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Orbifold Diffeomorphism Groups

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Abstract. Orbifolds are a generalization of manifolds. They arise naturally in different areas of mathematics and physics, e.g.:

– Spaces of symplectic reduction are orbifolds,
– Orbifolds may be used to construct a conformal field theory model.

In [10], we considered the diffeomorphism group of a paracompact, non-compact smooth reduced orbifold. Our main result is the construction of an infinite-dimensional Lie-group structure on the diffeomorphism group and several interesting subgroups. The aim of these notes is to sketch the main ingredients of the proof. Furthermore, we will consider the special case of an orbifold with a global chart.

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1. Orbifolds in local charts

We recall the notion of an orbifold defined via local charts and their morphisms. Our exposition of orbifolds follows [4]:

Definition 1. Let Q be a paracompact Hausdorff topological space with $d \in \mathbb{N}_0$.

1. A (*reduced*) *orbifold chart* of dimension d on Q is a triple (V, G, φ) where V is a connected paracompact n-dimensional manifold without boundary, G is a finite subgroup of $\mathrm{Diff}(V)$, and $\varphi\colon V \to Q$ is a map with open image $\varphi(V)$ inducing a homeomorphism from the orbit space V/G to $\varphi(V)$. Here the orbit space V/G is the set of all G-orbits with respect to the natural G-action on V. We endow V/G with the quotient topology with respect to the map sending $x \in V$ to its orbit $G.x$.
2. Two orbifold charts (V, G, φ), (W, H, ψ) on Q are called *compatible* if for each pair $(x, y) \in V \times W$ with $\varphi(x) = \psi(y)$ there are open connected neighborhoods V_x of x and W_y of y together with a C^∞-diffeomorphism $h\colon V_x \to W_y$ with $\psi \circ h = \varphi|_{V_x}$. The map h is called a *change of charts*.

3. A *reduced orbifold atlas* of dimension d on Q is a set of pairwise compatible reduced orbifold charts $\mathcal{V} = \{(V_i, G_i, \varphi_i) \mid i \in I\}$ of dimension d on Q such that $\bigcup_{i\in I} \varphi_i(V_i) = Q$. Two reduced orbifold atlases are equivalent if their union is a reduced orbifold atlas.
4. A *reduced orbifold* of dimension d is a pair $(Q, \mathcal{U})$ where $\mathcal{U}$ is an equivalence class of orbifold atlases of dimension d on Q.
5. For an orbifold $(Q, \mathcal{U})$, a point $x \in Q$ will be called *singular* if there is an orbifold chart (V, G, ψ), such that for any $y \in \psi^{-1}(x)$ the isotropy subgroup $G_y := \{g \in G \mid g.y = y\}$ is non-trivial. Otherwise x is called *regular*. This property is independent of choice of charts (see [7, p. 39]).

The term reduced refers to the fact that the finite group G is required to be a subgroup of $\mathrm{Diff}(V)$. Hence, each group G acts effectively on V. Every orbifold in these notes will be reduced, whence we drop the word "reduced" for the rest of this paper. We consider a class of orbifolds with global chart, which will serve as our main example. Notice that in general an orbifold will not admit a global orbifold chart.

Example 1. Let d be in $\mathbb{N}$ and $G \neq \{\mathrm{id}_{\mathbb{R}^d}\}$ be a finite subgroup of the orthogonal group $O(d) \subseteq \mathrm{Diff}(\mathbb{R}^d)$ such that:

(IS) The group G satisfies $G_x = \{\mathrm{id}_{\mathbb{R}^d}\}$ for all $x \in \mathbb{R}^d \setminus \{0\}$, i.e., 0 is the only singularity fixed jointly by all elements of G.

We remark the following:

1. For odd d only $G = \{\mathrm{id}_{\mathbb{R}^d}, -\mathrm{id}_{\mathbb{R}^d}\}$ is possible. For $d = 1$ we denote the reflection generating G by $r\colon \mathbb{R} \to \mathbb{R}, x \mapsto -x$.
2. If $d = 2$, then the group G may not contain reflections by condition (IS). In this case, G contains at least one rotation of $\mathbb{R}^2$ fixing the origin.

Let $\pi\colon \mathbb{R}^d \to \mathbb{R}^d/G$ be the quotient map onto the orbit space and $Q := \mathbb{R}^d/G$. Then $\{(\mathbb{R}^d, G, \pi)\}$ is an atlas for Q, turning the orbit space into an orbifold with a global chart. We identify for $d \in \{1, 2\}$ the orbit spaces with $[0, \infty[$ and respectively

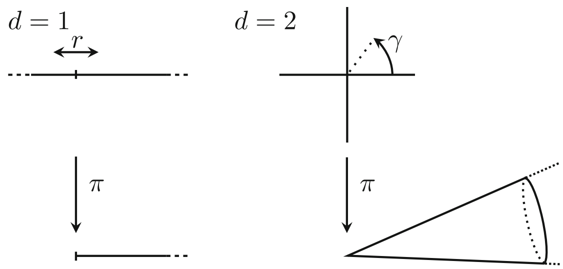

FIGURE 1. Cone shaped orbifolds. The element γ is a rotation which generates G for $d = 2$.

a cone: Each finite subgroup of $O(2)$ – which is not a dihedral group – is cyclic by [2, Ch. 5, Theorem 3.4]. Hence Figure 1 exhibits the general case for $d = 2$.

