Trends in Mathematics

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Editors

Geometric Methods in Physics

XXXII Workshop, Białowieża, Poland, June 30–July 6, 2013

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Contents

Part III: Groups and Non-commutative Structures

Part IV: Differential Equations and Special Functions

Part V: General Methods

Participants of the XXXII WGMP (Photo by Tomasz Goliński)

Preface

The Workshop on Geometric Methods in Physics is an annual conference organized by the Department of Mathematical Physics of the University of Białystok, Poland. The general scope of the conference is such that it is interesting for both theoretical physicists and mathematicians. The present volume contains selected papers based on the talks presented at the XXXII Workshop on Geometric Methods in Physics, held during the period June 30–July 6, 2013.

The scientific program of our workshops generally covers such subjects as quantization, integrable systems, coherent states, non-commutative geometry, Poisson and symplectic geometry, infinite-dimensional Lie groups and Lie algebras. During the recent workshops we were commemorating in special sessions the achievements of outstanding mathematical physicists. At the XXXII Workshop we had a special session devoted to Daniel Sternheimer in relation to his important contributions to the method of deformation quantization. Daniel Sternheimer gave a talk at our workshop and this volume contains a paper based on it.

The workshop was followed by the week long *School on Geometry and Physics*. The aim of the school was to present in an accessible way to students and young researchers some of the most important research topics in mathematical physics. The school consisted of several courses of 2 or 4 hours length and two one hour talks.

Białowieża – the traditional site of our workshops – deserves a special mentioning. It is a small village located in eastern Poland at the boundary of the "Białowieża Forest", which is the only remaining piece of ancient forests, which used to cover most of Europe. The beautiful and unique surroundings create a special atmosphere of mutual understanding and collaboration during all the activities of the workshop.

The organizers of the XXXII WGMP gratefully acknowledge the financial support from the following sources:

– Belgian Scientific Policy (BELSPO), IAP grant P7/18 DYGEST.
– The University of Białystok.

Finally, we would like to heartily thank the students and young researchers from the Department of Mathematics of the Białystok University for their enthusiastic help in the daily running of the workshop.

March 2014 The Editors

Part I: Deformation, Quantization: Scientific Landmarks of Daniel Sternheimer

Daniel Sternheimer in Białowieża

Geometric Methods in Physics. XXXII Workshop 2013
Trends in Mathematics, 3–5

Daniel Sternheimer

Pierre Bieliavsky and Martin Schlichenmaier

It was a great pleasure for the organizers to learn that Prof. Daniel Sternheimer who was invited as a plenary speaker to the XXXII Workshop on Geometric Methods in Physics will celebrate his 75th birthday during the conference together with us. On this happy occasion we decided to dedicate one day of the workshop to recent work in and around some of the topics Daniel is working on (see the program of this day appended).

Daniel is one of the fathers of an exact formulation of deformation quantization. For everybody working in the field, or in related topics his joint article [1] with Bayen, Flato, Frønsdal and Lichnerowicz is the basic reference. There they exemplified the basic importance of the concept of deformations in physics. Quantum mechanics should be considered as deformation of classical mechanics. The precise mathematical object giving the quantization is the deformation of the Poisson algebra of functions (with commutative point-wise product of the functions) into a non-commutative algebra, a star product.

Deformation quantization is not the only field of interest, research and competence of Daniel. Daniel was born in 1938 in Lyon, France. There he also started his university studies in mathematics. He went to Israel to work in a kibbutz. Luckily he was "ordered" to continue his studies in mathematics at the Hebrew University in Jerusalem, from which he received his master degree. Returning back in 1961 to Paris he first worked in analysis, e.g., in the theory of PDEs, operator theory, and symbol calculus. In 1968 he graduated with his *thèse de doctorat* with Bruhat and Demazure. Already in 1964 his extremely fruitful collaboration with Moshe Flato begun, whom he already met during his stay in Jerusalem. This collaboration suddenly ended by the unexpected passing away of Moshe Flato in 1998. With the collaboration with Moshe, Daniel shifted more and more to mathematical physics. Some of the topics he worked on (and on some he still continues working) are the fundamental symmetry properties of elementary particles, quantum gravity, foundations of quantum mechanics, conformal symmetry, quantum field theory, Lie algebras, general deformation concepts, quantum groups, Hopf algebras, cohomology, Nambu mechanics, AdS universe and singleton physics. In particular, in respect to the latter his interest has revived recently again.

He was a member of the CNRS (first in Paris then in Dijon) till his retirement in 2003. Furthermore, he was and still is a member of the Mathematics Institute of the Université du Bourgogne.

Since 2004 he spends at least half of the year in Japan. From 2004 to 2010 he was Visiting Professor at the University of Keio and since 2010 he is Visiting Researcher at the Rikkyo University in Tokyo. In 2004 he was appointed Honorary Professor of the University of Sankt Petersbourg, Russia.

He has always served the community. Together with Moshe Flato he initiated to create a mathematical physics association at the European level, which finally came into life as the *International Association of Mathematical Physics* (*IAMP*). He is editor of the *Letters in Mathematical Physics*, editor of several book series, organizer of several international conferences, evaluator of research proposals, and is involved in many more tasks.

His scientific influence was and is still very strong. Beside being an author of numerous publications (more than 90) he is frequently invited as speaker at international conferences (like the current one in Białowieża).

On the more personal level we enjoy very much his friendliness, openness and sense of humour. Daniel and Moshé were not only scientific influential figures but also always close friends to us whose help were invaluable. It is a great pleasure to wish Daniel a

Happy Birthday !!

References

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Program of the special day dedicated to Daniel Sternheimer (2.7. 2013)

Morning session
Chairman: Martin Schlichenmaier

- *Daniel Sternheimer* (*Rikkyo University Tokyo, Japan and Université de Bourgogne, France*)
  Altneuland in mathematical particle physics: back to the drawing board?
- *William Kirwin* (*Universität zu Köln, Germany*)
  Complex time flows in geometric quantization
- *Giovanni Landi* (*Università di Trieste, Italy*)
  The Weil algebra of a general Hopf algebra
- *Akira Yoshioka* (*Tokyo University of Science, Japan*)
  Star exponentials and applications

Afternoon session
Chairman: S. Twareque Ali

- *Martin Schlichenmaier* (*University of Luxembourg, Luxembourg*)
  Some naturally defined star products for Kähler manifolds
- *Stephen Sontz* (*Centro de Investigación en Matemáticas, Mexico*)
  Toeplitz quantization with non-commuting symbols
- *Pierre Bieliavsky* (*Université Catholique de Louvain, Belgium*)
  On deformation quantisation
- *Stéphane Korvers* (*Université Catholique de Louvain, Belgium*)
  On deformation quantizations of the Hermitian symmetric space $SU(1, n)/U(n)$.

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Geometric Methods in Physics. XXXII Workshop 2013
Trends in Mathematics, 7–37

"The Important Thing is not to Stop Questioning", Including the Symmetries on Which is Based the Standard Model

Daniel Sternheimer

To the memory of Moshe Flato and of Noriko Sakurai

Abstract. New fundamental physical theories can, so far a posteriori, be seen as emerging from existing ones via some kind of deformation. That is the basis for Flato's "deformation philosophy", of which the main paradigms are the physics revolutions from the beginning of the twentieth century, quantum mechanics (via deformation quantization) and special relativity. On the basis of these facts we describe two main directions by which symmetries of hadrons (strongly interacting elementary particles) may "emerge" by deforming in some sense (including quantization) the Anti de Sitter symmetry (AdS), itself a deformation of the Poincaré group of special relativity. The ultimate goal is to base on fundamental principles the dynamics of strong interactions, which originated half a century ago from empirically guessed "internal" symmetries. After a rapid presentation of the physical (hadrons) and mathematical (deformation theory) contexts, we review a possible explanation of photons as composites of AdS singletons (in a way compatible with QED) and of leptons as similar composites (massified by 5 Higgs, extending the electroweak model to 3 generations). Then we present a "model generating" multifaceted framework in which AdS would be deformed and quantized (possibly at root of unity and/or in manner not yet mathematically developed with noncommutative "parameters"). That would give (using deformations) a space-time origin to the "internal" symmetries of elementary particles, on which their dynamics were based, and either question, or give a conceptually solid base to, the Standard Model, in line with Einstein's quotation: *"The important thing is not to stop questioning. Curiosity has its own reason for existing."*

Mathematics Subject Classification (2010). Primary 81R50; Secondary 53D55, 17B37, 53Z05, 81S10.

Keywords. Symmetries of hadrons, models, Anti de Sitter, deformation theory, deformation quantization, singletons, quantum groups at root of unity, "quantum deformations".

1. Introduction: the deformation philosophy and the present proposal

1.1. Why deformations?

However seductive the idea may be, the notion of "Theory of Everything" is to me unrealistic. In physics, knowingly or not, one makes approximations in order to have as manageable a theory (or model) as possible. That happens in particular when the aim is to describe the reality known at the time, even if one suspects that a more elaborate reality is yet to be discovered. The question is how to discover that reality. We claim, on the basis of past experience, that one should not extrapolate but rather "deform."

Indeed physical theories have their domain of applicability defined, e.g., by the relevant distances, velocities, energies, etc. involved. But the passages from one domain (of distances, etc.) to another do not happen in an uncontrolled way: experimental phenomena appear that cause a paradox and contradict accepted theories, in line with the famous quote by Fermi [24]:

> *There are two possible outcomes: if the result confirms the hypothesis, then you've made a measurement. If the result is contrary to the hypothesis, then you've made a discovery.*

Eventually a new fundamental constant enters, causing the formalism to be modified: the attached structures (symmetries, observables, states, etc.) *deform* the initial structure to a new structure which in the limit, when the new parameter goes to zero, "contracts" to the previous formalism. The problem is that (at least until there is no other way out) the physics community is gregarious. Singing "It ain't necessarily so" (which I am doing in this paper) is not well received.

A first example of the "deformation" phenomenon can be traced back to the Antiquity, when it was gradually realized that the earth is not flat. [Yet nowadays some still dispute the fact!] In mathematics the first instances of deformations can be traced to the nineteenth century with Riemann surface theory, though the main developments happened a century later, in particular with the seminal analytic geometry works of Kodaira and Spencer [56] (and their lesser known interpretation by Grothendieck [45], where one can see in watermark his "EGA" that started a couple of years later). These deep geometric works were in some sense "linearized" in the theory of deformations of algebras by Gerstenhaber [44].

The realization that deformations are fundamental in the development of physics happened a couple of years later in France, when it was noticed that the Galilean invariance ($SO(3) \cdot \mathbb{R}^3 \cdot \mathbb{R}^4$) of Newtonian mechanics is deformed, in the Gerstenhaber sense [44], to the Poincaré group of special relativity ($SO(3,1) \cdot \mathbb{R}^4$). In spite of the fact that the composition law of symbols of pseudodifferential operators, essential in the Atiyah–Singer index theorem developed at that time (to the exposition of which I took part in Paris in the Séminaire Cartan–Schwartz 1963/64), was in effect a deformation of their abelian product, it took another ten years or so to develop the tools which enabled us to make explicit, rigorous and convincing, what was in the back of the mind of many: quantum mechanics is a deformation of classical mechanics. That developed into what became known as *deformation quantization* and its manifold avatars and more generally into the realization that quantization is deformation. This stumbling block being removed, the paramount importance of deformations in theoretical physics became clear [26], giving rise to what I call “Flato’s deformation philosophy”.

This paper being aimed at both physicists and mathematicians and dealing with so many topics, we may look overly schematic (even trivial) in many parts for readers coming from one or the other community. More details can be found in relatively recent reviews ([21, 76, 77] by myself, and many more by others) and references quoted therein. The hope is that both communities will get the flavor of (and maybe contribute to) the framework for models proposed here. It is based on developments I have witnessed since the early 1960s and in which Flato and I took part, sometimes in a controversial way. In numerous discussions I had with scientists around these ideas, especially in the past two years, I was surprised to notice that many had often only a vague idea of a number of the topics involved, not going beyond the views given in textbooks and/or educated popularizations. I am probably one of the very few who can (and dare) deal with all the topics involved in that unconventional manner. Part of the mathematical aspect is virgin territory and in any case requires an approach (which can be called “mathematical engineering”) dealing more with specific examples than with very abstract developments. On the other hand the basic physical approach is unconventional, and some of the physical issues and models questioned here have for years been taught as facts in courses and presented as such in the literature.

1.2. A brief overlook of the paper

Towards the end of the nineteenth century many believed that, in particular with Newtonian mechanics (and gravitation) and electromagnetism, physics was well understood. Yet the best was to come. In the first half of last century appeared relativity and quantum mechanics, which we now can interpret as deformations. On the fundamental side the second half of last century was dominated by the interactions between elementary particles, classified (in increasing order of strength) as gravitational, weak, electromagnetic and strong. Quantum electrodynamics (QED), developed in the 1940s, explained electromagnetic interactions with an extremely high level of accuracy (even if the theory is not yet fully mathematically rigorous). In the 1970s it was combined with weak interactions in the electroweak model, which required the Higgs boson that was (most likely) now discovered in CERN.

After an outlook of the physical and mathematical context we shall indicate how, using AdS symmetry (a deformation of Poincaré) we can explain the photon (the basis of QED) as composite of two “singletons”, massless particles in a $2+1$ space-time (themselves composites of two harmonic oscillators). Then an extension of the electroweak model to the presently known 3 generations of leptons could explain how, in AdS, these can also be composites of singletons, massified by 5 Higgs.

It is therefore tempting to try and obtain the symmetries of hadrons, on which their dynamics has been built, by deforming further AdS. That cannot be done in the category of Lie groups but can, e.g., in that of Hopf algebras (quantum groups). It turns out that these, at root of unity (often called "restricted quantum groups") are finite-dimensional vector spaces, and have finite-dimensional UIRs (unitary irreducible representations), an important feature of the presently used simple unitary symmetries. There are of course many other problems to address, which cannot be ignored, but if that direction produces a model which could fit experimental data, a revolution in our understanding of physics might follow.

That could be too much to hope for and more general deformations might be needed, in particular (also at roots of unity) multiparameter (e.g., parameters in the group algebra of $\mathbb{Z}/n\mathbb{Z}$, denoted in the following by $\mathbb{Z}_{(n)}$), or a novel theory of deformations, not yet developed mathematically, with non-commutative "deformation parameter" (especially quaternions or belonging to the group algebra of $\mathbb{S}_n$, the permutation group of n elements, e.g., $n = 3$).

Both are largely virgin mathematical territory, and if successful we might have to "go back to the drawing board," for the theory and for the interpretation of many raw experimental data. That is a challenge worthy of the future generations, which in any case should give nontrivial mathematics.

2. A very schematic glimpse on the context: hadrons and their symmetries

In the fifties the number of known elementary particles increased so dramatically that Fermi quipped one day [24]:

> *Young man, if I could remember the names of these particles, I would have been a botanist.*

Clearly, already then, the theoretical need was felt, to bring some order into that fast increasing [8] flurry of particles. Two (related) natural ideas appeared: To apply in particle physics "spectroscopy" methods that were successful in molecular spectroscopy, in particular group theory [83]. And to try and treat some particles as "more elementary", considering others as composite.

A seldom mentioned caveat: In molecular spectroscopy, e.g., when a crystalline structure breaks rotational symmetry, which (for trigonal and tetragonal crystals) was the subject of Flato's M.Sc. Thesis [55] under Racah (defended in 1960 and still frontier when its main part was published in 1965 as his French "second thesis"), we know the forces, and their symmetries give the spectra (energy levels). In particle physics things occurred in reverse order: one guessed symmetries from the observed spectra, interpreted experimental data on that basis and developed dynamics compatible with them.

In the beginning, in order to explain the similar behavior of proton p and neutron n under strong interactions, a quantum number (isospin) was introduced in the 1930s, related to a $SU(2)$ symmetry. In the 1950s new particles were discovered in cosmic rays, that behaved "strangely" (e.g., they lived much longer than expected). So a new quantum number (strangeness) was introduced, which would be conserved by strong and electromagnetic interactions, but not by weak interactions. One of these is the baryon Λ. In 1956 Shoichi Sakata [71], extending an earlier proposal by Fermi and Yang (involving only protons and neutrons) came with the "Sakata model" according to which p, n and Λ are "more elementary" and the other particles are composites of these 3 and their antiparticles. This conceptually appealing model (maybe not as "sexy" as Yoko Sakata, a top model who was not born then) had a strong impact [66], in spite of the fact that a number of the experimental predictions it gave turned out to be wrong.

In the beginning of 1961 an idea (that was in the making before) appeared: since we have 2 quantum numbers (isospin and strangeness, we would now say "two generations") conserved in strong interactions, we should try a rank 2 compact Lie group to "put into nice boxes" the many particles we had. In particular three papers were written then: An elaborate paper [7] in which, "since it is as yet too early to establish a definite symmetry of the strong interactions," all 3 groups (types A_2, $B_2 = C_2$ and G_2), and more, were systematically studied. And two [43, 65], in which only the simplest ($SU(3)$, type A_2) was proposed. The known "octets" of 8 baryons of spin $\frac{1}{2}$ and of the 8 scalar (spin 0) and vector (spin 1) mesons fitted nicely in the 8-dimensional adjoint representation. (Hence the name "the eightfold way" coined by Gell-Mann, an allusion to the "Noble Eightfold Path of Buddhism".) In 1962 Lev Okun proposed "hadrons" as a common name for strongly interacting particles, the half-integer spin (fermions) baryons, usually heavier, and the integer spin (bosons) mesons. The 9 then known baryons of spin $\frac{3}{2}$ ($4\ \Delta + 3\ \Sigma^* + 2\ \Xi^*$) were associated with the 10-dimensional representation: the missing one (Ω^-) in the "decuplet" was discovered in 1964 with roughly the properties predicted by Gell-Mann in 1962. Big success! (Even if anyone can guess that after 4, 3, 2 comes 1...)

But what to do with the basic (3-dimensional) representations of $SU(3)$, which can give (by tensor product and reduction into irreducible components) all other representations? In 1964 Murray Gell-Mann, and independently George Zweig, suggested that they could be associated with 3 entities (the same number as in the Sakata model) and their antiparticles. Zweig proposed to call them "aces" but Gell-Mann, with his feeling for a popular name, called them "quarks", a nonsense word which he imagined and shortly afterward found was used by James Joyce in "Finnegans Wake:"

Three quarks for Muster Mark!
Sure he has not got much of a bark
And sure any he has it's all beside the mark.

Now, how could such "confined" quarks, which would have spin $\frac{1}{2}$ (not to mention fractional charge), coexist in a hadron, something forbidden by the Pauli exclusion principle? That same year O.W. Greenberg (and Y. Nambu) proposed to give them different "colors", now labeled blue, green, and red. Eventually, since the 1970s, that gave rise to QCD (quantum chromodynamics) in parallel with QED but with nonabelian "gauge group" $SU(3)$ instead of the abelian group $U(1)$ in QED. In order to keep them together "gluons" were introduced, which carried the strong force. From that time on, the development of particle physics followed essentially a ballistic trajectory, and eventually its theory became more and more phenomenology-oriented – with the caveat that many raw experimental data are interpreted within the prevalent models.

In 1964 quarks came in 3 "flavors" (up, down and strange) but the same year a number of people, in particular Sheldon Glashow, proposed a fourth flavor (named charm) for a variety of reasons, which became gradually more convincing until in 1974 a "charmed" meson J/Ψ was discovered, completing the 2 generations of quarks, in parallel with the 2 generations of leptons (e and μ) and their associated neutrinos. The number of supposed quark flavors grew to the current six in 1973, when Makoto Kobayashi and Toshihide Maskawa noted that an experimental observation (CP violation) could be explained if there were another pair of quarks, eventually named bottom and top by Haim Harrari, and "observed" (with much heavier mass[1] than expected for the top) at Fermilab in 1977 and 1995 (resp.). In parallel, in 1974–1977, the existence of a heavier lepton τ was experimentally found, and its neutrino discovered in 2000. Kobayashi and Maskawa shared the 2008 Nobel prize in physics with Yoichiro Nambu who, already in 1960, described the mechanism of spontaneous symmetry breaking in particle physics. They were also awarded in 1985 the first J.J. Sakurai prize for Theoretical Particle Physics established, after JJ's premature death in 1982, with the American Physical Society (by his widow Noriko Sakurai, who in 2008 became my wife [25]); Nambu had received the J.J. Sakurai prize in 1994.

So now we have 3 generations of leptons and 3 of quarks (in 6 flavors and 3 colors). $SU(3)$ is back in, with a different meaning than originally. Eventually the electroweak model was incorporated and elaborate dynamics built on that basis of empirical origin, and everything seems to fit.

In a series of recent papers (see [13] and references therein) Alain Connes and coworkers showed that "noncommutative geometry provides a promising framework for unification of all fundamental interactions including gravity." In the last paper, assuming that "space-time is a noncommutative space formed as a product of a continuous four-dimensional manifold times a finite space" he develops a quite personal attempt to predict the Standard Model (possibly with 4 colors).

But what if the Standard Model was a colossus with clay feet (as in the interpretation by prophet Daniel of Nebuchadnezzar's dream: Book of Daniel, Chapter 2, verses 31–36)? What if it were "*all beside the mark*"?

[1] The quark masses are not measurements, but parameters used in theoretical models and compatible with raw experimental data.

3. The mathematical context: Deformation theory and quantization

In this section, for the sake of self-completeness, we shall give a very brief summary of what can be found with more details in a number of books, papers and reviews (in particular [21, 77]). Since quantization is a main paradigm for our "deformation philosophy", the idea is to give readers who would not know these already, some rudiments of deformation theory, of how quantum mechanics and field theory can be realized as a deformation of their classical counterparts, and of applications to symmetries (in particular the quantum group "avatar"). Educated readers or those who do not care too much about mathematical details may (at least for the time being...) only browse through this section. Note however that deformation quantization (as it is now known), introduced in the "founding papers" [6], is more than a mere reformulation of usual quantum mechanics; in particular it goes beyond canonical quantization (on $\mathbb{R}^{2\ell}$) and applies to general phase spaces (symplectic or Poisson manifolds).

3.1. The Gerstenhaber theory of deformations of algebras

A concise formulation of a Gerstenhaber deformation (over the field $\mathbb{K}[[\nu]]$ of formal series in a parameter ν with coefficients in a field $\mathbb{K}$) of an algebra (associative, Lie, bialgebra, etc.) over $\mathbb{K}$ is [10, 44]:

Definition 1. A deformation of an algebra A over $\mathbb{K}$ is an algebra $\tilde{A}$ over $\mathbb{K}[[\nu]]$ such that $\tilde{A}/\nu\tilde{A} \approx A$. Two deformations $\tilde{A}$ and $\tilde{A}'$ are said equivalent if they are isomorphic over $\mathbb{K}[[\nu]]$ and $\tilde{A}$ is said trivial if it is isomorphic to the original algebra A considered by base field extension as a $\mathbb{K}[[\nu]]$-algebra.

For associative (resp. Lie) algebras, the above definition tells us that there exists a new product $*$ (resp. bracket $[\cdot,\cdot]$) such that the new (deformed) algebra is again associative (resp. Lie). Denoting the original composition laws by ordinary product (resp. Lie bracket $\{\cdot,\cdot\}$) this means, for $u_1, u_2 \in A$ (we can extend this to $A[[\nu]]$ by $\mathbb{K}[[\nu]]$-linearity), that we have the formal series expansions:

$$u_1 * u_2 = u_1 u_2 + \sum_{r=1}^{\infty} \nu^r C_r(u_1, u_2) \tag{1}$$

$$[u_1, u_2] = \{u_1, u_2\} + \sum_{r=1}^{\infty} \nu^r B_r(u_1, u_2) \tag{2}$$

where the bilinear maps ($A \times A \to A$) C_r and (skew-symmetric) B_r are what are called 2-cochains in the respective cohomologies (Hochschild and Chevalley–Eilenberg), satisfying (resp.) $(u_1 * u_2) * u_3 = u_1 * (u_2 * u_3) \in \tilde{A}$ and $\mathcal{S}[[u_1, u_2], u_3] = 0$, for $u_1, u_2, u_3 \in A$, $\mathcal{S}$ denoting summation over cyclic permutations, the leading term (resp. C_1 or B_1) being necessarily a 2-cocycle (the coefficient of ν in the preceding conditions may be taken as a definition of that term).

For a (topological) *bialgebra* (an associative algebra A where we have in addition a coproduct $\Delta : A \longrightarrow A \otimes A$ and the obvious compatibility relations), denoting by $\otimes_\nu$ the tensor product of $\mathbb{K}[[\nu]]$-modules, we can identify $\tilde{A}\hat{\otimes}_\nu\tilde{A}$ with $(A\hat{\otimes}A)[[\nu]]$, where $\hat{\otimes}$ denotes the algebraic tensor product completed with respect to some topology (e.g., projective for Fréchet nuclear topology on A). Then we have also a deformed coproduct $\tilde{\Delta} = \Delta + \sum_{r=1}^{\infty}\nu^r D_r$, $D_r \in \mathcal{L}(A, A\hat{\otimes}A)$ satisfying $\tilde{\Delta}(u_1 * u_2) = \tilde{\Delta}(u_1) * \tilde{\Delta}(u_2)$. In this context appropriate cohomologies can be introduced and there are natural additional requirements for Hopf algebras (see, e.g., [11]).

3.2. Deformation quantization

The above abstract definition should become less abstract when applied to an algebra $N(W)$ of (differentiable) functions on a symplectic or Poisson manifold W, in particular those over phase space $\mathbb{R}^{2\ell}$ (with coordinates $p, q \in \mathbb{R}^\ell$) endowed with the Poisson bracket P of two functions u_1 and u_2, defined on a Poisson manifold W as $P(u_1, u_2) = \imath(\Lambda)(du_1 \wedge du_2)$ (where $\imath$ denotes the interior product, here of the 2-form $du_1 \wedge du_2$ with the 2-tensor Λ defining the Poisson structure on W, which in the case of a symplectic manifold is everywhere nonzero with for inverse a closed nondegenerate 2-form ω). For $W = \mathbb{R}^{2\ell}$, P can be written by setting $r = 1$ in the formula for the rth power ($r \geq 1$) of the bidifferential operator P (we sum over repeated indices):

$$P^r(u_1, u_2) = \Lambda^{i_1 j_1} \ldots \Lambda^{i_r j_r}(\partial_{i_1 \ldots i_r} u_1)(\partial_{j_1 \ldots j_r} u_2) \tag{3}$$

with $i_k, j_k = 1, \ldots, 2\ell$, $k = 1, \ldots, r$ and $(\Lambda^{i_k j_k}) = \left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)$. We can write deformations of the usual product of functions (deformations driven by the Poisson bracket) and of the Poisson bracket as what are now called the Moyal ("star") product and bracket, resp.

$$u_1 *_M u_2 = \exp(\nu P)(u_1, u_2) = u_1 u_2 + \sum_{r=1}^{\infty} \frac{\nu^r}{r!} P^r(u_1, u_2). \tag{4}$$

$$M(u_1, u_2) = \nu^{-1}\sinh(\nu P)(u_1, u_2) = P(u_1, u_2) + \sum_{r=1}^{\infty} \frac{\nu^{2r}}{(2r+1)!} P^{2r+1}(u_1, u_2). \tag{5}$$

These correspond (resp.) to the product and commutator of operators in the "canonical" quantization on $\mathbb{R}^{2\ell}$ of a function $H(q, p)$ with inverse Fourier transform $\tilde{H}(\xi, \eta)$, given by (that formula was found by Hermann Weyl [80] as early as 1927 when the weight is $\varpi = 1$):

$$H \mapsto \hat{H} = \Omega_\varpi(H) = \int_{\mathbb{R}^{2\ell}} \tilde{H}(\xi, \eta) \exp(i(\hat{p}.\xi + \hat{q}.\eta)/\hbar)\varpi(\xi, \eta) d^\ell \xi d^\ell \eta \tag{6}$$

which maps the classical function H into an operator on $L^2(\mathbb{R}^{2\ell})$, the "kernel" $\exp(i(\hat{p}.\xi + \hat{q}.\eta)/\hbar)$ being the corresponding unitary operator in the (projectively unique) representation of the Heisenberg group with generators $\hat{p_\alpha}$ and $\hat{q_\beta}$ ($\alpha, \beta = 1, \ldots, \ell$) satisfying the canonical commutation relations $[\hat{p_\alpha}, \hat{q_\beta}] = i\hbar\delta_{\alpha,\beta} I$. An inverse formula to that of the Weyl quantization formula was found in 1932 by Eugene Wigner [81] and maps an operator into what mathematicians call its symbol by a kind of trace formula: Ω_1 defines an isomorphism of Hilbert spaces between $L^2(\mathbb{R}^{2\ell})$ and Hilbert–Schmidt operators on $L^2(\mathbb{R}^\ell)$ with inverse given by

$$u = (2\pi\hbar)^{-\ell}\,\mathrm{Tr}[\Omega_1(u)\exp((\xi.\hat{p} + \eta.\hat{q})/i\hbar)]. \tag{7}$$

It is important to remember that "star products" exist as deformations (the skew-symmetric part of the leading term being the Poisson bracket P) of the ordinary product of functions in $N(W)$ for any W [21], including when there are no Weyl or Wigner maps and no obvious Hilbert space treatment of quantization. They can also be defined for algebraic varieties, "manifolds with singularities", and (with some care) infinite-dimensional manifolds.

We refer, e.g., to [6, 21, 77] and especially references therein for more developments on deformation quantization and its many avatars. These include the notion of covariance of star products and the "star representations" (without operators) it permits. They include also quantum groups, which appeared in Leningrad around 1980 for entirely different reasons [23] but, especially after the seminal works of Drinfel'd [22] (who coined the name) and of Jimbo [53] (for quantized enveloping algebras) in the early 1980s, can be viewed as deformations of (topological) Hopf algebras (see, e.g., [10, 11, 75]).

In this connection it is worth remembering a prophetic general statement by Dirac [19], which applies to many situations in physics:

> *Two points of view may be mathematically equivalent, and you may think for that reason if you understand one of them you need not bother about the other and can neglect it. But it may be that one point of view may suggest a future development which another point does not suggest, and although in their present state the two points of view are equivalent they may lead to different possibilities for the future. Therefore, I think that we cannot afford to neglect any possible point of view for looking at Quantum Mechanics and in particular its relation to Classical Mechanics.*

What Dirac had then in mind is certainly the quantization of constrained systems which he developed shortly afterward and by now can be viewed as a special case of deformation quantization. But the principle applies to many contexts and is even a most fruitful strategy to extend a framework beyond its initial context. A wonderful example is given by noncommutative geometry [14], now a frontier domain of mathematics with a wide variety of developments ranging from number theory to various areas of physics.

In order to show that important and concrete problems in physics can be treated in an *autonomous* manner using deformation quantization, without the need to introduce a Hilbert space (which for most physicists is still considered as a requirement of quantum theories) we treated in [6] a number of important problems, first and foremost the harmonic oscillator (the basic paradigm in many approaches), but also angular momentum, the hydrogen atom, and in general the definition of spectrum inside deformation quantization, without needing a Hilbert space. Not so many further applications have been developed since but the approach should eventually prove fruitful (even necessary) in many domains in which quantum phenomena play a role (including quantum computing). True, in many concrete examples we need (at least implicitly) "auxiliary conditions" to limit the possibly excessive freedom coming, in particular for spectra, from the absence of that Procrustean bed, the Hilbert space. But there also, too much freedom is better than not enough.

3.3. Further important deformations (and contractions)

3.3.1. An instance of multiparameter deformation quantization. A natural question is whether "the buck stops there", i.e., whether, like for Gerstenhaber deformations of simple Lie groups or algebras, the structure obtained is rigid, or whether some further deformations are possible. An answer to that question, looking for further deformations of $N(W)$ with another parameter β (in addition to $\nu = i\frac{\hbar}{2}$), was given in [5] and applied to statistical mechanics and the so-called KMS states (with parameter $\beta = 1/kT$, T denoting the absolute temperature). It turns out that there is some intertwining which is not an equivalence of deformations: As a ν-deformation, the two-parameter "star product" is driven by a "conformal Poisson bracket" with conformal factor of the form $\exp(-\frac{1}{2}\beta H)$ for some Hamiltonian H.

3.3.2. Brief survey of a few aspects of quantum groups. The literature on quantum groups (and Hopf algebras) is so vast, diversified (and growing) that we shall refer the interested reader to his choice among the textbooks and papers dealing with the many aspects of that notion, often quite algebraic. A two-pages primer can be found in [61].

Roughly speaking quantum groups can often be considered [22] as deformations (in the sense of Definition 1) of an algebra of functions on a Poisson-Lie group (a Lie group G equipped with a Poisson bracket compatible with the group multiplication, e.g., a semi-simple Lie group), or the "dual aspect" [22, 53] of a deformation of (some closure of) its enveloping algebra $\mathcal{U}(\mathfrak{g})$ equipped with its natural Hopf algebra structure, which is how the whole thing started in Leningrad around 1980. The first example was $\mathcal{U}_t(\mathfrak{sl}(2))$, an algebra with generators e, f, h as for $\mathfrak{sl}(2)$ but with "deformed" commutation relations that can be written somewhat formally (the deformation parameter $t = 0$ for $\mathfrak{sl}(2)$):

$$[h, e] = 2e,\ [h, f] = -2f,\ [e, f] = \sinh(th)/\sinh t \tag{8}$$

or more traditionally, as an algebra with generators E, F, K, K^{-1} (one often writes $K^{\pm 1} = q^{\pm H}$) and relations

$$KK^{-1} = K^{-1}K = 1,\ KEK^{-1} = q^2E,\ KFK^{-1} = q^{-2}F, \tag{9}$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}. \tag{10}$$

For simplicity we shall not write here the expressions for the coproduct, counit and antipode, needed to show the Hopf algebra structure. For higher rank simple

Lie algebras one has in addition trilinear relations (the deformed Serre relations), which complicate matters. All these can be found in the literature.

Note that the algebraic dual of a Hopf algebra is also a Hopf algebra only when these are finite-dimensional vector spaces, which is quite restrictive a requirement. In particular the Hopf algebras considered in quantum groups (except at root of unity), e.g., those of differentiable functions over a Poisson-Lie group, are (finitely generated) infinite dimensional vector spaces; but one can [10, 11] define on these spaces natural topologies (e.g., Fréchet nuclear) which in particular express the duality between them and “quantized enveloping algebras”. Remember that for any connected Lie group G with Lie algebra $\mathfrak{g}$ the elements of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ can be considered as differential operators over G, i.e., as distributions with support at any point in G (e.g., the identity $e \in G$), which lie in the topological dual of the space of differentiable functions with compact support. That exhibits a “hidden group structure” [10] in Drinfeld’s quantum groups, which [22] are not groups (and not always quantum ...)

3.3.3. About quantum groups at root of 1. As was noticed around 1990, in particular by Lusztig [59, 60] (see also, e.g., [17, 70]) the situation changes drastically when the deformation parameter is a root of unity. Then the Hopf algebras $\mathcal{U}_q(\mathfrak{g})$ become finite dimensional.

The case of $\mathcal{U}_q(\mathfrak{sl}(2))$ is well understood. For q a $2p$th root of unity, the finite-dimensionality of the algebra comes from the fact that, in addition to (9) and (10), one has the relations

$$K^{2p} = 1,\ E^p = 0,\ F^p = 0. \tag{11}$$

In particular (see, e.g., [85]) all finite-dimensional indecomposable representations have been determined, as well as the indecomposable decomposition of tensor products [57]. The higher rank case is still largely virgin mathematical territory. It seems that one either needs a new approach, or to restrict oneself to particular cases, or both.

3.3.4. Multiparameter quantum groups. More generally it is natural to try and look from the start at deformations with several scalar parameters. That question seems to have been tackled for the first time, in the context of quantum groups, by Manin et al. [18, 62], who called “nonstandard” these multiparameter deformations, and Reshetikhin [69], and later by Frønsdal [39, 41]. But the notion does not seem to have drawn the attention it deserves, certainly not much in comparison with the many works on more traditional aspects of quantum groups. And more sophisticated questions such as what happens (outside the generic case), e.g., “at roots of unity” (whether the same root for all parameters or not) do not seem to have ever been considered for multiparameter quantum groups.

3.3.5. Nonscalar deformation “parameter”. Other deformations, more general than those of Gerstenhaber type, were considered by Pinczon [68] and his student Nadaud [63, 64], in which the “parameter” acts on the algebra (on the left, on the right, or both) instead of being a scalar. For instance one can have [68], for

$\tilde{a} = \sum_n a_n \lambda^n$, $a_n \in A$, a left multiplication by λ of the form $\lambda \cdot \tilde{a} = \sum_n \sigma(a_n)\lambda^{n+1}$ where σ is an endomorphism of A. A similar deformation theory can be done in this case, with appropriate cohomologies, which gives new and interesting results.

In particular [68], while the Weyl algebra W_1 (generated by the Heisenberg Lie algebra $\mathfrak{h}_1$) is known to be Gerstenhaber-rigid, it can be nontrivially deformed in such a *supersymmetric deformation theory* to the supersymmetry enveloping algebra $\mathcal{U}(\mathfrak{osp}(1,2))$. Shortly thereafter [64], on the polynomial algebra $\mathbb{C}[x,y]$ in 2 variables, Moyal-like products of a new type were discovered; a more general situation was studied, where the relevant Hochschild cohomology is still valued in the algebra but with "twists" on both sides for the action of the deformation parameter on the algebra.

3.3.6. Contractions. Curiously, it is the (less precisely defined) inverse notion of *contraction* of symmetries that was first introduced in mathematical physics [51, 72]. Contractions, "limits of Lie algebras" as they were called in the first examples, can be viewed as an inverse of deformations – but not necessarily of Gerstenhaber-type deformations. We shall not expand on that "inverse" notion (see [78] for a more elaborate study) but give its flavor since it makes it easier to grasp the deformations of symmetries which are important in our presentation. A (finite-dimensional) Lie algebra $\mathfrak{g}$ can be described in a given basis L_i $(i = 1, \ldots, n)$ by its structure constants $C^k_{i,j}$. The equations governing the skew-symmetry of the Lie bracket and the Jacobi identity ensure that the set of all structure constants lies on an algebraic variety in that n^3-dimensional space [58]. A contraction is obtained, e.g., when one makes a simple basis change of the form $L'_i = \varepsilon L_i$ on *some* of the basis elements, and lets $\varepsilon \to 0$. Take for example $n = 3$ and restrict to the 3-dimensional subspace of the algebraic variety of 3-dimensional Lie algebras with commutation relations $[L_1, L_2] = c_3 L_3$ and cyclic permutations. The semi-simple algebras $\mathfrak{so}(3)$ and $\mathfrak{so}(2,1)$ are obtained in the open set $c_1c_2c_3 \neq 0$. A contraction gives the Euclidean algebras, where one c_i is 0. The "coordinate axes" (two of the c_i's are 0) give the Heisenberg algebra $\mathfrak{h}_1$ and the origin is the Abelian Lie algebra. That is of course a partial picture (e.g., solvable algebras are missing) but it is characteristic.

The above-mentioned passage from the Poincaré Lie algebra to the Galilean is a higher-dimensional version of such contractions of Lie algebras (multiply the "Lorentz boosts" generators M_{0j} by ε). A similar "trick" on the AdS_4 Lie algebra $\mathfrak{so}(3,2)$ gives the Poincaré Lie algebra. A traditional basis for the Poincaré Lie algebra is $M_{\mu\nu}$ for the Lorentz Lie algebra $\mathfrak{so}(3,1)$ and P_μ for the space-time translations (momentum generators), with $\mu, \nu \in \{0,1,2,3\}$. The commutation relations for the conformal Lie algebra $\mathfrak{so}(4,2)$ in the basis $M_{\mu\nu}$ with $\mu,\nu \in \{0,1,2,3,5,6\}$ can be written (see, e.g., [2]) $[M_{\mu\nu}, M_{\mu'\nu'}] = \eta_{\nu\mu'}M_{\mu\nu'} + \eta_{\mu\nu'}M_{\nu\mu'} - \eta_{\mu\mu'}M_{\nu\nu'} - \eta_{\nu\nu'}M_{\mu\mu'}$ with diagonal metric tensor $\eta_{\mu\mu}$ (equal, e.g., to $+1$ for $\mu = 0,5$ and -1 for $\mu = 1,2,3,6$). One can identify a Poincaré subalgebra by setting, e.g., $P_\mu = M_{\mu 5} + M_{\mu 6}$. If one omits $\mu = 6$ one obtains the AdS_4 Lie algebra $\mathfrak{so}(3,2)$ where the role of the Poincaré space translations is taken over by M_{j5} $(j \in \{1,2,3\})$

and that of the time translations by M_{05}. How to realize a contraction from AdS_4 to Poincaré for the massless representations used in Section 5 is described in [2]. Since it may be easier for physicists to grasp the notion of contraction, we mentioned it here and explained the examples we use in familiar notations (possibly old fashioned, but with bases on $\mathbb{R}$).

4. On the connection between internal and external symmetries

We shall not extend our desire to be as self contained as possible to describing in full detail the Poincaré group and its UIRs (unitary irreducible representations), known since on the instigation of Dirac (his "famous brother-in-law" as he liked to call him) Wigner [82] published the paper that started the study of UIRs of non compact Lie groups. Those associated with free particles are usually denoted by $D(m, s)$, where $m \geq 0$ is the mass and s the spin of the particle (for $m > 0$) or its helicity (for $m = 0$), associated with the "squared mass Casimir operator" (in the center of the Poincaré enveloping algebra) $P_\mu P^\mu$ and with the inducing representation of the "little group" ($SO(3)$ and $SO(2) \cdot \mathbb{R}^2$, resp.). [The letter D, coming from the German "Darstellung", is often used to denote representations.]

In the early 1960s a natural question appeared: Is there any connection between the "internal symmetry" used in the classification of interacting elementary particles (tentatively $SU(3)$ at the time), and the Poincaré symmetry whose UIRs are associated with free particles? The question is not innocent since 3 octets of different spins were associated with the same representation of $SU(3)$ (the 8-dimensional adjoint representation). And the various families within an octet (the same applies to the decuplet) exhibit a mass spectrum. If there is a connection, one ought to describe a mechanism permitting all that. Of course an important issue is how to formulate mathematically the question.

4.1. No-go theorems, objections, counter-examples and generalizations

4.1.1. A Lie algebra no-go and counterexamples. In the "particle spectroscopy" spirit of the time, it was natural to look for a Lie algebra containing both symmetries (internal and external). In 1965, a year after quarks and color were proposed, appeared a "no-go theorem" as physicists like to call such results, due to L. O'Raifeartaigh [67]. It boiled down to the fact that, since the momentum generators P_μ are nilpotent in the Poincaré Lie algebra, they are nilpotent in any simple Lie algebra containing it, which forbids a discrete mass spectrum. Hence in order to have a mass spectrum the connection must be a direct sum. Almost everybody was happy, except that two trouble makers in France said: "It ain't necessarily so". Thanks to Isidor Rabi (then president of the APS, who remembered Moshe Flato as a most brilliant student who often asked difficult questions during the course he gave for a quarter at the Hebrew University, invited by Giulio Racah) our objection was published shortly afterward [33] in the provocative form desired by Moshe. It was followed by counterexamples [34, 36]. The problem with the "proof" in [67] is that it implicitly assumed the existence of a common invariant domain of differentiable vectors for the whole Lie algebra, something which Wigner was careful to state as an assumption in [82] and was proved later for Banach–Lie group representations by (in Wigner's own words) "a Swedish gentleman" [42]. Eventually the statement of [67] could be proved within the context of UIRs of finite-dimensional Lie groups [54] and was further refined by several authors (especially L. O'Raifeartaigh). However we showed in [36] that a mass spectrum is possible when assuming only the Poincaré part to be integrable to a UIR, and there is no a priori reason why the additional observables should close to a finite-dimensional group UIR. We gave also counterexamples [37] with a natural infinite-dimensional group and even showed [35] that it is possible to obtain any desired mass spectrum in the framework of finitely-generated infinite-dimensional associative algebras and unitary groups.

Like with many physical "theorems", a main issue is to decide what assumptions and what heuristic developments can be considered as "natural". While some flexibility can be accepted in proving positive results, no-go theorems should be taken with many grains of salt.

4.1.2. Subsequent developments and relative importance of the question. As we indicated in [36], such considerations apply also to a more sophisticated no-go theorem [12], formulated in the context of symmetries of the S-matrix, which could be applied also to infinite-dimensional groups. This very nice piece of work is still considered by most physicists (especially those who learned it at university) as definitely proving the direct sum connection (under hypotheses easily forgotten, some of which are even hidden in the apparently natural notations).

In retrospect one can say that a main impact of the latter result came through an attempt to get a supersymmetric extension [46], which showed one might get around the no-go in the supersymmetry context. That gave a big push to the latter. Incidentally it is generally considered that the "super-Poincaré group" of Wess and Zumino [79] is practically the first instance of supersymmetry. That is not quite correct. In particular already in 1967 (in CRAS) we introduced what we called "a Poincaré-like group", semi-direct product of the Lorentz group and $\mathbb{R}^8$ consisting of both vector and spinor translations, but (for fear of Pauli) we did not dare introduce anticommutators for spinorial translations together with commutators for space-time translations, so we remained in the Lie algebra framework. However one can find in [31] a physical application of that group in which the spinorial translations are multiplied by an operator F anticommuting with itself. That was in effect the first realization of the super-Poincaré group. Both Wess and Zumino told me some years ago that they were unaware of the fact and it seems that (except for Frønsdal) not many noticed it either.

5. Singleton physics

5.1. Singletons as "square roots" of massless particles

The contraction of AdS to Poincaré (in both structure and representations) is (cf. [2]) one of the justifications for calling "massless" some minimal weight UIRs of $Sp(\mathbb{R}^2)$ (the double covering of $SO(3,2)$). These are denoted by $D(E_0, s)$, the parameters being the lowest values of the energy and spin (resp.) for the compact subgroup $SO(2) \times SO(3)$. These irreducible representations are unitary provided $E_0 \geq s+1$ for $s \geq 1$ and $E_0 \geq s+\frac{1}{2}$ for $s = 0$ and $s = \frac{1}{2}$. The massless representations of $SO(3,2)$ are thus defined (for $s \geq \frac{1}{2}$) as $D(s+1,s)$ and (for helicity zero) $D(1,0) \oplus D(2,0)$. At the limit of unitarity (when going down in the values of E_0 for fixed s) the Harish–Chandra module $D(E_0, s)$ becomes indecomposable and the physical UIR appears as a quotient, a hall-mark of gauge theories. For $s \geq 1$ we get in the limit an indecomposable representation $D(s+1,s) \rightsquigarrow D(s+2,s-1)$, where $\rightsquigarrow$ ("leaking into") is a shorthand notation [28] for what mathematicians would write as a short exact sequence of modules.

A complete classification of the UIRs of the (covering of) $SO(p,2)$ can be found, e.g., in [1]. For $p = 3$ the classification had been completed when Dirac [20] introduced the most degenerate "singleton" representations. The latter are irreducible and massless on a subgroup, the Poincaré subgroup of a $2+1$-dimensional space-time, of which AdS is the conformal group. That is why (on the pattern of Dirac's "bra" and "ket") we call these representations $\mathrm{Di} = D(1, \frac{1}{2})$ and $\mathrm{Rac} = D(\frac{1}{2}, 0)$ for (resp.) the spinorial and scalar representations. The singleton representations have a fundamental property:

$$(\mathrm{Di} \oplus \mathrm{Rac}) \otimes (\mathrm{Di} \oplus \mathrm{Rac}) = (D(1,0) \oplus D(2,0)) \oplus 2 \bigoplus_{s=\frac{1}{2}}^{\infty} D(s+1,s). \qquad (12)$$

The representations appearing in the decomposition are what we call massless representations of the AdS group, for a variety of good reasons [2]. For $s = 0$ a split occurs because time is compact in AdS and one does not distinguish between positive and negative helicity, which is also the reason for the factor 2 in front of the sum. An extension to the conformal group $SO(4,2)$, which is operatorially unique for massless representations of the Poincaré group and (once a helicity sign is chosen) for those of $SO(3,2)$, or a contraction of the latter to the Poincaré group, restores the distinction between both helicity signs and provide other good reasons for calling massless representations of AdS those in the right-hand side of (12).

Thus, in contradistinction with flat space, in AdS_4, massless states are "composed" of two singletons. The flat space limit of a singleton is a vacuum and, even in AdS_4, the singletons are very poor in states: their (E, j) diagram has a single trajectory (hence the name given to them by Dirac), and is not a lattice like, e.g., for massless particles in AdS. In normal units a singleton with angular momentum j has energy $E = (j + \frac{1}{2})\rho$, where ρ is the curvature of the AdS_4 universe. This means that only a laboratory of cosmic dimensions can detect a j large enough for E to be measurable: one can say that the singletons are "naturally confined".

Like the $\mathrm{AdS}_n/\mathrm{CFT}_{n-1}$ correspondence, the symmetry part of which states essentially that $SO(n-1,2)$ is the conformal group for $n-1$-dimensional space-time, singletons exist in any space-time of dimension $n \geq 3$ [3], $n = 4$ being somewhat special. For $n = 3$ the analogue of (12) writes $(HO) \otimes (HO) = \mathrm{Di} \oplus \mathrm{Rac}$ where (HO) denotes the harmonic oscillator representation of the metaplectic group (double covering of $SL(2,\mathbb{R})$, itself a double covering of $SO(2,1)$) which is the sum of the discrete series representation $D(\frac{3}{4})$ and of the complementary series representation $D(\frac{1}{4})$. One thus has a kind of "dimensional reduction" by which ultimately massless particles can be considered as arising from the interaction of harmonic oscillators. I leave it to the reader to derive consequences from that fact, and maybe study connections with the challenging suggestions of Gerard 't Hooft (see, e.g., [49, 50] and references therein) dealing with "quantum determinism" and based in particular on cellular automata.

Remark 5.1.1. Phase space realization. Quadratic polynomials in 2ℓ real variables p_α and q_β ($\alpha, \beta \in \{1, \ldots, \ell\}$), satisfying the Heisenberg canonical commutation relations (CCR) $[p_\alpha, q_\beta] = \delta_{\alpha\beta} I$, generate a realization of the symplectic Lie algebra $\mathfrak{sp}(\mathbb{R}^{2\ell})$. Together with the linear polynomials they close to an irreducible realization of the corresponding superalgebra ($\mathbb{Z}_{(2)}$-graded). [$\mathrm{Di} \oplus \mathrm{Rac}$ and (HO) are special cases of the phenomenon for $\ell = 2, 1$ (resp.).] That fact allowed us in [6] to write a power series expansion in t of what we call the "star exponential" $\mathrm{Exp} * (tH/i\hbar)$ (corresponding in Weyl quantization to the unitary evolution operator) of the harmonic oscillator Hamiltonian $H = \frac{1}{2}(p^2 + q^2)$, while a theorem of Harish–Chandra states that the character of a UIR (which in our formalism is the integral over phase space of the star-exponential) always has a singularity at the origin: the singularities for the two components in (HO) cancel at the origin, a true miracle which puzzles many specialists of Lie group representation theory!

5.2. AdS_4 dynamics

Until now we were concerned mainly with what can be called the "kinematical aspect" of the question, i.e., symmetries. However at some point one cannot avoid looking at the dynamics involved. In particular covariant field equations and Lagrangians will have to be studied. And indeed many papers were written, especially in the 1980s and 1990s by Flato, Frønsdal and coworkers, developing various aspects of singleton physics. These include BRST symmetry, conformal aspects and related indecomposable representations (in particular of the Gupta–Bleuler type), etc. Our purpose here is to build on these and on the "deformation philosophy" and not to give an extensive account of all these works, references to many of which can be found in Flato's last paper [29]. In the next two subsections we shall give a brief account of the two papers that are the most important from the point of view developed here, composite QED [28] and what can be called an electroweak model extended to 3 generations of leptons [40].

5.2.1. The Flato–Frønsdal "singletonic QED". Dynamics require in particular the consideration of field equations, initially at the first quantized level, in particular the analogue of the Klein–Gordon equation in AdS_4 for the Rac. There, as can be expected of massless (in 1+2 space) representations, gauges appear, and the physical states of the singletons are determined by the value of their fields on the cone at infinity of AdS_4 (see below; we have here a phenomenon of holography [48], in this case an $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence).

We thus have to deal with indecomposable representations, triple extensions of UIR, as in the Gupta–Bleuler (GB) theory, and their tensor products. [It is also desirable to take into account conformal covariance at these GB-triplets level, which in addition permits distinguishing between positive and negative helicities (in AdS_4, the time variable being compact, the massless representations of $SO(2,3)$ of helicity $s > 0$ contract (resp. extend in a unique way) to massless representations of helicity $\pm s$ of the Poincaré (resp. conformal) group.] The situation gets therefore much more involved, quite different from the flat space limit, which makes the theory even more interesting.

In order to test the procedure it is necessary to make sure that it is compatible with conventional Quantum Electrodynamics (QED), the best understood quantum field theory, at least at the physical level of rigor.

One is therefore led to see whether QED is compatible with a massless photon composed of two scalar singletons. For reasons explained, e.g., in [29] and references quoted therein, we consider for the Rac, the dipole equation $(\Box - \frac{5}{4}\rho)^{2}\phi = 0$ with the boundary conditions $r^{1/2}\phi < \infty$ as $r \to \infty$, which carries the indecomposable representation $D(\frac{1}{2},0) \rightsquigarrow D(\frac{5}{2},0)$. A remarkable fact is that this theory is a *topological field theory*; that is [27], the physical solutions manifest themselves only by their boundary values at $r \to \infty$: $\lim r^{1/2}\phi$ defines a field on the 3-dimensional boundary at infinity. There, on the boundary, gauge invariant interactions are possible and make a 3-dimensional conformal field theory (CFT).

However, if massless fields (in four dimensions) are singleton composites, then singletons must come to life as 4-dimensional objects, and this requires the introduction of unconventional statistics (neither Bose–Einstein nor Fermi–Dirac). The requirement that the bilinears have the properties of ordinary (massless) bosons or fermions tells us that the statistics of singletons must be of another sort. The basic idea is [28] that we can decompose the Rac field operator as $\phi(x) = \sum_{-\infty}^{\infty} \phi^j(x)a_j$ in terms of positive energy creation operators $a^{*j} = a_{-j}$ and annihilation operators a_j (with $j > 0$) without so far making any assumptions about their commutation relations. The choice of commutation relations comes later, when requiring that photons, considered as 2-Rac fields, be Bose–Einstein quanta, i.e., their creation and annihilation operators satisfy the usual canonical commutation relations (CCR). The singletons are then subject to unconventional statistics (which is perfectly admissible since they are naturally confined), the total algebra being an interesting infinite-dimensional Lie algebra of a new type, a kind of "square root" of the CCR. An appropriate Fock space can then be built. Based on these principles, a (conformally covariant) composite QED theory was constructed [28], with all the good features of the usual theory – however about 40 years after QED was developed by Schwinger, Feynman, Tomonaga and Dyson.

Remark 5.2.1.1. Classical Electrodynamics as a covariant nonlinear PDE approach to coupled Maxwell–Dirac equations. Only relatively recently was classical electrodynamics (on 4-dimensional flat space-time) rigorously understood. By this we mean the proof of asymptotic completeness and global existence for the coupled Maxwell–Dirac equations, and a study of the infrared problem. That was done [32] with the third aspect of our trilogy (complementing deformation quantization and singleton physics), based on a theory of nonlinear group representations, plus a lot of hard analysis using spaces of initial data suggested by the linear group representations. The deformation quantization of that classical electrodynamics (e.g., on an infinite-dimensional phase space of initial conditions) remains to be done.

5.2.2. Composite leptons, Frønsdal's extended elecroweak model. After QED the natural step is to introduce compositeness in electroweak theory. Along the lines described above, that would require finding a kind of "square root of an infinite-dimensional superalgebra," with both CAR (canonical anticommutation relations) and CCR included: The creation and annihilation operators for the naturally confined Di or Rac need not satisfy CAR or CCR; they can be subject to unusual statistics, provided that the two-singleton states satisfy Fermi–Dirac or Bose–Einstein statistics depending on their nature. We would then have a (possibly $\mathbb{Z}$-)graded algebra where only the two-singleton states creation and annihilation operators satisfy CCR or CAR. That has yet to be done. Some steps in that direction have been initiated but the mathematical problems are formidable, even more so since now the three generations of leptons have to be considered.

But here a more pragmatic approach can be envisaged [40], triggered by experimental data showing oscillations between various generations of neutrinos. The latter can thus no more be considered as massless. This is not as surprising as it seems from the AdS point of view, because one of the attributes of masslessness is the presence of gauges. These are group theoretically associated with the limit of unitarity in the representations diagram, and the neutrino is above that limit in AdS: the Di is at the limit. Thus, all nine leptons can be treated on an equal footing.

It is then natural [40] to arrange them in a square table (L^A_β), the rows being the 3 generations of leptons, each of which carry the Glashow representation of the 'weak group' $S_W = SU(2) \otimes U(1)$ and to consider the 9 leptons L^A_β (ν_e, e_L, e_R and similarly for the two other generations μ and τ) as composites, $L^A_\beta = R^A D_\beta$ ($A = N, L, R; \beta = \epsilon, \mu, \tau$). We assume that the R^As are Racs and carry the Glashow representation of S_W, while the D_βs are Dis, insensitive to S_W but transforming as a Glashow triplet under a 'flavor group' S_F isomorphic to S_W. To be more economical we also assume that the two $U(1)$s are identified, the same hypercharge group acting on both Dis and Racs. As explained in [40], the leptons are initially massless (as Di-Rac composites) and massified by (in effect, five) Higgs fields $K^{\alpha\beta}_{AB}$

that (like in the electroweak model) have a Yukawa coupling to the leptons. The model predicts, in parallel to the $W^{\pm}$ and Z bosons, two new bosons $C^{\pm}$ and C^3 (hard to detect due to the large mass differences between the 3 generations of leptons) and explains the neutrino masses. It is qualitatively promising but the presence of too many free parameters limits its quantitative predictive power.

One could then be tempted to add to the picture a deformation induced by the strong force and 18 quarks, which (with the 9 leptons) could be written in a cube and also considered composite (of maybe three constituents when the strong force is introduced). That might make this "composite Standard Model" more predictive. But introducing the hadrons brings in a significant quantitative change that should require a qualitative change, e.g., some further deformation of the AdS symmetry.

6. Hadrons and quantized Anti de Sitter

6.1. A beginning of a new picture

Instead of a "totalitarian" approach aiming towards a "theory of everything," at least inasmuch as elementary particles are concerned, we shall adopt an approach which is both more pragmatic and based on fundamental principles. Since symmetries were the starting point from which what is now the Standard Model emerged, and since free particles are governed by UIRs of the Poincaré group (the symmetry of special relativity), we shall start from the latter and proceed using our "deformation philosophy" as a guideline.

6.1.1. Photons, Leptons and Hadrons. As we have seen, assuming that in the "microworld", i.e., at some scale (to be made more precise eventually) the universe is endowed with a small negative curvature, the Poincaré group is deformed to AdS and the massless photon states can be dynamically (in a manner compatible with QED) considered as a 2-Rac state [28]. Then, extending the electroweak theory to the empirically discovered 3 generations, the 9 leptons can be considered, using the AdS deformation of the Poincaré group, as initially massless Di-Rac states, massified by 5 Higgs bosons [40].

The "tough cookie" is then how to explain hadrons and strong interactions. The deformation philosophy suggests to try and deform AdS, which is not possible as a group but can be done as a Hopf algebra, to a "quantum group", qAdS. In the "generic case" the obtained representation theory will not be very different from the AdS case. That is essentially due to the "Drinfeld twist" which intertwines between AdS and qAdS, even if that is not an equivalence of deformations (it is a kind of "outer automorphism"). But at root of unity the situation becomes drastically different: the Hopf algebra becomes finite-dimensional (which is not the case of the generic qAdS) and there are only a finite number of irreducible representations.

6.1.2. Remarks on the mathematical context. The fact that only a few irreducible representations may be relevant is both an encouraging feature and a restrictive one, even more so since for quantum groups at root of 1, the theory of tensor products of such representations (needed in order to consider these as describing interacting particles), which is "nice" in the generic case, is not straightforward. In particular the tensor products are usually indecomposable, extensions of direct sums of irreducibles defined by some cocycles (which could however be related to the "width" of the observed resonances). The phenomenon appears already in rank one (quantized $\mathfrak{sl}(2)$ at root of unity) [57], where the category of representations is not braided and the tensor products $R \otimes S$ and $S \otimes R$ of two representations can sometimes be different. The general theory for higher ranks seems hopeless. But for physical applications we do not need a general theory, possibly on the contrary: If only a few representations behave "nicely", and it turns out that Nature selects these, so much the better. [Even for the Poincaré group only half of the UIRs are physically relevant, those of positive mass, and of zero mass and discrete helicity.] The case of rank 2 (in particular qAdS) seems more within reach, though the mathematics is highly non trivial. Some works are in progress in that direction, in particular by Jun Murakami who recently [16] studied quantum $6j$-symbols for $SL(2,\mathbb{C})$. But a lot remains to be done in order to clarify the mathematical background.

6.2. Quantized AdS (in particular at some root of 1) and generalizations

6.2.1. Some ideas, problems and results around qAdS representations. Over 20 years ago appeared a concise and interesting paper [30], written, without sacrificing rigor, using a language (e.g., Bose creation and annihilation operators, supersymmetry, Fock space) that can appeal to physicists (but maybe less to mathematicians...). In that paper, on the basis of a short panorama of singleton and massless representations of $\mathfrak{so}(3,2)$, their supersymmetric extensions and (for the massless) imbedding in the conformal Lie algebra $\mathfrak{u}(2,2)$, the authors dealt with q-deformations of that picture, especially q-singletons, q-massless representations, and the imbedding therein of q-deformations of $\mathfrak{sl}(2)$. They studied both the case of generic q and the case when q is an even root of unity. A main purpose was, in the latter case, to write explicitly, in the case of $\mathcal{U}_q(\mathfrak{so}(3,2))$, defining relations similar to those of (9) and (10) (including now the "q-Serre relations") and to express in a more physical language the fact (discovered a few years before by Lusztig [60]) that one gets then finite-dimensional unitary representations.

Incidentally the "unitarization" of irreducible representations can be important for possible physical applications. For quantum groups at root of unity that has been studied, in particular for AdS, in [30, 73] and for many series of simple noncompact Lie algebras in [74], where in addition it is shown that the unitary highest weight modules of the classical case are recovered in the limit $q \to 1$.

Guided by the "deformation philosophy" we are thus led to look at what happens when AdS, the deformation of the Poincaré group when we assume a (tiny) negative curvature in some regions, is further deformed to the quantum group qAdS. The idea is that "internal symmetries" might arise as such deformations, which seems especially appealing at root of unity because we have then finite-dimensional unitary representations.

Now if we want to try and assign multiplets of particles to (irreducible) representations of qAdS at root of 1, a first step is to know what are the dimensions of these representations. These dimensions have been found (by Jun Murakami, work in progress) for sixth root of 1, to be: 1; 4, 5; 10, 14, 16; 35, 40; 81. We chose here $p = 3$ in the $2p$th root of 1, physically because there are 3 generations, and mathematically because for a variety of reasons one must take $p \geq 3$, so that is the first case.

The first nontrivial representations are, as can be expected in the case of a Lie algebra of type $B_2 \equiv C_2$, of dimensions 4 and 5 (that was 3 in the case of $\mathfrak{su}(3)$, hence quarks). Thus if we want to mimic what has been done for unitary symmetries, we might have to replace, e.g., the basic octet by two "quartets", unless a doubling of the dimension can be justified by some mathematical or physical reasons (maybe looking at a corresponding supersymmetry).

Note that while all (compact) simple groups of rank 2 were studied in detail in [7] from the point of view of strong interaction symmetries and eventually type A_2 emerged, and while the (finite-dimensional) representation theory of generic quantum groups is similar to the classical case, the restricted quantum groups are so different that the B_2 type cannot be excluded a priori. A notable difference is that the restricted quantum groups have only a finite number of finite-dimensional representations, which might be an advantage.

But the knowledge of the dimensions is only the beginning of the beginning. In particular we need to study the tensor products of the representations we want to use, which helps to describe what happens when two particles interact strongly and eventually produce other particles. In the case of roots of unity these tensor products typically give rise to indecomposable representations, essentially extensions of irreducibles given by some cocycles (somewhat like in the Gupta–Bleuler formalism for the electromagnetic field). The fact might here be related to the widths of the resonances produced, but that is so far only a conjecture.

