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On Complex Analytic 1|2- and 1|3-dimensional Supermanifolds Associated with $\mathbb{CP}^1$

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Abstract. We obtain a classification up to isomorphism of complex analytic supermanifolds with underlying space $\mathbb{CP}^1$ of dimension 1|2 and of dimension 1|3 with retract (k, k, k), where $k \in \mathbb{Z}$. More precisely, we prove that classes of isomorphic complex analytic supermanifolds of dimension 1|3 with retract (k, k, k) are in one-to-one correspondence with points of the following set:

$$\mathbf{Gr}_{4k-4,3} \cup \mathbf{Gr}_{4k-4,2} \cup \mathbf{Gr}_{4k-4,1} \cup \mathbf{Gr}_{4k-4,0}$$

for $k \geq 2$. For $k < 2$ all such supermanifolds are isomorphic to their retract (k, k, k). In addition, we show that classes of isomorphic complex analytic supermanifolds of dimension 1|2 with retract (k_1, k_2) are in one-to-one correspondence with points of $\mathbb{CP}^{k_1+k_2-4}$ for $k_1 + k_2 \geq 5$. For $k_1 + k_2 < 5$ all such supermanifolds are isomorphic to their retract (k_1, k_2).

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1. Introduction.

We can assign the holomorphic vector bundle, so-called retract, to each complex analytic supermanifold. If underlying space of a complex analytic supermanifold is $\mathbb{CP}^1$, by the Birkhoff–Grothendieck Theorem the corresponding vector bundle is isomorphic to the direct sum of m line bundles: $\mathbf{E} \simeq \bigoplus_{i=1}^{m} L(k_i)$, where $k_i \in \mathbb{Z}$ and m is the odd dimension of the supermanifold. We obtain a classification up to isomorphism of complex analytic supermanifolds of dimension 1|3 with underlying space $\mathbb{CP}^1$ and with retract $L(k) \oplus L(k) \oplus L(k)$, where $k \in \mathbb{Z}$. In addition, we give a classification up to isomorphism of complex analytic supermanifolds of dimension 1|2 with underlying space $\mathbb{CP}^1$.

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The paper is structured as follows. In Section 2 we explain the idea of the classification. In Section 3 we do all necessary preparations. The classification up to isomorphism of complex analytic supermanifolds of dimension $1|3$ with underlying space $\mathbb{CP}^1$ and with retract (k,k,k) is obtained in Section 4. The last section is devoted to the classification up to isomorphism of complex analytic supermanifolds of dimension $1|2$ with underlying space $\mathbb{CP}^1$.

The study of complex analytic supermanifolds with underlying space $\mathbb{CP}^1$ was started in [3]. There the classification of homogeneous complex analytic supermanifolds of dimension $1|m$, $m \leq 3$, up to isomorphism was given. It was proven that in the case $m = 2$ there exists only one non-split homogeneous supermanifold constructed by P. Green [5] and V.P. Palamodov [1]. For $m = 3$ it was shown that there exists a series of non-split homogeneous supermanifolds, parameterized by $k = 0, 2, 3, \dots$

In [7] we studied even-homogeneous supermanifold, i.e., supermanifolds which possess transitive actions of Lie groups. It was shown that there exists a series of non-split even-homogeneous supermanifolds, parameterized by elements in $\mathbb{Z} \times \mathbb{Z}$, three series of non-split even-homogeneous supermanifolds, parameterized by elements of $\mathbb{Z}$, and finite set of exceptional supermanifolds.

2. Classification of supermanifolds, main definitions

We will use the word "supermanifold" in the sense of Berezin–Leites [2, 6], see also [3]. All the time, we will be interested in the complex analytic version of the theory. We begin with main definitions.

Recall that a *complex superdomain of dimension* $n|m$ is a $\mathbb{Z}_2$-graded ringed space of the form $(U, \mathcal{F}_U \otimes \bigwedge(m))$, where $\mathcal{F}_U$ is the sheaf of holomorphic functions on an open set $U \subset \mathbb{C}^n$ and $\bigwedge(m)$ is the exterior (or Grassmann) algebra with m generators.

Definition 2.1. A *complex analytic supermanifold* of dimension $n|m$ is a $\mathbb{Z}_2$-graded locally ringed space that is locally isomorphic to a complex superdomain of dimension $n|m$.

