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Complex Manifold Structure and Algebroid of the Partially Invertible Elements Groupoid of a W^*-algebra

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Abstract. The goal of the present note based on [4] is a description of complex manifold structure of the groupoid $\mathcal{G}(\mathfrak{M})$ of partially invertible elements of W^*-algebra $\mathfrak{M}$. We also describe Banach–Lie algebroid $\mathcal{A}(\mathfrak{M})$ of the groupoid $\mathcal{G}(\mathfrak{M})$.

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1. Introduction

During the last few decades one can observe a progress in the study of Lie groupoids and algebroids which play a significant role in differential geometry. As a consequence their impact in mathematical physics is also increasing, see, e.g., [3, 8] and references therein. Similar situation occurs in the operator algebras theory where the convolution C^*-algebras of functions on locally compact groupoids equipped with a left Haar system are considered, see [6].

In this note, following [4] and [5], we describe Banach–Lie groupoids and algebroids related in the canonical way to the structure of a W^*-algebra (von Neumann algebra).

The most detailed description of the subject and motivation for this kind of investigations one can find in [4] and [5].

2. Groupoid of partially invertible elements of W^*-algebra

Let us begin with recalling the basic definitions.

A *groupoid* over base set B (see, e.g., [3, 8]) is a set $\mathcal{G}$ equipped with maps:

(i) a *source map* $\mathbf{s} : \mathcal{G} \to B$ and a *target map* $\mathbf{t} : \mathcal{G} \to B$

(ii) a *product* $m : \mathcal{G}^{(2)} \to \mathcal{G}$

$$m(g,h) =: gh,$$

defined on *the set of composable pairs*

$$\mathcal{G}^{(2)} := \{(g,h) \in \mathcal{G} \times \mathcal{G} : \ \mathbf{s}(g) = \mathbf{t}(h)\},$$

(iii) an injective *identity section* $\varepsilon : B \to \mathcal{G}$,

(iv) an *inverse map* $\iota : \mathcal{G} \to \mathcal{G}$, which are subject to the following compatibility conditions:

$$\mathbf{s}(gh) = \mathbf{s}(h), \quad \mathbf{t}(gh) = \mathbf{t}(g), \tag{1}$$

$$k(gh) = (kg)h, \tag{2}$$

$$\varepsilon(\mathbf{t}(g))g = g = g\varepsilon(\mathbf{s}(g)), \tag{3}$$

$$\iota(g)g = \varepsilon(\mathbf{s}(g)), \qquad g\iota(g) = \varepsilon(\mathbf{t}(g)), \tag{4}$$

where $g, k, h \in \mathcal{G}$.

For a groupoid $\mathcal{G}$ over a base B we will use the notation $\mathcal{G} \rightrightarrows B$.

Remark 1. Equivalently one can define a groupoid $\mathcal{G} \rightrightarrows B$ as a small category in which all morphisms are invertible, see for example [2].

Let us recall that C^*-algebra $\mathfrak{M}$ is called W^**-algebra* (or *von Neumann algebra*) if there exists a Banach space $\mathfrak{M}_*$ such that

$$(\mathfrak{M}_*)^* = \mathfrak{M},$$

i.e., $\mathfrak{M}$ possesses a predual Banach space $\mathfrak{M}_*$. If $\mathfrak{M}_*$ exists it is defined in a unique way by the structure of W^*-algebra $\mathfrak{M}$, see [7].

Element $p \in \mathfrak{M}$ is called a (*orthogonal*) *projection* if $p^* = p = p^2$. We will denote the lattice of projections of the W^*-algebra $\mathfrak{M}$ by $\mathcal{L}(\mathfrak{M})$. Element $u \in \mathfrak{M}$ is called a *partial isometry* if uu^* (or equivalently u^*u) is a projection. We will denote the set of partial isometries of the W^*-algebra $\mathfrak{M}$ by $\mathcal{U}(\mathfrak{M})$.

