak{B}} \circ f_{\mathfrak{B}} = g_{\mathfrak{C}} \circ f_{\mathfrak{C}}$.

Examples of Fraïssé classes include:

- all finite structures over an empty signature, i.e., finite sets,
- finite total orders,
- all finite structures over a signature with a single binary relation symbol, i.e., directed graphs,
- finite Boolean algebras,
- finite groups,
- finite fields of characteristic p.

Classes that are not Fraïssé include:

- total orders of size at most 7 – due to lack of amalgamation,
- all finite fields – due to lack of joint embedding property.

Some Fraïssé classes admit a stronger version of amalgamation property. We say that a class $\mathcal{K}$ has *strong amalgamation* if it has amalgamation and moreover, $g_{\mathfrak{B}} \circ f_{\mathfrak{B}}(\mathfrak{A}) = g_{\mathfrak{C}} \circ f_{\mathfrak{C}}(\mathfrak{A}) = g_{\mathfrak{B}}(\mathfrak{B}) \cap g_{\mathfrak{C}}(\mathfrak{C})$. It means that we can make amalgamation without identifying any more points than absolutely necessary.

Example 3.2. All the Fraïssé classes listed above, except for the class of finite fields of characteristic p, have the strong amalgamation property.

The *age* of a structure $\mathfrak{U}$ is the class $\mathcal{K}$ of all structures isomorphic to finitely generated substructures of $\mathfrak{U}$. A structure $\mathfrak{U}$ is *homogeneous* if any isomorphism between finitely generated substructures of $\mathfrak{U}$ extends to an automorphism of $\mathfrak{U}$. The following theorem says that for a Fraïssé class $\mathcal{K}$ there exists a so-called *universal* homogeneous structure of age $\mathcal{K}$. We shall refer to it also as the *Fraïssé limit* of the class $\mathcal{K}$ (see e.g. [Hod93]).

Theorem 3.3. *For any Fraïssé class $\mathcal{K}$ there exists a unique, up to isomorphism, countable (finite or infinite) structure $\mathfrak{U}_{\mathcal{K}}$ such that $\mathcal{K}$ is the age of $\mathfrak{U}_{\mathcal{K}}$ and $\mathfrak{U}_{\mathcal{K}}$ is homogeneous.*

Example 3.4. The Fraïssé limit of the class of finite total orders is $\langle \mathbb{Q}, \leq \rangle$. For finite Boolean algebras it is the countable atomless Boolean algebra.

A structure $\mathfrak{U}$ is called *weakly homogeneous* if for any two finitely generated substructures $\mathfrak{A}, \mathfrak{B}$ of $\mathfrak{U}$, such that $\mathfrak{A} \subseteq \mathfrak{B}$, any embedding $f_{\mathfrak{A}} \colon \mathfrak{A} \to \mathfrak{U}$ extends to an embedding $f_{\mathfrak{B}} \colon \mathfrak{B} \to \mathfrak{U}$. It turns out that a countable structure $\mathfrak{U}$ is homogeneous if and only if it is weakly homogeneous (see [Hod93]). Hence, one way to obtain a Fraïssé class $\mathcal{K}$ is to take a weakly homogeneous, countable structure $\mathfrak{U}$ and simply consider its age.

Fact 3.5. *Every countable, weakly homogeneous structure $\mathfrak{U}$ is a Fraïssé limit of its age.*

Example 3.6. Consider an algebraic signature with a single binary relation symbol $\leq$, and two unary function symbols $+1$ and -1. It is not difficult to see that the structure $\langle \mathbb{Q}, \leq, +1, -1 \rangle$ is countable and weakly homogeneous. Therefore it is the Fraïssé limit of its age. Observe that its automorphism group contains precisely those permutations of rational numbers which are monotone and preserve the successor funtion $x \mapsto x + 1$.

From a Fraïssé class $\mathcal{K}$ we obtain an atom symmetry $(\mathbb{A}_{\mathcal{K}}, G_{\mathcal{K}})$, where $\mathbb{A}_{\mathcal{K}}$ is the carrier of $\mathfrak{U}_{\mathcal{K}}$ and $G_{\mathcal{K}} = \mathrm{Aut}(\mathfrak{U}_{\mathcal{K}})$ is its group of automorphisms. Such an atom symmetry is called a *Fraïssé symmetry.*

For simplicity we frequently identify the elements of age $\mathcal{K}$ with finitely generated substructures of $\mathfrak{U}_{\mathcal{K}}$.

Example 3.7. All symmetries in Example 2.3 are Fraïssé symmetries. The equality symmetry arises from the class of all finite sets, the total order symmetry – from the class of finite total orders and the timed symmetry – from the class of all finitely generated substructures of $\langle \mathbb{Q}, \leq, +1, -1 \rangle$ (see Example 3.6).

The timed symmetry was originally defined based on a structure without the unary function -1. In the context of [BL12] adding -1 to the signature does not make any difference since the automorphism groups of both structures are the same. As we will show, thanks to this slight modification the timed symmetry satisfies all the conditions of our representation theorem.

3.2 Least Supports

From now on, we focus on G-sets for groups arising from Fraïssé symmetries. Consider such a symmetry $(\mathbb{A}_\mathcal{K}, G_\mathcal{K})$ and a $G_\mathcal{K}$-set X. By $\pi|_C$ we denote the restriction of a permutation π to a subset C of its domain.