Let $\mathcal{M} = (\mathcal{M}_0, \mathcal{O}_\mathcal{M})$ be a supermanifold and

$$\mathcal{J}_\mathcal{M} = (\mathcal{O}_\mathcal{M})_{\bar{1}} + ((\mathcal{O}_\mathcal{M})_{\bar{1}})^2$$

be the subsheaf of ideals in $\mathcal{O}_\mathcal{M}$ generated by the subsheaf $(\mathcal{O}_\mathcal{M})_{\bar{1}}$ of odd elements. We put $\mathcal{F}_\mathcal{M} := \mathcal{O}_\mathcal{M}/\mathcal{J}_\mathcal{M}$. Then $(\mathcal{M}_0, \mathcal{F}_\mathcal{M})$ is a usual complex analytic manifold, it is called the *reduction* or *underlying space* of $\mathcal{M}$. Usually we will write $\mathcal{M}_0$ instead of $(\mathcal{M}_0, \mathcal{F}_\mathcal{M})$. Denote by $\mathcal{T}_\mathcal{M}$ the *tangent sheaf* or the *sheaf of vector fields* of $\mathcal{M}$. In other words, $\mathcal{T}_\mathcal{M}$ is the sheaf of derivations of the structure sheaf $\mathcal{O}_\mathcal{M}$. Since the sheaf $\mathcal{O}_\mathcal{M}$ is $\mathbb{Z}_2$-graded, the tangent sheaf $\mathcal{T}_\mathcal{M}$ also possesses the induced $\mathbb{Z}_2$-grading, i.e., there is the natural decomposition $\mathcal{T}_\mathcal{M} = (\mathcal{T}_\mathcal{M})_{\bar{0}} \oplus (\mathcal{T}_\mathcal{M})_{\bar{1}}$.

Let $\mathcal{M}_0$ be a complex analytic manifold and let $\mathbf{E}$ be a holomorphic vector bundle over $\mathcal{M}_0$. Denote by $\mathcal{E}$ the sheaf of holomorphic sections of $\mathbf{E}$. Then the ringed space $(\mathcal{M}_0, \bigwedge \mathcal{E})$ is a supermanifold. In this case $\dim \mathcal{M} = n|m$, where $n = \dim \mathcal{M}_0$ and m is the rank of $\mathbf{E}$.

Definition 2.2. A supermanifold $(\mathcal{M}_0, \mathcal{O}_{\mathcal{M}})$ is called *split* if $\mathcal{O}_{\mathcal{M}} \simeq \bigwedge \mathcal{E}$ (as supermanifolds) for a locally free sheaf $\mathcal{E}$.

It is known that any real (smooth or real analytic) supermanifold is split. The structure sheaf $\mathcal{O}_{\mathcal{M}}$ of a split supermanifold possesses a $\mathbb{Z}$-grading, since $\mathcal{O}_{\mathcal{M}} \simeq \bigwedge \mathcal{E}$ and $\bigwedge \mathcal{E} = \bigoplus\limits_p \bigwedge^p \mathcal{E}$ is naturally $\mathbb{Z}$-graded. In other words, we have the decomposition $\mathcal{O}_{\mathcal{M}} = \bigoplus\limits_p (\mathcal{O}_{\mathcal{M}})_p$. This $\mathbb{Z}$-grading induces the $\mathbb{Z}$-grading in $\mathcal{T}_{\mathcal{M}}$ in the following way:

$$(\mathcal{T}_{\mathcal{M}})_p := \{v \in \mathcal{T}_{\mathcal{M}} \mid v((\mathcal{O}_{\mathcal{M}})_q) \subset (\mathcal{O}_{\mathcal{M}})_{p+q} \text{ for all } q \in \mathbb{Z}\}. \tag{1}$$

We have the decomposition:

$$\mathcal{T}_{\mathcal{M}} = \bigoplus_{p=-1}^{m} (\mathcal{T}_{\mathcal{M}})_p.$$

Therefore the superspace $H^0(\mathcal{M}_0, \mathcal{T}_{\mathcal{M}})$ is also $\mathbb{Z}$-graded. Consider the subspace

$$\operatorname{End} \mathbf{E} \subset H^0(\mathcal{M}_0, (\mathcal{T}_{\mathcal{M}})_0).$$