The least projection $l(x) \in \mathcal{L}(\mathfrak{M}))$ in $\mathfrak{M}$, such that

$$l(x)x = x \qquad (\text{respectively } x\, r(x) = x) \tag{5}$$

is called the *left support* (respectively *right support*) of $x \in \mathfrak{M}$.

If $x \in \mathfrak{M}$ is self adjoint, then *support* of x is a projection

$$s(x) := l(x) = r(x).$$

The *polar decomposition* of $x \in \mathfrak{M}$ is a representation

$$x = u|x|, \tag{6}$$

where $u \in \mathfrak{M}$ is partial isometry and $|x| := \sqrt{x^*x} \in \mathfrak{M}^+$ such that

$$l(x) = s(|x^*|) = uu^*, \qquad r(x) = s(|x|) = u^*u.$$

We define the set $\mathcal{G}(\mathfrak{M})$ of *partially invertible* elements of $\mathfrak{M}$ as follows

$$\mathcal{G}(\mathfrak{M}) := \{x \in \mathfrak{M}; \ \ |x| \in G(p\mathfrak{M}p), \text{ where } p = s(|x|)\},$$

where $G(p\mathfrak{M}p)$ is the group of all invertible elements of W^*-subalgebra $p\mathfrak{M}p \subset \mathfrak{M}$.

Remark 2. $\mathcal{G}(\mathfrak{M}) \subsetneq \mathfrak{M}$.

We can define the groupoid structure $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ with $\mathcal{G}(\mathfrak{M})$ being the set of invertible morphisms and $\mathcal{L}(\mathfrak{M})$ as the base set. The groupoid maps for $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ are defined as follows:

(i) the source and target maps $\mathbf{s}, \mathbf{t} : \mathcal{G}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ are

$$\mathbf{s}(x) := r(x), \qquad \mathbf{t}(x) := l(x),$$

(ii) the product is the product in $\mathfrak{M}$ restricted to

$$\mathcal{G}(\mathfrak{M})^{(2)} := \{(x,y) \in \mathcal{G}(\mathfrak{M}) \times \mathcal{G}(\mathfrak{M}); \quad \mathbf{s}(x) = \mathbf{t}(y)\},$$

(iii) the identity section $\varepsilon : \mathcal{L}(\mathfrak{M}) \hookrightarrow \mathcal{G}(\mathfrak{M})$ as the embedding,

(iv) the inverse map $\iota : \mathcal{G}(\mathfrak{M}) \to \mathcal{G}(\mathfrak{M})$ is

$$\iota(x) := |x|^{-1} u^*.$$

The subset of partial isometries $\mathcal{U}(\mathfrak{M}) \subset \mathcal{G}(\mathfrak{M})$ inherits the groupoid structure $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ from $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$. Let us note here that for $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ the source and target maps $\mathbf{s}, \mathbf{t} : \mathcal{G}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ are:

$$\mathbf{s}(u) = u^* u, \qquad \mathbf{t}(u) = uu^*,$$

and inverse map $\iota : \mathcal{U}(\mathfrak{M}) \to \mathcal{U}(\mathfrak{M})$ is expressed by the involution:

$$\iota(u) = u^*.$$

Remark 3. The groupoid $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ is a wide subgroupoid of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$.

For details we address to [4].

3. Banach–Lie groupoid structure of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$

One properly defines the complex Banach manifold structure on the groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ and shows that the groupoid maps are consistent with the structure, i.e., the groupoid of partially invertible elements is a Banach–Lie groupoid, see [4].

For any projection $p \in \mathcal{L}(\mathfrak{M})$ we define (following [4]) the subset $\Pi_p \subset \mathcal{L}(\mathfrak{M})$ by

$$q \in \Pi_p \quad \text{iff} \quad \mathfrak{M} = q\mathfrak{M} \oplus (1-p)\mathfrak{M} \tag{7}$$

and maps $\quad \sigma_p : \Pi_p \to \mathfrak{M}p, \qquad \varphi_p : \Pi_p \xrightarrow{\sim} (1-p)\mathfrak{M}p$ by

$$\sigma_p(q) := x, \qquad \varphi_p(q) := y, \tag{8}$$

where $p = x - y$ is consistent with the splitting (7). Note that $l \circ \sigma_p = id_{\Pi_p}$ and φ_p defines a bijections between Π_p and the Banach space $(1-p)\mathfrak{M}p$.