Notice that the chart mappings of an orbifold will in general be non-invertible. To define a "smooth morphism of orbifolds" we have to provide smooth lifts in charts. However, these lifts should be "smoothly related" to obtain a well-behaved notion of orbifold morphism. In this note we understand orbifold morphisms as maps in the sense of Pohl [9]:

Definition 2. A *representative of an orbifold map* from $(Q,\mathcal{U})$ to $(Q',\mathcal{U}')$ is a tuple $\hat{f} = (f, \{f_i\}_{i\in I}, [P,\nu])$ where

R1 $f\colon Q \to Q'$ is a continuous map,

R2 $\forall i \in I$, f_i is a smooth local lift of f with respect to $(V_i, G_i, \pi_i) \in \mathcal{U}$, $(V_i', G_i', \pi_i') \in \mathcal{U}'$ such that the (V_i, G_i, π_i) cover Q

R3 the lifts are smoothly related to each other, i.e., for certain change of charts $\lambda\colon V_i \supseteq U \to V_j, i, j \in I$ (contained in the set P), there is a change of charts $\nu(\lambda)$, such that $f_j \circ \lambda = \nu(\lambda) \circ f_i|_{\mathrm{dom}\lambda}$ holds. This compatibility condition is encoded by the pair (P,ν).

We will not give details in these notes concerning the pair (P,ν) and the axioms they satisfy (cf. [9, Definition 4.4]). It turns out that these data are naturally fixed for most types of mappings considered in these notes.

An *orbifold map* (or *morphism of orbifolds*) $[\hat{f}]$ is an equivalence class of representatives. The equivalence relation is obtained by identifying representatives which arise by refinements of orbifold atlases. Again, we omit the details here (which are recorded in [9]) and remark only:
Orbifolds and orbifold morphisms form a category denoted by **Orb**.

Definition 3. A morphism of orbifolds $[\hat{f}] \in \mathbf{Orb}\left((Q_1,\mathcal{U}_1),(Q_2,\mathcal{U}_2)\right)$ is called an *orbifold diffeomorphism* if it is an isomorphism in **Orb**. Define the *orbifold diffeomorphism group* $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ to be the subset of all orbifold diffeomorphisms contained in $\mathbf{Orb}((Q,\mathcal{U}),(Q,\mathcal{U}))$.

The following result shows that we may forget the compatibility condition R3 mentioned in Definition 2 for orbifold diffeomorphisms:

Proposition 1 ([10, Corollary 2.1.12]). *For an orbifold map $[\hat{f}]$ the following are equivalent:*

1. *$[\hat{f}]$ is an orbifold diffeomorphism,*
2. *there is a representative $\hat{f} = (f, \{f_j\}_{j\in J}, [P,\nu])$ of $[\hat{f}]$ such that f is a homeomorphism and each f_j is a diffeomorphism.*

In particular, an orbifold diffeomorphism is uniquely determined by its lifts.

Example 2. Consider an orbifold with global chart as in Example 1. Let $\tilde{h}\colon \mathbb{R}^d \to \mathbb{R}^d$ be a homeomorphism. If there is a group automorphism $\alpha\colon G \to G$ with $\tilde{h}\circ\gamma = \alpha(\gamma).\tilde{h}$ for all $\gamma \in G$, we call $\tilde{h}$ a *weak equivalence.* For a weak equivalence $\tilde{h}$ the map $h\colon \mathbb{R}^d/G \to \mathbb{R}^d/G, x \mapsto \pi \circ \tilde{h} \circ \pi^{-1}(x)$ makes sense and is a homeomorphism.

Following Proposition 1, each diffeomorphism $\tilde{h}\colon \mathbb{R}^d \to \mathbb{R}^d$ which is a weak equivalence induces an orbifold diffeomorphism $[\hat{h}] \in \mathrm{Diff}_{\mathrm{Orb}}\left(\mathbb{R}^d/G, \left\{(\mathbb{R}^d, G, \pi)\right\}\right)$. A representative $\hat{h}$ of $[\hat{h}]$ is uniquely determined by the smooth lift $\tilde{h}$.

