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States in Deformation Quantization

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Abstract. We consider several tests to check whether a function defined on the phase space of a system represents a quantum state. Our criteria have been obtained from theorems holding for a density operator in the Hilbert space formulation of quantum mechanics. The tests are based on a notion of trace and follow from their Hilbert space counterparts through the Stratonovich–Weyl correspondence.

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1. Introduction

Physics is an experimental science. Thus its mathematical structure has to respect measurements, i.e., contain a class of physically important quantities – observables and predict results of a single measurement as well as a mean value of a series of observations.

There is no formal definition of an observable. We assume that in the phase space formulation of quantum mechanics [1–3] measurable quantities are represented by smooth real functions on a symplectic manifold, but other functions and even generalized functions can be considered. In the Hilbert space formulation of quantum theory the observables are identified with linear self-adjoint operators [4, 5].

For a given observable the result of a single observation or the average of a series of measurements depend on a state. Hence in the phase space quantum mechanics information about the state is contained in a linear functional satisfying some extra conditions. We will consider this problem in the next paragraph. In the Hilbert space version of quantum physics normalizable pure states are represented by vectors of the unit length from a Hilbert space $\mathbf{H}$ of the system. An arbitrary state belongs to a convex set spanned by the pure states [6].

The strict relationship between a set of observables and a space of states looks as follows. We choose an algebra $\mathbf{A}$ with involution $*$. The algebra $\mathbf{A}$ contains some subset of the set of observables. Quantum states are represented by positive linear functionals $f \in \mathbf{F}$ over the $*$ -algebra $\mathbf{A}$.

$$\forall\, A \in \mathbf{A} \quad f(A^+ A) \geq 0. \tag{1}$$

Moreover, the states obey the normalization condition

$$f(\mathbf{1}) = 1.$$

By $\mathbf{1}$ we denote the unity of the algebra $\mathbf{A}$.

Thus the expected value of an observable $A \in \mathbf{A}$ in a state $f \in \mathbf{F}$ is equal to

$$\langle A \rangle := f(A).$$

Another approach to states is based on the following idea. We distinguish a special class of linear normalized positive functionals called pure states and then we build a convex set spanned by these special states.

Let us consider the Hilbert space formulation of quantum mechanics. The $*$ -algebra $\mathbf{A}$ in this case is the algebra $\mathbf{B}(\mathbf{H})$ of bounded linear operators whose domain is the whole Hilbert space $\mathbf{H}$. The functional is defined as

$$f(A) := \mathrm{Tr}(\hat{\varrho}\hat{A})\ \forall\ \hat{A} \in \mathbf{A}. \tag{2}$$

The symbol $\hat{\varrho}$ denotes the density operator.

The positivity condition (1) is hardly testable. The same about convexity of a set. Thus we propose several applicable criteria of checking if a given functional really represents a physical state. These criteria are divided in two groups: tests for density operators and for Wigner functions. Their complete list with proofs can be found in our paper [7]. In this contribution only the tests based on the notion of trace are presented.

A square matrix C of a dimension $\dim \mathbf{H} \times \dim \mathbf{H}$ is symbolized by $[\langle \varphi_i | C | \varphi_j \rangle]_1^{\dim \mathbf{H}}$ while an element of this matrix is represented as $\langle \varphi_i | C | \varphi_j \rangle$.

2. Hilbert space version of quantum mechanics

Let us consider a quantum system modeled on a separable Hilbert space $\mathbf{H}$. By $\{|\varphi_j\rangle\}_{j=1}^{\dim \mathbf{H}}$ we mean a complete set of orthonormal vectors in $\mathbf{H}$. We do not know the state vector of the system but only a probability of detecting the system in each of the states $|\varphi_j\rangle$.

As it was postulated by von Neumann [8], in such cases the state of the quantum system is characterized by a positive functional determined by a density operator.