In order to construct transitions maps

$$\varphi_p \circ \varphi_{p'}^{-1} : \varphi_{p'}(\Pi_p \cap \Pi_{p'}) \to \varphi_p(\Pi_p \cap \Pi_{p'})$$

in the case $\Pi_p \cap \Pi_{p'} \neq \emptyset$, let us take for $q \in \Pi_p \cap \Pi_{p'}$ the following splittings

$$\begin{aligned} \mathfrak{M} &= q\mathfrak{M} \oplus (1-p)\mathfrak{M} = p\mathfrak{M} \oplus (1-p)\mathfrak{M}, \\ \mathfrak{M} &= q\mathfrak{M} \oplus (1-p')\mathfrak{M} = p'\mathfrak{M} \oplus (1-p')\mathfrak{M}. \end{aligned} \tag{9}$$

Splittings (9) lead to the corresponding decompositions of p and p'

$$\begin{aligned} p &= x - y & p &= a + b \\ p' &= x' - y' & 1 - p &= c + d \end{aligned} \tag{10}$$

where $x \in q\mathfrak{M}p$, $y \in (1-p)\mathfrak{M}p$, $x' \in q\mathfrak{M}p'$, $y' \in (1-p')\mathfrak{M}p'$, $a \in p'\mathfrak{M}p$, $b \in (1-p')\mathfrak{M}p$, $c \in p'\mathfrak{M}(1-p)$ and $d \in (1-p')\mathfrak{M}(1-p)$. Using (10) we get the formula

$$y' = (\varphi_{p'} \circ \varphi_p^{-1})(y) = (b + dy)\iota(a + cy).$$

Theorem 4. *The family of charts*

$$(\Pi_p, \varphi_p) \quad p \in \mathcal{L}(\mathfrak{M})$$

defines a complex analytic atlas on $\mathcal{L}(\mathfrak{M})$. This atlas is modeled on the family of Banach spaces $(1-p)\mathfrak{M}p$, where $p \in \mathcal{L}(\mathfrak{M})$.

Remark 5. For equivalent projections $p \sim p'$ there exists a partial isometry $u \in \mathcal{U}(\mathfrak{M})$ such that $uu^* = p$ and $u^*u = p'$, so that one has $(1-p)\mathfrak{M}p \cong (1-p')\mathfrak{M}p'$.

In order to introduce the complex analytic structure on $\mathcal{G}(\mathfrak{M})$ we define for any $\tilde{p}, p \in \mathcal{L}(\mathfrak{M})$ the set

$$\Omega_{\tilde{p}p} := \mathbf{t}^{-1}(\Pi_{\tilde{p}}) \cap \mathbf{s}^{-1}(\Pi_p).$$

If $\Omega_{\tilde{p}p} \neq \emptyset$ we define the map

$$\psi_{\tilde{p}p} : \Omega_{\tilde{p}p} \to (1-\tilde{p})\mathfrak{M}\tilde{p} \oplus \tilde{p}\mathfrak{M}p \oplus (1-p)\mathfrak{M}p$$

by

$$\psi_{\tilde{p}p}(x) := (\varphi_{\tilde{p}}(\mathbf{t}(x)), \iota(\sigma_{\tilde{p}}(\mathbf{t}(x)))x\sigma_p(\mathbf{s}(x)), \varphi_p(\mathbf{s}(x)))\,, \tag{11}$$

which is a bijection of $\Omega_{\tilde{p}p}$ onto an open subset of the direct sum of the Banach subspaces $(1-\tilde{p})\mathfrak{M}\tilde{p} \oplus \tilde{p}\mathfrak{M}p \oplus (1-p)\mathfrak{M}p$ of the W^*-algebra $\mathfrak{M}$. The inverse map $\psi_{\tilde{p}p}^{-1} : \psi_{\tilde{p}p}(\Omega_{\tilde{p}p}) \to \Omega_{\tilde{p}p}$ has the form