Definition 3.8. A set $C \subseteq \mathbb{A}_\mathcal{K}$ *supports* an element $x \in X$ if $x \cdot \pi = x$ for all $\pi \in G_\mathcal{K}$ such that $\pi|_C = \mathrm{id}|_C$. A $G_\mathcal{K}$-set is *nominal* in the symmetry $(\mathbb{A}_\mathcal{K}, G_\mathcal{K})$ if every element in the set is supported by the carrier of a finitely generated substructure $\mathfrak{A}$ of $\mathfrak{U}_\mathcal{K}$. We call $\mathfrak{A}$ a *finitely generated support* of x.

Nominal $G_\mathcal{K}$-sets and equivariant functions between them form a category $G_\mathcal{K}$-**Nom** which is a full subcategory of $G_\mathcal{K}$-**Set**. When the symmetry $(\mathbb{A}_\mathcal{K}, G_\mathcal{K})$ under consideration is the equality symmetry, the category $G_\mathcal{K}$-**Nom** coincides with the category **Nom** defined in [Pit13].

Example 3.9. For any Fraïssé symmetry $(\mathbb{A}_\mathcal{K}, G_\mathcal{K})$ the sets $\mathbb{A}_\mathcal{K}$ and $\mathbb{A}_\mathcal{K}^n$ are nominal. A tuple $(d_1, ..., d_n)$ is supported by the structure generated by its elements.

Lemma 3.10. *The following conditions are equivalent:*
(1) *C supports an element $x \in X$;*
(2) *for any $\pi, \sigma \in G_\mathcal{K}$ if $\pi|_C = \sigma|_C$ then $x \cdot \pi = x \cdot \sigma$.*

Proof. For the implication (1) $\Longrightarrow$ (2), notice that if $\pi|_C = \sigma|_C$, then $\pi\sigma^{-1}$ acts as identity on C, hence $x \cdot \pi\sigma^{-1} = x$ and $x \cdot \pi = x \cdot \sigma$, as required. The opposite implication follows immediately from the definition if we take $\sigma = \mathrm{id}$.

It is easy to see that if an element $x \in X$ has a finitely generated support $\mathfrak{A}$ then it is also supported by the finite set C of its generators. Thus we can equivalently require x to be finitely supported.

Fact 3.11. *A $G_\mathcal{K}$-set is nominal if and only if its every element has a finite support.*

Example 3.12. Consider the structure $\langle \mathbb{Q}, \leq, +1 \rangle$. It is countable and weakly homogeneous, and therefore gives rise to a Fraïssé symmetry. This symmetry is almost the same as the timed symmetry (the carriers and automorphim groups of both $\langle \mathbb{Q}, \leq, +1 \rangle$ and $\langle \mathbb{Q}, \leq, +1, -1 \rangle$ are the same). It has, though, some unwanted properties. Notice that an automorphism π of $\langle \mathbb{Q}, \leq, +1 \rangle$ which preserves an atom $a \in \mathbb{Q}$ necessarily preserves also $a + i$ for any integer i. Therefore, if an element x of a nominal set is supported by a substructure generated e.g. by $\{1, 30\frac{1}{2}, 100\frac{5}{7}\}$ it is also supported by its proper substructure generated by $\{1000, 300\frac{1}{2}, 105\frac{5}{7}\}$. Hence, in this case for any finitely generated support $\mathfrak{A}$ of an element x one can find a finitely generated substructure $\mathfrak{B}$, which is properly contained in $\mathfrak{A}$ and still supports x.

An element of a nominal set has many supports. In particular, supports are closed under adding atoms. If every element of a nominal set X has a unique least finitely generated support, we say that X is *supportable*. As shown in Example 3.12 it is not always the case. It turns out that to check if a single-orbit nominal set is supportable, one just needs to find out if any element of the set has the least finitely generated support.

Lemma 3.13. *If $\mathfrak{A} \subseteq \mathfrak{U}_{\mathcal{K}}$ is the least finitely generated support of an element $x \in X$, then $\mathfrak{A} \cdot \pi$ is the least finitely generated support of $x \cdot \pi$ for any $\pi \in G_{\mathcal{K}}$.*

Proof. First we prove that $\mathfrak{A} \cdot \pi$ supports $x \cdot \pi$. Indeed, if an arbitrary $\rho \in G_{\mathcal{K}}$ is an identity on $\mathfrak{A} \cdot \pi$, then $\pi\rho\pi^{-1}$ is an identity on $\mathfrak{A}$, hence $x \cdot (\pi\rho\pi^{-1}) = x$. As a result $(x \cdot \pi) \cdot \rho = x \cdot \pi$, as required.

Now let $\mathfrak{B} \subseteq \mathfrak{U}_{\mathcal{K}}$ be any finitely generated support of $x \cdot \pi$. We need to show that $\mathfrak{A} \cdot \pi \subseteq \mathfrak{B}$. A reasoning similar to the one above shows that $\mathfrak{B} \cdot \pi^{-1}$ supports x, from which we obtain $\mathfrak{A} \subseteq \mathfrak{B} \cdot \pi^{-1}$. Therefore, since π is a bijection, $\mathfrak{A} \cdot \pi \subseteq \mathfrak{B}$.

Definition 3.14. A Fraïssé symmetry $(\mathbb{A}_{\mathcal{K}}, G_{\mathcal{K}})$ is *supportable* if every nominal $G_{\mathcal{K}}$-set is supportable.

We call a structure $\mathfrak{U}$ *locally finite* if all its finitely generated substructures are finite. Notice that if the universal structure $\mathfrak{U}_K$ is locally finite then being supportable is equivalent to finitely generated supports being closed under finite intersections. The same holds under the weaker assumption that any finitely generated structure has only finitely many finitely generated substructures.