The general study of such tensor products (beyond the rank 1 case, where it is already complicated) is nontrivial mathematically. As a "warm up exercise" for rank 2, it may be worth to start with the A_2 case, i.e., $\mathcal{U}_q(\mathfrak{sl}(3))$.

If we want to have for strongly interacting particles a picture similar to what has been done so far with unitary symmetries, we could first want to assign particle multiplets with some low-dimensional representations of qAdS. But since now we have a connection (via deformations) between the free (Poincaré) and the strongly interacting symmetry (qAdS), which conceptually is an advantage, we should imagine a mechanism explaining why to assign some spins (traditionally associated with the Poincaré group) to such multiplets, and how can we have a mass spectrum inside the multiplet.

Assuming we solve these (hard) "mathematical and physical homeworks", the physical task ahead of us is even more formidable: Re-examine critically half a century of particle physics, first from the phenomenological and experimental points of view on the basis of the new symmetries. “Going back to the drawing board,” we should then re-examine the present phenomenology in the new framework, including interpretations of raw experimental data. These were so far made in the context of the standard model and the quarks hypothesis, starting from nucleons and a few other particles and explaining inductively the observations in accelerators and cosmic rays within that framework.

Note that, as we have seen in Section 5, a main success of the present theory (the electroweak model) is preserved by deforming Poincaré to AdS. The speculations in this subsection are natural extensions of that in order to try and describe strong interactions using our deformation philosophy.

Inasmuch as we would “simply” replace the internal symmetries by some qAdS at root of 1, we should also, from the theoretical point of view, re-examine the various aspects (e.g., QCD) of the dynamics that was built on the possibly “clay feet” of simple unitary symmetries. That is a colossal task, but not as much as it may seem because we are not starting from scratch. A lot of the sophisticated notions introduced and theoretical advances made in the past decades might be adapted to our “deformed” view of symmetries. That could include many parts of the string framework.

Even more so since in the same spirit, it is possible that more sophisticated (and largely unexplored) mathematics would require less drastic a departure from the present puzzle, the pieces of which fit so well (so far). We shall explain that in the following.

6.2.2. Generalizations: Multiparameter, superizations and affinizations. An essential part in the representation theory of the traditional internal symmetries like $SU(n)$ (we take usually $n = 3$) boils down to questions of number theory, around the Weyl group ($\mathbb{S}_n$ for $SU(n)$) and the center of the group ($\mathbb{Z}_{(n)}$ in that case). One is therefore led to study what can be said of quantum deformations of $\mathcal{U}(\mathfrak{so}(3,2))$ when we take for deformation parameter an element of the group algebra of the center, $\mathbb{C}\mathbb{Z}_{(n)}$. That is, if we want to remain in the context already studied (see the next section for a more daring suggestion) of multiparameter deformations. While the generic case is relatively well understood (cf. Section 3.3.4), the case of root of 1 seems not to have been considered. It is not even clear whether one could (or should) take the same root of 1 for all the generators of the center. The “warping effect” of the roots of 1, which manifests itself already in (11), could play tricks. The same procedure can be applied to quantizing the superalgebra $\mathfrak{osp}(1,4)$ obtained [30, 38] from the realization of $\mathfrak{so}(3,2)$) as quadratic homogeneous polynomials in 4 real variables $(p_1, p_2; q_1, q_2)$ by adding the linear terms (endowed with anticommutators).

In view of possibly incorporating dynamics into the symmetry picture, and in the spirit of the string framework of “blowing up” points, often a cause of singularities, into, e.g., strings, one may then want to consider loop algebras (maps from a closed string S^1 to the symmetry in question, e.g., $\mathfrak{so}(3,2)$), and their quantization. In the same vein one may want to consider a number of infinite-dimensional algebras (Kac–Moody, Virasoro, etc.) built on that pattern. That is a very active topic (cf., e.g., [47, 52]) in which many results are available. But the kinds of specific examples we would need here have not been much studied and the root of unity case even less (mildly speaking). And then one may want to say something about the very hard question of maps from something more general than S^1 (e.g., a K_3 surface or a Calabi–Yau complex 3-fold) into some groups or algebras, and the possible quantization of such structures: these are totally virgin territory.

All these generalizations, even in the specific cases we would need here for possible applications, are at least valid mathematical questions.

6.2.3. "Quantum deformations" (with noncommutative "parameter"). The "best of both worlds" might however result from a challenging idea. Internal symmetries could emerge from deforming those of space-time in a more general sense that would include the use made of unitary symmetries like $SU(n)$. A fringe benefit might even be to give a conceptually beautiful explanation to the fact that we observe 3 generations.

The idea would be to "quantize" the Gerstenhaber definition (1) of deformations of algebras, not simply by considering a parameter that acts on the algebra (as indicated in Section 3.3.5) but by trying to develop a similar theory with a noncommutative "parameter". That has not been done and is far from obvious. It is not even clear what (if any) cohomology would be needed.

In particular one may think of deformations with a quaternionic deformation parameter. The field $\mathbb{H}$ of quaternions is the only number field extending that of complex numbers $\mathbb{C}$, but it is nonabelian. So we could speak of such a theory as a "quantum deformation" since one often calls "quantum" mathematical notions that extend existing ones by "plugging in" noncommutativity. As is well known, elements of $\mathbb{H}$ can be written in the form $a + bi + cj + dk$ with $a, b.c.d \in \mathbb{R}$, $i^2 = j^2 = k^2 = -1$ and i, j, k anticommuting (like Pauli matrices). Interchanging the roles of i, j, k would give a symmetry ($\mathbb{S}_3$ or $SO(3)$) which might explain why we have 3 (and only 3) generations. And deforming $\mathcal{U}(\mathfrak{so}(3,2))$ (the choice of that real form could be important) using such a (so far, hypothetical) quantum deformation theory, especially in some sense at root of unity, might give rise to internal symmetries for which the role of the Weyl group $\mathbb{S}_3$ of $SU(3)$ (and possibly of $SU(3)$) would permit to re-derive, this time on a fundamental basis, all what has been done with the representations of $SU(3)$. We would still have to explain, e.g., spin assignments to multiplets, and much more. But now we would have a frame for that, and a subtle nontrivial connection between internal and external symmetries could be developed, with all its implications (especially concerning the dynamics involved), with a relatively modest adaptation of the present empirical models.

A variant of that would be to develop a theory of deformations parametrized by the group algebra of $\mathbb{S}_n$. Except for the fact that we would have to assume, e.g., $n = 3$ and not "explain" why we have only 3 generations, the general idea would be similar, so we shall not repeat the above speculations.

More generally, a mathematical study of such "quantum deformations" should be of independent interest, even if that uses very abstract tools that may not make it directly applicable to the physical problems we started from.

6.2.4. Remark: Quantized AdS space and related cosmology. One should not only try and develop a theory of the fundamental constituents of the matter we know (even if it constitutes only about 4% of the universe) but also explain how that small part is created, and if possible how comes that we see so little antimatter. To this end also, quantizing AdS space-time might help.

Of course, in line with recent observational cosmology, our universe is probably, "in the large", asymptotically de Sitter, with positive curvature and invariance group $SO(4,1)$, the other simple group deformation of the Poincaré group which however, unlike AdS, does not give room to a positive energy operator. At our scale, for most practical purposes, we can treat it as Minkowskian (flat). Focussing "deeper" we would then discover that it can be considered as Anti de Sitter. There one can explain photons and leptons as composites of singletons that live in AdS space-time. It is thus natural to try and quantize that AdS space-time. And in fact, in [9], we showed how to build such "quantized hyperbolic spheres", i.e., noncommutative spectral triples à la Connes, but in a Lorentzian context, which induce in (an open orbit in) AdS space-time a pseudo-Riemannian deformation triple similar (except for the compactness of the resolvent) to the triples developed for quantized spheres by Connes et al. (see, e.g., [15]). Such a "quantized AdS space" has a horizon which permits to consider it as a black hole (similar to the BTZ black holes [4], which exist for all AdS_n when $n \geq 3$). [A kind of groupoid structure might be needed if one wants to treat all 3 regions.]

For q an even root of unity, since the corresponding quantum AdS group has finite-dimensional UIRs, such a quantized AdS black hole could be considered as "q-compact" in a sense to be made precise. As we mention in [9, 76], in some regions of our universe, our Minkowski space-time could be, at very small distances, both deformed to anti de Sitter and quantized, to qAdS. These regions would appear as black holes which might be found at the edge of our expanding universe, a kind of "stem cells" of the initial singularity dispersed at the Big Bang. From these (that is so far mere speculation) might emerge matter, possibly first some kind of singletons that couple and become massified by interaction with, e.g., dark matter and/or dark energy. Such a scheme could be responsible, at very large distances, for the observed positive cosmological constant – and might bring us a bit closer to quantizing gravity, the Holy Grail of modern physics, whether or not that is a relevant question (even if very recent and well publicized observations of gravitational waves might indicate that quantizing gravity is needed).

7. Epilogue and a tentative "road map"

After such a long overview involving a fireworks of fundamental mathematical and physical notions, many of which need to be developed, a natural question (which most physicists will probably ask after going through the abstract) is: why argue with success?

After all, the Standard Model is considered (so far) as the ultimate description of particle physics. We were looking for a key to knowledge under a lantern (bei der Laterne), found one which turned out to open a door nearby (not the one sought initially but never mind) through which a blue angel M led us to a beautiful avenue with many ramifications. But what if M was Fata Morgana and that avenue eventually turns out to be a dead end? We would then need a powerful flashlight (deformations maybe) to find in a dark corner a strange key with which, with much effort, a hidden door can be opened and lead us to an avenue where part of our questions can be answered.

Still, physicists tend to have a rather positivist attitude and, in most areas of physics, one takes for granted some experimental facts without trying to explain them on the basis of fundamental principles. Why would particle physics be different? A first answer is given by Einstein: "Curiosity has its own reason for existing." Theoretical particle physics is certainly an area in which very fundamental questions can be asked, and answered, if needed with the help of sophisticated mathematics to be developed.

So, at the risk of being considered simple minded, I am asking the question of *why* the symmetries on which is based the standard model are what they are in the model, and not only *what* are they and *how* do they work. In other words, the question is: *Is it necessarily so??*

Another reason (more pragmatic) is that often in mathematics, questions originating in Nature tend to be more seminal than others imagined "out of the blue". That is, if I may say so, an experimental fact. It is therefore worth developing the new mathematical tools we have indicated, and those that their development will suggest. If that solves physical problems (those intended or others), so much the better. In any case that should give nontrivial mathematics.

We express the general approach as follows:

Conjecture 2 (The Deformation Conjecture). *Internal symmetries of elementary particles emerge from their relativistic counterparts by some form of deformation (possibly generalized, including quantization).*

In particular we would like to realize a *Quantum Deformation Dream*:

- The above-mentioned "Quantum Deformations" can be defined, then permit to define "QQgroups," including a "restricted" version thereof (at roots of unity) which would be finite dimensional algebras, and the tensor products of their representations can be studied.
- Such a procedure can be applied to $\mathcal{U}(\mathfrak{so}(3,2))$ (and $\mathcal{U}(\mathfrak{so}(1,2))$ as a toy model), if needed along with a supersymmetric extension and maybe some kind of "affinization", and serve as a starting point for a well-based theory of strong interactions.

Of course one cannot rule out that symmetries, one of the bases of “the unreasonable effectiveness of mathematics in the natural sciences” [84], turn out to have little role in particle physics. I strongly doubt it.

In the list of problems we have encountered, the “simplest” on the mathematical side seems to be what happens with tensor products of representations of qAdS (or of other rank 2 quantum groups) at root of unity. In a similar spirit it would be possibly more interesting to study representations of multiparameter deformations of AdS, including their tensor products.

After some mathematical and theoretical progress we should then try and see how that knowledge can be used to interpret the known strong interactions, step by step, starting with the earliest known particles. That requires both theoretical and phenomenological studies, and possibly also renewed experiments to check the theoretical results.

A more appealing approach is to try and develop a “quantum deformations” theory, in particular with quaternions. Then, if and when we have such a theory, would come the question to apply it to AdS both in mathematics and in possible theoretical physics applications.

Ultimately one would have to study in details the phenomenological implications of these developments. If they differ somewhat from the present interpretation of the raw experimental data, we would need to revise that interpretation and possibly to re-do some experiments.

Note that a “fringe benefit” of any such revision is that it can be done using the present experimental tools, if needed (as far as more ancient data are concerned) with the refinement of modern technology. The civil society is not likely to give us significantly more powerful accelerators.

All these are problems worthy of attack. It can be expected that they will prove their worth by hitting back. Starting from a primary question, I have asked many more, combined that with many notions and results developed during half a century of research in physical mathematics, and indicated avenues along which some of these might be answered. At 75 I leave it to the next generations to enter that promised land and tackle the many problems that will follow.

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Bargmann–Fock Realization of the Noncommutative Torus

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Abstract. We give an interpretation of the Bargmann transform as a correspondence between state spaces that is analogous to commonly considered intertwiners in representation theory of finite groups. We observe that the non-commutative torus is nothing else that the range of the star-exponential for the Heisenberg group within the Kirillov's orbit method context. We deduce from this a realization of the non-commutative torus as acting on a Fock space of entire functions.

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1. Introduction

For systems of finite degrees of freedom, there are two main quantization procedures, the others being variants of them. The first one consisting in pseudo-differential calculus (see, e.g., [4]). The second one relies on geometric quantization (see, e.g., [6]). The main difference between the two lies in the types of polarizations on which they are based. Pseudo-differential calculus is based on the existence of a "real polarization", while geometric quantization often uses "complex polarizations". There are no systematic ways to compare the two. Although in some specific situations, this comparison is possible. This is what is investigated in the present work.

The aim of this small note is threefold. First we give an interpretation of the Bargmann transform as a correspondence between state spaces that is analogous to commonly considered intertwiners in representation theory of finite groups. Second, we observe that the non-commutative torus is nothing else that the range of the star-exponential for the Heisenberg group within the Kirillov's orbit method context. Third, we deduce from this a realization of the non-commutative torus as acting on a Fock space of entire functions. The latter relates the classical approach to the non-commutative torus in the context of Weyl quantization to its realization, frequent in the physics literature, in terms of the canonical quantization.

2. Remarks on the geometric quantization of co-adjoint orbits

We let $\mathcal{O}$ be a co-adjoint orbit of a connected Lie group G in the dual $\mathfrak{g}^\star$ of its Lie algebra $\mathfrak{g}$. We fix a base point ξ_o in $\mathcal{O}$ and denote by $K =: G_{\xi_o}$ its stabilizer in G (w.r.t. the co-adjoint action). We assume K to be connected. Denote by $\mathfrak{k}$ the Lie sub-algebra of K in $\mathfrak{g}$. Consider the $\mathbb{R}$-linear map $\xi_o : \mathfrak{k} \to \mathfrak{u}(1) = \mathbb{R} : Z \mapsto \langle \xi_o, Z\rangle$. Since the two-form $\delta\xi_o : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R} : (X,Y) \mapsto \langle \xi_o, [X,Y]\rangle$ identically vanishes on $\mathfrak{k}\times\mathfrak{k}$, the above mapping is a character of $\mathfrak{k}$. Assume the above character exponentiates to K as a unitary character (Kostant's condition): $\chi : K \to U(1) \quad (\chi_{\star e} = i\xi_o)$. One then has an action of K on $U(1)$ by group automorphisms: $K\times U(1) \to U(1) : (k,z) \mapsto \chi(k)z$. The associated circle bundle $Y := G\times_K U(1)$ is then naturally a $U(1)$-principal bundle over the orbit $\mathcal{O}$: $\pi : Y \to \mathcal{O} : [g,z] \mapsto \mathtt{Ad}^\flat_g(\xi_o)$ (indeed, one has the well-defined $U(1)$-right action $[g,z].z_0 := [g, zz_0]$).

The data of the character yields a connection one-form ϖ in Y. Indeed, the following formula defines a left-action of G on Y: $g_0.[g,z] := [g_0 g, z]$.

When $\xi_o|_{\mathfrak{k}}$ is non-trivial, the latter is transitive: $\mathbf{C}_g(k).[g,z] = [gk,z] = [g,\chi(k)z]$ and $\pi(g_0.[g,z]) = g_0.\pi(g)$. We then set

$$\varpi_{[g,z]}(X^\star_{[g,z]}) := -\langle \mathtt{Ad}^\flat_g \xi_o, X\rangle \tag{1}$$

with, for every $X\in\mathfrak{g}$: $X^\star_{[g,z]} := \frac{\mathrm{d}}{\mathrm{d}t}|_0 \exp(-tX).[g,z]$. The above formula (1) defines a 1-form. Indeed, an element $X\in\mathfrak{g}$ is such that $X^\star_{[g,z]} = 0$ if and only if $\mathtt{Ad}_{g^{-1}}X \in \ker(\xi_0)\cap\mathfrak{k}$. It is a connection form because for every $z_0\in U(1)$, one has $(z_0^\star\varpi)_{[g,z]}(X^\star) = \varpi_{[g,zz_0]}(z_{0\star_{[g,z]}}X^\star) = \varpi_{[g,zz_0]}(X^\star) = -\langle \mathtt{Ad}^\flat_g\xi_o, X\rangle = \varpi_{[g,z]}(X^\star)$. At last, denoting by $\iota_y := \frac{\mathrm{d}}{\mathrm{d}t}|_0 y.e^{it}$ the infinitesimal generator of the circle action on Y, one has $\iota_{[g,z]} = -(\mathtt{Ad}_g E)^\star_{[g,z]}$ where $E\in\mathfrak{k}$ is such that $\langle\xi_0, E\rangle = 1$. Indeed, $\frac{\mathrm{d}}{\mathrm{d}t}|_0[g,z].e^{it} = \frac{\mathrm{d}}{\mathrm{d}t}|_0[g,e^{it}z] = \frac{\mathrm{d}}{\mathrm{d}t}|_0[g\exp(tE),z] = \frac{\mathrm{d}}{\mathrm{d}t}|_0[\exp(t\mathtt{Ad}_g E)g, z]$. Therefore: $\varpi_{[g,z]}(\iota) = \langle \mathtt{Ad}^\flat_g\xi_0, \mathtt{Ad}_g E\rangle = 1$.

The curvature $\Omega^\varpi := \mathrm{d}\varpi + [\varpi,\varpi] = \mathrm{d}\varpi$ of that connection equals to the lift $\pi^\star(\omega^{\mathcal{O}})$ to Y of the KKS-symplectic structure $\omega^{\mathcal{O}}$ on $\mathcal{O}$ because $X^\star_{[g,z]}.\varpi(Y^\star) = \langle \mathtt{Ad}^\flat_g\xi_o, [X,Y]\rangle$.

2.1. Real polarizations

We now relate Kirillov's polarizations to Souriau's Planck condition. We consider a **partial polarization** affiliated to ξ_o, i.e., a sub-algebra $\mathfrak{b}$ of $\mathfrak{g}$ that is normalized by $\mathfrak{k}$ and maximal for the property of being isotropic w.r.t. the two-form $\delta\xi_o$ on $\mathfrak{g}$ defined as $\delta\xi_o(X,Y) := \langle\xi_o,[X,Y]\rangle$. We assume that the analytic (i.e., connected) Lie sub-group $\mathbf{B}$ of G whose Lie algebra is $\mathfrak{b}$ is closed. We denote by $Q := G/\mathbf{B}$ the corresponding quotient manifold. Note that one necessarily has $K\subset\mathbf{B}$, hence the fibration $p : \mathcal{O} \to Q : \mathtt{Ad}^\flat_g(\xi_o) \mapsto g\mathbf{B}$. The distribution $\mathfrak{L}$ in $T\mathcal{O}$ tangent to the fibers is isotropic w.r.t. the KKS form. Its ϖ-horizontal lift $\overline{\mathfrak{L}}$ in $T(Y)$, being integrable, induces a **Planck foliation** of Y (cf. p. 337 in Souriau's book [5]).

Usually, Kirillov's representation space consists in a space of sections of the associated complex line bundle $\mathbb{E}_\chi := G \times_\chi \mathbb{C} \to Q$ where χ is viewed as a unitary character of $\mathbf{B}$. While the Kostant–Souriau representation space rather consists in a space of sections of the line bundle $Y \times_{U(1)} \mathbb{C} \to \mathcal{O}$. One therefore looks for a morphism between these spaces. To this end we first observe that the circle bundle $G \times_\chi U(1) \to Q$ is a principal $U(1)$-bundle similarly as Y is over $\mathcal{O}$. Second, we note the complex line bundle isomorphism over Q:

$$(G \times_\chi U(1)) \times_{U(1)} \mathbb{C} \to G \times_\chi \mathbb{C} : [[g_0, z_0], z] \mapsto [g, z_0 z] \, . \tag{2}$$

Third, we have the morphism of $U(1)$-bundle:

$$\begin{array}{ccc} Y & \xrightarrow{\tilde{p}} & G \times_\chi U(1) \\ \downarrow & & \downarrow \\ \mathcal{O} & \xrightarrow{p} & Q \end{array}$$

where $\tilde{p}([gk, \chi(k^{-1})z_0]) := [gb, \chi(b^{-1})z_0]$. This induces a linear map between equivariant functions: $\tilde{p}^\star : C^\infty(G \times_\chi U(1), \mathbb{C})^{U(1)} \to C^\infty(Y, \mathbb{C})^{U(1)}$ which, red through the isomorphism (2), yields a natural G-equivariant linear embedding $\Gamma^\infty(Q, \mathbb{E}_\chi) \to \Gamma^\infty(\mathcal{O}, Y \times_{U(1)} \mathbb{C})$ whose image coincides with the Planck space.

2.2. Complex polarizations

We first note that for every element $X \in \mathfrak{g}$, the ϖ-horizontal lift of $X^\star_\xi$ at $y = [g, z] \in Y$ is given by $h_y(X^\star_\xi) := X^\star_y + \langle \xi, X \rangle \iota_y$. Indeed, $\pi_{\star y}(X^\star_y) = X^\star_\xi$ and $\varpi_y(X^\star_y + \langle \xi, X \rangle \iota_y) = -\langle \xi, X \rangle + \langle \xi, X \rangle = 0$.

Therefore, for every smooth section φ of the associated complex line bundle $\mathbb{F} := Y \times_{U(1)} \mathbb{C} \to \mathcal{O}$, denoting by ∇ the covariant derivative in $\mathbb{F}$ associated to ϖ and by $\hat{\varphi}$ the $U(1)$-equivariant function on Y representing φ, one has $\widehat{\nabla_{X^\star}\varphi}(y) = X^\star_y \hat{\varphi} - i \langle \xi, X \rangle \hat{\varphi}(y)$. Now let us assume that our orbit $\mathcal{O}$ is *pseudo-Kähler* in the sense that it is equipped with a G-invariant $\omega^{\mathcal{O}}$-compatible (i.e., $J \in \mathrm{Sp}(\omega^{\mathcal{O}})$) almost complex structure J. Let us denote by $T_\xi(\mathcal{O})^{\mathbb{C}} = T^0_\xi(\mathcal{O}) \oplus T^1_\xi(\mathcal{O})$ the $(-1)^{0,1}i$-eigenspace decomposition of the complexified tangent space $T_\xi(\mathcal{O})$ w.r.t. J_ξ. One observes the following descriptions $T^{0,1}_\xi(\mathcal{O}) = \rangle X + (-1)^{0,1} iJX \langle_{X \in T_\xi(\mathcal{O})}$ where the linear span is taken over the complex numbers. We note also that $\dim_{\mathbb{C}} T^{0,1}_\xi(\mathcal{O}) = \frac{1}{2} \dim_{\mathbb{R}} \mathcal{O}$.

In that context, a smooth section φ of $\mathbb{F}$ is called **polarized** when $\nabla_Z \varphi = 0$ for every $Z \in T^0(\mathcal{O})$. The set $\mathrm{hol}(\mathbb{F})$ of polarized sections is a complex sub-space of $\Gamma^\infty(\mathbb{F})$. Moreover, it carries a natural linear action of G. Indeed, the group G acts on $\Gamma^\infty(\mathbb{F})$ via $\widehat{U(g)\varphi} := (g^{-1})^\star \hat{\varphi}$. The fact that both φ and J are G-invariant then implies that $\mathrm{hol}(\mathbb{F})$ is a U-invariant sub-space of $\Gamma^\infty(\mathbb{F})$. The linear representation $U : G \to \mathtt{End}(\mathrm{hol}(\mathbb{F}))$ is called the **Bargmann–Fock representation**.

3. The Heisenberg group

We consider a symplectic vector space (V,Ω) of dimension $2n$ and its associated Heisenberg Lie algebra: $\mathfrak{h}_n := V\oplus\mathbb{R}E$ whose Lie bracket is given by $[v,v'] := \Omega(v,v')E$ for all $v,v'\in V$, the element E being central. The corresponding connected simply connected Lie group $\mathbf{H}_n$ is modeled on $\mathfrak{h}_n$ with group law given by $g.g' := g+g'+\frac{1}{2}[g.g']$.

Within this setting, one observes that the exponential mapping is the identity map on $\mathfrak{h}_n$. The symplectic structure defines an isomorphism $^\flat : V\to V^\star$ by $^\flat v(v') := \Omega(v,v')$. The latter extends to a linear isomorphism $^\flat : \mathfrak{h}_n\to\mathfrak{h}_n^\star$ where we set $^\flat E(v+zE) := z$.

Now one has

$$\mathtt{Ad}^\flat_{v+zE}(^\flat v_0+\mu^\flat E) = {}^\flat(v_0-\mu v)+\mu^\flat E\ . \tag{3}$$

Indeed, in the case of the Heisenberg group, the exponential mapping coincides with the identity mapping. Hence, for every $v_1+z_1E\in\mathfrak{h}_n$:

$$\begin{aligned}\langle\mathtt{Ad}^\flat_{v+zE}(^\flat v_0+\mu^\flat E), v_1+z_1E\rangle &= \langle^\flat v_0+\mu^\flat E, \mathtt{Ad}_{-v-zE}(v_1+z_1E)\rangle\\ &= \frac{\mathrm{d}}{\mathrm{d}t}|_0\langle^\flat v_0+\mu^\flat E, (-v-zE).(tv_1+tz_1E).(v+zE)\rangle\ ,\end{aligned}$$

which in view of the above expression for the group law immediately yields (3).

Generic orbits $\mathcal{O}$ are therefore affine hyperplanes parametrized by $\mu\in\mathbb{R}_0$. Setting $\xi_0 := \mu^\flat E$, every real polarization $\mathfrak{b}$ corresponds to a choice of a Lagrangian sub-space $\mathfrak{L}$ in V: $\mathfrak{b}=\mathfrak{L}\oplus\mathbb{R}E$. Note in particular, that the polarization is an ideal in $\mathfrak{h}_n$. Choosing a Lagrangian sub-space $\mathfrak{q}$ in duality with $\mathfrak{L}$ in V determines an Abelian sub-group $Q=\exp(\mathfrak{q})$ in $\mathbf{H}_n$ which splits the exact sequence $\mathbf{B}\to\mathbf{H}_n\to\mathbf{H}_n/\mathbf{B} =: Q$, i.e., $\mathbf{H}_n = Q.\mathbf{B}$. The stabilizers all coincide (in the generic case) with the center $K := \mathbb{R}E$ of $\mathbf{H}_n$ and one has the global trivialization $\mathcal{O}\to\mathbf{H}_n$: $^\flat v+\xi_0\mapsto v$.

3.1. Representations from real and complex polarizations

This yields the linear isomorphism $\Gamma^\infty(\mathbb{E}_\chi)\to C^\infty(Q) : u\mapsto\hat{u}|_Q =: \tilde{u}$ under which the $\mathbf{H}_n$-action reads $U_{\mathrm{KW}}(qb)\tilde{u}(q_0) = e^{i\mu(z+\Omega(q-q_0,p))}\tilde{u}(q_0-q)$ with $b=p+zE$, $p\in\exp(\mathfrak{L})$. The latter induces a unitary representation on $L^2(Q)$.

The isomorphism $\mathbb{C}^n = \mathfrak{q}\oplus i\mathfrak{L}\to V : Z = q+ip\mapsto q+p$ yields an $\mathbf{H}_n$-equivariant global complex coordinate system on the orbit $\mathcal{O}$ through $Z\mapsto \mathtt{Ad}^\flat_{q+p}\xi_0$. The map $V\times U(1)\to Y : (v_0,z_0)\mapsto[v_0,z_0]$ consists in a global trivialization of the bundle $Y\to\mathcal{O}$ under the isomorphism $V\simeq\mathcal{O}$ described above. Hence the linear isomorphism $\Gamma^\infty(\mathbb{F})\to C^\infty(V) : \varphi\mapsto\tilde{\varphi} := \hat{\varphi}(\,.\,,1)$. At the level of the trivialization the left-action of $\mathbf{H}_n$ on Y reads: $g.(v_0,z_0) = (v_0+q+p, e^{i\mu(z+\frac{1}{2}\Omega(q,p)+\frac{1}{2}\Omega(q+p,v_0))}z_0)$. Also, the representation is given by $g.\tilde{\varphi}(v_0) = e^{i\mu(z+\frac{1}{2}\Omega(q,p)+\frac{1}{2}\Omega(q+p,v_0))}\tilde{\varphi}(v_0-p-q)$. Choosing a basis $\{f_j\}$ of $\mathfrak{q}$ and setting $\{e_j\}$ for the corresponding dual basis of $\mathfrak{L}$, one has $\partial_{\overline{Z}^j} = -\frac{1}{2}(f_j^\star+ie_j^\star)$. Within the trivialization, the connection form corresponds to $\varpi = \frac{\mu}{2}(p^j\mathrm{d}q_j-q^j\mathrm{d}p_j)+\iota_\star$. A simple computation then yields: $\mathrm{hol}(\mathbb{F}) = \{\varphi_f : \mathbb{C}^n \to \mathbb{C} : z \mapsto e^{-\frac{\mu}{4}|z|^2} f(z) \quad (f \text{ entire})\}$. Note that provided $\mu > 0$, the space $\mathrm{hol}(\mathbb{F})$ naturally contains the pre-Hilbert space: $\mathcal{L}^2_{\mathrm{hol}}(\mathbb{F}) := \{\varphi_f : <\varphi_f, \varphi_f> := \int_{\mathbb{C}^n} e^{-\frac{\mu}{2}|z|^2} |f(z)|^2 \,\mathrm{d}q\,\mathrm{d}p < \infty\}$. The above sub-space turns out to be invariant under the representation U of H_n. The latter is seen to be unitary and irreducible. For negative μ, one gets a unitary representation by considering anti-polarized sections corresponding to anti-holomorphic functions. Note that within complex coordinates, the action reads $U_{\mathrm{BF}}(g)\tilde{\varphi}(Z_0) := g.\tilde{\varphi}(Z_0) = e^{i\mu(z+\frac{1}{2}\mathrm{Im}(\frac{1}{2}Z^2+\overline{Z}Z_0))}\tilde{\varphi}(Z_0 - Z)$.

3.2. Intertwiners and the Bargmann transform

We know (cf. Stone–von Neumann's theorem) that in the case of the Heisenberg group, representations constructed either via complex or via real polarizations are equivalent. In order to exhibit intertwiners, we make the following observation.

Proposition 1. *Let G be a Lie group with left-invariant Haar measure $\mathrm{d}g$ and $(\mathcal{H}, \rho)$ and $(\mathcal{H}', \rho')$ be square-integrable unitary representations. Assume furthermore the continuity of the associated bilinear forms $\mathcal{H} \times \mathcal{H} \to L^2(G) : (\varphi_1, \varphi_2) \to [g \mapsto \langle\varphi_1|\rho(g)\varphi_2\rangle]$ and $\mathcal{H}' \times \mathcal{H}' \to L^2(G) : (\varphi'_1, \varphi'_2) \to [g \mapsto \langle\varphi'_1|\rho'(g)\varphi'_2\rangle]$. Fix "mother states" $|\eta\rangle \in \mathcal{H}$ and $|\eta'\rangle \in \mathcal{H}'$. For every element $g \in G$, set $|\eta_g\rangle := \rho(g)|\eta\rangle$ and $|\eta'_g\rangle := \rho'(g)|\eta'\rangle$. Then the following formula*

$$T := \int_G |\eta'_g\rangle\langle\eta_g| \,\mathrm{d}g \tag{4}$$

formally defines an intertwiner from $(\mathcal{H}, \rho)$ to $(\mathcal{H}', \rho')$.

Proof. First let us observe that square-integrability and Cauchy–Schwartz inequality on $L^2(G)$ imply that for all $\varphi \in \mathcal{H}$ and $\varphi' \in \mathcal{H}'$ the element $[g \mapsto \langle\varphi'|\eta'_g\rangle\langle\eta_g|\varphi\rangle]$ is well defined as an element of $L^1(G)$. Moreover the continuity of the above-mentioned bilinear forms insures the continuity of the bilinear map $\mathcal{H} \times \mathcal{H}' \to \mathbb{C}$ defined by integrating the later. This is in this sense that we understand formula (4). Now for all $|\varphi\rangle \in \mathcal{H}$ and $g_0 \in G$, one has $T|\varphi_{g_0}\rangle = \int |\eta'_g\rangle\langle\eta_g|\varphi_{g_0}\rangle \,\mathrm{d}g = \int |\eta'_g\rangle\langle\eta_{g_0^{-1}g}|\varphi\rangle \,\mathrm{d}g = \int |\eta'_{g_0 g}\rangle\langle\eta_g|\varphi\rangle \,\mathrm{d}g = \int \rho'(g_0)|\eta'_g\rangle\langle\eta_g|\varphi\rangle \,\mathrm{d}g = \rho'(g_0)T|\varphi\rangle$. □

In our present context of the Heisenberg group, the integration over G should rather be replaced by an integration over the orbit (which does not correspond to a sub-group of $\mathbf{H}_n$). But the above argument essentially holds the same way up to a slight modification by a pure phase. Namely,

Proposition 2. *Fix $\tilde{\varphi}^0 \in \mathcal{L}^2_{hol}(\mathbb{F})$ and $\tilde{u}^0 \in L^2(Q)$. For every $v \in V$, set $\tilde{\varphi}^0_v := U_{\mathrm{BF}}(v)\tilde{\varphi}^0$ and $\tilde{u}^0_v := U_{\mathrm{KW}}(v)\tilde{u}^0$. Then, setting*

$$T_{\varphi^0, u^0} := \int_V |\tilde{\varphi}^0_v\rangle\langle\tilde{u}^0_v| \,\mathrm{d}v$$

formally defines a V-intertwiner between U_{KW} and U_{BF}.

Proof. Observe first that for every $w \in V$, one has $\tilde{\varphi}^0_{wv} = e^{\frac{i\mu}{2}\Omega(w,v)}\tilde{\varphi}^0_{w+v}$ and similarly for $\tilde{u}^0$. Also for every $\tilde{u} \in L^2(Q)$, one has

$$\begin{aligned}
T_{\varphi^0,u^0}U_{\mathrm{KW}}(w)\tilde{u} &= \int_V |\tilde{\varphi}^0_v\rangle\langle\tilde{u}^0_v|\tilde{u}_w\rangle\,\mathrm{d}v = \int_V |\tilde{\varphi}^0_v\rangle\langle\tilde{u}^0_{w^{-1}v}|\tilde{u}\rangle\,\mathrm{d}v\\
&= \int_V |\tilde{\varphi}^0_v\rangle\langle e^{-\frac{i\mu}{2}\Omega(w,v)}\tilde{u}^0_{v-w}|\tilde{u}\rangle\,\mathrm{d}v\\
&= \int_V e^{\frac{i\mu}{2}\Omega(w,v)}|\tilde{\varphi}^0_v\rangle\langle\tilde{u}^0_{v-w}|\tilde{u}\rangle\,\mathrm{d}v\\
&= \int_V e^{\frac{i\mu}{2}\Omega(w,v+w)}|\tilde{\varphi}^0_{v+w}\rangle\langle\tilde{u}^0_v|\tilde{u}\rangle\,\mathrm{d}v\\
&= \int_V e^{\frac{i\mu}{2}\Omega(w,v)}|\tilde{\varphi}^0_{v+w}\rangle\langle\tilde{u}^0_v|\tilde{u}\rangle\,\mathrm{d}v\\
&= \int_V |\tilde{\varphi}^0_{wv}\rangle\langle\tilde{u}^0_v|\tilde{u}\rangle\,\mathrm{d}v = U_{\mathrm{BF}}(w)T_{\varphi^0,u^0}\tilde{u}\,.
\end{aligned}$$

Now, one needs to check whether the above definition makes actual sense. The special choices $\tilde{\varphi}^0(Z) := \varphi_1(Z) = e^{-\frac{\mu}{4}|Z|^2}$ and $\tilde{u}^0(q) := e^{-\alpha q^2}$ reproduce the usual Bargmann transform.

Indeed, first observe that $\langle\tilde{u}^0_v,\tilde{u}\rangle = e^{\frac{-i\mu}{2}qp}\int_Q e^{i\mu q_0 p-\alpha(q_0-q)^2}\tilde{u}(q_0)\mathrm{d}q_0$ and $\tilde{\varphi}^0_v(v_1) = e^{i\frac{\mu}{2}(qp_1-q_1p)-\frac{\mu}{4}((q_1-q)^2+(p_1-p)^2)}$. This leads to

$$\begin{aligned}
T\tilde{u}(v_1) = \int \mathrm{d}q_0\mathrm{d}q\mathrm{d}p\; &e^{\frac{i\mu}{2}((p-p_1)(2q_0-q_1-q)+(2q_0-q_1)p_1)}\\
&\times e^{-\alpha(q_0-q)^2-\frac{\mu}{4}((q_1-q)^2+(p_1-p)^2)}\tilde{u}(q_0).
\end{aligned}$$

From the fact that $\int e^{-ixy}e^{-\frac{x^2}{2\sigma^2}}\mathrm{d}x = e^{-\frac{\sigma^2}{2}y^2}$, we get:

$$\begin{aligned}
T\tilde{u}(v_1) = \left(2\sqrt{\frac{\pi}{\mu}}\right)^n &\int \mathrm{d}q_0\mathrm{d}q\\
&\times e^{\frac{i\mu}{2}(2q_0-q_1)p_1}e^{-\alpha(q_0-q)^2-\frac{\mu}{4}(q_1-q)^2}e^{-\frac{\mu}{4}(2q_0-q_1-q)^2}\tilde{u}(q_0)\,.
\end{aligned}$$

The formula $\int e^{-\frac{2}{\sigma^2}(q-q_0)^2}\mathrm{d}q = (\frac{\sqrt{2\pi}}{\sigma})^n$ yields

$$T\tilde{u}(v_1) = \left(2\pi\sqrt{\frac{\alpha+\frac{\mu}{4}}{\mu}}\right)^n \int \mathrm{d}q_0\; e^{\frac{i\mu}{2}(2q_0-q_1)p_1}e^{-\frac{\mu}{2}((q_1-q_0)^2+\frac{1}{2}q_0^2)}\tilde{u}(q_0)\,.$$

Setting $Z_1 := q_1 + ip_1$ leads to

$$T\tilde{u}(v_1) = e^{-\frac{\mu}{4}|Z_1|^2}\left(2\pi\sqrt{\frac{\alpha+\frac{\mu}{4}}{\mu}}\right)^n \int \mathrm{d}q_0\; e^{-\frac{\mu}{4}(Z_1-q_0)(Z_1-3q_0)}\tilde{u}(q_0)\,. \qquad \square$$

Remark 1. The usual Bargmann transform (see Folland [3] page 40) equals the latter when $\frac{\alpha}{3} = \frac{\pi}{2} = \frac{\mu}{4}$.

4. Star-exponentials and noncommutative tori

4.1. Recalls on Weyl calculus

It is well known that the Weyl–Moyal algebra can be seen as a by-product of the Kirillov–Weyl representation. In [2], this fact is realized in terms of the natural symmetric space structure on the coadjoint orbits of the Heisenberg group. This construction is based on the fact that the Euclidean centered symmetries on $V = \mathbb{R}^{2n} \simeq \mathcal{O}$ naturally "quantize as phase symmetries". More precisely, at the level of the Heisenberg group the flat symmetric space structure on V is encoded by the involutive automorphism

$$\sigma : \mathbf{H}_n \to \mathbf{H}_n : v + zE \mapsto -v + zE\ .$$

The latter yields an involution of the equivariant function space

$$\sigma^\star : C^\infty(\mathbf{H}_n, \mathbb{C})^B \to C^\infty(\mathbf{H}_n, \mathbb{C})^B$$

which induces by restriction to Q the unitary phase involution:

$$\Sigma : L^2(Q) \to L^2(Q) : \tilde{u} \mapsto [q \mapsto \tilde{u}(-q)]\ .$$

Observing that the associated map

$$\mathbf{H}_n \to U(L^2(Q)) : g \mapsto U_{\mathrm{KW}}(g)\, \Sigma\, U_{\mathrm{KW}}(g^{-1})$$

is constant on the lateral classes of the stabilizer group $\mathbb{R}E$, one gets an $\mathbf{H}_n$-equivariant mapping:

$$\Xi_\mu : V \simeq \mathbf{H}_n/\mathbb{R}E \to U(L^2(Q)) : v \mapsto U_{\mathrm{KW}}(v)\, \Sigma\, U_{\mathrm{KW}}(v^{-1})$$

which at the level of functions yields the so-called "Weyl correspondence" valued in the $C^\star$-algebra of bounded operators on $L^2(Q)$:

$$\Xi_\mu : L^1(V) \to \mathcal{B}(L^2(Q)) : F \mapsto \int_V F(v)\, \Xi_\mu(v)\, \mathrm{d}v\ .$$

The above mapping uniquely extends from the space of compactly supported functions $C_c^\infty(V)$ to an injective continuous map defined on the Laurent Schwartz B-space $\mathcal{B}(V)$ of complex-valued smooth functions on V whose partial derivatives at every order are bounded:

$$\Xi_\mu : \mathcal{B}(V) \to \mathcal{B}(L^2(Q))\ ,$$

expressing in particular that the quantum operators associated to classical observables in the B-space are $L^2(Q)$-bounded in accordance with the classical Calderòn–Vaillancourt theorem.

It turns out that the range of the above map is a sub-algebra of $\mathcal{B}(L^2(Q))$. The Weyl product $\star_\mu$ on $\mathcal{B}(V)$ is then defined by structure transportation:

$$F_1 \star_\mu F_2 \ := \ \Xi_\mu^{-1}\left(\Xi_\mu(F_1) \circ \Xi_\mu(F_2)\right),$$

whose asymptotics in terms of powers of $\theta := \frac{1}{\mu}$ consists in the usual formal Moyal-star-product:

$$F_1 \star_\mu F_2 \sim F_1 \star^0_\theta F_2 := \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{\theta}{2i}\right)^k \Omega^{i_1 j_1} \ldots \Omega^{i_k j_k} \partial^k_{i_1 \ldots i_k} F_1 \, \partial^k j_1 \ldots j_k F_2 \, .$$

The resulting associative algebra $(\mathcal{B}(V), \star_\mu)$ is then both Fréchet (w.r.t. the natural Fréchet topology on $\mathcal{B}(V)$) and pre-$C^\star$ (by transporting the operator norm from $\mathcal{B}(L^2(Q))$).

4.2. Star-exponentials

Combining the above-mentioned results of [2] and well-known results on star-exponentials (see, e.g., [1]), we observe that the heuristic consideration of the series:

$$\mathrm{Exp}_\theta(F) := \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{i}{\theta} F\right)^{\star_\mu k}$$

yields a well-defined group homomorphism:

$$\mathcal{E}_\theta : \mathbf{H}_n \to (\mathcal{B}(V), \star_\mu) : g \mapsto \mathrm{Exp}_\theta(\lambda_{\log g})$$

where λ denotes the classical linear moment:

$$\mathfrak{h}_n \to C^\infty(V) : X \mapsto [v \mapsto \langle \mathtt{Ad}^\flat_v \xi_0, X\rangle] \, .$$

Indeed, in this case, if F depends only either on the q-variable or on the p-variable then the above star-exponential just coincides with the usual exponential: $\mathrm{Exp}_\theta(F) = \exp\left(\frac{i}{\theta} F\right)$. In particular, for x either in Q or $\exp \mathfrak{L}$ we find:

$$\left(\mathcal{E}_\theta(x)\right)(v) = e^{\frac{i}{\theta^2}\Omega(x,v)} \, ;$$

while for $x = zE$, we find the constant function:

$$\left(\mathcal{E}_\theta(zE)\right)(v) = e^{\frac{zi}{\theta^2}} \, .$$

From which we conclude that $\mathcal{E}_\theta$ is indeed valued in $\mathcal{B}(V)$.

4.3. An approach to the non-commutative torus

For simplicity, restrict to the case $n = 1$ and consider Ω-dual basis elements e_q of $\mathfrak{q}$ and e_p of $\mathfrak{L}$. From what precedes we observe that the group elements

$$U := \mathcal{E}_\theta(\exp(\theta^2 e_q)) = e^{ip} \quad \text{and} \quad V := \mathcal{E}_\theta(\exp(\theta^2 e_p)) = e^{-iq}$$

behave inside the image group $\mathcal{E}_\theta(\mathbf{H}_1) \subset \mathcal{B}(\mathbb{R}^2)$ as

$$U\,V = e^{i\theta} V\,U$$

(where we omitted to write $\star_\mu$).

In particular, we may make the following

Definition 1. Endowing $(\mathcal{B}(\mathbb{R}^2), \star_\mu)$ with its pre-$C^\star$-algebra structure (coming from Ξ_μ), the complex linear span inside $\mathcal{B}(\mathbb{R}^2)$ of the sub-group of $\mathcal{E}_\theta(\mathbf{H}_1)$ generated by elements U and V underlies a pre-$C^\star$-algebra $\mathbb{T}^\circ_\theta$ that completes as the non-commutative 2-torus $\mathbb{T}_\theta$.

4.4. Bargmann–Fock realization of the non-commutative torus

Identifying elements of $\mathbb{R}^2 = V \subset \mathbf{H}_1$ with complex numbers as before, we compute that

$$T \circ \Sigma \;=\; -\mathtt{id}^\star \circ T \quad \text{and} \quad \mathrm{BF}_\mu(Z)\tilde{\varphi}(Z_0) \;:=\; T\Xi_\mu(Z)T^{-1}\tilde{\varphi}(Z_0) \;=\; e^{i\mu\,\mathrm{Im}(\overline{Z}Z_0)}\tilde{\varphi}(2Z - Z_0)\,.$$

By structure transportation, we define the following correspondence:

$$\mathrm{BF}_\mu : \mathcal{B}(\mathbb{C}) \to \mathcal{B}(L^2_{\mathrm{hol}}(\mathbb{F}))$$

as the unique continuous linear extension of

$$C^\infty_c(\mathbb{C}) \to \mathcal{B}(L^2_{\mathrm{hol}}(\mathbb{F})) : F \mapsto \int_{\mathbb{C}} F(Z)\,\mathrm{BF}_\mu(Z)\,\mathrm{d}Z\,.$$

Proposition 3. *Applied to an element in* $\mathcal{E}_\theta(\mathbf{H}_1)$ *of the form* $F_X(Z) = e^{i\alpha\,\mathrm{Im}(\overline{Z}X)}$ *with* $X \in \mathbb{C}$ *and* $\alpha \in \mathbb{R}$*, one has:*

$$\mathrm{BF}_\mu(F_X)\tilde{\varphi}(Z_0) \;=\; e^{\frac{i\alpha}{2}\,\mathrm{Im}(\overline{Z_0}X)}\tilde{\varphi}(\mu Z_0 + \alpha X)\,.$$

Proof. A small computation leads to the formula:

$$\mathrm{BF}_\mu(F)\tilde{\varphi}(Z_0) \;=\; \frac{1}{4}\int_{\mathbb{C}} F\left(\frac{1}{2}(Z + Z_0)\right) e^{\frac{i}{2}\mu\,\mathrm{Im}(\overline{Z}Z_0)}\,\tilde{\varphi}(Z)\,\mathrm{d}Z\,.$$

Applied to an element in $\mathcal{E}_\theta(\mathbf{H}_1)$ of the form $F_X(Z) = e^{i\alpha\,\mathrm{Im}(\overline{Z}X)}$ with $X \in \mathbb{C}$ and $\alpha \in \mathbb{R}$, the above formula yields:

$$\mathrm{BF}_\mu(F_X)\tilde{\varphi}(Z_0) \;=\; e^{\frac{i\alpha}{2}\,\mathrm{Im}(\overline{Z_0}X)}\mathcal{F}_{\mathbb{C}}(\tilde{\varphi})(\mu Z_0 + \alpha X),$$

where $\mathcal{F}_{\mathbb{C}}(\tilde{\varphi})(Z_0) := C\int_{\mathbb{C}} e^{\frac{i}{2}\,\mathrm{Im}(\overline{Z}Z_0)}\tilde{\varphi}(Z)\,\mathrm{d}Z$. The limit $X_0 \to 0$ yields $\tilde{\varphi} = \mathcal{F}_{\mathbb{C}}(\tilde{\varphi})$, hence:

$$\mathrm{BF}_\mu(F_X)\tilde{\varphi}(Z_0) \;=\; e^{\frac{i\alpha}{2}\,\mathrm{Im}(\overline{Z_0}X)}\tilde{\varphi}(\mu Z_0 + \alpha X)\,.$$ □

5. Conclusions

We now summarize what has been done in the present work. First, we establish a way to systematically produce explicit formulae for intertwiners of group unitary representations. Second, applying the above intertwiner in the case of the Bargmann–Fock and Kirillov–Weyl realizations of the unitary dual of the Heisenberg groups, we realized the non-commutative torus as the range of the star-exponential for the Heisenberg group. And third, we then deduced from this a realization of the non-commutative torus as acting on a Fock space of entire functions.

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Berezin Transform and a Deformation Decomposition

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Abstract. We present a new form for a deformation decomposition of the Berezin transform in polynomial quantization on para-Hermitian symmetric spaces. For rank one spaces, we write a full deformation decomposition explicitly

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In construction of quantizations in the spirit of Berezin on symplectic spaces G/H the main role belongs to the Berezin transform. Program for such a construction is the following: (a) to express it in terms of Laplace operators on G/H, in fact it is the same as to determine the Plancherel formula for a canonical representation on G/H; (b) to write its asymptotic decomposition when $h \to 0$ (h being the Planck constant). Two first terms of the decomposition give the corresponding principle. Berezin carried out it for Hermitian symmetric spaces G/K, see [1, 2]. We succeeded in solving these problems for para-Hermitian symmetric spaces of rank one. Moreover, for *polynomial quantization* we can write a *full* asymptotic decomposition explicitly.

1. Para-Hermitian symmetric spaces

Let G/H be a *semisimple symmetric space*. Here G is a connected semisimple Lie group with an involutive automorphism $\sigma \neq 1$, and H is an open subgroup of G^σ, the subgroup of fixed points of σ.

We consider that groups act on their homogeneous spaces *from the right*, so that G/H consists of right cosets Hg.

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Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and of H respectively. Let $B_{\mathfrak{g}}$ be the Killing form of G. There is a decomposition of $\mathfrak{g}$ into direct sum of $+1$, -1-eigenspaces of σ:

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{q}.$$

The subspace $\mathfrak{q}$ is invariant with respect to H in the adjoint representation Ad. It can be identified with the tangent space to G/H at the point $x^0 = He$.

The dimension of Cartan subspaces of $\mathfrak{q}$ (maximal Abelian subalgebras in $\mathfrak{q}$ consisting of semisimple elements) is called the rank of G/H.

Now let G/H be a *symplectic* manifold. Then $\mathfrak{h}$ has a non-trivial center $Z(\mathfrak{h})$. For simplicity we assume that G/H is an orbit $\mathrm{Ad}G \cdot Z_0$ of an element $Z_0 \in \mathfrak{g}$. In particular, then $Z_0 \in Z(\mathfrak{h})$.

Further, we can also assume that G is *simple*. Such spaces G/H are divided into 4 classes (see [3, 4]):

(a) Hermitian symmetric spaces;
(b) semi-Kählerian symmetric spaces;
(c) para-Hermitian symmetric spaces;
(d) complexifications of spaces of class (a).

Dimensions of $Z(\mathfrak{h})$ are 1,1,1,2, respectively. Spaces of class (a) are Riemannian, of other three classes are pseudo-Riemannian (not Riemannian).

We focus on spaces of class (c). Here the center $Z(\mathfrak{h})$ is one-dimensional, so that $Z(\mathfrak{h}) = \mathbb{R}Z_0$, and Z_0 can be normalized so that the operator $I = (\mathrm{adZ_0})_{\mathfrak{q}}$ on $\mathfrak{q}$ has eigenvalues ± 1. A symplectic structure on G/H is defined by the bilinear form $\omega(X,Y) = B_{\mathfrak{g}}(X, IY)$ on $\mathfrak{q}$.

The ± 1-eigenspaces $\mathfrak{q}^{\pm} \subset \mathfrak{q}$ of I are Lagrangian, H-invariant, and irreducible. They are Abelian subalgebras of $\mathfrak{g}$. So $\mathfrak{g}$ becomes a graded Lie algebra:

$$\mathfrak{g} = \mathfrak{q}^- + \mathfrak{h} + \mathfrak{q}^+,$$

with commutation relations $[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{h},\mathfrak{q}^-] \subset \mathfrak{q}^-$, $[\mathfrak{h},\mathfrak{q}^+] \subset \mathfrak{q}^+$.

The pair $(\mathfrak{q}^+, \mathfrak{q}^-)$ is a Jordan pair [5] with multiplication

$$\{XYZ\} = (1/2)\,[[X,Y],\,Z].$$

Let r and $\varkappa$ be the rank and the genus of this Jordan pair. This rank r coincides with the rank of G/H.

Set $Q^{\pm} = \exp \mathfrak{q}^{\pm}$. The subgroups $P^{\pm} = HQ^{\pm} = Q^{\pm}H$ are maximal parabolic subgroups of G. One has the following decompositions:

$$G = \overline{Q^+HQ^-} \tag{1}$$
$$= \overline{Q^-HQ^+}, \tag{2}$$

where bar means closure and the sets under the bar are open and dense in G. Let us call (1) and (2) the *Gauss decomposition* and (allowing some slang) the *anti-Gauss decomposition* respectively. For an element in G all three factors in (1) and (2) are defined uniquely.

These decompositions generate actions of G on $\mathfrak{q}^{\pm}$, namely, $\xi \mapsto \widetilde{\xi} = \xi \bullet g$ on $\mathfrak{q}^-$ and $\eta \mapsto \widehat{\eta} = \eta \circ g$ on $\mathfrak{q}^+$ by (1) and (2) respectively:

$$\exp\xi \cdot g = \exp Y \cdot \widetilde{h} \cdot \exp\widetilde{\xi}, \tag{3}$$

$$\exp\eta \cdot g = \exp X \cdot \widehat{h} \cdot \exp\widehat{\eta}, \tag{4}$$

where $X \in \mathfrak{q}^-$, $Y \in \mathfrak{q}^+$. These actions are defined on open and dense sets depending on g. Therefore, G acts on $\mathfrak{q}^- \times \mathfrak{q}^+ : (\xi, \eta) \mapsto (\widetilde{\xi}, \widehat{\eta})$. The stabilizer of the point $(0,0) \in \mathfrak{q}^- \times \mathfrak{q}^+$ is $P^+ \cap P^- = H$, so that we get an embedding

$$\mathfrak{q}^- \times \mathfrak{q}^+ \hookrightarrow G/H. \tag{5}$$

It is defined on an open and dense set, its image is also an open and dense set. Therefore, we can consider ξ, η as coordinates on G/H, let us call them *horospherical coordinates*.

Take $\xi \in \mathfrak{q}^-$, $\eta \in \mathfrak{q}^+$ and decompose $\exp\xi \cdot \exp(-\eta)$ (the "anti-Gauss") according to the "Gauss":

$$\exp\xi \cdot \exp(-\eta) = \exp Y \cdot h \cdot \exp X,$$

where $X \in \mathfrak{q}^-$, $Y \in \mathfrak{q}^+$. The obtained $h \in H$ depends on ξ and η only, denote it by $h(\xi, \eta)$.

The determinant of $\mathrm{Ad}h(\xi,\eta)$ to power -1 is a polynomial in ξ, η. Moreover, it is the $\varkappa$th power of an irreducible polynomial $N(\xi,\eta)$ of degree r in ξ and in η separately:

$$\left\{\det\left(\mathrm{Ad}h(\xi,\eta)\right)|_{\mathfrak{q}^+}\right\}^{-1} = N(\xi,\eta)^{\varkappa}.$$

The G-invariant measure on G/H is:

$$dx = dx(\xi,\eta) = |N(\xi,\eta)|^{-\varkappa}\, d\xi\, d\eta$$

where $d\xi$, $d\eta$ are Euclidean measures on $\mathfrak{q}^-$, $\mathfrak{q}^+$, respectively.

2. Maximal degenerate series representations

For $\lambda \in \mathbb{C}$, we take the character of H:

$$\omega_\lambda(h) = \left|\det(Ad\, h)|_{\mathfrak{q}^+}\right|^{-\lambda/\varkappa}$$

and extend this character to the subgroups $P^{\pm}$, setting it equal to 1 on $Q^{\pm}$. We consider induced representations of G:

$$\pi_\lambda^{\pm} = \mathrm{Ind}_{P^{\mp}}^{G}\, \omega_{\mp\lambda}.$$

Let $\mathcal{D}_\lambda^{\pm}(G)$ be the space of functions $f \in C^\infty(G)$ satisfying the uniformity property

$$f(pg) = \omega_{\mp\lambda}(p) f(g), \quad p \in P^{\mp}.$$

The representation $\pi_\lambda^{\pm}$ acts on it by translations from the right:

$$\left(\pi_\lambda^{\pm}(g) f\right)(g_1) = f(g_1 g).$$

Realize them in the *noncompact picture*: we restrict functions from $\mathcal{D}^{\pm}_{\lambda}(G)$ to the subgroups $Q^{\pm}$ and identify them (as manifolds) with $\mathfrak{q}^{\pm}$, we obtain

$$\left(\pi^{-}_{\lambda}(g)f\right)(\xi) = \omega_{\lambda}(\widetilde{h})f(\widetilde{\xi}), \quad \left(\pi^{+}_{\lambda}(g)f\right)(\eta) = \omega_{\lambda}(\widehat{h}^{-1})f(\widehat{\eta}),$$

where $\widetilde{\xi}$, $\widetilde{h}$, $\widehat{\eta}$, $\widehat{h}$ are taken from decompositions (3), (4).

Let us write intertwining operators. Introduce operators A_{λ} and B_{λ} by:

$$(A_{\lambda}\varphi)(\eta) = \int_{\mathfrak{q}^{-}} |N(\xi,\eta)|^{-\lambda-\varkappa}\, \varphi(\xi)\, d\xi,$$

$$(B_{\lambda}\varphi)(\xi) = \int_{\mathfrak{q}^{+}} |N(\xi,\eta)|^{-\lambda-\varkappa}\, \varphi(\eta)\, d\eta.$$

The operator A_{λ} intertwines π^{-}_{λ} with $\pi^{+}_{-\lambda-\varkappa}$ and the operator B_{λ} intertwines π^{+}_{λ} with $\pi^{-}_{-\lambda-\varkappa}$.

Their composition is a scalar operator:

$$B_{\lambda}A_{-\lambda-\varkappa} = c(\lambda)^{-1} \cdot \mathrm{id}, \tag{6}$$

where $c(\lambda)$ is a meromorphic function.

We can extend $\pi^{\pm}_{\lambda}$, A_{λ} and B_{λ} to distributions on $\mathfrak{q}^{-}$ and $\mathfrak{q}^{-}$.

3. Symbols and transforms

In this section we give main constructions of a quantization in the spirit of Berezin on *para-Hermitian* symmetric spaces G/H, see [6]. We consider a variant of the quantization, which we call the *polynomial quantization*. We introduce two types of symbols of operators: covariant and contravariant ones, the Berezin transform etc.

As a (an analog of) supercomplete system we take the kernel of the intertwining operators from Section 2, i.e., the function

$$\Phi(\xi,\eta) = \Phi(\xi,\eta)_{\lambda} = |N(\xi,\eta)|^{\lambda}.$$

It has a reproducing property, which is formula (6) written in another form:

$$\varphi(s) = c(\lambda) \int_{G/H} \frac{\Phi(\xi,\eta)}{\Phi(u,v)}\, \varphi(u)\, dx(u,v).$$

The role of the Fock space is played by a space of functions $\varphi(\xi)$ depending on one of horospherical coordinates ξ, η.

For an initial algebra of operators we take the algebra $\pi^{-}_{\lambda}(\mathrm{Env}(\mathfrak{g}))$, where $\mathrm{Env}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. For an operator $D = \pi^{-}_{\lambda}(X)$, $X \in \mathrm{Env}(\mathfrak{g})$), the function

$$F(\xi,\eta) = \frac{1}{\Phi(\xi,\eta)}\, (\pi^{-}_{\lambda}(X) \otimes 1)\Phi(\xi,\eta)$$

is called the *covariant symbol* of D. Since ξ, η are horospherical coordinates on G/H, covariant symbols becomes functions on G/H and, moreover, *polynomials* on

$G/H \subset \mathfrak{g}$. It is why we call this variant of quantization by *polynomial* quantization. For generic λ, the space of covariant symbols is the space of all polynomials on G/H.

In particular, the covariant symbol of the identity operator is the function on G/H equal to 1 identically. For the operator $\pi_\lambda^-(X)$, corresponding to an element X of the Lie algebra $\mathfrak{g}$, its covariant symbol is a linear function $B_\mathfrak{g}(X, x)$ of $x \in G/H \subset \mathfrak{g}$ with coordinates ξ, η, up to a factor depending on λ.

The operator D is recovered by its covariant symbol F:

$$(D\varphi)(\xi) = c \int_{G/H} F(\xi, v) \, \frac{\Phi(\xi, v)}{\Phi(u, v)} \, \varphi(u) \, dx(u, v), \tag{7}$$

where $c = c(\lambda)$ is taken from (6).

The multiplication of operators gives rise to the multiplication of covariant symbols. Namely, let F_1, F_2 be covariant symbols of operators D_1, D_2, respectively. Then the covariant symbol $F_1 * F_2$ of the product $D_1 D_2$ is

$$F(\xi, \eta) = \frac{1}{\Phi(\xi, \eta)} \, (D_1 \otimes 1)(\Phi(\xi, \eta) F_2(\xi, \eta)),$$

or

$$(F_1 * F_2)(\xi, \eta) = \int_{G/H} F_1(\xi, v) F_2(u, \eta) \, \mathcal{B}(\xi, \eta; u, v) \, dx(u, v),$$

where

$$\mathcal{B}(\xi, \eta; u, v) = c \, \frac{\Phi(\xi, v)\Phi(u, \eta)}{\Phi(\xi, \eta)\Phi(u, v)} \, .$$

Let us call this function $\mathcal{B}$ the *Berezin kernel*. It can be regarded as a function $\mathcal{B}(x, y)$ on $G/H \times G/H$. It is invariant with respect to G:

$$\mathcal{B}(\mathrm{Ad}\, g \cdot x, \, \mathrm{Ad} g \cdot y) = \mathcal{B}(x, y).$$

Now we define *contravariant symbols*. A function (a polynomial) $F(\xi, \eta)$ is the contravariant symbols for the following operator A (acting on functions $\varphi(\xi)$):

$$(A\varphi)(\xi) = c \int_{G/H} F(u, v) \, \frac{\Phi(\xi, v)}{\Phi(u, v)} \, \varphi(u) \, dx(u, v) \tag{8}$$

(notice that (8) differs from (7) by the first argument of F only). This operator is a Toeplitz type operator.

Thus, we have two maps: $D \mapsto F$(“co”) and $F \mapsto A$(“contra”), connecting polynomials on G/H and operators acting on functions $\varphi(\xi)$.

If a polynomial F on G/H is the covariant symbol of an operator $D = \pi_\lambda^-(X)$, $X \in \mathrm{Env}(\mathfrak{g})$, and the contravariant symbol of an operator A simultaneously, then $A = \pi_{-\lambda-\varkappa}^-(X^\vee)$, where $X \mapsto X^\vee$ is the transform of $\mathrm{Env}(\mathfrak{g})$, generated by $g \mapsto g^{-1}$ in the group G. Therefore, A is obtained from D by the conjugation with respect to the bilinear form

$$(F, f) = \int_{\mathfrak{q}^-} F(\xi) \, f(\xi) \, d\xi.$$

In terms of kernels, it means that the kernel $L(\xi, u)$ of the operator A is obtained from the kernel $K(\xi, u)$ of the operator D by the transposition of arguments and the change of λ by $-\lambda - \varkappa$. So, the composition $\mathcal{O} : D \mapsto A$(“contra” ∘ “co”) is

$$\mathcal{O} : \ \pi_\lambda^-(X) \longmapsto \pi_{-\lambda-\varkappa}^-(X^\vee).$$

This map commutes with the adjoint representation. Such a map was absent in Berezin’s theory for Hermitian symmetric spaces.