Definition 1. The operator given by

$$\hat{\varrho} := \mathrm{u}-\lim_{n \to \dim \mathbf{H}} \sum_{j=1}^{n} p_j |\varphi_j\rangle\langle\varphi_j|, \quad \forall j \ \ p_j \geq 0 \ \ , \quad \sum_{j=1}^{\dim \mathbf{H}} p_j = 1$$

is called a **density operator**. Each number p_j, $j = 1, 2, \dots, \dim \mathbf{H}$ is equal to the probability of observing the system in the state represented by the ket $|\varphi_j\rangle$. If one of these numbers is equal to 1 we say that the system is in a **pure state**. Otherwise the system is in a **mixed state**.

The symbol u$-$ denotes the uniform convergence of a sequence of operators. An equivalent formulation of Definition 1 is the following.

Definition 2. An operator $\hat{\varrho} : \mathbf{H} \to \mathbf{H}$ is a **density operator** if it is:

1. positive, i.e., $\langle\phi\big|\hat{\varrho}\big|\phi\rangle \geq 0 \; \forall \big|\phi\rangle \in \mathbf{H}$,
2. self-adjoint $\hat{\varrho}^+ = \hat{\varrho}$,
3. $\mathrm{Tr}\,\hat{\varrho} = 1$.

As it was mentioned in the Introduction, the functional action is determined by the formula (2). From Definitions 1 and 2 we can deduce several properties of the density operator.

- $\mathrm{Tr}\,\hat{\varrho}^2 \leq 1$. Thus the density operator is a Hilbert–Schmidt operator. Its Hilbert–Schmidt norm, given by
$$||\hat{\varrho}||_2 := \sqrt{\mathrm{Tr}(\hat{\varrho}^+ \hat{\varrho})},$$
is not greater than 1. Moreover, $||\hat{\varrho}||_2 = 1$ if and only if the density operator represents a pure state.
- As eigenvalues of the density operator $\hat{\varrho}$ are nonnegative, the density operator is a trace class operator and its trace norm
$$||\hat{\varrho}||_1 := \mathrm{Tr}\,\sqrt{\hat{\varrho}^+ \hat{\varrho}} = \mathrm{Tr}\,|\hat{\varrho}| = \mathrm{Tr}\,\hat{\varrho} = 1.$$
For each trace class operator $\hat{A}$ the following estimation holds
$$||\hat{A}|| \leq ||\hat{A}||_2 \leq ||\hat{A}||_1.$$
- The density operator is positive. Hence for every operator $\hat{A} \in \mathbf{B}(\mathbf{H})$ the mean value of the product $\hat{A}\hat{A}^+$ obeys
$$\langle \hat{A}\hat{A}^+ \rangle = \mathrm{Tr}(\hat{\varrho}\hat{A}\hat{A}^+) \geq 0.$$

Assume that a matrix representation $[\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle]_1^{\dim \mathbf{H}}$ of an operator $\hat{\varrho}$ is known. We intend to settle whether this matrix represents a physical state of a quantum system. We consider finite- and infinite-dimensional separable Hilbert spaces and discuss mixed as well as pure states.

2.1. A finite-dimensional Hilbert space

At the beginning we consider a finite-dimensional Hilbert space. In this case the matrix $[\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle]_1^{\dim \mathbf{H}}$ completely determines the operator $\hat{\varrho}$, which is defined on the whole space $\mathbf{H}$.

Applying elementary linear algebra we propose the following algorithm.

1. First we test if the matrix is symmetric
$$\forall\, i, j \quad \langle\varphi_i|\hat{\varrho}|\varphi_j\rangle = \overline{\langle\varphi_j|\hat{\varrho}|\varphi_i\rangle}.$$

2. If the answer is positive, we calculate its trace $\sum_{i=1}^{\dim \mathbf{H}} \langle\varphi_i|\hat{\varrho}|\varphi_i\rangle$.
3. If the trace is equal to 1 we go to the last step, in which we find the principal minors and decide whether the matrix is positive.

A symmetric matrix of trace 1 with all the principal minors nonnegative is a density matrix.