It consists of all endomorphisms of the vector bundle $\mathbf{E}$ inducing the identity morphism on $\mathcal{M}_0$. Denote by $\operatorname{Aut} \mathbf{E} \subset \operatorname{End} \mathbf{E}$ the group of automorphisms of $\mathbf{E}$, i.e., the group of all invertible endomorphisms. We define the action Int of $\operatorname{Aut} \mathbf{E}$ on $\mathcal{T}_{\mathcal{M}}$ by

$$\operatorname{Int} A : v \mapsto A v A^{-1}.$$

Clearly, the action Int preserves the $\mathbb{Z}$-grading (1), therefore, we have the action of $\operatorname{Aut} \mathbf{E}$ on $H^1(\mathcal{M}_0, (\mathcal{T}_{\mathcal{M}})_2)$.

There is a functor denoting by gr from the category of supermanifolds to the category of split supermanifolds. Let us describe this construction. Let $\mathcal{M}$ be a supermanifold and let as above $\mathcal{J}_M \subset \mathcal{O}_{\mathcal{M}}$ be the subsheaf of ideals generated by odd elements of $\mathcal{O}_{\mathcal{M}}$. Then by definition $\operatorname{gr}(\mathcal{M}) = (\mathcal{M}_0, \operatorname{gr} \mathcal{O}_{\mathcal{M}})$, where

$$\operatorname{gr} \mathcal{O}_{\mathcal{M}} = \bigoplus_{p \geq 0} (\operatorname{gr} \mathcal{O}_{\mathcal{M}})_p, \quad (\operatorname{gr} \mathcal{O}_{\mathcal{M}})_p = \mathcal{J}_{\mathcal{M}}^p / \mathcal{J}_{\mathcal{M}}^{p+1}, \quad \mathcal{J}_{\mathcal{M}}^0 := \mathcal{O}_{\mathcal{M}}.$$

In this case $(\operatorname{gr} \mathcal{O}_{\mathcal{M}})_1$ is a locally free sheaf and there is a natural isomorphism of $\operatorname{gr} \mathcal{O}_{\mathcal{M}}$ onto $\bigwedge (\operatorname{gr} \mathcal{O}_{\mathcal{M}})_1$. If $\psi = (\psi_{\mathrm{red}}, \psi^*) : (M, \mathcal{O}_{\mathcal{M}}) \to (N, \mathcal{O}_{\mathcal{N}})$ is a morphism of supermanifolds, then $\operatorname{gr}(\psi) = (\psi_{\mathrm{red}}, \operatorname{gr}(\psi^*))$, where $\operatorname{gr}(\psi^*) : \operatorname{gr} \mathcal{O}_{\mathcal{N}} \to \operatorname{gr} \mathcal{O}_{\mathcal{M}}$ is defined by

$$\operatorname{gr}(\psi^*)(f + \mathcal{J}_{\mathcal{N}}^p) := \psi^*(f) + \mathcal{J}_{\mathcal{M}}^p \text{ for } f \in (\mathcal{J}_{\mathcal{N}})^{p-1}.$$

Recall that by definition every morphism ψ of supermanifolds is even and as consequence sends $\mathcal{J}_{\mathcal{N}}^p$ into $\mathcal{J}_{\mathcal{M}}^p$.

Definition 2.3. The supermanifold $\operatorname{gr}(\mathcal{M})$ is called the *retract* of $\mathcal{M}$.

To classify supermanifolds, we use the following corollary of the well-known Green Theorem (see [3, 5] or [4] for more details):

Theorem 2.4 (Green). *Let* $\mathcal{N} = (\mathcal{N}_0, \bigwedge \mathcal{E})$ *be a split supermanfold of dimension* $n|m$, *where* $m \leq 3$. *The classes of isomorphic supermanifolds* $\mathcal{M}$ *such that* $\operatorname{gr} \mathcal{M} = \mathcal{N}$ *are in bijection with orbits of the action* Int *of the group* $\operatorname{Aut} \mathbf{E}$ *on* $H^1(\mathcal{M}_0, (\mathcal{T}_{\mathcal{N}})_2)$.