$$\psi_{\tilde{p}p}^{-1}(\tilde{y}, z, y) := \sigma_{\tilde{p}}(\tilde{q})z\iota(\sigma_p(q)) = (\tilde{p} + \tilde{y})z\iota(p + y) \tag{12}$$

where $\tilde{q} = l(\tilde{p}+\tilde{y})$ and $q = l(p+y)$ are left supports of $\tilde{p}+\tilde{y}$ and $p+y$ respectively. The transition maps

$$\psi_{\tilde{p}'p'} \circ \psi_{\tilde{p}p}^{-1} : \psi_{\tilde{p}p}(\Omega_{\tilde{p}'p'} \cap \Omega_{\tilde{p}p}) \to \psi_{\tilde{p}'p'}(\Omega_{\tilde{p}'p'} \cap \Omega_{\tilde{p}p})$$

for $(\tilde{y}, z, y) \in \psi_{\tilde{p}p}(\Omega_{\tilde{p}'p'} \cap \Omega_{\tilde{p}p})$ are given by

$$(\psi_{\tilde{p}'p'} \circ \psi_{\tilde{p}p}^{-1})(\tilde{y}, z, y) := (\tilde{y'}, z', y'), \tag{13}$$

where

$$\tilde{y'} = (\varphi_{\tilde{p}'} \circ \varphi_{\tilde{p}}^{-1})(\tilde{y}) = (\tilde{b} + \tilde{d}\tilde{y})\iota(\tilde{a} + \tilde{c}\tilde{y}) \tag{14}$$

$$y' = (\varphi_{p'} \circ \varphi_p^{-1})(y) = (b + dy)\iota(a + cy) \tag{15}$$

and

$$z' = \iota(\tilde{p}' + \tilde{y}')(\tilde{p} + \tilde{y})z\iota(p + y)(p' + y'). \tag{16}$$

We note that all maps in (14), (15) and (16) are complex analytic.

Thus we derive

Theorem 6.

(i) *The family of charts*

$$(\Omega_{\tilde{p}p}, \psi_{\tilde{p}p}),$$

where $(p, \tilde{p}) \in \mathcal{L}(\mathfrak{M}) \times \mathcal{L}(\mathfrak{M})$ *are pairs of equivalent projections, defines a complex analytic atlas on the groupoid* $\mathcal{G}(\mathfrak{M})$ *(in the sense of* [1]*). This atlas is modeled on the family of Banach spaces* $(1 - \tilde{p})\mathfrak{M}\tilde{p} \oplus \tilde{p}\mathfrak{M}p \oplus (1 - p)\mathfrak{M}p$ *indexed by the pair of equivalent projections* $p, \tilde{p} \in \mathcal{L}(\mathfrak{M})$.

(ii) *All groupoid structure maps and the groupoid product are complex analytic with respect to the above Banach manifold structure.*

Following [5] we present now an example of the subgroupoid of the groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$. By $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ we denote the transitive subgroupoid of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$, where

$$\mathcal{L}_{p_0}(\mathfrak{M}) := \{l(x) : \quad x \in \mathcal{G}(\mathfrak{M}), \quad r(x) = p_0\} \tag{17}$$

and

$$\mathcal{G}_{p_0}(\mathfrak{M}) := l^{-1}(\mathcal{L}_{p_0}(\mathfrak{M})) \cap r^{-1}(\mathcal{L}_{p_0}(\mathfrak{M})). \tag{18}$$

Let G_0 be the group of invertible elements of W^*-subalgebra $p_0\mathfrak{M}p_0 \subset \mathfrak{M}$. By P_0 we denote the intersection $\mathcal{G}_{p_0}(\mathfrak{M}) \cap \mathfrak{M}p_0$ of $\mathcal{G}_{p_0}(\mathfrak{M})$ with the left W^*-ideal $\mathfrak{M}p_0$. From the subsequent (see [5])

Proposition 7.