For a pure state this procedure becomes simpler. A matrix $[\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle]_1^{\dim \mathbf{H}}$ represents a pure state if it is symmetric, its trace is equal to 1 and the square of it is the same matrix

$$\forall\, i,j \quad \sum_{k=1}^{\dim \mathbf{H}} \langle\varphi_i|\hat{\varrho}|\varphi_k\rangle\langle\varphi_k|\hat{\varrho}|\varphi_j\rangle = \langle\varphi_i|\hat{\varrho}|\varphi_j\rangle.$$

2.2. An infinite-dimensional Hilbert space

The case of an infinite-dimensional space is more complicated. It may happen that in a given orthonormal basis $\{|\varphi_j\rangle\}_{j=1}^{\infty}$ the matrix $\left[\langle\varphi_i|\hat{A}|\varphi_j\rangle\right]_1^{\infty}$ of an operator $\hat{A}$ exists but does not uniquely characterize this operator.

Thus to know whether a matrix $[\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle]_1^{\infty}$ can represent a density operator, we propose to check first if $\hat{\varrho}$ is a Hilbert–Schmidt operator, i.e.,

$$\sum_{i,j=1}^{\infty} |\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle|^2 < \infty.$$

Every Hilbert–Schmidt operator $\hat{A}$ is bounded and is defined on the whole space $\mathbf{H}$. Moreover, the matrix $\left[\langle\varphi_i|\hat{A}|\varphi_j\rangle\right]_1^{\infty}$ completely characterizes the operator $\hat{A}$. In addition, if the matrix $\left[\langle\varphi_i|\hat{A}|\varphi_j\rangle\right]_1^{\infty}$ is symmetric, we conclude that the operator $\hat{A}$ is self adjoint. Hence all of our criteria will start from checking, if a given operator is of the Hilbert–Schmidt class and if it is symmetric.

In the next step we have to test if the matrix $[\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle]_1^{\infty}$ represents a positive operator. This operation is the most complicated and can be done in many ways. Finally one calculates the trace of the matrix $[\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle]_1^{\infty}$.

Several realizations of tests are formulated below. We write down only hints at which proofs of these statements are based. More detailed explanation can be found in [7].

Theorem 1. *A matrix $[\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle]_1^{\infty}$ represents a quantum state iff:*

1. $\sum_{i,j=1}^{\infty} |\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle|^2 \leq 1$,
2. $\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle = \overline{\langle\varphi_j|\hat{\varrho}|\varphi_i\rangle}\ \forall\ 1 \leq i,j < \infty$,
3. $$\frac{1}{2}\sum_{i=1}^{\infty}\langle\varphi_i|\hat{\varrho}^2|\varphi_i\rangle + \sum_{l=2}^{\infty}\frac{(-1)^{l+1}}{l!}\frac{(2l-3)!!}{2^l}\sum_{r=0}^{l-1}(-1)^r\binom{l}{r} \times \sum_{i=1}^{\infty}\langle\varphi_i|\hat{\varrho}^{2(l-r)}|\varphi_i\rangle = 1, \qquad (3)$$
4. $\sum_{i=1}^{\infty}\langle\varphi_i|\hat{\varrho}|\varphi_i\rangle = 1$.

Formula (3) comes from the observation that for a density operator the square root $\sqrt{\hat{\varrho}^2}$ defined by the series must be equal to the operator $\hat{\varrho}$. Thus $\mathrm{Tr}\,\sqrt{\hat{\rho}^2} = 1$.

Theorem 2. *A matrix* $[\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle]_1^\infty$ *represents a quantum state iff:*

1. $\sum_{i,j=1}^{\infty}|\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle|^2 \leq 1$,
2. $\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle = \overline{\langle\varphi_j|\hat{\varrho}|\varphi_i\rangle}\ \forall\ 1 \leq i,j < \infty$,
3. *for every natural number* n *the sum is nonnegative*
$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{i=1}^{\infty}\langle\varphi_i|\hat{\varrho}^{k+1}|\varphi_i\rangle \geq 0, \tag{4}$$
4. $\sum_{i=1}^{\infty}\langle\varphi_i|\hat{\varrho}|\varphi_i\rangle = 1$.

In the conditions 3 and 4 we check if all of eigenvalues of the matrix $[\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle]_1^\infty$ are nonnegative and they do not exceed 1.