Remark 2.5. This theorem allows to classify supermanifolds $\mathcal{M}$ such that $\operatorname{gr} \mathcal{M}$ is fixed up to isomorphisms that induce the identity morphism on $\operatorname{gr} \mathcal{M}$.

3. Supermanifolds associated with $\mathbb{CP}^1$

In what follows we will consider supermanifolds with the underlying space $\mathbb{CP}^1$.

3.1. Supermanifolds with underlying space $\mathbb{CP}^1$

Let $\mathcal{M}$ be a supermanifold of dimension $1|m$. Denote by U_0 and U_1 the standard charts on $\mathbb{CP}^1$ with coordinates x and $y = \frac{1}{x}$ respectively. By the Birkhoff–Grothendieck Theorem we can cover $\operatorname{gr} \mathcal{M}$ by two charts

$$(U_0, \mathcal{O}_{\operatorname{gr} \mathcal{M}}|_{U_0}) \text{ and } (U_1, \mathcal{O}_{\operatorname{gr} \mathcal{M}}|_{U_1})$$

with local coordinates x, $\xi_1, \ldots, \xi_m$ and y, $\eta_1, \ldots, \eta_m$, respectively, such that in $U_0 \cap U_1$ we have

$$y = x^{-1}, \quad \eta_i = x^{-k_i} \xi_i, \ i = 1, \ldots, m,$$

where k_i are integers. Note that a permutation of k_i induces the automorphism of $\operatorname{gr} \mathcal{M}$.

We will identify $\operatorname{gr} \mathcal{M}$ with the set $(k_1, \ldots, k_m)$, so we will say that a supermanifold has the retract $(k_1, \ldots, k_m)$. In this paper we study two cases $m = 2$ and $m = 3$ for $k_1 = k_2 = k_3 =: k$. From now on we use the notation $\mathcal{T} = \bigoplus \mathcal{T}_p$ for the tangent sheaf of $\operatorname{gr} \mathcal{M}$.

3.2. A basis of $H^1(\mathbb{CP}^1, \mathcal{T}_2)$.

Assume that $m = 3$ and that $\mathcal{M} = (k_1, k_2, k_3)$ is a split supermanifold with the underlying space $\mathcal{M}_0 = \mathbb{CP}^1$. Let $\mathcal{T}$ be its tangent sheaf. In [3] the following decomposition

$$\mathcal{T}_2 = \sum_{i<j} \mathcal{T}_2^{ij} \tag{2}$$

was obtained. The sheaf $\mathcal{T}_2^{ij}$ is a locally free sheaf of rank 2; its basis sections over $(U_0, \mathcal{O}_{\mathcal{M}}|_{U_0})$ are:

$$\xi_i \xi_j \frac{\partial}{\partial x}, \quad \xi_i \xi_j \xi_l \frac{\partial}{\partial \xi_l}; \tag{3}$$

where $l \neq i, j$. In $U_0 \cap U_1$ we have

$$\begin{aligned} \xi_i\xi_j\frac{\partial}{\partial x} &= -y^{2-k_i-k_j}\eta_i\eta_j\frac{\partial}{\partial y} - k_l y^{1-k_i-k_j}\eta_i\eta_j\eta_l\frac{\partial}{\partial \eta_l},\\ \xi_i\xi_j\xi_l\frac{\partial}{\partial \xi_l} &= y^{-k_i-k_j}\eta_i\eta_j\eta_l\frac{\partial}{\partial \eta_l}. \end{aligned} \tag{4}$$

The following theorem was proven in [7]. For completeness we reproduce it here.

Theorem 3.1. *Assume that $i < j$ and $l \neq i, j$.*

1. *For $k_i + k_j > 3$ the basis of $H^1(\mathbb{CP}^1, \mathcal{T}_2^{ij})$ is:*

$$\begin{aligned} &x^{-n}\xi_i\xi_j\frac{\partial}{\partial x}, \quad n = 1, \ldots, k_i + k_j - 3,\\ &x^{-n}\xi_i\xi_j\xi_l\frac{\partial}{\partial \xi_l}, \quad n = 1, \ldots, k_i + k_j - 1; \end{aligned} \tag{5}$$