The composition $\mathcal{B}$(“co” ∘ “contra”) maps the contravariant symbol of an operator D to its covariant symbol. Let us call $\mathcal{B}$ the *Berezin transform.* The kernel of this transform is just the Berezin kernel.

Let us formulate unsolved problems for spaces of arbitrary rank (for $r > 1$):

1) to express the Berezin transform $\mathcal{B}$ in terms of Laplacians $\Delta_1, \ldots, \Delta_r$ (in fact, it is the same that to decompose a canonical representation into irreducible constituents);
2) to compute eigenvalues of $\mathcal{B}$ on irreducible constituents;
3) to find a full asymptotics of $\mathcal{B}$ when $\lambda \to -\infty$ (an analog of the Planck constant is $h = -1/\lambda$).

These problems are solved for spaces of rank one, see Section 4.

4. Polynomial quantization on rank one spaces

In this section we lean on [7]. We consider here the spaces G/H, where $G = \mathrm{SL}\,(n, \mathbb{R})$, $H = \mathrm{GL}\,(n-1, \mathbb{R})$. They have dimension $2n-2$, rank $r = 1$ and genus $\varkappa = n$. These spaces G/H exhaust all para-Hermitian symmetric spaces of rank one up to the covering. Further we assume $n \geqslant 3$.

Let $\mathrm{Mat}(n, \mathbb{R})$ denote the space of real $n \times n$ matrices x. The Lie algebra $\mathfrak{g}$ of G consists of x with $\mathrm{tr}\, x = 0$. By Section 1, the space G/H is a G-orbit in $\mathfrak{g}$.

But now it is more convenient for us to change a little the realization of G/H.

The group G acts on $\mathrm{Mat}\,(n, \mathbb{R})$ by $x \mapsto g^{-1}xg$. Let us write matrices x in the block form according to the partition $n = (n-1) + 1$:

$$x = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

where $\alpha \in \mathrm{Mat}\,(n-1, \mathbb{R})$, β is a vector-column in $\mathbb{R}^{n-1}$, γ is a vector-row in $\mathbb{R}^{n-1}$ and δ is a number.

Let x^0 be the following matrix:

$$x^0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The G-orbit of x^0 is just G/H. This manifold is the set of matrices x whose trace and rank are equal to 1. The stabilizer H of x^0 consists of matrices $\mathrm{diag}\{a, b\}$, where $a \in \mathrm{GL}\,(n-1, \mathbb{R})$, $b = (\det\, a)^{-1}$, so that $H = \mathrm{GL}\,(n-1, \mathbb{R})$.

Subalgebras $\mathfrak{q}^-$ and $\mathfrak{q}^+$ consist respectively of matrices

$$X = \begin{pmatrix} 0 & 0 \\ \xi & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & \eta \\ 0 & 0 \end{pmatrix},$$

where ξ is a row $(\xi_1, \ldots, \xi_{n-1})$, and η is a column $(\eta_1, \ldots, \eta_{n-1})$ in $\mathbb{R}^{n-1}$. Embedding (5) is

$$x = \frac{1}{N(\xi,\eta)} \begin{pmatrix} -\eta\xi & -\eta \\ \xi & 1 \end{pmatrix},$$

where $N(\xi,\eta) = 1 - \xi\eta = 1 - (\xi_1\eta_1 + \cdots + \xi_{n-1}\eta_{n-1})$.

A G-invariant metric ds^2 on G/H up to a factor is $\mathrm{tr}\,(dx^2)$. It generates the measure dx, the Laplace–Beltrami operator Δ, the symplectic form ω and the Poisson bracket $\{f,h\}$. In coordinates ξ, η we have:

$$\begin{aligned} ds^2 &= -2N(\xi,\eta)^{-2}\Big\{\sum \xi_i\, d\eta_i \sum \eta_i d\xi_i + N(\xi,\eta) \sum d\xi_i\, d\eta_i\Big\}. \\ dx &= |N(\xi,\eta)|^{-n}\ d\xi\, d\eta \quad (d\xi = d\xi_1 \ldots d\xi_{n-1}), \\ \Delta &= N(\xi,\eta) \sum (\delta_{ij} - \xi_i\eta_j)\, \frac{\partial^2}{\partial\xi_i\, \partial\eta_j}, \\ \omega &= \frac{1}{N(\xi,\eta)} \sum \left(\delta_{ij} + \frac{1}{N(\xi,\eta)}\, \eta_i\, \xi_j\right) d\xi_i \wedge d\eta_j, \\ \{f,h\} &= N(\xi,\eta) \sum (\delta_{ij} - \xi_i\eta_j) \left(\frac{\partial f}{\partial \eta_i}\, \frac{\partial h}{\partial \xi_j} - \frac{\partial f}{\partial \xi_i}\, \frac{\partial h}{\partial \eta_j}\right). \end{aligned}$$

The Berezin kernel is

$$\mathcal{B}(x,y) = c(\lambda)\, \frac{\Phi(\xi,v)\Phi(u,\eta)}{\Phi(\xi,\eta)\Phi(u,v)} \ = c(\lambda)\, |\mathrm{tr}(xy)|^{\lambda},$$

where

$$c(\lambda) = \Big\{2^{n+1}\pi^{n-2}\Gamma(-\lambda - n + 1)\Gamma(\lambda+1) \left[\cos\left(\lambda + \frac{n}{2}\right)\pi - \cos\frac{n\pi}{2}\right]\Big\}^{-1}.$$

The Berezin transform is written in terms of the Laplace–Beltrami operator Δ as follows

$$\mathcal{B} = \frac{\Gamma(-\lambda+\sigma)\,\Gamma(-\lambda-\sigma-n+1)}{\Gamma(-\lambda)\,\Gamma(-\lambda-n+1)}, \tag{9}$$

the right-hand side should be regarded as a function of $\Delta = \sigma(\sigma + n - 1)$.

Now let $\lambda \to -\infty$. Then (9) gives

$$\mathcal{B} \sim 1 - \frac{1}{\lambda}\,\Delta.$$

Hence we have

$$F_1 * F_2 \sim F_1 F_2 - \frac{1}{\lambda}\, N^2\, \frac{\partial F_1}{\partial \xi}\, \frac{\partial F_2}{\partial \eta},$$

so that for $\lambda \to -\infty$ we have

$$F_1 * F_2 \longrightarrow F_1 F_2, \tag{10}$$

$$-\lambda\,(F_1 * F_2 - F_2 * F_1) \longrightarrow \{F_1,\, F_2\}, \tag{11}$$

in the right-hand sides of (10) and (11) the pointwise multiplication and the Poisson bracket stand, respectively. Relations (10) and (11) show that for the family of algebras of covariant symbols the *correspondence principle* is true. As the Planck constant, one has to take $h = -1/\lambda$.

Moreover, we can write not only two terms of the asymptotics but also a full asymptotic decomposition (a deformation decomposition) of $\mathcal{B}$ explicitly. In order to have a transparent formula, one has to expand not in powers of $h = -1/\lambda$ but use "generalized powers" of $-\lambda - n$. Then decomposition turns out to be a series terminating on any irreducible subspace of polynomials on G/H.

Namely, we have the following decomposition of the Berezin transform:

$$\mathcal{B} = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{\Delta\,[\Delta - 1\cdot n]\,[\Delta - 2\cdot(n+1)]\,\ldots\,[\Delta - (k-1)(k-2+n)]}{(-\lambda - n)^{(k)}}\,,$$

where

$$a^{(m)} = a(a-1)\ldots(a-m+1).$$

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Toeplitz Quantization without Measure or Inner Product

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Abstract. This note is a follow-up to a recent paper by the author. Most of that theory is now realized in a new setting where the vector space of symbols is not necessarily an algebra nor is it equipped with an inner product, although it does have a conjugation. As in the previous paper one does not need to put a measure on this vector space. A Toeplitz quantization is defined and shown to have most of the properties as in the previous paper, including creation and annihilation operators. As in the previous paper this theory is implemented by densely defined Toeplitz operators which act in a Hilbert space, where there is an inner product, of course. Planck's constant also plays a role in the canonical commutation relations of this theory. Proofs are given in order to provide a self-contained paper.

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1. Introduction

In a recent paper [6] of mine I developed a theory of a Toeplitz quantization whose symbols lie in a possibly non-commutative algebra which has an inner product. At that time I was motivated by previous papers ([4] and [5]) of mine that had symbols in a non-commutative algebra. In those cases there was also an inner product available which served more than anything as a part of a formula defining a projection operator. And that projection operator was used in the standard way to define Toeplitz operators in that setting. But now I have realized that there is another way to arrive at most of the results of [6] without supposing that the complex vector space (no longer assumed to be an algebra) of symbols has an inner product, though I still require that it have a conjugation to get more interesting results.

While the paper [6] presented a viable quantization scheme that did not involve a measure, the objection could be made that an inner product is some sort of mild generalization of a measure, that it is a 'measure in disguise' or some such criticism. However, in this note there is neither measure nor inner product on the 'classical' space of symbols. Of necessity there is an inner product in the quantum Hilbert space.

The references for this short note are deliberately kept to just a very few. For further background and motivation on this topic see [6], consult the references found there and continue recursively.

2. The new setting

We have a new setting that has some things in common with that in [6]. So, to facilitate this presentation I will use the same notation as in [6]. Here are the exact structures to be considered in this note together with their notations. They involve three vectors spaces (denoted by $\mathcal{A}$, $\mathcal{H}$ and $\mathcal{P}$) over the field $\mathbb{C}$ of complex numbers. These spaces are required to satisfy these eight conditions:

1. $\mathcal{H}$ is a Hilbert space.
2. $\mathcal{A}$ has a conjugation denoted by g^* for all $g \in \mathcal{A}$. A conjugation is by definition an anti-linear, involutive mapping of a vector space to itself.
3. $\mathcal{P}$ is a dense subspace of $\mathcal{H}$.
4. $\mathcal{P}$ is a vector subspace of $\mathcal{A}$.
5. $\mathcal{P}$ is an associative algebra with unit 1 satisfying $1^* = 1$. Note that $\mathcal{P}$ is not necessarily commutative.
6. There is a left action of $\mathcal{P}$ on $\mathcal{A}$. This means that there is a unital algebra morphism $\mathcal{P} \to \mathrm{End}(\mathcal{A})$, since $\mathrm{End}(\mathcal{A})$ acts by convention on the left of $\mathcal{A}$. In particular we assume that this action (thought of as a bilinear map $\mathcal{P} \times \mathcal{A} \to \mathcal{A}$) restricts to the multiplication map $\mathcal{P} \times \mathcal{P} \to \mathcal{P}$ of the algebra $\mathcal{P}$. The notation is $(\phi, g) \mapsto \phi g$ for $(\phi, g) \in \mathcal{P} \times \mathcal{A}$.
7. There is a linear map $P : \mathcal{A} \to \mathcal{P} \subset \mathcal{A}$ which satisfies $P^2 = P$ and with range $\mathrm{Ran}\, P = \mathcal{P}$. (The co-domain of P is taken to be either $\mathcal{P}$ or $\mathcal{A}$, as convenience dictates.) One immediately has that the restriction of P to $\mathcal{P}$ is the identity map on $\mathcal{P}$.
8. $\langle T_g \phi_1, \phi_2 \rangle_{\mathcal{H}} = \langle \phi_1, T_{g^*} \phi_2 \rangle_{\mathcal{H}}$ for all $\phi_1, \phi_2 \in \mathcal{P}$ and $g \in \mathcal{A}$ where T_g, the Toeplitz operator with symbol g, will be defined below. This condition means $T_{g^*} \subset (T_g)^*$, the adjoint of T_g.

I do not assume that there is an inner product on $\mathcal{A}$, but of course we do have an inner product, denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, on the Hilbert space $\mathcal{H}$. And this restricts to an inner product on $\mathcal{P}$ thereby making it a pre-Hilbert space. In [6] the vector space $\mathcal{A}$ of symbols was assumed to be an algebra. We retain the notation, but not that hypothesis, for this space. The conjugation on $\mathcal{A}$ typically will not leave $\mathcal{P}$ invariant. All that we can say in general is that $\mathcal{P}^* \subset \mathcal{A}$. A natural way to define an inner product on $\mathcal{P}^*$ is

$$\langle \psi_1, \psi_2 \rangle_{\mathcal{P}^*} := \langle \psi_2^*, \psi_1^* \rangle_{\mathcal{H}}$$

for all $\psi_1, \psi_2 \in \mathcal{P}^*$. With this inner product $\mathcal{P}^*$ becomes a pre-Hilbert space, which is anti-unitarily equivalent to $\mathcal{P}$ via the map $\phi \to \phi^*$ for all $\phi \in \mathcal{P}^*$. The completion of $\mathcal{P}^*$ is denoted by $\mathcal{H}^*$. Bearing in mind typical examples from classical analysis, one sees that $\mathcal{H}$ corresponds to a Hilbert space of holomorphic functions while $\mathcal{P}$ corresponds to its subspace of holomorphic polynomials. Similarly, $\mathcal{H}^*$ and $\mathcal{P}^*$ are their anti-holomorphic counterparts. Given this intuition behind these structures, one sees that the requirement $\mathcal{P} \cap \mathcal{P}^* = \mathbb{C}1$ is quite natural. However, it is not needed for the present theory, nor was it used in [6]. So, we will not make any assumption on $\mathcal{P} \cap \mathcal{P}^*$.

The main differences from the setting in [6] are that $\mathcal{A}$ no longer need be an algebra nor need it have an inner product defined on it. However, its subspace $\mathcal{P}$ has the restriction of the inner product of $\mathcal{H}$. Condition 6 is new in its details, but preserves the idea of the assumption as given in [6] that $\mathcal{P}$ is a subalgebra of the algebra $\mathcal{A}$. Condition 7 was a consequence of other assumptions given in [6] about the existence of a certain subset Φ of $\mathcal{P}$. Here it is simply taken as an additional assumption that replaces the assumptions about that subset Φ.

In Condition 8 we require the consistency of the conjugation in $\mathcal{A}$ and the adjoint operation of operators. In [6] this was a consequence of an identity that itself was assumed as a hypothesis. (See Theorem 3.3, Part 4.) Here we take this property itself itself as a hypothesis. Of course, Toeplitz operators will be defined presently without using Condition 8.

The theory in [6] satisfies these eight conditions. So, the theory in this new setting generalizes the theory in [6]. But we see no way to define an inner product on $\mathcal{A}$ nor to extract the set Φ in this new setting. Also, $\mathcal{A}$ in this note need not be an algebra. So it seems safe to say that this note has a strict generalization of the theory presented in [6]. Nonetheless, most of the results in [6] remain true in this new setting.

3. Definitions and basic results

We now present and prove all those results in [6] which are still valid in this new setting. First, here are some definitions almost identical to those in [6]. These are simply the natural definitions of Toeplitz operator and Toeplitz quantization in this new setting.

Definition 1. For any $g \in \mathcal{A}$ define $M_g : \mathcal{P} \to \mathcal{A}$ by $M_g \phi := \phi g$ for all $\phi \in \mathcal{P}$. (Recall ϕg is the left action of $\phi \in \mathcal{P}$ on $g \in \mathcal{A}$.) Then define the *Toeplitz operator* $T_g : \mathcal{P} \to \mathcal{P}$ *associated to the symbol* $g \in \mathcal{A}$ by $T_g := P M_g$.

We let $\mathrm{End}(\mathcal{P})$ denote the vector space of all linear maps $\mathcal{P} \to \mathcal{P}$. The linear map $T : \mathcal{A} \to \mathrm{End}(\mathcal{P})$ defined by $T : g \mapsto T_g$ is called the *Toeplitz quantization.*

We also consider T_g as a densely defined linear operator defined in (but not on) the Hilbert space $\mathcal{H}$ as follows:

$$\mathcal{P} \xrightarrow{T_g} \mathcal{P} \subset \mathcal{H}.$$

Viewed this way the domain of T_g is given by $\mathrm{Dom}(T_g) = \mathcal{P}$.

So each Toeplitz operator in this setting is defined in the same dense subspace $\mathcal{P}$, which is invariant under the action of T_g. Consequently the composition of the Toeplitz operators T_g and T_h is an operator in $\mathrm{End}(\mathcal{P})$ though it need not be itself a Toeplitz operator. Whether a Toeplitz operator is bounded depends on more specific information about the symbol. Some light is already cast on these considerations by the next theorem, which is a standard, expected result for Toeplitz operators.

Theorem 1. *The Toeplitz quantization has the following properties:*

1. $T_1 = I_{\mathcal{P}}$*, the identity map of* $\mathcal{P}$.
2. $g \in \mathcal{P}$ *implies that* $T_g = M_g$.
3. *If* $g \in \mathcal{A}$ *and* $\psi \in \mathcal{P}$*, then* $T_g T_\psi = T_{\psi g}$.

Proof. We let $\phi \in \mathcal{P}$ be arbitrary throughout the proof.

For Part 1 we calculate $T_1\phi = PM_1\phi = P(\phi 1) = P(\phi) = \phi$, since P acts as the identity on $\mathcal{P}$.

For Part 2 we have $T_g\phi = PM_g\phi = P(\phi g) = \phi g = M_g\phi$, where we used that $\phi g \in \mathcal{P}$, which follows from ϕ, $g \in \mathcal{P}$.

For Part 3 we let $g \in \mathcal{A}$ and $\psi \in \mathcal{P}$. Then we calculate

$$\begin{aligned} T_g T_\psi \phi &= PM_g PM_\psi \phi = PM_g(P(\phi\psi)) = PM_g(\phi\psi) \\ &= P(\phi\psi g) = PM_{\psi g}\phi = T_{\psi g}\phi. \end{aligned}$$

Here we used $P(\phi\psi) = \phi\psi$, since $\mathcal{P}$ is an algebra and so $\phi\psi \in \mathcal{P}$ □

Part 1 shows that a Toeplitz operator can be bounded yet not compact. And Part 3 shows that the composition of two Toeplitz operators can itself be a Toeplitz operator, in which case the symbol of the composition is given by a simple formula involving the symbols of the factors, that is, the symbol calculus is rather straightforward in this case.

As promised Condition 8 was not used in the definition of a Toeplitz operator. Also Condition 8 implies that T_g is a symmetric operator if g is a self-adjoint element of $\mathcal{A}$, namely $g = g^*$. Whether this symmetric operator has any self-adjoint extensions and, in particular, whether it is essentially self-adjoint, are in general delicate questions that can be addressed with functional analysis. However, $T_1 = I_{\mathcal{P}}$ trivially has a self-adjoint extension, namely $I_{\mathcal{H}}$.

Theorem 2. *Each Toeplitz operator* T_g *is closable and its closure, denoted by* $\overline{T_g}$*, satisfies*

$$\overline{T_g} = (T_g)^{**} \subset (T_{g^*})^*$$

for every $g \in \mathcal{P}$.

Proof. By functional analysis an operator R is closable if and only if $\mathrm{Dom}\, R^*$ is dense. However $\mathrm{Dom}(T_g)^* \supset \mathrm{Dom}\, T_{g^*} = \mathcal{P}$ and $\mathcal{P}$ is dense in $\mathcal{H}$. So, $\mathrm{Dom}(T_g)^*$ is itself a dense subspace and therefore T_g is closable. Then by functional analysis $\overline{T_g} = (T_g)^{**}$. Finally, $(T_g)^{**} \subset (T_{g^*})^*$ follows by taking the adjoint of $T_{g^*} \subset (T_g)^*$. (See [3] for the functional analysis results.) □

Because this is a rather specific setting, one could expect a more explicit description of the closure of a Toeplitz operator. However, we leave this as a consideration for future research.

Theorem 3.2 in [6] that identifies the kernel of T does not go over to this setting; neither do its consequences. However, we can see that $g \in \ker T$ if $g \in \mathcal{P}$ and $M_g = 0$, the zero operator. Also, Condition 8 implies that the subspace $\ker T$ is closed under conjugation. We do have the following direct consequence of the definitions, although a more computable result clearly would be desirable.

Proposition 1. *$g \in \ker T$ if and only if $\mathrm{Ran}\, M_g \subset \ker P$.*

4. Creation and annihilation operators

We have creation and annihilation operators in this setting.

Definition 2. Let $g \in \mathcal{P}$ be given. Then the *creation operator* associated to g is defined to be

$$A^*(g) := T_g$$

and the *annihilation operator* associated to g is defined to be

$$A(g) := T_{g^*}.$$

These are reasonable definitions since they agree with the usual formulas for these operators as found, for example, in [5]. Notice that $g \mapsto A^*(g)$ is linear while $g \mapsto A(g)$ is anti-linear. Also $A^*(g) = T_g = M_g$ holds, because $g \in \mathcal{P}$. Since $A^*(1) = A(1) = T_1 = I_{\mathcal{P}}$, we see that $I_{\mathcal{P}}$ is both a creation and an annihilation operator. More generally, for any $g \in \mathcal{P} \cap \mathcal{P}^*$, one has $T_g = A^*(g) = A(g^*)$ and so T_g is both a creation and an annihilation operator.

One of the important contributions of Bargmann's seminal paper [1] is that it realizes the creation and annihilation operators introduced by Fock as adjoints of each other with respect to the inner product on the Hilbert space which is nowadays called the Segal–Bargmann space. In the present setting the creation operator $A^*(g)$ and the annihilation operator $A(g)$ also have this relation, modulo domain considerations, as we have already seen in Condition 8. Whether each is *exactly* the adjoint of the other is an open question if $\mathcal{P}$ has infinite dimension, but is true for finite-dimensional $\mathcal{P}$.

In this setting, unlike that in [6], there is only one definition possible for an anti-Wick quantization.

Definition 3. We say that T is an *anti-Wick quantization* if

$$T_{hg^*} = T_{g^*} T_h$$

for all $g, h \in \mathcal{P}$. Notice that hg^* makes sense since it is the left action of $h \in \mathcal{P}$ on an element of $\mathcal{A}$.

Notice that on the right side of this definition we have the product of an annihilation operator T_{g^*} to the left of a creation operator T_h. And so the right side is in what is known as *anti-Wick order.*

In [6] we defined T an *alternative anti-Wick quantization* if the equation $T_{g^*h} = T_{g^*} T_h$ is satisfied for all $g, h \in \mathcal{P}$. But in this setting the expression g^*h has not even been defined. So this concept does not apply here.

Theorem 3. *The Toeplitz quantization T is an anti-Wick quantization.*

Proof. Take $g, h \in \mathcal{P}$. Then $T_{hg^*} = T_{g^*} T_h$, where we have used Part 3 in Theorem 1. □

This proof replaces the rather lengthy proofs by explicit calculations given in [4] and [5].

Corollary 1. *If $\mathcal{A} = \mathcal{P}\mathcal{P}^*$, then one can write any Toeplitz operator as a finite sum of terms in anti-Wick order.*

Proof. Let $f \in \mathcal{A}$ be a symbol. The hypothesis means that we can write f as a finite sum, $f = \sum_k h_k g_k^*$ with $g_k, h_k \in \mathcal{P}$, where $h_k g_k^*$ is the left action of $h_k \in \mathcal{P}$ on an element of $\mathcal{A}$. So, $T_f = \sum_k T_{g_k^*} T_{h_k}$. □

To show more clearly that our definition of anti-Wick ordering compares well with the discussion of this topic in Theorem 8.2 in [2] we prove the next result. But first we need a definition that is a modification for this setting of a definition given in [6].

Definition 4. We say that $\mathcal{P}$ is $*$-friendly if $\mathcal{P}^*$ is an algebra and if its multiplication satisfies $(p_1 \cdots p_n)^* = p_n^* \cdots p_1^*$ for all $p_1, \ldots, p_n \in \mathcal{P}$.

One point of this definition is that we do not require $(p_1 \cdots p_n)^*$ to be an element in $\mathcal{P}$. If $\mathcal{A}$ is a $*$-algebra, then $\mathcal{P}$ is $*$-friendly where the multiplication on $\mathcal{P}^*$ is the restriction of that on $\mathcal{A}$.

The Toeplitz quantization is a linear map whose co-domain is an algebra and whose domain contains an algebra, namely $\mathcal{P}$. And in the $*$-friendly case its domain also contains the algebra $\mathcal{P}^*$.

Theorem 4. *Suppose that $g_1, \ldots, g_n, h_1, \ldots h_m \in \mathcal{P}$. Then*

1. $T_{g_1 \cdots g_n} = T_{g_n} \cdots T_{g_1}$.
2. $T_{h_1^* \cdots h_m^*} = T_{h_m^*} \cdots T_{h_1^*}$ *if $\mathcal{P}$ is a $*$-friendly.*
3. $T_{(g_1 \cdots g_n)(h_1^* \cdots h_m^*)} = T_{h_m^*} \cdots T_{h_1^*} T_{g_n} \cdots T_{g_1}$ *if $\mathcal{P}$ is $*$-friendly.*

Proof. For Part 1 we use induction. The case $n = 1$ is trivial, while the case $n = 2$ follows from Part 3 in Theorem 1. For $n \geq 3$ we have that

$$T_{g_1 g_2 \cdots g_n} = T_{g_1(g_2 \cdots g_n)} = T_{g_2 \cdots g_n} T_{g_1} = T_{g_n} \cdots T_{g_2} T_{g_1},$$

where we used Part 3 in Theorem 1 for the second equality and the induction hypothesis for $n-1$ for the third equality.

For the proof of Part 2 we take the notation T_f^* for any $f \in \mathcal{A}$ to mean the restriction of the adjoint $(T_f)^*$ of T_f to the algebra $\mathcal{P}$. So, $T_f^* = T_{f^*}$ follows form Condition 8. We then note that

$$T_{h_m^*} \cdots T_{h_1^*} = T_{h_m}^* \cdots T_{h_1}^* = (T_{h_1} \cdots T_{h_m})^* = (T_{h_m \cdots h_1})^* = T_{(h_m \cdots h_1)^*} = T_{h_1^* \cdots h_m^*}$$

where we used Part 1 in the third equality and that $\mathcal{P}$ is a $*$-friendly in the last equality.

For Part 3 we first remark that $(g_1 \cdots g_n)(h_1^* \cdots h_m^*)$ exists since it is the left action of the element $g_1 \cdots g_n \in \mathcal{P}$ on the element $h_1^* \cdots h_m^* \in \mathcal{P}^* \subset \mathcal{A}$. Then we have that

$$T_{h_m^*} \cdots T_{h_1^*} T_{g_n} \cdots T_{g_1} = T_{h_1^* \cdots h_m^*} T_{g_1 \cdots g_n} = T_{(g_1 \cdots g_n)(h_1^* \cdots h_m^*)}$$

by applying Parts 1 and 2 in the first equality and Part 3 of Theorem 1 in the second equality, using $g_1 \cdots g_n \in \mathcal{P}$. □

5. Canonical commutation relations

We now consider the canonical commutation relations which are satisfied by the creation and annihilation operators. However, our approach here is the opposite of the usual approach in which one starts with some deformation of the standard canonical commutation relations, and then one looks for representations of those relations by operators in some Hilbert space. Here we ask what are the appropriate canonical commutation relations that are associated with a given Toeplitz quantization. So, the operators acting in a Hilbert space are given first. This section only contains definitions and a discussion of them. It is basically the framework of a program for future research.

Definition 5. The subalgebra of $\mathrm{End}(\mathcal{P})$ generated by all the creation and annihilation operators is defined to be the algebra of canonical commutation relations and is denoted by $\mathcal{CCR}(P)$.

We define $\mathcal{F} = \mathbb{C}\{\mathcal{P} \cup \mathcal{P}^*\}$ to be the free algebra over $\mathbb{C}$ generated by the *set* $\mathcal{P} \cup \mathcal{P}^*$. Notice that $\mathbb{C}1 \subset \mathcal{P} \cap \mathcal{P}^*$. To avoid confusion, we will write the algebra generators of $\mathcal{F}$ as G_f for $f \in \mathcal{P} \cup \mathcal{P}^*$. So $\mathcal{F}$ is the complex vector space with a basis given by the monomials $G_{f_1} G_{f_2} \cdots G_{f_n}$ of degree n, where $f_j \in \mathcal{P} \cup \mathcal{P}^*$ for each j. We define the algebra morphism $\pi : \mathcal{F} \to \mathcal{CCR}(P)$ by $\pi(G_f) := T_f$ for all $f \in \mathcal{P} \cup \mathcal{P}^*$. Since the algebra $\mathcal{F}$ is free on the G_f's, this defines π uniquely. Also since the elements T_f for $f \in \mathcal{P} \cup \mathcal{P}^*$ are algebra generators for the algebra $\mathcal{CCR}(P)$, we see that π is an epimorphism. We define the *ideal of canonical commutation* *relations* in $\mathcal{F}$ to be $\mathcal{R} := \ker \pi$. Any minimal set of algebra generators of $\mathcal{R}$ is called a *set of canonical commutation relations.* Notice that such a set will not be unique in general.

The standard canonical quantum mechanical commutation relations, when written as ideal generators given by $a_j a_k^* - a_k^* a_j - \hbar \delta_{j,k} 1$, have the property that for $j \neq k$ they are homogeneous in the variables a_j and a_k^* and do not include any quantum effect due to Planck's constant $\hbar$. In this case they correspond to the commutativity of classical mechanical variables. However, for $j = k$ they are not homogeneous in the variables, and they do include $\hbar$. Moreover, in this case the classical relation is obtained by dropping the lower-order 'quantum correction'. These remarks motivate the following definition.

Definition 6. We say that a homogeneous element in $\mathcal{R} \subset \mathcal{F}$ is a classical relation and that a non-homogeneous element in $\mathcal{R}$ is a quantum relation.

Suppose $R \in \mathcal{R}$ is a non-zero relation. Then we can write R uniquely as $R = R_0 + R_1 + \cdots + R_n$, where each R_j is homogeneous with deg $R_j = j$ for each $j = 0, 1, \ldots, n$ and $R_n \neq 0$. Then we say that R_n is the classical relation associated to R.

Of course, R_n is actually a classical relation. Both of the cases $R_n \in \mathcal{R}$ and $R_n \notin \mathcal{R}$ can occur as the example before this definition shows. What we are doing intuitively to get the classical relation R_n from R is to discard the 'quantum corrections' $R_0, R_1, \ldots, R_{n-1}$ in R. We next define

$$\mathcal{R}_{cl} := \langle R_n \mid R_n \text{ is the classical relation associated to some } R \in \mathcal{R} \rangle,$$

where the brackets $\langle \cdot \rangle$ indicate that we are taking the two-sided ideal in $\mathcal{F}$ generated by the elements inside the brackets.

Definition 7. The dequantized algebra associated to $\mathcal{A}$ is defined to be

$$\mathcal{DQ} := \mathcal{F}/\mathcal{R}_{cl}.$$

Note that $\mathcal{DQ}$ need not be commutative. We can realize $\mathcal{DQ}$ as the case $\hbar = 0$ of a family of algebras parameterized by $\hbar \in \mathbb{C}$ and with $\hbar = 1$ corresponding to $\mathcal{CCR}(P)$. Based on this we can now define the associated *$\hbar$-deformed relations* to be

$$\mathcal{R}_\hbar := \langle \hbar^{n/2} R_0 + \hbar^{(n-1)/2} R_1 + \cdots + \hbar^{1/2} R_{n-1} + R_n \mid R \in \mathcal{R} \rangle \tag{1}$$

$$= \langle R_0 + \hbar^{-1/2} R_1 + \cdots + \hbar^{-(n-1)/2} R_{n-1} + \hbar^{-n/2} R_n \mid R \in \mathcal{R} \rangle, \tag{2}$$

using the notation $R = R_0 + R_1 + \cdots + R_n$ as given above. Next we define

$$\mathcal{CCR}_\hbar(P) := \mathcal{F}/\mathcal{R}_\hbar.$$

The second expression (2) has the virtue that the powers of $\hbar^{-1/2}$ are the degrees of homogeneity of the terms. On the other hand, in the first expression (1) each of the homogeneous terms has a coefficient giving its intuitively correct degree of 'quantumness'. The expression (1) also indicates formally what happens when one takes the limit $\hbar \to 0$. For $\hbar \neq 0$ the two expressions (1) and (2) are clearly equivalent, but for $\hbar = 0$ only the definition (1) makes sense. In physics one considers $\hbar > 0$ to be Planck's constant, but here we can take $\hbar \in \mathbb{C}$ to be arbitrary.

We have included $\hbar$ in part to emphasize that this theory has semi-classical behavior (more precisely, what happens to $\mathcal{CCR}_\hbar(P)$ when $\hbar$ tends to zero) as well as a classical counterpart $\mathcal{DQ}$ (that is, what happens when we put $\hbar$ equal to zero). However, the developments of the semi-classical theory and the classical counterpart theory remain for future research.

Also, it is important to remark that this theory includes both Planck's constant as well as a Hilbert space where creation and annihilation operators are defined. These are some of the important characteristics of a quantization relevant to physics.

The Toeplitz algebra, defined as the subalgebra of $\mathrm{End}(\mathcal{P})$ generated by the Toeplitz operators, is also a quantum algebra of interest in itself.

6. Concluding remarks

The point of this note is to develop much of the theory in [6] by starting from a different set of assumptions. The inference is that this theory is quite general and probably even more general than has been worked out so far. While non-trivial examples exist in [4] and [5], there remains more work to find other applications of this theory. Again, the absence of a measure in this approach distinguishes it sharply from other approaches, such as the coherent state quantization, and so one expects to find examples of this sort of Toeplitz quantization in settings where other approaches do not give results. I hope that this is not only useful in such mathematical physics contexts, but that applications of these ideas from mathematical physics will be useful in the study of the non-commutative 'spaces' of non-commutative geometry (such as quantum groups, among others) as well as of 'spaces' that are even more general. Also, several open problems were raised during the course of this short note. So this is very much a report of work in process.

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States in Deformation Quantization

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Abstract. We consider several tests to check whether a function defined on the phase space of a system represents a quantum state. Our criteria have been obtained from theorems holding for a density operator in the Hilbert space formulation of quantum mechanics. The tests are based on a notion of trace and follow from their Hilbert space counterparts through the Stratonovich–Weyl correspondence.

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1. Introduction

Physics is an experimental science. Thus its mathematical structure has to respect measurements, i.e., contain a class of physically important quantities – observables and predict results of a single measurement as well as a mean value of a series of observations.

There is no formal definition of an observable. We assume that in the phase space formulation of quantum mechanics [1–3] measurable quantities are represented by smooth real functions on a symplectic manifold, but other functions and even generalized functions can be considered. In the Hilbert space formulation of quantum theory the observables are identified with linear self-adjoint operators [4, 5].

For a given observable the result of a single observation or the average of a series of measurements depend on a state. Hence in the phase space quantum mechanics information about the state is contained in a linear functional satisfying some extra conditions. We will consider this problem in the next paragraph. In the Hilbert space version of quantum physics normalizable pure states are represented by vectors of the unit length from a Hilbert space **H** of the system. An arbitrary state belongs to a convex set spanned by the pure states [6].

The strict relationship between a set of observables and a space of states looks as follows. We choose an algebra $\mathbf{A}$ with involution $*$. The algebra $\mathbf{A}$ contains some subset of the set of observables. Quantum states are represented by positive linear functionals $f \in \mathbf{F}$ over the $*$ -algebra $\mathbf{A}$.

$$\forall\, A \in \mathbf{A} \quad f(A^{+}A) \geq 0. \tag{1}$$

Moreover, the states obey the normalization condition

$$f(\mathbf{1}) = 1.$$

By $\mathbf{1}$ we denote the unity of the algebra $\mathbf{A}$.

Thus the expected value of an observable $A \in \mathbf{A}$ in a state $f \in \mathbf{F}$ is equal to

$$\langle A \rangle := f(A).$$

Another approach to states is based on the following idea. We distinguish a special class of linear normalized positive functionals called pure states and then we build a convex set spanned by these special states.

Let us consider the Hilbert space formulation of quantum mechanics. The $*$ -algebra $\mathbf{A}$ in this case is the algebra $\mathbf{B}(\mathbf{H})$ of bounded linear operators whose domain is the whole Hilbert space $\mathbf{H}$. The functional is defined as

$$f(A) := \mathrm{Tr}(\hat{\varrho}\hat{A})\ \forall\, \hat{A} \in \mathbf{A}. \tag{2}$$

The symbol $\hat{\varrho}$ denotes the density operator.

The positivity condition (1) is hardly testable. The same about convexity of a set. Thus we propose several applicable criteria of checking if a given functional really represents a physical state. These criteria are divided in two groups: tests for density operators and for Wigner functions. Their complete list with proofs can be found in our paper [7]. In this contribution only the tests based on the notion of trace are presented.

A square matrix C of a dimension $\dim \mathbf{H} \times \dim \mathbf{H}$ is symbolized by $[\langle \varphi_i | C | \varphi_j \rangle]_1^{\dim \mathbf{H}}$ while an element of this matrix is represented as $\langle \varphi_i | C | \varphi_j \rangle$.

2. Hilbert space version of quantum mechanics

Let us consider a quantum system modeled on a separable Hilbert space $\mathbf{H}$. By $\{|\varphi_j\rangle\}_{j=1}^{\dim \mathbf{H}}$ we mean a complete set of orthonormal vectors in $\mathbf{H}$. We do not know the state vector of the system but only a probability of detecting the system in each of the states $|\varphi_j\rangle$.

As it was postulated by von Neumann [8], in such cases the state of the quantum system is characterized by a positive functional determined by a density operator.

Definition 1. The operator given by

$$\hat{\varrho} := \mathrm{u}-\lim_{n \to \dim \mathbf{H}} \sum_{j=1}^{n} p_j |\varphi_j\rangle\langle\varphi_j|, \quad \forall\, j \;\; p_j \geq 0 \;\; , \quad \sum_{j=1}^{\dim \mathbf{H}} p_j = 1$$

is called a **density operator**. Each number p_j, $j = 1, 2, \ldots, \dim \mathbf{H}$ is equal to the probability of observing the system in the state represented by the ket $|\varphi_j\rangle$. If one of these numbers is equal to 1 we say that the system is in a **pure state**. Otherwise the system is in a **mixed state**.

The symbol u$-$ denotes the uniform convergence of a sequence of operators. An equivalent formulation of Definition 1 is the following.

Definition 2. An operator $\hat{\varrho} : \mathbf{H} \to \mathbf{H}$ is a **density operator** if it is:

1. positive, i.e., $\langle\phi|\hat{\varrho}|\phi\rangle \geq 0 \; \forall \, |\phi\rangle \in \mathbf{H}$,
2. self-adjoint $\hat{\varrho}^+ = \hat{\varrho}$,
3. $\mathrm{Tr}\, \hat{\varrho} = 1$.

As it was mentioned in the Introduction, the functional action is determined by the formula (2). From Definitions 1 and 2 we can deduce several properties of the density operator.

- $\mathrm{Tr}\, \hat{\varrho}^2 \leq 1$. Thus the density operator is a Hilbert–Schmidt operator. Its Hilbert–Schmidt norm, given by
$$||\hat{\varrho}||_2 := \sqrt{\mathrm{Tr}(\hat{\varrho}^+ \hat{\varrho})},$$
is not greater than 1. Moreover, $||\hat{\varrho}||_2 = 1$ if and only if the density operator represents a pure state.
- As eigenvalues of the density operator $\hat{\varrho}$ are nonnegative, the density operator is a trace class operator and its trace norm
$$||\hat{\varrho}||_1 := \mathrm{Tr}\, \sqrt{\hat{\varrho}^+ \hat{\varrho}} = \mathrm{Tr}\, |\hat{\varrho}| = \mathrm{Tr}\, \hat{\varrho} = 1.$$
For each trace class operator $\hat{A}$ the following estimation holds
$$||\hat{A}|| \leq ||\hat{A}||_2 \leq ||\hat{A}||_1.$$
- The density operator is positive. Hence for every operator $\hat{A} \in \mathbf{B}(\mathbf{H})$ the mean value of the product $\hat{A}\hat{A}^+$ obeys
$$\langle \hat{A}\hat{A}^+ \rangle = \mathrm{Tr}(\hat{\varrho}\hat{A}\hat{A}^+) \geq 0.$$

Assume that a matrix representation $[\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle]_1^{\dim \mathbf{H}}$ of an operator $\hat{\varrho}$ is known. We intend to settle whether this matrix represents a physical state of a quantum system. We consider finite- and infinite-dimensional separable Hilbert spaces and discuss mixed as well as pure states.

2.1. A finite-dimensional Hilbert space

At the beginning we consider a finite-dimensional Hilbert space. In this case the matrix $[\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle]_1^{\dim \mathbf{H}}$ completely determines the operator $\hat{\varrho}$, which is defined on the whole space $\mathbf{H}$.

Applying elementary linear algebra we propose the following algorithm.

1. First we test if the matrix is symmetric
$$\forall \, i, j \quad \langle\varphi_i|\hat{\varrho}|\varphi_j\rangle = \overline{\langle\varphi_j|\hat{\varrho}|\varphi_i\rangle}.$$

2. If the answer is positive, we calculate its trace $\sum_{i=1}^{\dim \mathbf{H}} \langle \varphi_i | \hat{\varrho} | \varphi_i \rangle$.
3. If the trace is equal to 1 we go to the last step, in which we find the principal minors and decide whether the matrix is positive.

A symmetric matrix of trace 1 with all the principal minors nonnegative is a density matrix.

For a pure state this procedure becomes simpler. A matrix $[\langle \varphi_i | \hat{\varrho} | \varphi_j \rangle]_1^{\dim \mathbf{H}}$ represents a pure state if it is symmetric, its trace is equal to 1 and the square of it is the same matrix

$$\forall\, i,j \quad \sum_{k=1}^{\dim \mathbf{H}} \langle \varphi_i | \hat{\varrho} | \varphi_k \rangle \langle \varphi_k | \hat{\varrho} | \varphi_j \rangle = \langle \varphi_i | \hat{\varrho} | \varphi_j \rangle .$$

2.2. An infinite-dimensional Hilbert space

The case of an infinite-dimensional space is more complicated. It may happen that in a given orthonormal basis $\{|\varphi_j\rangle\}_{j=1}^{\infty}$ the matrix $\left[\langle \varphi_i | \hat{A} | \varphi_j \rangle\right]_1^{\infty}$ of an operator $\hat{A}$ exists but does not uniquely characterize this operator.

Thus to know whether a matrix $[\langle \varphi_i | \hat{\varrho} | \varphi_j \rangle]_1^{\infty}$ can represent a density operator, we propose to check first if $\hat{\varrho}$ is a Hilbert–Schmidt operator, i.e.,

$$\sum_{i,j=1}^{\infty} |\langle \varphi_i | \hat{\varrho} | \varphi_j \rangle|^2 < \infty .$$

Every Hilbert–Schmidt operator $\hat{A}$ is bounded and is defined on the whole space $\mathbf{H}$. Moreover, the matrix $\left[\langle \varphi_i | \hat{A} | \varphi_j \rangle\right]_1^{\infty}$ completely characterizes the operator $\hat{A}$. In addition, if the matrix $\left[\langle \varphi_i | \hat{A} | \varphi_j \rangle\right]_1^{\infty}$ is symmetric, we conclude that the operator $\hat{A}$ is self adjoint. Hence all of our criteria will start from checking, if a given operator is of the Hilbert–Schmidt class and if it is symmetric.

In the next step we have to test if the matrix $[\langle \varphi_i | \hat{\varrho} | \varphi_j \rangle]_1^{\infty}$ represents a positive operator. This operation is the most complicated and can be done in many ways. Finally one calculates the trace of the matrix $[\langle \varphi_i | \hat{\varrho} | \varphi_j \rangle]_1^{\infty}$.

Several realizations of tests are formulated below. We write down only hints at which proofs of these statements are based. More detailed explanation can be found in [7].

Theorem 1. *A matrix $[\langle \varphi_i | \hat{\varrho} | \varphi_j \rangle]_1^{\infty}$ represents a quantum state iff:*

1. $\sum_{i,j=1}^{\infty} |\langle \varphi_i | \hat{\varrho} | \varphi_j \rangle|^2 \leq 1$,
2. $\langle \varphi_i | \hat{\varrho} | \varphi_j \rangle = \overline{\langle \varphi_j | \hat{\varrho} | \varphi_i \rangle}\ \forall\ 1 \leq i,j < \infty$,
3. $$\frac{1}{2} \sum_{i=1}^{\infty} \langle \varphi_i | \hat{\varrho}^2 | \varphi_i \rangle + \sum_{l=2}^{\infty} \frac{(-1)^{l+1}}{l!} \frac{(2l-3)!!}{2^l} \sum_{r=0}^{l-1} (-1)^r \binom{l}{r} \times \sum_{i=1}^{\infty} \langle \varphi_i | \hat{\varrho}^{2(l-r)} | \varphi_i \rangle = 1, \tag{3}$$
4. $\sum_{i=1}^{\infty} \langle \varphi_i | \hat{\varrho} | \varphi_i \rangle = 1$.

Formula (3) comes from the observation that for a density operator the square root $\sqrt{\hat{\varrho}^2}$ defined by the series must be equal to the operator $\hat{\varrho}$. Thus $\mathrm{Tr}\, \sqrt{\hat{\rho}^2} = 1$.

Theorem 2. *A matrix* $[\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle]_1^\infty$ *represents a quantum state iff:*

1. $\sum_{i,j=1}^\infty |\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle|^2 \leq 1$,
2. $\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle = \overline{\langle\varphi_j|\hat{\varrho}|\varphi_i\rangle}\ \forall\ 1 \leq i,j < \infty$,
3. *for every natural number* n *the sum is nonnegative*
$$\sum_{k=0}^n (-1)^k \binom{n}{k} \sum_{i=1}^\infty \langle\varphi_i|\hat{\varrho}^{k+1}|\varphi_i\rangle \geq 0, \tag{4}$$
4. $\sum_{i=1}^\infty \langle\varphi_i|\hat{\varrho}|\varphi_i\rangle = 1$.

In the conditions 3 and 4 we check if all of eigenvalues of the matrix $[\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle]_1^\infty$ are nonnegative and they do not exceed 1.

Theorem 3. *A matrix* $[\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle]_1^\infty$ *represents a quantum state iff:*

1. $\sum_{i,j=1}^\infty |\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle|^2 \leq 1$,
2. $\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle = \overline{\langle\varphi_j|\hat{\varrho}|\varphi_i\rangle}\ \forall\ 1 \leq i,j < \infty$,
3. *the limit*
$$\lim_{n\to\infty} \sum_{k=0}^n (-1)^k \binom{n}{k} \sum_{i=1}^\infty \langle\varphi_i|\hat{\varrho}^{k+1}|\varphi_i\rangle = 0,$$
4. $\sum_{i=1}^\infty \langle\varphi_i|\hat{\varrho}|\varphi_i\rangle = 1$.

In fact this theorem is a consequence of Theorem 2 because for nonnegative eigenvalues $e_1, e_2, \ldots$ of the matrix $[\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle]_1^\infty$ the sum $\sum_{i=1}^\infty e_i(1-e_i)^n$ tends to 0 as $n \to \infty$.

For pure states the testing procedure can be simplified.

Theorem 4. *A matrix* $[\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle]_1^\infty$ *represents a pure quantum state iff:*

1. $\sum_{i,j=1}^\infty |\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle|^2 = 1$,
2. $\langle\varphi_i|\hat{\varrho}|\varphi_j\rangle = \overline{\langle\varphi_j|\hat{\varrho}|\varphi_i\rangle}\ \forall\ 1 \leq i,j < \infty$,
3. $\sum_{j=1}^\infty \langle\varphi_i|\hat{\varrho}|\varphi_j\rangle\langle\varphi_j|\hat{\varrho}|\varphi_k\rangle = \langle\varphi_i|\hat{\varrho}|\varphi_k\rangle$,
4. $\sum_{i=1}^\infty \langle\varphi_i|\hat{\varrho}|\varphi_i\rangle = 1$.

This theorem states that the density operator of a pure state must be a projective operator.

3. Phase space version of quantum mechanics

When the phase space formulation of quantum mechanics is considered, two fundamental elements: a phase space and a $*$-product must be taken into account. We restrict ourselves to problems, in which phase spaces are differentiable symplectic manifolds.

On every symplectic manifold $(\mathbf{M}, \omega)$ there exists a nontrivial $*$-product. We assume that the $*$-product is local and in its differential form is of the Weyl type.

Definition 3. An **observable** on a phase space $(\mathbf{M},\omega)$ is any smooth real function on $\mathbf{M}$ being a formal series in the Planck constant $\hbar$

$$C^{\infty}(\mathbf{M})[[\hbar]] \ni A(q^1,\ldots,q^{2n}) = \sum_{i=0}^{\infty} \hbar^i A_i(q^1,\ldots,q^{2n}). \tag{5}$$

As it was explained in the Introduction, construction of the space of states is based on some algebra with an involution. This algebra contains a subset of the set of observables. In deformation quantization the $*$-algebra $\mathbf{A}$ consists of all smooth functions on $(\mathbf{M},\omega)$, which are formal series in $\hbar$ and have compact supports. The involution '$*$' is realized by the complex conjugation. The product in the algebra $\mathbf{A}$ is a Weyl type $*$-product.

According to the general definition, quantum states are positive linear functionals over the algebra $\mathbf{A}$ satisfying the normalization condition. Every such functional f can be written in the following form

$$f\Big(A(q^1,\ldots,q^{2n})\Big) = \int_{\mathbf{M}} A(q^1,\ldots,q^{2n}) * W(q^1,\ldots,q^{2n}) t(q^1,\ldots,q^{2n})\omega^n \tag{6}$$
$$= \int_{\mathbf{M}} W(q^1,\ldots,q^{2n}) * A(q^1,\ldots,q^{2n}) t(q^1,\ldots,q^{2n})\omega^n, \ \forall\, A(q^1,\ldots,q^{2n}) \in \mathbf{A}.$$

A real function

$$C^{\infty}(\mathbf{M})[[\hbar]] \ni t(q^1,\ldots,q^{2n}) = \sum_{i=0}^{\infty} \hbar^i t_i(q^1,\ldots,q^{2n})$$

is called a trace density. The trace density ensures that the integral (6) possesses a trace property

$$\int_{\mathbf{M}} A(q^1,\ldots,q^{2n}) * B(q^1,\ldots,q^{2n}) t(q^1,\ldots,q^{2n})\omega^n$$
$$= \int_{\mathbf{M}} B(q^1,\ldots,q^{2n}) * A(q^1,\ldots,q^{2n}) t(q^1,\ldots,q^{2n})\omega^n,$$
$$\forall\, A(q^1,\ldots,q^{2n}),\ B(q^1,\ldots,q^{2n}) \in \mathbf{A}.$$

The trace density is determined by a symplectic curvature tensor and its derivatives [9, 10].

A function $W(q^1,\ldots,q^{2n})$ contains information about the state and is called a **Wigner function**. We will restrict ourselves to the functions $W(q^1,\ldots,q^{2n})$ which are $*$- square integrable, i.e.,

$$\int_{\mathbf{M}} W(q^1,\ldots,q^{2n}) * W(q^1,\ldots,q^{2n}) t(q^1,\ldots,q^{2n})\omega^n < \infty$$

for each fixed positive value of the deformation parameter $\hbar$.

The integral

$$\frac{1}{(2\pi\hbar)^n} \int_{\mathbf{M}} A(q^1,\ldots,q^{2n}) t(q^1,\ldots,q^{2n})\omega^n \tag{7}$$

is often called a **trace** as is a classical counterpart of the trace of operator.

3.1. The Stratonovich–Weyl correspondence

Considerations from the current section are based on belief that the Hilbert space formulation of quantum mechanics and its phase space version are equivalent. This equivalence is expressed by the Stratonovich–Weyl correspondence $SW : \mathbf{A_H} \to \mathbf{A}$ between an algebra of operators $\mathbf{A_H}$ and an algebra of functions $\mathbf{A}$. Although a general form of this relationship is not known, it should satisfy a few natural requirements. First of all the SW mapping is one to one. The choice of the algebra of operators $\mathbf{A_H}$ determines the choice of the algebra of functions $\mathbf{A}$. On the other hand we know, that this choice is not unique and for different algebras $\mathbf{A_H}$ or equivalently $\mathbf{A}$ we obtain the same quantum mechanics.

Next, the SW correspondence is linear. Moreover, $SW(\hat{A}^{+}) = \overline{SW(\hat{A})}$. The image $SW(\hat{1}) = 1$, i.e., the constant function equal 1 on the whole symplectic manifold $\mathbf{M}$ represents the identity operator. If an operator $\hat{A}$ is self adjoint then $SW(\hat{A})$ is a real function. Finally, the $\cdot$ -product of operators is represented by the $*$-multiplication of functions,

$$SW(\hat{A} \cdot \hat{B}) = SW(\hat{A}) * SW(\hat{B})$$

and the trace of an operator is equal to

$$\operatorname{Tr} \hat{A} = \frac{1}{(2\pi\hbar)^n} \int_{\mathbf{M}} SW(\hat{A})(q^1, \ldots, q^{2n}) t(q^1, \ldots, q^{2n}) \omega^n.$$

3.2. Wigner function

The Stratonovich–Weyl correspondence establishes a correspondence between a density operator $\hat{\varrho}$ and a Wigner function $W(q^1, \ldots, q^{2n})$

$$SW\left(\frac{1}{(2\pi\hbar)^n}\hat{\varrho}\right) = W(q^1, \ldots, q^{2n}).$$

Applying the Stratonovich–Weyl mapping to a density operator we find properties of the Wigner function on an arbitrary symplectic manifold.

Integration of the Wigner function yields

$$\int_{\mathbf{M}} W(q^1, \ldots, q^{2n}) t(q^1, \ldots, q^{2n}) \omega^n = 1.$$

This result is an immediate consequence of the fact that $\operatorname{Tr} \hat{\varrho} = 1$. Moreover, every Wigner function, as a counterpart of the self-adjoint operator, is real. Since a density operator is of a Hilbert–Schmidt type, the integral satisfies the estimation

$$\int_{\mathbf{M}} W^{*2}(q^1, \ldots, q^{2n})\, t(q^1, \ldots, q^{2n}) \omega^n \le \frac{1}{(2\pi\hbar)^n}.$$

The equality holds only for pure states.

Applying the Stratonovich–Weyl correspondence we present criteria to test, if a given function $W(q^1, \ldots, q^{2n})$ on the phase space of a system represents a physical state. It is easily to see that these criteria are phase space counterparts of Theorems 1–4.

Theorem 5. *A function $W(q^1,\dots,q^{2n})$ defined on a symplectic manifold $(\mathbf{M},\omega)$ is a Wigner function iff:*

1. $\int_{\mathbf{M}} W^{*2}(q^1,\dots,q^{2n})\, t(q^1,\dots,q^{2n})\omega^n \le \frac{1}{(2\pi\hbar)^n}$,
2. *the function is real,*
3. $\frac{1}{2}\int_{\mathbf{M}} W^{*2}(q^1,\dots,q^{2n})\, t(q^1,\dots,q^{2n})\omega^n + \sum_{l=2}^{\infty} \frac{(-1)^{l+1}}{l!}\frac{(2l-3)!!}{2^l}$
   $\times \sum_{r=0}^{l-1}(-1)^r\binom{l}{r}(2\pi\hbar)^{2n(l-r-1)}\int_{\mathbf{M}} W^{*2(l-r)}(q^1,\dots,q^{2n})\, t(q^1,\dots,q^{2n})\omega^n$
   $= \frac{1}{(2\pi\hbar)^n}$, (8)
4. $\int_{\mathbf{M}} W(q^1,\dots,q^{2n})\, t(q^1,\dots,q^{2n})\omega^n = 1$.

Theorem 6. *A function $W(q^1,\dots,q^{2n})$ defined on a symplectic manifold $(\mathbf{M},\omega)$ is a Wigner function of a quantum state iff:*

1. $\int_{\mathbf{M}} W^{*2}(q^1,\dots,q^{2n})\, t(q^1,\dots,q^{2n})\omega^n \le \frac{1}{(2\pi\hbar)^n}$,
2. *the function is real,*
3. *for every natural number m the sum is nonnegative*
   $\sum_{k=0}^{m}(-1)^k\binom{m}{k}(2\pi\hbar)^{nk}\int_{\mathbf{M}} W^{*(k+1)}(q^1,\dots,q^{2n})\, t(q^1,\dots,q^{2n})\omega^n \ge 0,$ (9)
4. $\int_{\mathbf{M}} W(q^1,\dots,q^{2n})\, t(q^1,\dots,q^{2n})\omega^n = 1$.

Theorem 7. *A function $W(q^1,\dots,q^{2n})$ defined on a symplectic manifold $(\mathbf{M},\omega)$ represents a quantum state iff:*

1. $\int_{\mathbf{M}} W^{*2}(q^1,\dots,q^{2n})\, t(q^1,\dots,q^{2n})\omega^n \le \frac{1}{(2\pi\hbar)^n}$,
2. *the function is real,*
3. $\lim_{m\to\infty} \sum_{k=0}^{m}(-1)^k\binom{m}{k}(2\pi\hbar)^{nk}\int_{\mathbf{M}} W^{*(k+1)}(q^1,\dots,q^{2n})$
   $\times t(q^1,\dots,q^{2n})\omega^n = 0,$ (10)
4. $\int_{\mathbf{M}} W(q^1,\dots,q^{2n})\, t(q^1,\dots,q^{2n})\omega^n = 1$.

An identification method of pure states is based on the following statement.

Theorem 8. *A function $W(q^1,\dots,q^{2n})$ defined on a symplectic manifold $(\mathbf{M},\omega)$ represents a pure quantum state iff:*

1. $\int_{\mathbf{M}} W^{*2}(q^1,\dots,q^{2n})\, t(q^1,\dots,q^{2n})\omega^n = \frac{1}{(2\pi\hbar)^n}$,
2. *the function is real,*
3. $W^{*2}(q^1,\dots,q^{2n}) = \frac{1}{(2\pi\hbar)^n}W(q^1,\dots,q^{2n})$
4. $\int_{\mathbf{M}} W(q^1,\dots,q^{2n})\, t(q^1,\dots,q^{2n})\omega^n = 1$.

As an illustration of the presented criteria we examine a function considered by Tatarskij [1].

The function

$$W(p,q) = \frac{2}{3}W_0(p,q) + \frac{2}{3}W_1(p,q) - \frac{1}{3}W_2(p,q) \tag{11}$$

is defined on the phase space $\mathbb{R}^2$. By $W_0(p,q)$, $W_1(p,q)$ and $W_2(p,q)$ we denote Wigner functions of mutually orthogonal states, i.e.,

$$\int_{\mathbb{R}^2} W_i(p,q) * W_j(p,q)dpdq = \frac{1}{2\pi\hbar}\delta_{ij}.$$

As it can be seen from (11), the function $W(p,q)$ is not a Wigner function, because one of its eigenvalues is negative. The mth star power of this function is equal

$$W^{*m}(p,q) = \frac{1}{(2\pi\hbar)^{m-1}}\left(\left(\frac{2}{3}\right)^m W_0(p,q) + \left(\frac{2}{3}\right)^m W_1(p,q) + \left(-\frac{1}{3}\right)^m W_2(p,q)\right).$$

The function $W(p,q)$ is real and the integral $\int_{\mathbb{R}^2} W(p,q)dpdq = 1$. Moreover, $\int_{\mathbb{R}^2} W^{*2}(p,q)dpdq = \frac{1}{2\pi\hbar}$ so the function $W(p,q)$ satisfies the conditions (i), (ii) and (iv) of Theorems 5–7.

However, for the function (11) the sum (8) is equal to $\frac{5}{3}\cdot\frac{1}{2\pi\hbar} \quad \left(\neq \frac{1}{2\pi\hbar}\right)$ so this function is not a Wigner function.

Applying Theorem 6 we see that the sum (9) is equal to $2\pi\hbar$ for $m=0$, 0 for $m=1$ and $-\frac{4}{9}\cdot 2\pi\hbar \ \ (<0)$ for $m=2$. Therefore after taking three steps we conclude that $W(p,q)$ does not represent any state.

The limit (10) is equal to $-\infty \ \ (\neq 0)$ so the tested function obviously cannot be a Wigner function.

4. Conclusions

As criteria considered in our contribution require calculation of arbitrary powers of density matrices or arbitrary $*$-powers of Wigner functions, there might be some doubts about usefulness of this approach. However, it seems that at this moment there is no other constructive method of recognition of physical states. Moreover, tests of positivity are quite general so the presented results can be applied in different problems requiring analysis of positivity.

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n-ary Star Product: Construction and Integral Representation

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Abstract. This paper addresses a construction of an n-ary star product. Relevant identities are given. Besides, the formalism is illustrated by a computation of eigenvalues and eigenfunctions for a physical system of coupled oscillators in an n-dimensional phase space.

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1. Introduction

Several works were devoted to generalizations of Lie algebras to various types of n-ary algebras. To cite a few, see the works by Filippov, Hanlon, Vinogradov, Takhtajan and collaborators in [1–3]. In the same time, and intended to physical applications, the new algebraic structures were considered in the case of the algebra $C^\infty(M)$ of functions on a C^∞-manifold M, under the assumption that the operation is a derivation of each entry separately. In this way one got the Nambu–Poisson brackets, see, e.g., [4]. The same versatility was observed for generalized Poisson brackets in [5] (and references therein) providing unexpected algebraic structures on vector fields, which played an essential role in the construction of universal enveloping algebras of Filippov algebras (n-Lie algebras). See, for instance, [6] and references therein. For many other applications, especially to theoretical physics, see a nice and interesting survey of n-ary analogues of Lie algebras written by Azcárraga and Izquierdo [7].

This work intends to provide a construction of an n-ary star product, to investigate some identities related to it, and to give a concrete illustration on a physical system of coupled oscillators in an n-dimensional phase space.

The paper is organized as follows. In Section 2, we focus on the study of a 3-ary star product in a 3-dimensional Euclidean space. We prove that the 3-ary star product is distributive, associative and satisfies the Jacobi identity. We also construct its integral representation which should likely allow to establish new classes of solvable actions in the context of field theories. Section 3 is devoted to the generalization of this star product in higher dimension, i.e., for $n > 3$. In Section 4 we provide a simple application of such a star product on a physical system of coupled oscillators for which the eignenvalues and eigenfunctions are explicitly computed. In Section 5, we give some concluding remarks.

2. 3-ary star product

2.1. Definition and properties

We start by the following definition:

Definition 1. Consider $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Let σ_c be the cyclic permutation over the set $\{1, 2, 3\}$ such that $\sigma_c(1) = 2$, $\sigma_c(2) = 3$, $\sigma_c(3) = 1$, and $\mathcal{A} = \Big(\mathcal{S}\Big(\mathbb{R}^3\Big), \star\Big)$ be the Schwartz space of (smooth, rapidly decreasing, together with all their derivatives, faster than the reciprocal of any polynomial at ∞) real-valued functions on $\mathbb{R}^3$, endowed with a 3-ary star product defined at a point x as follows:

$$\Big(f \overset{g}{\star} h\Big)(x) := \mathbf{m}\Big[e^{\mathcal{P}(\theta_1,\theta_2,\theta_3)}(f \otimes g \otimes h)(x)\Big], \tag{1}$$

where

$$\mathcal{P}(\theta_1, \theta_2, \theta_3) = \sum_{k=1}^{3} \frac{i\theta_k}{2}\Big(\partial_k \otimes \partial_{\sigma_c(k)} \otimes \partial_{\sigma_c\circ\sigma_c(k)} - \partial_k \otimes \partial_{\sigma_c\circ\sigma_c(k)} \otimes \partial_{\sigma_c(k)}\Big), \tag{2}$$

$\mathbf{m}(f \otimes g \otimes h) = fgh$; the parameters θ_i, $i = 1, 2, 3$, are real numbers.

The conjugate of $\Big(f \overset{g}{\star} h\Big)(x)$, denoted by $\overline{\Big(f \overset{g}{\star} h\Big)(x)}$ is

$$\overline{\Big(f \overset{g}{\star} h\Big)(x)} := \mathbf{m}\Big[e^{-\mathcal{P}(\theta_1,\theta_2,\theta_3)}(f \otimes g \otimes h)(x)\Big]. \tag{3}$$

Note that, if the space $\mathbb{R}^3$ is the space of position coordinates, then the parameters θ_i must be of order of magnitude of the Planck volume, i.e., $[\theta_i] = [L^3]$. There exist several constructions of n-ary star products in the literature. For instance in [8, see Remark (1.1)], where a more general star product with tensor $\Theta^{\mu_1\mu_2\cdots\mu_n}$ is examined. In (2), we adopt a specific choice of Θ motivated by the fact that the resulting n-ary product has nice properties like the associativity, the distributivity, a consistent Jacobi identity, and so on. Moreover, it well behaves in concrete applications like for the example of coupled oscillators exhibited in this work.