Example 3.15. If we have only relation symbols in the signature it is obvious that any finitely generated structure is finite. One can prove that in the equality symmetry the intersection of two supports is a support itself. Hence the equality symmetry is supportable. The same holds for the total order symmetry. Both facts are proved e.g. in [BKL].

From Example 3.12 we learned that the symmetry arising from the structure $\langle \mathbb{Q}, \leq, +1 \rangle$ is not supportable (even though the finitely generated supports are closed under finite intersections). In the structure $\langle \mathbb{Q}, \leq, +1, -1 \rangle$ all the elements $a + i$ are bound together and, as a result, we obtain a Fraïssé symmetry that is supportable.

Proposition 3.16. *The timed symmetry is supportable.*

Proof. Notice that any finitely generated substructure of $\langle \mathbb{Q}, \leq, +1, -1 \rangle$ has only finitely many substructures. Hence it is enough to show that finitely generated supports are closed under finite intersections.

Take any two finitely generated substructures $\mathfrak{A}$, $\mathfrak{B}$ of $\langle \mathbb{Q}, \leq, +1, -1 \rangle$. Let A and B be the sets of elements of $\mathfrak{A}$ and $\mathfrak{B}$ that are contained in the interval $[0, 1)$. These are (finite) sets of generators. Moreover, the structure $\mathfrak{A} \cap \mathfrak{B}$ is generated by $A \cap B$. Hence, it is enough to show that if an automorphism π acts as identity on $A \cap B$, then π can be decomposed as

$$\pi = \sigma_1 \tau_1 \sigma_2 \tau_2 ... \sigma_n \tau_n,$$

where σ_i acts as identity on A and τ_i acts as identity on B. Indeed, since each σ_i, τ_i acts as identity on $\mathfrak{A}$ and $\mathfrak{B}$ respectively, we have $x \cdot \sigma_i = x$ and $x \cdot \tau_i = x$. As a result $x \cdot \pi = x$.

Let l be the smallest and h the biggest element of the set $A \cup B$. Notice that $h - l < 1$. Take two different open intervals (l_A, h_A), (l_B, h_B) of length 1 such that

$$[l, h] \subseteq (l_A, h_A) \text{ and } [l, h] \subseteq (l_B, h_B).$$

Now, consider sets $A' = A \cup \{l_A, h_A\}$, $B' = B \cup \{l_B, h_B\}$. Take an automorphism π that acts as identity on $A \cap B = A' \cap B'$. Obviously π is a monotone bijection of the set of rational numbers. Therefore, since the total order symmetry is supportable,

$$\pi = \sigma'_1 \tau'_1 \sigma'_2 \tau'_2 ... \sigma'_n \tau'_n,$$

where σ'_i, τ'_i are monotone bijections of $\mathbb{Q}$ and σ'_i act as identity on A', τ'_i act as identity on B'. For each of the permutations σ'_i, τ'_i take an automorphism σ_i, τ_i of the universal structure $\langle \mathbb{Q}, \leq, +1, -1 \rangle$, such that

$$\sigma'_i|_{(l_A,h_A)} = \sigma_i|_{(l_A,h_A)}, \quad \tau'_i|_{(l_B,h_B)} = \tau_i|_{(l_B,h_B)}.$$

Then σ_i act as identity on A and τ_i act as identity on B. Moreover $\pi = \sigma_1 \tau_1 \sigma_2 \tau_2 ... \sigma_n \tau_n$, as required.

4 Structure Representation

For any $C \subseteq \mathbb{A}$ and $G \leq \mathrm{Sym}(\mathbb{A})$, the restriction of G to C is defined by

$$G|_C = \{\pi|_C \mid \pi \in G,\ C \cdot \pi = C\} \leq \mathrm{Sym}(C).$$

Lemma 4.1. *Let $\mathfrak{A} \in \mathcal{K}$ be a finitely generated structure. The set of embeddings $u \colon \mathfrak{A} \to \mathfrak{U}_K$ with the $G_{\mathcal{K}}$-action defined by composition:*

$$u \cdot \pi = u\pi$$

is a single-orbit nominal set.

Proof. First notice that any embedding $u\colon \mathfrak{A} \to \mathfrak{U}_K$ is supported by its image $u(\mathfrak{A})$. Indeed, if an automorphism $\pi \in G_\mathcal{K}$ is an identity on $u(\mathfrak{A})$ then obviously $u \cdot \pi = u$. Hence the set of embeddings is a nominal set. Now take any two embeddings u and v. The images $u(\mathfrak{A})$, $v(\mathfrak{A})$ are finitely generated isomorphic substructures of $\mathfrak{U}_\mathcal{K}$. By extending any isomorphism between $u(\mathfrak{A})$ and $v(\mathfrak{A})$, we obtain an automorphism $\pi \in G_\mathcal{K}$ such that $u \cdot \pi = v$.

As we shall show now, in a supportable symmetry $(\mathbb{A}_\mathcal{K}, G_\mathcal{K})$ every single-orbit nominal set is isomorphic to one of the above form, quotiented by some equivariant equivalence relation.

Notice that the quotient of a G-set by an equivariant equivalence relation R has a natural structure of a G-set, with the action defined as follows:

$$[x]_R \cdot \pi = [x \cdot \pi]_R.$$

It is easy to see that if X has one orbit, then so does the quotient X/R. Moreover, any support C of an element $x \in X$ supports the equivalence class $[x]_R$, hence if X is nominal then X/R is also nominal.