Theorem 3. *A matrix* $[\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle]_1^\infty$ *represents a quantum state iff:*

1. $\sum_{i,j=1}^{\infty}|\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle|^2 \leq 1$,
2. $\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle = \overline{\langle\varphi_j|\hat{\varrho}|\varphi_i\rangle}\ \forall\ 1 \leq i,j < \infty$,
3. *the limit*
$$\lim_{n\to\infty}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{i=1}^{\infty}\langle\varphi_i|\hat{\varrho}^{k+1}|\varphi_i\rangle = 0,$$
4. $\sum_{i=1}^{\infty}\langle\varphi_i|\hat{\varrho}|\varphi_i\rangle = 1$.

In fact this theorem is a consequence of Theorem 2 because for nonnegative eigenvalues $e_1, e_2, \ldots$ of the matrix $[\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle]_1^\infty$ the sum $\sum_{i=1}^{\infty} e_i(1-e_i)^n$ tends to 0 as $n \to \infty$.

For pure states the testing procedure can be simplified.

Theorem 4. *A matrix* $[\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle]_1^\infty$ *represents a pure quantum state iff:*

1. $\sum_{i,j=1}^{\infty}|\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle|^2 = 1$,
2. $\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle = \overline{\langle\varphi_j|\hat{\varrho}|\varphi_i\rangle}\ \forall\ 1 \leq i,j < \infty$,
3. $\sum_{j=1}^{\infty}\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle\langle\varphi_j|\hat{\varrho}|\varphi_k\rangle = \langle\varphi_i|\hat{\varrho}|\varphi_k\rangle$,
4. $\sum_{i=1}^{\infty}\langle\varphi_i|\hat{\varrho}|\varphi_i\rangle = 1$.

This theorem states that the density operator of a pure state must be a projective operator.

3. Phase space version of quantum mechanics

When the phase space formulation of quantum mechanics is considered, two fundamental elements: a phase space and a $*$-product must be taken into account. We restrict ourselves to problems, in which phase spaces are differentiable symplectic manifolds.

On every symplectic manifold $(\mathbf{M}, \omega)$ there exists a nontrivial $*$-product. We assume that the $*$-product is local and in its differential form is of the Weyl type.

Definition 3. An **observable** on a phase space $(\mathbf{M}, \omega)$ is any smooth real function on $\mathbf{M}$ being a formal series in the Planck constant $\hbar$

$$C^\infty(\mathbf{M})[[\hbar]] \ni A(q^1, \ldots, q^{2n}) = \sum_{i=0}^{\infty} \hbar^i A_i(q^1, \ldots, q^{2n}). \tag{5}$$

As it was explained in the Introduction, construction of the space of states is based on some algebra with an involution. This algebra contains a subset of the set of observables. In deformation quantization the $*$-algebra $\mathbf{A}$ consists of all smooth functions on $(\mathbf{M}, \omega)$, which are formal series in $\hbar$ and have compact supports. The involution '$*$' is realized by the complex conjugation. The product in the algebra $\mathbf{A}$ is a Weyl type $*$-product.

According to the general definition, quantum states are positive linear functionals over the algebra $\mathbf{A}$ satisfying the normalization condition. Every such functional f can be written in the following form

$$f\Big(A(q^1, \ldots, q^{2n})\Big) = \int_{\mathbf{M}} A(q^1, \ldots, q^{2n}) * W(q^1, \ldots, q^{2n}) t(q^1, \ldots, q^{2n}) \omega^n \tag{6}$$
$$= \int_{\mathbf{M}} W(q^1, \ldots, q^{2n}) * A(q^1, \ldots, q^{2n}) t(q^1, \ldots, q^{2n}) \omega^n, \ \forall\, A(q^1, \ldots, q^{2n}) \in \mathbf{A}.$$

A real function

$$C^\infty(\mathbf{M})[[\hbar]] \ni t(q^1, \ldots, q^{2n}) = \sum_{i=0}^{\infty} \hbar^i t_i(q^1, \ldots, q^{2n})$$

is called a trace density. The trace density ensures that the integral (6) possesses a trace property

$$\int_{\mathbf{M}} A(q^1, \ldots, q^{2n}) * B(q^1, \ldots, q^{2n}) t(q^1, \ldots, q^{2n}) \omega^n$$
$$= \int_{\mathbf{M}} B(q^1, \ldots, q^{2n}) * A(q^1, \ldots, q^{2n}) t(q^1, \ldots, q^{2n}) \omega^n,$$
$$\forall\, A(q^1, \ldots, q^{2n}),\ B(q^1, \ldots, q^{2n}) \in \mathbf{A}.$$

The trace density is determined by a symplectic curvature tensor and its derivatives [9, 10].