2. *for $k_i + k_j = 3$ the basis of $H^1(\mathbb{CP}^1, \mathcal{T}_2^{ij})$ is:*

$$x^{-1}\xi_i\xi_j\xi_l\frac{\partial}{\partial \xi_l}, \quad x^{-2}\xi_i\xi_j\xi_l\frac{\partial}{\partial \xi_l};$$

3. *for $k_i + k_j = 2$ and $k_l = 0$ the basis of $H^1(\mathbb{CP}^1, \mathcal{T}_2^{ij})$ is:*

$$x^{-1}\xi_i\xi_j\xi_l\frac{\partial}{\partial \xi_l};$$

4. *if $k_i + k_j = 2$ and $k_l \neq 0$ or $k_i + k_j < 2$, we have $H^1(\mathbb{CP}^1, \mathcal{T}_2^{ij}) = \{0\}$.*

Proof. We use the Čech cochain complex of the cover $\mathfrak{U} = \{U_0, U_1\}$, hence, 1-cocycle with values in the sheaf $\mathcal{T}_2^{ij}$ is a section v of $\mathcal{T}_2^{ij}$ over $U_0 \cap U_1$. We are looking for *basis cocycles*, i.e., cocycles such that their cohomology classes form a basis of $H^1(\mathfrak{U}, \mathcal{T}_2^{ij}) \simeq H^1(\mathbb{CP}^1, \mathcal{T}_2^{ij})$. Note that if $v \in Z^1(\mathfrak{U}, \mathcal{T}_2^{ij})$ is holomorphic in U_0 or U_1 then the cohomology class of v is equal to 0. Obviously, any $v \in Z^1(\mathfrak{U}, \mathcal{T}_2^{ij})$ is a linear combination of vector fields (3) with holomorphic in $U_0 \cap U_1$ coefficients. Further, we expand these coefficients in a Laurent series in x and drop the summands x^n, $n \geq 0$, because they are holomorphic in U_0. We see that v can be replaced by

$$v = \sum_{n=1}^{\infty} a_{ij}^n x^{-n}\xi_i\xi_j\frac{\partial}{\partial x} + \sum_{n=1}^{\infty} b_{ij}^n x^{-n}\xi_i\xi_j\xi_l\frac{\partial}{\partial \xi_l}, \tag{6}$$

where $a_{ij}^n, b_{ij}^n \in \mathbb{C}$. Using (4), we see that the summands corresponding to $n \geq k_i + k_j - 1$ in the first sum of (6) and the summands corresponding to $n \geq k_i + k_j$ in the second sum of (6) are holomorphic in U_1. Further, it follows from (4) that

$$x^{2-k_i-k_j}\xi_i\xi_j\frac{\partial}{\partial x} \sim -k_l x^{1-k_i-k_j}\xi_i\xi_j\xi_l\frac{\partial}{\partial \xi_l}.$$

Hence the cohomology classes of the following cocycles

$$x^{-n}\xi_i\xi_j\frac{\partial}{\partial x},\qquad n=1,\dots,k_i+k_j-3,$$
$$x^{-n}\xi_i\xi_j\xi_l\frac{\partial}{\partial \xi_l},\quad n=1,\dots,k_i+k_j-1,$$

generate $H^1(\mathbb{CP}^1,\mathcal{T}_2^{ij})$. If we examine linear combination of these cocycles which are cohomological trivial, we get the result. □

Remark 3.2. Note that a similar method can be used for computation of a basis of $H^1(\mathbb{CP}^1,\mathcal{T}_q)$ for any odd dimension m and any q.

In the case $k_1=k_2=k_3=k$, from Theorem 3.1, it follows:

Corollary 3.3. *Assume that $i<j$ and $l\neq i,j$.*

1. *For $k\geq 2$ the basis of $H^1(\mathbb{CP}^1,\mathcal{T}_2^{ij})$ is*

$$\begin{aligned} &x^{-n}\xi_i\xi_j\frac{\partial}{\partial x},\qquad n=1,\dots,2k-3,\\ &x^{-n}\xi_i\xi_j\xi_l\frac{\partial}{\partial \xi_l},\quad n=1,\dots,2k-1.\end{aligned}\tag{7}$$

2. *If $k<2$, we have $H^1(\mathbb{CP}^1,\mathcal{T}_2)=\{0\}$.*

3.3. The group Aut E

This section is devoted to the calculation of the group of automorphisms $\operatorname{Aut}\mathbf{E}$ of the vector bundle $\mathbf{E}$ in the case (k,k,k). Here $\mathbf{E}$ is the vector bundle corresponding to the split supermanifold (k,k,k).