Before we state the main result on the diffeomorphism group of an orbifold, we need to introduce mappings associated to the *tangent orbifold*. Analogous to the construction of a tangent manifold for a smooth manifold, one can construct a tangent orbifold $(\mathcal{T}Q, \mathcal{T}\mathcal{U})$ for an orbifold $(Q, \mathcal{U})$ (see [1, Section 1.3] or [10, Section 3.1]). The orbifold tangent space $\mathcal{T}_x Q$ for a singular point $x \in Q$ will not support a natural vector space structure, but it contains a unique zero-element 0_x. Moreover, there is a well-defined orbifold morphism $\pi_{\mathcal{T}(Q,\mathcal{U})}\colon (\mathcal{T}Q, \mathcal{T}\mathcal{U}) \to (Q, \mathcal{U})$, the so-called "bundle projection". Right inverses of the projection are sections to the tangent orbifold:

Definition 4.

1. A map of orbifolds $[\hat{\sigma}] \in \mathbf{Orb}\left((Q,\mathcal{U}), \mathcal{T}(Q,\mathcal{U})\right)$ is called *orbisection* if it satisfies $\pi_{\mathcal{T}(Q,\mathcal{U})} \circ [\hat{\sigma}] = \mathrm{id}_{(Q,\mathcal{U})}$. Here $\mathrm{id}_{(Q,\mathcal{U})}$ is the identity morphism of $(Q,\mathcal{U})$. Denote the *set of all orbisections* for $(Q,\mathcal{U})$ by $\mathfrak{X}_{\mathrm{Orb}}(Q)$.
2. For $[\hat{\sigma}] \in \mathfrak{X}_{\mathrm{Orb}}(Q)$ the *support* $\mathrm{supp}[\hat{\sigma}]$ of $[\hat{\sigma}]$ is the closure of the set $\{x \in Q \mid \sigma(x) \neq 0_x\} \subseteq Q$. If $\mathrm{supp}[\hat{\sigma}] \subseteq K$ holds for some compact subset $K \subseteq Q$, then $[\hat{\sigma}] \in \mathfrak{X}_{\mathrm{Orb}}(Q)$ is called *compactly supported* (*in* K). Let $\mathfrak{X}_{\mathrm{Orb}}(Q)_c$ be the *set of all compactly supported orbisections in* $\mathfrak{X}_{\mathrm{Orb}}(Q)$.

It turns out that orbisections are uniquely determined by their lifts. Even more, an orbisection possesses a unique lift in each chart, which we call a *canonical lift*. Notice that in general orbifold morphisms need not posses lifts in a prescribed orbifold chart. We obtain the following characterization for the compactly supported orbisections:

Theorem 2 ([10, Proposition 3.2.9 and Section 3.3]). *Let* $\{(U_i, G_i, \varphi_i) \mid i \in I\}$ *be any orbifold atlas for* $(Q,\mathcal{U})$. *Denote by* $\mathfrak{X}(U_i)$ *the space of all smooth vector fields on the manifold* U_i. *The set* $\mathfrak{X}_{\mathrm{Orb}}(Q)_c$ *is in bijection with all families of vector fields* $(\sigma_i)_{i\in I} \in \bigoplus_{i \in I} \mathfrak{X}(U_i)$ *which satisfy the compatibility condition:*

$$T\lambda \circ \sigma_i|_{\mathrm{dom}\,\lambda} = \sigma_j \circ \lambda, \quad \forall \lambda\colon V_i \supseteq U \to V_j \text{ change of charts}, i, j \in I$$

The embedding $\mathfrak{X}_{\mathrm{Orb}}(Q)_c \hookrightarrow \bigoplus_{i\in I} \mathfrak{X}(U_i)$ *turns the compactly supported orbisections into a locally convex space.*

Example 3. Consider an orbifold as in Example 1. By Theorem 2, the space of compactly supported orbisections $\mathfrak{X}_{\mathrm{Orb}}\left(\mathbb{R}^d/G\right)_c$ corresponds to the compactly supported vector fields in $\mathfrak{X}\left(\mathbb{R}^d\right)$ which satisfy $X \circ \lambda = T\lambda \circ X|_{\mathrm{dom}\lambda}$ for all change of charts λ. Then [7, Lemma 2.11] implies that this condition is equivalent to $X \circ \gamma = T\gamma \circ X$ for all $\gamma \in G$. In particular the space $\mathfrak{X}_{\mathrm{Orb}}\left(\mathbb{R}^d/G\right)_c$ is isomorphic to the subset of all compactly supported and equivariant vector fields

$$\mathfrak{X}\left(\mathbb{R}^d\right)_c^G := \left\{X \in \mathfrak{X}\left(\mathbb{R}^d\right) \,\middle|\, \mathrm{supp}\, X \text{ is compact}, X \circ \gamma = T\gamma \circ X, \forall \gamma \in G\right\}.$$