A function $W(q^1, \ldots, q^{2n})$ contains information about the state and is called a **Wigner function**. We will restrict ourselves to the functions $W(q^1, \ldots, q^{2n})$ which are $*$- square integrable, i.e.,

$$\int_{\mathbf{M}} W(q^1, \ldots, q^{2n}) * W(q^1, \ldots, q^{2n}) t(q^1, \ldots, q^{2n}) \omega^n < \infty$$

for each fixed positive value of the deformation parameter $\hbar$.

The integral

$$\frac{1}{(2\pi\hbar)^n} \int_{\mathbf{M}} A(q^1, \ldots, q^{2n}) t(q^1, \ldots, q^{2n}) \omega^n \tag{7}$$

is often called a **trace** as is a classical counterpart of the trace of operator.

3.1. The Stratonovich–Weyl correspondence

Considerations from the current section are based on belief that the Hilbert space formulation of quantum mechanics and its phase space version are equivalent. This equivalence is expressed by the Stratonovich–Weyl correspondence $SW : \mathbf{A_H} \to \mathbf{A}$ between an algebra of operators $\mathbf{A_H}$ and an algebra of functions $\mathbf{A}$. Although a general form of this relationship is not known, it should satisfy a few natural requirements. First of all the SW mapping is one to one. The choice of the algebra of operators $\mathbf{A_H}$ determines the choice of the algebra of functions $\mathbf{A}$. On the other hand we know, that this choice is not unique and for different algebras $\mathbf{A_H}$ or equivalently $\mathbf{A}$ we obtain the same quantum mechanics.

Next, the SW correspondence is linear. Moreover, $SW(\hat{A}^{+}) = \overline{SW(\hat{A})}$. The image $SW(\hat{1}) = 1$, i.e., the constant function equal 1 on the whole symplectic manifold $\mathbf{M}$ represents the identity operator. If an operator $\hat{A}$ is self adjoint then $SW(\hat{A})$ is a real function. Finally, the $\cdot$ -product of operators is represented by the $*$-multiplication of functions,

$$SW(\hat{A} \cdot \hat{B}) = SW(\hat{A}) * SW(\hat{B})$$

and the trace of an operator is equal to

$$\operatorname{Tr} \hat{A} = \frac{1}{(2\pi\hbar)^n} \int_{\mathbf{M}} SW(\hat{A})(q^1, \ldots, q^{2n}) t(q^1, \ldots, q^{2n}) \omega^n .$$

3.2. Wigner function

The Stratonovich–Weyl correspondence establishes a correspondence between a density operator $\hat{\varrho}$ and a Wigner function $W(q^1, \ldots, q^{2n})$

$$SW\left(\frac{1}{(2\pi\hbar)^n} \hat{\varrho}\right) = W(q^1, \ldots, q^{2n}).$$

Applying the Stratonovich–Weyl mapping to a density operator we find properties of the Wigner function on an arbitrary symplectic manifold.

Integration of the Wigner function yields

$$\int_{\mathbf{M}} W(q^1, \ldots, q^{2n}) t(q^1, \ldots, q^{2n}) \omega^n = 1.$$

This result is an immediate consequence of the fact that $\operatorname{Tr} \hat{\varrho} = 1$. Moreover, every Wigner function, as a counterpart of the self-adjoint operator, is real. Since a density operator is of a Hilbert–Schmidt type, the integral satisfies the estimation

$$\int_{\mathbf{M}} W^{*2}(q^1, \ldots, q^{2n})\, t(q^1, \ldots, q^{2n}) \omega^n \leq \frac{1}{(2\pi\hbar)^n}.$$

The equality holds only for pure states.