Definition 4.2. A *structure representation* is a finitely generated structure $\mathfrak{A} \in \mathcal{K}$ together with a group of automorphisms $S \leq \mathrm{Aut}(\mathfrak{A})$ (the *local symmetry*). Its *semantics* $[\mathfrak{A}, S]$ is the set of embeddings of $u\colon \mathfrak{A} \to \mathfrak{U}_K$, quotiented by the equivalence relation:

$$u \equiv_S v \Leftrightarrow \exists \tau \in S \ \tau u = v.$$

A $G_\mathcal{K}$-action on $[\mathfrak{A}, S]$ is defined by composition:

$$[u]_S \cdot \pi = [u\pi]_S.$$

Proposition 4.3. *(1)* $[\mathfrak{A}, S]$ *is a single-orbit nominal* $G_\mathcal{K}$*-set. (2) If a Fraïssé symmetry* $(\mathbb{A}_\mathcal{K}, G_\mathcal{K})$ *is supportable then every single-orbit nominal* $G_\mathcal{K}$*-set* X *is isomorphic to some* $[\mathfrak{A}, S]$.

Proof. For (1), use Lemma 4.1. The set of embeddings $u\colon \mathfrak{A} \to \mathfrak{U}_K$ is a single-orbit nominal $G_\mathcal{K}$-set, and so is the quotient $[\mathfrak{A}, S]$.

For (2), take a single-orbit nominal set X and let $H \leq G_\mathcal{K}$ be the stabilizer of some element $x \in X$. Put $S = H|_\mathfrak{A}$ where $\mathfrak{A} \in \mathcal{K}$ is the least finitely generated support of x. Define $f\colon X \to [\mathfrak{A}, S]$ by $f(x \cdot \pi) = [\pi|_\mathfrak{A}]_S$. The function f is well defined: if $x \cdot \pi = x \cdot \sigma$ then $\pi\sigma^{-1} \in H$. As $\mathfrak{A} \cdot \pi\sigma^{-1}$ is the least finitely generated support of $x \cdot \pi\sigma^{-1} = x$, we obtain $\mathfrak{A} \cdot \pi\sigma^{-1} = \mathfrak{A}$. Therefore for $\tau = (\pi\sigma^{-1})|_\mathfrak{A} \in S$ we have $\tau\sigma|_\mathfrak{A} = \pi|_\mathfrak{A}$, hence $[\pi|_\mathfrak{A}]_S = [\sigma|_\mathfrak{A}]_S$. It is easy to check that f is equivariant.

It remains to show that f is bijective. For injectivity, assume $f(x\cdot\pi) = f(x\cdot\sigma)$. This means that there exists $\tau \in S$ such that $\tau\sigma|_\mathfrak{A} = \pi|_\mathfrak{A}$, then $(\pi\sigma^{-1})|_\mathfrak{A} \in S$, hence $(\pi\sigma^{-1})|_\mathfrak{A} = \rho|_\mathfrak{A}$ for some $\rho \in H$. Therefore $x \cdot \pi\sigma^{-1} = x \cdot \rho = x$, from which we obtain $x \cdot \pi = x \cdot \sigma$. For surjectivity of f, note that by universality of the structure $\mathfrak{U}_K$ any embedding $u\colon \mathfrak{A} \to \mathfrak{U}_K$ can be extended to an automorphism π of $\mathfrak{U}_K$, for which we have $f(x \cdot \pi) = [u]_S$.

Structures over signatures with no function symbols are called *relational structures.* Structure representation was defined by Bojańczyk et al. in the special case of $\mathfrak{U}_K$ being a relational structure. The proposition above generalizes Proposition 11.7 of [BKL].

Example 4.4. Consider the universal structure $\langle \mathbb{Q}, \leq, +1, -1\rangle$ and its substructure $\mathfrak{A}$ generated by $\{\frac{1}{3}, \frac{1}{2}, \frac{3}{4}\}$. Notice that mapping one of the generators, say $\frac{1}{2}$, to any element of $\mathfrak{A}$, say $\frac{1}{2} \mapsto 3\frac{3}{4}$, uniquely determines an automorphism π of $\mathfrak{A}$. The automorphism can be seen as a shift. It maps $\frac{1}{3}$ to $3\frac{1}{2}$ and $\frac{3}{4}$ to $4\frac{1}{3}$. This observation leads to the conclusion that $\mathrm{Aut}(\mathfrak{A}) = \mathbb{Z}$. Any subgroup S of $\mathrm{Aut}(\mathfrak{A})$ is therefore isomorphic to $\mathbb{Z}$ and generated by a single automorphism π of the form described above. The same holds for any finitely generated substructure $\mathfrak{A}$. In the case of timed symmetry Proposition 4.3 provides a very nice finite representation of single-orbit nominal sets.

4.1 Fungibility

Even if the symmetry is supportable it may happen that some finitely generated structure is not the least finitely generated support of anything. Now we will introduce a condition which ensures that any finitely generated structure is the least finitely generated support of some element of some nominal set.

Definition 4.5. A finitely generated substructure $\mathfrak{A}$ of $\mathfrak{U}_\mathcal{K}$ is *fungible* if for every finitely generated substructure $\mathfrak{B} \subsetneq \mathfrak{A}$, there exists $\pi \in G_\mathcal{K}$ such that:

- $\pi|_\mathfrak{B} = \mathrm{id}|_\mathfrak{B}$,
- $\pi(\mathfrak{A}) \neq \mathfrak{A}$.

A Fraïssé symmetry $(\mathbb{A}_\mathcal{K}, G_\mathcal{K})$ is fungible if every finitely generated substructure $\mathfrak{A}$ of $\mathfrak{U}_\mathcal{K}$ is fungible.