Let $f_i, g_i, h_i \in \mathcal{A}$, $i = 1, 2, 3$. The following properties are satisfied for the defined 3-ary star product:

Property 2 (Distributivity). The distributivity of the 3-ary star product (1) is given by the following three relations:

$$(f_1 + f_2) \overset{g_1}{\star} h_1 = f_1 \overset{g_1}{\star} h_1 + f_2 \overset{g_1}{\star} h_1, \tag{4}$$

$$f_1 \overset{(g_1+g_2)}{\star} h_1 = f_1 \overset{g_1}{\star} h_1 + f_1 \overset{g_2}{\star} h_1, \tag{5}$$

$$f_1 \overset{g_1}{\star} (h_1 + h_2) = f_1 \overset{g_1}{\star} h_1 + f_1 \overset{g_1}{\star} h_2. \tag{6}$$

Property 3 (Associativity). The associativity of the 3-ary star product (1) can be defined as:

$$(f_1 \overset{g_1}{\star} h_1) \overset{g_2}{\star} h_2 = f_1 \overset{g_1}{\star} (h_1 \overset{g_2}{\star} h_2). \tag{7}$$

Besides, it appears natural to define a 3-ary star bracket as below:

Definition 4 (3-ary star bracket).

$$\{f, h\}_{\star,g} := f \overset{g}{\star} h - h \overset{g}{\star} f \tag{8}$$

with the following properties:

Property 5 (Skew-symmetry).

$$\{f, h\}_{\star,g} = -\{h, f\}_{\star,g} \tag{9}$$

Property 6 (Jacobi identity).

$$\begin{aligned}\{l, \{f, h\}_{\star,g_1}\}_{\star,g_2} &+ \{l, \{f, h\}_{\star,g_2}\}_{\star,g_1} + \{h, \{l, f\}_{\star,g_1}\}_{\star,g_2} \\ &+ \{h, \{l, f\}_{\star,g_2}\}_{\star,g_1} + \{f, \{h, l\}_{\star,g_1}\}_{\star,g_2} + \{f, \{h, l\}_{\star,g_2}\}_{\star,g_1} = 0.\end{aligned} \tag{10}$$

The proof of the Jacobi identity stems from the relation

$$\{l, \{f, h\}_{\star,g_1}\}_{\star,g_2} = (l \overset{g_2}{\star} f) \overset{g_1}{\star} h - (l \overset{g_2}{\star} h) \overset{g_1}{\star} f - (f \overset{g_1}{\star} h) \overset{g_2}{\star} l + (h \overset{g_1}{\star} f) \overset{g_2}{\star} l. \tag{11}$$

A thorough analysis of the defined star product and bracket properties is not included here and it will be in the core of a forthcoming work. Let us now sketch some interesting computations with this star product in the Euclidean coordinates. Indeed, the star composition of functions in $\mathcal{A}$ with coordinate functions gives rise to the following results:

Proposition 1 (3-ary star product of two functions in $\mathcal{A}$ and one coordinate function).

$$x_k \overset{f}{\star} g = x_k f g + \frac{i\theta_k}{2}\Big(\partial_{\sigma_c(k)} f \partial_{\sigma_c \circ \sigma_c(k)} g - \partial_{\sigma_c \circ \sigma_c(k)} f \partial_{\sigma_c(k)} g\Big) \tag{12}$$

$$g \overset{x_k}{\star} f = x_k f g - \frac{i\theta_{\sigma_c(k)}}{2} \partial_{\sigma_c(k)} g \partial_{\sigma_c \circ \sigma_c(k)} f + \frac{i\theta_{\sigma_c \circ \sigma_c(k)}}{2} \partial_{\sigma_c \circ \sigma_c(k)} g \partial_{\sigma_c(k)} f \tag{13}$$

$$f \overset{g}{\star} x_k = x_k f g + \frac{i\theta_{\sigma_c(k)}}{2} \partial_{\sigma_c(k)} f \partial_{\sigma_c \circ \sigma_c(k)} g - \frac{i\theta_{\sigma_c \circ \sigma_c(k)}}{2} \partial_{\sigma_c \circ \sigma_c(k)} f \partial_{\sigma_c(k)} g. \tag{14}$$

Proposition 2 (3-ary star product of one function in $\mathcal{A}$ and two coordinate functions).

$$x_k \overset{x_{\sigma_c(k)}}{\star} f = x_k x_{\sigma_c(k)} f + \frac{i\theta_k}{2}\partial_{\sigma_c\circ\sigma_c(k)} f \tag{15}$$

$$x_k \overset{f}{\star} x_{\sigma_c(k)} = x_k x_{\sigma_c(k)} f - \frac{i\theta_k}{2}\partial_{\sigma_c\circ\sigma_c(k)} f \tag{16}$$

$$x_k \overset{x_{\sigma_c\circ\sigma_c(k)}}{\star} f = x_k x_{\sigma_c\circ\sigma_c(k)} f - \frac{i\theta_k}{2}\partial_{\sigma_c(k)} f \tag{17}$$

$$x_k \overset{f}{\star} x_{\sigma_c\circ\sigma_c(k)} = x_k x_{\sigma_c\circ\sigma_c(k)} f + \frac{i\theta_k}{2}\partial_{\sigma_c(k)} f. \tag{18}$$

Therefore the following interesting results hold for a 3-ary star product of any function $f \in \mathcal{A}$ with two coordinate functions:

Proposition 3 (3-ary star product complex conjugation). *Provided* (1)*, we have,* $\forall f \in \mathcal{A}$,

$$\overline{x_k \overset{x_{\sigma_c(k)}}{\star} f} = x_k \overset{f}{\star} x_{\sigma_c(k)}, \quad \overline{x_k \overset{x_{\sigma_c\circ\sigma_c(k)}}{\star} f} = x_k \overset{f}{\star} x_{\sigma_c\circ\sigma_c(k)}. \tag{19}$$

In opposite, when any two functions $f, g \in \mathcal{A}$ enter in the 3-ary star product with a unique coordinate function, the star non commutativity is clearly expressed, i.e.,

$$x_k \overset{g}{\star} f \neq f \overset{g}{\star} x_k, \quad x_k \overset{g}{\star} f \neq f \overset{g}{\star} x_k, \quad g \overset{x_k}{\star} f \neq f \overset{x_k}{\star} g.$$

Furthermore, introducing complex variables a_{kl} and their conjugate $\bar{a}_{kl}$ by

$$a_{kl} = \frac{x_k + ix_l}{\sqrt{2}}, \quad \bar{a}_{kl} = \frac{x_k - ix_l}{\sqrt{2}}, \quad k, l = 1, 2, 3, \ l \neq k \tag{20}$$

and using the equations (12), (13) and (14), we establish the relations given in the three next propositions:

Proposition 4 (3-ary star product of two functions in $\mathcal{A}$ and one complex coordinate function).

$$\begin{aligned} a_{ij} \overset{f}{\star} g = a_{ij} f g &+ \frac{i\theta_i}{4}\Big(\partial_{\sigma_c(i)} f \partial_{\sigma_c^2(i)} g - \partial_{\sigma_c^2(i)} f \partial_{\sigma_c(i)} g\Big) \\ &- \frac{\theta_j}{4}\Big(\partial_{\sigma_c(j)} f \partial_{\sigma_c^2(j)} g - \partial_{\sigma_c^2(j)} f \partial_{\sigma_c(j)} g\Big) \end{aligned} \tag{21}$$

$$\begin{aligned} \bar{a}_{ij} \overset{f}{\star} g = \bar{a}_{ij} f g &+ \frac{i\theta_i}{4}\Big(\partial_{\sigma_c(i)} f \partial_{\sigma_c^2(i)} g - \partial_{\sigma_c^2(i)} f \partial_{\sigma_c(i)} g\Big) \\ &+ \frac{\theta_j}{4}\Big(\partial_{\sigma_c(j)} f \partial_{\sigma_c^2(j)} g - \partial_{\sigma_c^2(j)} f \partial_{\sigma_c(j)} g\Big) \end{aligned} \tag{22}$$

Proposition 5 (3-ary star product of two functions in $\mathcal{A}$ and one complex coordinate function).

$$
\begin{aligned}
g \overset{f}{\star} a_{ij} &= a_{ij} f g + \frac{i\theta_{\sigma_c(i)}}{4}\partial_{\sigma_c(i)} g \partial_{\sigma_c^2(i)} f - \frac{i\theta_{\sigma_c^2(i)}}{4}\partial_{\sigma_c^2(i)} g \partial_{\sigma_c(i)} f \\
&\quad - \frac{\theta_{\sigma_c(j)}}{4}\partial_{\sigma_c(j)} g \partial_{\sigma_c^2(j)} f + \frac{\theta_{\sigma_c^2(j)}}{4}\partial_{\sigma_c^2(j)} g \partial_{\sigma_c(j)} f
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
g \overset{f}{\star} \bar{a}_{ij} &= \bar{a}_{ij} f g + \frac{i\theta_{\sigma_c(i)}}{4}\partial_{\sigma_c(i)} f \partial_{\sigma_c^2(i)} f - \frac{i\theta_{\sigma_c^2(i)}}{4}\partial_{\sigma_c^2(i)} g \partial_{\sigma_c(i)} f \\
&\quad + \frac{\theta_{\sigma_c(j)}}{4}\partial_{\sigma_c(j)} g \partial_{\sigma_c^2(j)} f - \frac{\theta_{\sigma_c^2(j)}}{4}\partial_{\sigma_c^2(j)} g \partial_{\sigma_c(j)} f.
\end{aligned} \tag{24}
$$

Proposition 6 (3-ary star product of two functions in $\mathcal{A}$ and one complex coordinate function).

$$
\begin{aligned}
f \overset{a_{ij}}{\star} g &= a_{ij} f g - \frac{i\theta_{\sigma_c(i)}}{4}\partial_{\sigma_c(i)} f \partial_{\sigma_c^2(i)} g + \frac{i\theta_{\sigma_c^2(i)}}{4}\partial_{\sigma_c^2(i)} f \partial_{\sigma_c(i)} g \\
&\quad + \frac{\theta_{\sigma_c(j)}}{4}\partial_{\sigma_c(j)} f \partial_{\sigma_c^2(j)} g - \frac{\theta_{\sigma_c^2(j)}}{4}\partial_{\sigma_c^2(j)} f \partial_{\sigma_c(j)} g
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
f \overset{\bar{a}_{ij}}{\star} g &= \bar{a}_{ij} f g - \frac{i\theta_{\sigma_c(i)}}{4}\partial_{\sigma_c(i)} f \partial_{\sigma_c^2(i)} g + \frac{i\theta_{\sigma_c^2(i)}}{4}\partial_{\sigma_c^2(i)} f \partial_{\sigma_c(i)} g \\
&\quad - \frac{\theta_{\sigma_c(j)}}{4}\partial_{\sigma_c(j)} f \partial_{\sigma_c^2(j)} g + \frac{\theta_{\sigma_c^2(j)}}{4}\partial_{\sigma_c^2(j)} f \partial_{\sigma_c(j)} g.
\end{aligned} \tag{26}
$$

Evidently, any two of these last results cannot be straightforwardly obtained by complex conjugation of each other. Indeed, we well get:

$$
\overline{a_{ij} \overset{f}{\star} g} \neq \bar{a}_{ij} \overset{f}{\star} g, \quad \overline{g \overset{f}{\star} a_{ij}} \neq g \overset{f}{\star} \bar{a}_{ij}, \quad a_{ij} \overset{f}{\star} g \neq g \overset{f}{\star} a_{ij}, \quad \overline{f \overset{a_{ij}}{\star} g} \neq f \overset{\bar{a}_{ij}}{\star} g.
$$

2.2. Integral representation

To construct the integral representation of the 3-ary star product (1), consider $s, x \in \mathbb{R}^3$ and the plane wave function of the form:

$$
\exp(isx) = \exp[i(s_1 x_1 + s_2 x_2 + s_3 x_3)]. \tag{27}
$$

Then, their 3-ary star product gives

$$
e^{Ike} \overset{e^{iqx}}{\star} e^{irx} = e^{\sum_{j=1}^{3} \frac{\theta_j}{2}\left(k_j q_{\sigma_c(j)} r_{\sigma_c^2(j)} - k_j q_{\sigma_c^2(j)} r_{\sigma_c(j)}\right) + i(k+q+r)x}. \tag{28}
$$

Defining the quantity Ω_j^{qr} by

$$
\Omega_j^{qr} = q_{\sigma_c(j)} r_{\sigma_c^2(j)} - q_{\sigma_c^2(j)} r_{\sigma_c(j)}, \tag{29}
$$

which satisfies the conditions

$$
\Omega_j^{qr} = -\Omega_j^{rq}, \; p\Omega^{qr} = r\Omega^{pq} = q\Omega^{rp}, \tag{30}
$$

the integral representation of the 3-ary star product of functions can be expressed as follows:

$$(f \overset{g}{\star} h)(x) = \int d^3k\, d^3q\, d^3r\, \tilde{f}(k)\tilde{g}(q)\tilde{h}(r) \left(e^{ikx} \overset{e^{iqx}}{\star} e^{irx}\right)$$
$$= \frac{1}{(2\pi)^9} \int d^3k\, d^3q\, d^3r\, d^3y\, d^3z\, d^3t\, f(y)g(z)h(t)$$
$$\times\, e^{\frac{1}{2}\sum_j \theta_j k_j \Omega_j^{qr}} e^{ik(x-y)} e^{iq(x-z)} e^{ir(x-t)}. \qquad (31)$$

It can be simplified, using the identity

$$\int d^3k\, e^{ik(x-y-\frac{i\theta}{2}\Omega^{qr})} = (2\pi)^3 \delta^{(3)}\Big(x - y - \frac{i\theta}{2}\Omega^{qr}\Big), \qquad (32)$$

into the form

$$(f \overset{g}{\star} h)(x) = \frac{1}{(2\pi)^6} \int d^3q\, d^3r\, d^3y\, d^3z\, d^3t\, f(y)g(z)h(t)$$
$$\times\, \delta^{(3)}(x - y - \frac{i}{2} q\Omega^{r\theta}) e^{iq(x-z)} e^{ir(x-t)}. \qquad (33)$$

3. Generalization to n-ary star product

In this section, we deal with the generalization of the 3-ary star product (1) into an n-ary star product.

Definition 7. Consider $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Let σ_c be the cyclic permutation over the set $\{1, 2, \dots, d\}$, i.e., $\sigma_c(k) = k + 1$, $k + 1 \le n$, and $\sigma_c(n) = 1$, and let $\mathcal{A} = \Big(\mathcal{S}\Big(\mathbb{R}^n\Big), \star\Big)$ be the Schwartz space of (smooth, rapidly decreasing, together with all their derivatives, faster than the reciprocal of any polynomial at ∞) real-valued functions on $\mathbb{R}^n$, endowed with an n-ary star product defined at a point x as follows:

$$\star\{\cdot, \cdot, \cdots, \cdot\} : \underbrace{\mathcal{A} \times \mathcal{A} \times \cdots \times \mathcal{A}}_{n \text{ times}} \longrightarrow \mathcal{A}$$
$$(f_1, f_2, \dots, f_n) \longmapsto \star\{f_1, f_2, \dots, f_n\} := \star\{f_i\}_{i=1}^n \qquad (34)$$

where

$$\star\{f_i\}_{i=1}^n(x) = \mathbf{m}\Big[e^{\mathcal{P}(\theta_1, \theta_2, \dots, \theta_n)} (f_1 \otimes f_2 \otimes \cdots \otimes f_n)(x)\Big], \qquad (35)$$

$$\mathbf{m}(f_1 \otimes f_2 \otimes \cdots \otimes f_n) = \prod_{i=1}^n f_i, \qquad (36)$$

and

$$\mathcal{P}(\theta_1, \theta_2, \dots, \theta_n) = \sum_{k=1}^n \frac{i\theta_k}{2} \Big(\partial_k \otimes \partial_{\sigma_c(k)} \otimes \cdots \otimes \partial_{\sigma_c^n(k)} - \partial_k \otimes \partial_{\sigma_c^{n-1}(k)} \otimes \cdots \otimes \partial_{\sigma_c^1(k)}\Big). \qquad (37)$$

For $f_i, g_i \in \mathcal{A}$, $i \in \mathbb{N}$, we obtain:

Proposition 7 (n-ary star product of functions in $\mathcal{A}$ and coordinate functions).

$$\begin{aligned} &\star \{f_i, x_p, g_j\}_{i=1,\ldots,m-1, j=m+1,\ldots,n-1} \\ &\quad = x_p \prod_{i=1}^{m-1} f_i \prod_{j=m+1}^{n} g_j + \frac{i\theta_{p-m+1}}{2} \prod_{i=1}^{m-1} \partial_{\sigma_c^{i-1}(i)} f_i \prod_{i=m+1}^{n} \partial_{\sigma_c^{i-1}(i)} g_i \\ &\qquad - \frac{i\theta_{p+m-n-1}}{2} \prod_{i=1}^{m-1} \partial_{\sigma_c^{n-i+1}(i)} f_i \prod_{i=m+1}^{n} \partial_{\sigma_c^{n-i+1}(i)} g_i. \end{aligned} \tag{38}$$

Therefore, for $p, q \in \{1, 2, \ldots, n\}$, these relations show that any two of them cannot be given by complex conjugation. In fact, we have:

$$\begin{aligned} \overline{\star\{a_{pq}, f_i\}}_{i=1,\ldots,n-1} &\neq \star\{\bar{a}_{pq}, f_i\}_{i=1,\ldots,n-1}, \\ \overline{\star\{f_i, a_{pq}\}_{i=1,\ldots,n-1}} &\neq \star\{f_i, \bar{a}_{pq}\}_{i=1,\ldots,n-1} \\ \overline{\star\{f_i, a_{pq}, g_j\}}_{i=1,\ldots,m-1, j=m+1,\ldots,n-1} &\neq \star\{f_i, \bar{a}_{pq}, g_j\}_{i=1,\ldots,m-1, j=m+1,\ldots,n-1}. \end{aligned}$$

The n-ary star product integral representation, for n arbitrary points, can be also computed by the same method as in the previous section 3. By considering the plane wave functions

$$e^{isx} = e^{i(s_1 x_1 + \cdots + s_n x_n)}, \quad s, x \in \mathbb{R}^n, \tag{39}$$

the following result can be proved:

Proposition 8 (n-ary star product integral representation).

$$\begin{aligned} \star\{f_i\}_{i=1}^{n}(x) = \frac{1}{(2\pi)^{2n}} \int \prod_{j=1}^{n-1} d^n q_j \prod_{j=1}^{n} d^n y_j \, f(y_j) \\ \times \delta^{(n)}(x - y - \frac{i}{2} q \Omega^{r\theta}) \prod_{j=1}^{n-1} e^{iq(x-y_j)}. \end{aligned} \tag{40}$$

4. Application

Consider a physical system described by the Hamiltonian model:

$$\begin{aligned} H = \sum_{j=1}^{n} x_j^2 + \sum_{i<j=1}^{n} (\epsilon_{ij} \lambda_{ij} x_i x_j) + \sum_{i<j<k<l=1}^{n} (\epsilon_{ijkl} \lambda_{ijk} x_i x_j x_k x_l) + \cdots \\ + \sum_{i_1<i_2<\cdots<i_n=1}^{n} (\epsilon_{i_1 i_2 \cdots i_n} \lambda_{i_1 \cdots i_n} x_{i_1} x_{i_2} \cdots x_{i_k}) \end{aligned} \tag{41}$$

where $\lambda_{i_1 \cdots i_k}, k \leq n$, are the coupling constants and $\epsilon_{i_1 i_2 \cdots i_k}$ the Levi-Civita tensor of rank k; $k = n$ if n is even and $n - 1$ if n is odd. Using the orthogonal transformation $\mathcal{R}$ such that $\mathcal{R}_{kl}x_l = X_k$, the Hamiltonian (41) can be re-expressed in the new coordinates X as follows:

$$H = \sum_{i=1}^{n} \lambda_i^{(0)} X_i^2 + \sum_{i=1}^{n} \lambda_i^{(2)} X_i^4 + \cdots \tag{42}$$

allowing a re-formulation in terms of previous quantities a_{ij} and $\bar{a}_{ij}$, defined in (20), as follows:

$$H = \sum_{i,j=1, i\neq j}^{n} \sum_{p=0}^{n-1} \Big(\lambda_i^{(2p)} a_{ij}\bar{a}_{ij}\Big)^{p+1}. \tag{43}$$

For $\psi_k^m \in \mathcal{A}$, the eigenvalue problem is given by the system of equations:

$$\star\{H, \psi_1^m, \ldots, \psi_{n-1}^m\} = E_{1,\bar{n}}\Big[\star\{1, \psi_1^m, \ldots, \psi_{n-1}^m\}\Big] \tag{44}$$

$$\star\{\psi_1^m, H, \psi_2^m, \ldots, \psi_{n-1}^m\} = E_{2,\bar{n}}\Big[\star\{\psi_1^m, 1, \psi_2^m, \ldots, \psi_{n-1}^m\}\Big] \tag{45}$$

$$\vdots$$

$$\star\{\psi_1^m, \psi_2^m, \ldots, \psi_{n-1}^m, H\} = E_{n,\bar{n}}\Big[\star\{\psi_1^m, \psi_2^m, \ldots, \psi_{n-1}^m, 1\}\Big]. \tag{46}$$

$\bar{n} \in \mathbb{N}^n$ is an n-vector characterizing the quantum number associated to the Hamiltonian (41) while ψ_j^m, $j = 1, 2, \ldots, n-1$; $m = 1, 2, \ldots, n$, are the eigenstates diagonalizing it. The ground state satisfies the equation

$$\star\{a_{ij}, \psi_1^0, \ldots, \psi_{n-1}^0\} = 0 \tag{47}$$

which can be explicitly solved to give:

$$\psi_k^0 = Ce^{-|x|^2/2} H_k(|x|^2/2) f(\lambda_k, |x|), \quad C \in \mathbb{R}, \tag{48}$$

where H_k is the Hermite polynomial; the functions $f(\lambda_k, |x|) := f_k$ are orthogonal with the orthogonality condition

$$\star\{f_{k_1}, f_{k_2}, \ldots, f_{k_n}\} = C' \delta_{k_1k_2}\delta_{k_2k_3}\cdots\delta_{k_{n-1}k_n}; \quad C' = C^n \in \mathbb{R}. \tag{49}$$

The case where $f_k = 1$, $\forall k = 1, 2, \ldots, n$, is solution of relation (49).

The excited states can be computed by using the well-known harmonic oscillator algebraic method performed with the raising operator, acting here as follows:

$$\star\{\bar{a}_{ij}, \psi_1^0, \ldots, \psi_{n-1}^0\} = f_1(n)\big(\star\{1, \psi_1^1, \ldots, \psi_{n-1}^1\}\big), \tag{50}$$

where $f_1(n)$ is a function depending on the parameter n.

They result in the following expressions for the eigenvalues and eigenfunctions characterizing the considered physical model:

$$E_{k,\bar{n}} = \theta_k\Big(\lambda_k^{(0)}|\bar{n}| + \sum_{i=1}^{n} \lambda_i^{(k)}|\bar{n}|^2 + \sum_{i,j=1}^{n} \lambda_i^{(k)}\lambda_j^{(k)}|\bar{n}|^3 + \cdots + \frac{n}{2}\Big) \tag{51}$$

and

$$\psi_k^n = Ce^{-|x|^2/2} H_k^{|n|}(|x|^2/2) f(\lambda_k, |x|). \tag{52}$$

$H_k^{|n|}$ stands for the (n,k)-order Hermite polynomial.

5. Concluding remarks

In this work, we have given a method of constructing an n-ary star product. Relevant identities have been provided and discussed. A physical problem of coupled oscillators has been treated. The associated eigenvalues and eigenfunctions have been explicitly computed.

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Quantum Mechanics and Geometry on the Siegel–Jacobi Disk

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Abstract. We present some geometric properties of the Siegel–Jacobi disk $\mathcal{D}^J_1 = \mathbb{C} \times \mathcal{D}_1$ obtained using the coherent states attached to the Jacobi group $G^J_1 = H_3 \rtimes \mathrm{SU}(1,1)$, where $\mathcal{D}_1$ denotes the Siegel disk and H_3 is the Heisenberg group.

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1. Introduction

The coherent states offer the possibility to study the interplay between classical and quantum mechanics [1]. In the group-theoretic approach of Perelomov [2] are defined coherent states vectors e_z in a Hilbert space $\mathfrak{H}$ indexed by the points of a homogeneous manifold $M = G/H$. In a series of papers [3–8] it was shown how the coherent states permit to find different geometrical objects on M such as: geodesics, the conjugate locus, the cut locus, Calabi's diastasis, the Kodaira embedding theorem, and more. In the above quoted papers the coherent state manifolds M are hermitian symmetric spaces DS of compact and non compact type, while the standard Glauber coherent states attached to the Heisenberg group H_{2n+1} are defined on $\mathbb{C}^n$. It is natural to look for coherent states based on manifolds which have simultaneously parts of both types, $\mathbb{C}^n$ and DS. The simplest examples of such spaces are the Siegel–Jacobi spaces $\mathcal{D}^J_n$ [9–12], which are non symmetric domains. The points of $\mathcal{D}^J_n$ are in the set $\mathbb{C}^n \times \mathcal{D}_n$, where $\mathcal{D}_n$ denotes the Siegel ball. The Siegel–Jacobi domains are homogeneous manifolds associated to the Jacobi group $G^J_n = H_{2n+1} \rtimes \mathrm{Sp}(n, \mathbb{R})$ [13, 14].

We have defined coherent states attached to the Jacobi group G^J_n based on $\mathcal{D}^J_n$ [15, 16]. The Jacobi G^J_n groups are the simplest nontrivial example of groups of coherent type [17, 18]. The holomorphic irreducible representations of the Jacobi group based on the Siegel–Jacobi domains have been studied in [14, 19–21].

Besides its interest for mathematicians, the Jacobi group is relevant in physics, being responsible for the squeezed states in quantum optics [1, 22–24]. Other references on applications of the Jacobi group in physics can be found in [25–30].

The coherent states attached to the Jacobi group G^J_1 were the subject of the papers [25, 29], where we have constructed a holomorphic representation of the Jacobi Lie algebra $\mathfrak{g}^J_1$ indexed by an integer k coming from the positive discrete series representation of group $\mathrm{SU}(1,1)$ [31]. In the present paper we introduce besides k another parameter $\mu\in\mathbb{R}$ which characterizes the part coming from the Heisenberg group H_3 in the Jacobi group G^J_1. The standard realization in quantum mechanics of the position and momentum operators $\hat{\boldsymbol{q}}=q,\ \hat{\boldsymbol{p}}=-\mathrm{i}\hbar\frac{\partial}{\partial q}$ in $\mathfrak{H}=L^2(\mathbb{R},\mathrm{d}\,x)$ corresponds to the choice $\mu=\frac{1}{\hbar}$.

With this preparation, we proceed to investigate some geometric properties of the Siegel–Jacobi disk using the coherent states attached to the Jacobi group G^J_1. The paper is laid out as follows. After the Introduction, we collect in Section 2 some general definitions of coherent states [2], some notions related with Berezin's quantization [32–35], and also some notions used in our approach of differential geometry via coherent states initiated in [3], as Kobayashi embedding [36], Cayley distance [6, 37] and Cauchy formula [6, 8], all summarized in Remark 1. In Proposition 1 from Section 3 we reformulate some of our previous results referring to the reproducing kernel $K(z,w)_{k\mu}$ and the base of orthonormal polynomials on $\mathcal{D}^J_1$ established in [25], this time with both parameters k and μ describing the holomorphic representation of the Lie algebra $\mathfrak{g}^J_1$. The main results are contained in Proposition 2 in Section 4: a description of the Jacobi group G^J_1 as unimodular, non-reductive, algebraic group of Harish–Chandra type. The Siegel–Jacobi disk is a reductive, non-symmetric manifold associated to the Jacobi group by the generalized Harish–Chandra embedding. $\mathcal{D}^J_1$ is a coherent state type quantizable manifold generated by the coherent type Jacobi group G^J_1. The significance in the context of coherent states of the transform which realizes the fundamental conjecture [38, 39] for the Siegel–Jacobi disk proved in [29] is emphasized. Also the Ricci form and the scalar curvature of the Siegel–Jacobi disk are presented. Only short indications of the proofs are sketched. More details will be published [40].

2. Geometry via coherent states

The starting point in Perelomov's construction of coherent states is the triplet $(G,\pi,\mathfrak{H})$, where π is a continuous, unitary, irreducible representation of the Lie group G on the separable complex Hilbert space $\mathfrak{H}$ [2]. The coherent states are based on a complex homogeneous manifold $M\cong G/H$ [2], where H is the isotropy group. We restrict ourselves to manifolds M which are CS-*orbits*, i.e., which admit a holomorphic embedding $\iota_M : M\hookrightarrow\mathbb{P}(\mathcal{H}^\infty)$ [17, 18, 41].

If X is an element of the Lie algebra $\mathfrak{g}$ of G, we denote $\boldsymbol{X}:=\mathrm{d}\,\pi(X)$.

Let us introduce the normalized (unnormalized) vectors $\underline{e}_x$ (respectively, e_z) defined on G/H

$$\underline{e}_x = \exp\left(\sum_{\varphi\in\Delta_+} x_\varphi \boldsymbol{X}^+_\varphi - \bar{x}_\varphi \boldsymbol{X}^-_\varphi\right) e_0,\ e_z = \exp\left(\sum_{\varphi\in\Delta_+} z_\varphi \boldsymbol{X}^+_\varphi\right) e_0, \qquad (1)$$

where e_0 is the extremal weight vector of the representation π, Δ_+ are the positive roots of the Lie algebra $\mathfrak{g}$ of G, and $X_\varphi, \varphi \in \Delta$, are the generators [2, 41]. The vectors $e_{\bar{z}} \in \bar{\mathfrak{H}}$ indexed by the points of the manifold M are called *Perelomov coherent state vectors.*

We adopt [41] Berezin's approach to quantization of Kähler manifolds with the supercomplete sets of vectors [32–35], in the formulation of Rawnsley [42, 43]. We consider the space $\mathcal{F}_{\mathfrak{H}} = L^2_{\rm hol}(M, \mathrm{d}\,\nu_M)$ of holomorphic, square integrable functions with respect to the scalar product

$$(f,g)_{\mathcal{F}_{\mathfrak{H}}} = \int_M \bar{f}(z) g(z)\,\mathrm{d}\,\nu_M(z,\bar{z}),\ \mathrm{d}\,\nu_M(z,\bar{z}) = \frac{\Omega_M(z,\bar{z})}{K_M(z,\bar{z})}, \qquad (2)$$

where Ω_M is the normalized G-invariant volume form, and the Bergman kernel is obtained as the scalar product $K_M(z,\bar{z}) = (e_{\bar{z}}, e_{\bar{z}})$. Above we have used the map $\Phi : \mathfrak{H}^\star \to \mathcal{F}_{\mathfrak{H}},\ f_\psi(z) = \Phi(\psi)(z) = (e_{\bar{z}}, \psi)_{\mathfrak{H}},\ z \in M$ [41]. The G-invariant Kähler two-form ω on the $2n$-dimensional manifold $M = G/H$ is obtained from the Kähler potential $f(z,\bar{z}) = \ln K_M(z,\bar{z})$:

$$\omega_M(z) = \mathrm{i} \sum_{\alpha\in\Delta_+} h_{\alpha,\bar{\beta}}(z)\,\mathrm{d}\,z_\alpha \wedge \mathrm{d}\,\bar{z}_\beta,\ h_{\alpha,\bar{\beta}} = \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} \ln(e_{\bar{z}}, e_{\bar{z}}). \qquad (3)$$

If $\{\Phi\} = \{\varphi_n(z)\}_{n=1,\dots,}$ is an orthonormal base of functions of $\mathcal{F}_{\mathfrak{H}}$, then the Bergman kernel $K_M(z,\bar{w}) = (e_{\bar{z}}, e_{\bar{w}})$ admits the series expansion

$$K_M(z,\bar{w}) = \sum_n^\infty \varphi_n(z)\bar{\varphi}_n(w). \qquad (4)$$

Let us denote by FC the change of variables $x \to z$ in (1) such that

$$\underline{e}_x = \tilde{e}_z,\ \tilde{e}_z := (e_z, e_z)^{-\frac{1}{2}} e_z,\ z = FC(x). \qquad (5)$$

In [3–8] we have analyzed the deep relationship between geometry and quantum mechanics via the coherent states. We have underlined a geometric meaning of the transition amplitudes $(\tilde{e}_z, \tilde{e}_{z'})$, where z, z' belong to the coherent state manifold M [7], and of the different distances and angles in quantum mechanics for coherent states [4, 8] using the Kobayashi embedding [36]. Most of our results in [3–8] refer to hermitian symmetric coherent state manifolds M. We have proved that *for symmetric spaces the dependence* $z(t) = FC(tX)$ *from* (5) *gives geodesics in* M *with the property that* $z(0) = p$ *and* $\dot{z}(0) = X$ [6]. We have investigated in particular the conjugate locus and cut locus on the complex Grassmann manifold using the machinery of coherent states defined on such spaces [5]. Having in mind to highlight similar behavior on non symmetric spaces as the Siegel–Jacobi disk, we recall some notation and results from [3–8].

Let $\xi : \mathfrak{H}\backslash 0 \rightarrow \mathbb{P}(\mathfrak{H})$ be the canonical projection $\xi(\boldsymbol{z}) = [\boldsymbol{z}]$. The Fubini-Study metric [36] is

$$\mathrm{d}\, s^2|_{FS}([z]) = \frac{(\mathrm{d}\,\boldsymbol{z}, \mathrm{d}\,\boldsymbol{z})(\boldsymbol{z}, \boldsymbol{z}) - (\mathrm{d}\,\boldsymbol{z}, \boldsymbol{z})(\boldsymbol{z}, \mathrm{d}\,\boldsymbol{z})}{(\boldsymbol{z}, \boldsymbol{z})^2}. \tag{6}$$

The elliptic *Cayley distance* [37] between two points in the projective Hilbert space $\mathbb{P}(\mathfrak{H})$ is defined as [6]

$$\mathrm{d}_C([z_1], [z_2]) = \arccos \frac{|(z_1, z_2)|}{||z_1||||z_2||}. \tag{7}$$

If the Kähler manifold M admits a holomorphic embedding [36]

$$\iota_M : M \hookrightarrow \mathbb{CP}^\infty, \iota_M(z) = [\varphi_0(z) : \varphi_1(z) : \dots], \tag{8}$$

then [6–8, 40]:

Remark 1. *The Hermitian metric on M is the pullback of the Fubini-Study metric* (6) *via the embedding* (8)*, i.e.:*

$$\mathrm{d}\, s^2_M(z) = \iota^*_M \,\mathrm{d}\, s^2_{FS}(z) = \mathrm{d}\, s^2_{FS}(\iota_M(z)). \tag{9}$$

The angle defined by the normalized Bergman kernel can be expressed via the embedding (8) *as function of the Cauchy distance* (7)

$$\arccos |(\tilde{e}_{z_1}, \tilde{e}_{z_2})_M| = d_C(\iota_M(z_1), \iota_M(z_2)). \tag{10}$$

The following (*Cauchy*) *formula is true*

$$(\tilde{e}_{z_1}, \tilde{e}_{z_2})_M = (\iota_M(z_1), \iota_M(z_2))_{\mathbb{CP}^\infty}. \tag{11}$$

3. Coherent states attached to the Jacobi group G^J_1

In the present work, the coherent state representation of the Jacobi group G^J_1 is indexed with two parameters: μ, describing the part coming from the Heisenberg group, and k, characterizing the positive discrete series representation of $\mathrm{SU}(1,1)$. We follow the prescription of [25], where the case $\mu = 1$ was considered. Only the results of the calculation are presented. More details will be published [40].

The Jacobi algebra is the semi-direct sum $\mathfrak{g}^J_1 := \mathfrak{h}_3 \rtimes \mathfrak{su}(1,1)$ [25, 27], where the Heisenberg algebra $\mathfrak{h}_3$ is generated by the boson operators $a, a^\dagger$ and 1, $[a, a^\dagger] = 1$, $\mathfrak{su}(1,1)$ is generated by $K_{\pm,0}$, and

$$\begin{aligned} &[a, K_+] = a^\dagger, \left[K_-, a^\dagger\right] = a, \left[K_+, a^\dagger\right] = [K_-, a] = 0, \\ &\left[K_0, a^\dagger\right] = \frac{1}{2}a^\dagger,\ [K_0, a] = -\frac{1}{2}a. \end{aligned} \tag{12}$$

We impose to the cyclic vector e_0 to verify simultaneously the conditions

$$\boldsymbol{a}e_0 = 0,\ \boldsymbol{K}_-e_0 = 0,\ \boldsymbol{K}_0e_0 = ke_0;\ k > 0, 2k = 2, 3, \dots, \tag{13}$$

and k indexes the positive discrete series representations D^+_k of $\mathrm{SU}(1,1)$ [31].

Perelomov's coherent state vectors

$$e_{z,w} := \mathrm{e}^{\sqrt{\mu} z \boldsymbol{a}^\dagger + w \boldsymbol{K}_+} e_0, \ z, w \in \mathbb{C}, \ |w| < 1. \tag{14}$$

are vectors in the Hilbert space of the representation of the group G_1^J, based on the Siegel–Jacobi disk

$$\mathcal{D}_1^J := H_1/\mathbb{R} \times \mathrm{SU}(1,1)/\mathrm{U}(1) = \mathbb{C} \times \mathcal{D}_1. \tag{15}$$

It can be proved that [40]:

Remark 2. *The standard realization* $\hat{\boldsymbol{q}} = q, \ \hat{\boldsymbol{p}} = -\mathrm{i}\hbar\frac{\partial}{\partial q}$ *in* $\mathfrak{H} = L^2(\mathbb{R}, \mathrm{d}\, x)$ *of the position and momentum operators, where* $\boldsymbol{a} = \lambda(\hat{\boldsymbol{q}} + \mathrm{i}\hat{\boldsymbol{p}}), \ \boldsymbol{a}^\dagger = \lambda(\hat{\boldsymbol{q}} - \mathrm{i}\hat{\boldsymbol{p}})$, *corresponds to the choice* $\mu\hbar = 1, \ 2\hbar\lambda^2 = 1$.

We consider the *squeezed* CS vector $\underline{e}_{\alpha,w} := D_\mu(\alpha) S(w) e_0$ [44], where

$$D_\mu(\alpha) = \exp \sqrt{\mu}(\alpha \boldsymbol{a}^\dagger - \bar{\alpha}\boldsymbol{a}) = \exp(-\mu\frac{1}{2}|\alpha|^2)\exp(\sqrt{\mu}\alpha\boldsymbol{a}^\dagger)\exp(-\sqrt{\mu}\bar{\alpha}\boldsymbol{a}),$$
$$\underline{S}(z) = \exp(z\boldsymbol{K}_+ - \bar{z}\boldsymbol{K}_-); \ S(w) = \exp(w\boldsymbol{K}_+)\exp(\eta\boldsymbol{K}_0)\exp(-\bar{w}\boldsymbol{K}_-),$$
$$\underline{S}(z) = S(w); \ w = \frac{z}{|z|}\tanh(|z|), \eta = \log(1 - w\bar{w}).$$

We have also [25, 40]:

Proposition 1. *With the notation* $\zeta := (z, w) \in (\mathbb{C} \times \mathcal{D}_1)$, *the reproducing kernel* $K : \mathcal{D}_1^J \times \mathcal{D}_1^J \to \mathbb{C}$, $K_{k\mu}(\zeta; \bar{\zeta}') := (e_{\bar{z},\bar{w}}, e_{\bar{z}',\bar{w}'})$ *is*

$$K_{k\mu}(\zeta, \bar{\zeta}') = (1 - w\bar{w}')^{-2k} \exp \mu \frac{2\bar{z}'z + z^2\bar{w}' + \bar{z}'^2 w}{2(1 - w\bar{w}')}, \tag{16}$$

and it admits the series expansion (4) *in the base* $\{\Phi(z, w)\} = \{f_{nk'm}(z, w)\}$, $k = k' + \frac{1}{4}, 2k' \in \mathbb{Z}_+$ *of orthonormal polynomials:*

$$f_{nk'm}(z, w) = f_{k'm}(w)\frac{P_n(\sqrt{\mu}z, w)}{\sqrt{n!}}, \tag{17}$$

$$f_{kn}(w) = \sqrt{\frac{\Gamma(n + 2k)}{n!\Gamma(2k)}} w^n; \ P_n(z, w) = n! \sum_{p=0}^{[\frac{n}{2}]} \left(\frac{w}{2}\right)^p \frac{z^{n-2p}}{p!(n - 2p)!}. \tag{18}$$

In particular, $K = (e_{\bar{z},\bar{w}}, e_{\bar{z},\bar{w}}) : \mathcal{D}_1^J \times \bar{\mathcal{D}}_1^J \to \mathbb{C}$ *has the expression*

$$K_{k\mu}(z, w) = P^{-2k} \exp \mu \frac{2z\bar{z} + z^2\bar{w} + \bar{z}^2 w}{2P}, \ P = 1 - w\bar{w}. \tag{19}$$

The normalized squeezed state vector and the unnormalized Perelomov's coherent state vector are related by the relation

$$\underline{e}_{\eta,w} = P^k \exp(-\frac{\bar{\eta}}{2}z)e_{z,w}, \ \eta = \frac{z + \bar{z}w}{P}, \tag{20}$$

where the change of variables $(\eta, w) \mapsto (z, w)$ *in* (20)

$$(z, w) = FC(\eta, w), \ z = \eta - w\bar{\eta} \tag{21}$$

is an FC-transform in the sense of (5) *for coherent states defined on the Siegel–Jacobi disk* $\mathcal{D}_1^J$.

The action of $(g,\alpha) \in G_1^J$ *on* $(z,w) \in \mathcal{D}_1^J$ *is given by*

$$z_1 = \frac{\alpha - \bar{\alpha} w + z}{\bar{b} w + \bar{a}}; \ w_1 = \frac{aw + b}{\bar{b} w + \bar{a}}, \ g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \ |a|^2 - |b|^2 = 1. \tag{22}$$

4. Geometric properties of the Siegel–Jacobi disk

In this section we present several results on the geometry of the Siegel–Jacobi disk. Some of the formulae obtained when $\mu = 1$ have been already published in our papers [25, 29]. Formulae (27) and (28) have been obtained previously with a different technique in [45]. Only a sketch of the proof is given. More details will be published [40].

Proposition 2. *The Jacobi group* G_1^J *is an unimodular, non-reductive, algebraic group of Harish–Chandra type. The Siegel–Jacobi disk is a reductive, non-symmetric manifold associated to the Jacobi group by the generalized Harish–Chandra embedding.*

The scalar product of functions from the space $\mathfrak{F}_{k\mu} = L^2_{\rm hol}(\mathcal{D}_1^J, \rho_{k\mu})$ *corresponding to the kernel* $K_{k\mu}$ *defined by* (19) *on the Siegel–Jacobi disk is:*

$$(\phi,\psi)_{k\mu} = \int_{\mathcal{D}_1^J} \bar{f}_\phi(z,w) f_\psi(z,w) \rho_{k\mu} \,\mathrm{d}\,\nu, \ \rho_{k\mu} = \frac{\Lambda}{K_{k\mu}(z,w)}, \tag{23}$$

$$\mathrm{d}\,\nu = \mu \frac{\mathrm{d}\,\Re w \,\mathrm{d}\,\Im w}{P^3} \mathrm{d}\,\Re z \,\mathrm{d}\,\Im z, \ \Lambda = \frac{4k-3}{2\pi^2}. \tag{24}$$

The Kähler two-form $\omega_{k\mu}$ *on* $\mathcal{D}_1^J$, G_1^J*-invariant to the action* (22), *is:*

$$-\mathrm{i}\omega_{k\mu}(z,w) = 2k \frac{\mathrm{d}\,w \wedge \mathrm{d}\,\bar{w}}{P^2} + \mu \frac{A \wedge \bar{A}}{P}, \ A = \mathrm{d}\,z + \bar{\eta}\,\mathrm{d}\,w, \ \eta = \frac{z + \bar{z} w}{P}. \tag{25}$$

The FC-transform (21), $FC(\eta,w) = (z,w)$, $z = \eta - w\bar{\eta}$, *is a homogeneous Kähler diffeomorphism, i.e.,* $FC^* \omega_{k\mu}(z,w) = \omega_{k\mu}(\eta,w)$, *where*

$$\omega_{k\mu}(\eta,w) = \omega_k(w) + \omega_\mu(\eta), \tag{26}$$

$$-\mathrm{i}\omega_\mu(z) = \mu \,\mathrm{d}\,z \wedge \mathrm{d}\,\bar{z}, \ z \in \mathbb{C}; \ -\mathrm{i}\omega_k(w) = 2k \frac{\mathrm{d}\,w \wedge \mathrm{d}\,\bar{w}}{P^2}, \ w \in \mathcal{D}_1.$$

The Ricci form is

$$\rho_{\mathcal{D}_1^J}(z,w) = -3\mathrm{i} \frac{\mathrm{d}\,w \wedge \mathrm{d}\,\bar{w}}{P^2}, \tag{27}$$

and $\mathcal{D}_1^J$ *is not a Kähler–Einstein manifold.*

The scalar curvature has the value

$$s_{\mathcal{D}_1^J}(p) = -\frac{3}{2k}, p \in \mathcal{D}_1^J. \tag{28}$$

The Jacobi group G_1^J *is a coherent-state type group and the Siegel–Jacobi disk* $\mathcal{D}_1^J$ *is a quantizable Kähler coherent state manifold. The Kählerian embedding*

$\iota_{\mathcal{D}_1^J} : \mathcal{D}_1^J \hookrightarrow \mathbb{CP}^\infty$ (8), $\iota_{\mathcal{D}_1^J} = \{\Phi(z,w)\}$, *is realized by the base of orthonormal functions* (17), *and the Kähler two-form* (25) *is the pullback of the Fubini-Study Kähler two-form* (6) *on* $\mathbb{CP}^\infty$,

$$\omega_{k\mu} = \iota^*_{\mathcal{D}_1^J}\omega_{FS}|_{\mathbb{CP}^\infty},\ \omega_{k\mu}(z,w) = \omega_{FS}([\Phi(z,w)]).$$

A Cauchy formula of the type (11) *is verified by the normalized vectors* $\tilde{e}_{z,w}$ *obtained from the Perelomov coherent state vectors* (14) *which have the scalar product* (16), *for the embedding* $\iota_{\mathcal{D}_1^J}$.

Proof. The fact that $\mathcal{D}_1^J$ is a reductive domain in the sense of Nomizu [46] follows from the definition (12) of the Lie algebra $\mathfrak{g}_1^J$. See also [25]. In [27] it was proved that the Siegel–Jacobi spaces admit a Harish–Chandra embedding.

The realization (23) of the resolution of unit (2) on $\mathcal{D}_1^J$ was proved in [25] in the case $\mu = 1$.

With the Kähler potential furnished by (19), we get the metric coefficients in (3)

$$h_{z\bar{z}} = \frac{\mu}{P},\ h_{z\bar{w}} = \mu\frac{\eta}{P},\ h_{w\bar{w}} = \mu\frac{|\eta|^2}{P} + \frac{2k}{P^2},\ P = (1 - w\bar{w}), \tag{29}$$

which are used to obtain the expression (25) of the homogeneous Kähler two-form on $\mathcal{D}_1^J$.

The fundamental conjecture expressed in (26) for the Siegel–Jacobi disk was proved in [16, 29] in the case $\mu = 1$. Here we just emphasize the significance of the FC-transform as the transform (5).

In order to calculate the Ricci form of the Bergman metric (cf. p. 90 in [47]), we calculate the determinant of metric coefficients (29) on the Siegel–Jacobi disk as $G(z,w) = \frac{2k\mu}{(1-w\bar{w})^3}$.

The scalar curvature at a point $p \in M$ of coordinates (z,w) in (27) is obtained with the last formula in Note 3 at p. 294 in [48].

As a consequence of the fact that the Kähler potential was taken as the logarithm of the scalar product of coherent state vectors, the Siegel–Jacobi disk is a quantizable [42] Kähler manifold. $\mathcal{D}_1^J$ is a coherent state manifold in accord with Remark 1 realized by the embedding (8) via the orthonormal base of polynomials (17). □

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Quantum Resonances: Theory and Models

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Abstract. Along this paper, we give a short review of some interesting aspects of a formulation of quantum resonances. In particular how and why to characterize quantum resonances through Gamow state vectors as functionals of spaces constructed using Hardy functions on a half-plane. In addition, we give a couple of quite distinct interesting examples of resonance models. Here, we limit ourselves to the non-relativistic case.

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Keywords. Quantum resonances, Gamow states, one-dimensional models.

1. Introduction

This article contains a brief review in some interesting aspects of quantum resonances and their relation with some lines of research in modern quantum theory like time asymmetric quantum mechanics (TAQM) [1]. In the present summary, our discussion will be preferentially focused on non relativistic resonances. Explicit examples of resonance models will be also discussed.

It is reasonable to begin our discussion with a presentation of the different definitions of quantum resonances. Although, they are not equivalent in all cases, there exists an account in the literature of some sufficient conditions among them. This account is far to be complete and a research on this direction is probably worthy to carry out. For this purpose, we address the interested reader to the standard literature on the subject [2–8].

There are some techniques to obtain the parameters of resonances in given models. Rigorous mathematical methods like complex scaling and the use of Krein formula are of order here. Nevertheless, in the case of the very illustrative one-dimensional models, these methods are usually too sophisticated. Instead, the use of the so-called purely outgoing boundary conditions gives resonances in one-dimensional models, as solutions of a transcendental algebraic equation. This can

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usually be solved numerically, to give the parameters of a finite number of resonances, with a reasonable level of accuracy. We can also study the behavior of a limited number of resonances as well as bound and antibound states with the variation of some given parameters, using this method. An example of such a procedure will be given in the last section.

Resonances in quantum mechanics describe unstable quantum states. When considered as pure quantum states, they are described by a vector state. Resonance vector states split into a sum of two contributions, one that decays exponentially with time and other that produces deviations of this exponential decay for very short and very long values of time. The range of observation usually covers this exponential decay being the other modes not easily observable. This characteristic and the fact that the vector state for the exponentially decay mode can be often constructed explicitly, permits the identification of it with the resonance state. This vector state is usually called the Gamow vector. It has, however, a basic difficulty as it cannot be given by a normalizable vector in the usual Hilbert space. Then, the use of extensions of the Hilbert space to rigged Hilbert spaces (RHS) is necessary for this description of quantum resonances. In this context, Gamow vectors are realized as objects in a space of functionals that admit the Hilbert space as a subspace.

It is often convenient to describe resonances as produced in resonance scattering. By an appropriate choice of the RHS based on the use of Hardy functions on a half-plane, we can split this scattering into a preparation and a registration processes. This is used as a basis for the formulation of a Time Asymmetric Quantum Mechanics, which requires a simple refinement of the concept of pure state [1].

In the study of resonances, textbooks often propose an identification between the width of a bump in the cross section, which is characteristic of resonance phenomena and the inverse of the mean life. However, this identification suffers some structural problems like measuring difficulties [9]. This identification can be better understood in the context of our formalism based in RHS of Hardy functions.

The Friedrichs model and its refinements and generalizations [10] give a good laboratory to investigate basic properties of resonances, Gamow vectors, etc, having applications to a wide set of realistic physical systems. Also, one-dimensional models can be proposed for the study of resonance as well as bound and antibound states behavior, which can be somehow unexpected.

Finally, we are summarizing here a formalism of non-relativistic resonance and have included two models that we consider as interesting. The extension of this formalism to relativistic resonances as well as unstable interaction of quantum fields has been discussed elsewhere. See [11] and references therein.

2. Definitions and characterizations of quantum resonances

A quantum resonance may be caused by the action of an interaction on an otherwise free particle. As the interaction is usually produced by a potential, it is customary to consider two Hamiltonians in the production of resonance phenomena, a so-called "free" Hamiltonian H_0 under which the particle is supposed to move freely and a total or interaction Hamiltonian $H = H_0 + V$, where V is the potential. The physical effect of V is the creation of a metastable state in which V retains the studied particle in a bounded region a time which is much larger than the time the particle would stay in this region should the interaction not exist. Then, for resonances to be produced, we need a *Hamiltonian pair* $\{H_0, H\}$.

This situation can be better understood in the context of *resonance scattering*, which is a scattering process that produces resonances. Let us assume that the potential V is of compact support just to give a better intuitive image of the process. In the remote past, a state ψ^{in} is prepared in a *preparation apparatus* and evolves under the action of the free Hamiltonian H_0. This state is captured inside the interacting region (the support of the potential) and stays for a long delay time, i.e., it forms a resonance. In the far future it becomes ψ^{out} and again its time evolution is governed by H_0. Both "in" and "out" state vectors are related through the S-matrix, $\psi^{\text{out}} = S\psi^{\text{in}}$.

The definitions of quantum resonances resonances most popular in the standard literature are discussed below.

2.1. Definitions of resonances from the mathematical point of view

Assume that both H_0 and H are defined on a separable infinite-dimensional Hilbert space $\mathcal{H}$ and have an absolutely continuous spectrum $\mathbb{R}^+ := [0, \infty)$, which is the same for both operators. This is a very common situation. Then, the first definition of a resonance produced by the Hamiltonian pair $\{H_0, H\}$ is the following [7]:

Definition 1. Assume that there is a dense set of vectors $\mathcal{D}$ in $\mathcal{H}$ such that for $\psi \in \mathcal{D}$, both

$$R_{0\psi}(\lambda) = \langle\psi|(H_0 - \lambda)^{-1}\psi\rangle\,, \quad R_\psi(\lambda) = \langle\psi|(H - \lambda)^{-1}\psi\rangle \tag{1}$$

have analytic continuation through the positive real axis. Assume that $R_{0\psi}(\lambda)$ is analytic at $z_R = E_R - i\Gamma/2$ for any ψ, but there exists a $\psi \in \mathcal{D}$ for which $R_\psi(\lambda)$ shows a pole. Then, we say that z_R is a resonance of the Hamiltonian pair $\{H_0, H\}$.

We should stress that both $R_{0\psi}(\lambda)$ and $R_\psi(\lambda)$ are analytic functions on the complex plane of λ with a branch cut on the positive semiaxis $\mathbb{R}^+$. Their possible isolated singularities lie on their analytic continuations through the cut. In the language of Riemann surfaces, these poles appear on the second sheet. Resonance poles may not be unique and in fact, in most realistic models they appear in an infinite number. Also resonance poles appear in complex conjugate pairs of the same multiplicity, each pair of resonance poles represent the same resonance.

A second definition of quantum resonance is the celebrated *pair of complex conjugate poles* of the analytic continuation of the S-matrix:

Definition 2. Let $S(k)$ and $S(E)$ the S-matrix in the momentum and energy representations, respectively ($E = \hbar^2 k^2/2m$). Assume that $S(k)$ can be analytically continued to a meromorphic function on the whole complex plane $\mathbb{C}$. Then, a resonance is defined as one of these equivalent forms:

i) Pairs of poles of the analytic continuation of $S(k)$ located symmetrically with respect to the negative imaginary axis;
ii) Pairs of complex conjugate poles of the analytic continuation of $S(E)$ across the positive real axis.

In the language of Riemann surfaces, these poles lie on the second sheet of the Riemann surface corresponding to the transformation $k = \sqrt{E}$. These pairs are located at: $z_R = E_R - i\Gamma/2$ and $z_R^* = E_R + i\Gamma/2$, $E_R, \Gamma > 0$. Each of these pairs of resonance poles may have a multiplicity bigger than one, which is the same for each member of the pair. This multiplicity is preserved when we change from the momentum representation $S(k)$ to the energy representation $S(E)$ and vice-versa.

The existence of these analytic continuations is usually related to the verification of certain *causality conditions* [3].

2.2. Definition of resonances from the physicists point of view

We here mention a few definitions that come from the resonance scattering. From this point of view, we can define resonances by one of these usually equivalent choices [3, 5]:

i) Large delay times. This is the difference of times that an incident particle would stay in the interacting region with or without interaction. Delay times are measurable [5].
ii) Sudden bump in the cross section around a given energy E_R and with width Γ. The bump's width is associated to the mean life $\tau = 1/\Gamma$.
iii) Sudden change of the phase shift $\delta_\ell(E)$, around E_R, in the energy representation.
iv) The scattering amplitude $\psi(E)$ for the decaying state has a Lorentzian shape:

$$|\psi(E)|^2 \approx N \frac{\Gamma}{(E - E_R)^2 + \Gamma^2/4} \,. \tag{2}$$

Physics determines the meaning of the constants E_R and Γ, respectively, the real and imaginary parts of the resonance poles of the definitions in the previous subsection. E_R means the energy at which the resonance is produced and Γ is the width of the bump in the cross section that detects the resonance.

Concerning the physical meaning of the imaginary part $\Gamma/2$. The usual identification between the width Γ and the inverse of the mean life τ [5] is far from being trivial. First of all, the width is often quite difficult to be measured with precision. Sometimes it is not possible to measure both for a decaying process. Thus, this identification is often ambiguous [9].

Probably the best characterization of a resonance state is that it should have a scattering amplitude (proportional to the square of the modulus of the wave function in the energy representation) of Breit–Wigner type as in (2). This unifies both meanings of Γ as the width and as the inverse of the mean life. In fact, Γ is the width of (2) and its Fourier transform, that gives the decay mode, an exact decaying exponential [12]. On the other hand, no vector in Hilbert space may have an energy distribution of Breit–Wigner type. It can be the case of a Gamow vector constructed, as explained below, with the help of spaces of Hardy functions.

2.3. On the decay of a quantum state

Let us consider a vector state ϕ in the absolutely continuous Hilbert space for the total Hamiltonian H and consider the decay amplitude given by $|\langle\phi|e^{-itH}\phi\rangle|^2$, with $t > 0$. As a consequence of the Riemann–Lebesgue lemma for integrable functions, one has that $\lim_{t\mapsto\infty} |\langle\phi|e^{-itH}\phi\rangle| = 0$. The state ϕ can be considered as a vector state describing a resonance if for a large range of values of time, neither close to zero nor very large, the function $|\langle\phi|e^{-itH}\phi\rangle|^2$ is approximately proportional to $e^{-\Gamma t}$ with $\Gamma > 0$, i.e., ϕ decays exponentially for this time range. However, as a consequence of the semiboundedness of the Hamiltonian H, no vector state may decay exponentially for all positive values of time [6]. In fact, due to the properties of the Fourier transform, the amplitude of such a state in the energy representation must be proportional to (1), which is only possible if the spectrum of H is the whole real line. Deviations of the exponential law decay are attributed to the interaction of the resonance with the external media (background) or other effects like re-scattering [6].

A possible cure is the split of the decaying state into a term that decays exponentially for all values of time $t > 0$, plus another term that justifies the deviations (background term), $\phi = \psi^D + \psi^B$. But then, neither ψ^D nor ψ^B can be Hilbert space normalizable vector states. However, such a cure is possible and it is quite reasonable, as we shall discuss in the sequel.

2.4. Determination of resonances

The purpose of this paragraph is to present a very brief review on the most usual methods for the determination of the resonances.

These methods are based in different definitions of resonances. The complex scaling method and the use of the Krein formula start from the definition of resonances as singularities of the analytic extensions of the resolvent. The Krein formula relates different self-adjoint extensions of symmetric operators with finite equal deficiency indices.

The second type of methods comes from the consideration of resonances as poles of the analytic continuation of the S-matrix. In general, it is not easy to find the explicit form of the S-matrix, and therefore, we have to resort to indirect methods. This is quite feasible for one-dimensional systems as we shall discuss later.

Also, resonances poles are often looked as generalized complex eigenvalues of the total Hamiltonian $H\psi^D = z_R\psi^D$, with $z_R = E_R - i\Gamma/2$. The self-adjointness of H shows that the corresponding eigenvector ψ^D cannot belong to the Hilbert space $\mathcal{H}$ of the states of the system, but instead it is a functional which belongs to an extension of $\mathcal{H}$, as shall discussed later. The problem here is that in these extensions, the discrete spectrum of H may be even a whole complex half-plane. The point is how to isolate the eigenvalues of H, which are resonance poles. This problem has been solved by H. Baumgärtel [13]. The vector ψ^D is then the decaying Gamow vector. Along $H\psi^D = z_R\psi^D$, we also have the solution to the eigenvalue problem $H\psi^G = z_R^*\psi^G$ and ψ^G is called the growing Gamow vector, which is nothing else that the time reversal of ψ^D.

2.4.1. The Complex Scaling Method and the Krein formula. Here, we include a few comments on the methods that derive from the consideration of resonances as poles of the analytic continuation of the resolvent.

The Complex Scaling Method [7, 14] requires potentials belonging to the so-called class of dilation analytic potentials (DAP) [7, 14]. One starts with the transformation $U(\theta)\psi(\mathbf{x}) = e^{3\theta/2}\psi(e^\theta\,\mathbf{x})$. When $V(\mathbf{x})$ is a DAP, then, $H(\theta) := U(\theta)[H_0 + V]U^{-1}(\theta) = e^{-2\theta}\,H_0 + V(\theta)$ admits an analytic continuation for complex values of θ in a strip of the complex plane. The spectrum $\sigma(H(\theta))$ of $H(\theta)$, only depends on the imaginary part of θ and has two components: i.) a complex continuous spectrum, which is the semiaxis $e^{-2\theta}\,\lambda$, with $\lambda \in [0,\infty)$ and ii.) a discrete spectrum of complex eigenvalues having zero as the only possible limit point. These eigenvalues are the resonance poles (in the sense of the above definition making use of the resolvents) [7, 14] and do not depend on θ (although the number eigenvectors of $H(\theta)$ does depend on θ). Each of the resonances, say z_n, satisfies an eigenvalue equation of the type $H(\theta)\psi_n(\theta) = z_n\psi(\theta)$.

This method is quite suggesting. In fact is like a curtain with a rail (the continuous spectrum for $H(\theta)$) with the origin as a fixed point were being moved downwards with angle θ, disclosing the resonance poles. In addition the eigenvectors $\psi_n(\theta)$ are normalizable, i.e., vectors in the Hilbert space $\mathcal{H}$. However, they depend on the value of θ and therefore cannot be used as Gamow vectors, i.e., vector states for resonances.

The Krein formula as stated before, gives us the relations between the resolvents of two different self-adjoint extensions of a symmetric operator [15]. This can be useful to obtain resonances produced by point potentials, since these potentials are often defined by this type of self-adjoint extensions, as is the case of a delta type perturbation. In such a case, H_0 and $H = H_0 + V$ may be two different self-adjoint extensions of the same symmetric operators and therefore their resolvents be easily comparable through the Krein formula. This formula is easily computable when the deficiency indices are $(1,1)$, becomes computationally more involved when they are $(2,2)$ and difficult or even intractable in most cases for higher deficiency indices.

2.4.2. One-dimensional resonance scattering. The one-dimensional resonance scattering is a laboratory friendly to user for the study of resonance behavior. A particularly common situation arises when both H_0 (usually $H_0 = \mathbf{p}^2/(2m)$) and $V(\mathbf{x})$

are spherically symmetric. When $\ell = 0$, where ℓ is the orbital angular momentum, the Schrödinger equation is an ordinary differential equation in the radial variable $r \geq 0$. Let us denote by $\chi(r;E)$ an arbitrary solution withe energy $E > 0$. The asymptotic form of $\chi(r;E)$ far from the region in which the potential acts has the following form [2]:

$$\chi(r;E) = \mathcal{F}_1(k)\, e^{ikr} + \mathcal{F}_2(k)\, e^{-ikr}\,, \quad k = \sqrt{2mE/\hbar^2}\,. \tag{3}$$

Observe that e^{-ikr} denotes a free incoming wave and e^{ikr} a free outgoing wave, so that the S-matrix has the form

$$S(k) = -\frac{\mathcal{F}_1(k)}{\mathcal{F}_2(k)}\,. \tag{4}$$

Thus, the search for resonances as poles of the S-matrix is equivalent to the search of complex zeros of $\mathcal{F}_2(k)$. This gives in general a transcendental function that should be solved numerically. Resonances exist for many known models and its number is often infinite. If for a complex value k_R, we have that $\mathcal{F}_2(k_R) = 0$, write $z_R := k_R^2\hbar^2/(2m)$, then, for large values of r, the wave function has the form:

$$\chi(r;z_R) \approx \mathcal{F}_1(k_R)\, e^{ik_Rr}\,. \tag{5}$$

We see that there is only an outgoing wave function without incoming wave function. This situation is a consequence of imposing the condition $\mathcal{F}_2(k) = 0$, the *purely outgoing boundary condition*. Since $\chi(r;E)$ is a solution of the Schrödinger equation $H\chi(r;E) = E\chi(r;E)$, we must have $H\chi(r;z_R) = z_R\chi(r;z_R)$, i.e., $\chi(r;z_R)$ is the decaying Gamow vector (or Gamow function). Since the imaginary part of k_R is negative [3], the decaying Gamow function grows exponentially for large values of r. Note that this Gamow vector (or Gamow function) should fulfill the boundary condition $\chi(0;z_R) = 0$.