(i) *Group* G_0 *is an open subset of the Banach space* $p_0\mathfrak{M}p_0$*. So,* G_0 *is a Banach–Lie group whose Lie algebra is* $p_0\mathfrak{M}p_0$*.*

(ii) *The subset* $P_0 \subset \mathfrak{M}p_0$ *is open in the Banach space* $\mathfrak{M}p_0$*. Thus the tangent bundle* TP_0 *can be identified with the trivial bundle* $P_0 \times \mathfrak{M}p_0$*.*

(iii) *One has a free right action of* G_0 *on* $P_0 \times P_0$ *defined by*

$$P_0 \times P_0 \times G_0 \ni (\eta, \xi, g) \;\mapsto\; (\eta g, \xi g) \in P_0 \times P_0. \tag{19}$$

It follows that $P_0\,(\mathcal{L}_{p_0}(\mathfrak{M}),\ G_0,\ l)$ is a principal bundle with P_0 as the total space, $\mathcal{L}_{p_0}(\mathfrak{M})$ as the bundle base, and the left support $l : P_0 \to \mathcal{L}_{p_0}(\mathfrak{M})$ as the canonical projection. Thus we obtain the gauge groupoid $\frac{P_0 \times P_0}{G_0} \rightrightarrows P_0/G_0$ of the above principal bundle. For the definition of the gauge groupoid see for example [3].

In [5] we show that Banach–Lie groupoids $\frac{P_0 \times P_0}{G_0} \rightrightarrows P_0/G_0$ and $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ are isomorphic.

4. Algebroid of the groupoid $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$

The Atiyah sequence of the principal bundle $P_0\left(\mathcal{L}_{p_0}(\mathfrak{M}),\ G_0,\ l\right)$ is the following one

$$0 \to p_0\mathfrak{M}p_0 \times_{Ad_{G_0}} P_0 \overset{\iota}{\hookrightarrow} TP_0/G_0 \overset{a}{\to} T(P_0/G_0) \to 0, \tag{20}$$

where $p_0\mathfrak{M}p_0$ is the Lie algebra of the group G_0. The vector bundle morphisms ι and a are defined by the sequence

$$0 \to T^V P_0/G_0 \overset{\iota}{\to} TP_0/G_0 \overset{\pi}{\to} TP_0/TG_0 \to 0 \tag{21}$$

and isomorphisms

$$TP_0/TG_0 \cong T\left(P_0/G_0\right), \tag{22}$$

$$p_0\mathfrak{M}p_0 \times_{AdG_0} P_0 \cong T^V P_0/G_0, \tag{23}$$

where $T^V P_0$ is the vertical bundle of $P_0\left(\mathcal{L}_{p_0}(\mathfrak{M}),\ G_0,\ l\right)$. We can see from (20) that $TP_0/G_0 \to P_0/G_0$ is the algebroid of the gauge groupoid $\frac{P_0 \times P_0}{G_0} \rightrightarrows P_0/G_0$ which as we have shown above is isomorphic to $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$. Hence we conclude that the gauge algebroid $\frac{TP_0}{G_0} \to \frac{P_0}{G_0}$ is isomorphic to the algebroid $\mathcal{A}_{p_0}(\mathfrak{M}) \to \mathcal{L}_{p_0}(\mathfrak{M})$ of the groupoid $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$. Using this isomorphism we find that the Lie bracket of $\mathfrak{X}_1, \mathfrak{X}_2 \in \Gamma(T^V P_0/G_0) \cong \Gamma\mathcal{A}_{p_0}(\mathfrak{M})$ is given by the following expression

$$[\mathfrak{X}_1, \mathfrak{X}_2](\eta) = \left(\left\langle \frac{\partial\vartheta_2}{\partial\eta}(\eta), \vartheta_1(\eta)\right\rangle - \left\langle \frac{\partial\vartheta_1}{\partial\eta}(\eta), \vartheta_2(\eta)\right\rangle\right)\frac{\partial}{\partial\eta}, \tag{24}$$

where

$$\mathfrak{X}(\eta) = \vartheta(\eta)\frac{\partial}{\partial\eta} \tag{25}$$

is G_0-invariant vector field on P_0, i.e., $\vartheta : P_0 \to \mathfrak{M}p_0$ satisfies

$$\vartheta(\eta g) = \vartheta(\eta)g, \tag{26}$$

where $\eta \in P_0,\ g \in G_0$. The notation (25) means that

$$(\mathfrak{X}f)(\eta) = \left\langle \frac{\partial f}{\partial\eta}(\eta), \vartheta(\eta)\right\rangle$$

for any $f \in C^\infty(P_0)$.