Example 4.6. The equality, total order and timed symmetries are all fungible. The symmetry obtained from the universal structure $\langle \mathbb{Q}, \leq, +1\rangle$ is not fungible. Take a structure $\mathfrak{A}$ generated by $\{0\}$ and its substructure $\mathfrak{B}$ generated by $\{1\}$. Obviously if an automorphism π acts as identity on $\mathfrak{B}$ then it acts as identity also on $\mathfrak{A}$.

In general, being supportable and being fungible are independent properties of symmetries. Examples are given in [BKL]. The following result generalizes Lemma 10.8. of [BKL].

Lemma 4.7. *(1) If* $(\mathbb{A}_\mathcal{K}, G_\mathcal{K})$ *is supportable then every finitely generated fungible* $\mathfrak{A} \subseteq \mathfrak{U}_\mathcal{K}$ *is the least finitely generated support of* $[id|_\mathfrak{A}]_S$*, for any* $S \leq \mathrm{Aut}(\mathfrak{A})$.

(2) If $(\mathbb{A}_\mathcal{K}, G_\mathcal{K})$ *is fungible then every finitely generated* $\mathfrak{A} \subseteq \mathfrak{U}_\mathcal{K}$ *is the least finitely generated support of* $[id|_\mathfrak{A}]_S$*, for any* $S \leq \mathrm{Aut}(\mathfrak{A})$.

Proof. For (1), recall from Lemma 4.1 that an embedding $u\colon \mathfrak{A} \to \mathfrak{U}_{\mathcal{K}}$ is supported by its image. Therefore $\mathfrak{A}$ supports $\mathrm{id}\,|_{\mathfrak{A}}$ and hence also $[\mathrm{id}\,|_{\mathfrak{A}}]_S$. Now consider any finitely generated structure $\mathfrak{B}$ properly contained in $\mathfrak{A}$. Since $\mathfrak{A}$ is fungible there exists an automorphism π from the Definition 4.5. The automorphism π acts as identity on $\mathfrak{B}$, but $[\mathrm{id}\,|_{\mathfrak{A}}]_S \cdot \pi = [\pi|_{\mathfrak{A}}]_S \neq [\mathrm{id}\,|_{\mathfrak{A}}]_S$ as the image of π is not $\mathfrak{A}$.

For (2), we first show that $\mathfrak{A}$ supports $[\mathrm{id}\,|_{\mathfrak{A}}]_S$ as in (1) above. Then let $\mathfrak{B}$ be another support of $[\mathrm{id}\,|_{\mathfrak{A}}]_S$ and assume $\mathfrak{A}$ is not contained in $\mathfrak{B}$, i.e., there exists some $a \in \mathfrak{A} \setminus \mathfrak{B}$. Since the structure $\mathfrak{C}$ generated by $\mathfrak{A} \cup \mathfrak{B}$ is fungible, there exists an automorphism π such that $\pi|_{\mathfrak{B}} = \mathrm{id}\,|_{\mathfrak{B}}$ and $\pi(\mathfrak{C}) \neq \mathfrak{C}$, which means that also $\pi(\mathfrak{A}) \neq \mathfrak{A}$. Hence $[\mathrm{id}\,|_{\mathfrak{A}}]_S \cdot \pi = [\pi|_{\mathfrak{A}}]_S \neq [\mathrm{id}\,|_{\mathfrak{A}}]_S$ and we obtain a contradiction as it turns out that $\mathfrak{B}$ does not support $[\mathrm{id}\,|_{\mathfrak{A}}]_S$.

Let us focus for a moment on relational structures. In this case to obtain a fungible symmetry it is enough to require an existence of π that is not an identity on $\mathfrak{A}$.

Definition 4.8. A finitely generated substructure $\mathfrak{A}$ of $\mathfrak{U}_{\mathcal{K}}$ is *weakly fungible* if for every finitely generated substructure $\mathfrak{B} \subsetneq \mathfrak{A}$, there exists $\pi \in G_{\mathcal{K}}$ such that:

- $\pi|_{\mathfrak{B}} = \mathrm{id}|_{\mathfrak{B}}$,
- $\pi|_{\mathfrak{A}} \neq \mathrm{id}|_{\mathfrak{A}}$.

A Fraïssé symmetry $(\mathbb{A}_{\mathcal{K}}, G_{\mathcal{K}})$ is weakly fungible if every finitely generated substructure $\mathfrak{A}$ of $\mathfrak{U}_{\mathcal{K}}$ is weakly fungible.

On the other hand, if we restrict ourselves to relational structures, we can also equivalently require an existence of automorphisms π that satisfy a stronger condition.

Definition 4.9. A finitely generated substructure $\mathfrak{A}$ of $\mathfrak{U}_{\mathcal{K}}$ is *strongly fungible* if for every finitely generated substructure $\mathfrak{B} \subsetneq \mathfrak{A}$, there exists $\pi \in G_{\mathcal{K}}$ such that:

- $\pi|_{\mathfrak{B}} = \mathrm{id}|_{\mathfrak{B}}$,
- $\pi(\mathfrak{A}) \cap \mathfrak{A} = \mathfrak{B}$.

A Fraïssé symmetry $(\mathbb{A}_{\mathcal{K}}, G_{\mathcal{K}})$ is strongly fungible if every finitely generated substructure $\mathfrak{A}$ of $\mathfrak{U}_{\mathcal{K}}$ is strongly fungible.

Fact 4.10. *Let $(\mathbb{A}_{\mathcal{K}}, G_{\mathcal{K}})$ be a Fraïssé symmetry over a signature containing only relation symbols. The following conditions are equivalent:*
(1) $(\mathbb{A}_{\mathcal{K}}, G_{\mathcal{K}})$ is weakly fungible,
(2) $(\mathbb{A}_{\mathcal{K}}, G_{\mathcal{K}})$ is fungible,
(3) $(\mathbb{A}_{\mathcal{K}}, G_{\mathcal{K}})$ is strongly fungible.