However, the interesting range of one-dimensional models covers more situations in which resonances play an interesting role. Examples are the finite square well potential and the semi oscillator with or without point potentials. This model has resonances with sometimes unexpected behavior and we shall describe it briefly later. In such case, resonances are found by imposing again purely outgoing boundary conditions, as we shall do in the proposed example.

2.5. Resonance scattering

Here, we are considering a resonance scattering situation as described in the beginning of Section 2, with the Hamiltonian pair $\{H_0, H\}$. Assume that the incoming free state is $\psi^{\rm in}$ and the outgoing free state is $\psi^{\rm out}$. However, after the scattering we cannot detect the whole state $\psi^{\rm out}$ but instead its projection into the region occupied by a *registration apparatus*. The projection of $\psi^{\rm out}$ into this region is a state vector here denoted as $\phi^{\rm out}$. The main object in our formalism is the *transition amplitude* between the scattered state and the registered state.

$$\langle\phi^{\rm out}|\psi^{\rm out}\rangle = \langle\phi^{\rm out}|S\psi^{\rm in}\rangle = \int_0^\infty [\phi^{\rm out}(E)]^*\, S(E)\, \psi^{\rm in}(E)\, dE\,. \tag{6}$$

For simplicity, we are assuming that H_0 has a simple absolutely continuous spectrum which is $\mathbb{R}^+ = [0, \infty)$. Then, it is unitarily equivalent to the multiplication operator by E on $L^2(\mathbb{R}^+)$. This variable E is the energy. On the other hand, since the S-matrix commutes with H_0, on $L^2(\mathbb{R}^+)$ is represented by a function $S(E)$ of the energy. We have already seen that $S(E)$ is often analytically continuable and its analytic continuation admits as support the Riemann surface associated the square root. If we choose $[\phi^{\rm out}(E)]^*$ and $\psi^{\rm in}(E)$ to be analytically continuable functions from above to below on the lower half-plane, we can find an interesting decomposition for the integral in (6):

$$\int_\gamma [\phi^{\rm out}(z^*)]^* \, S_{II}(z) \, \psi^{\rm in}(z) \, dz - 2\pi i \sum \text{Residues} \, [[\phi^{\rm out}(z^*)]^* \, S_{II}(z) \, \psi^{\rm in}(z)] \, . \quad (7)$$

Here, $S_{II}(z)$ is the analytic continuation of $S(E)$ beyond its cut $\mathbb{R}^+ = [0, \infty)$, supported on the second sheet of the Riemann surface. The functions $[\phi^{\rm out}(E)]^*$ and $\psi^{\rm in}(E)$ should be defined on the upper rim of the cut, so that the analytic continuation is supported on the lower half-plane in the second sheet. The contour γ lies on this half-plane, although under some conditions it can be moved to the negative semiaxis of the second sheet [16]. This integral is the *background integral.*

When sufficient conditions for the existence and asymptotic completeness of the Møller wave operators are fulfilled [27, 28, 30], the S-matrix can be written as the product $S = \Omega^\dagger_{\rm OUT} \Omega_{\rm IN}$, where $\Omega_{\rm IN}$ and $\Omega_{\rm OUT}$ are these Møller wave operators. Let us write $\phi^+ := \Omega_{\rm OUT} \phi^{\rm out}$ and $\psi^- := \Omega_{\rm IN} \psi^{\rm in}$. From (6)–(7), one finds that $\langle \phi^{\rm out} | S \psi^{\rm in} \rangle$ is equal to

$$\langle \phi^+ | \psi^- \rangle = \text{background} - 2\pi i \sum \text{Residues} \, [[\phi^{\rm out}(z^*)]^* \, S_{II}(z) \, \psi^{\rm in}(z)] \, . \quad (8)$$

The vectors ϕ^+, ψ^- belong to the locally convex spaces Φ^+ and Φ^-, respectively, to be defined in the next section. Now, for arbitrary vectors $\phi^+ \in \Phi^+$ and $\psi^- \in \Phi^-$, let us define the following maps:

$$\phi^+ \longrightarrow [\phi^{\rm out}(z_R^*)]^* = \langle \phi^+ | \psi^D \rangle \quad \text{and} \quad \psi^- \longrightarrow [\psi^{\rm in}(z_R^*)]^* = \langle \psi^- | \psi^G \rangle \, . \quad (9)$$

These maps define functionals $|\psi^D\rangle$ and $|\psi^G\rangle$ on the spaces Φ^+ and Φ^-, respectively (we use the notations ψ^D and the Dirac version $|\psi^D\rangle$ indistinctly). These maps are anti-linear and continuous [12] and therefore, elements of the respective duals of these spaces. Then, for the case of having one unique resonance, or include one unique resonance between the contour γ and the positive semi-axis, (9) becomes

$$\langle \phi^+ | \psi^- \rangle = \text{background} - 2\pi i \langle \phi^+ | \psi^D \rangle s_1 \langle \psi^G | \psi^- \rangle \, . \quad (10)$$

We can write the background term in the form $\langle \phi^+ | \text{bgk} \rangle$, where $|\text{bgk}\rangle$ is a continuous antilinear functional on Φ^+ [12]. The extension of this formula to more resonance poles is straightforward. Then, if we omit the arbitrary $\phi^+ \in \Phi^+$, we have:

$$\psi^- = |\text{bgk}\rangle - 2\pi i \, |\psi^D\rangle s_1 \langle \psi^G | \psi^- \rangle = |\text{bgk}\rangle + c |\psi^D\rangle \, . \quad (11)$$

This equation is an identity in the dual space of Φ^+. Now assume that the resonance pole is at $z_R = E_R - i\Gamma/2$. Then the functional $|\psi^D\rangle$ has the following properties:

$$H|\psi^D\rangle = z_R|\psi^D\rangle\,, \quad e^{-itH}|\psi^D\rangle = e^{-iz_Rt}|\psi^D\rangle = e^{-iE_Rt}\,e^{\Gamma t/2}\,|\psi^D\rangle\,. \tag{12}$$

According to the previous definitions and discussion on the Gamow vectors, $|\psi^D\rangle$ is the decaying Gamow vector. The formalism thus far summarized manages to separate the exponential decay of an unstable quantum state from the background, supposedly responsible of the deviations of the exponential decay and their consequences [12, 16].

3. Mathematical interlude

Gamow vectors are generalized eigenvectors of the total Hamiltonian with given complex eigenvalues. Since the Hamiltonian is self adjoint, no solution of this eigenvalue equation is given by a Hilbert space. To give meaning to them, we need to extend the action of the Hamiltonian beyond the Hilbert space vector, for which we need rigged Hilbert spaces (RHS). As is well known, a RHS is a triplet of spaces

$$\Phi \subset \mathcal{H} \subset \Phi^\times\,, \tag{13}$$

where: i.) $\mathcal{H}$ is an infinite-dimensional separable Hilbert space that contains all the pure states of a given physical system. In the case of a scattering process with two Hamiltonians $\{H_0, H\}$, both must be defined and be self adjoint on $\mathcal{H}$; ii.) Φ is a subspace dense in $\mathcal{H}$ endowed with a locally convex topology finer that the topology on $\mathcal{H}$. Although not strictly necessary, Φ is often chosen to be a nuclear space [17]. Finally, iii.) $\Phi^\times$ is the antidual space of Φ, the space of continuous antilinear functionals on Φ. It is endowed with any topology compatible with the dual pair $(\Phi, \Phi^\times)$ [18].

We need to find a RHS such that H can be extended into an operator to $\Phi^\times$. One possible form is to choose Φ such that for any $\varphi \in \Phi$, $H\varphi \in \Phi$, so that Φ is stable under the action of H. If H is self adjoint, we can always find Φ with this property and also being continuous on Φ [19–22]. In this case, we can extend uniquely H to $\Phi^\times$ using the so-called duality formula. If the action of $F \in \Phi^\times$ on $\varphi \in \Phi$ is represented by $\langle\varphi|F\rangle$, then, the extension is defined with the property:

$$\langle H\varphi|F\rangle = \langle\varphi|HF\rangle\,, \tag{14}$$

where the extension is also denoted by the same letter H. If H is continuous on Φ, with its local convex topology, then its extension is continuous on $\Phi^\times$ endowed with the weak topology of the dual pair $(\Phi, \Phi^\times)$.

A *Hardy function* $\mathcal{H}^2_+$ on the upper complex half-plane is a complex analytic function on $\mathbb{C}^+ := \{z \in \mathbb{C}\,,\ \mathrm{Im}\, z > 0\}$, which is square integrable along every line parallel to the real axis and such that there exists a positive constant K such that $(z = x + iy)$

$$\int_{-\infty}^{\infty} |f(x+iy)|^2\, dy < K\,. \tag{15}$$

As a consequence the function $f(x)$ formed by the boundary values of $f(z)$ on the real line is well defined, i.e., it is a square integrable function. The values of $f(x)$ on the whole real line and on the positive semiaxis $[0,\infty)$ reproduce the values of $f(z)$ for all $\mathbb{C}^+$. Similarly, we define the space of Hardy functions on the lower half-plane, $\mathcal{H}^2_-$. Properties of these functions are summarized in [12, 16]. For a systematic study, see [29].

Now, take the Hamiltonian pair $\{H_0, H\}$ fulfilling the following conditions: i.) Their absolutely continuous spectra are simple and given by $\mathbb{R}^+ \equiv [0,\infty)$. The condition of being simple is not necessary, but simplifies discussion and notation. ii.) The Møller wave operators $\Omega_{\rm IN}$ and $\Omega_{\rm OUT}$ exist and are asymptotically complete. iii.) The realization of the S-matrix as a function in terms of the energy, $S(E)$, is analytic and can analytically be continued through the branch cut $[0,\infty)$. This continuation has pairs of conjugate poles that are identified with resonances.

As a consequence of i.) and the spectral theorem for self-adjoint operators [23], there exists a unitary operator $U : \mathcal{H}_{\rm ac} \longmapsto L^2(\mathbb{R}^+)$ such that for any ψ in the domain of H_0 and $\psi(E) := U\psi$, $U^{-1}H_0U\psi = E\psi(E)$, where $\mathcal{H}_{\rm ac}$ is the absolutely continuous subspace of H_0 [23].

Now, let $\mathcal{H}^2_+ \cap S$ and $\mathcal{H}^2_- \cap S$ be the intersections of these spaces with the Schwartz space S of infinitely differentiable functions such that they and their derivatives go to zero at the infinity faster than the inverse of any polynomial. The spaces of the restrictions of these functions to $\mathbb{R}^+$ are called $\Psi^+ := (\mathcal{H}^2_+ \cap S)\Big|_{\mathbb{R}^+}$ and $\Psi^- := (\mathcal{H}^2_- \cap S)\Big|_{\mathbb{R}^+}$, respectively.

A Hardy function on either the upper or the lower half-plane is uniquely determined by its boundary values on the positive semiaxis $\mathbb{R}^+$ [24]. Then, the mappings $j_\pm : \mathcal{H}^2_\pm \cap S \longrightarrow \Psi^\pm$ that associate a function in $\mathcal{H}^2_\pm$ to its restriction to $\mathbb{R}^+$ are one to one and onto (bijection). If we consider the topology induced by the topology of the Schwartz space on $\mathcal{H}^2_\pm \cap S$ and then, transport it by the action of $j_\pm$, we obtain as a consequence that

$$\Psi^\pm \subset L^2(\mathbb{R}^+) \subset (\Psi^\pm)^\times \tag{16}$$

are a new pair of RHS. If we anew define $\Phi^+ := \Omega_{\rm OUT}U^{-1}\Psi^+$ and $\Phi^- := \Omega_{\rm IN}U^{-1}\Psi^-$ and again endow these spaces with the topology transported by the bijections $\Omega_{\rm OUT}U^{-1}$ and $\Omega_{\rm IN}U^{-1}$, we have a new RHS:

$$\Phi^\pm \subset \mathcal{H}_{\rm ac}(H) \subset (\Phi^\pm)^\times\,, \tag{17}$$

where $\mathcal{H}_{\rm ac}(H)$ is the absolutely continuous subspace of H.

By construction, $H^n\,\Phi^\pm \subset \Phi^\pm,\;\; n = 1,2,\dots$ and H^n is continuous on $\Phi^\pm$. Note that the duality formula (14) extends H into both antiduals $(\Phi^\pm)^\times$:

$$\langle H\phi^\pm | F^\pm\rangle = \langle \phi^\pm | H\,F^\pm\rangle\,, \quad \forall\,\phi^\pm \in \Phi^\pm\,,\;\; \forall\,F^\pm \in (\Phi^\pm)^\times\,. \tag{18}$$

This formula (18) permits the definition of Gamow vectors. The above spaces and their relations can be summarized in the following diagram:

$$
\begin{array}{ccccc}
\Phi^+ & \xrightarrow{i} & \mathcal{H}_{\rm ac}(H) & \xrightarrow{i} & (\Phi^+)^\times \\
\Big\downarrow \Omega_{\rm OUT}^{-1} & & \Big\downarrow \Omega_{\rm OUT}^{-1} & & \Big\downarrow (\Omega_{\rm OUT}^{-1})^\times \\
\Omega_{\rm OUT}^{-1}\Phi^+ & \xrightarrow{i} & \mathcal{H}_{\rm ac}(H_0) & \xrightarrow{i} & \left(\Omega_{\rm OUT}^{-1}\Phi^+\right)^\times \\
\Big\downarrow U & & \Big\downarrow U & & \Big\downarrow (U)^\times \\
\Psi^+ & \xrightarrow{i} & L^2(\mathbb{R}^+) & \xrightarrow{i} & (\Psi^+)^\times \\
\Big\downarrow j_+^{-1} & & \Big\downarrow j_+^{-1} & & \Big\downarrow (j_+^{-1})^\times \\
\mathcal{H}_+^2 \cap S & \xrightarrow{i} & \mathcal{H}_+^2 & \xrightarrow{i} & (\mathcal{H}_+^2 \cap S)^\times
\end{array}
$$

Here, i denotes canonical injection. Note that i is continuous in each case. There is an analogous diagram for Φ^-, etc and $\Omega_{\rm IN}$ [12]. From the previous diagram, one concludes that $[j_+^{-1}\, U\, \Omega_{\rm OUT}^{-1}]\Phi^+ = \mathcal{H}_+^2 \cap S$. Therefore, the mapping $j_+^{-1}\, U\, \Omega_{\rm OUT}^{-1}$ transforms $\phi^+ \in \Phi$ into an analytic function $\phi^+(E)$ on the upper half-plane. For any z in the open lower half-plane, the mapping

$$\phi^+ \longrightarrow [\phi^+(z^*)]^* \tag{19}$$

defines a continuous antilinear functional on Φ^- that we shall denote as $|z\rangle$. Note that the complex conjugate of a function in $\mathcal{H}_\pm^2$ is a function in $\mathcal{H}_\mp^2$.

If we have a resonance pole located at the point $z_R = E_R - i\Gamma/2$, its corresponding decaying Gamow vector is given by

$$\phi^+ \longrightarrow [\phi^-(z_R^*)]^* = \langle \Phi^+|\psi^D\rangle\,, \tag{20}$$

with the following properties:

$$H|\psi^D\rangle = z_R|\psi^D\rangle\,, \qquad e^{-itH}|\psi^D\rangle = e^{-itE_R}\, e^{-\Gamma t/2}\, |\psi^D\rangle\,. \tag{21}$$

Along the decaying Gamow vector, we also have the growing Gamow vector: Let $\phi^-(E) := [j_-^{-1}\, U\, \Omega_{\rm IN}]\, \phi^-$ for all $\phi^- \in \Phi^-$. Then, we define the following continuous antilinear functional $|\psi^G\rangle$ on Φ^-:

$$\phi^- \longrightarrow [\phi^+(z_R^*)]^* = \langle \Phi^+|\psi^G\rangle\,. \tag{22}$$

The growing Gamow vector $|\psi^G\rangle$ has the following properties: i.) $H|\psi^G\rangle = z_R^*|\psi^G\rangle$ and ii.) $e^{-itH}|\psi^G\rangle = e^{-E_R t}e^{\Gamma t}|\psi^G\rangle$ for $t < 0$ [12, 16].

It is important to remark that, since $|\psi^D\rangle \in (\Phi^+)^\times$, the above diagram gives [12, 16]:

$$(j_+^{-1})^\times U^\times (\Omega_{\rm OUT}^{-1})^{-1}|\psi^D\rangle = \frac{N}{(E - E_R)^2 + \Gamma^2/4} \in (\mathcal{H}_+^2 \cap S)^\times\,, \tag{23}$$

where N is a normalization constant. In this sense, *the Gamow vector* $|\psi^D\rangle$ *has a Breit–Wigner energy distribution* [16].

4. Time asymmetric quantum mechanics

Here, we just want to call the attention to an application of the above mathematical model to a very important attempt to understand the asymmetric nature of quantum mechanics, which may have an important consequence in order to understand the existence of a time arrow at the microscopical level. It is not the objective of this section to discuss the important physical as well as philosophical implications of this asymmetry in quantum mechanics, but just give a brief notion of its existence. Then, the interested reader can go to the original sources for further information [1, 25, 26].

The notion of time asymmetric quantum mechanics (TAQM) comes from the idea according to which the process of creation of a resonance in resonance scattering is not just the time reversal of the process of decay.

According to this idea, one divides a scattering process into two parts:

1. **Preparation:** *States* are prepared by the *preparation apparatus*. Thus, in a scattering experiment a state is identified with an *incoming state* $\psi^{\rm in}$. A resonance is produced.

2. **Registration:** *Observables* are detected in the *registration apparatus*, which registers and measures the result of the decay of the resonance. *Detected outgoing states* $|\phi^{out}(t)\rangle\langle\phi^{out}(t)|$ *are indeed observables*, according to this principle.

The detected outgoing state $|\phi^{\rm out}(t)\rangle\langle\phi^{\rm out}(t)|$ cannot be registered before the incoming state has been completely prepared (*causality principle*). If this preparation is complete at a time t_0, this is taken as origin of times, $t_0 = 0$. The Born probability of measuring $|\phi^{\rm out}(t)\rangle\langle\phi^{\rm out}(t)|$ in the state $\rho(t) = |\psi^{\rm in}(t)\rangle\langle\psi^{\rm in}(t)|$ is given by $(t > 0)$

$$\begin{aligned} \mathcal{P}_{\rho(t)}(|\phi^{\rm out}\rangle\langle\phi^{\rm out}|) &= |\langle\phi^{\rm out}|\ \psi^{\rm in}(t)\rangle|^2 \\ &= {\rm Tr}\{\ |\phi^{\rm out}\rangle\langle\phi^{\rm out}|[e^{-iHt}|\psi^{\rm in}\rangle\langle\psi^{\rm in}|e^{iHt}] \\ &= {\rm Tr}\{\ [e^{iHt}|\phi^{\rm out}\rangle\langle\phi^{\rm out}|e^{-iHt}]\ |\psi^{\rm in}\rangle\langle\psi^{\rm in}|] \\ &= |\langle\phi^{\rm out}(t)|\ \psi^{\rm in}\rangle| = \mathcal{P}_{\rho}(|\phi^{\rm out}(t)\rangle\langle\phi^{\rm out}(t)|)\,. \end{aligned} \tag{24}$$

This somehow justifies the idea of being $|\phi^{\rm out}\rangle\langle\phi^{\rm out}|$ an observable, since it would evolve following the Heisenberg evolution. Whenever we use the realization of wave functions in the energy representation by Hardy functions as explained before, time evolution of observables follows a semigroup law.

The fundamentals of TAQM are based on a new axiom to be added to quantum mechanics, which is relevant in the scattering processes. This new axiom refers to the choice of the relevant wave functions in the processes of preparation and registration. At this point it should be remarked that, the procedure of taking the wave functions from a dense subspace in Hilbert space is indistinguishable itself from functions in the Hilbert space, as the error in any measurement can be made arbitrarily small.

This new axiom can be formulated as follows:

Preparation: For $t = 0$ states are prepared and given in the energy representation by functions in $\mathcal{H}^2_- \cap S\Big|_{\mathbb{R}^+}$. Note that $\mathcal{H}^2_- \cap S\Big|_{\mathbb{R}^+}$ is dense in $L^2(\mathbb{R}^+)$, so that any $\varphi(E) \in L^2(\mathbb{R}^+)$ can be approximated by a $\phi(E) \in \mathcal{H}^2_- \cap S\Big|_{\mathbb{R}^+}$.

Registration: Observables are $|\psi\rangle\langle\psi|$, where the ψ are approximated by functions in $\mathcal{H}^2_+ \cap S|_{\mathbb{R}^+}$.

The *time arrow* goes from the preparation apparatus to the registration apparatus [25].

5. Models of resonances

5.1. The Friedrichs model

The basic Friedrichs model has just one resonance. Nevertheless, it contains all features of resonance scattering and provides a framework for understanding resonance phenomena in realistic systems. Here, the Hamiltonian pair $\{H_0, H\}$ is given by

$$H_0 = \omega_0\, |1\rangle\langle 1| + \int_0^\infty \omega\, |\omega\rangle\langle\omega|\, d\omega\,, \tag{25}$$

where $|1\rangle$ is an eigenvector of H_0 with eigenvalue ω_0, $H_0|1\rangle = \omega_0|1\rangle$ and $|\omega\rangle$ are generalized eigenvectors of H with eigenvalues ω in the absolutely continuous spectrum of H_0, which is the positive semi-axis $[0,\infty)$, $H_0|\omega\rangle = \omega|\omega\rangle$. The total Hamiltonian is $H = H_0 + \lambda V$, where λ is a real coupling constant and V is

$$V = \int_0^\infty f(\omega)\, [|\omega\rangle\langle 1| + |1\rangle\langle\omega|]\, d\omega\,. \tag{26}$$

Here $f(\omega)$ is a function, usually taken square integrable [31], called the *form factor*.

Resonances are here obtained using Definition 1 in 2.1. The conclusion is that they are poles of the analytic continuation of the following function (sometimes called the reduced resolvent):

$$\frac{1}{\eta(z)} := \langle 1|\frac{1}{H - zI}|1\rangle\,, \tag{27}$$

with

$$\eta(z) = z - \omega_0 - \lambda^2 \int_0^\infty \frac{|f(\omega)|^2}{z-\omega}\, d\omega\,. \tag{28}$$

The function $\eta(z)$ is analytic on the complex plane except for a branch cut on the positive semiaxis $[0,\infty)$, with no zeroes. It admits an analytic continuation through the cut, both from above to below or from below to above, that can be supported by the two sheeted Riemann surface generated by the square root.

The analytic continuation has a pair of complex conjugate zeroes located at the following points:

$$z_R = \omega_0 + \lambda^2 \int_0^\infty \frac{|f(\omega)|^2}{z_R - \omega + i0}\, d\omega = \omega_0 + \lambda^2 \int_0^\infty \frac{|f(\omega)|^2}{\omega_0 - \omega + i0}\, d\omega + o(\lambda^4)\,, \quad (29)$$

$$z_R^* = \omega_0 + \lambda^2 \int_0^\infty \frac{|f(\omega)|^2}{z_R - \omega - i0}\, d\omega = \omega_0 + \lambda^2 \int_0^\infty \frac{|f(\omega)|^2}{\omega_0 - \omega - i0}\, d\omega + o(\lambda^4)\,. \quad (30)$$

The meaning of $\pm i0$ in the denominator is the usual in the theory of distributions [32]. These zeroes are poles of $\eta^{-1}(z)$ and consequently show the existence of a resonance. Note on the dependence on the coupling constant λ of the resonance poles of the reduced resolvent. If $\lambda \longmapsto 0$, then, both resonance poles go to ω_0 the eigenvalue of H_0. The usual interpretation says that as the consequence of the interaction, the bound state of H_0 becomes unstable and as a result it is a resonance with resonance poles as in (29)–(30).

On the spaces Φ^+ and Φ^- as in (17), respectively, the growing and decaying Gamow vectors are functionals that can be written explicitly as

$$|\psi^D\rangle = |1\rangle + \int_0^\infty \frac{\lambda\, f(\omega)}{z_R - \omega + i0}\, |\omega\rangle\, d\omega\,, \quad (31)$$

$$|\psi^G\rangle = |1\rangle + \int_0^\infty \frac{\lambda\, f(\omega)}{z_R^* - \omega - i0}\, |\omega\rangle\, d\omega\,. \quad (32)$$

In addition, the Hamiltonian admits respective diagonalizations as operators on $\mathcal{L}\{\Phi^-, (\Phi^+)^\times\}$ and $\mathcal{L}\{\Phi^+, (\Phi^-)^\times\}$, where $\mathcal{L}\{\Phi, \Psi\}$ is the space of continuous operators from the locally convex space Φ into the locally convex space Ψ, of the following form (on the duals $(\Phi^\pm)^\times$ we consider the weak topology):

$$H = z_R|\psi^D\rangle\langle\psi^G| + \text{background}\,, \quad (33)$$

$$H = z_R^*|\psi^G\rangle\langle\psi^D| + \text{background}\,. \quad (34)$$

The word background here denotes an integral term, which physically would correspond to the existence of the background part.

Finally, we remark that objects like Møller wave operators and the S-matrix exist for the Friedrichs model. See [31]. Poles of the S-matrix coincide with the resonance poles obtained by the method of the resolvent.

5.1.1. Double resonances. Causality conditions do not forbid the existence of resonance poles with multiplicity bigger than one. Assume for instance the existence of a resonance represented by a pair of complex conjugate poles of the analytic extension of the S-matrix $S(E)$. In this case, the decaying state as in (11) should be written as [33]

$$\psi^- = |\text{bgk}\rangle + \sum_{k=0}^{N-1} c_k|\psi_k^D\rangle\,. \quad (35)$$

The first term of the sum in (35) (excluding $|\text{bgk}\rangle$) is nothing else than the previously defined Gamow vector $|\psi_0^D\rangle = |\psi^D\rangle$. Thus, the vectors in the sum in (35) are functionals on Φ^- and satisfy the condition

$$H|\psi_k^D\rangle = z_R|\psi_k^D\rangle + k|\psi_{k-1}^D\rangle\,, \quad k = 0, 1, \dots, N-1\,. \tag{36}$$

Vectors $|\psi_k^D\rangle$ are the Gamow vectors for multiple pole (degenerated) resonances, also called Jordan–Gamow vectors [5]. We can project the extension to $(\Phi^-)^\times$ of H to the Nth-dimensional subspace of $(\Phi^-)^\times$ spanned by these vectors. In the basis given by the Jordan–Gamow vectors the restriction of the Hamiltonian has a typical Jordan block form:

$$H = \begin{pmatrix} z_R & 1 & 0 & \dots & 0 \\ 0 & z_R & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & 1 \\ 0 & 0 & 0 & \dots & z_R \end{pmatrix} \tag{37}$$

and the time evolution e^{-itH} is the corresponding exponentiation of $-itH$ with H as in (37).

In the Friedrichs model, we can produce a resonance characterized by a pair of double poles. To do it, we need a clever choice of the form factor. This is [33]:

$$f(\omega) := \frac{\sqrt{\omega}}{P(\omega)}\,, \quad P(\omega) = (\omega - \alpha)(\omega - \alpha^*)\,. \tag{38}$$

In this case, the Gamow–Jordan vectors are given by: $|\psi^D\rangle$ as in (31) and

$$|\psi_1^D\rangle = -\int_0^\infty \frac{f(\omega)}{(z_R - \omega + i0)^2}\,|\omega\rangle\, d\omega\,. \tag{39}$$

Other models showing a resonance with a double pole are the following:

1. The following Hamiltonian on $L^2(\mathbb{R})$ [34]:

$$H = -\frac{d^2}{dx^2} + \frac{\pi}{\alpha}\,\delta(x-a) + \frac{\pi}{\beta}\,\delta(x-b)\,. \tag{40}$$

2. The following Hamiltonian on $L^2(\mathbb{R}^+)$, i.e., the potential is infinite for $x \le 0$ [35]:

$$H = -\frac{d^2}{dx^2} + \alpha\,\delta(x-a) + \beta\,\delta(x-b)\,, \quad \alpha\,,\beta\,,a\,,b > 0\,. \tag{41}$$

The double pole of the analytic continuation of $S(E)$ can be found for some values of the constants a, b, α and β only [34, 35].

5.2. A one-dimensional model

An interesting one-dimensional model with resonances having a great richness of features is the half oscillator with a point potential at the origin and possibly a mass jump at the same point. In principle, the interest of this model was essentially pedagogical with the presence of an infinite number of resonances that under certain limit process (the coefficient of a delta perturbation at the origin going to the infinity) become the odd bound states of the harmonic oscillator [36]. In addition, this model has well-defined S-matrix, scattering operators, transmission and reflections coefficients, etc. However, its particular interest lies on the presence of some unexpected features [37], which opens the interest for the search for resonance models with unusual behavior.

The Hamiltonian for the model under consideration is the following:

$$H = -\frac{\hbar^2}{2m}\,\frac{d^2}{dx^2} + V_1(x) + V_2(x) \tag{42}$$

with

$$V_1(x) := \begin{cases} \frac{1}{2}\,m\omega^2 x^2 & \text{if } x < 0 \\ \\ 0 & \text{if } x > 0 \end{cases}, \qquad V_2(x) = a\delta(x) + b\delta'(x)\,. \tag{43}$$

In order that Hamiltonian (42) be well defined and self adjoint, we need to resort to the theory of self-adjoint extensions of symmetric operators with equal deficiency indices [15], according to which we need to construct the domain of the operator. In our case, this construction should include functions that have a jump at the origin. This mass jump cannot be arbitrary, but in any case, we should give a prescription that define the products of $\delta(x)$ and $\delta'(x)$ with functions showing a discontinuity at the origin. Let us write the solutions of the corresponding Schrödinger equation as $\psi(x) = \psi_1(x)H(-x) + \psi_2(x)H(x)$, where $H(x)$ is the Heaviside step function. Then, this prescription for an arbitrary function $\psi(x)$ is

$$\delta(x)\psi(x) = \frac{\psi_1(0) + \psi_2(0)}{2}\,\delta(x)\,, \tag{44}$$

$$\delta'(x)\psi(x) = \frac{\psi_1(0) + \psi_2(0)}{2}\,\delta'(x) - \frac{\psi_1'(0) + \psi_2'(0)}{2}\,\delta(x)\,. \tag{45}$$

Note that these products coincide with the usual ones when both $\psi(x)$ and its derivative $\psi'(x)$ are continuous at the origin. The resulting Schrödinger equation is then an equation for distributions.

The domain of self-adjointness of the Hamiltonian (42) is the space of functions $\psi(x) = \psi_1(x)H(-x) + \psi_2(x)H(x)$ in the Sobolev space $W_2^2(\mathbb{R}\backslash\{0\})$ with the additional condition ($\hbar = 1$)

$$\begin{pmatrix} \psi_2(0) \\ \psi_2'(0) \end{pmatrix} = \begin{pmatrix} \frac{1+mb}{1-mb} & 0 \\ \frac{2ma}{1-m^2b^2} & \frac{1-mb}{1+mb} \end{pmatrix} \begin{pmatrix} \psi_1(0) \\ \psi_1'(0) \end{pmatrix} \tag{46}$$

Although these matching conditions do not apply in the case $b = \pm 1/m$, self-adjoint extensions can be defined for this particular case [37].

In order to obtain the resonances, we use the earlier mentioned method of the purely outgoing boundary conditions, according to which there is no incoming wave, so that it must be equal to zero. This gives a transcendental equation for which the solutions not only give the resonances, but also bound and antibound states. This transcendental equation can be numerically solved, after some algebra, with the aid of a package like Mathematica.

A thorough description of the results obtained is reported in [37]. Let us mention here some of the most relevant in order to understand the interest of the model.

In absence of a mass jump at the origin:

1. There are a countably infinite number of resonances even in the absence of any point potential at the origin (in this case $V_2(x) \equiv 0$). When we switch on $V_2(x) = a\delta(x)$, all resonances save for one, have a smaller imaginary part (higher mean life) no matter if a is either positive or negative. For $a > 0$, it appears an extra resonance (which did not exist for $V_2(x) \equiv 0$) which does not follow the general pattern. When a is very small and goes to zero, its real and imaginary parts go to $+\infty$ and $-\infty$ respectively to disappear in the limit $a = 0$. For higher values of a, this resonance has the smaller real part and otherwise behaves like the others. Because of the unusual behavior of this resonance, we have named it as the *maverick resonance*. It does not exist for $a < 0$.
2. There exists one and only one bound state for $a < 0$ and below a certain threshold. Between zero and this threshold, we do not have bound states but instead one and only one antibound state.
3. When we switch on the term $b\delta'(x)$, we obtain analogous results except for the limit values $b = \pm 1/m$ where each resonance collapse into a bound state.

In presence of a mass jump at the origin.

1. Assume that the mass is m_1 if $x < 0$ and m_2 if $x > 0$. Then, the relevant parameter is $r = m_2/m_1$. The maverick resonance still exists, but its presence is only observable near $r = 1$, i.e., the limit of equal masses. All other features remain essentially equal except one:
2. There are two critical points for b, which are $b = -1/r$ and $b = -(1 + r)/2$. When the value of b lies on one of these two critical points, all resonances collapse into bound states. If $a = 0$, the energy of these bound states coincide for both critical points and is given by the even energy levels of the oscillator. If $a \neq 0$, the energy levels corresponding to both critical points are slightly different, but all them have the form $A + Bn$, where $n = 0, 1, 2, \ldots$, B is always close to 2 and A depends on a and r.

In general, it is possible to plot the eigenfunctions of the Hamiltonian with complex eigenvalues which are resonance poles, i.e., Gamow vectors. One sees that for large values of $x > 0$, these eigenfunctions have an approximate exponential grow.

One final remark: Formula (45) shows that the contribution to the potential given by $a\delta'(x)$ behaves like the derivative of the delta. This delta prime perturbation has been given by the particular self-adjoint choice of the Hamiltonian given by the matching conditions (45). It may be surprising to say that the determination of a delta prime type perturbation is not unique. In fact, there are other possible matching conditions determining other self-adjoint determinations of the Hamiltonian that also give a delta prime term. In all cases, the operational behavior of the term $a\delta'(x)$ is the same for functions with continuous derivative at the origin, but the self-adjoint extension that determines this perturbation is different [38, 39].

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Part III: Groups and Non-commutative Structures

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Symplectic Dual Pair Related to Deformed $\mathfrak{so}(5)$

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Abstract. We investigate the Hamiltonian systems defined by one-parameter deformation $\mathfrak{so}_{0,\alpha}(5)$ of the orthogonal Lie algebra $\mathfrak{so}(5)$. We discuss related symplectic dual pair. The possible physical applications are also discussed.

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Keywords. Integrable Hamiltonian systems, Lie algebra deformation, symplectic dual pair, momentum map.

1. Introduction

One of the most effective methods of finding the integrable Hamiltonian systems is the construction of bi-Hamiltonian structure on Lie–Poisson space, e.g., see [1, 4]. Following [3] in this presentation we consider such a structure on vector space $\mathcal{L}_+(5)$ of the upper triangular real 5×5 matrices. We also show how to integrate in quadratures the Hamiltonian equations obtained in this way. At the end we present possible applications of the obtained results in nonlinear wave optics. We also construct symplectic dual pair for Lie–Poisson space $\mathfrak{p}(1,1)^*$ and $\mathcal{L}_+(5)$ (see Proposition 1).

2. Bi-Hamiltonian structure related to $\mathfrak{so}_{\lambda,\alpha}(5)$

In paper [4] we introduced and investigated the bi-Hamiltonian systems defined by the infinite-parameter deformation of the Lie algebra of Hilbert–Schmidt operators. The case has its finite-dimensional version which for dimension $d = 5$ leads to the Lie algebra $\mathfrak{so}_{\lambda,\alpha}(5)$ consisting of real 5×5 matrices

$$\begin{pmatrix} 0 & \alpha b & \alpha\lambda\vec{u}^\top \\ -b & 0 & \lambda\vec{w}^\top \\ -\vec{u} & -\vec{w} & \boldsymbol{\delta} \end{pmatrix}, \quad b \in \mathbb{R}, \quad \vec{u}, \vec{w} \in \mathbb{R}^3, \quad \boldsymbol{\delta} \in \mathfrak{so}(3) \tag{1}$$

1.	$\lambda = 1 \quad \wedge \quad \alpha = 1$ $\lambda > 0 \quad \wedge \quad \alpha > 0$	$\mathfrak{so}(5)$ $\simeq \mathfrak{so}(5)$
2.	$\lambda = -1 \quad \wedge \quad \alpha = 1$ $\lambda < 0 \quad \wedge \quad \alpha > 0$	$\mathfrak{so}(3,2) \simeq \mathfrak{sp}(2,\mathbb{R})$ $\simeq \mathfrak{so}(3,2)) \simeq \mathfrak{sp}(2,\mathbb{R})$
3.	$\lambda = -1 \quad \wedge \quad \alpha = -1$ $\lambda < 0 \quad \wedge \quad \alpha < 0$	$\mathfrak{so}(1,4)$ $\simeq \mathfrak{so}(1,4)$
4.	$\lambda = -1 \quad \wedge \quad \alpha = 0$ $\lambda < 0 \quad \wedge \quad \alpha = 0$	$\mathfrak{p}(1,3)$ (Poincaré algebra) $\simeq \mathfrak{p}(1,3)$
5.	$\lambda = 0 \quad \wedge \quad \alpha = 0$	Galilean algebra
6.	$\lambda = 1 \quad \wedge \quad \alpha = 0$ $\lambda > 0 \quad \wedge \quad \alpha = 0$	$\mathfrak{e}(4)$ (Euclidean algebra) $\simeq \mathfrak{e}(4)$
7.	$\lambda = 0 \quad \wedge \quad \alpha \neq 0$	$(\mathfrak{so}_\alpha(2) \times \mathfrak{so}(3)) \ltimes \mathrm{Mat}_{3\times 2}(\mathbb{R})$

Table 1.

with Lie bracket defined by matrix commutator and $\alpha, \lambda \in \mathbb{R}$. For certain values of the parameters α and λ in the family of Lie algebras $\mathfrak{so}_{\lambda,\alpha}(5)$ one finds almost all physically important Lie algebras (see Table 1 taken from [3]). It also follows from Table 1 that one can consider $SO_{\lambda,\alpha}(5)$ as a two-parameter deformation of the orthogonal group $SO(5)$. In Ref. [3] we integrated the related Hamiltonian system for $\alpha \neq 0$ and $\lambda \neq 0$. The case $\alpha = 0$ was considered in [2]. In this presentation we discuss the case $\lambda = 0$.

Let $\mathcal{L}_+(5)$ be the vector space of strictly upper triangular real 5×5 matrices. Hence in block notation $\rho \in \mathcal{L}_+(5)$ has the form

$$\rho = \left(\begin{array}{cc|c} 0 & a & \vec{x}^\top \\ 0 & 0 & \vec{y}^\top \\ \hline \mathbf{0} & \mathbf{0} & \boldsymbol{\mu} \end{array} \right), \tag{2}$$

where $a \in \mathbb{R}$, $\vec{x} = (x_1, x_2, x_3)^\top$, $\vec{y} = (y_1, y_2, y_3)^\top \in \mathbb{R}^3$ and

$$\boldsymbol{\mu} := \begin{pmatrix} 0 & \mu_3 & -\mu_2 \\ 0 & 0 & \mu_1 \\ 0 & 0 & 0 \end{pmatrix} \in \mathrm{Mat}_{3\times 3}(\mathbb{R}). \tag{3}$$

Then the pairing

$$\langle \boldsymbol{X}, \rho \rangle = \mathrm{Tr}(\rho \boldsymbol{X}), \tag{4}$$

between $\rho \in \mathcal{L}_+(5)$ and $\boldsymbol{X} \in \mathfrak{so}_{\lambda,\alpha}(5)$ defines isomorphism of $\mathcal{L}_+(5)$ with the dual $\mathfrak{so}_{\lambda,\alpha}(5)^*$ of Lie algebra $\mathfrak{so}_{\lambda,\alpha}(5)$.

Using this isomorphism we obtain for $f, g \in C^\infty(\mathcal{L}_+(5))$ the Lie–Poisson bracket

$$\begin{aligned}
\{f,g\}_{\lambda,\alpha} =& \ \operatorname{Tr}\left(\rho\left[\frac{\partial f}{\partial\rho},\frac{\partial g}{\partial\rho}\right]\right) \\
=& \lambda a\left(\frac{\partial f}{\partial\vec{x}}\cdot\frac{\partial g}{\partial\vec{y}}-\frac{\partial f}{\partial\vec{y}}\cdot\frac{\partial g}{\partial\vec{x}}\right) \\
&+\vec{\delta}\cdot\left(\alpha\lambda\left(\frac{\partial f}{\partial\vec{x}}\times\frac{\partial g}{\partial\vec{x}}\right)+\lambda\left(\frac{\partial f}{\partial\vec{y}}\times\frac{\partial g}{\partial\vec{y}}\right)+\left(\frac{\partial f}{\partial\vec{\delta}}\times\frac{\partial g}{\partial\vec{\delta}}\right)\right) \\
&+\frac{\partial g}{\partial a}\vec{x}\cdot\frac{\partial f}{\partial\vec{y}}-\frac{\partial f}{\partial a}\vec{x}\cdot\frac{\partial g}{\partial\vec{y}}-\alpha\frac{\partial g}{\partial a}\vec{y}\cdot\frac{\partial f}{\partial\vec{x}}+\alpha\frac{\partial f}{\partial a}\vec{y}\cdot\frac{\partial g}{\partial\vec{x}} \\
&+\vec{x}\cdot\left(\frac{\partial f}{\partial\vec{x}}\times\frac{\partial g}{\partial\vec{\mu}}+\frac{\partial f}{\partial\vec{\mu}}\times\frac{\partial g}{\partial\vec{x}}\right)+\vec{y}\cdot\left(\frac{\partial f}{\partial\vec{y}}\times\frac{\partial g}{\partial\vec{\mu}}+\frac{\partial f}{\partial\vec{\mu}}\times\frac{\partial g}{\partial\vec{y}}\right),
\end{aligned} \tag{5}$$

where $\vec{\mu}=(\mu_1,\mu_2,\mu_3)^\top$, which is compatible with the Poisson bracket $\{\cdot,\cdot\}_{\epsilon,\alpha}$ on $\mathcal{L}_+(5)$ defined by $\mathfrak{so}_{\epsilon,\alpha}(5)$ (see [3]). Hence the Casimir functions

$$h_1 = \vec{x}^2+\alpha\vec{y}^2+\alpha\epsilon\vec{\mu}^2+\epsilon a^2, \tag{6}$$

$$h_2 = \alpha\epsilon(\vec{\mu}\cdot\vec{y})^2+\epsilon(\vec{\mu}\cdot\vec{x})^2+\left(\epsilon a\vec{\mu}-\vec{x}\times\vec{y}\right)^2, \tag{7}$$

of the bracket $\{\cdot,\cdot\}_{\epsilon,\alpha}$ are integrals of motion in involution with respect to $\{\cdot,\cdot\}_{\lambda,\alpha}$.

The Hamilton equations on $(\mathcal{L}_+(5),\{\cdot,\cdot\}_{\lambda,\alpha})$ generated by the Hamiltonian

$$H=\gamma h_1+\nu h_2, \quad \gamma,\nu\in\mathbb{R}, \tag{8}$$

assume the following form

$$\frac{da}{dt}=0, \quad \frac{d\vec{\mu}}{dt}=0, \tag{9}$$

$$\begin{aligned}
\frac{d\vec{x}}{dt} =& 2(\lambda-\epsilon)\left(\gamma\alpha\left(a\vec{y}+\vec{x}\times\vec{\mu}\right)+\nu\left(\alpha\vec{\mu}\times((\vec{x}\times\vec{y})\times\vec{y})\right.\right. \\
&\left.\left.+\ \alpha\epsilon a\vec{\mu}^2\vec{y}+\epsilon a^2\vec{x}\times\vec{\mu}+a(\vec{x}\times\vec{y})\times\vec{x}\right)\right),
\end{aligned} \tag{10}$$

$$\begin{aligned}
\frac{d\vec{y}}{dt} =& 2(\lambda-\epsilon)\left(\gamma\left(-a\vec{x}+\alpha\vec{y}\times\vec{\mu}\right)+\nu\left(\vec{\mu}\times((\vec{y}\times\vec{x})\times\vec{x})\right.\right. \\
&\left.\left.-\ \epsilon a\vec{\mu}^2\vec{x}+\epsilon a^2\vec{y}\times\vec{\mu}+a(\vec{x}\times\vec{y})\times\vec{y}\right)\right).
\end{aligned} \tag{11}$$

These equations depend on λ through $\lambda-\epsilon$. Thus as long as $\lambda\neq\epsilon$ the form of their solutions is independent on the choice of value of the parameter λ including the case $\lambda=0$.

Therefore in this presentation we will fix our attention on the geometry as well as physical interpretation of the system in the case when $\lambda=0$. For $\lambda\neq 0$ these questions were discussed in Refs. [2] and [3].

3. The symplectic dual pair for λ=0

In order to find a physical interpretation of the Hamiltonian dynamics defined by Hamiltonian (8) let us consider the canonical phase space $T^*\mathbb{R}^5$ with the symplectic action of the group $SO_{0,\alpha}(5) := \exp\left(\mathfrak{so}_{0,\alpha}(5)\right)$ defined for $g \in SO_{0,\alpha}(5)$ and $(p,q) \in T^*\mathbb{R}^5 \cong \left(\mathbb{R}^5\right)^* \times \mathbb{R}^5$ by

$$\Phi_g(p,q) := (pg^{-1}, gq). \tag{12}$$

We will use the following block notations for

$$p = (p_{-1}, p_0, \vec{p}^\top), \quad q = \begin{pmatrix} q_{-1} \\ q_0 \\ \vec{q} \end{pmatrix}, \tag{13}$$

where $p_{-1}, p_0, q_{-1}, q_0 \in \mathbb{R}$ and $\vec{p}, \vec{q} \in \mathbb{R}^3$. In consistence with (13) we will describe $g \in SO_{0,\alpha}(5)$ in the following way

$$g = \left(\begin{array}{c|c} \Lambda & 0 \\ \hline U & R \end{array}\right), \tag{14}$$

where $R \in O(3)$, $U = (\vec{t}\,\vec{v}) \in Mat_{3\times 2}(\mathbb{R})$, and $\Lambda \in Mat_{2\times 2}(\mathbb{R})$ satisfies

$$\Lambda^\top \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \quad \text{and} \quad \det\Lambda \det R = 1. \tag{15}$$

We note here that in the case $\alpha = 0$ one obtains Galilean group and the action (12) gives transitions between Galilean inertial reference frames.

Using the notation (14) we find that the coadjoint action of $SO_{0,\alpha}(5)$ on $\mathcal{L}_+(5)$ is given by

$$Ad^*_{g^{-1}} \begin{pmatrix} \vec{x}^\top \\ \vec{y}^\top \end{pmatrix} = \Lambda \begin{pmatrix} (R\vec{x})^\top \\ (R\vec{y})^\top \end{pmatrix}, \tag{16}$$

$$Ad^*_{g^{-1}}(\vec{\mu}) = R\vec{\mu} - (R\vec{x}) \times \vec{t} - (R\vec{y}) \times \vec{v}, \tag{17}$$

$$Ad^*_{g^{-1}}(a) = \det\Lambda \left(a + \alpha\vec{t}^\top R\vec{y} - \vec{v}^\top R\vec{x}\right). \tag{18}$$

The map $\mathcal{J}_\alpha : T^*\mathbb{R}^5 \to \mathcal{L}_+(5)$ defined by

$$\begin{aligned} a &= \alpha p_{-1}q_0 - p_0 q_{-1}, & \vec{x} &= -q_{-1}\vec{p}, \\ \vec{y} &= -q_0\vec{p}, & \vec{\mu} &= \vec{p} \times \vec{q}, \end{aligned} \tag{19}$$

is $SO_{0,\alpha}(5)$-equivariant

$$\mathcal{J}_\alpha \circ \Phi_g = Ad^*_{g^{-1}} \circ \mathcal{J}_\alpha. \tag{20}$$

Poisson map of canonical phase space $\left(T^*\mathbb{R}^5, \{\cdot,\cdot\}\right)$, where

$$\{f,g\}(p,q) = \sum_{i=-1}^{3} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right), \tag{21}$$

for $f,g \in C^\infty\left(T^*\mathbb{R}^5\right)$, into the Lie–Poisson space $(\mathcal{L}_+(5), \{\cdot,\cdot\}_{0,\alpha})$.

In such a way we have obtained an integrable Hamiltonian system on $T^*\mathbb{R}^5$ whose Hamiltonian is given by

$$H_\alpha = H \circ \mathcal{J}_\alpha = \gamma\Big(\alpha\epsilon\vec{q}^2\vec{p}^2 + \left(q_{-1}^2 + \alpha q_0^2\right)\left(\vec{p}^2 + \epsilon p_0^2 + \alpha\epsilon p_{-1}^2\right) - \alpha\epsilon\left(q_{-1}p_{-1} + q_0p_0 + \vec{q}\cdot\vec{p}\right)^2 + 2\alpha\epsilon\left(q_{-1}p_{-1} + q_0p_0\right)\left(\vec{q}\cdot\vec{p}\right)\Big) + \nu\epsilon^2\left(\alpha p_{-1}q_0 - p_0q_{-1}\right)^2\left(\vec{p}\times\vec{q}\right)^2 \tag{22}$$

and corresponding Hamilton equations are the following

$$\frac{dp_{-1}}{dt} = 2\gamma\left(\left(\vec{p}^2 + \epsilon p_0^2\right)q_{-1} - \alpha\epsilon q_0p_0p_{-1}\right) - - 2\nu\epsilon^2\left(\alpha p_{-1}q_0 - p_0q_{-1}\right)\left(\vec{p}\times\vec{q}\right)^2 p_0, \tag{23}$$

$$\frac{dp_0}{dt} = 2\gamma\alpha\left(\left(\vec{p}^2 + \alpha\epsilon p_{-1}^2\right)q_0 - \epsilon q_{-1}p_{-1}p_0\right) + + 2\nu\alpha\epsilon^2\left(\alpha p_{-1}q_0 - p_0q_{-1}\right)\left(\vec{p}\times\vec{q}\right)^2 p_{-1}, \tag{24}$$

$$\frac{dq_{-1}}{dt} = -2\gamma\alpha\left(\alpha\epsilon q_0^2p_{-1} - \epsilon q_0p_0q_{-1}\right) - - 2\alpha\nu\epsilon^2\left(\alpha p_{-1}q_0 - p_0q_{-1}\right)\left(\vec{p}\times\vec{q}\right)^2 q_0, \tag{25}$$

$$\frac{dq_0}{dt} = -2\gamma\left(\epsilon q_{-1}^2p_0 - \alpha\epsilon q_{-1}p_{-1}q_0\right) + + 2\nu\epsilon^2\left(\alpha p_{-1}q_0 - p_0q_{-1}\right)\left(\vec{p}\times\vec{q}\right)^2 q_{-1}, \tag{26}$$

$$\frac{d\vec{p}}{dt} = 2\gamma\alpha\left(\epsilon\vec{p}^2\vec{q} - \epsilon\vec{q}\cdot\vec{p}\,\vec{p}\right) + + 2\nu\epsilon^2\left(\alpha p_{-1}q_0 - p_0q_{-1}\right)^2\left(\vec{p}^2\vec{q} - \left(\vec{p}\cdot\vec{q}\right)\vec{p}\right), \tag{27}$$

$$\frac{d\vec{q}}{dt} = -2\gamma\left(\left(\alpha\epsilon\vec{q}^2 + \alpha q_0^2 + q_{-1}^2\right)\vec{p} - \alpha\epsilon\vec{q}\cdot\vec{p}\vec{q}\right) - - 2\nu\epsilon^2\left(\alpha p_{-1}q_0 - p_0q_{-1}\right)^2\left(\vec{q}^2\vec{p} - \left(\vec{q}\cdot\vec{p}\right)\vec{q}\right). \tag{28}$$

Beside from the integrals of motion $a = \alpha q_{-1}p_0 - q_0p_{-1}$ and $\vec{\mu} = \vec{q}\times\vec{p}$ the Hamiltonian (22) has the following three additional integrals of motion

$$d_+ = \vec{p}^2, \tag{29}$$

$$d_- = q_{-1}^2 + \alpha q_0^2, \tag{30}$$

$$d_3 = q_{-1}p_{-1} + q_0p_0 + (\vec{q}\cdot\vec{p}). \tag{31}$$

They form the Lie algebra

$$\{d_+, d_-\} = 0, \tag{32}$$

$$\{d_3, d_+\} = 2d_+, \tag{33}$$

$$\{d_3, d_-\} = -2d_- \tag{34}$$

which is the Lie algebra $\mathfrak{p}(1,1)$ of Poincaré group $P(1,1)$ for the (1+1)-dimensional space-time.

In subsequent considerations we will use the following realizations of $\mathfrak{p}(1,1)$ and $P(1,1)$:

$$\mathfrak{p}(1,1) := \left\{ \begin{pmatrix} \delta\sigma_1 & \vec{\delta} \\ 0 & 0 \end{pmatrix} : \quad \delta \in \mathbb{R} \quad \text{and} \quad \vec{\delta} \in \mathbb{R}^2 \right\}, \tag{35}$$

and

$$P(1,1) := \left\{ A = \begin{pmatrix} \Delta(\delta) & \vec{r} \\ 0 & 1 \end{pmatrix} : \ \Delta(\delta) = \begin{pmatrix} \cosh\delta & \sinh\delta \\ \sinh\delta & \cosh\delta \end{pmatrix} \text{ and } \vec{r} = \Delta\vec{\delta} \right\}, \tag{36}$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Using the pairing

$$\langle \chi, \varrho \rangle := \operatorname{Tr}\left(\begin{pmatrix} \frac{1}{2}d_3\sigma_1 & 0 \\ \vec{d}^\top & 0 \end{pmatrix} \begin{pmatrix} \delta\sigma_1 & \vec{\delta} \\ 0 & 0 \end{pmatrix} \right) = d_3\delta + \vec{d}^\top\vec{\delta} \tag{37}$$

we will identify $\mathfrak{p}(1,1)^*$ with

$$\mathfrak{p}(1,1)^* = \left\{ \begin{pmatrix} \frac{1}{2}d_3\sigma_1 & 0 \\ \vec{d}^\top & 0 \end{pmatrix} : \quad d_3 \in \mathbb{R} \quad \text{and} \quad \vec{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \in \mathbb{R}^2 \right\}. \tag{38}$$

Hence the Lie Poisson bracket of $f, g \in C^\infty(\mathfrak{p}(1,1)^*)$ is given by

$$\{f,g\}_{\mathfrak{p}(1,1)^*} = d_1\left(\frac{\partial f}{\partial d_3}\frac{\partial g}{\partial d_1} - \frac{\partial f}{\partial d_1}\frac{\partial g}{\partial d_3} \right) + d_2\left(\frac{\partial f}{\partial d_3}\frac{\partial g}{\partial d_2} - \frac{\partial f}{\partial d_2}\frac{\partial g}{\partial d_3} \right) \tag{39}$$

From (29)–(31) one find that $\mathcal{I}_\alpha : T^*\mathbb{R}^5 \to \mathfrak{p}(1,1)^*$

$$\mathcal{I}_\alpha(p,q) := \begin{pmatrix} \frac{1}{2}d_3(p,q)\sigma_1 & 0 \\ \vec{d}^\top(p,q) & 0 \end{pmatrix}, \tag{40}$$

where $d_1 := \frac{1}{2}\left(d_+(p,q) + d_-(p,q)\right)$ and $d_2 := \frac{1}{2}\left(d_+(p,q) - d_-(p,q)\right)$ is an equivariant Poisson map, i.e.,

$$\{f \circ \mathcal{I}_\alpha, g \circ \mathcal{I}_\alpha\} = \{f,g\}_{\mathfrak{p}(1,1)^*} \circ \mathcal{I}_\alpha \tag{41}$$

and

$$\mathcal{I}_\alpha \circ \Psi_A = Ad^*_{A^{-1}} \circ \mathcal{I}_\alpha, \tag{42}$$

where one has

$$Ad^*_{A^{-1}} \begin{pmatrix} d_3 \\ \vec{d} \end{pmatrix} = \begin{pmatrix} d_3 + \vec{\delta}^\top\sigma_1\vec{d} \\ \left(\Delta^{-1}\right)^\top \vec{d} \end{pmatrix}, \tag{43}$$

$$\Psi_A \begin{pmatrix} p^\top \\ q \end{pmatrix} = \begin{pmatrix} e^{-\frac{\delta}{2}} \left(p_{-1} + \frac{1}{2}(\delta_2 - \delta_1) q_{-1}\right) \\ e^{-\frac{\delta}{2}} \left(p_0 + \frac{1}{2}(\delta_2 - \delta_1) q_0\right) \\ e^{-\frac{\delta}{2}} \vec{p} \\ e^{\frac{\delta}{2}} q_{-1} \\ e^{\frac{\delta}{2}} q_0 \\ e^{\frac{\delta}{2}} \left(\vec{q} + \frac{1}{2}(\delta_1 + \delta_2) \vec{p}\right) \end{pmatrix}. \tag{44}$$

We summarize obtained results in the following

Proposition 1.

(i) *One has a symplectic dual pair*

$$\begin{array}{ccc} & T^*\mathbb{R}^5 & \\ {}^{\mathcal{I}_\alpha}\swarrow & & \searrow^{\mathcal{J}_\alpha} \\ \mathfrak{p}(1,1)^* & & \mathcal{L}_+(5) \end{array} ,$$

i.e., $\mathcal{I}_\alpha$ *and* $\mathcal{J}_\alpha$ *are Poisson maps and*

$$\{\mathcal{J}_\alpha^*(C^\infty(\mathcal{L}_+)), \mathcal{I}_\alpha^*(C^\infty(\mathfrak{p}(1,1)^*))\} = 0. \tag{45}$$

(ii) *The maps* $\mathcal{I}_\alpha$ *and* $\mathcal{J}_\alpha$ *are equivariant momentum maps with respect to the commuting*

$$\forall_{g\in SO_{0,\alpha}(5),\ A\in P(1,1)} \qquad \Psi_A \circ \Phi_g = \Phi_g \circ \Psi_A \tag{46}$$

actions Ψ *and* Φ.

See [6] for the definition of the dual symplectic pair and importance of this notion in Poisson geometry.

4. Reduction to $\mathcal{I}_\alpha^{-1}\left(d_3, \vec{d}\right)$ and physical applications

Now we reduce the Hamiltonian system $(T^*\mathbb{R}^5, H_\alpha)$ to the fiber $\mathcal{I}_\alpha^{-1}\left(d_3, \vec{d}\right)$ of the momentum map $\mathcal{I}_\alpha : T^*\mathbb{R}^5 \to \mathfrak{p}(1,1)^*$. We also discuss possible applications of obtained Hamiltonian dynamics in nonlinear wave optics.

The Casimirs for $\{\cdot,\cdot\}_{0,\alpha}$ and $\{\cdot,\cdot\}_{\mathfrak{p}(1,1)^*}$ are the following

$$c_1 = \vec{x}^2 + \alpha \vec{y}^2, \tag{47}$$

$$c_2 = (\vec{x} \times \vec{y}\,)^2 \tag{48}$$

and

$$c = d_+ d_-, \tag{49}$$

respectively.

One has the equalities:

$$h_\alpha := c_1 \circ \mathcal{J}_\alpha = c \circ \mathcal{I}_\alpha = \left(q_{-1}^2 + \alpha q_0^2\right) \vec{p}^2, \tag{50}$$

$$c_2 \circ \mathcal{J}_\alpha = 0. \tag{51}$$

The flow

$$\sigma_t^{h_\alpha} \begin{pmatrix} p^\top \\ q \end{pmatrix} = \begin{pmatrix} 2d_+ q_{-1} t + p_{-1} \\ 2\alpha d_+ q_0 t + p_0 \\ \vec{p} \\ q_{-1} \\ q_0 \\ -2d_- \vec{p} t + \vec{q} \end{pmatrix}, \tag{52}$$

defined by $h_\alpha \in C^\infty \left(T^*\mathbb{R}^5\right)$ preserves the fiber $\mathcal{I}_\alpha^{-1}\left(d_3, \vec{d}\right) \subset T^*\mathbb{R}^5$.

The fiber $\mathcal{I}_\alpha^{-1}\left(d_3, \vec{d}\right)$ is also invariant with respect to the action (12) of the group $S0_{0,\alpha}(5)$ and its image $\mathcal{J}_\alpha\left(\mathcal{I}_\alpha^{-1}\left(d_3, \vec{d}\right)\right)$ in $\mathcal{L}_+(5)$ is described by the conditions

$$\vec{x}^2 + \alpha \vec{y}^2 = c_1, \quad \vec{x} \times \vec{y} = 0, \quad \vec{x} \cdot \vec{\mu} = 0. \tag{53}$$

Assuming the value of the Casimir function (47) to be $c_1 \neq 0$ we find that $\mathcal{J}_\alpha\left(\left(d_3, \vec{d}\right)\right)$ is a 6-dimensional $Ad^*_{S0_{0,\alpha}(5)}$ – orbit $\mathcal{O}_\alpha \subset \mathcal{L}_+(5)$. Now let us introduce the new variable χ by

$$\chi q_{-1} = q_0 \tag{54}$$

which for $c_1 \neq 0$ satisfies $1 + \alpha\chi^2 \neq 0$. Using this variable we find that equations (53) are equivalent to

$$\vec{x}^2 = \frac{c_1}{1 + \alpha\chi^2}, \quad \vec{y} = \chi \vec{x}, \quad \vec{x} \cdot \vec{\mu} = 0 \tag{55}$$

and for $(p, q) \in T^*\mathbb{R}^5$ one has

$$\begin{aligned} q_{-1} &= \pm\sqrt{\frac{d_-}{1 + \alpha\chi^2}}, \\ q_0 &= \pm\chi\sqrt{\frac{d_-}{1 + \alpha\chi^2}}, \\ p_{-1} &= \mp\frac{-d_3 - \chi a + \vec{p} \cdot \vec{q}}{\sqrt{d_-(1 + \alpha\chi^2)}}, \\ p_0 &= \mp\frac{-\alpha\chi d_3 + a + \alpha\chi \vec{p} \cdot \vec{q}}{\sqrt{d_-(1 + \alpha\chi a^2)}}, \end{aligned} \tag{56}$$

We see from (55) that one can use the variables $a, \vec{x}, \vec{\mu} - \frac{\vec{x}\cdot\vec{\mu}}{\vec{x}\cdot\vec{x}}\vec{x}$ as a coordinate system on $\mathcal{O}_\alpha$. On the other hand we see from (56) that the variables $a, \chi, \vec{q}$ and $\vec{p}$, where $\vec{p}^2 = d_+$ can be used as coordinates on $\mathcal{I}_\alpha^{-1}\left(d_3, \vec{d}\right)$. The canonical form

$$\gamma = p_{-1} dq_{-1} + p_0 dq_0 + \vec{p} \cdot d\vec{q}. \tag{57}$$

of $T^*\mathbb{R}^5$ after restriction to $\mathcal{I}_\alpha^{-1}\left(d_3, \vec{d}\right)$ in these coordinates is given by

$$\gamma\big|_{\mathcal{I}_\alpha^{-1}\left(d_3,\vec{d}\right)} = -\frac{a}{1+\alpha\chi^2}d\chi + \vec{p}\cdot d\vec{q}. \tag{58}$$

The flow (52) preserves a and χ and it transforms the 1-form $\vec{p}\cdot d\vec{q}$ to the 1-form $\vec{p}\cdot d\vec{q} - td_-d(d_+)$. Thus $\gamma\big|_{\mathcal{I}_\alpha^{-1}\left(d_3,\vec{d}\right)}$ defines a symplectic form on the reduced phase space $\mathcal{I}_\alpha^{-1}\left(d_3, \vec{d}\right)/\{\sigma_t^{h_\alpha}\}$.