The anchor map $a : \mathcal{A}_{p_0}(\mathfrak{M}) \to T\mathcal{L}_{p_0}(\mathfrak{M})$ for $\mathcal{A}_{p_0}(\mathfrak{M})$ is given by

$$a := Tl, \tag{27}$$

where $l : \mathcal{G}_{p_0}(\mathfrak{M}) \to \mathcal{L}_{p_0}(\mathfrak{M})$ is the left support map.

5. An example

We conclude our note presenting an example. Let us take $\mathfrak{M} = L^\infty(\mathcal{H})$, where $\mathcal{H}$ is a separable complex Hilbert space with a fixed orthonormal basis $\{|e_k\rangle\}_{k=0}^{\infty}$. Setting $p_0 = |e_0\rangle\langle e_0|$ (we use the Dirac notation) we find that

$$\mathcal{L}_{p_0}(L^\infty(\mathcal{H})) = \left\{ \frac{|\eta\rangle\langle\eta|}{\langle\eta|\eta\rangle} : \quad \eta \in \mathcal{H} \setminus \{0\} \right\} \cong \mathbb{CP}(\mathcal{H}) \tag{28}$$

and

$$\mathcal{G}_{p_0}(L^\infty(\mathcal{H})) = \left\{ \frac{|\eta\rangle\langle\xi|}{\langle\xi|\xi\rangle} : \quad \eta, \xi \in \mathcal{H} \setminus \{0\} \right\}. \tag{29}$$

5.1. The groupoid $\mathcal{G}_{p_0}(L^\infty(\mathcal{H})) \rightrightarrows \mathcal{L}_{p_0}(L^\infty(\mathcal{H}))$

The structure maps in this case are as follows:

(i) the source and target maps $\mathbf{s}, \mathbf{t} : \mathcal{G}_{p_0}(L^\infty(\mathcal{H})) \to \mathcal{L}_{p_0}(L^\infty(\mathcal{H}))$ are of the form:

$$\mathbf{s}\left(\frac{|\eta\rangle\langle\xi|}{\langle\xi|\xi\rangle}\right) = \frac{|\xi\rangle\langle\xi|}{\langle\xi|\xi\rangle}, \qquad \mathbf{t}\left(\frac{|\eta\rangle\langle\xi|}{\langle\xi|\xi\rangle}\right) = \frac{|\eta\rangle\langle\eta|}{\langle\eta|\eta\rangle}, \tag{30}$$

(ii) the product of elements $\frac{|\eta\rangle\langle\xi|}{\langle\xi|\xi\rangle}, \frac{|\xi\rangle\langle\lambda|}{\langle\lambda|\lambda\rangle} \in \mathcal{G}_{p_0}(L^\infty(\mathcal{H}))$ is:

$$\frac{|\eta\rangle\langle\xi|}{\langle\xi|\xi\rangle}\frac{|\xi\rangle\langle\lambda|}{\langle\lambda|\lambda\rangle} = \frac{|\eta\rangle\langle\lambda|}{\langle\lambda|\lambda\rangle}, \tag{31}$$

(iii) the identity section $\varepsilon : \mathcal{L}_{p_0}(L^\infty(\mathcal{H})) \to \mathcal{G}_{p_0}(L^\infty(\mathcal{H}))$ is the embedding,

(iv) the inverse map $\iota : \mathcal{G}_{p_0}(L^\infty(\mathcal{H})) \to \mathcal{G}_{p_0}(L^\infty(\mathcal{H}))$ is given by

$$\iota\left(\frac{|\eta\rangle\langle\xi|}{\langle\xi|\xi\rangle}\right) = \frac{|\xi\rangle\langle\eta|}{\langle\eta|\eta\rangle}. \tag{32}$$