The general picture is more complicated. When we introduce function symbols, the notions of weak fungibility, fungibility and strong fungibility differ from each other. Before showing this let us notice that the condition of strong fungibility is in fact equivalent to the strong amalgamation property.

Proposition 4.11. *A Fraïssé symmetry* $(\mathbb{A}_{\mathcal{K}}, G_{\mathcal{K}})$ *is strongly fungible if and only if the age* $\mathcal{K}$ *of the universal structure* $\mathfrak{U}_{\mathcal{K}}$ *has the strong amalgamation property.*

Proof. The *if* part is easily proved using homogeneity. For the *only if* part take any finitely generated substructures $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ of $\mathfrak{U}_{\mathcal{K}}$ and embeddings $f_{\mathfrak{B}}\colon \mathfrak{A} \to \mathfrak{B}$, $f_{\mathfrak{C}}\colon \mathfrak{A} \to \mathfrak{C}$. Thanks to amalgamation there exists a finitely generated substructure $\mathfrak{D}$ of $\mathfrak{U}_{\mathcal{K}}$ together with two embeddings $g_{\mathfrak{B}}\colon \mathfrak{B} \to \mathfrak{D}$ and $g_{\mathfrak{C}}\colon \mathfrak{C} \to \mathfrak{D}$ such that $g_{\mathfrak{B}} \circ f_{\mathfrak{B}}(\mathfrak{A}) = g_{\mathfrak{C}} \circ f_{\mathfrak{C}}(\mathfrak{A}) = \mathfrak{A}'$. Take $\pi \in G_{\mathcal{K}}$ for which $\pi|_{\mathfrak{A}'} = \mathrm{id}|_{\mathfrak{A}'}$ and $\pi(\mathfrak{D}) \cap \mathfrak{D} = \mathfrak{A}'$. Let $\mathfrak{D}'$ be a substructure generated by $\mathfrak{D} \cup \pi(\mathfrak{D})$. The embeddings $g_{\mathfrak{B}}$ and $g'_{\mathfrak{C}} = \pi \circ g_{\mathfrak{C}}$ into $\mathfrak{D}'$ are as needed:

$$g_{\mathfrak{B}} \circ f_{\mathfrak{B}}(\mathfrak{A}) = g'_{\mathfrak{C}} \circ f_{\mathfrak{C}}(\mathfrak{A}) = g_{\mathfrak{B}}(\mathfrak{B}) \cap g'_{\mathfrak{C}}(\mathfrak{C}).$$

Corollary 4.12. *A Fraïssé symmetry* $(\mathbb{A}_{\mathcal{K}}, G_{\mathcal{K}})$ *over a signature containing only relation symbols is fungible if and only if the age* $\mathcal{K}$ *of the universal structure* $\mathfrak{U}_{\mathcal{K}}$ *has the strong amalgamation property.*

Example 4.13. Consider an algebraic signature with unary function symbols F and G. For any integer i let $\mathbb{A}_i$ be the set of all infinite, binary sequences $\langle a_n \rangle$ defined for $n \geq i$ and equal 0 almost everywhere. Take $\mathbb{A} = \bigcup \mathbb{A}_i$ and define a structure $\mathfrak{U}$ with a carrier $\mathbb{A}$, where

$$F(\langle a_i, a_{i+1}, a_{i+2}, ...\rangle) = \langle a_{i+1}, a_{i+2}, ...\rangle, \quad G(0w) = 1w, \quad G(1w) = 0w.$$

Since the structure is weakly homogeneous, we obtain a Fraïssé symmetry. The symmetry is weakly fungible, but it is not fungible, as the structure generated by $\{0w, 1w\}$ is not fungible for any $w \in \mathbb{A}$.

Example 4.14. Consider an algebraic signature with a single unary function symbol F. For any integer i let $\mathbb{A}_i$ be the set of all infinite sequences $\langle a_n \rangle$ of natural numbers defined for $n \geq i$ and equal 0 almost everywhere. Take $\mathbb{A} = \bigcup \mathbb{A}_i$ and define a structure $\mathfrak{U}$ with a carrier $\mathbb{A}$, where

$$F(\langle a_i, a_{i+1}, a_{i+2}, ...\rangle) = \langle a_{i+1}, a_{i+2}, ...\rangle.$$

Notice that the age of $\mathfrak{U}$ is the class $\mathcal{K}$ of all finitely generated structures that satisfy the following axioms

- for any a, b there exist $m, n \in \mathbb{N}$ such that $F^m(a) = F^n(b)$,
- there are no loops, i.e., $F^n(a) \neq a$ for all $n \in \mathbb{N}$.

Since the structure is weakly homogeneous, we obtain a Fraïssé symmetry $(\mathbb{A}_{\mathcal{K}}, G_{\mathcal{K}})$. It is easy to check that the symmetry is fungible.

Now, take any nonempty finitely generated substructure $\mathfrak{A}$ of $\mathfrak{U}$ and the empty substructure $\emptyset \subseteq \mathfrak{A}$. For any automorphism π of $\mathfrak{U}$ and $a \in \mathfrak{A}$ there exist $m, n \in \mathbb{N}$ for which $F^m(a) = F^n(a \cdot \pi)$. Hence there is no π for which $\pi(\mathfrak{A}) \cap \mathfrak{A} = \emptyset$ and the structure $\mathfrak{A}$ is not strongly fungible. Therefore the symmetry $(\mathbb{A}_{\mathcal{K}}, G_{\mathcal{K}})$ is not strongly fungible.