Let (ξ_i) be a local basis of $\mathbf{E}$ over U_0 and A be an automorphism of $\mathbf{E}$. Assume that $A(\xi_j)=\sum a_{ij}(x)\xi_i$. In U_1 we have

$$A(\eta_j)=A(y^{k_j}\xi_j)=\sum y^{k_j-k_i}a_{ij}(y^{-1})\eta_i.$$

Therefore, $a_{ij}(x)$ is a polynomial in x of degree $\leq k_j-k_i$, if $k_j-k_i\geq 0$ and is 0, if $k_j-k_i<0$. We denote by b_{ij} the entries of the matrix $B=A^{-1}$. The entries are also polynomials in x of degree $\leq k_j-k_i$. We need the following formulas:

$$\begin{aligned} A(\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_k})A^{-1} &=\det(A)\sum_s b_{ks}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_s};\\ A(\xi_i\xi_j\frac{\partial}{\partial x})A^{-1} &=\det(A)\sum_{k<s}(-1)^{l+r}b_{lr}\xi_k\xi_s\frac{\partial}{\partial x}+\det(A)\sum_s b'_{ls}\xi_i\xi_j\xi_l\frac{\partial}{\partial\xi_s},\end{aligned}\tag{8}$$

where $i<j$, $l\neq i,j$ and $r\neq k,s$. Here we use the notation $b'_{ls}=\frac{\partial}{\partial x}(b_{ls})$. By (8), in the case $k_1=k_2=k_3=k$, we have:

Proposition 3.4. *Assume that $k_1=k_2=k_3=k$.*

1. *We have*

$$\operatorname{Aut}\mathbf{E}\simeq \mathrm{GL}_3(\mathbb{C}).$$

In other words

$$\operatorname{Aut}\mathbf{E} = \{(a_{ij}) \mid a_{ij} \in \mathbb{C},\ \det(a_{ij}) \neq 0\}.$$

2. *The action of* $\operatorname{Aut}\mathbf{E}$ *on* $\mathcal{T}_2$ *is given in* U_0 *by the following formulas:*

$$\begin{aligned} A(\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_k})A^{-1} &= \det(A)\xi_1\xi_2\xi_3\sum_s b_{ks}\frac{\partial}{\partial\xi_s};\\ A(\xi_i\xi_j\frac{\partial}{\partial x})A^{-1} &= \det(A)\sum_{k<s}(-1)^{l+r}b_{lr}\xi_k\xi_s\frac{\partial}{\partial x}, \end{aligned} \tag{9}$$

where $i<j$, $l\neq i,j$ *and* $r\neq k,s$. *Here* $B=(b_{ij})=A^{-1}$

4. Classification of supermanifolds with retract (k,k,k)

In this section we give a classification up to isomorphism of complex analytic supermanifolds with underlying space $\mathbb{CP}^1$ and with retract (k,k,k) using Theorem 2.4. In previous section we have calculated the vector space $H^1(\mathbb{CP}^1,\mathcal{T}_2)$, the group $\operatorname{Aut}\mathbf{E}$ and the action Int of $\operatorname{Aut}\mathbf{E}$ on $H^1(\mathbb{CP}^1,\mathcal{T}_2)$, see Theorem 3.3 and Proposition 3.4. Our objective in this section is to calculate the orbit space corresponding to the action Int:

$$H^1(\mathbb{CP}^1,\mathcal{T}_2)/\operatorname{Aut}\mathbf{E}. \tag{10}$$

By Theorem 2.4 classes of isomorphic supermanifolds are in one-to-one correspondence with points of the set (10).