Applying the Stratonovich–Weyl correspondence we present criteria to test, if a given function $W(q^1, \ldots, q^{2n})$ on the phase space of a system represents a physical state. It is easily to see that these criteria are phase space counterparts of Theorems 1–4.

Theorem 5. *A function $W(q^1,\ldots,q^{2n})$ defined on a symplectic manifold $(\mathbf{M},\omega)$ is a Wigner function iff:*

1. $\int_{\mathbf{M}} W^{*2}(q^1,\ldots,q^{2n})\, t(q^1,\ldots,q^{2n})\omega^n \leq \frac{1}{(2\pi\hbar)^n}$,
2. *the function is real,*
3. $$\frac{1}{2}\int_{\mathbf{M}} W^{*2}(q^1,\ldots,q^{2n})\, t(q^1,\ldots,q^{2n})\omega^n + \sum_{l=2}^{\infty} \frac{(-1)^{l+1}}{l!}\frac{(2l-3)!!}{2^l} \times \sum_{r=0}^{l-1}(-1)^r \binom{l}{r}(2\pi\hbar)^{2n(l-r-1)} \int_{\mathbf{M}} W^{*2(l-r)}(q^1,\ldots,q^{2n})\, t(q^1,\ldots,q^{2n})\omega^n = \frac{1}{(2\pi\hbar)^n}, \qquad (8)$$
4. $\int_{\mathbf{M}} W(q^1,\ldots,q^{2n})\, t(q^1,\ldots,q^{2n})\omega^n = 1.$

Theorem 6. *A function $W(q^1,\ldots,q^{2n})$ defined on a symplectic manifold $(\mathbf{M},\omega)$ is a Wigner function of a quantum state iff:*

1. $\int_{\mathbf{M}} W^{*2}(q^1,\ldots,q^{2n})\, t(q^1,\ldots,q^{2n})\omega^n \leq \frac{1}{(2\pi\hbar)^n}$,
2. *the function is real,*
3. *for every natural number m the sum is nonnegative*
$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}(2\pi\hbar)^{nk}\int_{\mathbf{M}} W^{*(k+1)}(q^1,\ldots,q^{2n})\, t(q^1,\ldots,q^{2n})\omega^n \geq 0, \qquad (9)$$
4. $\int_{\mathbf{M}} W(q^1,\ldots,q^{2n})\, t(q^1,\ldots,q^{2n})\omega^n = 1.$

Theorem 7. *A function $W(q^1,\ldots,q^{2n})$ defined on a symplectic manifold $(\mathbf{M},\omega)$ represents a quantum state iff:*

1. $\int_{\mathbf{M}} W^{*2}(q^1,\ldots,q^{2n})\, t(q^1,\ldots,q^{2n})\omega^n \leq \frac{1}{(2\pi\hbar)^n}$,
2. *the function is real,*
3. $$\lim_{m\to\infty}\sum_{k=0}^{m}(-1)^k\binom{m}{k}(2\pi\hbar)^{nk}\int_{\mathbf{M}} W^{*(k+1)}(q^1,\ldots,q^{2n}) \times t(q^1,\ldots,q^{2n})\omega^n = 0, \qquad (10)$$
4. $\int_{\mathbf{M}} W(q^1,\ldots,q^{2n})\, t(q^1,\ldots,q^{2n})\omega^n = 1.$

An identification method of pure states is based on the following statement.

Theorem 8. *A function $W(q^1,\ldots,q^{2n})$ defined on a symplectic manifold $(\mathbf{M},\omega)$ represents a pure quantum state iff:*

1. $\int_{\mathbf{M}} W^{*2}(q^1,\ldots,q^{2n})\, t(q^1,\ldots,q^{2n})\omega^n = \frac{1}{(2\pi\hbar)^n}$,
2. *the function is real,*
3. $W^{*2}(q^1,\ldots,q^{2n}) = \frac{1}{(2\pi\hbar)^n}W(q^1,\ldots,q^{2n})$
4. $\int_{\mathbf{M}} W(q^1,\ldots,q^{2n})\, t(q^1,\ldots,q^{2n})\omega^n = 1.$

As an illustration of the presented criteria we examine a function considered by Tatarskij [1].