4.2 Representation of Functions

For any finitely generated substructure $\mathfrak{A}$ of $\mathfrak{U}_{\mathcal{K}}$ and any $S \leq \mathrm{Aut}(\mathfrak{A})$, the $G_{\mathcal{K}}$-extension of S is

$$ext_{G_{\mathcal{K}}}(S) = \{\pi \in G_{\mathcal{K}} \mid \pi|_{\mathfrak{A}} \in S\} \leq G_{\mathcal{K}}.$$

Notice that $ext_{G_{\mathcal{K}}}(S)$ is exactly the stabilizer of $[\mathrm{id}\,|_{\mathfrak{A}}]_S$ in $G_{\mathcal{K}}$.

Lemma 4.15. *For each embedding $u\colon \mathfrak{A} \to \mathfrak{U}_{\mathcal{K}}$ the group $ext_{G_{\mathcal{K}}}(u^{-1}Su)$, where $u^{-1}Su \leq \mathrm{Aut}(u(\mathfrak{A}))$, is the stabilizer of an element $[u]_S \in [\mathfrak{A}, S]$.*

Proof. For any $\pi \in G_{\mathcal{K}}$ that extends u we have $[u]_S = [\mathrm{id}\,|_{\mathfrak{A}}]_S \cdot \pi$. Hence, by Lemma 2.6, the stabilizer of $[u]_S$ is $\pi^{-1}ext_{G_{\mathcal{K}}}(S)\pi$. It is easy to check that

$$\pi^{-1}ext_{G_{\mathcal{K}}}(S)\pi = ext_{G_{\mathcal{K}}}(u^{-1}Su).$$

Lemma 4.16. *For any supportable and fungible Fraïssé symmetry $(\mathbb{A}_{\mathcal{K}}, G_{\mathcal{K}})$ let $\mathfrak{A}, \mathfrak{B}$ be finitely generated substructures of $\mathfrak{U}_{\mathcal{K}}$ and let $S \leq \mathrm{Aut}(\mathfrak{A})$, $T \leq \mathrm{Aut}(\mathfrak{B})$, then $ext_{G_{\mathcal{K}}}(S) \leq ext_{G_{\mathcal{K}}}(T)$ if and only if $\mathfrak{B} \subseteq \mathfrak{A}$ and $S|_{\mathfrak{B}} \leq T$.*

Proof. The *if* part is obvious. For the *only if* part, we first prove that $\mathfrak{B} \subseteq \mathfrak{A}$. Notice that if $\pi|_{\mathfrak{A}} = \mathrm{id}\,|_{\mathfrak{A}}$ then $\pi \in ext_{G_{\mathcal{K}}}(S)$ and hence $\pi \in ext_{G_{\mathcal{K}}}(T)$, which is the stabilizer of $[\mathrm{id}\,|_{\mathfrak{B}}]_T$. Therefore $\mathfrak{A}$ supports $[\mathrm{id}\,|_{\mathfrak{B}}]_T$. By Lemma 4.7 (2) the least support of $[\mathrm{id}\,|_{\mathfrak{B}}]_T$ is $\mathfrak{B}$. Hence $\mathfrak{B} \subseteq \mathfrak{A}$. Then we have

$$ext_{G_{\mathcal{K}}}(S) \leq ext_{G_{\mathcal{K}}}(T)$$

$$\Updownarrow$$

$$\forall \pi \in G_{\mathcal{K}} \quad \pi|_{\mathfrak{A}} \in S \Longrightarrow \pi|_{\mathfrak{B}} \in T$$

$$\Updownarrow$$

$$\forall \pi \in G_{\mathcal{K}} \quad \pi|_{\mathfrak{A}} \in S \Longrightarrow (\pi|_{\mathfrak{A}})|_{\mathfrak{B}} \in T$$

$$\Updownarrow$$

$$\forall \tau \in S \quad \tau|_{\mathfrak{B}} \in T.$$

Similar facts about finite substructures of a universal relational structure $\mathfrak{U}_{\mathcal{K}}$ were proven in [BKL]. The following proposition generalizes Proposition 11.8.

Proposition 4.17. *For any supportable and fungible Fraïssé symmetry $(\mathbb{A}_{\mathcal{K}}, G_{\mathcal{K}})$ let $X = [\mathfrak{A}, S]$ and $Y = [\mathfrak{B}, T]$ be single-orbit nominal sets. The set of equivariant functions from X to Y is in one to one correspondence with the set of embeddings $u\colon \mathfrak{B} \to \mathfrak{A}$, for which $uS \subseteq Tu$, quotiented by $\equiv_T$.*

Proof. By Proposition 2.7 and Lemma 4.15 equivariant functions from $[\mathfrak{A}, S]$ to $[\mathfrak{B}, T]$ are in bijective correspondence with those elements $[u]_T \in [\mathfrak{B}, T]$ for which

$$ext_{G_\mathcal{K}}(S) \leq ext_{G_\mathcal{K}}(u^{-1}Tu).$$

Hence, by Lemma 4.16, equivariant functions from $[\mathfrak{A}, S]$ to $[\mathfrak{B}, T]$ correspond to those elements $[u]_T \in [\mathfrak{B}, T]$ for which

$$u(\mathfrak{B}) \subseteq \mathfrak{A} \quad \text{and} \quad S|_{u(\mathfrak{B})} \leq u^{-1}Tu,$$

which means that u is an embedding from $\mathfrak{B}$ to $\mathfrak{A}$ and $uS \subseteq Tu$, as required.