Let us fix the following basis of $H^1(\mathbb{CP}^1,\mathcal{T}_2)$:

$$\begin{array}{lll} v_{11}=x^{-1}\xi_2\xi_3\frac{\partial}{\partial x}, & v_{12}=-x^{-1}\xi_1\xi_3\frac{\partial}{\partial x}, & v_{13}=x^{-1}\xi_1\xi_2\frac{\partial}{\partial x},\\ \cdots & \cdots & \cdots\\ v_{p1}=x^{-p}\xi_2\xi_3\frac{\partial}{\partial x}, & v_{p1}=-x^{-p}\xi_1\xi_3\frac{\partial}{\partial x}, & v_{p3}=x^{-p}\xi_1\xi_2\frac{\partial}{\partial x}, \end{array} \tag{11}$$

$$\begin{array}{lll} v_{p+1,1}=x^{-1}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_1}, & \cdots & v_{p+1,3}=x^{-1}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_3},\\ \cdots & \cdots & \cdots\\ v_{q1}=x^{-a}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_1}, & \cdots & v_{q3}=x^{-a}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_3}, \end{array} \tag{12}$$

where $p=2k-3$, $a=2k-1$ and $q=p+a=4k-4$. (Compare with Theorem 3.3.) Let us take $A\in\operatorname{Aut}\mathbf{E}\simeq\mathrm{GL}_3(\mathbb{C})$, see Proposition 3.4. We get that in the basis (11)–(12) the automorphism $\operatorname{Int}A$ is given by

$$\operatorname{Int}A(v_{is})=\frac{1}{\det B}\sum_j b_{sj}v_{ij}.$$

Note that for any matrix $C\in\mathrm{GL}_3(\mathbb{C})$ there exists a matrix B such that

$$C=\frac{1}{\det\, B}B.$$

Indeed, we can put $B=\frac{1}{\sqrt{\det\, C}}C$. We summarize these observations in the following proposition:

Proposition 4.1. *Assume that* $k_1 = k_2 = k_3 = k$. *Then*

$$H^1(\mathbb{CP}^1, \mathcal{T}_2) \simeq \mathrm{Mat}_{3\times(4k-4)}(\mathbb{C})$$

and the action Int *of* Aut $\mathbf{E}$ *on* $H^1(\mathbb{CP}^1, \mathcal{T}_2)$ *is equivalent to the standard action of* $\mathrm{GL}_3(\mathbb{C})$ *on* $\mathrm{Mat}_{3\times(4k-4)}(\mathbb{C})$. *More precisely,* Int *is equivalent to the following action:*

$$D \longmapsto (W \longmapsto DW), \tag{13}$$

where $D \in \mathrm{GL}_3(\mathbb{C})$, $W \in \mathrm{Mat}_{3\times(4k-4)}(\mathbb{C})$ *and* DW *is the usual matrix multiplication.*

Now we prove our first main result.

Theorem 4.2. *Let* $k \geq 2$. *Complex analytic supermanifolds with underlying space* $\mathbb{CP}^1$ *and retract* (k, k, k) *are in one-to-one correspondence with points of the following set:*

$$\bigcup_{r=0}^{3} \mathbf{Gr}_{4k-4,r},$$

where $\mathbf{Gr}_{4k-4,r}$ *is the Grassmannian of type* $(4k-4, r)$, *i.e., it is the set of all* r*-dimensional subspaces in* $\mathbb{C}^{4k-4}$.

In the case $k < 2$ *all supermanifolds with underlying space* $\mathbb{CP}^1$ *and retract* (k, k, k) *are split and isomorphic to their retract* (k, k, k).

Proof. Assume that $k \geq 2$. Clearly, the action (13) preserves the rank r of matrices from $\mathrm{Mat}_{3\times(4k-4)}(\mathbb{C})$ and $r \leq 3$. Therefore, matrices with different rank belong to different orbits of this action. Furthermore, let us fix $r \in \{0, 1, 2, 3\}$. Denote by $\mathrm{Mat}^r_{3\times(4k-4)}(\mathbb{C})$ all matrices with rank r. Clearly, we have

$$\mathrm{Mat}_{3\times(4k-4)}(\mathbb{C}) = \bigcup_{r=0}^{3} \mathrm{Mat}^r_{3\times(4k-4)}(\mathbb{C}).$$