We note that for $p_0 = |e_0\rangle\langle e_0|$ one has

$$(L^\infty(\mathcal{H}))p_0 = \{|\vartheta\rangle\langle e_0| : \quad \vartheta \in \mathcal{H}\} \cong \mathcal{H}, \tag{33}$$

$$P_0 = \{|\eta\rangle\langle e_0| : \quad \eta \in \mathcal{H} \setminus \{0\}\} \cong \mathcal{H} \setminus \{0\}, \tag{34}$$

and

$$G_0 = G\left(p_0(L^\infty(\mathcal{H}))p_0\right) \cong \mathbb{C} \setminus \{0\}. \tag{35}$$

So, the groupoid $\mathcal{G}_{p_0}(L^\infty(\mathcal{H})) \rightrightarrows \mathcal{L}_{p_0}(L^\infty(\mathcal{H}))$ is isomorphic to the gauge groupoid of the complex Hopf bundle

$$\begin{array}{ccc} \mathbb{C} \setminus \{0\} & \longrightarrow & \mathcal{H} \setminus \{0\} \\ & & \Big\downarrow l \\ & & \mathbb{CP}(\mathcal{H})\,. \end{array} \tag{36}$$

5.2. The complex manifold structure of $\mathcal{G}_{p_0}(L^\infty(\mathcal{H})) \rightrightarrows \mathcal{L}_{p_0}(L^\infty(\mathcal{H}))$

In order to introduce the differential structure of the groupoid $\mathcal{G}_{p_0}(L^\infty(\mathcal{H})) \rightrightarrows \mathcal{L}_{p_0}(L^\infty(\mathcal{H}))$ we should notice that for the orthonormal projections $p_k := |e_k\rangle\langle e_k|$, $k \in \mathbb{N} \cup \{0\}$, the sets $\Pi_k := \Pi_{p_k}$ defined in (7) are the following

$$\Pi_k = \left\{ q = \frac{|\xi\rangle\langle\xi|}{\langle\xi|\xi\rangle}; \quad \xi_k \neq 0, \text{where } \xi = \sum_{k=o}^{\infty} \xi_k |e_k\rangle \right\}. \tag{37}$$

The maps $\sigma_k : \Pi_k \to (L^\infty(\mathcal{H}))p_k$ and $\varphi_k : \Pi_k \to (1 - p_k)(L^\infty(\mathcal{H}))p_k$, see (8), are given by

$$\sigma_k(q) = \frac{1}{\xi_k} |\xi\rangle\langle e_k|, \tag{38}$$

and

$$\varphi_k(q) = \frac{1}{\xi_k} |\xi\rangle\langle e_k| - |e_k\rangle\langle e_k| = \mathbf{y}_k, \tag{39}$$

respectively. Let us note here that we can write $\mathbf{y}_k \in (1 - p_k)(L^\infty(\mathcal{H}))p_k$ in the form

$$\mathbf{y}_k = \sum_{l\neq k} \frac{\xi_l}{\xi_k} |e_l\rangle\langle e_k|. \tag{40}$$

So, $\frac{\xi_l}{\xi_k} =: y_k^l$, where $k \neq l \in \mathbb{N} \cup \{0\}$, are the homogeneous coordinates of $q \in \Pi_k$.

The charts

$$\psi_{km} : l^{-1}(\Pi_k) \cap r^{-1}(\Pi_m) \to (1 - p_k)(L^\infty(\mathcal{H}))p_k \oplus p_k(L^\infty(\mathcal{H}))p_m \oplus (1 - p_m)(L^\infty(\mathcal{H}))p_m$$

of the atlas (11) for $\mathcal{G}_{p_0}(L^\infty(\mathcal{H})) \rightrightarrows \mathcal{L}_{p_0}(L^\infty(\mathcal{H}))$ are given by

$$\psi_{km}(g) = \big(\varphi_k(l(g)), (\sigma_k(l(g)))^{-1} g \sigma_m(r(g)), \varphi_m(r(g))\big) = (\mathbf{y}_k, \mathbf{z}_{km}, \mathbf{y}_m). \tag{41}$$

The coordinates $\mathbf{y}_k$ and $\mathbf{y}_m$ in (41) are defined in (39) and the coordinate $\mathbf{z}_{km}$ is given by

$$\mathbf{z}_{km} = z_{km} |e_k\rangle\langle e_m|, \tag{42}$$

where $z_{km} := \frac{\eta_k}{\xi_m}$.