The space of compactly supported orbisections will be the modeling space for the Lie group structure on $\mathrm{Diff}_{\mathrm{Orb}}\left(\mathbb{R}^d/G, \left\{(\mathbb{R}^d, G, \pi)\right\}\right)$. To construct a chart, we use results from Riemannian geometry on orbifolds. The leading idea is to construct objects from Riemannian geometry on charts and to glue them via change of charts. It is well known that one can construct a *Riemannian orbifold metric* for an orbifold (see [7, Proposition 2.20]). Moreover, one can construct for each element in $\mathcal{T}Q$ a unique maximal orbifold geodesic (cf. [10, Section 4.1]). Let Ω be the subset of elements in $\mathcal{T}Q$ whose associated maximal geodesic exists at least up to time 1. Then the map $\exp_{\mathrm{Orb}}\colon \mathcal{T}Q \supseteq \Omega \to Q$ sending an element to its associated orbifold geodesic evaluated at time 1 induces a morphism of orbifolds $[\exp_{\mathrm{Orb}}]$ (cf. [10, Section 4.2]). This morphism is called *Riemannian orbifold exponential map.*

Theorem 3 ([10, Theorem 5.2.4]). *The group* $\mathrm{Diff}_{\mathrm{Orb}}(Q, \mathcal{U})$ *can be made into an infinite-dimensional Lie group* (*in the sense of* [8]) *such that:*

For some Riemannian orbifold metric ρ, *let* $[\exp_{\mathrm{Orb}}]$ *be the Riemannian orbifold exponential map with domain* Ω. *There exists an open zero-neighborhood in* $\mathfrak{X}_{\mathrm{Orb}}(Q)_c$ *such that*

$$[\hat{\sigma}] \mapsto [\exp_{\mathrm{Orb}}] \circ [\hat{\sigma}]|^{\Omega}$$

is a C^∞*-diffeomorphism onto an open submanifold of* $\mathrm{Diff}_{\mathrm{Orb}}(Q, \mathcal{U})$. *The condition is then satisfied for every Riemannian orbifold metric on* $(Q, \mathcal{U})$.

Proposition 4 ([10, Theorem 5.3.1]). *The Lie algebra of* $\mathrm{Diff}_{\mathrm{Orb}}(Q, \mathcal{U})$ *is given by* $(\mathfrak{X}_{\mathrm{Orb}}(Q)_c, [\cdot,\cdot])$. *The Lie bracket of two orbisections* $[\hat{\sigma}], [\hat{\tau}]$ *is the orbisection whose canonical lift on a chart* (U, G, φ) *is*

$$[\sigma_U, \tau_U] \ (\textit{Lie bracket in } \mathfrak{X}(U)).$$

Here σ_U *and* τ_U *are the canonical lifts of* $[\hat{\sigma}]$ *and* $[\hat{\tau}]$, *respectively.*

In the rest of this note, we will apply the results to the orbifolds considered in Example 1. We will see that for these orbifolds, Theorem 3 induces Lie group structures for certain subgroups of $\mathrm{Diff}(\mathbb{R}^d)$. In particular, these Lie group structures will coincide with closed Lie subgroups of $\mathrm{Diff}(\mathbb{R}^d)$ (see [5] for the construction of the Lie group $\mathrm{Diff}(\mathbb{R}^d)$).

2. Application to equivariant diffeomorphism groups

For this section, we use the notation introduced in Examples 1 and 2.

5. Denote by $\mathrm{Diff}^G(\mathbb{R}^d)$ the subset of all weak equivalences in $\mathrm{Diff}(\mathbb{R}^d)$. As in Example 2, we let $[\hat{h}]$ be the orbifold diffeomorphism associated to $\tilde{h} \in \mathrm{Diff}^G(\mathbb{R}^d)$. Consider the map

$$D\colon \mathrm{Diff}^G(\mathbb{R}^d) \to \mathrm{Diff}_{\mathrm{Orb}}\left(\mathbb{R}^d/G, \left\{\mathbb{R}^d, G, \pi\right\}\right), \tilde{f} \mapsto [\hat{f}].$$

Each orbifold diffeomorphisms in the image of D is induced by a lift in the global chart. Since orbifold diffeomorphisms are uniquely determined by their lifts, the composition of these lifts induces the composition of orbifold diffeomorphisms.

Thus $D(\tilde{h}^{-1})$ coincides with $D(\tilde{h})^{-1}$ (the inverse in $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$) by [10, Corollary 2.1.12]. Summing up, D is a group homomorphism.