The function
$$W(p,q) = \frac{2}{3}W_0(p,q) + \frac{2}{3}W_1(p,q) - \frac{1}{3}W_2(p,q) \qquad (11)$$

is defined on the phase space $\mathbb{R}^2$. By $W_0(p,q)$, $W_1(p,q)$ and $W_2(p,q)$ we denote Wigner functions of mutually orthogonal states, i.e.,

$$\int_{\mathbb{R}^2} W_i(p,q) * W_j(p,q) dpdq = \frac{1}{2\pi\hbar}\delta_{ij}.$$

As it can be seen from (11), the function $W(p,q)$ is not a Wigner function, because one of its eigenvalues is negative. The mth star power of this function is equal

$$W^{*m}(p,q) = \frac{1}{(2\pi\hbar)^{m-1}}\left(\left(\frac{2}{3}\right)^m W_0(p,q) + \left(\frac{2}{3}\right)^m W_1(p,q) + \left(-\frac{1}{3}\right)^m W_2(p,q)\right).$$

The function $W(p,q)$ is real and the integral $\int_{\mathbb{R}^2} W(p,q)dpdq = 1$. Moreover, $\int_{\mathbb{R}^2} W^{*2}(p,q)dpdq = \frac{1}{2\pi\hbar}$ so the function $W(p,q)$ satisfies the conditions (i), (ii) and (iv) of Theorems 5–7.

However, for the function (11) the sum (8) is equal to $\frac{5}{3}\cdot\frac{1}{2\pi\hbar}$ $\left(\neq \frac{1}{2\pi\hbar}\right)$ so this function is not a Wigner function.

Applying Theorem 6 we see that the sum (9) is equal to $2\pi\hbar$ for $m = 0$, 0 for $m = 1$ and $-\frac{4}{9}\cdot 2\pi\hbar$ (< 0) for $m = 2$. Therefore after taking three steps we conclude that $W(p,q)$ does not represent any state.

The limit (10) is equal to $-\infty$ $(\neq 0)$ so the tested function obviously cannot be a Wigner function.

4. Conclusions

As criteria considered in our contribution require calculation of arbitrary powers of density matrices or arbitrary $*$-powers of Wigner functions, there might be some doubts about usefulness of this approach. However, it seems that at this moment there is no other constructive method of recognition of physical states. Moreover, tests of positivity are quite general so the presented results can be applied in different problems requiring analysis of positivity.

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References

[1] Tatarskij V.I., *The Wigner representation of quantum mechanics*, Usp. Fiz. Nauk **139** (1983), 311–327.

[2] Hillery M., O'Connell R.F., Scully M.O. and Wigner E.P., *Distribution functions in physics: fundamentals*, Phys. Rep. **106** (1984), 121–167.

[3] Lee H.W., *Theory and application of the quantum phase-space distribution functions*, Phys. Rep. **259** (1995), 147–211.

[4] Bohm A., *Quantum Mechanics: Foundations and Applications*, 3rd Edition, Springer-Verlag, New York, 1993.

[5] Prugovečki E., *Quantum mechanics in Hilbert space*, 2nd Edition, Academic Press, Inc. New York, 1981.

[6] Bengtsson I. and Życzkowski K., *Geometry of Quantum States*, Cambridge University Press, New York 2006.

[7] Tosiek, J., Brzykcy, P., *States in the Hilbert space formulation and in the phase space formulation of quantum mechanics*, Ann. Phys. **332** (2013), 1–15.

[8] Neumann von J., *Wahrscheinlichkeitstheoretischer Aufbau der Quantenmechanik* (*German*) [*Probabilistic structure of quantum mechanics*] Göttinger Nachr. (1927), 245–272 and *Thermodynamik quantenmechanischer Gesamtheiten* (*German*) [*Thermodynamics of quantum-mechanical populations*] Göttinger Nachr. (1927), 273–291.

[9] Fedosov B., *Deformation quantization and index theory*, Akademie Verlag, Berlin 1996.

[10] Fedosov B., *On the Trace Density in Deformation Quantization* in *Deformation Quantization*, ed. G. Halbout, IRMA Lectures Maths. Theor. Phys., Walter de Gruyter, Berlin 2002, 67.

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