Let $G_\mathcal{K}$-$\mathbf{Nom}^\mathbf{1}$ denote the category of single-orbit nominal sets and equivariant functions. Propositions 4.3 and 4.17 can be phrased in the language of category theory:

Proposition 4.18. *In a supportable and fungible Fraïssé symmetry, the category* $G_\mathcal{K}$-$\mathbf{Nom}^\mathbf{1}$ *is equivalent to the category with:*

- *as objects, pairs* $(\mathfrak{A}, S)$ *where* $\mathfrak{A} \in \mathcal{K}$ *and* $S \leq \mathrm{Aut}(\mathfrak{A})$,
- *as morphisms from* $(\mathfrak{A}, S)$ *to* $(\mathfrak{B}, T)$*, those embeddings* $u\colon \mathfrak{B} \to \mathfrak{A}$ *for which* $uS \subseteq Tu$*, quotiented by* $\equiv_T$*.*

Since a nominal set is a disjoint union of single-orbit sets, this representation extends to orbit-finite sets in an obvious way:

Theorem 4.19. *In a supportable and fungible Fraïssé symmetry, the category* $G_\mathcal{K}$-$\mathbf{Nom}$ *is equivalent to the category with:*

- *as objects, finite sets of pairs* $(\mathfrak{A}_i, S_i)$ *where* $\mathfrak{A}_i \in \mathcal{K}$ *and* $S_i \leq \mathrm{Aut}(\mathfrak{A}_i)$,
- *as morphisms from* $\{(\mathfrak{A}_1, S_1), \ldots, (\mathfrak{A}_n, S_n)\}$ *to* $\{(\mathfrak{B}_1, T_m), \ldots, (\mathfrak{B}_m, T_m)\}$*, pairs* $(f, \{[u_i]_{T_{f(i)}}\}_{i=1,\ldots,n})$*, where* $f\colon \{1, \ldots, n\} \to \{1, \ldots, m\}$ *is a function and each* u_i *is an embedding* $u_i\colon \mathfrak{B}_{f(i)} \to \mathfrak{A}_i$ *such that* $u_i S_i \subseteq T_{f(i)} u_i$*.*

In the special case of relational structures the above theorem was formulated and proved in [BKL].

5 Conclusions and Future Work

Orbit-finite nominal sets can be used to model devices, such as automata or Turing machines, which operate over infinite alphabets. This approach makes sense only if one can treat objects with atoms as data structures and manipulate them using algorithms. To do so the existence of a finite representation of orbit-finite nominal sets is crucial.

In this paper we have generalized the representation theorem due to Bojańczyk et al. to cover atoms with algebraic structure. The result is however not entirely satisfying. Our representation uses automorphism groups of finitely generated substructures of the atoms. If such groups are finitely presentable Theorem 4.19

indeed provides a concrete, finite representation of orbit-finite nominal sets (the timed symmetry being an example). But is it always the case? So far we do not know and we regard it as a field for a further research effort.

Another thing left to be done is a characterization of "well-behaved" atom symmetries in terms of Fraïssé classes that induce them. One might think of algebraic atoms that could be potentially interesting from the point of view of computation theory: strings with the concatenation operator, binary vectors with addition, etc. Yet checking the technical conditions, such as supportability and fungibility, needed for the representation theorem to hold requires each time a lot of effort. This is because these conditions are formulated in terms of Fraïssé limits, and these are not always easy to construct. It would be desirable to have more natural criteria that would be easier to verify.

Acknowledgments. I would like to thank Mikołaj Bojańczyk, Bartek Klin, Sławomir Lasota and Szymon Toruńczyk for inspiring discussions and helpful remarks on earlier drafts of this paper, and the anonymous referees for many insightful comments.

References

[BKL] Bojańczyk, M., Klin, B., Lasota, S.: Automata Theory in Nominal Sets (to appear)

[BKL11] Bojańczyk, M., Klin, B., Lasota, S.: Automata with Group Actions. In: Proc. LICS 2011, pp. 355–364 (2011)

[BL12] Bojańczyk, M., Lasota, S.: A Machine-independent Characterization of Timed Languages. In: Czumaj, A., Mehlhorn, K., Pitts, A., Wattenhofer, R. (eds.) ICALP 2012, Part II. LNCS, vol. 7392, pp. 92–103. Springer, Heidelberg (2012)

[BKLT13] Bojańczyk, M., Klin, B., Lasota, S., Toruńczyk, S.: Turing Machines with Atoms. In: Proc. LICS 2013, pp. 183–192 (2013)

[FS06] Fiore, M., Staton, S.: Comparing Operational Models of Name-passing Process Calculi. Inf. Comput. 204, 524–560 (2006)

[GP02] Gabbay, M.J., Pitts, A.M.: A new approach to abstract syntax with variable binding. Formal Aspects of Computing 13, 341–363 (2002)

[GMM06] Gadducci, F., Miculan, M., Montanari, U.: About Permutation Algebras (Pre)Sheaves and Named Sets. Higher Order Symbol. Comput. 19, 283–304 (2006)

[Hod93] Hodges, W.: Model theory. Cambridge University Press (1993)

[KF94] Kaminski, M., Francez, N.: Finite-memory Automata. Theor. Comput. Sci. 134, 329–363 (1994)

[Pis99] Pistore, M.: History Dependent Automata. PhD thesis, Università di Pisa, Dipartimento di Informatica. available at University of Pisa as PhD Thesis TD-5/99 (1999)

[Pit13] Pitts, A.M.: Nominal Sets: Names and Symmetry in Computer Science. Cambridge Tracts in Theoretical Computer Science, vol. 57. Cambridge University Press (2013)