The map D is not injective, as elements of $\mathrm{Diff}^G(U)$ which differ only up to composition with an element of G are mapped to the same diffeomorphism of orbifolds. From [7, Lemma 2.11] we deduce that the kernel of D coincides with G. Hence D induces an injective group homomorphism:

$$\Delta\colon \mathrm{Diff}^G(\mathbb{R}^d)/G \to \mathrm{Diff}_{\mathrm{Orb}}\left(\mathbb{R}^d/G, \left\{\mathbb{R}^d, G, \pi\right\}\right)$$

We will show in the next proposition that Δ is surjective, i.e., each orbifold diffeomorphisms of the orbifold $(\mathbb{R}^d/G, \left\{\mathbb{R}^d, G, \pi\right\})$ corresponds to diffeomorphism of $\mathbb{R}^d$, which is a weak equivalence with respect to the G-action.

Proposition 6. *Consider* $[\hat{h}] \in \mathrm{Diff}_{\mathrm{Orb}}\left(\mathbb{R}^d/G, \left\{\mathbb{R}^d, G, \pi\right\}\right)$ *with representative* $(h, \{h_i\}_{i\in I}, (P,\nu)) \in [\hat{h}]$. *The map* h *lifts to a diffeomorphism* $\tilde{h}$ *of* $\mathbb{R}^d$ *which is a weak equivalence, with respect to the* G*-action.*

Proof. We shall construct at first a lift on the set of non-singular points. By condition **(IS)** of Example 1, there is only one singular point. The origin in $\mathbb{R}^d$ is jointly fixed by all elements of G. Hence $\mathbb{R}^d \setminus \{0\}$ corresponds to the set of non-singular points and we set $Q_{\mathrm{reg}} := Q \setminus \{0\}$. It is easy to see that $q := \pi|_{\mathbb{R}^d\setminus\{0\}}^{Q_{\mathrm{reg}}}$ is a covering.

Diffeomorphisms of orbifolds preserve singular points by [10, Proposition 2.1.5] and thus the homeomorphism $h\colon Q \to Q$ satisfies $f\pi(0) = \pi(0)$. The restriction $h|_{Q_{\mathrm{reg}}}^{Q_{\mathrm{reg}}}$ yields a homeomorphism.

If $d = 1$ **holds**, then the space $\mathbb{R} \setminus \{0\}$ is disconnected. Then the mapping $q|_{]0,\infty[}\colon]0,\infty[\to Q_{\mathrm{reg}}$ is a homeomorphism and we obtain a well-defined homeomorphism $h^+ := (q|_{]0,\infty[})^{-1} hq|_{]0,\infty[}$, mapping $]0,\infty[$ to itself. This mapping extends to a homeomorphism via

$$h_{\mathrm{reg}}\colon \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}, x \mapsto \begin{cases} h^+(x) & x > 0 \\ r \circ h^+ \circ r(x) = -h^+(-x) & x < 0. \end{cases}$$

By construction, h_{reg} and also its inverse are equivariant maps with respect to $G = \langle r \rangle$. Thus h_{reg} extends to an equivariant homeomorphism $\tilde{h}\colon \mathbb{R} \to \mathbb{R}$ by setting $\tilde{h}(0) = 0$.

If $d \geq 2$ **holds**, then the space $\mathbb{R}^d \setminus \{0\}$ is (path-)connected. We have to construct a lift $f_{\mathrm{reg}}\colon \mathbb{R}^d \setminus \{0\} \to \mathbb{R}^d \setminus \{0\}$.

For $d \geq 3$, the space $\mathbb{R}^d \setminus \{0\}$ is simply connected, path-connected and locally path-connected. Choose $x_0 \in \mathbb{R}^d \setminus \{0\}$ and $y_0 \in q^{-1}hq(x_0)$. Then by [6, Proposition 1.33], there is a unique lift $h_{\mathrm{reg}}\colon \mathbb{R}^d \setminus \{0\} \to \mathbb{R}^d \setminus \{0\}$ of $h|^{Q_{\mathrm{reg}}} \circ q$ which maps x_0 to y_0.

For $d = 2$, the space $\mathbb{R}^2 \setminus \{0\}$ is *not* simply connected. However, it is path-connected and locally path-connected. We may still apply [6, Proposition 1.33] if the fundamental group $\pi_1(\mathbb{R}^2 \setminus \{0\}, x_0)$ satisfies:

$$(h|^{Q_{\mathrm{reg}}} \circ q)_*(\pi_1(\mathbb{R}^2 \setminus \{0\}, x_0)) \subseteq q_*(\pi_1(\mathbb{R}^2 \setminus \{0\}, y_0)) \quad (1)$$

Recall that the fundamental group $\pi_1(\mathbb{R}^2 \setminus \{0\}, x_0)$ can be identified with $\mathbb{Z}$ (cf. [6, Example 1.15]). Moreover, as $G \subseteq \mathrm{SO}(2)$ holds, the subgroup $G \subseteq O(2)$ must be a cyclic group, generated by a rotation γ of order $m \in \mathbb{N}$. As we have already seen, Q is homeomorphic to a cone and Q_{reg} may be identified with a cone whose tip has been removed. In particular, the space Q_{reg} is homeomorphic to $\mathbb{R}^2 \setminus \{0\}$.