For $\alpha > 0$ we can consider $(a, \varphi = -\frac{1}{\sqrt{\alpha}}\mathrm{arctg}(\sqrt{\alpha}\chi), \vec{q}, \vec{p})$ as a canonically conjugated coordinates on $\mathcal{I}_\alpha^{-1}\left(d_3, \vec{d}\right)/\{\sigma_t^{h_\alpha}\}$. One can treat the case $\alpha \leqslant 0$ in a similarly way. The Hamiltonian (22) in the coordinates $(a, \varphi, \vec{q}, \vec{p})$ is given by

$$H_\alpha = \gamma h_\alpha + \gamma\epsilon(a^2 + \alpha\vec{\mu}^2) + \nu\epsilon^2 a^2\vec{\mu}^2. \tag{59}$$

We see from (59) that Hamiltonian H_α as well as $\vec{P} := \vec{p}$ and $\vec{Q} := \vec{q} - (\vec{p}\cdot\vec{q})\,\vec{p}$ are invariant with respect to the flow $\sigma_t^{h_\alpha}$. Assuming $d_+ = 1$ we use $(a, \varphi, \vec{Q}, \vec{P})$ as a canonical coordinates on the reduced phase space $\mathcal{I}_\alpha^{-1}\left(d_3, \vec{d}\right)/\{\sigma_t^{h_\alpha}\}$. In these coordinates the reduced Hamiltonian H_α is

$$H_\alpha = \gamma d_-\vec{P}^2 + \gamma\epsilon a^2 + \epsilon(\gamma\alpha + \nu\epsilon a^2)\vec{Q}^2\vec{P}^2 \tag{60}$$

and the corresponding Hamilton equations are

$$\frac{da}{dt}(t) = 0, \tag{61}$$

$$\frac{d\varphi}{dt}(t) = 2\epsilon a(\gamma + \nu\epsilon\vec{Q}^2), \tag{62}$$

$$\frac{d\vec{Q}}{dt}(t) = -2\epsilon(\gamma\alpha + \nu\epsilon a^2)\vec{Q}^2\vec{P}, \tag{63}$$

$$\frac{d\vec{P}}{dt}(t) = 2\epsilon(\gamma\alpha + \nu\epsilon a^2)\vec{Q}. \tag{64}$$

We see from (64) that a, $\vec{P}^2$, $\vec{Q}^2$ are integrals of motion of this Hamiltonian system. Thus one can reduce Eqs. (63)–(64) to the linear equations. The above allows us to integrate them in quadrature.

Rewriting the Hamiltonians (22) and (59) in the complex coordinates $z_k := q_k + ip_k$, $k = -1, 0, 1, 2, 3$, and $z := a + i\chi$, $\vec{z} := \vec{q} + i\vec{p}$, respectively we obtain the Hamiltonians which can be used for the description of nonlinear effects for the running plane waves in a nonlinear dielectric medium. For a detailed discussion of such problems see [3] and [5]. The Hamiltonian system defined by (8) after reduction to the vector bundle cotangent to de Sitter space among other describes geodesic flow on this manifold (see [3]).

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Poisson Reduction

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Abstract. In this paper we develop a theory of reduction for classical systems with Poisson Lie groups symmetries using the notion of momentum map introduced by Lu. The local description of Poisson manifolds and Poisson Lie groups and the properties of Lu's momentum map allow us to define a Poisson reduced space.

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1. Introduction

In this paper we prove a generalization of the Marsden–Weinstein reduction to the general case of an arbitrary Poisson Lie group action on a Poisson manifold. Reduction procedures are known in many different settings. In particular, a reduction theory is known in the case of Poisson Lie groups acting on symplectic manifolds [10] and in the case of Lie groups acting on Poisson manifolds [14, 18]. An important generalization to the Dirac setting has been studied in [2].

The theory of symplectic reduction plays a key role in classical mechanics. The phase space of a system of n particles is described by a symplectic or more generally Poisson manifold. Given a symmetry group of dimension k acting on a mechanical system, the dimension of the phase space can be reduced by $2k$. Marsden–Weinstein reduction formalizes this feature. Recall roughly the notion of Hamiltonian actions in this setting. Given a Poisson manifold M there are natural Hamiltonian vector fields $\{f, \cdot\}$ on M. Let G be a Lie group acting on M by Φ; the action is Hamiltonian if the vector fields defined by the infinitesimal generator of Φ are Hamiltonian. More precisely, let G be a Lie group acting on a Poisson manifold (M, π). The action $\Phi : G \times M \to M$ is canonical if it preserves the Poisson structure π. Suppose that there exists a linear map $H : \mathfrak{g} \to C^\infty(M)$ such that the infinitesimal generator Φ_X for $X \in \mathfrak{g}$ of the canonical action is induced by H by

$$\Phi_X = \{H_X, \cdot\}.$$

A canonical action induced by H is said Hamiltonian if H is a Lie algebra homomorphism. We can define a map $\boldsymbol{\mu} : M \to \mathfrak{g}^*$, called momentum map, by $H_X(m) = \langle \boldsymbol{\mu}(m), X \rangle$ for $m \in M$. It is equivariant if the corresponding H is a Lie algebra homomorphism. Given an Hamiltonian action, under certain assumptions, the reduced space has been defined as $M//G := \boldsymbol{\mu}^{-1}(u)/G_u$ and it has been proved that it is a Poisson manifold [15].

In this paper we are interested in analyzing the case in which one has an extra structure on the Lie group, a Poisson structure making it a Poisson Lie group. Poisson Lie groups are very interesting objects in mathematical physics. They may be regarded as classical limit of quantum groups [5] and they have been studied as carrier spaces of dynamical systems [9]. It is believed that actions of Poisson Lie groups on Poisson manifolds should be used to understand the "hidden symmetries" of certain integrable systems [19]. Moreover, the study of classical systems with Poisson Lie group symmetries may give information about the corresponding quantum group invariant system (an attempt can be found in [6, 7]).

The purpose of this paper is to prove that, given a Poisson manifold acted by a Poisson Lie group, under certain conditions, we can also reduce this phase space to another Poisson manifold.

The paper is organized as follows. In Section 2 we recall some basic elements of Poisson geometry: Poisson manifolds and their local description, Lie bialgebras and Poisson Lie groups. A nice review of these results can be found in [20] and [17]. Section 3 is devoted to Poisson actions and associated momentum maps and we discuss dressing actions and their properties. In Section 4 we present the main result of this paper, the Poisson reduction, and we discuss an example.

2. Poisson manifolds, Poisson Lie groups and Lie bialgebras

In this section we introduce the notion of Poisson manifolds and their local description, we give some background about Poisson Lie groups and Lie bialgebras which will be used in the paper. For more details on this subject, see [5, 10, 17, 20, 21].

2.1. Poisson manifolds and symplectic foliation

A Poisson structure on a smooth manifold M is a Lie bracket $\{\cdot, \cdot\}$ on the space $C^\infty(M)$ of smooth functions on M which satisfies the Leibniz rule. This bracket is called Poisson bracket and a manifold M equipped with such a bracket is called Poisson manifold. Therefore, a bivector field π on M such that the bracket

$$\{f, g\} := \langle \pi, df \wedge dg \rangle$$

is a Poisson bracket is called Poisson tensor or Poisson bivector field. A Poisson tensor can be regarded as a bundle map $\pi^\sharp : T^*M \to TM$:

$$\langle \alpha, \pi^\sharp(\beta) \rangle = \pi(\alpha, \beta)$$

Definition 1. A mapping $\phi : (M_1, \pi_1) \to (M_2, \pi_2)$ between two Poisson manifolds is called a Poisson mapping if $\forall f, g \in C^\infty(M_2)$ one has

$$\{f \circ \phi, g \circ \phi\}_1 = \{f, g\}_2 \circ \phi \tag{1}$$

The structure of a Poisson manifold is described by the splitting theorem of Alan Weinstein [21], which shows that locally a Poisson manifold is a direct product of a symplectic manifold with another Poisson manifold whose Poisson tensor vanishes at a point.

Theorem 1 (Weinstein). *On a Poisson manifold (M, π), any point $m \in M$ has a coordinate neighborhood with coordinates $(q_1, \ldots, q_k, p_1, \ldots, p_k, y_1, \ldots, y_l)$ centered at m, such that*

$$\pi = \sum_i \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \frac{1}{2} \sum_{i,j} \phi_{ij}(y) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j} \qquad \phi_{ij}(0) = 0. \tag{2}$$

The rank of π at m is $2k$. Since ϕ depends only on the y_i's, this theorem gives a decomposition of the neighborhood of m as a product of two Poisson manifolds: one with rank $2k$, and the other with rank 0 at m.

The term

$$\frac{1}{2} \sum_{i,j} \phi_{ij}(y) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j} \tag{3}$$

is called transverse Poisson structure and it is evident that the equations $y_i = 0$ determine the symplectic leaf through m.

2.2. Lie bialgebras and Poisson Lie groups

Definition 2. A Poisson Lie group (G, π_G) is a Lie group equipped with a multiplicative Poisson structure π_G, i.e., such that the multiplication map $G \times G \to G$ is a Poisson map.

Let G be a Lie group with Lie algebra $\mathfrak{g}$. The linearization $\delta := d_e \pi_G : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ of π_G at e defines a Lie algebra structure on the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ and, for this reason, it is called cobracket. The pair $(\mathfrak{g}, \mathfrak{g}^*)$ is called Lie bialgebra. The relation between Poisson Lie groups and Lie bialgebras has been proved by Drinfeld [5]:

Theorem 2. *If (G, π_G) is a Poisson Lie group, then the linearization of π_G at e defines a Lie algebra structure on $\mathfrak{g}^*$ such that $(\mathfrak{g}, \mathfrak{g}^*)$ form a Lie bialgebra over $\mathfrak{g}$, called the tangent Lie bialgebra to (G, π_G). Conversely, if G is connected and simply connected, then every Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ over $\mathfrak{g}$ defines a unique multiplicative Poisson structure π_G on G such that $(\mathfrak{g}, \mathfrak{g}^*)$ is the tangent Lie bialgebra to the Poisson Lie group (G, π_G).*

From this theorem it follows that there is a unique connected and simply connected Poisson Lie group (G^*, π_{G^*}), called the dual of (G, π_G), associated to the Lie bialgebra $(\mathfrak{g}^*, \delta)$. If G is connected and simply connected, then the dual of G^* is G.

Example 1 ($\mathfrak{g} = ax + b$). Consider the Lie algebra $\mathfrak{g}$ spanned by X and Y with commutator

$$[X, Y] = Y \tag{4}$$

and cobracket given by

$$\delta(X) = 0 \quad \delta(Y) = X \wedge Y. \tag{5}$$

The Lie bracket on $\mathfrak{g}^*$ is given by

$$[X^*, Y^*] = Y^*.$$

A matrix representation of $\mathfrak{g}$ is the Lie algebra $\mathfrak{gl}(2, \mathbb{R})$ via

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and

$$X^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad Y^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with the metric $\gamma(a, b) = \mathrm{tr}(aJbJ)$ and $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

The corresponding Poisson Lie group G and dual G^* are subgroups of $GL(2, \mathbb{R})$ of matrices with positive determinant are given by

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ \xi & \eta \end{pmatrix} : \eta > 0 \right\} \qquad G^* = \left\{ \begin{pmatrix} s & t \\ 0 & 1 \end{pmatrix} : s > 0 \right\} \tag{6}$$

3. Poisson actions and momentum maps

In this section we first introduce the concept of Poisson action of a Poisson Lie group on a Poisson manifold, which generalizes the canonical action of a Lie group on a symplectic manifold. We define momentum maps associated to such actions and finally we consider the particular case of a Poisson Lie group G acting on its dual G^* by dressing transformations. This allows us to study the symplectic leaves of G that are exactly the orbits of the dressing action. These topics can be found, e.g., in [10, 11, 19].

From now on we assume that G is connected and simply connected.

Definition 3. The action $\Phi : G \times M \to M$ of a Poisson Lie group (G, π_G) on a Poisson manifold (M, π) is called Poisson action if Φ is a Poisson map, where $G \times M$ is a Poisson manifold with structure $\pi_G \oplus \pi$.

This definition generalizes the notion of canonical action; indeed, if G carries the trivial Poisson structure $\pi_G = 0$, the action Φ is Poisson if and only if it preserves π, i.e., if it is canonical. In general, the structure π is not invariant with respect to the action Φ. The easiest examples of Poisson actions are given by the left and right actions of G on itself.

For an action $\Phi : G \times M \to M$ we use $\Phi : \mathfrak{g} \to \mathit{Vect}\, M : X \mapsto \Phi_X$ to denote the Lie algebra anti-homomorphism which defines the infinitesimal generators of this action. The proof of the following theorem can be found in [12].

Theorem 3. *The action $\Phi : G \times M \to M$ is a Poisson action if and only if*

$$L_{\Phi_X}(\pi) = (\Phi \wedge \Phi)\delta(X) \tag{7}$$

for any $X \in \mathfrak{g}$, where L denotes the Lie derivative and δ is the derivative of π_G at e.

Let $\Phi : G \times M \to M$ be a Poisson action of (G, π_G) on (M, π). Let G^* be the dual Poisson Lie group of G and let Φ_X be the vector field on M which generates the action Φ. In this formalism the definition of momentum map reads (Lu [10, 11]):

Definition 4. A momentum map for the Poisson action $\Phi : G \times M \to M$ is a map $\boldsymbol{\mu} : M \to G^*$ such that

$$\Phi_X = \pi^\sharp(\boldsymbol{\mu}^*(\theta_X)) \tag{8}$$

where θ_X is the left invariant 1-form on G^* defined by the element $X \in \mathfrak{g} = (T_eG^*)^*$ and $\boldsymbol{\mu}^*$ is the cotangent lift $T^*G^* \to T^*M$.

In other words, the momentum map generates the vector field Φ_X via the construction

$$X \in \mathfrak{g} \to \theta_X \in T^*G^* \to \alpha_X = \boldsymbol{\mu}^*(\theta_X) \in T^*M \to \pi^\sharp(\alpha_X) \in TM$$

It is important to remark that Noether's theorem still holds in this general context.

Theorem 4. *Let $\Phi : G \times M \to M$ a Poisson action with momentum map $\boldsymbol{\mu} : M \to G^*$. If $H \in C^\infty(M)$ is G-invariant, then $\boldsymbol{\mu}$ is an integral of the Hamiltonian vector field associated to H.*

It is important to point out that in this setting the vector field Φ_X is not Hamiltonian, unless the Poisson structure on G is trivial. In this case $G^* = \mathfrak{g}^*$, the differential 1-form θ_X is the constant 1-form X on $\mathfrak{g}^*$, and

$$\boldsymbol{\mu}^*(\theta_X) = d(H_X), \quad \text{where} \quad H_X(m) = \langle \boldsymbol{\mu}(m), X \rangle. \tag{9}$$

This implies that the momentum map is the canonical one and

$$\Phi_X = \pi^\sharp(dH_X) = \{H_X, \cdot\}. \tag{10}$$

In other words, Φ_X is the Hamiltonian vector field with Hamiltonian $H_X \in C^\infty(M)$. We observe that, when π_G is not trivial, θ_X is a Maurer–Cartan form, hence $\boldsymbol{\mu}^*(\theta_X)$ can not be written as a differential of a Hamiltonian function. In the following we give an example for the infinitesimal generator in this general case.

3.1. Dressing transformations

One of the most important example of Poisson action is the dressing action of G on G^*. The name "dressing" comes from the theory of integrable systems and was introduced in this context in [19]. Interesting examples can be found in [1]. We remark that, given a Poisson Lie group (G, π_G), the left (right) invariant 1-forms on G^* form a Lie algebra with respect to the bracket:

$$[\alpha, \beta] = L_{\pi^\sharp(\alpha)}\beta - L_{\pi^\sharp(\beta)}\alpha - d(\pi(\alpha, \beta)).$$

For $X \in \mathfrak{g}$, let θ_X be the left invariant 1-form on G^* with value X at e. Let us define the vector field on G^*

$$l(X) = \pi^{\sharp}_{G^*}(\theta_X). \tag{11}$$

The map $l : \mathfrak{g} \to TG^* : X \mapsto l(X)$ is a Lie algebra anti-homomorphism. We call l the left infinitesimal dressing action of $\mathfrak{g}$ on G^*; its linearization at e is the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$. Similarly we can define the right infinitesimal dressing action.

Let $l(X)$ (resp. $r(X)$) a left (resp. right) dressing vector field on G^*. If all the dressing vector fields are complete, we can integrate the $\mathfrak{g}$-action into an action of G on G^* called the dressing action and we say that the dressing actions consist of dressing transformations.

Definition 5. A multiplicative Poisson tensor π_G on G is complete if each left (equiv. right) dressing vector field is complete on G.

From the definition of dressing action follows (the proof can be found in [19]) that the orbits of the right or left dressing action of G^* (resp. G) are the symplectic leaves of G (resp. G^*).

It can be proved (see [10]) that if π_G is complete, both left and right dressing actions are Poisson actions with momentum map given by the identity.

Assume that G is a complete Poisson Lie group. We denote respectively the left (resp. right) dressing action of G on its dual G^* by $g \mapsto l_g$ (resp. $g \mapsto r_g$).

Definition 6. A momentum map $\boldsymbol{\mu} : M \to G^*$ for a left (resp. right) Poisson action Φ is called G-equivariant if it is such with respect to the left dressing action of G on G^*, that is, $\boldsymbol{\mu} \circ \Phi_g = \lambda_g \circ \boldsymbol{\mu}$ (resp. $\boldsymbol{\mu} \circ \Phi_g = \rho_g \circ \boldsymbol{\mu}$)

It is important to remark that a momentum map is G-equivariant if and only if it is a Poisson map, i.e., $\boldsymbol{\mu}_* \pi = \pi_{G^*}$.

Definition 7. An action $\Phi : G \times M \to M$ of a Poisson Lie group (G, π_G) on a Poisson manifold (M, π) is said Hamiltonian if it is a Poisson action generated by an equivariant momentum map.

4. Poisson reduction

In this section we present the main result of this paper. We show that, given a Hamiltonian action Φ, as defined above, we can define a reduced manifold in terms of momentum map and prove that it is a Poisson manifold. The approach used is a generalization of the orbit reduction [13] in symplectic geometry. Recall that, under certain conditions, the orbit space of Φ is a smooth manifold and it carries a Poisson structure. First, we give an alternate proof of this claim. Then, we consider a generic orbit $\mathcal{O}_u$ of the dressing action of G on G^*, for $u \in G^*$, and we prove that the set $\boldsymbol{\mu}^{-1}(\mathcal{O}_u)/G$ is a regular quotient manifold with Poisson structure induced by the Poisson structure on M. Similarly to the symplectic case, this reduced space is isomorphic to the space $\boldsymbol{\mu}^{-1}(u)/G_u$ which will be regarded as the Poisson reduced space.

4.1. Poisson structure on M/G

Consider a Hamiltonian action of a connected and simply connected Poisson Lie group (G, π_G) on a Poisson manifold (M, π). It is known that, if the action is proper and free, the orbit space M/G is a smooth manifold, it carries a Poisson structure such that the natural projection $M \to M/G$ is a Poisson map (a proof of this result can be found in [19]). In this section we give an alternate proof of this result, by introducing an explicit formulation for the infinitesimal generator of the Hamiltonian action, in terms of local coordinates.

As discussed in the previous section, a Hamiltonian action is a Poisson action induced by an equivariant momentum map $\boldsymbol{\mu} : M \to G^*$ by formula (8). In other words, the map

$$\alpha : \mathfrak{g} \to \Omega^1(M) : X \mapsto \alpha_X = \boldsymbol{\mu}^*(\theta_X)$$

is a Lie algebra homomorphism such that

$$\Phi_X = \pi^\sharp(\alpha_X)$$

The dual map of α defines a $\mathfrak{g}^*$-valued 1-form on M, still denoted by α, satisfying Maurer–Cartan equation (as proved in [10])

$$d\alpha + \frac{1}{2}[\alpha, \alpha]_{\mathfrak{g}^*} = 0.$$

In particular,

$$\{\alpha_X : X \in \mathfrak{g}\}$$

defines a foliation $\mathcal{F}$ on M.

Lemma 1. *The space of G-invariant functions on M is closed under Poisson bracket. Hence π defines a Poisson structure on M/G.*

Proof. Let H_i, $i = 1, \ldots, n$ be local coordinates on M such that

$$\mathcal{F} = \mathrm{Ker}\{dH_1, \ldots, dH_n\}.$$

Then

$$\alpha_X = \sum_i c_i(X) dH_i \tag{12}$$

and

$$\Phi_X[f] = \pi^\sharp(\alpha_X) = \sum_i c_i(X)\{H_j, f\}_M. \tag{13}$$

This implies that a function $f \in C^\infty(M)$ is G-invariant ($\Phi_X[f] = 0$) if and only if $\{H_i, f\} = 0$ for any i. If f, g are G-invariant functions on M, we have $\{H_i, f\} = \{H_i, g\} = 0$ for any i. Then, using the Jacobi identity we get $\{H_i, \{f, g\}\} = 0$. Since G is connected, the result follows. □

4.2. Poisson reduced space

Assume that G is connected, simply connected and complete. In order to define a reduced space and to prove that it is a Poisson manifold we consider a generic orbit $\mathcal{O}_u$ of the dressing orbit of G on G^* passing through $u \in G^*$. First, we prove the following:

Lemma 2. *Let $\Phi : G \times M \to M$ be a free and Hamiltonian action of a compact Poisson Lie group (G, π_G) on a Poisson manifold (M, π). Then:*

(i) *$\mathcal{O}_u$ is closed and the Poisson structure π_{G^*} does not depend on the transversal coordinates on $\mathcal{O}_u$.*

(ii) *$\boldsymbol{\mu}^{-1}(\mathcal{O}_u)/G$ is a smooth manifold.*

Proof. (i) If G is compact, any G-action is automatically proper. This implies that, given $u \in G^*$ the generic orbit $\mathcal{O}_u$ of the dressing action is closed. From section 3.1 we know that $\mathcal{O}_u$ is the symplectic leaf through u. Using the local description of Poisson manifolds introduced in Theorem 1 it is evident that π_{G^*} restricted to $\mathcal{O}_u$ does not depend on the transversal coordinates y_i.

(ii) If the action Φ is free, the momentum map $\boldsymbol{\mu} : M \to G^*$ is a submersion onto some open subset of G^*. This implies that $\boldsymbol{\mu}^{-1}(u)$ is a closed submanifold of M. As $\boldsymbol{\mu}$ is equivariant, it follows that $\boldsymbol{\mu}^{-1}(u)$ is G-invariant. Free and proper actions of G on M restrict to free and proper G-actions on G-invariant submanifolds. In particular, the action of G on $\boldsymbol{\mu}^{-1}(u)$ is still proper, then $G \cdot \boldsymbol{\mu}^{-1}(u)$ is closed. Using the equivariance we have that $G \cdot \boldsymbol{\mu}^{-1}(u) = \boldsymbol{\mu}^{-1}(\mathcal{O}_u)$, which is still G-invariant. The action of G on $\boldsymbol{\mu}^{-1}(\mathcal{O}_u)$ is proper and free, so we can conclude that the orbit space $\boldsymbol{\mu}^{-1}(\mathcal{O}_u)/G$ is a smooth manifold. □

We aim to prove that the manifold $N/G := \boldsymbol{\mu}^{-1}(\mathcal{O}_u)/G$ carries a Poisson structure. In the previous Lemma we stated that π_{G^*} restricted to $\mathcal{O}_u$ does not depend on the transversal coordinates y_i's; if x_i are local coordinates along $N = \boldsymbol{\mu}^{-1}(\mathcal{O}_u)$ and H_i are pullback of the transversal coordinates y_i's by

$$H_i := y_i \circ \boldsymbol{\mu} \tag{14}$$

we can easily deduce that the Poisson structure π on M involves derivatives in H_i only in the combination

$$\partial_{x_i} \wedge \partial_{H_i}$$

This is evident because the differential $d\boldsymbol{\mu}$ between $TM|_N/TN$ and $TG^*/T\mathcal{O}_u$ is a bijective map. Moreover, since $\{y_i, y_j\}$ vanishes on the orbit $\mathcal{O}_u$, $\{H_i, H_j\}$ vanishes on the preimage N and dH_i's are in the span of $\{\alpha_X : X \in \mathfrak{g}\}$.

Now we introduce the ideal $\mathcal{I}$ generated by H_i and prove some properties.

Lemma 3. *Let $\mathcal{I} = \{f \in C^\infty(M) : f|_N = 0\}$.*

(i) *$\mathcal{I}$ is defined in an open G-invariant neighborhood U of N.*

(ii) *$\mathcal{I}$ is closed under Poisson bracket.*

Proof. (i) The coordinates H_i are locally defined but we can show that $\mathcal{I}$ is globally defined. Considering a different neighborhood on the orbit of G^* we have transversal coordinates y'_i and their pullback to M will be $H'_i = y'_i \circ \boldsymbol{\mu}$. The coordinates H'_i are defined in a different open neighborhood V of N, but we can see that the ideal $\mathcal{I}$ generated by H_i coincides with $\mathcal{I}'$ generated by H'_i on the intersection of U and V, then it is globally defined.

(ii) Since $\boldsymbol{\mu}$ is a Poisson map we have:

$$\{H_i, H_j\}_M = \{y_i \circ \boldsymbol{\mu}, y_j \circ \boldsymbol{\mu}\}_M = \{y_i, y_j\}_{G^*} \circ \boldsymbol{\mu}.$$

Hence the ideal $\mathcal{I}$ is closed under Poisson brackets. □

Motivated by this lemma we use the following identification

$$C^\infty(N/G) \simeq (C^\infty(U)/\mathcal{I})^G.$$

Lemma 4. *Suppose that N/G is an embedded submanifold of the smooth manifold M/G, then*

$$(C^\infty(U)/\mathcal{I})^G \simeq (C^\infty(U)^G + \mathcal{I})/\mathcal{I}. \tag{15}$$

Proof. Let f be a smooth function on U satisfying $[f] \in (C^\infty(U)/\mathcal{I})^G$. As the equivalence class $[f]$ is G-invariant, we have

$$f(G \cdot m) = f(m) + i(m), \tag{16}$$

where $i \in \mathcal{I}$ and $G \cdot m$ is a generic orbit of the Hamiltonian action of G on M. It is clear that $f|_N$ is G-invariant and hence it defines a smooth function $\bar{f} \in C^\infty(N/G)$. Since N/G is a k-dimensional embedded submanifold of the n-dimensional smooth manifold M/G, the inclusion map $\iota : N/G \to M/G$ has local coordinates representation:

$$(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, c_{k+1}, \ldots, c_n) \tag{17}$$

where c_i are constants. Hence we can extend $\bar{f}$ to a smooth function ϕ on M/G by setting $\bar{f}(x_1, \ldots, x_k) = \phi(x_1, \ldots, x_k, 0, \ldots, 0)$. The pullback $\tilde{f}$ of ϕ by $\mathrm{pr} : M \to M/G$ is G-invariant and satisfies

$$\tilde{f} - f|_N = 0, \tag{18}$$

hence $\tilde{f} - f \in \mathcal{I}$. □

Using these results we can prove the following:

Theorem 5. *Let $\Phi : G \times M \to M$ be a free Hamiltonian action of a compact Poisson Lie group (G, π_G) on a Poisson manifold (M, π) with momentum map $\boldsymbol{\mu} : M \to G^*$. The orbit space N/G has a Poisson structure induced by π.*

Proof. First we prove that the Poisson bracket of M induces a well-defined Poisson bracket on $(C^\infty(U)^G + \mathcal{I})/\mathcal{I}$. In fact, for any $f + i \in C^\infty(U)^G/\mathcal{I}$ and $j \in \mathcal{I}$ the Poisson bracket $\{f + i, j\}$ still belongs to the ideal $\mathcal{I}$. Since the ideal $\mathcal{I}$ is closed under Poisson brackets, $\{i, j\}$ belongs to $\mathcal{I}$. The function j, by definition on the ideal $\mathcal{I}$, can be written as a linear combination of H_i, so $\{f, j\} = \sum_i a_i \{f, H_i\}$. By Lemma 1, we have $\{f, H_i\} = 0$, hence $\{f + i, j\} \in \mathcal{I}$ as stated. Finally, using the isomorphism proved in Lemma 4 and the identification $C^\infty(N/G) \simeq (C^\infty(U)/\mathcal{I})^G$, the claim is proved. □

Finally, we observe that there is a natural isomorphism

$$\boldsymbol{\mu}^{-1}(u)/G_u \simeq \boldsymbol{\mu}^{-1}(\mathcal{O}_u)/G. \tag{19}$$

We refer to $\boldsymbol{\mu}^{-1}(u)/G_u$ as the Poisson reduced space.

5. An example

In this section we discuss a concrete example of Poisson reduction. Consider the Lie bialgebra $\mathfrak{g} = ax + b$ discussed in Example 1. The Poisson tensor on the dual Poisson Lie group G^* is given, in the coordinates (s,t) introduced in the matrix representation, by

$$\pi_{G^*} = st\partial_s \wedge \partial_t. \tag{20}$$

It is clear that (s,t) are global coordinates on G^*. First, we need to study the orbits of the dressing action. Remember that the dressing orbits $\mathcal{O}_u$ through a point $u \in G^*$ are the same as the symplectic leaves, hence it is clear that they are determined by the equation $t = 0$. The symplectic foliation of the manifold G^* in this case is given by two open orbits, determined by the conditions $t > 0$ and $t < 0$, respectively, and a closed orbit given by $t = 0$ and $a \in \mathbb{R}^+$.

Consider a Hamiltonian action $\Phi : G \times M \to M$ of G on a generic Poisson manifold M induced by the equivariant momentum map $\boldsymbol{\mu} : M \to G^*$. Its pullback

$$\boldsymbol{\mu}^* : C^\infty(G^*) \longrightarrow C^\infty(M) \tag{21}$$

maps the coordinates s and t on G^* to

$$x(u) = s(\boldsymbol{\mu}(u)) \qquad y(u) = t(\boldsymbol{\mu}(u)).$$

It is important to underline that we have no information on the dimension of M, so x and y are just a pair of the possible coordinates. Nevertheless, since $\boldsymbol{\mu}$ is a Poisson map, we have

$$\{x, y\} = xy \tag{22}$$

on M. The infinitesimal generators of the action Φ can be written in terms of these coordinates (x, y) as

$$\Phi(X) = x\{y, \cdot\} \quad \Phi(Y) = x\{x^{-1}, \cdot\}. \tag{23}$$

In the following, we discuss the Poisson reduction case by case, by considering the different dressing orbits studied above.

Case 1: ($t > 0$). Consider the dressing orbit $\mathcal{O}_u$ determined by the condition $t > 0$. Since s and t are both positive, we can put

$$x = e^p, \quad y = e^q. \tag{24}$$

Since $\{x, y\} = xy$ we have

$$\{p, q\} = 1. \tag{25}$$

For this reason the preimage of the dressing orbit can be split as $N = \mathbb{R}^2 \times M_1$ and $C^\infty(N)$ is given explicitly by the set of functions generated by y^{-1}. The infinitesimal generators are given by

$$\Phi(X) = e^p\{e^q, \cdot\} \qquad \Phi(Y) = e^p\{e^{-p}, \cdot\} \tag{26}$$

which is the action of G on the plane. Hence the Poisson reduction in this case is given by

$$(C^\infty(M)[y^{-1}])^G. \tag{27}$$

Case 2: ($t < 0$). This case is similar, with the only difference that $y = -e^q$.

Case 3: ($t = 0$). The orbit $\mathcal{O}_u$ is given by fixed points on the line $t = 0$, then we choose the point $s = 1$. Consider the ideal $\mathcal{I} = \langle x - 1, y\rangle$ of functions vanishing on N. It is easy to check that it is G-invariant, hence the Poisson reduction in this case is simply given by

$$(C^\infty(M)/\mathcal{I})^G. \tag{28}$$

6. Questions and future directions

The theory of Poisson reduction can be further developed, as it has been obtained under the assumption that the orbit space M/G is a smooth manifold. This result could be proved under weaker hypothesis, for instance requiring that M/G is an orbifold.

As stated in the introduction, the idea of momentum map and Poisson reduction can be also used for the study of symmetries in quantum mechanics. In particular, the approach of deformation quantization would provide a relation between classical and quantum symmetries. A notion of quantum momentum map has been defined in [6, 7] and it can be used to define the quantization of the Poisson reduction.

At classical level, Poisson reduction could be generalized to actions of Dirac Lie groups [16] on Dirac manifolds [3]. Finally, a possible development of this theory is its integration to symplectic groupoids by means of the theories on the integrability of Poisson brackets [4] and Poisson Lie group actions [8].

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Complex Manifold Structure and Algebroid of the Partially Invertible Elements Groupoid of a W^*-algebra

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Abstract. The goal of the present note based on [4] is a description of complex manifold structure of the groupoid $\mathcal{G}(\mathfrak{M})$ of partially invertible elements of W^*-algebra $\mathfrak{M}$. We also describe Banach–Lie algebroid $\mathcal{A}(\mathfrak{M})$ of the groupoid $\mathcal{G}(\mathfrak{M})$.

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1. Introduction

During the last few decades one can observe a progress in the study of Lie groupoids and algebroids which play a significant role in differential geometry. As a consequence their impact in mathematical physics is also increasing, see, e.g., [3, 8] and references therein. Similar situation occurs in the operator algebras theory where the convolution C^*-algebras of functions on locally compact groupoids equipped with a left Haar system are considered, see [6].

In this note, following [4] and [5], we describe Banach–Lie groupoids and algebroids related in the canonical way to the structure of a W^*-algebra (von Neumann algebra).

The most detailed description of the subject and motivation for this kind of investigations one can find in [4] and [5].

2. Groupoid of partially invertible elements of W^*-algebra

Let us begin with recalling the basic definitions.

A *groupoid* over base set B (see, e.g., [3, 8]) is a set $\mathcal{G}$ equipped with maps:

(i) a *source map* $\mathbf{s} : \mathcal{G} \to B$ and a *target map* $\mathbf{t} : \mathcal{G} \to B$

(ii) a *product* $m : \mathcal{G}^{(2)} \to \mathcal{G}$

$$m(g,h) =: gh,$$

defined on *the set of composable pairs*

$$\mathcal{G}^{(2)} := \{(g,h) \in \mathcal{G} \times \mathcal{G} : \ \mathbf{s}(g) = \mathbf{t}(h)\},$$

(iii) an injective *identity section* $\varepsilon : B \to \mathcal{G}$,

(iv) an *inverse map* $\iota : \mathcal{G} \to \mathcal{G}$, which are subject to the following compatibility conditions:

$$\mathbf{s}(gh) = \mathbf{s}(h), \quad \mathbf{t}(gh) = \mathbf{t}(g), \tag{1}$$

$$k(gh) = (kg)h, \tag{2}$$

$$\varepsilon(\mathbf{t}(g))g = g = g\varepsilon(\mathbf{s}(g)), \tag{3}$$

$$\iota(g)g = \varepsilon(\mathbf{s}(g)), \qquad g\iota(g) = \varepsilon(\mathbf{t}(g)), \tag{4}$$

where $g, k, h \in \mathcal{G}$.

For a groupoid $\mathcal{G}$ over a base B we will use the notation $\mathcal{G} \rightrightarrows B$.

Remark 1. Equivalently one can define a groupoid $\mathcal{G} \rightrightarrows B$ as a small category in which all morphisms are invertible, see for example [2].

Let us recall that C^*-algebra $\mathfrak{M}$ is called W^**-algebra* (or *von Neumann algebra*) if there exists a Banach space $\mathfrak{M}_*$ such that

$$(\mathfrak{M}_*)^* = \mathfrak{M},$$

i.e., $\mathfrak{M}$ possesses a predual Banach space $\mathfrak{M}_*$. If $\mathfrak{M}_*$ exists it is defined in a unique way by the structure of W^*-algebra $\mathfrak{M}$, see [7].

Element $p \in \mathfrak{M}$ is called a (*orthogonal*) *projection* if $p^* = p = p^2$. We will denote the lattice of projections of the W^*-algebra $\mathfrak{M}$ by $\mathcal{L}(\mathfrak{M})$. Element $u \in \mathfrak{M}$ is called a *partial isometry* if uu^* (or equivalently u^*u) is a projection. We will denote the set of partial isometries of the W^*-algebra $\mathfrak{M}$ by $\mathcal{U}(\mathfrak{M})$.

The least projection $l(x) \in \mathcal{L}(\mathfrak{M}))$ in $\mathfrak{M}$, such that

$$l(x)x = x \qquad (\text{respectively } x\, r(x) = x) \tag{5}$$

is called the *left support* (respectively *right support*) of $x \in \mathfrak{M}$.

If $x \in \mathfrak{M}$ is self adjoint, then *support* of x is a projection

$$s(x) := l(x) = r(x).$$

The *polar decomposition* of $x \in \mathfrak{M}$ is a representation

$$x = u|x|, \tag{6}$$

where $u \in \mathfrak{M}$ is partial isometry and $|x| := \sqrt{x^*x} \in \mathfrak{M}^+$ such that

$$l(x) = s(|x^*|) = uu^*, \qquad r(x) = s(|x|) = u^*u.$$

We define the set $\mathcal{G}(\mathfrak{M})$ of *partially invertible* elements of $\mathfrak{M}$ as follows

$$\mathcal{G}(\mathfrak{M}) := \{x \in \mathfrak{M}; \quad |x| \in G(p\mathfrak{M}p), \ \text{where } p = s(|x|)\},$$

where $G(p\mathfrak{M}p)$ is the group of all invertible elements of W^*-subalgebra $p\mathfrak{M}p \subset \mathfrak{M}$.

Remark 2. $\mathcal{G}(\mathfrak{M}) \subsetneq \mathfrak{M}$.

We can define the groupoid structure $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ with $\mathcal{G}(\mathfrak{M})$ being the set of invertible morphisms and $\mathcal{L}(\mathfrak{M})$ as the base set. The groupoid maps for $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ are defined as follows:

(i) the source and target maps $\mathbf{s}, \mathbf{t} : \mathcal{G}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ are

$$\mathbf{s}(x) := r(x), \qquad \mathbf{t}(x) := l(x),$$

(ii) the product is the product in $\mathfrak{M}$ restricted to

$$\mathcal{G}(\mathfrak{M})^{(2)} := \{(x, y) \in \mathcal{G}(\mathfrak{M}) \times \mathcal{G}(\mathfrak{M}); \quad \mathbf{s}(x) = \mathbf{t}(y)\},$$

(iii) the identity section $\varepsilon : \mathcal{L}(\mathfrak{M}) \hookrightarrow \mathcal{G}(\mathfrak{M})$ as the embedding,

(iv) the inverse map $\iota : \mathcal{G}(\mathfrak{M}) \to \mathcal{G}(\mathfrak{M})$ is

$$\iota(x) := |x|^{-1} u^*.$$

The subset of partial isometries $\mathcal{U}(\mathfrak{M}) \subset \mathcal{G}(\mathfrak{M})$ inherits the groupoid structure $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ from $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$. Let us note here that for $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ the source and target maps $\mathbf{s}, \mathbf{t} : \mathcal{G}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ are:

$$\mathbf{s}(u) = u^* u, \qquad \mathbf{t}(u) = u u^*,$$

and inverse map $\iota : \mathcal{U}(\mathfrak{M}) \to \mathcal{U}(\mathfrak{M})$ is expressed by the involution:

$$\iota(u) = u^*.$$

Remark 3. The groupoid $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ is a wide subgroupoid of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$.

For details we address to [4].

3. Banach–Lie groupoid structure of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$

One properly defines the complex Banach manifold structure on the groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ and shows that the groupoid maps are consistent with the structure, i.e., the groupoid of partially invertible elements is a Banach–Lie groupoid, see [4].

For any projection $p \in \mathcal{L}(\mathfrak{M})$ we define (following [4]) the subset $\Pi_p \subset \mathcal{L}(\mathfrak{M})$ by

$$q \in \Pi_p \quad \text{iff} \quad \mathfrak{M} = q\mathfrak{M} \oplus (1 - p)\mathfrak{M} \tag{7}$$

and maps $\quad \sigma_p : \Pi_p \to \mathfrak{M}p, \qquad \varphi_p : \Pi_p \tilde{\to} (1 - p)\mathfrak{M}p$ by

$$\sigma_p(q) := x, \qquad \varphi_p(q) := y, \tag{8}$$

where $p = x - y$ is consistent with the splitting (7). Note that $l \circ \sigma_p = id_{\Pi_p}$ and φ_p defines a bijections between Π_p and the Banach space $(1 - p)\mathfrak{M}p$.

In order to construct transitions maps

$$\varphi_p \circ \varphi_{p'}^{-1} : \varphi_{p'}(\Pi_p \cap \Pi_{p'}) \to \varphi_p(\Pi_p \cap \Pi_{p'})$$

in the case $\Pi_p \cap \Pi_{p'} \neq \emptyset$, let us take for $q \in \Pi_p \cap \Pi_{p'}$ the following splittings

$$\begin{aligned} \mathfrak{M} &= q\mathfrak{M} \oplus (1-p)\mathfrak{M} = p\mathfrak{M} \oplus (1-p)\mathfrak{M}, \\ \mathfrak{M} &= q\mathfrak{M} \oplus (1-p')\mathfrak{M} = p'\mathfrak{M} \oplus (1-p')\mathfrak{M}. \end{aligned} \tag{9}$$

Splittings (9) lead to the corresponding decompositions of p and p'

$$\begin{aligned} p &= x - y \qquad & p &= a + b \\ p' &= x' - y' \qquad & 1 - p &= c + d \end{aligned} \tag{10}$$

where $x \in q\mathfrak{M}p$, $y \in (1-p)\mathfrak{M}p$, $x' \in q\mathfrak{M}p'$, $y' \in (1-p')\mathfrak{M}p'$, $a \in p'\mathfrak{M}p$, $b \in (1-p')\mathfrak{M}p$, $c \in p'\mathfrak{M}(1-p)$ and $d \in (1-p')\mathfrak{M}(1-p)$. Using (10) we get the formula

$$y' = (\varphi_{p'} \circ \varphi_p^{-1})(y) = (b + dy)\iota(a + cy).$$

Theorem 4. *The family of charts*

$$(\Pi_p, \varphi_p) \quad p \in \mathcal{L}(\mathfrak{M})$$

defines a complex analytic atlas on $\mathcal{L}(\mathfrak{M})$. This atlas is modeled on the family of Banach spaces $(1-p)\mathfrak{M}p$, where $p \in \mathcal{L}(\mathfrak{M})$.

Remark 5. For equivalent projections $p \sim p'$ there exists a partial isometry $u \in \mathcal{U}(\mathfrak{M})$ such that $uu^* = p$ and $u^*u = p'$, so that one has $(1-p)\mathfrak{M}p \cong (1-p')\mathfrak{M}p'$.

In order to introduce the complex analytic structure on $\mathcal{G}(\mathfrak{M})$ we define for any $\tilde{p}, p \in \mathcal{L}(\mathfrak{M})$ the set

$$\Omega_{\tilde{p}p} := \mathbf{t}^{-1}(\Pi_{\tilde{p}}) \cap \mathbf{s}^{-1}(\Pi_p).$$

If $\Omega_{\tilde{p}p} \neq \emptyset$ we define the map

$$\psi_{\tilde{p}p} : \Omega_{\tilde{p}p} \to (1-\tilde{p})\mathfrak{M}\tilde{p} \oplus \tilde{p}\mathfrak{M}p \oplus (1-p)\mathfrak{M}p$$

by

$$\psi_{\tilde{p}p}(x) := \left(\varphi_{\tilde{p}}(\mathbf{t}(x)), \iota(\sigma_{\tilde{p}}(\mathbf{t}(x)))x\sigma_p(\mathbf{s}(x)), \varphi_p(\mathbf{s}(x))\right), \tag{11}$$

which is a bijection of $\Omega_{\tilde{p}p}$ onto an open subset of the direct sum of the Banach subspaces $(1-\tilde{p})\mathfrak{M}\tilde{p} \oplus \tilde{p}\mathfrak{M}p \oplus (1-p)\mathfrak{M}p$ of the W^*-algebra $\mathfrak{M}$. The inverse map $\psi_{\tilde{p}p}^{-1} : \psi_{\tilde{p}p}(\Omega_{\tilde{p}p}) \to \Omega_{\tilde{p}p}$ has the form

$$\psi_{\tilde{p}p}^{-1}(\tilde{y}, z, y) := \sigma_{\tilde{p}}(\tilde{q})z\iota(\sigma_p(q)) = (\tilde{p} + \tilde{y})z\iota(p + y) \tag{12}$$

where $\tilde{q} = l(\tilde{p}+\tilde{y})$ and $q = l(p+y)$ are left supports of $\tilde{p}+\tilde{y}$ and $p+y$ respectively. The transition maps

$$\psi_{\tilde{p}'p'} \circ \psi_{\tilde{p}p}^{-1} : \psi_{\tilde{p}p}(\Omega_{\tilde{p}'p'} \cap \Omega_{\tilde{p}p}) \to \psi_{\tilde{p}'p'}(\Omega_{\tilde{p}'p'} \cap \Omega_{\tilde{p}p})$$

for $(\tilde{y}, z, y) \in \psi_{\tilde{p}p}(\Omega_{\tilde{p}'p'} \cap \Omega_{\tilde{p}p})$ are given by

$$(\psi_{\tilde{p}'p'} \circ \psi_{\tilde{p}p}^{-1})(\tilde{y}, z, y) := (\tilde{y'}, z', y'), \tag{13}$$

where

$$\tilde{y'} = (\varphi_{\tilde{p}'} \circ \varphi_{\tilde{p}}^{-1})(\tilde{y}) = (\tilde{b} + \tilde{d}\tilde{y})\iota(\tilde{a} + \tilde{c}\tilde{y}) \tag{14}$$

$$y' = (\varphi_{p'} \circ \varphi_p^{-1})(y) = (b + dy)\iota(a + cy) \tag{15}$$

and

$$z' = \iota(\tilde{p'} + \tilde{y'})(\tilde{p} + \tilde{y})z\iota(p + y)(p' + y'). \tag{16}$$

We note that all maps in (14), (15) and (16) are complex analytic.

Thus we derive

Theorem 6.

(i) *The family of charts*

$$(\Omega_{\tilde{p}p}, \psi_{\tilde{p}p}),$$

where $(p, \tilde{p}) \in \mathcal{L}(\mathfrak{M}) \times \mathcal{L}(\mathfrak{M})$ *are pairs of equivalent projections, defines a complex analytic atlas on the groupoid* $\mathcal{G}(\mathfrak{M})$ *(in the sense of* [1]*). This atlas is modeled on the family of Banach spaces* $(1 - \tilde{p})\mathfrak{M}\tilde{p} \oplus \tilde{p}\mathfrak{M}p \oplus (1 - p)\mathfrak{M}p$ *indexed by the pair of equivalent projections* $p, \tilde{p} \in \mathcal{L}(\mathfrak{M})$*.*

(ii) *All groupoid structure maps and the groupoid product are complex analytic with respect to the above Banach manifold structure.*

Following [5] we present now an example of the subgroupoid of the groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$. By $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ we denote the transitive subgroupoid of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$, where

$$\mathcal{L}_{p_0}(\mathfrak{M}) := \{l(x) : \quad x \in \mathcal{G}(\mathfrak{M}), \quad r(x) = p_0\} \tag{17}$$

and

$$\mathcal{G}_{p_0}(\mathfrak{M}) := l^{-1}(\mathcal{L}_{p_0}(\mathfrak{M})) \cap r^{-1}(\mathcal{L}_{p_0}(\mathfrak{M})). \tag{18}$$

Let G_0 be the group of invertible elements of W^*-subalgebra $p_0\mathfrak{M}p_0 \subset \mathfrak{M}$. By P_0 we denote the intersection $\mathcal{G}_{p_0}(\mathfrak{M}) \cap \mathfrak{M}p_0$ of $\mathcal{G}_{p_0}(\mathfrak{M})$ with the left W^*-ideal $\mathfrak{M}p_0$. From the subsequent (see [5])

Proposition 7.

(i) *Group* G_0 *is an open subset of the Banach space* $p_0\mathfrak{M}p_0$*. So,* G_0 *is a Banach–Lie group whose Lie algebra is* $p_0\mathfrak{M}p_0$*.*

(ii) *The subset* $P_0 \subset \mathfrak{M}p_0$ *is open in the Banach space* $\mathfrak{M}p_0$*. Thus the tangent bundle* TP_0 *can be identified with the trivial bundle* $P_0 \times \mathfrak{M}p_0$*.*

(iii) *One has a free right action of* G_0 *on* $P_0 \times P_0$ *defined by*

$$P_0 \times P_0 \times G_0 \ni (\eta, \xi, g) \;\mapsto\; (\eta g, \xi g) \in P_0 \times P_0. \tag{19}$$

It follows that $P_0\,(\mathcal{L}_{p_0}(\mathfrak{M}),\ G_0,\ l)$ is a principal bundle with P_0 as the total space, $\mathcal{L}_{p_0}(\mathfrak{M})$ as the bundle base, and the left support $l : P_0 \to \mathcal{L}_{p_0}(\mathfrak{M})$ as the canonical projection. Thus we obtain the gauge groupoid $\frac{P_0 \times P_0}{G_0} \rightrightarrows P_0/G_0$ of the above principal bundle. For the definition of the gauge groupoid see for example [3].

In [5] we show that Banach–Lie groupoids $\frac{P_0 \times P_0}{G_0} \rightrightarrows P_0/G_0$ and $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ are isomorphic.

4. Algebroid of the groupoid $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$

The Atiyah sequence of the principal bundle $P_0\left(\mathcal{L}_{p_0}(\mathfrak{M}),\ G_0,\ l\right)$ is the following one

$$0 \to p_0\mathfrak{M}p_0 \times_{Ad_{G_0}} P_0 \overset{\iota}{\hookrightarrow} TP_0/G_0 \overset{a}{\to} T(P_0/G_0) \to 0, \tag{20}$$

where $p_0\mathfrak{M}p_0$ is the Lie algebra of the group G_0. The vector bundle morphisms ι and a are defined by the sequence

$$0 \to T^V P_0/G_0 \overset{\iota}{\to} TP_0/G_0 \overset{\pi}{\to} TP_0/TG_0 \to 0 \tag{21}$$

and isomorphisms

$$TP_0/TG_0 \cong T\left(P_0/G_0\right), \tag{22}$$

$$p_0\mathfrak{M}p_0 \times_{AdG_0} P_0 \cong T^V P_0/G_0, \tag{23}$$

where $T^V P_0$ is the vertical bundle of $P_0\left(\mathcal{L}_{p_0}(\mathfrak{M}),\ G_0,\ l\right)$. We can see from (20) that $TP_0/G_0 \to P_0/G_0$ is the algebroid of the gauge groupoid $\frac{P_0\times P_0}{G_0} \rightrightarrows P_0/G_0$ which as we have shown above is isomorphic to $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$. Hence we conclude that the gauge algebroid $\frac{TP_0}{G_0} \to \frac{P_0}{G_0}$ is isomorphic to the algebroid $\mathcal{A}_{p_0}(\mathfrak{M}) \to \mathcal{L}_{p_0}(\mathfrak{M})$ of the groupoid $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$. Using this isomorphism we find that the Lie bracket of $\mathfrak{X}_1, \mathfrak{X}_2 \in \Gamma(T^V P_0/G_0) \cong \Gamma\mathcal{A}_{p_0}(\mathfrak{M})$ is given by the following expression

$$[\mathfrak{X}_1, \mathfrak{X}_2](\eta) = \left(\left\langle \frac{\partial \vartheta_2}{\partial \eta}(\eta), \vartheta_1(\eta)\right\rangle - \left\langle \frac{\partial \vartheta_1}{\partial \eta}(\eta), \vartheta_2(\eta)\right\rangle\right)\frac{\partial}{\partial \eta}, \tag{24}$$

where

$$\mathfrak{X}(\eta) = \vartheta(\eta)\frac{\partial}{\partial \eta} \tag{25}$$

is G_0-invariant vector field on P_0, i.e., $\vartheta : P_0 \to \mathfrak{M}p_0$ satisfies

$$\vartheta(\eta g) = \vartheta(\eta) g, \tag{26}$$

where $\eta \in P_0,\ g \in G_0$. The notation (25) means that

$$(\mathfrak{X} f)(\eta) = \left\langle \frac{\partial f}{\partial \eta}(\eta), \vartheta(\eta)\right\rangle$$

for any $f \in C^\infty(P_0)$.

The anchor map $a : \mathcal{A}_{p_0}(\mathfrak{M}) \to T\mathcal{L}_{p_0}(\mathfrak{M})$ for $\mathcal{A}_{p_0}(\mathfrak{M})$ is given by

$$a := Tl, \tag{27}$$

where $l : \mathcal{G}_{p_0}(\mathfrak{M}) \to \mathcal{L}_{p_0}(\mathfrak{M})$ is the left support map.

5. An example

We conclude our note presenting an example. Let us take $\mathfrak{M} = L^\infty(\mathcal{H})$, where $\mathcal{H}$ is a separable complex Hilbert space with a fixed orthonormal basis $\{|e_k\rangle\}_{k=0}^{\infty}$. Setting $p_0 = |e_0\rangle\langle e_0|$ (we use the Dirac notation) we find that

$$\mathcal{L}_{p_0}(L^\infty(\mathcal{H})) = \left\{ \frac{|\eta\rangle\langle\eta|}{\langle\eta|\eta\rangle} : \quad \eta \in \mathcal{H} \setminus \{0\} \right\} \cong \mathbb{CP}(\mathcal{H}) \tag{28}$$

and

$$\mathcal{G}_{p_0}(L^\infty(\mathcal{H})) = \left\{ \frac{|\eta\rangle\langle\xi|}{\langle\xi|\xi\rangle} : \quad \eta, \xi \in \mathcal{H} \setminus \{0\} \right\}. \tag{29}$$

5.1. The groupoid $\mathcal{G}_{p_0}(L^\infty(\mathcal{H})) \rightrightarrows \mathcal{L}_{p_0}(L^\infty(\mathcal{H}))$

The structure maps in this case are as follows:

(i) the source and target maps $\mathbf{s}, \mathbf{t} : \mathcal{G}_{p_0}(L^\infty(\mathcal{H})) \to \mathcal{L}_{p_0}(L^\infty(\mathcal{H}))$ are of the form:

$$\mathbf{s}\left(\frac{|\eta\rangle\langle\xi|}{\langle\xi|\xi\rangle}\right) = \frac{|\xi\rangle\langle\xi|}{\langle\xi|\xi\rangle}, \qquad \mathbf{t}\left(\frac{|\eta\rangle\langle\xi|}{\langle\xi|\xi\rangle}\right) = \frac{|\eta\rangle\langle\eta|}{\langle\eta|\eta\rangle}, \tag{30}$$

(ii) the product of elements $\frac{|\eta\rangle\langle\xi|}{\langle\xi|\xi\rangle}, \frac{|\xi\rangle\langle\lambda|}{\langle\lambda|\lambda\rangle} \in \mathcal{G}_{p_0}(L^\infty(\mathcal{H}))$ is:

$$\frac{|\eta\rangle\langle\xi|}{\langle\xi|\xi\rangle} \frac{|\xi\rangle\langle\lambda|}{\langle\lambda|\lambda\rangle} = \frac{|\eta\rangle\langle\lambda|}{\langle\lambda|\lambda\rangle}, \tag{31}$$

(iii) the identity section $\varepsilon : \mathcal{L}_{p_0}(L^\infty(\mathcal{H})) \to \mathcal{G}_{p_0}(L^\infty(\mathcal{H}))$ is the embedding,

(iv) the inverse map $\iota : \mathcal{G}_{p_0}(L^\infty(\mathcal{H})) \to \mathcal{G}_{p_0}(L^\infty(\mathcal{H}))$ is given by

$$\iota\left(\frac{|\eta\rangle\langle\xi|}{\langle\xi|\xi\rangle}\right) = \frac{|\xi\rangle\langle\eta|}{\langle\eta|\eta\rangle}. \tag{32}$$

We note that for $p_0 = |e_0\rangle\langle e_0|$ one has

$$(L^\infty(\mathcal{H}))p_0 = \{|\vartheta\rangle\langle e_0| : \quad \vartheta \in \mathcal{H}\} \cong \mathcal{H}, \tag{33}$$

$$P_0 = \{|\eta\rangle\langle e_0| : \quad \eta \in \mathcal{H} \setminus \{0\}\} \cong \mathcal{H} \setminus \{0\}, \tag{34}$$

and

$$G_0 = G\left(p_0(L^\infty(\mathcal{H}))p_0\right) \cong \mathbb{C} \setminus \{0\}. \tag{35}$$

So, the groupoid $\mathcal{G}_{p_0}(L^\infty(\mathcal{H})) \rightrightarrows \mathcal{L}_{p_0}(L^\infty(\mathcal{H}))$ is isomorphic to the gauge groupoid of the complex Hopf bundle

$$\begin{array}{ccc} \mathbb{C} \setminus \{0\} & \longrightarrow & \mathcal{H} \setminus \{0\} \\ & & \big\downarrow l \\ & & \mathbb{CP}(\mathcal{H})\,. \end{array} \tag{36}$$

5.2. The complex manifold structure of $\mathcal{G}_{p_0}(L^\infty(\mathcal{H})) \rightrightarrows \mathcal{L}_{p_0}(L^\infty(\mathcal{H}))$

In order to introduce the differential structure of the groupoid $\mathcal{G}_{p_0}(L^\infty(\mathcal{H})) \rightrightarrows \mathcal{L}_{p_0}(L^\infty(\mathcal{H}))$ we should notice that for the orthonormal projections $p_k := |e_k\rangle\langle e_k|$, $k \in \mathbb{N} \cup \{0\}$, the sets $\Pi_k := \Pi_{p_k}$ defined in (7) are the following

$$\Pi_k = \left\{ q = \frac{|\xi\rangle\langle\xi|}{\langle\xi|\xi\rangle}; \quad \xi_k \neq 0, \text{where } \xi = \sum_{k=o}^{\infty} \xi_k |e_k\rangle \right\}. \tag{37}$$

The maps $\sigma_k : \Pi_k \to (L^\infty(\mathcal{H}))p_k$ and $\varphi_k : \Pi_k \to (1 - p_k)(L^\infty(\mathcal{H}))p_k$, see (8), are given by

$$\sigma_k(q) = \frac{1}{\xi_k} |\xi\rangle\langle e_k|, \tag{38}$$

and

$$\varphi_k(q) = \frac{1}{\xi_k} |\xi\rangle\langle e_k| - |e_k\rangle\langle e_k| = \mathbf{y}_k, \tag{39}$$

respectively. Let us note here that we can write $\mathbf{y}_k \in (1 - p_k)(L^\infty(\mathcal{H}))p_k$ in the form

$$\mathbf{y}_k = \sum_{l \neq k} \frac{\xi_l}{\xi_k} |e_l\rangle\langle e_k|. \tag{40}$$

So, $\frac{\xi_l}{\xi_k} =: y_k^l$, where $k \neq l \in \mathbb{N} \cup \{0\}$, are the homogeneous coordinates of $q \in \Pi_k$.

The charts

$$\psi_{km} : l^{-1}(\Pi_k) \cap r^{-1}(\Pi_m) \to (1 - p_k)(L^\infty(\mathcal{H}))p_k \oplus p_k(L^\infty(\mathcal{H}))p_m \oplus (1 - p_m)(L^\infty(\mathcal{H}))p_m$$

of the atlas (11) for $\mathcal{G}_{p_0}(L^\infty(\mathcal{H})) \rightrightarrows \mathcal{L}_{p_0}(L^\infty(\mathcal{H}))$ are given by

$$\psi_{km}(g) = \big(\varphi_k(l(g)), (\sigma_k(l(g)))^{-1} g \sigma_m(r(g)), \varphi_m(r(g))\big) = (\mathbf{y}_k, \mathbf{z}_{km}, \mathbf{y}_m). \tag{41}$$

The coordinates $\mathbf{y}_k$ and $\mathbf{y}_m$ in (41) are defined in (39) and the coordinate $\mathbf{z}_{km}$ is given by

$$\mathbf{z}_{km} = z_{km} |e_k\rangle\langle e_m|, \tag{42}$$

where $z_{km} := \frac{\eta_k}{\xi_m}$.

So, as one can expect, the complex analytic manifold structure of $\mathcal{G}_{p_0}(L^\infty(\mathcal{H}))$ is consistent with the complex analytic structure of the complex Hopf bundle (36).

5.3. The algebroid $\mathcal{A}_{p_0}(L^\infty(\mathcal{H}))$ of $\mathcal{G}_{p_0}(L^\infty(\mathcal{H})) \rightrightarrows \mathcal{L}_{p_0}(L^\infty(\mathcal{H}))$

Using the algebroid isomorphism $\mathcal{A}_{p_0}(\mathfrak{M}) \cong \frac{TP_0}{G_0}$ for the case $\mathfrak{M} = L^\infty(\mathcal{H})$ and $p_0 = |e_0\rangle\langle e_0|$ by virtue of (33)–(35) we obtain the isomorphism

$$\mathcal{A}_{|e_0\rangle\langle e_0|}(L^\infty(\mathcal{H})) \cong \frac{\mathcal{H} \times (\mathcal{H} \setminus \{0\})}{\mathbb{C} \setminus \{0\}}. \tag{43}$$

Hence the sections of the algebroid $\mathcal{A}_{p_0}(L^\infty(\mathcal{H}))$ in the coordinates $(\mathbf{y}_k,\ \mathbf{z}_{km})$ have the following form

$$\mathfrak{X}(\mathbf{y}_k, \mathbf{z}_{km}) = \sum_{l \neq k} a^l(\mathbf{y}_k) \frac{\partial}{\partial y_k^l} + b(\mathbf{y}_k) z_{km} \frac{\partial}{\partial z_{km}} \tag{44}$$

and the algebroid Lie bracket (24) of sections $\mathfrak{X}_1, \mathfrak{X}_2 \in \Gamma \mathcal{A}_{|e_0\rangle\langle e_0|}(L^\infty(\mathcal{H}))$ is

$$\begin{aligned}[]
[\mathfrak{X}_1, \mathfrak{X}_2](\mathbf{y}_k, \mathbf{z}_{km}) = \sum_{s \neq k} \sum_{l \neq k} \left(a_1^l(\mathbf{y}_k) \frac{\partial a_1^s}{\partial y_k^l} - a_2^l(\mathbf{y}_k) \frac{\partial a_1^s}{\partial y_k^l} \right) \frac{\partial}{\partial y_k^s} \\
+ \left(\sum_{l \neq k} a_1^l(\mathbf{y}_k) \frac{\partial b_2}{\partial y_k^l} - \sum_{l \neq k} a_2^l(\mathbf{y}_k) \frac{\partial b_1}{\partial y_k^l} \right) z_{km} \frac{\partial}{\partial z_{km}}.
\end{aligned} \tag{45}$$

The anchor $a : \mathcal{A}_{|e_0\rangle\langle e_0|}(L^\infty(\mathcal{H})) \to T\mathbb{CP}(\mathcal{H})$ acts on the section

$$\mathfrak{X} \in \Gamma \mathcal{A}_{|e_0\rangle\langle e_0|}(L^\infty(\mathcal{H}))$$

according to the formula

$$a(\mathfrak{X}) = \sum_{l \neq k} a^l(\mathbf{y}_k) \frac{\partial}{\partial y_k^l}. \tag{46}$$

Finally let us note that $b(\mathbf{y}_k) z_{km} \frac{\partial}{\partial z_{km}}$ proves to be the vertical vector field of the complex Hopf bundle.

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Orbifold Diffeomorphism Groups

Alexander Schmeding

Abstract. Orbifolds are a generalization of manifolds. They arise naturally in different areas of mathematics and physics, e.g.:

– Spaces of symplectic reduction are orbifolds,
– Orbifolds may be used to construct a conformal field theory model.

In [10], we considered the diffeomorphism group of a paracompact, non-compact smooth reduced orbifold. Our main result is the construction of an infinite-dimensional Lie-group structure on the diffeomorphism group and several interesting subgroups. The aim of these notes is to sketch the main ingredients of the proof. Furthermore, we will consider the special case of an orbifold with a global chart.