A matrix $W \in \mathrm{Mat}^r_{3\times(4k-4)}(\mathbb{C})$ defines the r-dimensional subspace V_W in $\mathbb{C}^{4k-4}$. This subspace is the linear combination of lines of W. (We consider lines of a matrix $X \in \mathrm{Mat}_{3\times(4k-4)}(\mathbb{C})$ as vectors from $\mathbb{C}^{4k-4}$.) Therefore, we have defined the map F_r:

$$W \longmapsto F_r(W) = V_W \in \mathbf{Gr}_{4k-4,r}.$$

The map F_r is surjective. Indeed, in any r-dimensional subspace $V \in \mathbf{Gr}_{4k-4,r}$ we can take 3 vectors generating V and form the matrix $W_V \in \mathrm{Mat}^r_{3\times(4k-4)}(\mathbb{C})$. In this case the matrix W_V is of rank r and $F_r(W_V) = V$. Clearly, $F_r(W) = F_r(DW)$, where $D \in \mathrm{GL}_3(\mathbb{C})$. Conversely, if W and $W' \in F_r^{-1}(V_W)$, then there exists a matrix $D \in \mathrm{GL}_3(\mathbb{C})$ such that $DW = W'$. It follows that orbits of $\mathrm{GL}_3(\mathbb{C})$ on $\mathrm{Mat}^r_{3\times(4k-4)}(\mathbb{C})$ are in one to one correspondence with points of $\mathbf{Gr}_{4k-4,r}$. Therefore, orbits of $\mathrm{GL}_3(\mathbb{C})$ on

$$\mathrm{Mat}_{3\times(4k-4)}(\mathbb{C}) = \bigcup_{r=0}^{3} \mathrm{Mat}^r_{3\times(4k-4)}(\mathbb{C})$$

are in one-to-one correspondence with points of $\bigcup_{r=0}^{3} \mathbf{Gr}_{4k-4,r}$. The proof is complete. □

5. Classification of supermanifolds with underlying space $\mathbb{CP}^1$ of odd dimension 2

In this section we study the case $m = 2$ and $\operatorname{gr}\mathcal{M} = (k_1, k_2)$, where k_1, k_2 are any integers. Let us compute the 1-cohomology with values in the tangent sheaf $\mathcal{T}_2$. The sheaf $\mathcal{T}_2$ is a locally free sheaf of rank 1. Its basis section over $(U_0, \mathcal{O}_{\mathcal{M}}|_{U_0})$ is $\xi_1\xi_2\frac{\partial}{\partial x}$. The transition functions in $U_0 \cap U_1$ are given by the following formula:

$$\xi_1\xi_2\frac{\partial}{\partial x} = -y^{2-k_1-k_2}\eta_1\eta_2\frac{\partial}{\partial y}.$$

Therefore, a basis of $H^1(\mathbb{CP}^1, \mathcal{T}_2)$ is

$$x^{-n}\xi_1\xi_2\frac{\partial}{\partial x},\ n = 1, \dots, k_1 + k_2 - 3.$$

Let (ξ_i) be a local basis of $\mathbf{E}$ over U_0 and A be an automorphism of $\mathbf{E}$. As in the case $m = 3$, we have that $a_{ij}(x)$ is a polynomial in x of degree $\leq k_j - k_i$, if $k_j - k_i \geq 0$ and is 0, if $k_j - k_i < 0$. We need the following formulas:

$$A(x^{-n}\xi_1\xi_2\frac{\partial}{\partial x})A^{-1} = (\det A)x^{-n}\xi_1\xi_2\frac{\partial}{\partial x}.$$

Denote

$$v_n = x^{-n}\xi_1\xi_2\frac{\partial}{\partial x}.$$

We see that the action Int is equivalent to the action of $\mathbb{C}^*$ on $\mathbb{C}^{k_1+k_2-3}$, therefore, the quotient space is $\mathbb{CP}^{k_1+k_2-4}$. We have proven the following theorem:

Theorem 5.1. *Assume that $k_1 + k_2 \geq 5$. Complex analytic supermanifolds with underlying space $\mathbb{CP}^1$ and retract (k_1, k_2) are in one-to-one correspondence with points of*

$$\mathbb{CP}^{k_1+k_2-4} \cup \{\mathrm{pt}\}.$$

In the case $k_1 + k_2 < 5$ all supermanifolds with underlying space $\mathbb{CP}^1$ and retract (k_1, k_2) are split and isomorphic to their retract (k_1, k_2).

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