Consider a generator $[e]$ of the fundamental group $\pi_1(\mathbb{R}^2 \setminus \{0\}, x_0)$, where e is chosen as a circle around the origin passing through x_0. If γ is a rotation of order m, then we have $q_*[e] = [q \circ e]$ is a loop in Q_{reg}, which passes m times through $\pi(y_0)$. The next picture illustrates this behavior:

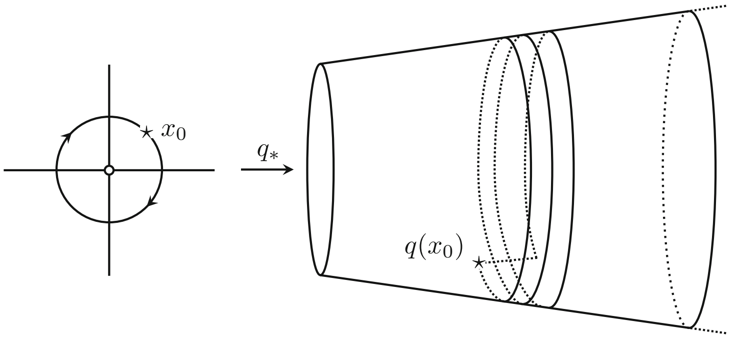

FIGURE 2. Image of a loop generating $\pi_1(\mathbb{R}^2 \setminus \{0\}, x_0)$ with respect to q_*. The loop displayed in Q_{reg} is a curve homotopic to the image of the closed loop for $m = 3$.

Note that $\pi_1(Q_{\mathrm{reg}}, q(y_0))$ is isomorphic to $\mathbb{Z}$ and let $[f]$ be the generator of $\pi_1(Q_{\mathrm{reg}}, q(x_0))$. By abuse of notation we let $[f]$ be the generator of each fundamental group for points in Q_{reg}. From the arguments above, we deduce $q_*(\pi_1(\mathbb{R}^2 \setminus \{0\}, x_0)) = \langle m[f] \rangle$ and thus

$$(h|^{Q_{\mathrm{reg}}} \circ q)_*([e]) = (h|_{Q_{\mathrm{reg}}}^{Q_{\mathrm{reg}}})_*(m[f]) = m([h \circ f]) \in \langle m[f] \rangle = \operatorname{Im} q_*.$$

Therefore property (1) is satisfied and we obtain a unique lift

$$h_{\mathrm{reg}} \colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$$

of $h|_{Q_{\mathrm{reg}}}^{Q_{\mathrm{reg}}}$ mapping x_0 to y_0.

Analogous arguments allow the construction of a unique lift $(h^{-1})_{\mathrm{reg}}$ for $h^{-1}|^{Q \setminus \{0\}} \circ q$ and $d \geq 2$, which maps y_0 to x_0. We claim that $(h^{-1})_{\mathrm{reg}}$ is the inverse of h_{reg}. If this is true, then h_{reg} is a homeomorphism. To prove the claim, consider the map $f := h_{\mathrm{reg}} \circ (h^{-1})_{\mathrm{reg}}$ and compute

$$q \circ f = q \circ h_{\mathrm{reg}} \circ (h^{-1})_{\mathrm{reg}} = h \circ q \circ (h^{-1})_{\mathrm{reg}} = q.$$

Hence f is a lift of $\mathrm{id}_{Q_{\mathrm{reg}}}$ taking y_0 to y_0, and so is the map $\mathrm{id}_{\mathbb{R}^d \setminus \{0\}}$. By the uniqueness of lifts between pointed spaces (see [6, Proposition 1.34]), $h_{\mathrm{reg}} \circ (h^{-1})_{\mathrm{reg}} =$

$f = \mathrm{id}_{\mathbb{R}^d\setminus\{0\}}$. Likewise, $(h^{-1})_{\mathrm{reg}} \circ h_{\mathrm{reg}} = \mathrm{id}_{\mathbb{R}^d\setminus\{0\}}$. Summing up, h_{reg} is a homeomorphism with inverse $(h^{-1})_{\mathrm{reg}}$.