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1. Orbifolds in local charts

We recall the notion of an orbifold defined via local charts and their morphisms. Our exposition of orbifolds follows [4]:

Definition 1. Let Q be a paracompact Hausdorff topological space with $d \in \mathbb{N}_0$.

1. A (*reduced*) *orbifold chart* of dimension d on Q is a triple (V, G, φ) where V is a connected paracompact n-dimensional manifold without boundary, G is a finite subgroup of $\mathrm{Diff}(V)$, and $\varphi\colon V \to Q$ is a map with open image $\varphi(V)$ inducing a homeomorphism from the orbit space V/G to $\varphi(V)$. Here the orbit space V/G is the set of all G-orbits with respect to the natural G-action on V. We endow V/G with the quotient topology with respect to the map sending $x \in V$ to its orbit $G.x$.
2. Two orbifold charts (V, G, φ), (W, H, ψ) on Q are called *compatible* if for each pair $(x, y) \in V \times W$ with $\varphi(x) = \psi(y)$ there are open connected neighborhoods V_x of x and W_y of y together with a C^∞-diffeomorphism $h\colon V_x \to W_y$ with $\psi \circ h = \varphi|_{V_x}$. The map h is called a *change of charts*.

3. A *reduced orbifold atlas* of dimension d on Q is a set of pairwise compatible reduced orbifold charts $\mathcal{V} = \{(V_i, G_i, \varphi_i) \mid i \in I\}$ of dimension d on Q such that $\bigcup_{i\in I} \varphi_i(V_i) = Q$. Two reduced orbifold atlases are equivalent if their union is a reduced orbifold atlas.
4. A *reduced orbifold* of dimension d is a pair $(Q, \mathcal{U})$ where $\mathcal{U}$ is an equivalence class of orbifold atlases of dimension d on Q.
5. For an orbifold $(Q, \mathcal{U})$, a point $x \in Q$ will be called *singular* if there is an orbifold chart (V, G, ψ), such that for any $y \in \psi^{-1}(x)$ the isotropy subgroup $G_y := \{g \in G \mid g.y = y\}$ is non-trivial. Otherwise x is called *regular*. This property is independent of choice of charts (see [7, p. 39]).

The term reduced refers to the fact that the finite group G is required to be a subgroup of $\mathrm{Diff}(V)$. Hence, each group G acts effectively on V. Every orbifold in these notes will be reduced, whence we drop the word "reduced" for the rest of this paper. We consider a class of orbifolds with global chart, which will serve as our main example. Notice that in general an orbifold will not admit a global orbifold chart.

Example 1. Let d be in $\mathbb{N}$ and $G \neq \{\mathrm{id}_{\mathbb{R}^d}\}$ be a finite subgroup of the orthogonal group $O(d) \subseteq \mathrm{Diff}(\mathbb{R}^d)$ such that:

(IS) The group G satisfies $G_x = \{\mathrm{id}_{\mathbb{R}^d}\}$ for all $x \in \mathbb{R}^d \setminus \{0\}$, i.e., 0 is the only singularity fixed jointly by all elements of G.

We remark the following:

1. For odd d only $G = \{\mathrm{id}_{\mathbb{R}^d}, -\mathrm{id}_{\mathbb{R}^d}\}$ is possible. For $d = 1$ we denote the reflection generating G by $r\colon \mathbb{R} \to \mathbb{R}, x \mapsto -x$.
2. If $d = 2$, then the group G may not contain reflections by condition (IS). In this case, G contains at least one rotation of $\mathbb{R}^2$ fixing the origin.

Let $\pi\colon \mathbb{R}^d \to \mathbb{R}^d/G$ be the quotient map onto the orbit space and $Q := \mathbb{R}^d/G$. Then $\{(\mathbb{R}^d, G, \pi)\}$ is an atlas for Q, turning the orbit space into an orbifold with a global chart. We identify for $d \in \{1, 2\}$ the orbit spaces with $[0, \infty[$ and respectively

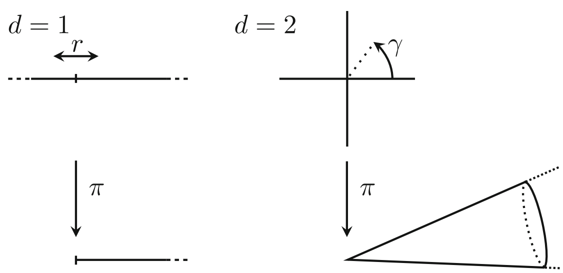

FIGURE 1. Cone shaped orbifolds. The element γ is a rotation which generates G for $d = 2$.

a cone: Each finite subgroup of $O(2)$ – which is not a dihedral group – is cyclic by [2, Ch. 5, Theorem 3.4]. Hence Figure 1 exhibits the general case for $d = 2$.

Notice that the chart mappings of an orbifold will in general be non-invertible. To define a "smooth morphism of orbifolds" we have to provide smooth lifts in charts. However, these lifts should be "smoothly related" to obtain a well-behaved notion of orbifold morphism. In this note we understand orbifold morphisms as maps in the sense of Pohl [9]:

Definition 2. A *representative of an orbifold map* from $(Q,\mathcal{U})$ to $(Q',\mathcal{U}')$ is a tuple $\hat{f} = (f, \{f_i\}_{i\in I}, [P,\nu])$ where

R1 $f\colon Q \to Q'$ is a continuous map,

R2 $\forall i \in I$, f_i is a smooth local lift of f with respect to $(V_i, G_i, \pi_i) \in \mathcal{U}$, $(V_i', G_i', \pi_i') \in \mathcal{U}'$ such that the (V_i, G_i, π_i) cover Q

R3 the lifts are smoothly related to each other, i.e., for certain change of charts $\lambda\colon V_i \supseteq U \to V_j, i, j \in I$ (contained in the set P), there is a change of charts $\nu(\lambda)$, such that $f_j \circ \lambda = \nu(\lambda) \circ f_i|_{\mathrm{dom}\lambda}$ holds. This compatibility condition is encoded by the pair (P, ν).

We will not give details in these notes concerning the pair (P, ν) and the axioms they satisfy (cf. [9, Definition 4.4]). It turns out that these data are naturally fixed for most types of mappings considered in these notes.

An *orbifold map* (or *morphism of orbifolds*) $[\hat{f}]$ is an equivalence class of representatives. The equivalence relation is obtained by identifying representatives which arise by refinements of orbifold atlases. Again, we omit the details here (which are recorded in [9]) and remark only:
Orbifolds and orbifold morphisms form a category denoted by **Orb**.

Definition 3. A morphism of orbifolds $[\hat{f}] \in \mathbf{Orb}((Q_1,\mathcal{U}_1),(Q_2,\mathcal{U}_2))$ is called an *orbifold diffeomorphism* if it is an isomorphism in **Orb**. Define the *orbifold diffeomorphism group* $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ to be the subset of all orbifold diffeomorphisms contained in $\mathbf{Orb}((Q,\mathcal{U}),(Q,\mathcal{U}))$.

The following result shows that we may forget the compatibility condition R3 mentioned in Definition 2 for orbifold diffeomorphisms:

Proposition 1 ([10, Corollary 2.1.12]). *For an orbifold map $[\hat{f}]$ the following are equivalent:*

1. *$[\hat{f}]$ is an orbifold diffeomorphism,*
2. *there is a representative $\hat{f} = (f, \{f_j\}_{j\in J}, [P,\nu])$ of $[\hat{f}]$ such that f is a homeomorphism and each f_j is a diffeomorphism.*

In particular, an orbifold diffeomorphism is uniquely determined by its lifts.

Example 2. Consider an orbifold with global chart as in Example 1. Let $\tilde{h}\colon \mathbb{R}^d \to \mathbb{R}^d$ be a homeomorphism. If there is a group automorphism $\alpha\colon G \to G$ with $\tilde{h}\circ\gamma = \alpha(\gamma).\tilde{h}$ for all $\gamma \in G$, we call $\tilde{h}$ a *weak equivalence*. For a weak equivalence $\tilde{h}$ the map $h\colon \mathbb{R}^d/G \to \mathbb{R}^d/G, x \mapsto \pi\circ\tilde{h}\circ\pi^{-1}(x)$ makes sense and is a homeomorphism.

Following Proposition 1, each diffeomorphism $\tilde{h}\colon \mathbb{R}^d \to \mathbb{R}^d$ which is a weak equivalence induces an orbifold diffeomorphism $[\hat{h}] \in \mathrm{Diff}_{\mathrm{Orb}}\left(\mathbb{R}^d/G, \left\{(\mathbb{R}^d, G, \pi)\right\}\right)$. A representative $\hat{h}$ of $[\hat{h}]$ is uniquely determined by the smooth lift $\tilde{h}$.

Before we state the main result on the diffeomorphism group of an orbifold, we need to introduce mappings associated to the *tangent orbifold*. Analogous to the construction of a tangent manifold for a smooth manifold, one can construct a tangent orbifold $(\mathcal{T}Q, \mathcal{T}\mathcal{U})$ for an orbifold $(Q,\mathcal{U})$ (see [1, Section 1.3] or [10, Section 3.1]). The orbifold tangent space $\mathcal{T}_xQ$ for a singular point $x \in Q$ will not support a natural vector space structure, but it contains a unique zero-element 0_x. Moreover, there is a well-defined orbifold morphism $\pi_{\mathcal{T}(Q,\mathcal{U})}\colon (\mathcal{T}Q, \mathcal{T}\mathcal{U}) \to (Q,\mathcal{U})$, the so-called "bundle projection". Right inverses of the projection are sections to the tangent orbifold:

Definition 4.

1. A map of orbifolds $[\hat{\sigma}] \in \mathbf{Orb}\left((Q,\mathcal{U}), \mathcal{T}(Q,\mathcal{U})\right)$ is called *orbisection* if it satisfies $\pi_{\mathcal{T}(Q,\mathcal{U})} \circ [\hat{\sigma}] = \mathrm{id}_{(Q,\mathcal{U})}$. Here $\mathrm{id}_{(Q,\mathcal{U})}$ is the identity morphism of $(Q,\mathcal{U})$. Denote the *set of all orbisections* for $(Q,\mathcal{U})$ by $\mathfrak{X}_{\mathrm{Orb}}(Q)$.
2. For $[\hat{\sigma}] \in \mathfrak{X}_{\mathrm{Orb}}(Q)$ the *support* $\mathrm{supp}[\hat{\sigma}]$ of $[\hat{\sigma}]$ is the closure of the set $\{x \in Q \mid \sigma(x) \neq 0_x\} \subseteq Q$. If $\mathrm{supp}[\hat{\sigma}] \subseteq K$ holds for some compact subset $K \subseteq Q$, then $[\hat{\sigma}] \in \mathfrak{X}_{\mathrm{Orb}}(Q)$ is called *compactly supported* (*in* K). Let $\mathfrak{X}_{\mathrm{Orb}}(Q)_c$ be the *set of all compactly supported orbisections in* $\mathfrak{X}_{\mathrm{Orb}}(Q)$.

It turns out that orbisections are uniquely determined by their lifts. Even more, an orbisection possesses a unique lift in each chart, which we call a *canonical lift*. Notice that in general orbifold morphisms need not posses lifts in a prescribed orbifold chart. We obtain the following characterization for the compactly supported orbisections:

Theorem 2 ([10, Proposition 3.2.9 and Section 3.3]). *Let $\{(U_i, G_i, \varphi_i) \mid i \in I\}$ be any orbifold atlas for $(Q,\mathcal{U})$. Denote by $\mathfrak{X}(U_i)$ the space of all smooth vector fields on the manifold U_i. The set $\mathfrak{X}_{\mathrm{Orb}}(Q)_c$ is in bijection with all families of vector fields $(\sigma_i)_{i\in I} \in \bigoplus_{i\in I} \mathfrak{X}(U_i)$ which satisfy the compatibility condition:*

$$T\lambda \circ \sigma_i|_{\mathrm{dom}\,\lambda} = \sigma_j \circ \lambda, \quad \forall \lambda\colon V_i \supseteq U \to V_j \text{ change of charts}, i,j \in I$$

The embedding $\mathfrak{X}_{\mathrm{Orb}}(Q)_c \hookrightarrow \bigoplus_{i\in I} \mathfrak{X}(U_i)$ turns the compactly supported orbisections into a locally convex space.

Example 3. Consider an orbifold as in Example 1. By Theorem 2, the space of compactly supported orbisections $\mathfrak{X}_{\mathrm{Orb}}\left(\mathbb{R}^d/G\right)_c$ corresponds to the compactly supported vector fields in $\mathfrak{X}\left(\mathbb{R}^d\right)$ which satisfy $X \circ \lambda = T\lambda \circ X|_{\mathrm{dom}\lambda}$ for all change of charts λ. Then [7, Lemma 2.11] implies that this condition is equivalent to $X \circ \gamma = T\gamma \circ X$ for all $\gamma \in G$. In particular the space $\mathfrak{X}_{\mathrm{Orb}}\left(\mathbb{R}^d/G\right)_c$ is isomorphic to the subset of all compactly supported and equivariant vector fields

$$\mathfrak{X}\left(\mathbb{R}^d\right)_c^G := \left\{X \in \mathfrak{X}\left(\mathbb{R}^d\right) \,\middle|\, \mathrm{supp}\, X \text{ is compact}, X \circ \gamma = T\gamma \circ X, \forall \gamma \in G\right\}.$$

The space of compactly supported orbisections will be the modeling space for the Lie group structure on $\mathrm{Diff}_{\mathrm{Orb}}\left(\mathbb{R}^d/G, \left\{(\mathbb{R}^d, G, \pi)\right\}\right)$. To construct a chart, we use results from Riemannian geometry on orbifolds. The leading idea is to construct objects from Riemannian geometry on charts and to glue them via change of charts. It is well known that one can construct a *Riemannian orbifold metric* for an orbifold (see [7, Proposition 2.20]). Moreover, one can construct for each element in $\mathcal{T}Q$ a unique maximal orbifold geodesic (cf. [10, Section 4.1]). Let Ω be the subset of elements in $\mathcal{T}Q$ whose associated maximal geodesic exists at least up to time 1. Then the map $\exp_{\mathrm{Orb}} \colon \mathcal{T}Q \supseteq \Omega \to Q$ sending an element to its associated orbifold geodesic evaluated at time 1 induces a morphism of orbifolds $[\exp_{\mathrm{Orb}}]$ (cf. [10, Section 4.2]). This morphism is called *Riemannian orbifold exponential map.*

Theorem 3 ([10, Theorem 5.2.4]). *The group* $\mathrm{Diff}_{\mathrm{Orb}}(Q, \mathcal{U})$ *can be made into an infinite-dimensional Lie group* (*in the sense of* [8]) *such that:*

For some Riemannian orbifold metric ρ, let $[\exp_{\mathrm{Orb}}]$ be the Riemannian orbifold exponential map with domain Ω. There exists an open zero-neighborhood in $\mathfrak{X}_{\mathrm{Orb}}(Q)_c$ such that

$$[\hat{\sigma}] \mapsto [\exp_{\mathrm{Orb}}] \circ [\hat{\sigma}]|^{\Omega}$$

is a C^∞-diffeomorphism onto an open submanifold of $\mathrm{Diff}_{\mathrm{Orb}}(Q, \mathcal{U})$. *The condition is then satisfied for every Riemannian orbifold metric on $(Q, \mathcal{U})$.*

Proposition 4 ([10, Theorem 5.3.1]). *The Lie algebra of* $\mathrm{Diff}_{\mathrm{Orb}}(Q, \mathcal{U})$ *is given by* $(\mathfrak{X}_{\mathrm{Orb}}(Q)_c, [\cdot,\cdot])$. *The Lie bracket of two orbisections $[\hat{\sigma}], [\hat{\tau}]$ is the orbisection whose canonical lift on a chart (U, G, φ) is*

$$[\sigma_U, \tau_U] \ \ (\textit{Lie bracket in } \mathfrak{X}(U)).$$

Here σ_U and τ_U are the canonical lifts of $[\hat{\sigma}]$ and $[\hat{\tau}]$, respectively.

In the rest of this note, we will apply the results to the orbifolds considered in Example 1. We will see that for these orbifolds, Theorem 3 induces Lie group structures for certain subgroups of $\mathrm{Diff}(\mathbb{R}^d)$. In particular, these Lie group structures will coincide with closed Lie subgroups of $\mathrm{Diff}(\mathbb{R}^d)$ (see [5] for the construction of the Lie group $\mathrm{Diff}(\mathbb{R}^d)$).

2. Application to equivariant diffeomorphism groups

For this section, we use the notation introduced in Examples 1 and 2.

5. Denote by $\mathrm{Diff}^G(\mathbb{R}^d)$ the subset of all weak equivalences in $\mathrm{Diff}(\mathbb{R}^d)$. As in Example 2, we let $[\hat{h}]$ be the orbifold diffeomorphism associated to $\tilde{h} \in \mathrm{Diff}^G(\mathbb{R}^d)$. Consider the map

$$D \colon \mathrm{Diff}^G(\mathbb{R}^d) \to \mathrm{Diff}_{\mathrm{Orb}}\left(\mathbb{R}^d/G, \left\{\mathbb{R}^d, G, \pi\right\}\right), \tilde{f} \mapsto [\hat{f}].$$

Each orbifold diffeomorphisms in the image of D is induced by a lift in the global chart. Since orbifold diffeomorphisms are uniquely determined by their lifts, the composition of these lifts induces the composition of orbifold diffeomorphisms.

Thus $D(\tilde{h}^{-1})$ coincides with $D(\tilde{h})^{-1}$ (the inverse in $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$) by [10, Corollary 2.1.12]. Summing up, D is a group homomorphism.

The map D is not injective, as elements of $\mathrm{Diff}^G(U)$ which differ only up to composition with an element of G are mapped to the same diffeomorphism of orbifolds. From [7, Lemma 2.11] we deduce that the kernel of D coincides with G. Hence D induces an injective group homomorphism:

$$\Delta\colon \mathrm{Diff}^G(\mathbb{R}^d)/G \to \mathrm{Diff}_{\mathrm{Orb}}\left(\mathbb{R}^d/G, \left\{\mathbb{R}^d, G, \pi\right\}\right)$$

We will show in the next proposition that Δ is surjective, i.e., each orbifold diffeomorphisms of the orbifold $(\mathbb{R}^d/G, \left\{\mathbb{R}^d, G, \pi\right\})$ corresponds to diffeomorphism of $\mathbb{R}^d$, which is a weak equivalence with respect to the G-action.

Proposition 6. *Consider* $[\hat{h}] \in \mathrm{Diff}_{\mathrm{Orb}}\left(\mathbb{R}^d/G,\ \left\{\mathbb{R}^d, G, \pi\right\}\right)$ *with representative* $(h, \{h_i\}_{i\in I}, (P,\nu)) \in [\hat{h}]$. *The map* h *lifts to a diffeomorphism* $\tilde{h}$ *of* $\mathbb{R}^d$ *which is a weak equivalence, with respect to the* G*-action.*

Proof. We shall construct at first a lift on the set of non-singular points. By condition **(IS)** of Example 1, there is only one singular point. The origin in $\mathbb{R}^d$ is jointly fixed by all elements of G. Hence $\mathbb{R}^d \setminus \{0\}$ corresponds to the set of non-singular points and we set $Q_{\mathrm{reg}} := Q \setminus \{0\}$. It is easy to see that $q := \pi|_{\mathbb{R}^d\setminus\{0\}}^{Q_{\mathrm{reg}}}$ is a covering.

Diffeomorphisms of orbifolds preserve singular points by [10, Proposition 2.1.5] and thus the homeomorphism $h\colon Q \to Q$ satisfies $f\pi(0) = \pi(0)$. The restriction $h|_{Q_{\mathrm{reg}}}^{Q_{\mathrm{reg}}}$ yields a homeomorphism.

If $d = 1$ **holds**, then the space $\mathbb{R} \setminus \{0\}$ is disconnected. Then the mapping $q|_{]0,\infty[}\colon\ ]0,\infty[\to Q_{\mathrm{reg}}$ is a homeomorphism and we obtain a well-defined homeomorphism $h^+ := (q|_{]0,\infty[})^{-1} hq|_{]0,\infty[}$, mapping $]0,\infty[$ to itself. This mapping extends to a homeomorphism via

$$h_{\mathrm{reg}}\colon \mathbb{R}\setminus\{0\} \to \mathbb{R}\setminus\{0\}, x \mapsto \begin{cases} h^+(x) & x > 0 \\ r\circ h^+ \circ r(x) = -h^+(-x) & x < 0. \end{cases}$$

By construction, h_{reg} and also its inverse are equivariant maps with respect to $G = \langle r\rangle$. Thus h_{reg} extends to an equivariant homeomorphism $\tilde{h}\colon \mathbb{R} \to \mathbb{R}$ by setting $\tilde{h}(0) = 0$.

If $d \geq 2$ **holds**, then the space $\mathbb{R}^d \setminus \{0\}$ is (path-)connected. We have to construct a lift $f_{\mathrm{reg}}\colon \mathbb{R}^d \setminus \{0\} \to \mathbb{R}^d \setminus \{0\}$.

For $d \geq 3$, the space $\mathbb{R}^d\setminus\{0\}$ is simply connected, path-connected and locally path-connected. Choose $x_0 \in \mathbb{R}^d\setminus\{0\}$ and $y_0 \in q^{-1}hq(x_0)$. Then by [6, Proposition 1.33], there is a unique lift $h_{\mathrm{reg}}\colon \mathbb{R}^d \setminus \{0\} \to \mathbb{R}^d \setminus \{0\}$ of $h|^{Q_{\mathrm{reg}}} \circ q$ which maps x_0 to y_0.

For $d = 2$, the space $\mathbb{R}^2\setminus\{0\}$ is *not* simply connected. However, it is path-connected and locally path-connected. We may still apply [6, Proposition 1.33] if the fundamental group $\pi_1(\mathbb{R}^2 \setminus \{0\}, x_0)$ satisfies:

$$(h|^{Q_{\mathrm{reg}}} \circ q)_*(\pi_1(\mathbb{R}^2 \setminus \{0\}, x_0)) \subseteq q_*(\pi_1(\mathbb{R}^2 \setminus \{0\}, y_0)) \tag{1}$$

Recall that the fundamental group $\pi_1(\mathbb{R}^2 \setminus \{0\}, x_0)$ can be identified with $\mathbb{Z}$ (cf. [6, Example 1.15]). Moreover, as $G \subseteq \mathrm{SO}(2)$ holds, the subgroup $G \subseteq O(2)$ must be a cyclic group, generated by a rotation γ of order $m \in \mathbb{N}$. As we have already seen, Q is homeomorphic to a cone and Q_{reg} may be identified with a cone whose tip has been removed. In particular, the space Q_{reg} is homeomorphic to $\mathbb{R}^2 \setminus \{0\}$.

Consider a generator $[e]$ of the fundamental group $\pi_1(\mathbb{R}^2 \setminus \{0\}, x_0)$, where e is chosen as a circle around the origin passing through x_0. If γ is a rotation of order m, then we have $q_*[e] = [q \circ e]$ is a loop in Q_{reg}, which passes m times through $\pi(y_0)$. The next picture illustrates this behavior:

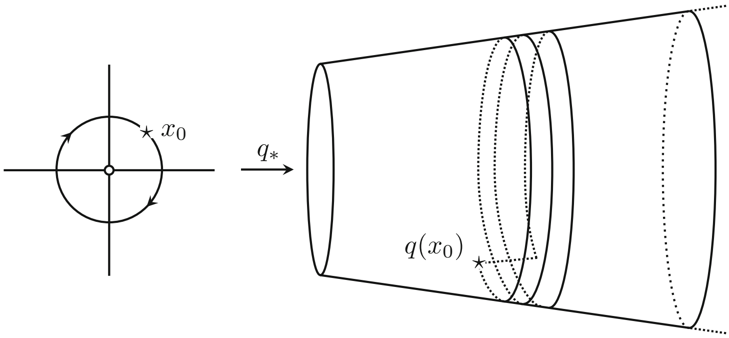

FIGURE 2. Image of a loop generating $\pi_1(\mathbb{R}^2 \setminus \{0\}, x_0)$ with respect to q_*. The loop displayed in Q_{reg} is a curve homotopic to the image of the closed loop for $m = 3$.

Note that $\pi_1(Q_{\mathrm{reg}}, q(y_0))$ is isomorphic to $\mathbb{Z}$ and let $[f]$ be the generator of $\pi_1(Q_{\mathrm{reg}}, q(x_0))$. By abuse of notation we let $[f]$ be the generator of each fundamental group for points in Q_{reg}. From the arguments above, we deduce $q_*(\pi_1(\mathbb{R}^2 \setminus \{0\}, x_0)) = \langle m[f] \rangle$ and thus

$$(h|^{Q_{\mathrm{reg}}} \circ q)_*([e]) = (h|^{Q_{\mathrm{reg}}}_{Q_{\mathrm{reg}}})_*(m[f]) = m([h \circ f]) \in \langle m[f] \rangle = \operatorname{Im} q_*.$$

Therefore property (1) is satisfied and we obtain a unique lift

$$h_{\mathrm{reg}} \colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$$

of $h|^{Q_{\mathrm{reg}}}_{Q_{\mathrm{reg}}}$ mapping x_0 to y_0.

Analogous arguments allow the construction of a unique lift $(h^{-1})_{\mathrm{reg}}$ for $h^{-1}|^{Q \setminus \{0\}} \circ q$ and $d \geq 2$, which maps y_0 to x_0. We claim that $(h^{-1})_{\mathrm{reg}}$ is the inverse of h_{reg}. If this is true, then h_{reg} is a homeomorphism. To prove the claim, consider the map $f := h_{\mathrm{reg}} \circ (h^{-1})_{\mathrm{reg}}$ and compute

$$q \circ f = q \circ h_{\mathrm{reg}} \circ (h^{-1})_{\mathrm{reg}} = h \circ q \circ (h^{-1})_{\mathrm{reg}} = q.$$

Hence f is a lift of $\mathrm{id}_{Q_{\mathrm{reg}}}$ taking y_0 to y_0, and so is the map $\mathrm{id}_{\mathbb{R}^d \setminus \{0\}}$. By the uniqueness of lifts between pointed spaces (see [6, Proposition 1.34]), $h_{\mathrm{reg}} \circ (h^{-1})_{\mathrm{reg}} =$

$f = \mathrm{id}_{\mathbb{R}^d\setminus\{0\}}$. Likewise, $(h^{-1})_{\mathrm{reg}} \circ h_{\mathrm{reg}} = \mathrm{id}_{\mathbb{R}^d\setminus\{0\}}$. Summing up, h_{reg} is a homeomorphism with inverse $(h^{-1})_{\mathrm{reg}}$.

We show that the homeomorphism h_{reg} commutes with the G-action via an automorphism of G. To prove this, consider $g \in G$ and $x \in \mathbb{R}^d \setminus \{0\}$. We have

$$q \circ h_{\mathrm{reg}} \circ g \circ h_{\mathrm{reg}}^{-1}(x)) = hh^{-1}q(x) = q(x).$$

Hence $h_{\mathrm{reg}} \circ g \circ h_{\mathrm{reg}}^{-1}$ is a lift of $\mathrm{id}_{\mathbb{R}^d\setminus\{0\}}$ and so there is an unique element $\alpha(g) \in G$ such that $h_{\mathrm{reg}} \circ g \circ h_{\mathrm{reg}}^{-1}(x_0) = \alpha(g)(x_0)$. By uniqueness of lifts, $h_{\mathrm{reg}} \circ g \circ h_{\mathrm{reg}}^{-1} = \alpha(g)$ on $\mathbb{R}^d \setminus \{0\}$. Repeat this construction for each $g \in G$ to obtain a map $\alpha\colon G \to G$ with $h_{\mathrm{reg}} \circ g = \alpha(g) \circ h_{\mathrm{reg}}$ on $\mathbb{R}^d \setminus \{0\}$ for each $g \in G$. Since $\alpha(gk) \circ h_{\mathrm{reg}} = h_{\mathrm{reg}} \circ (gk) = \alpha(g).h_{\mathrm{reg}} \circ k = \alpha(g).\alpha(k).h_{\mathrm{reg}}$ holds and h_{reg} is a homeomorphism, the map α is an injective group homomorphism. As G is finite, α is thus a group automorphism with $h_{\mathrm{reg}} \circ g = \alpha(g).h_{\mathrm{reg}}$ for each $g \in G$. We extend this map h_{reg} to a homeomorphism $\tilde{h}\colon \mathbb{R}^d \to \mathbb{R}^d$ via $\tilde{h}(0) = 0$. This map satisfies $\tilde{h} \circ g = \alpha(g).\tilde{h}$. We conclude that $\tilde{h}$ is indeed a weak equivalence. Using the definition of orbifold morphisms, one can show that $\tilde{h}$ is a smooth diffeomorphism (see [10, Proposition 6.0.5]). □

Corollary 7. *For an orbifold* $(Q,\mathcal{U})$ *as in* 1*, the mapping* D *is surjective. In particular, the induced map* $\Delta\colon \mathrm{Diff}^G(\mathbb{R}^d)/G \to \mathrm{Diff}_{\mathrm{Orb}}\,(Q,\mathcal{U})$ *is a group isomorphism.*

8. Endow $\mathrm{Diff}^G(\mathbb{R}^d)/G$ with the Lie group structure making Δ an isomorphism of Lie groups. Now we consider the subgroup of $\mathrm{Diff}^G(\mathbb{R}^d)$ whose elements coincide with the identity off some compact subset:

$$\mathrm{Diff}^G_c(\mathbb{R}^d) := \left\{ f \in \mathrm{Diff}^G(\mathbb{R}^d) \,\middle|\, \exists K \subseteq \mathbb{R}^d \text{ compact}, f|_{\mathbb{R}^d\setminus K} = \mathrm{id}_{\mathbb{R}^d\setminus K} \right\}$$

On the level of orbifold diffeomorphisms, we may also consider diffeomorphisms which coincide off some compact set with the identity. These diffeomorphisms form an open Lie subgroup $\mathrm{Diff}_{\mathrm{Orb}}\left(\mathbb{R}^d/G, \left\{(\mathbb{R}^d, G, \pi)\right\}\right)_c$ of all orbifold diffeomorphisms (cf. [10, Remark 5.2.7]). By construction, D maps $\mathrm{Diff}^G_c(\mathbb{R}^d)$ into the open Lie subgroup $\mathrm{Diff}_{\mathrm{Orb}}\left(\mathbb{R}^d/G, \left\{(\mathbb{R}^d, G, \pi)\right\}\right)_c$. Recall that $G \cap \mathrm{Diff}^G_c(U) = \{\mathrm{id}_{\mathbb{R}^d}\}$. Thus for the orbifolds in Example 1, the map D restricts to an injective group homomorphism.

$$\Delta_c\colon \mathrm{Diff}^G_c(U) \to \mathrm{Diff}_{\mathrm{Orb}}\left(\mathbb{R}^d/G, \left\{(\mathbb{R}^d, G, \pi)\right\}\right)_c$$

Lemma 9. *The map* $\Delta_c\colon \mathrm{Diff}^G_c(\mathbb{R}^d) \to \mathrm{Diff}_{\mathrm{Orb}}\left(\mathbb{R}^d/G, \left\{(\mathbb{R}^d, G, \pi)\right\}\right)_c$ *introduced in* 8 *is an isomorphism of groups.*

Proof. Consider $[\hat{h}] \in \mathrm{Diff}_{\mathrm{Orb}}\,(Q,\mathcal{U})_c$ with a representative $(h, \{\tilde{h}\}, [P,\nu])$. Here the lift $\tilde{h}\colon \mathbb{R}^d \to \mathbb{R}^d$ has been chosen with $\tilde{h} \in D^{-1}([\hat{h}])$ (which is possible by Proposition 7). Let $K \subseteq Q$ be a compact set with $h|_{Q\setminus K} \equiv \mathrm{id}_{Q\setminus K}$. As $\pi\colon \mathbb{R}^d \to Q$ is a proper map, the set $\pi^{-1}(K)$ is compact. Choose a compact set $L \subseteq \mathbb{R}^d$ with $\pi^{-1}(K) \subseteq L$ and $\mathbb{R}^d \setminus L$ being connected if $d \geq 2$. If $d = 1$, we may assume that $0 \in L$ and $\mathbb{R} \setminus L$ contains exactly two connected components. Recall from the proof of Proposition 6 that the lift $\tilde{h}$ has been constructed with respect to an arbitrary pair $x_0 \in \mathbb{R}^d \setminus \{0\}$ and $y_0 \in \pi^{-1}h\pi(x_0)$ such that $\tilde{h}(x_0) = y_0$ (if $d \geq 2$). Without loss of generality, choose $x_0 \in \mathbb{R}^d \setminus L$. Since $h|_{Q\setminus\pi(L)} \equiv \mathrm{id}_{Q\setminus\pi(L)}$ holds, one can set $y_0 = x_0$. We claim that the lift $\tilde{h}$ with respect to these choices is contained in $\mathrm{Diff}_c^G(\mathbb{R}^d)$. If this is true, then $\Delta_c(\tilde{h}) = D(\tilde{h}) = [\hat{h}]$ follows and Δ_c is a group isomorphism.

To prove the claim, it suffices to prove that $\tilde{h}$ coincides with $\mathrm{id}_{\mathbb{R}^d}$ outside the compact set L. We distinguish two cases: If $d \geq 2$, then $\tilde{h}$ is a lift of the identity on the connected set $\mathbb{R}^d \setminus L$ which takes x_0 to x_0 and so is $\mathrm{id}_{\mathbb{R}^d\setminus L}$. Hence, $\tilde{h}|_{\mathbb{R}^d\setminus L} = \mathrm{id}_{\mathbb{R}^d\setminus L}$ by uniqueness of lifts (cf. [6, Proposition 1.34]). Hence $\tilde{h} \in \mathrm{Diff}_c^G(\mathbb{R}^d)$ follows.

If $d = 1$, by choice of L the space $\mathbb{R} \setminus L$ contains two connected components C_1, C_2. Now [7, Lemma 2.11] yields $\tilde{h}|_{C_i} = g_i|_{C_i}$ for some $g_i \in G$ and $i \in \{1,2\}$. By construction of $\tilde{h}$, we have $\tilde{h}(]0,\infty[) \subseteq]0,\infty[$ and $\tilde{h}(]-\infty,0[) \subseteq]-\infty,0[$, whence $g_1 = g_2 = \mathrm{id}_{\mathbb{R}}$ and thus $\tilde{h} \in \mathrm{Diff}_c^G(\mathbb{R})$. □

We can thus endow the group $\mathrm{Diff}_c^G(\mathbb{R}^d)$ with the unique topology turning Δ_c into an isomorphism of Lie groups. In this section, we have seen that for the class of orbifolds introduced in Example 1, the following holds:

- The Lie group of orbifold diffeomorphisms
$$\mathrm{Diff}_{\mathrm{Orb}}\left(\mathbb{R}^d/G, \left\{(\mathbb{R}^d, G, \pi)\right\}\right)$$
is isomorphic to $\mathrm{Diff}^G(\mathbb{R}^d)/G$. In particular, all orbifold diffeomorphisms are induced by diffeomorphisms of $\mathbb{R}^d$ which are weak equivalences with respect to the G-action.
- The Lie group of all compactly supported orbifold diffeomorphisms
$$\mathrm{Diff}_{\mathrm{Orb}}\left(\mathbb{R}^d/G, \left\{(\mathbb{R}^d, G, \pi)\right\}\right)_c$$
is isomorphic to $\mathrm{Diff}_c^G(\mathbb{R}^d)$.

  Thus compactly supported orbifold diffeomorphisms correspond bijectively to compactly supported weak equivalences of $\mathbb{R}^d$.

Finally, we would like to clarify how the Lie group structures obtained in this section relate to Lie group structures already constructed on these groups. In [5, Theorem 6.5] a Lie group structure for $\mathrm{Diff}(\mathbb{R}^d)$ has been constructed. This Lie group contains $\mathrm{Diff}^G(\mathbb{R}^d)$ as a closed subgroup modeled on the space $\mathfrak{X}\left(\mathbb{R}^d\right)_c^G$. By a general construction principle for Lie groups (see [3, III. §1 9. Proposition 18]), the Lie group $\mathrm{Diff}_c^G(\mathbb{R}^d)$ also induces a Lie group structure on $\mathrm{Diff}^G(\mathbb{R}^d)$. This Lie group then contains $\mathrm{Diff}_c^G(\mathbb{R}^d)$ as an open subgroup. Furthermore, notice that this structure turns G into a discrete normal subgroup of $\mathrm{Diff}^G(\mathbb{R}^d)$. We have shown in [10, Remark 6.0.8] that both Lie group structures coincide. Thus the Lie group structures constructed in this section coincide with the structures obtained by the traditional construction.

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On Complex Analytic 1|2- and 1|3-dimensional Supermanifolds Associated with $\mathbb{CP}^1$

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Abstract. We obtain a classification up to isomorphism of complex analytic supermanifolds with underlying space $\mathbb{CP}^1$ of dimension 1|2 and of dimension 1|3 with retract (k, k, k), where $k \in \mathbb{Z}$. More precisely, we prove that classes of isomorphic complex analytic supermanifolds of dimension 1|3 with retract (k, k, k) are in one-to-one correspondence with points of the following set:

$$\mathbf{Gr}_{4k-4,3} \cup \mathbf{Gr}_{4k-4,2} \cup \mathbf{Gr}_{4k-4,1} \cup \mathbf{Gr}_{4k-4,0}$$

for $k \geq 2$. For $k < 2$ all such supermanifolds are isomorphic to their retract (k, k, k). In addition, we show that classes of isomorphic complex analytic supermanifolds of dimension 1|2 with retract (k_1, k_2) are in one-to-one correspondence with points of $\mathbb{CP}^{k_1+k_2-4}$ for $k_1 + k_2 \geq 5$. For $k_1 + k_2 < 5$ all such supermanifolds are isomorphic to their retract (k_1, k_2).

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1. Introduction.

We can assign the holomorphic vector bundle, so-called retract, to each complex analytic supermanifold. If underlying space of a complex analytic supermanifold is $\mathbb{CP}^1$, by the Birkhoff–Grothendieck Theorem the corresponding vector bundle is isomorphic to the direct sum of m line bundles: $\mathbf{E} \simeq \bigoplus_{i=1}^{m} L(k_i)$, where $k_i \in \mathbb{Z}$ and m is the odd dimension of the supermanifold. We obtain a classification up to isomorphism of complex analytic supermanifolds of dimension 1|3 with underlying space $\mathbb{CP}^1$ and with retract $L(k) \oplus L(k) \oplus L(k)$, where $k \in \mathbb{Z}$. In addition, we give a classification up to isomorphism of complex analytic supermanifolds of dimension 1|2 with underlying space $\mathbb{CP}^1$.

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The paper is structured as follows. In Section 2 we explain the idea of the classification. In Section 3 we do all necessary preparations. The classification up to isomorphism of complex analytic supermanifolds of dimension $1|3$ with underlying space $\mathbb{CP}^1$ and with retract (k,k,k) is obtained in Section 4. The last section is devoted to the classification up to isomorphism of complex analytic supermanifolds of dimension $1|2$ with underlying space $\mathbb{CP}^1$.

The study of complex analytic supermanifolds with underlying space $\mathbb{CP}^1$ was started in [3]. There the classification of homogeneous complex analytic supermanifolds of dimension $1|m$, $m \leq 3$, up to isomorphism was given. It was proven that in the case $m = 2$ there exists only one non-split homogeneous supermanifold constructed by P. Green [5] and V.P. Palamodov [1]. For $m = 3$ it was shown that there exists a series of non-split homogeneous supermanifolds, parameterized by $k = 0, 2, 3, \ldots$

In [7] we studied even-homogeneous supermanifold, i.e., supermanifolds which possess transitive actions of Lie groups. It was shown that there exists a series of non-split even-homogeneous supermanifolds, parameterized by elements in $\mathbb{Z} \times \mathbb{Z}$, three series of non-split even-homogeneous supermanifolds, parameterized by elements of $\mathbb{Z}$, and finite set of exceptional supermanifolds.

2. Classification of supermanifolds, main definitions

We will use the word "supermanifold" in the sense of Berezin–Leites [2, 6], see also [3]. All the time, we will be interested in the complex analytic version of the theory. We begin with main definitions.

Recall that a *complex superdomain of dimension* $n|m$ is a $\mathbb{Z}_2$-graded ringed space of the form $(U, \mathcal{F}_U \otimes \bigwedge(m))$, where $\mathcal{F}_U$ is the sheaf of holomorphic functions on an open set $U \subset \mathbb{C}^n$ and $\bigwedge(m)$ is the exterior (or Grassmann) algebra with m generators.

Definition 2.1. A *complex analytic supermanifold* of dimension $n|m$ is a $\mathbb{Z}_2$-graded locally ringed space that is locally isomorphic to a complex superdomain of dimension $n|m$.

Let $\mathcal{M} = (\mathcal{M}_0, \mathcal{O}_\mathcal{M})$ be a supermanifold and

$$\mathcal{J}_\mathcal{M} = (\mathcal{O}_\mathcal{M})_{\bar{1}} + ((\mathcal{O}_\mathcal{M})_{\bar{1}})^2$$

be the subsheaf of ideals in $\mathcal{O}_\mathcal{M}$ generated by the subsheaf $(\mathcal{O}_\mathcal{M})_{\bar{1}}$ of odd elements. We put $\mathcal{F}_\mathcal{M} := \mathcal{O}_\mathcal{M}/\mathcal{J}_\mathcal{M}$. Then $(\mathcal{M}_0, \mathcal{F}_\mathcal{M})$ is a usual complex analytic manifold, it is called the *reduction* or *underlying space* of $\mathcal{M}$. Usually we will write $\mathcal{M}_0$ instead of $(\mathcal{M}_0, \mathcal{F}_\mathcal{M})$. Denote by $\mathcal{T}_\mathcal{M}$ the *tangent sheaf* or the *sheaf of vector fields* of $\mathcal{M}$. In other words, $\mathcal{T}_\mathcal{M}$ is the sheaf of derivations of the structure sheaf $\mathcal{O}_\mathcal{M}$. Since the sheaf $\mathcal{O}_\mathcal{M}$ is $\mathbb{Z}_2$-graded, the tangent sheaf $\mathcal{T}_\mathcal{M}$ also possesses the induced $\mathbb{Z}_2$-grading, i.e., there is the natural decomposition $\mathcal{T}_\mathcal{M} = (\mathcal{T}_\mathcal{M})_{\bar{0}} \oplus (\mathcal{T}_\mathcal{M})_{\bar{1}}$.

Let $\mathcal{M}_0$ be a complex analytic manifold and let $\mathbf{E}$ be a holomorphic vector bundle over $\mathcal{M}_0$. Denote by $\mathcal{E}$ the sheaf of holomorphic sections of $\mathbf{E}$. Then the ringed space $(\mathcal{M}_0, \bigwedge \mathcal{E})$ is a supermanifold. In this case $\dim \mathcal{M} = n|m$, where $n = \dim \mathcal{M}_0$ and m is the rank of $\mathbf{E}$.

Definition 2.2. A supermanifold $(\mathcal{M}_0, \mathcal{O}_{\mathcal{M}})$ is called *split* if $\mathcal{O}_{\mathcal{M}} \simeq \bigwedge \mathcal{E}$ (as supermanifolds) for a locally free sheaf $\mathcal{E}$.

It is known that any real (smooth or real analytic) supermanifold is split. The structure sheaf $\mathcal{O}_{\mathcal{M}}$ of a split supermanifold possesses a $\mathbb{Z}$-grading, since $\mathcal{O}_{\mathcal{M}} \simeq \bigwedge \mathcal{E}$ and $\bigwedge \mathcal{E} = \bigoplus_p \bigwedge^p \mathcal{E}$ is naturally $\mathbb{Z}$-graded. In other words, we have the decomposition $\mathcal{O}_{\mathcal{M}} = \bigoplus_p (\mathcal{O}_{\mathcal{M}})_p$. This $\mathbb{Z}$-grading induces the $\mathbb{Z}$-grading in $\mathcal{T}_{\mathcal{M}}$ in the following way:

$$(\mathcal{T}_{\mathcal{M}})_p := \{v \in \mathcal{T}_{\mathcal{M}} \mid v((\mathcal{O}_{\mathcal{M}})_q) \subset (\mathcal{O}_{\mathcal{M}})_{p+q} \text{ for all } q \in \mathbb{Z}\}. \tag{1}$$

We have the decomposition:

$$\mathcal{T}_{\mathcal{M}} = \bigoplus_{p=-1}^{m} (\mathcal{T}_{\mathcal{M}})_p.$$

Therefore the superspace $H^0(\mathcal{M}_0, \mathcal{T}_{\mathcal{M}})$ is also $\mathbb{Z}$-graded. Consider the subspace

$$\operatorname{End} \mathbf{E} \subset H^0(\mathcal{M}_0, (\mathcal{T}_{\mathcal{M}})_0).$$

It consists of all endomorphisms of the vector bundle $\mathbf{E}$ inducing the identity morphism on $\mathcal{M}_0$. Denote by $\operatorname{Aut} \mathbf{E} \subset \operatorname{End} \mathbf{E}$ the group of automorphisms of $\mathbf{E}$, i.e., the group of all invertible endomorphisms. We define the action Int of $\operatorname{Aut} \mathbf{E}$ on $\mathcal{T}_{\mathcal{M}}$ by

$$\operatorname{Int} A : v \mapsto A v A^{-1}.$$

Clearly, the action Int preserves the $\mathbb{Z}$-grading (1), therefore, we have the action of $\operatorname{Aut} \mathbf{E}$ on $H^1(\mathcal{M}_0, (\mathcal{T}_{\mathcal{M}})_2)$.

There is a functor denoting by gr from the category of supermanifolds to the category of split supermanifolds. Let us describe this construction. Let $\mathcal{M}$ be a supermanifold and let as above $\mathcal{J}_M \subset \mathcal{O}_{\mathcal{M}}$ be the subsheaf of ideals generated by odd elements of $\mathcal{O}_{\mathcal{M}}$. Then by definition $\operatorname{gr}(\mathcal{M}) = (\mathcal{M}_0, \operatorname{gr} \mathcal{O}_{\mathcal{M}})$, where

$$\operatorname{gr} \mathcal{O}_{\mathcal{M}} = \bigoplus_{p \geq 0} (\operatorname{gr} \mathcal{O}_{\mathcal{M}})_p, \quad (\operatorname{gr} \mathcal{O}_{\mathcal{M}})_p = \mathcal{J}_{\mathcal{M}}^p / \mathcal{J}_{\mathcal{M}}^{p+1}, \quad \mathcal{J}_{\mathcal{M}}^0 := \mathcal{O}_{\mathcal{M}}.$$

In this case $(\operatorname{gr} \mathcal{O}_{\mathcal{M}})_1$ is a locally free sheaf and there is a natural isomorphism of $\operatorname{gr} \mathcal{O}_{\mathcal{M}}$ onto $\bigwedge (\operatorname{gr} \mathcal{O}_{\mathcal{M}})_1$. If $\psi = (\psi_{\text{red}}, \psi^*) : (M, \mathcal{O}_{\mathcal{M}}) \to (N, \mathcal{O}_{\mathcal{N}})$ is a morphism of supermanifolds, then $\operatorname{gr}(\psi) = (\psi_{\text{red}}, \operatorname{gr}(\psi^*))$, where $\operatorname{gr}(\psi^*) : \operatorname{gr} \mathcal{O}_{\mathcal{N}} \to \operatorname{gr} \mathcal{O}_{\mathcal{M}}$ is defined by

$$\operatorname{gr}(\psi^*)(f + \mathcal{J}_{\mathcal{N}}^p) := \psi^*(f) + \mathcal{J}_{\mathcal{M}}^p \text{ for } f \in (\mathcal{J}_{\mathcal{N}})^{p-1}.$$

Recall that by definition every morphism ψ of supermanifolds is even and as consequence sends $\mathcal{J}_{\mathcal{N}}^p$ into $\mathcal{J}_{\mathcal{M}}^p$.

Definition 2.3. The supermanifold $\operatorname{gr}(\mathcal{M})$ is called the *retract* of $\mathcal{M}$.

To classify supermanifolds, we use the following corollary of the well-known Green Theorem (see [3, 5] or [4] for more details):

Theorem 2.4 (Green). *Let $\mathcal{N} = (\mathcal{N}_0, \bigwedge \mathcal{E})$ be a split supermanfold of dimension $n|m$, where $m \leq 3$. The classes of isomorphic supermanifolds $\mathcal{M}$ such that $\operatorname{gr}\mathcal{M} = \mathcal{N}$ are in bijection with orbits of the action* Int *of the group* $\operatorname{Aut}\mathbf{E}$ *on* $H^1(\mathcal{M}_0, (\mathcal{T}_{\mathcal{N}})_2)$.

Remark 2.5. This theorem allows to classify supermanifolds $\mathcal{M}$ such that $\operatorname{gr}\mathcal{M}$ is fixed up to isomorphisms that induce the identity morphism on $\operatorname{gr}\mathcal{M}$.

3. Supermanifolds associated with $\mathbb{CP}^1$

In what follows we will consider supermanifolds with the underlying space $\mathbb{CP}^1$.

3.1. Supermanifolds with underlying space $\mathbb{CP}^1$

Let $\mathcal{M}$ be a supermanifold of dimension $1|m$. Denote by U_0 and U_1 the standard charts on $\mathbb{CP}^1$ with coordinates x and $y = \frac{1}{x}$ respectively. By the Birkhoff–Grothendieck Theorem we can cover $\operatorname{gr}\mathcal{M}$ by two charts

$$(U_0, \mathcal{O}_{\operatorname{gr}\mathcal{M}}|_{U_0}) \text{ and } (U_1, \mathcal{O}_{\operatorname{gr}\mathcal{M}}|_{U_1})$$

with local coordinates x, $\xi_1, \ldots, \xi_m$ and y, $\eta_1, \ldots, \eta_m$, respectively, such that in $U_0 \cap U_1$ we have

$$y = x^{-1}, \quad \eta_i = x^{-k_i}\xi_i, \ i = 1, \ldots, m,$$

where k_i are integers. Note that a permutation of k_i induces the automorphism of $\operatorname{gr}\mathcal{M}$.

We will identify $\operatorname{gr}\mathcal{M}$ with the set $(k_1, \ldots, k_m)$, so we will say that a supermanifold has the retract $(k_1, \ldots, k_m)$. In this paper we study two cases $m = 2$ and $m = 3$ for $k_1 = k_2 = k_3 =: k$. From now on we use the notation $\mathcal{T} = \bigoplus \mathcal{T}_p$ for the tangent sheaf of $\operatorname{gr}\mathcal{M}$.

3.2. A basis of $H^1(\mathbb{CP}^1, \mathcal{T}_2)$.

Assume that $m = 3$ and that $\mathcal{M} = (k_1, k_2, k_3)$ is a split supermanifold with the underlying space $\mathcal{M}_0 = \mathbb{CP}^1$. Let $\mathcal{T}$ be its tangent sheaf. In [3] the following decomposition

$$\mathcal{T}_2 = \sum_{i<j} \mathcal{T}_2^{ij} \tag{2}$$

was obtained. The sheaf $\mathcal{T}_2^{ij}$ is a locally free sheaf of rank 2; its basis sections over $(U_0, \mathcal{O}_{\mathcal{M}}|_{U_0})$ are:

$$\xi_i\xi_j\frac{\partial}{\partial x}, \quad \xi_i\xi_j\xi_l\frac{\partial}{\partial \xi_l}; \tag{3}$$

where $l \neq i, j$. In $U_0 \cap U_1$ we have

$$\begin{aligned} \xi_i\xi_j\frac{\partial}{\partial x} &= -y^{2-k_i-k_j}\eta_i\eta_j\frac{\partial}{\partial y} - k_l y^{1-k_i-k_j}\eta_i\eta_j\eta_l\frac{\partial}{\partial \eta_l}, \\ \xi_i\xi_j\xi_l\frac{\partial}{\partial \xi_l} &= y^{-k_i-k_j}\eta_i\eta_j\eta_l\frac{\partial}{\partial \eta_l}. \end{aligned} \tag{4}$$

The following theorem was proven in [7]. For completeness we reproduce it here.

Theorem 3.1. *Assume that $i < j$ and $l \neq i, j$.*

1. *For $k_i + k_j > 3$ the basis of $H^1(\mathbb{CP}^1, \mathcal{T}_2^{ij})$ is:*

$$\begin{aligned} &x^{-n}\xi_i\xi_j\frac{\partial}{\partial x}, \quad n = 1, \ldots, k_i + k_j - 3, \\ &x^{-n}\xi_i\xi_j\xi_l\frac{\partial}{\partial \xi_l}, \quad n = 1, \ldots, k_i + k_j - 1; \end{aligned} \tag{5}$$

2. *for $k_i + k_j = 3$ the basis of $H^1(\mathbb{CP}^1, \mathcal{T}_2^{ij})$ is:*

$$x^{-1}\xi_i\xi_j\xi_l\frac{\partial}{\partial \xi_l}, \quad x^{-2}\xi_i\xi_j\xi_l\frac{\partial}{\partial \xi_l};$$

3. *for $k_i + k_j = 2$ and $k_l = 0$ the basis of $H^1(\mathbb{CP}^1, \mathcal{T}_2^{ij})$ is:*

$$x^{-1}\xi_i\xi_j\xi_l\frac{\partial}{\partial \xi_l};$$

4. *if $k_i + k_j = 2$ and $k_l \neq 0$ or $k_i + k_j < 2$, we have $H^1(\mathbb{CP}^1, \mathcal{T}_2^{ij}) = \{0\}$.*

Proof. We use the Čech cochain complex of the cover $\mathfrak{U} = \{U_0, U_1\}$, hence, 1-cocycle with values in the sheaf $\mathcal{T}_2^{ij}$ is a section v of $\mathcal{T}_2^{ij}$ over $U_0 \cap U_1$. We are looking for *basis cocycles*, i.e., cocycles such that their cohomology classes form a basis of $H^1(\mathfrak{U}, \mathcal{T}_2^{ij}) \simeq H^1(\mathbb{CP}^1, \mathcal{T}_2^{ij})$. Note that if $v \in Z^1(\mathfrak{U}, \mathcal{T}_2^{ij})$ is holomorphic in U_0 or U_1 then the cohomology class of v is equal to 0. Obviously, any $v \in Z^1(\mathfrak{U}, \mathcal{T}_2^{ij})$ is a linear combination of vector fields (3) with holomorphic in $U_0 \cap U_1$ coefficients. Further, we expand these coefficients in a Laurent series in x and drop the summands x^n, $n \geq 0$, because they are holomorphic in U_0. We see that v can be replaced by

$$v = \sum_{n=1}^{\infty} a_{ij}^n x^{-n}\xi_i\xi_j\frac{\partial}{\partial x} + \sum_{n=1}^{\infty} b_{ij}^n x^{-n}\xi_i\xi_j\xi_l\frac{\partial}{\partial \xi_l}, \tag{6}$$

where $a_{ij}^n, b_{ij}^n \in \mathbb{C}$. Using (4), we see that the summands corresponding to $n \geq k_i + k_j - 1$ in the first sum of (6) and the summands corresponding to $n \geq k_i + k_j$ in the second sum of (6) are holomorphic in U_1. Further, it follows from (4) that

$$x^{2-k_i-k_j}\xi_i\xi_j\frac{\partial}{\partial x} \sim -k_l x^{1-k_i-k_j}\xi_i\xi_j\xi_l\frac{\partial}{\partial \xi_l}.$$

Hence the cohomology classes of the following cocycles

$$x^{-n}\xi_i\xi_j\frac{\partial}{\partial x}, \quad n=1,\dots,k_i+k_j-3,$$
$$x^{-n}\xi_i\xi_j\xi_l\frac{\partial}{\partial \xi_l}, \quad n=1,\dots,k_i+k_j-1,$$

generate $H^1(\mathbb{CP}^1,\mathcal{T}_2^{ij})$. If we examine linear combination of these cocycles which are cohomological trivial, we get the result. □

Remark 3.2. Note that a similar method can be used for computation of a basis of $H^1(\mathbb{CP}^1,\mathcal{T}_q)$ for any odd dimension m and any q.

In the case $k_1=k_2=k_3=k$, from Theorem 3.1, it follows:

Corollary 3.3. *Assume that $i<j$ and $l\neq i,j$.*

1. *For $k\geq 2$ the basis of $H^1(\mathbb{CP}^1,\mathcal{T}_2^{ij})$ is*

$$\begin{aligned}&x^{-n}\xi_i\xi_j\frac{\partial}{\partial x}, \quad n=1,\dots,2k-3,\\ &x^{-n}\xi_i\xi_j\xi_l\frac{\partial}{\partial \xi_l}, \quad n=1,\dots,2k-1.\end{aligned} \tag{7}$$

2. *If $k<2$, we have $H^1(\mathbb{CP}^1,\mathcal{T}_2)=\{0\}$.*

3.3. The group Aut E

This section is devoted to the calculation of the group of automorphisms $\operatorname{Aut}\mathbf{E}$ of the vector bundle $\mathbf{E}$ in the case (k,k,k). Here $\mathbf{E}$ is the vector bundle corresponding to the split supermanifold (k,k,k).

Let (ξ_i) be a local basis of $\mathbf{E}$ over U_0 and A be an automorphism of $\mathbf{E}$. Assume that $A(\xi_j)=\sum a_{ij}(x)\xi_i$. In U_1 we have

$$A(\eta_j)=A(y^{k_j}\xi_j)=\sum y^{k_j-k_i}a_{ij}(y^{-1})\eta_i.$$

Therefore, $a_{ij}(x)$ is a polynomial in x of degree $\leq k_j-k_i$, if $k_j-k_i\geq 0$ and is 0, if $k_j-k_i<0$. We denote by b_{ij} the entries of the matrix $B=A^{-1}$. The entries are also polynomials in x of degree $\leq k_j-k_i$. We need the following formulas:

$$\begin{aligned}A(\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_k})A^{-1}&=\det(A)\sum_s b_{ks}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_s};\\ A(\xi_i\xi_j\frac{\partial}{\partial x})A^{-1}&=\det(A)\sum_{k<s}(-1)^{l+r}b_{lr}\xi_k\xi_s\frac{\partial}{\partial x}+\det(A)\sum_s b'_{ls}\xi_i\xi_j\xi_l\frac{\partial}{\partial\xi_s},\end{aligned} \tag{8}$$

where $i<j$, $l\neq i,j$ and $r\neq k,s$. Here we use the notation $b'_{ls}=\frac{\partial}{\partial x}(b_{ls})$. By (8), in the case $k_1=k_2=k_3=k$, we have:

Proposition 3.4. *Assume that $k_1=k_2=k_3=k$.*

1. *We have*

$$\operatorname{Aut}\mathbf{E}\simeq \mathrm{GL}_3(\mathbb{C}).$$

In other words

$$\operatorname{Aut}\mathbf{E} = \{(a_{ij}) \mid a_{ij} \in \mathbb{C},\ \det(a_{ij}) \neq 0\}.$$

2. *The action of* $\operatorname{Aut}\mathbf{E}$ *on* $\mathcal{T}_2$ *is given in* U_0 *by the following formulas:*

$$\begin{aligned} A(\xi_1\xi_2\xi_3\frac{\partial}{\partial \xi_k})A^{-1} &= \det(A)\xi_1\xi_2\xi_3\sum_s b_{ks}\frac{\partial}{\partial \xi_s}; \\ A(\xi_i\xi_j\frac{\partial}{\partial x})A^{-1} &= \det(A)\sum_{k<s}(-1)^{l+r}b_{lr}\xi_k\xi_s\frac{\partial}{\partial x}, \end{aligned} \tag{9}$$

where $i<j$, $l \neq i,j$ *and* $r \neq k,s$. *Here* $B = (b_{ij}) = A^{-1}$

4. Classification of supermanifolds with retract (k,k,k)

In this section we give a classification up to isomorphism of complex analytic supermanifolds with underlying space $\mathbb{CP}^1$ and with retract (k,k,k) using Theorem 2.4. In previous section we have calculated the vector space $H^1(\mathbb{CP}^1,\mathcal{T}_2)$, the group $\operatorname{Aut}\mathbf{E}$ and the action Int of $\operatorname{Aut}\mathbf{E}$ on $H^1(\mathbb{CP}^1,\mathcal{T}_2)$, see Theorem 3.3 and Proposition 3.4. Our objective in this section is to calculate the orbit space corresponding to the action Int:

$$H^1(\mathbb{CP}^1,\mathcal{T}_2)/\operatorname{Aut}\mathbf{E}. \tag{10}$$

By Theorem 2.4 classes of isomorphic supermanifolds are in one-to-one correspondence with points of the set (10).

Let us fix the following basis of $H^1(\mathbb{CP}^1,\mathcal{T}_2)$:

$$\begin{array}{lll} v_{11} = x^{-1}\xi_2\xi_3\frac{\partial}{\partial x}, & v_{12} = -x^{-1}\xi_1\xi_3\frac{\partial}{\partial x}, & v_{13} = x^{-1}\xi_1\xi_2\frac{\partial}{\partial x}, \\ \cdots & \cdots & \cdots \\ v_{p1} = x^{-p}\xi_2\xi_3\frac{\partial}{\partial x}, & v_{p1} = -x^{-p}\xi_1\xi_3\frac{\partial}{\partial x}, & v_{p3} = x^{-p}\xi_1\xi_2\frac{\partial}{\partial x}, \end{array} \tag{11}$$

$$\begin{array}{lll} v_{p+1,1} = x^{-1}\xi_1\xi_2\xi_3\frac{\partial}{\partial \xi_1}, & \cdots & v_{p+1,3} = x^{-1}\xi_1\xi_2\xi_3\frac{\partial}{\partial \xi_3}, \\ \cdots & \cdots & \cdots \\ v_{q1} = x^{-a}\xi_1\xi_2\xi_3\frac{\partial}{\partial \xi_1}, & \cdots & v_{q3} = x^{-a}\xi_1\xi_2\xi_3\frac{\partial}{\partial \xi_3}, \end{array} \tag{12}$$

where $p = 2k-3$, $a = 2k-1$ and $q = p+a = 4k-4$. (Compare with Theorem 3.3.) Let us take $A \in \operatorname{Aut}\mathbf{E} \simeq \mathrm{GL}_3(\mathbb{C})$, see Proposition 3.4. We get that in the basis (11)–(12) the automorphism $\operatorname{Int} A$ is given by

$$\operatorname{Int} A(v_{is}) = \frac{1}{\det B}\sum_j b_{sj}v_{ij}.$$

Note that for any matrix $C \in \mathrm{GL}_3(\mathbb{C})$ there exists a matrix B such that

$$C = \frac{1}{\det\, B}B.$$

Indeed, we can put $B = \frac{1}{\sqrt{\det\, C}}C$. We summarize these observations in the following proposition:

Proposition 4.1. *Assume that* $k_1 = k_2 = k_3 = k$. *Then*

$$H^1(\mathbb{CP}^1, \mathcal{T}_2) \simeq \mathrm{Mat}_{3\times(4k-4)}(\mathbb{C})$$

and the action Int *of* Aut $\mathbf{E}$ *on* $H^1(\mathbb{CP}^1, \mathcal{T}_2)$ *is equivalent to the standard action of* $\mathrm{GL}_3(\mathbb{C})$ *on* $\mathrm{Mat}_{3\times(4k-4)}(\mathbb{C})$. *More precisely,* Int *is equivalent to the following action:*

$$D \longmapsto (W \longmapsto DW), \tag{13}$$

where $D \in \mathrm{GL}_3(\mathbb{C})$, $W \in \mathrm{Mat}_{3\times(4k-4)}(\mathbb{C})$ *and* DW *is the usual matrix multiplication.*

Now we prove our first main result.

Theorem 4.2. *Let* $k \geq 2$. *Complex analytic supermanifolds with underlying space* $\mathbb{CP}^1$ *and retract* (k, k, k) *are in one-to-one correspondence with points of the following set:*

$$\bigcup_{r=0}^{3} \mathbf{Gr}_{4k-4,r},$$

where $\mathbf{Gr}_{4k-4,r}$ *is the Grassmannian of type* $(4k-4, r)$, *i.e., it is the set of all* r*-dimensional subspaces in* $\mathbb{C}^{4k-4}$.

In the case $k < 2$ *all supermanifolds with underlying space* $\mathbb{CP}^1$ *and retract* (k, k, k) *are split and isomorphic to their retract* (k, k, k).