So, as one can expect, the complex analytic manifold structure of $\mathcal{G}_{p_0}(L^\infty(\mathcal{H}))$ is consistent with the complex analytic structure of the complex Hopf bundle (36).

5.3. The algebroid $\mathcal{A}_{p_0}(L^\infty(\mathcal{H}))$ of $\mathcal{G}_{p_0}(L^\infty(\mathcal{H})) \rightrightarrows \mathcal{L}_{p_0}(L^\infty(\mathcal{H}))$

Using the algebroid isomorphism $\mathcal{A}_{p_0}(\mathfrak{M}) \cong \frac{TP_0}{G_0}$ for the case $\mathfrak{M} = L^\infty(\mathcal{H})$ and $p_0 = |e_0\rangle\langle e_0|$ by virtue of (33)–(35) we obtain the isomorphism

$$\mathcal{A}_{|e_0\rangle\langle e_0|}(L^\infty(\mathcal{H})) \cong \frac{\mathcal{H} \times (\mathcal{H} \setminus \{0\})}{\mathbb{C} \setminus \{0\}}. \tag{43}$$

Hence the sections of the algebroid $\mathcal{A}_{p_0}(L^\infty(\mathcal{H}))$ in the coordinates $(\mathbf{y}_k,\ \mathbf{z}_{km})$ have the following form

$$\mathfrak{X}(\mathbf{y}_k, \mathbf{z}_{km}) = \sum_{l \neq k} a^l(\mathbf{y}_k) \frac{\partial}{\partial y_k^l} + b(\mathbf{y}_k) z_{km} \frac{\partial}{\partial z_{km}} \tag{44}$$

and the algebroid Lie bracket (24) of sections $\mathfrak{X}_1, \mathfrak{X}_2 \in \Gamma\mathcal{A}_{|e_0\rangle\langle e_0|}(L^\infty(\mathcal{H}))$ is

$$\begin{aligned}[\mathfrak{X}_1, \mathfrak{X}_2](\mathbf{y}_k, \mathbf{z}_{km}) = \sum_{s \neq k} \sum_{l \neq k} \left(a_1^l(\mathbf{y}_k) \frac{\partial a_1^s}{\partial y_k^l} - a_2^l(\mathbf{y}_k) \frac{\partial a_1^s}{\partial y_k^l} \right) \frac{\partial}{\partial y_k^s} \\ + \left(\sum_{l \neq k} a_1^l(\mathbf{y}_k) \frac{\partial b_2}{\partial y_k^l} - \sum_{l \neq k} a_2^l(\mathbf{y}_k) \frac{\partial b_1}{\partial y_k^l} \right) z_{km} \frac{\partial}{\partial z_{km}}.\end{aligned} \tag{45}$$

The anchor $a : \mathcal{A}_{|e_0\rangle\langle e_0|}(L^\infty(\mathcal{H})) \to T\mathbb{CP}(\mathcal{H})$ acts on the section

$$\mathfrak{X} \in \Gamma\mathcal{A}_{|e_0\rangle\langle e_0|}(L^\infty(\mathcal{H}))$$

according to the formula

$$a(\mathfrak{X}) = \sum_{l \neq k} a^l(\mathbf{y}_k) \frac{\partial}{\partial y_k^l}. \tag{46}$$

Finally let us note that $b(\mathbf{y}_k) z_{km} \frac{\partial}{\partial z_{km}}$ proves to be the vertical vector field of the complex Hopf bundle.

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