We show that the homeomorphism h_{reg} commutes with the G-action via an automorphism of G. To prove this, consider $g \in G$ and $x \in \mathbb{R}^d \setminus \{0\}$. We have

$$q \circ h_{\mathrm{reg}} \circ g \circ h_{\mathrm{reg}}^{-1}(x)) = hh^{-1}q(x) = q(x).$$

Hence $h_{\mathrm{reg}} \circ g \circ h_{\mathrm{reg}}^{-1}$ is a lift of $\mathrm{id}_{\mathbb{R}^d\setminus\{0\}}$ and so there is an unique element $\alpha(g) \in G$ such that $h_{\mathrm{reg}} \circ g \circ h_{\mathrm{reg}}^{-1}(x_0) = \alpha(g)(x_0)$. By uniqueness of lifts, $h_{\mathrm{reg}} \circ g \circ h_{\mathrm{reg}}^{-1} = \alpha(g)$ on $\mathbb{R}^d \setminus \{0\}$. Repeat this construction for each $g \in G$ to obtain a map $\alpha \colon G \to G$ with $h_{\mathrm{reg}} \circ g = \alpha(g) \circ h_{\mathrm{reg}}$ on $\mathbb{R}^d \setminus \{0\}$ for each $g \in G$. Since $\alpha(gk) \circ h_{\mathrm{reg}} = h_{\mathrm{reg}} \circ (gk) = \alpha(g).h_{\mathrm{reg}} \circ k = \alpha(g).\alpha(k).h_{\mathrm{reg}}$ holds and h_{reg} is a homeomorphism, the map α is an injective group homomorphism. As G is finite, α is thus a group automorphism with $h_{\mathrm{reg}} \circ g = \alpha(g).h_{\mathrm{reg}}$ for each $g \in G$. We extend this map h_{reg} to a homeomorphism $\tilde{h} \colon \mathbb{R}^d \to \mathbb{R}^d$ via $\tilde{h}(0) = 0$. This map satisfies $\tilde{h} \circ g = \alpha(g).\tilde{h}$. We conclude that $\tilde{h}$ is indeed a weak equivalence. Using the definition of orbifold morphisms, one can show that $\tilde{h}$ is a smooth diffeomorphism (see [10, Proposition 6.0.5]). □

Corollary 7. *For an orbifold $(Q,\mathcal{U})$ as in* 1*, the mapping D is surjective. In particular, the induced map $\Delta \colon \mathrm{Diff}^G(\mathbb{R}^d)/G \to \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ is a group isomorphism.*

8. Endow $\mathrm{Diff}^G(\mathbb{R}^d)/G$ with the Lie group structure making Δ an isomorphism of Lie groups. Now we consider the subgroup of $\mathrm{Diff}^G(\mathbb{R}^d)$ whose elements coincide with the identity off some compact subset:

$$\mathrm{Diff}_c^G(\mathbb{R}^d) := \left\{ f \in \mathrm{Diff}^G(\mathbb{R}^d) \,\middle|\, \exists K \subseteq \mathbb{R}^d \text{ compact}, f|_{\mathbb{R}^d\setminus K} = \mathrm{id}_{\mathbb{R}^d\setminus K} \right\}$$

On the level of orbifold diffeomorphisms, we may also consider diffeomorphisms which coincide off some compact set with the identity. These diffeomorphisms form an open Lie subgroup $\mathrm{Diff}_{\mathrm{Orb}}\left(\mathbb{R}^d/G, \left\{(\mathbb{R}^d, G, \pi)\right\}\right)_c$ of all orbifold diffeomorphisms (cf. [10, Remark 5.2.7]). By construction, D maps $\mathrm{Diff}_c^G(\mathbb{R}^d)$ into the open Lie subgroup $\mathrm{Diff}_{\mathrm{Orb}}\left(\mathbb{R}^d/G, \left\{(\mathbb{R}^d, G, \pi)\right\}\right)_c$. Recall that $G \cap \mathrm{Diff}_c^G(U) = \{\mathrm{id}_{\mathbb{R}^d}\}$. Thus for the orbifolds in Example 1, the map D restricts to an injective group homomorphism.

$$\Delta_c \colon \mathrm{Diff}_c^G(U) \to \mathrm{Diff}_{\mathrm{Orb}}\left(\mathbb{R}^d/G, \left\{(\mathbb{R}^d, G, \pi)\right\}\right)_c$$

Lemma 9. *The map $\Delta_c \colon \mathrm{Diff}_c^G(\mathbb{R}^d) \to \mathrm{Diff}_{\mathrm{Orb}}\left(\mathbb{R}^d/G, \left\{(\mathbb{R}^d, G, \pi)\right\}\right)_c$ introduced in* 8 *is an isomorphism of groups.*

Proof. Consider $[\hat{h}] \in \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_c$ with a representative $(h, \{\tilde{h}\}, [P,\nu])$. Here the lift $\tilde{h} \colon \mathbb{R}^d \to \mathbb{R}^d$ has been chosen with $\tilde{h} \in D^{-1}([\hat{h}])$ (which is possible by Proposition 7). Let $K \subseteq Q$ be a compact set with $h|_{Q\setminus K} \equiv \mathrm{id}_{Q\setminus K}$. As $\pi \colon \mathbb{R}^d \to Q$ is a proper map, the set $\pi^{-1}(K)$ is compact. Choose a compact set $L \subseteq \mathbb{R}^d$ with $\pi^{-1}(K) \subseteq L$ and $\mathbb{R}^d \setminus L$ being connected if $d \geq 2$. If $d = 1$, we may assume that $0 \in L$ and $\mathbb{R} \setminus L$ contains exactly two connected components. Recall from the proof of Proposition 6 that the lift $\tilde{h}$ has been constructed with respect to an arbitrary pair $x_0 \in \mathbb{R}^d \setminus \{0\}$ and $y_0 \in \pi^{-1}h\pi(x_0)$ such that $\tilde{h}(x_0) = y_0$ (if $d \geq 2$). Without loss of generality, choose $x_0 \in \mathbb{R}^d \setminus L$. Since $h|_{Q\setminus\pi(L)} \equiv \mathrm{id}_{Q\setminus\pi(L)}$ holds, one can set $y_0 = x_0$. We claim that the lift $\tilde{h}$ with respect to these choices is contained in $\mathrm{Diff}_c^G(\mathbb{R}^d)$. If this is true, then $\Delta_c(\tilde{h}) = D(\tilde{h}) = [\hat{h}]$ follows and Δ_c is a group isomorphism.

To prove the claim, it suffices to prove that $\tilde{h}$ coincides with $\mathrm{id}_{\mathbb{R}^d}$ outside the compact set L. We distinguish two cases: If $d \geq 2$, then $\tilde{h}$ is a lift of the identity on the connected set $\mathbb{R}^d \setminus L$ which takes x_0 to x_0 and so is $\mathrm{id}_{\mathbb{R}^d\setminus L}$. Hence, $\tilde{h}|_{\mathbb{R}^d\setminus L} = \mathrm{id}_{\mathbb{R}^d\setminus L}$ by uniqueness of lifts (cf. [6, Proposition 1.34]). Hence $\tilde{h} \in \mathrm{Diff}_c^G(\mathbb{R}^d)$ follows.

If $d = 1$, by choice of L the space $\mathbb{R} \setminus L$ contains two connected components C_1, C_2. Now [7, Lemma 2.11] yields $\tilde{h}|_{C_i} = g_i|_{C_i}$ for some $g_i \in G$ and $i \in \{1, 2\}$. By construction of $\tilde{h}$, we have $\tilde{h}(]0,\infty[) \subseteq]0,\infty[$ and $\tilde{h}(]-\infty,0[) \subseteq]-\infty,0[$, whence $g_1 = g_2 = \mathrm{id}_{\mathbb{R}}$ and thus $\tilde{h} \in \mathrm{Diff}_c^G(\mathbb{R})$. □

We can thus endow the group $\mathrm{Diff}_c^G(\mathbb{R}^d)$ with the unique topology turning Δ_c into an isomorphism of Lie groups. In this section, we have seen that for the class of orbifolds introduced in Example 1, the following holds:

- The Lie group of orbifold diffeomorphisms

  $$\mathrm{Diff}_{\mathrm{Orb}}\left(\mathbb{R}^d/G, \left\{(\mathbb{R}^d, G, \pi)\right\}\right)$$

  is isomorphic to $\mathrm{Diff}^G(\mathbb{R}^d)/G$. In particular, all orbifold diffeomorphisms are induced by diffeomorphisms of $\mathbb{R}^d$ which are weak equivalences with respect to the G-action.
- The Lie group of all compactly supported orbifold diffeomorphisms

  $$\mathrm{Diff}_{\mathrm{Orb}}\left(\mathbb{R}^d/G, \left\{(\mathbb{R}^d, G, \pi)\right\}\right)_c$$

  is isomorphic to $\mathrm{Diff}_c^G(\mathbb{R}^d)$.

  Thus compactly supported orbifold diffeomorphisms correspond bijectively to compactly supported weak equivalences of $\mathbb{R}^d$.

Finally, we would like to clarify how the Lie group structures obtained in this section relate to Lie group structures already constructed on these groups. In [5, Theorem 6.5] a Lie group structure for $\mathrm{Diff}(\mathbb{R}^d)$ has been constructed. This Lie group contains $\mathrm{Diff}^G(\mathbb{R}^d)$ as a closed subgroup modeled on the space $\mathfrak{X}\left(\mathbb{R}^d\right)_c^G$. By a general construction principle for Lie groups (see [3, III. §1 9. Proposition 18]), the Lie group $\mathrm{Diff}_c^G(\mathbb{R}^d)$ also induces a Lie group structure on $\mathrm{Diff}^G(\mathbb{R}^d)$. This Lie group then contains $\mathrm{Diff}_c^G(\mathbb{R}^d)$ as an open subgroup. Furthermore, notice that this structure turns G into a discrete normal subgroup of $\mathrm{Diff}^G(\mathbb{R}^d)$. We have shown in [10, Remark 6.0.8] that both Lie group structures coincide. Thus the Lie group structures constructed in this section coincide with the structures obtained by the traditional construction.

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