Proof. Assume that $k \geq 2$. Clearly, the action (13) preserves the rank r of matrices from $\mathrm{Mat}_{3\times(4k-4)}(\mathbb{C})$ and $r \leq 3$. Therefore, matrices with different rank belong to different orbits of this action. Furthermore, let us fix $r \in \{0, 1, 2, 3\}$. Denote by $\mathrm{Mat}^r_{3\times(4k-4)}(\mathbb{C})$ all matrices with rank r. Clearly, we have

$$\mathrm{Mat}_{3\times(4k-4)}(\mathbb{C}) = \bigcup_{r=0}^{3} \mathrm{Mat}^r_{3\times(4k-4)}(\mathbb{C}).$$

A matrix $W \in \mathrm{Mat}^r_{3\times(4k-4)}(\mathbb{C})$ defines the r-dimensional subspace V_W in $\mathbb{C}^{4k-4}$. This subspace is the linear combination of lines of W. (We consider lines of a matrix $X \in \mathrm{Mat}_{3\times(4k-4)}(\mathbb{C})$ as vectors from $\mathbb{C}^{4k-4}$.) Therefore, we have defined the map F_r:

$$W \longmapsto F_r(W) = V_W \in \mathbf{Gr}_{4k-4,r}.$$

The map F_r is surjective. Indeed, in any r-dimensional subspace $V \in \mathbf{Gr}_{4k-4,r}$ we can take 3 vectors generating V and form the matrix $W_V \in \mathrm{Mat}^r_{3\times(4k-4)}(\mathbb{C})$. In this case the matrix W_V is of rank r and $F_r(W_V) = V$. Clearly, $F_r(W) = F_r(DW)$, where $D \in \mathrm{GL}_3(\mathbb{C})$. Conversely, if W and $W' \in F_r^{-1}(V_W)$, then there exists a matrix $D \in \mathrm{GL}_3(\mathbb{C})$ such that $DW = W'$. It follows that orbits of $\mathrm{GL}_3(\mathbb{C})$ on $\mathrm{Mat}^r_{3\times(4k-4)}(\mathbb{C})$ are in one to one correspondence with points of $\mathbf{Gr}_{4k-4,r}$. Therefore, orbits of $\mathrm{GL}_3(\mathbb{C})$ on

$$\mathrm{Mat}_{3\times(4k-4)}(\mathbb{C}) = \bigcup_{r=0}^{3} \mathrm{Mat}^r_{3\times(4k-4)}(\mathbb{C})$$

are in one-to-one correspondence with points of $\bigcup_{r=0}^{3} \mathbf{Gr}_{4k-4,r}$. The proof is complete. □

5. Classification of supermanifolds with underlying space $\mathbb{CP}^1$ of odd dimension 2

In this section we study the case $m = 2$ and $\operatorname{gr}\mathcal{M} = (k_1, k_2)$, where k_1, k_2 are any integers. Let us compute the 1-cohomology with values in the tangent sheaf $\mathcal{T}_2$. The sheaf $\mathcal{T}_2$ is a locally free sheaf of rank 1. Its basis section over $(U_0, \mathcal{O}_{\mathcal{M}}|_{U_0})$ is $\xi_1\xi_2\frac{\partial}{\partial x}$. The transition functions in $U_0 \cap U_1$ are given by the following formula:

$$\xi_1\xi_2\frac{\partial}{\partial x} = -y^{2-k_1-k_2}\eta_1\eta_2\frac{\partial}{\partial y}.$$

Therefore, a basis of $H^1(\mathbb{CP}^1, \mathcal{T}_2)$ is

$$x^{-n}\xi_1\xi_2\frac{\partial}{\partial x},\ n = 1, \dots, k_1 + k_2 - 3.$$

Let (ξ_i) be a local basis of $\mathbf{E}$ over U_0 and A be an automorphism of $\mathbf{E}$. As in the case $m = 3$, we have that $a_{ij}(x)$ is a polynomial in x of degree $\le k_j - k_i$, if $k_j - k_i \ge 0$ and is 0, if $k_j - k_i < 0$. We need the following formulas:

$$A(x^{-n}\xi_1\xi_2\frac{\partial}{\partial x})A^{-1} = (\det A)x^{-n}\xi_1\xi_2\frac{\partial}{\partial x}.$$

Denote

$$v_n = x^{-n}\xi_1\xi_2\frac{\partial}{\partial x}.$$

We see that the action Int is equivalent to the action of $\mathbb{C}^*$ on $\mathbb{C}^{k_1+k_2-3}$, therefore, the quotient space is $\mathbb{CP}^{k_1+k_2-4}$. We have proven the following theorem:

Theorem 5.1. *Assume that $k_1 + k_2 \ge 5$. Complex analytic supermanifolds with underlying space $\mathbb{CP}^1$ and retract (k_1, k_2) are in one-to-one correspondence with points of*

$$\mathbb{CP}^{k_1+k_2-4} \cup \{\mathrm{pt}\}.$$

In the case $k_1 + k_2 < 5$ all supermanifolds with underlying space $\mathbb{CP}^1$ and retract (k_1, k_2) are split and isomorphic to their retract (k_1, k_2).

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Complex Hulls of the Hyperboloid of One Sheet and Spherical Functions

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Abstract. For the hyperboloid of one sheet in $\mathbb{R}^3$, we construct complex hulls, determine spherical functions that can be continued analytically to these hulls. We write explicitly operators projecting onto subspaces where representations of single series act and the corresponding Cauchy–Szegö kernels

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Let $\mathcal{X}$ be the hyperboloid of one sheet in $\mathbb{R}^3$. The quasiregular representation U of the group $G = \mathrm{SO}_0(1,2)$ on $\mathcal{X}$ is decomposed into irreducible unitary representations of the continuous series with multiplicity 2 and the holomorphic and antiholomorphic series discrete series with multiplicity 1. This decomposition is equivalent to the decomposition of a delta function on $\mathcal{X}$ into spherical functions of these series.

We construct four complex hulls of the hyperboloid $\mathcal{X}$ and determine spherical functions on the hyperboloid that can be continued analytically on these hulls; to each hull correspond spherical functions of a series. We find projectors onto subspaces corresponding to hulls and write corresponding Cauchy–Szegö kernels. In particular, it solves the problem to characterize series by means of complex hulls (Gel'fand–Gindikin program).

For discrete series, this result was obtained in our earlier paper [3].

1. Hyperboloid of one sheet

Let G be the group $\mathrm{SO}_0(1,2)$; it is a connected group of linear transformations of $\mathbb{R}^3$, preserving the form

$$[x, y] = -x_1y_1 + x_2y_2 + x_3y_3\,. \tag{1}$$

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We consider that G acts on $\mathbb{R}^3$ from the right: $x \mapsto xg$. In accordance with this we write vectors in the row form.

The hyperboloid of one sheet $\mathcal{X}$ is defined by the equation $[x,\, x] = 1$. It is a homogeneous space of the group G with respect to translations $x \mapsto xg$. The stabilizer H of the point $x^0 = (0, 0, 1)$ is isomorphic to the group $\mathrm{SO}_0(1, 1)$. The hyperboloid $\mathcal{X}$ has a G-invariant measure dx:

$$dx = |x_3|^{-1}\, dx_1\, dx_2\, .$$

If M is a manifold, then $\mathcal{D}(M)$ denotes the Schwartz space of compactly supported infinitely differentiable $\mathbb{C}$-valued functions on M, with the usual topology, and $\mathcal{D}'(M)$ denotes the space of distributions on M – of antilinear continuous functionals on $\mathcal{D}(M)$. Let dx be a measure on M and

$$\langle F, f\rangle_M = \int_M F(x)\, \overline{f(x)}\, dx \tag{2}$$

the inner product in $L^2(M, dx)$. The space $\mathcal{D}(M)$ can be embedded into $\mathcal{D}'(M)$ by means of form (2), hence we denote by the same form (2) the value of the distribution $F \in \mathcal{D}'(M)$ at a test function $f \in \mathcal{D}(M)$.

The quasiregular representation U of the group G on the hyperboloid $\mathcal{X}$ acts in $L^2(\mathcal{X}, dx)$ by translations: $(U(g)f)(x) = f(xg)$, it is unitary.

2. Complex hulls

Let us extend the bilinear form $[x,\, y]$ to the space $\mathbb{C}^3$ by the same formula (1). The complexification $\mathcal{X}^{\mathbb{C}}$ of $\mathcal{X}$ is the set of points x in $\mathbb{C}^3$ satisfying the equation $[x,\, x] = 1$. Its complex dimension is equal to 2. The group G acts on $\mathcal{X}^{\mathbb{C}}$: $x \mapsto xg$, but of course not transitively.

In this section we determine some complex manifolds in $\mathcal{X}^{\mathbb{C}}$ of complex dimension 2, invariant with respect to G. They are maximal in some sense, the group G acts on them simply transitively, so that the G-orbits are diffeomorphic to G and have real dimension 3. The hyperboloid $\mathcal{X}$ is contained in the boundary of each of these manifolds. We call them the "complex hulls" of the hyperboloid $\mathcal{X}$.

We need the group $G_1 = \mathrm{SU}(1, 1)$ and its complexification $G_1^{\mathbb{C}} = \mathrm{SL}(2, \mathbb{C})$. They consist respectively of matrices:

$$g_1 = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}, \quad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

with the unit determinant. The group $\mathrm{SL}(2, \mathbb{C})$ acts on the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (the Riemann sphere) linear fractionally:

$$z \mapsto z \cdot g = \frac{\alpha z + \gamma}{\beta z + \delta}\, .$$

This action is transitive. But the subgroup $\mathrm{SU}(1, 1)$ has three orbits on $\overline{\mathbb{C}}$: the open disk $D : z\overline{z} < 1$, its exterior $D' : z\overline{z} > 1$, and the circle $S : z\overline{z} = 1$.

Let us identify the space $\mathbb{R}^3$ with the space of matrices

$$x = \begin{pmatrix} ix_1 & x_2 + ix_3 \\ x_2 - ix_3 & -ix_1 \end{pmatrix}.$$

The action $x \mapsto g^{-1}xg$ of the group G_1 on these matrices x is the action $x \mapsto xg$ of the group G on vectors $x \in \mathbb{R}^3$. It gives a homomorphism of G_1 onto G.

Let us introduce on $\mathcal{X}$ horospherical coordinates u, v: $(u, v) \in S \times S$, $u \neq v$, by

$$x = \left(\frac{u+v}{i(u-v)}\,,\ \frac{1-uv}{u-v}\,,\ \frac{1+uv}{i(u-v)} \right).$$

The inverse map $x \mapsto (u, v)$ is given by

$$u = \frac{x_3 + ix_2}{x_1 + i}\,, \quad v = \frac{x_3 + ix_2}{x_1 - i}\,. \tag{3}$$

It embeds the hyperboloid $\mathcal{X}$ into the torus $S \times S$, the image is the torus without the diagonal $\{u = v\}$, the diagonal is a boundary of the hyperboloid.

When a point $x \in \mathcal{X}$ is transformed by $g \in G$, its coordinates (u, v) are transformed by a fractional linear transformation (the same for u and v): $u \mapsto u \cdot g_1$, $v \mapsto v \cdot g_1$, where g_1 is an element in the group G_1 which goes to $g \in G$ under the homomorphism $G_1 \to G$ mentioned above.

Similarly we introduce horospherical coordinates z, w on $\mathcal{X}^{\mathbb{C}}$: a point $x \in \mathcal{X}^{\mathbb{C}}$ is

$$x = \left(\frac{z+w}{i(z-w)}\,,\ \frac{1-zw}{z-w}\,,\ \frac{1+zw}{i(z-w)} \right), \tag{4}$$

the variables z, w run over the extended complex plane $\overline{\mathbb{C}}$, with the condition $z \neq w$. The inverse map is given by

$$z = \frac{x_3 + ix_2}{x_1 + i}, \quad w = \frac{x_3 + ix_2}{x_1 - i}. \tag{5}$$

These formulae, defined originally for $x_1 \neq \pm i$, are extended by continuity to the whole $\mathcal{X}^{\mathbb{C}}$: the points $(i\,, i\lambda\,, \lambda)$, $(-i\,, i\lambda\,, \lambda)$, $\lambda \neq 0$, have horospherical coordinates $(0, i/\lambda)$, $(-i/\lambda\,, 0)$, respectively, and the points $(i\,, -i\lambda\,, \lambda)$, $(-i\,, -i\lambda\,, \lambda)$ have horospherical coordinates $(-i\lambda\,, \infty)$ and $(\infty\,, i\lambda)$, respectively.

Thus, formulae (5) give an embedding $X^{\mathbb{C}} \to \overline{\mathbb{C}} \times \overline{\mathbb{C}}$, its image is $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$ without the diagonal.

There are the following relations. Let the points x and y in $\mathcal{X}^{\mathbb{C}}$ have horospherical coordinates (z, w) and (λ, μ) respectively. Then

$$[x,\, y] - 1 = -\frac{2(z-\lambda)(w-\mu)}{(z-w)(\lambda-\mu)}\,, \tag{6}$$

$$[x,\, y] + 1 = -\frac{2(z-\mu)(w-\lambda)}{(z-w)(\lambda-\mu)}\,. \tag{7}$$

If a point x in $\mathcal{X}^{\mathbb{C}}$ has horospherical coordinates (z, w), then the point $\overline{x}$ (it belongs to $\mathcal{X}^{\mathbb{C}}$ too) has horospherical coordinates $(1/\overline{z},\, 1/\overline{w})$.

Together with (6), (7) this gives

$$[x,\overline{x}]-1=2\,\frac{(1-z\overline{z})(1-w\overline{w})}{|z-w|^2}\,, \tag{8}$$

$$[x,\overline{x}]+1=2\left|\frac{1-z\overline{w}}{z-w}\right|^2. \tag{9}$$

Moreover, for the imaginary parts we have

$$\mathrm{Im}\;x_1=-\frac{z\overline{z}-w\overline{w}}{|z-w|^2}\,, \tag{10}$$

$$\mathrm{Im}\;\frac{x_3}{x_2}=-\frac{1-z\overline{z}\cdot w\overline{w}}{|1-zw|^2}\,. \tag{11}$$

In the direct product $\overline{\mathbb{C}}\times\overline{\mathbb{C}}$ let us consider 4 complex manifolds:

$$D\times D,\ D'\times D',\ D\times D',\ D'\times D. \tag{12}$$

The torus $S\times S$ is contained in the boundary of each of them. The group G_1 acts on $\overline{\mathbb{C}}\times\overline{\mathbb{C}}$ diagonally: $(z,w)\mapsto(z\cdot g,w\cdot g)$. It preserves all these manifolds (12). But this action is not transitive. This can be seen already when we compare dimensions: dimension of G_1 is less than dimension of each manifold ($3<4$). Further, the group G_1 preserves $[x,\overline{x}]$, therefore, by (9), it preserves, for instance,

$$J=\frac{[x,x+1}{2}=\left|\frac{1-z\overline{w}}{z-w}\right|^2\,,$$

so that any G_1-orbit lies on the level surface $J=\mathrm{const}$.

Lemma 1. *The following pairs are representatives of* G_1*-orbits:*

$$\begin{array}{lll}(-i\mu,i\mu), & 0\leqslant\mu<1, & \text{for } D\times D,\\ (-i\mu,i\mu), & 1<\mu\leqslant\infty, & \text{for } D'\times D',\\ (-i\mu,i\mu^{-1}), & 0\leqslant\mu<1, & \text{for } D\times D',\\ (-i\mu,i\mu^{-1}), & 1<\mu\leqslant\infty, & \text{for } D'\times D.\end{array}$$

Proof. Let us consider $D\times D$. Since G_1 acts transitively on D, we can move the first element of a pair in $D\times D$ to zero. We obtain a pair $(0,\zeta)$, $\zeta\in D$. Now we can act on this pair by the centralizer of 0, i.e., the diagonal group K_1 of G_1. It consists of matrices

$$k_1=\begin{pmatrix}\mathrm{e}^{i\alpha} & 0\\ 0 & \mathrm{e}^{-i\alpha}\end{pmatrix}.$$

They act by rotations with angle 2α around zero. So we can move ζ to ir, $0\leqslant r<1$. Thus, any pair in $D\times D$ can be moved to a pair $(0,ir)$, $0\leqslant r<1$. This pair can be transferred to a pair $(-i\mu,\,i\mu)$ in the lemma by means of a matrix

$$g_1=\frac{1}{\sqrt{1-\mu^2}}\begin{pmatrix}1 & i\mu\\ -i\mu & 1\end{pmatrix},\quad r=\frac{2\mu}{\mu^2+1}.$$

Similarly we consider the other 3 cases. □

For all μ satisfying the strong inequalities in Lemma 1, i.e., $0 < \mu < 1$ or $1 < \mu < \infty$, the stabilizer of the pair indicated in the lemma is the center $\{\pm E\}$ of the group G_1, so that the G_1-orbits of these pairs are diffeomorphic to the group $G \simeq G_1/\{\pm E\}$ and have dimension three.

For $\mu = 0$ or $\mu = \infty$ the stabilizer of the pairs is the subgroup K_1 in G_1, so that the corresponding G_1-orbits are diffeomorphic to the Lobachevsky plane $\mathcal{L} = G_1/K_1$ and have dimension two. For $D \times D$ and $D' \times D'$ these two-dimensional orbits are the diagonals $\{z = w\}$, and for $D \times D'$ and $D' \times D$ they are the manifolds $\{z\overline{w} = 1\}$. Indeed, the matrix g_1 carries the pairs $(0, 0)$, (∞, ∞), $(0, \infty)$, $(\infty, 0)$ to the pairs (z, z), (w, w), (z, w), (w, z), respectively, where $z = \overline{b}/\overline{a}$, $w = a/b$, so that $z\overline{w} = 1$.

Let us delete these two-dimensional orbits from the manifolds (12) and denote the remaining manifolds by the same symbols with index 0, for example, $(D \times D)_0$ etc. For these manifolds, the representatives of the G_1-orbits are the pairs indicated in Lemma 1 with μ satisfying the inequalities $0 < \mu < 1$ or $1 < \mu < \infty$.

Let us go from $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$ to $\mathcal{X}^{\mathbb{C}}$ by (4) and (5). The images of $(D \times D)_0$, $(D' \times D')_0$, $(D \times D')_0$, $(D' \times D)_0$ will be denoted by $\mathcal{Y}^+$, $\mathcal{Y}^-$, Ω^+, Ω^- respectively. By (8)–(11) we get the following description of these sets (recall that they all lie in $\mathcal{X}^{\mathbb{C}} : [x, x] = 1$):

$$\begin{aligned} &\mathcal{Y}^+ : [x, \overline{x}] > 1, && \operatorname{Im} \frac{x_3}{x_2} < 0, \\ &\mathcal{Y}^- : [x, \overline{x}] > 1, && \operatorname{Im} \frac{x_3}{x_2} > 0, \\ &\Omega^+ : -1 < [x, \overline{x}] < 1, && \operatorname{Im} x_1 > 0, \\ &\Omega^- : -1 < [x, \overline{x}] < 1, && \operatorname{Im} x_1 < 0. \end{aligned}$$

The pairs $(-i\mu, i\mu)$ and $(-i\mu, i\mu^{-1})$ go to the points in $\mathcal{X}^{\mathbb{C}}$ lying on the curves

$$y_t = (0, i\sinh t, \cosh t), \quad \omega_t = (i\sin t, 0, \cos t), \tag{13}$$

where $\mu = \mathrm{e}^{-t}$, $\mu = \tan(\pi/4 - t/2)$, respectively. Representatives of the G_1-orbits are the points:

$$\begin{aligned} &y_t : && t > 0, && \text{for } \mathcal{Y}^+, \\ &y_t : && t < 0, && \text{for } \mathcal{Y}^-, \\ &\omega_t : && 0 < t < \pi/2, && \text{for } \Omega^+, \\ &\omega_t : && -\pi/2 < t < 0, && \text{for } \Omega^-. \end{aligned}$$

Now let us consider a complexification $G^{\mathbb{C}}$ of the group G. It consists of complex matrices of the third-order preserving the form $[x, y]$ in $\mathbb{C}^3$. Let us take the following basis in the Lie algebra $\mathfrak{g}$ of the group $G = \mathrm{SO}_0(1, 2)$:

$$L_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and consider the following one-parameter subgroups in $G^{\mathbb{C}}$:

$$\begin{aligned}
\gamma_t = \mathrm{e}^{itL_0} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh t & -i\sinh t \\ 0 & i\sinh t & \cosh t \end{pmatrix}, \\
\delta_t = \mathrm{e}^{itL_2} &= \begin{pmatrix} \cos t & 0 & i\sin t \\ 0 & 1 & 0 \\ i\sin t & 0 & \cos t \end{pmatrix}.
\end{aligned} \tag{14}$$

The curves (13) are obtained when we multiply the point $x^0 = (0, 0, 1) \in \mathcal{X}$ by these matrices, i.e., $y_t = x^0\gamma_t$, $\omega_t = x^0\delta_t$.

Therefore, any point x in $\mathcal{Y}^\pm$ is $x^0\gamma_t g$, where $t > 0$ for $\mathcal{Y}^+$ and $t < 0$ for $\mathcal{Y}^-$, and any point x in $\Omega^\pm$ is $x^0\delta_t g$, where $0 < t < \pi/2$ for Ω^+ and $-\pi/2 < t < 0$ for Ω^-. Here g ranges over G.

Let us return to the G_1-orbits of the pairs $(0, \infty)$ and $(\infty, 0)$ which were deleted from $D \times D'$ and $D' \times D$ respectively. Under the map (4) the pairs $(0, \infty)$ and $(\infty, 0)$ go respectively to the points $\omega_{\pi/2} = (i, 0, 0) = ix^1$ and $\omega_{-\pi/2} = (-i, 0, 0) = -ix^1$, where $x^1 = (1, 0, 0)$. Therefore, the map (4) carries these G_1-orbits to the G-orbits of the points ix^1 and $-ix^1$. Both points $\pm x^1$ belong to the hyperboloid $[x, x] = -1$. It consists of the two sheets $\mathcal{L}^\pm$, so that $x^1 \in \mathcal{L}^+$ and $-x^1 \in \mathcal{L}^-$. Therefore the G-orbits are $i\mathcal{L}^\pm$. They lie on the boundary of the manifolds $\Omega^\pm$ respectively. Each of them can be identified with the Lobachevsky plane $\mathcal{L} = G_1/K_1 = G/K$, where K is the subgroup of G consisting of matrices (a maximal compact subgroup of G):

$$\mathrm{e}^{tL_0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\, t & -\sin\, t \\ 0 & \sin\, t & \cos\, t \end{pmatrix}.$$

All four complex manifolds (of real dimension 4) $\mathcal{Y}^\pm$, $\Omega^\pm$ are adjoint to the one-sheeted hyperboloid $\mathcal{X}$ (of real dimension 2). On their turn, each of the two manifolds Ω^+ and Ω^- are adjoint to one sheet (to $i\mathcal{L}^+$ and $i\mathcal{L}^-$) of the two-sheeted hyperboloid $[x, x] = -1$ (of real dimension 2). Relative to the Lobachevsky plane $i\mathcal{L}^\pm$ the manifold $\Omega^\pm$ is a *"complex crown"* (after Akhiezer–Gindikin).

Let us assign to any point x in the manifolds $\mathcal{Y}^\pm$, $\Omega^\pm$ its third coordinate $x_3 \in \mathbb{C}$.

Lemma 2. *Under the map $x \mapsto x_3$ the image of the manifold $\mathcal{Y}^\pm$ is the whole complex plane $\mathbb{C}$ with the cut $[-1, 1]$, the image of the manifold $\Omega^\pm$ is the whole complex plane with cuts $(-\infty, 1]$ and $[1, \infty)$.*

Proof. For a point $x \in \mathcal{X}^{\mathbb{C}}$ with coordinates z, w, see (4), we have

$$\frac{x_3 + 1}{x_3 - 1} = \frac{1 + iz}{1 - iz} \cdot \frac{1 - iw}{1 + iw}. \tag{15}$$

The function

$$z \mapsto \zeta = \frac{1 + iz}{1 - iz}$$

maps the disk D onto the right half-plane $\mathrm{Re}\,\zeta > 0$, and its exterior D' onto the left half-plane $\mathrm{Re}\,\zeta < 0$. Therefore, if $(z,w) \in D \times D$ or $(z,w) \in D' \times D'$, then both fractions in the right-hand side of (15) range over either the left or the right half-plane. Their product ranges over the whole plane $\mathbb{C}$ with cut $(-\infty, 0]$. If in addition $z \neq w$, then this product is not equal to 1. Hence x_3 ranges over $\mathbb{C}$ without $[-1, 1]$.

If $(z,w) \in D \times D'$ or $(z,w) \in D' \times D$, then both fractions in the right-hand side of (15) range over different half-planes. Therefore, their product ranges over the whole of $\mathbb{C}$ with cut $[0, \infty)$, hence x_3 ranges over $\mathbb{C}$ without $(-\infty, -1]$ and $[1, \infty)$. Since we consider $\Omega^\pm$, but not $D\times D'$ and $D'\times D$, we have to exclude in (15) pairs (z,w) for which $w = 1/\overline{z}$. But it does not make the image smaller. Indeed, if the first fraction in the right-hand side of (15) has value $r\mathrm{e}^{i\alpha}$, $-\pi/2 < \alpha < \pi/2$, then the second fraction has value $-r^{-1}\mathrm{e}^{i\alpha}$, so that their product is equal to $-\mathrm{e}^{2i\alpha}$. The intersection of the sets of these points over all α is empty. □

Let $M(y)$ be a holomorphic function on the manifold $\mathcal{Y}^\pm$ and let $N(x)$ be its limit values at the hyperboloid $\mathcal{X}$:

$$M(x) = \lim_{y\to x} M(y), \;\; y \in \mathcal{Y}^\pm, \, x \in \mathcal{X}.$$

We shall assume that y tends to x "along the radius", i.e., if $y \in \mathcal{Y}^\pm$ and $x \in \mathcal{X}$ have horospherical coordinates (z,w) and (u,v) respectively, then

$$z = \mathrm{e}^{-t}u, \;\; w = \mathrm{e}^{-t}v \tag{16}$$

and $t \to \pm 0$. These equalities (16) give (for γ_t, see (14):

$$y = x\gamma_t. \tag{17}$$

Lemma 3. *Let $M(y)$ depend only on y_3: $M(y) = N(y_3)$. By Lemma* 2 *the function $N(\lambda)$ is analytic on the plane $\mathbb{C}$ with cut $[-1, 1]$. Then one has*

$$M(x) = N(x_3 \mp i0x_2).$$

Proof. Let y and x be connected by (16). Then by (17) we have

$$y_3 = -i\sinh t \cdot x_2 + \cosh t \cdot x_3.$$

Therefore, $y_3 = -it \cdot x_2 + x_3 + o(t)$, when $t \to 0$. Hence the lemma is proved. □

Now let $M(\omega)$ be a holomorphic function on the manifold $\Omega^\pm$ and let $M(x)$ be its limit values on the hyperboloid $\mathcal{X}$:

$$M(x) = \lim_{y\to x} M(\omega).$$

Here we assume similarly that ω tends to x "along the radius", i.e., if $\omega \in \Omega^\pm$ and $x \in \mathcal{X}$ have horospherical coordinates (z,w) and (u,v) respectively, then

$$z = \mathrm{e}^{-t}u, \quad w = \mathrm{e}^{t}v \tag{18}$$

and $t \to \pm 0$.

Lemma 4. *Let $M(\omega)$ depend only on ω_3: $M(\omega) = N(\omega_3)$. By Lemma* 2.2 *the function $N(\lambda)$ is analytic on the plane $\mathbb{C}$ with cuts $(-\infty, -1]$ and $[1, \infty)$. Then*

$$M(x) = N(x_3 \pm i0 \cdot x_1 x_3)$$

Proof. By (18) we have

$$\omega_3 = \frac{1 + uv}{i(\mathrm{e}^{-t}u - \mathrm{e}^{t}v)}\,.$$

Let us substitute here the expressions of u, v in terms of x, see (3). Taking into account the equality $x_1^2 + 1 = (x_3 + ix_2)(x_3 - ix_2)$, we obtain

$$\omega_3 = \frac{x_3}{\cosh t - i \sinh t \cdot x_1}\,.$$

When $t \to 0$, it behaves as $x_3(1 + itx_1)$ up to terms of order t^2. Hence the lemma is proved. □

It is convenient to represent it using a cone in $\mathbb{C}^4$. Let us equip $\mathbb{C}^4$ with the bilinear form

$$[[x,\, y]] = -x_0 y_0 - x_1 y_1 + x_2 y_2 + x_3 y_3$$

(we add to vectors x in $\mathbb{C}^3$ the coordinate x_0). Let $\mathcal{C}$ be the cone in $\mathbb{C}^4$ defined by $[[x,\, x]] = 0$, $x \neq 0$. Then the complex hyperboloid $\mathcal{X}^{\mathbb{C}}$ is the section of the cone $\mathcal{C}$ by the hyperplane $x_0 = 1$. Looking at (4), consider the set $\mathcal{Z}$ of points

$$\zeta = \frac{1}{2}\,\left(i(z - w)\,,\; z + w\,,\; i(1 - zw)\,,\; 1 + zw\right),$$

where $z, w \in \mathbb{C}$. It is the section of the cone $\mathcal{C}$ by the hyperplane $-ix_2 + x_3 = 1$, i.e., the hyperplane $[x,\, \xi^0] = 1$, where $\xi^0 = (0\,,\, 0\,,\, -i,\, 1)$. The map $\zeta \mapsto x = \zeta/\zeta_0$ maps $\mathcal{Z}\backslash\{z = w\}$ in $\mathcal{X}^{\mathbb{C}}$, it gives just the horospherical coordinates.

The manifolds (12) without the points corresponding to ∞, lie in $\mathcal{Z}$: in order to obtain $D \times D$ or $D' \times D'$, one has to add the inequality $[\zeta,\, \overline{\zeta}] > 0$ to the inequality $\mathrm{Im}\,(\zeta_3/\zeta_2) < 0$ or $\mathrm{Im}\,(\zeta_3/\zeta_2) > 0$, respectively, and in order to obtain $D \times D'$ or $D' \times D$, one has to the inequality $[\zeta,\, \overline{\zeta}] < 0$ to add the condition that the imaginary part of the determinant

$$\begin{vmatrix} \zeta_0 & \zeta_1 \\ \overline{\zeta_0} & \overline{\zeta_1} \end{vmatrix}$$

is less or greater than zero.

3. Representations of the group $\mathrm{SO}_0(1, 2)$

Recall some material about the principal non-unitary series of representations of the group $G = \mathrm{SO}_0(1, 2)$, see, for example, [7]. They are representations associated with the cone $\mathcal{C}$ in $\mathbb{R}^3$ defined by $[x, x] = 0$, $x_1 > 0$. Let $\mathcal{D}_\sigma(\mathcal{C})$ be the space of C^∞ functions φ on $\mathcal{C}$ homogeneous of degree σ: $\varphi(tx) = t^\sigma \varphi(x)$, $t > 0$. Let T_σ, $\sigma \in \mathbb{C}$, be the representation of G acting on this space by translations: $(T_\sigma(g)\varphi)(x) = \varphi(xg)$. Take the section S of $\mathcal{C}$ by the plane $x_1 = 1$, it is a circle consisting of points $s = (1, \sin\alpha, \cos\alpha)$. The Euclidean measure on S is $ds = d\alpha$. The representation T_σ

can be realized on the space $\mathcal{D}(S)$ as follows (index 1 indicates the first coordinate of a vector):

$$(T_\sigma(g)\varphi)(s) = \varphi\left(\frac{sg}{(sg)_1}\right)(sg)_1^\sigma.$$

The Hermitian form

$$\langle\psi,\varphi\rangle_S = \int_S \psi(s)\,\overline{\varphi(s)}\,ds \tag{19}$$

is invariant with respect to the pair $(T_\sigma, T_{-\overline{\sigma}-1})$, i.e.,

$$\langle T_\sigma(g)\psi,\varphi\rangle_S = \langle\psi, T_{-\overline{\sigma}-1}(g^{-1})\varphi\rangle_S. \tag{20}$$

The representation T_σ can be extended to the space $\mathcal{D}'(S)$ by formula (20).

Let us define an operator A_σ in $\mathcal{D}(S)$:

$$(A_\sigma\varphi)(s) = \int_S (-[s,u])^{-\sigma-1}\varphi(u)du.$$

The integral converges absolutely for $\operatorname{Re}\sigma < -1/2$ and can be continued to the whole σ-plane as a meromorphic function. The operator A_σ intertwines T_σ with $T_{-\sigma-1}$.

Take a basis $\psi_m(\alpha) = \mathrm{e}^{im\alpha}$, $m \in \mathbb{Z}$, in $\mathcal{D}(S)$. It consists of eigenfunctions of A_σ:

$$A_\sigma\,\psi_m = a(\sigma,m)\,\psi_m,$$

where

$$a(\sigma,m) = 2^{\sigma+2}\pi(-1)^m\frac{\Gamma(-2\sigma-1)}{\Gamma(-\sigma+m)\Gamma(-\sigma-m)}.$$

The composition $A_\sigma A_{-\sigma-1}$ is a scalar operator: it is the multiplication by $(8\pi\omega(\sigma))^{-1}$, where $\omega(\sigma)$ is a "Plancherel measure" (36).

If σ is not integer, then T_σ is irreducible and T_σ is equivalent to $T_{-\sigma-1}$.

Let $\sigma \in \mathbb{Z}$. Subspaces $V_{\sigma,+}$ and $V_{\sigma,+}$ spanned by ψ_m for which $m \geqslant -\sigma$ and $m \leqslant \sigma$, respectively are invariant. For $\sigma < 0$ they are irreducible and orthogonal to each other. For $\sigma \geqslant 0$ their intersection E_σ is irreducible and has dimension $2\sigma + 1$.

Let $n \in \mathbb{N} = \{0,1,2,\dots\}$. Denote by $T_{n,\pm}$ the factor representation of the representation T_n acting on the factor space $\mathcal{D}(S)/V_{n,\mp}$ (which is isomorphic to $V_{n,\pm}/E_n$), and by $T_{-n-1,\pm}$ the subrepresentation of the representation T_{-n-1} acting on $V_{-n-1,\pm}$.

There are four series of unitarizable irreducible representations of T_σ and their subquotiens: the *continuous series* consisting of T_σ with $\sigma = -(1/2) + i\rho$, $\rho \in \mathbb{R}$, the inner product is (19); the *complementary series* is consisting of T_σ with $-1 < \sigma < 0$, the inner product is $\langle A_\sigma\psi,\varphi\rangle_S$ with a factor; the *holomorphic* and *antiholomorphic* series is consisting of $T_{n,+}$ $(\sim T_{-n-1,+})$ and $T_{n,-}$ $(\sim T_{-n-1,-})$, respectively. Invariant inner products for $T_{n,\pm}$ are induced by $\langle A_n\psi,\varphi\rangle_S$.

4. Integration over the subgroup H

First recall some distributions (*linear* functionals) on the real line $\mathbb{R}$, see [2]. We use distributions $t_\pm^\lambda$, $(t\pm i0)^\lambda$, t^m, where $t\in\mathbb{R}$, $\lambda\in\mathbb{C}$, $m\in\mathbb{Z}$. Recall also distributions $t_\pm^{-1}$, they are defined as follows. Let $\varphi\in\mathcal{D}(\mathbb{R})$, then

$$(t_+^{-1},\varphi)=\int_0^1\frac{\varphi(t)-\varphi(0)}{t}\,dt+\int_1^\infty\frac{\varphi(t)}{t}\,dt,$$

and

$$(t_-^{-1},\varphi)=(t_+^{-1},\varphi(-t)),$$

so that

$$t_+^{-1}-t_-^{-1}=t^{-1}.$$

The latter is the Cauchy principal value:

$$(t^{-1},\varphi)=\lim_{\varepsilon\to 0}\int_{|t|>\varepsilon}t^{-1}\,\varphi(t)\,dt.$$

We also denote

$$t^{\lambda,m}=|t|^\lambda(\operatorname{sgn}t)^m=t_+^\lambda+(-1)^m t_-^\lambda\,.$$

In fact it depends only on m modulo 2. We use also the following notation: $Y(t)$ is the Heaviside function ($Y(t)=1$ for $t>0$, $Y(t)=0$ for $t<0$), $\delta(t)$ is the Dirac delta function on the real line (a *linear* functional), $\delta^{(k)}(t)$ its kth derivative.

The subgroup H preserves the third coordinate x_3 of the vector x, hence orbits of H on the hyperboloid $\mathcal{X}$ are contained in sections of the hyperboloid by planes $x_3=c$. Sections $c\neq\pm1$ give non singular H-orbits and $c=\pm1$ give singular ones. Namely, for $c\neq\pm1$ the set $x_3=c$ on $\mathcal{X}$ consists of two H-orbits (two branches of a hyperbola), and the set $x_3=\pm1$ consists of five H-orbits (the point $\pm x^0$ and four open rays).

For representatives of non singular H-orbits on $\mathcal{X}$ we can take points

$$(\pm\sqrt{c^2-1},\,0,\,c)\quad\text{for } |c|>1,$$
$$(0,\,\pm\sqrt{1-c^2},\,c)\quad\text{for } |c|<1.$$

Points $c=\pm1$ divide the real line $\mathbb{R}$ into three intervals: $I_1=(1,\infty)$, $I_2=(-1,1)$, $I_3=(-\infty,-1)$.

An integration over H gives six maps $M_j^\pm:\mathcal{D}(\mathcal{X})\to C^\infty(I_j)$, $j=1,2,3$, namely:

$$(M_j^\pm f)(c)=\int_{\mathcal{X}}f(x)\,\delta(x_3-c)\,dx,$$

where the integral is taken over the set $\{\pm x_1>0\}$ for $|x_3|>1$ and over the set $\{\pm x_2>0\}$ for $|x_3|<1$. These maps are reduced to the integration of functions in $\mathcal{D}(\mathbb{R}^2)$ over branches of hyperbolas $x_1^2-x_2^2=c^2-1$, see [2, 6].

Let us write some facts we need. Two lines $(t\,,\,\pm t\,,\,1)$, $t\in\mathbb{R}$, are generating lines of the hyperboloid passing through x^0. Introduce 4 linear functionals $R_{\pm,\pm}$ on $\mathcal{D}(\mathcal{X})$:

$$(R_{\pm,\pm}\,,\,f)=\bigl(t_+^{-1}\,,\,f(\pm t\,,\,\pm t,\,1)\bigr)\,.$$

First assume that the support of f lies in the domain $\mathcal{X}\cap\{x_3>-1\}$. Then $M_3^{\pm}f=0$ and the functions $M_j^{\pm}f$ with $j=1,2$ can be written in the following form:

$$(M_j^{\pm}f)(c)=u(c)\cdot\ln|c-1|+v_j^{\pm}(c), \tag{21}$$

where functions u and $v_j^{\pm}$ belong to $\mathcal{D}(\mathbb{R})$, their values at the point $c=1$ are:

$$u(1)=-f(x^0),$$

$$v_1^{\pm}(1)=(R_{\pm,+}\,,\,f)+(R_{\pm,-}\,,\,f)+f(x^0)\cdot\ln 2, \tag{22}$$

$$v_2^{\pm}(1)=(R_{+,\pm}\,,\,f)+(R_{-,\pm}\,,\,f)+f(x^0)\cdot\ln 2. \tag{23}$$

Let us emphasize that the function $u(c)$ is the same for all four functions $M_j^{\pm}f$.

Introduce two linear functionals (Cauchy principal values) $Z^{\pm}$ on $\mathcal{D}(\mathcal{X})$:

$$(Z^{\pm}\,,f)=\int_{-\infty}^{\infty}f(t,\pm t,1)\,\frac{dt}{t}\,. \tag{24}$$

They are integrals of the function f taken over generating lines passing through x^0 with respect to a H-invariant measure. From (22), (23) we see that

$$v_1^{+}(1)-v_1^{-}(1)=(Z^{+}\,,f)+(Z^{-}\,,f), \tag{25}$$

$$v_2^{+}(1)-v_2^{-}(1)=(Z^{+}\,,f)-(Z^{-}\,,f). \tag{26}$$

Similarly we consider the case when the support of f is situated in the domain $\mathcal{X}\cap\{x_3<1\}$. Then $M_1^{\pm}f=0$, and $M_j^{\pm}f$ with $j=2,3$ are written by (21) with $c-1$ replaced by $c+1$.

Finally, any function $f\in\mathcal{D}(\mathcal{X})$ can be decomposed into the sum of two functions with supports in these domains.

Let us also consider a map of $\mathcal{D}(\mathcal{X})$ to functions on $\mathbb{R}$ (it is the integration over sections $x_3=c$):

$$(Mf)(c)=\int_{\mathcal{X}}f(x)\,\delta(x_3-c)\,dx. \tag{27}$$

The function Mf can be written in the following form:

$$(Mf)(c)=\ln|c-1|\cdot\alpha(c)+\ln|c+1|\cdot\beta(c)+\gamma(c). \tag{28}$$

where $\alpha,\beta,\gamma\in\mathcal{D}(\mathbb{R})$. In particular,

$$\alpha(1)=-2f(x^0).$$

The function Mf is the sum of functions $M_j^{\pm}f$:

$$(Mf)(c)=\sum\nolimits_{\pm}(M_j^{\pm}f)(c),\quad c\in I_j\,.$$

5. *H*-invariants

In this section we determine distributions θ in $\mathcal{D}'(S)$ invariant with respect to the subgroup H under the representation T_σ:

$$T_\sigma(h)\theta = \theta, \quad h \in H, \tag{29}$$

as well as H-invariant elements in subquotients of $\mathcal{D}'(S)$ in the reducible case.

Theorem 1. *Dimension of the space of solutions of equation* (29) *is equal to* 2 *for* $\sigma \neq -n-1$, $n \in \mathbb{N}$, *and is equal to* 3 *for* $\sigma = -n-1$.

We consider two bases of H-invariants for $\sigma \notin \mathbb{Z}$. The first basis consists of two distributions on S:

$$\theta_{\sigma,\varepsilon} = s_3^{\sigma,\varepsilon} = [x^0, s]^{\sigma,\varepsilon},$$

where $\varepsilon = 0, 1$. For $\operatorname{Re}\sigma > -1$, the distribution $\theta_{\sigma,\varepsilon}$ is regular (a locally integrable function), analytic in σ on this half-plane, it can be continued to the whole σ-plane as a meromorphic function – with simple poles at points $\sigma \in -1 - \varepsilon - 2\mathbb{N}$. Its residue at $\sigma = -n-1$, $n \equiv \varepsilon \pmod 2$, is up to a factor the distribution $\delta^{(n)}(s_3)$ concentrated at two points $s = (1, \pm 1, 0)$.

The second basis consists of two distributions:

$$\theta_{\sigma,\pm} = (s_3 \pm i0)^\sigma.$$

Theorem 2. *Let* $\sigma \in \mathbb{Z}$. *Every irreducible subfactor for* T_σ *contains, up to a factor, precisely one* H*-invariant. Here is the list:*

$\theta_{n,n} = s_3^n$ *in* E_n;

the coset of $\theta_{n,n+1} = s_3^{n,n+1}$ *in* $\mathcal{D}'(S)/V'_{n,\mp}$;

$\theta^{\pm}_{-n-1} = (s_3 \mp i0 \cdot s_2)^{-n-1}$ *in* $V'_{-n-1,\pm}$;

the coset of $\delta^{(n)}(s_3)$ *in* $\mathcal{D}'(S)/\left(V^d_{-n-1}\right)'$.

Both theorems follow from values of distributions $\theta_{\sigma,\varepsilon}$, $\delta^{(n)}(s_3)\operatorname{sgn}^\varepsilon s_2$ at ψ_m.

6. Spherical functions

We use the Legendre functions $P_\sigma(z)$, $Q_\sigma(z)$, $\sigma \in \mathbb{C}$, see [1]. They are analytic in the complex z-plane with the cut $(-\infty, -1]$ for $P_\sigma(z)$ and $(-\infty, 1]$ for $Q_\sigma(z)$. For $n \in \mathbb{N}$, the function $Q_n(z)$ is analytic in the plane with the cut $[-1, -1]$, and $P_n(z)$ is a polynomial. At the cuts, we define functions $P_\sigma(c)$, $Q_\sigma(c)$ as half the sum of limit values from above and below. For $-1 < c < 1$ it coincides with [1].

First let $\sigma \notin \mathbb{Z}$. The basis $\theta_{\sigma,\varepsilon}$ $\varepsilon = 0, 1$, in the space of H-invariants generates 4 functions on the hyperboloid $\mathcal{X}$:

$$\Psi_{\sigma,\varkappa,\varepsilon}(x) = \langle \theta_{-\sigma-1,\varkappa}, T_{\overline{\sigma}}(g^{-1})\,\theta_{\overline{\sigma},\varepsilon}\rangle_S \int_S s_3^{-\sigma-1,\varkappa}\,[x,s]^{\sigma,\varepsilon}\,ds, \tag{30}$$

where $\varkappa, \varepsilon \in \{0, 1\}$, $g \in G$ is such that $x^0 g = x$.

We call these functions *spherical functions* corresponding to the representation T_σ (or $T_{-\sigma-1}$). They are locally integrable and H-invariant functions on $\mathcal{X}$. Integral (30) for $\varkappa = \varepsilon$ was computed in [7], for $\varkappa \neq \varepsilon$ it can be done similarly. These spherical functions have the following expressions in terms of the Legendre functions:

$$\Psi_{\sigma,\varepsilon,\varepsilon}(x) = -\frac{2\pi}{\sin\sigma\pi}\Big[P_\sigma(-x_3) + (-1)^\varepsilon P_\sigma(x_3)\Big],$$
$$\Psi_{\sigma,1-\varepsilon,\varepsilon}(x) = \begin{cases} k(\sigma,\varepsilon)\cdot \operatorname{sgn} x_1 \cdot (\operatorname{sgn} x_3)^{1-\varepsilon}\, P_\sigma(|x_3|), & |x_3| > 1,\\ 0, & |x_3| < 1,\end{cases}$$

where

$$k(\sigma,\varepsilon) = -\frac{2\pi}{\sin\sigma\pi}\, \left[(-1)^\varepsilon - \cos\sigma\pi\right] .$$

Another basis $\theta_{\sigma,\pm}$, generates 4 other spherical functions corresponding to T_σ (or $T_{-\sigma-1}$)

$$\begin{aligned}\Phi_{\sigma,\pm,\pm}(x) &= \langle \theta_{-\sigma-1,\pm}\, , T_{\overline{\sigma}}\,(g^{-1})\, \theta_{\overline{\sigma},\pm}\rangle_S \\ &= \int_S (s_3 \pm i0)^{-\sigma-1}([x,s] \mp i0)^\sigma\, ds,\end{aligned}$$

where $g \in G$ is such that $x^0 g = x$. They are linear combinations of functions $\Psi_{\sigma,\varkappa,\varepsilon}(x)$. To write them explicitly, let us introduce the following functions:

$$\begin{aligned} A_{\sigma,\pm}(x) &= P_\sigma(x_3) \mp Y(-x_3-1)\cdot i\sin\sigma\pi \cdot \operatorname{sgn} x_1 \cdot P_\sigma(-x_3),\\ B_{\sigma,\pm}(x) &= P_\sigma(-x_3) \mp Y(x_3-1)\cdot i\sin\sigma\pi \cdot \operatorname{sgn} x_1 \cdot P_\sigma(x_3).\end{aligned} \tag{31}$$

Functions Φ are expressed in terms of A, B as follows:

$$\begin{aligned}\Phi_{\sigma,\mp,\pm}(x) &= \pm\, 2\pi i \cdot A_{\sigma,\pm}(x),\\ \Phi_{\sigma,\pm,\pm}(x) &= \mp\, 2\pi i \cdot e^{\mp i\sigma\pi}\, B_{\sigma,\pm}(x),\end{aligned}$$

here the upper or lower signs "±" have to be taken.

Let now $\sigma \in \mathbb{N}$. Then we have 2 spherical functions $\Psi_{n,\pm}(x)$ corresponding to discrete series representations $T_{n,\pm}$ or $T_{-n-1,\pm}$. Namely,

$$\begin{aligned}\Psi_{n,\pm}(x) &= \langle \theta^{\pm}_{-n-1}\, , T_n\,(g^{-1})\, \theta_{n,n+1}\rangle_S \\ &= \int_S (s_3 \mp i0\cdot s_2)^{-n-1}\, [x,s]^{n,n+1}\, ds.\end{aligned}$$

Let us introduce functions $C_{n,\pm}(x)$ on $\mathcal{X}$:

$$C_{n,\pm}(x) = Q_n(x_3 \mp i0\cdot x_2) = Q_n(x_3) \pm Y(1-|x_3|)\cdot \frac{i\pi}{2}\cdot \operatorname{sgn} x_2 \cdot P_n(x_3). \tag{32}$$

Then

$$\Psi_{n,\pm}(x) = 4C_{n,\pm}(x).$$

Functions A, B, C are limit values on $\mathcal{X}$ of analytic functions on $\Omega^{\pm}$ and $\mathcal{Y}^{\pm}$, namely,

$$\begin{aligned} A_{\sigma,\pm}(x) &= \lim P_\sigma(\omega_3), \quad \omega \in \Omega^{\pm},\ \omega \to x, \\ B_{\sigma,\pm}(x) &= \lim P_\sigma(-\omega_3), \quad \omega \in \Omega^{\pm},\ \omega \to x, \\ C_{n,\pm}(x) &= \lim Q_n(y_3),\ y \in \mathcal{Y}^{\pm},\ y \to x, \end{aligned}$$

so that spherical functions $\Phi_{\sigma,\pm,\pm}(x)$ and $\Psi_{n,\pm}(x)$ can be analytically continued to complex hulls: functions $\Phi_{\sigma,+,\pm}(x)$, $\Phi_{\sigma,-,\pm}(x)$ to $\Omega^{\pm}$ and $\Psi_{n,\pm}(x)$ to $\mathcal{Y}^{\pm}$.

The Legendre functions $P_\sigma(\pm x_3)$ and $Q_n(x_3)$ are half the sums of functions A, B, C:

$$P_\sigma(x_3) = \frac{1}{2}\sum_{\pm} A_{\sigma,\pm}(x),$$

$$P_\sigma(-x_3) = \frac{1}{2}\sum_{\pm} B_{\sigma,\pm}(x). \tag{33}$$

$$Q_n(x_3) = \frac{1}{2}\sum_{\pm} C_{\sigma,\pm}(x). \tag{34}$$

7. Plancherel formula

The decomposition of the quasiregular representation U of the group G on the hyperboloid $\mathcal{X}$ into irreducible unitary representations (Plancherel formula) contains representations of the continuous series with multiplicity 2 and the discrete series with multiplicity 1, see, for example, [4].

Let δ be the delta function on $\mathcal{X}$ concentrated at the point x^0, i.e.,

$$(\delta, f) = \overline{f(x^0)}.$$

The Plancherel formula for $\mathcal{X}$ is equivalent to the decomposition of δ in terms of spherical functions. In [4] we used the first variant of spherical functions of the continuous series, i.e., $\Psi_{\sigma,\varepsilon,\varepsilon}$, so that this decomposition is:

$$\delta = \int_{-\infty}^{\infty} \omega(\sigma) \sum_{\varepsilon} \Psi_{\sigma,\varepsilon,\varepsilon}\Big|_{\sigma=-(1/2)+i\rho} d\rho + \sum_{n=0}^{\infty} \omega_n \sum_{\pm} \Psi_{n,\pm}, \tag{35}$$

where

$$\omega(\sigma) = \frac{1}{32\pi^2}\,(2\sigma+1)\cot\sigma\pi, \quad \omega_n = \frac{1}{16\pi^2}\,(2n+1). \tag{36}$$

Let us rewrite formula (35) replacing spherical functions by their expressions in terms of Legendre functions:

$$\delta(x) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{\rho\,\mathrm{sh}\,\rho\pi}{(\mathrm{ch}\,\rho\pi)^2}\, P_\sigma(-x_3)\, d\rho + \frac{1}{2\pi^2}\sum_{n=0}^{\infty} (2n+1)\, Q_n(x_3), \tag{37}$$

where $\sigma = -(1/2) + i\rho$. Now express Legendre functions here in terms of functions $B_{\sigma,\pm}(x)$ and $C_{\sigma,\pm}(x)$, see (33) and (34). We obtain the decomposition of δ into the sum of 4 distributions:

$$\delta(x) = E_c^+(x) + E_c^-(x) + E_d^+(x) + E_d^-(x), \tag{38}$$

where

$$E_c^{\pm}(x) = \frac{1}{8\pi} \int_{-\infty}^{\infty} \frac{\rho \operatorname{sh} \rho\pi}{(\operatorname{ch} \rho\pi)^2} \, B_{\sigma,\pm}(x) \, d\rho, \;\; \sigma = -(1/2) + i\rho, \tag{39}$$

$$E_d^{\pm}(x) = \frac{1}{4\pi^2} \sum_{n=0}^{\infty} (2n+1) \, Q_n(x_3 \mp i0 \cdot x_2). \tag{40}$$

Decomposition (38) corresponds to the decomposition of the space $L^2(\mathcal{X}, dx)$ into subspaces where separate series of representations act – continuous, holomorphic discrete, antiholomorphic discrete series, respectively:

$$L^2(\mathcal{X}, dx) = H_c^+ + H_c^- + H_d^+ + H_d^- . \tag{41}$$

Theorem 3. *Distributions $E_c^{\pm}$, $E_d^{\pm}$ are given by the following formulae:*

$$E_c^{\pm} = \frac{1}{4}\,\delta - \frac{1}{4\pi^2}\,(x_3 - 1)^{-1} \;\pm\; \frac{i}{4\pi}\,(L^+ + L^-), \tag{42}$$

$$E_d^{\pm} = \frac{1}{4}\,\delta + \frac{1}{4\pi^2}\,(x_3 - 1)^{-1} \;\pm\; \frac{i}{4\pi}\,(L^+ - L^-), \tag{43}$$

where $L^{\pm}$ are the following distributions on $\mathcal{X}$:

$$\langle L^{\pm}, f \rangle_{\mathcal{X}} = (Z^{\pm}, \overline{f}) = \int_{-\infty}^{\infty} \overline{f(t, \pm t, 1)} \; \frac{dt}{t}\,, \quad f \in \mathcal{D}(\mathcal{X}),$$

the linear functionals $Z^{\pm}$ are given by (24).

Proof. Let us take the distribution $N_{\lambda,\nu} = (x_3 - 1)^{\lambda,\nu}$, $\lambda \in \mathbb{C}$, $\nu = 0, 1$, on $\mathcal{X}$. Its value at $f \in \mathcal{D}(\mathcal{X})$ is written as

$$\langle N_{\lambda,\nu}\,,\, f \rangle_{\mathcal{X}} = \int_{-\infty}^{\infty} (c-1)^{\lambda,\nu} \, \overline{(Mf)(c)} \, dc,$$

for $(Mf)(c)$ see (27), (28). The integral converges absolutely for $\operatorname{Re}\lambda > -1$ and can be continued meromorphically in λ. Under the condition

$$-1 < \operatorname{Re}\lambda < -1/2$$

the distribution $N_{\lambda,\nu}$ decomposes in terms of the spherical functions $\Psi_{\sigma,\varepsilon,\varepsilon}$ and $\Psi_{n,\pm}$ as follows (see [4]):

$$N_{\lambda,\nu} = \int_{-\infty}^{\infty} \sum_{\varepsilon} \Omega(\lambda, \nu; \sigma, \varepsilon) \, \Psi_{\sigma,\varepsilon,\varepsilon} \Big|_{\sigma=-(1/2)+i\rho} d\rho + \sum_{n=0}^{\infty} \Omega(\lambda, \nu; n) \sum_{\pm} \Psi_{n,\pm}, \tag{44}$$

where

$$\Omega(\lambda,\nu;\sigma,\varepsilon) = -2^{\lambda-3}\pi^{-3}(2\sigma+1)\cot\sigma\pi\cdot[1-(-1)^\nu\cos\lambda\pi] \\ \times[(-1)^\varepsilon\sin\lambda\pi+(-1)^\nu\sin\sigma\pi]\times\Gamma^2(\lambda+1)\Gamma(-\lambda+\sigma)\Gamma(-\lambda-\sigma-1),$$

$$\Omega(\lambda,\nu;n) = 2^{\lambda-2}\pi^{-2}(2n+1)[1-(-1)^\nu\cos\lambda\pi]\times\frac{\Gamma^2(\lambda+1)\Gamma(-\lambda+n)}{\Gamma(\lambda+n+2)}.$$

Let us take here $\nu=0$. The distribution $N_{\lambda,0}$ has at $\lambda=-1$ a pole of the second order, the first Laurent coefficient is 4δ. It gives (35).

Let us take now $\nu=1$. The distribution $N_{\lambda,1}$ has no singularity at $\lambda=-1$. Decomposition (44) with $\nu=1$ at $\lambda=-1$ is:

$$(x_3-1)^{-1} = -\frac{\pi}{4}\int_{-\infty}^{\infty}\frac{\rho\,\mathrm{sh}\,\rho\pi}{(\mathrm{ch}\,\rho\pi)^2}\,P_\sigma(-x_3)\,d\rho+\frac{1}{2}\sum_{n=0}^{\infty}(2n+1)\,Q_n(x_3), \qquad (45)$$

where $\sigma=-(1/2)+i\rho$. The value of the distribution $(x_3-1)^{-1}$ at $f\in\mathcal{D}(\mathcal{X})$ is the Cauchy principal value:

$$\int_{-\infty}^{\infty}(c-1)^{-1}\,\overline{(Mf)(c)}\,dc.$$

From (37) and (45) we express the integral and the sum in these formulae separately:

$$\frac{\pi}{2}\int\cdots = \pi^2\,\delta-(x_3-1)^{-1}, \qquad (46)$$

$$\sum\cdots = \pi^2\,\delta+(x_3-1)^{-1}. \qquad (47)$$

Now let us compute $E_c^{\pm}(x)$, see (39). Let us substitute in (39) expression (31) for $B_{\sigma,\pm}(x)$. We obtain:

$$E_c^{\pm}(x) = \frac{1}{8\pi}\int\cdots\pm Y(x_3-1)\cdot\mathrm{sgn}\,x_1\cdot\frac{i}{8\pi}\int_{-\infty}^{\infty}\rho\,\mathrm{th}\,\rho\pi\cdot P_\sigma(x_3)\,d\rho, \qquad (48)$$

where $\sigma=-(1/2)+i\rho$. The first summand on the right-hand side of (48) is easily found by (46), it is equal to

$$\frac{1}{4}\,\delta-\frac{1}{4\pi^2}\,(x_3-1)^{-1}.$$

Let us compute the second summand. Let us apply it to a function $f\in\mathcal{D}(\mathcal{X})$. Because of the factor $\mathrm{sgn}\,x_1$, the result is equal to

$$\pm\frac{i}{8\pi}\int_{-\infty}^{\infty}\rho\,\mathrm{th}\,\rho\pi\;d\rho\int_1^{\infty}P_\sigma(c)\,\left[\overline{(M_1^+f)(c)}-\overline{(M_1^-f)(c)}\right]\,dc. \qquad (49)$$

For $\varphi\in\mathcal{D}(\mathbb{R})$, Mehler–Fock formula [1, 3.14 (8), (9)] gives

$$\int_{-\infty}^{\infty}\rho\,\mathrm{th}\,\rho\pi\;d\rho\int_1^{\infty}P_\sigma(c)\,\varphi(c)\,dc = 2\varphi(1), \quad \sigma=-\frac{1}{2}+i\rho.$$

Therefore, (49) is equal to

$$\pm \frac{i}{4\pi}\,\left[\overline{v_1^+(1)} - \overline{v_1^-(1)}\right] = \pm \frac{i}{4\pi}\,\left[\langle L^+\,,\,f\rangle + \langle L^-\,,\,f\rangle\right],$$

where $v_1^{\pm}$ are functions (22), see also (25). It proves (42).

Finally, let us compute $E_d^{\pm}(x)$, see (40). We substitute in (40) expression (32) for $Q_n(x_3 \mp i0 \cdot x_2)$ and obtain:

$$E_d^{\pm}(x) = \frac{1}{4\pi^2}\,\sum \cdots \pm Y(1 - |x_3|) \cdot \operatorname{sgn} x_2 \cdot \frac{i}{8\pi} \sum_{n=0}^{\infty} (2n+1)\, P_n(x_3). \tag{50}$$

The first summand on the right-hand side of (50) is easily found by (47), it is equal to

$$\frac{1}{4}\,\delta + \frac{1}{4\pi^2}\,(x_3 - 1)^{-1}\,.$$

Let us compute the second summand. Let us apply it to a function $f \in \mathcal{D}(\mathcal{X})$. Because of the factor $\operatorname{sgn} x_2$, the result is equal to

$$\pm \frac{i}{8\pi}\,\sum_{n=0}^{\infty} (2n+1) \int_{-1}^{1} P_n(c)\,\left[\overline{(M_2^+ f)(c)} - \overline{(M_2^- f)(c)}\right]\,dc. \tag{51}$$

The decomposition into Legendre polynomials on $[-1, 1]$ gives in particular (for a continuous function φ):

$$\sum_{n=0}^{\infty} (2n+1) \int_{-1}^{1} P_n(c)\,\varphi(c)\,dc = 2\varphi(1).$$

Therefore, (51) is equal to

$$\pm \frac{i}{4\pi}\,\left[\overline{v_2^+(1)} - \overline{v_2^-(1)}\right] = \pm \frac{i}{4\pi}\,\left[\langle L^+\,,\,f\rangle - \langle L^-\,,\,f\rangle\right],$$

where $v_2^{\pm}$ are functions (23), see also (26). It proves (43) and the theorem. □

Subspaces (41) can be described by means of Hardy spaces related to manifolds (12), see [5].

Distributions from (38) are the limit values of some functions on complex hulls, namely,

$$E_c^{\pm}(x) = \lim \frac{1}{4\pi^2}\,(1 - \omega_3)^{-1}\,, \tag{52}$$

$$E_d^{\pm}(x) = \lim \frac{1}{4\pi^2}\,(y_3 - 1)^{-1}\,, \tag{53}$$

where limits are taken when $\omega \to x$, $\omega \in \Omega^{\pm}$, and when $y \to x$, $y \in \mathcal{Y}^{\pm}$, respectively. These limit relations (52), (53) are understood in the following sense. First we continue similar functions with the exponent λ instead of -1 from complex hulls to $\mathcal{X}$ and then we put $\lambda = -1$:

$$E_c^{\pm}(x) = \left[\lim \frac{1}{4\pi^2}\,(1-\omega_3)^{\lambda}\right]_{\lambda=-1},$$
$$E_d^{\pm}(x) = \left[\lim \frac{1}{4\pi^2}\,(y_3-1)^{\lambda}\right]_{\lambda=-1}.$$

8. Projectors to series, Cauchy–Szegö kernels

Denote by $\Pi_c^{\pm}$, $\Pi_d^{\pm}$ operators in $L^2(\mathcal{X}, dx)$, projecting onto subspaces $H_c^{\pm}$, $H_d^{\pm}$, respectively. Their explicit expression we obtain applying (42), (43) to a shifted complex conjugate function $\overline{U(g)f}$. Namely, for $f \in \mathcal{D}(\mathcal{X})$ we have

$$\begin{aligned}\left(\Pi_c^{\pm} f\right)(x) &= \frac{1}{4}\, f(x) - \frac{1}{4\pi^2} \int_{\mathcal{X}} ([x,u]-1)^{-1}\, f(u)\, du \\ &\quad \pm \frac{i}{4\pi} \left\{(W^+ f)(x) + (W^- f)(x)\right\},\\ \left(\Pi_d^{\pm} f\right)(x) &= \frac{1}{4}\, f(x) + \frac{1}{4\pi^2} \int_{\mathcal{X}} ([x,u]-1)^{-1}\, f(u)\, du \\ &\quad \pm \frac{i}{4\pi} \left\{(W^+ f)(x) - (W^- f)(x)\right\},\end{aligned}$$

where $W^{\pm}$ are the following operators (they can be considered as analogues of the Hilbert transform):

$$\left(W^{\pm} f\right)(x) = \int_{-\infty}^{\infty} f(x + t\cdot e_x^{\pm})\, \frac{dt}{t},$$

vectors $e_x^{\pm} \in S$ are directing vectors of generating lines passing through x, we obtain them translating vectors $(1, \pm 1, 0)$ by means of an element $g \in G$ such that $x = x^0 g$, namely,

$$e_x^{\pm} = \left(1, \frac{x_1x_2 \mp x_3}{x_1^2+1}, \frac{x_1x_3 \pm x_2}{x_1^2+1}\right).$$

Values $(W^{\pm} f)(x)$ at point x are integrals of the function f taken over generating lines passing through the point x with respect to a measure invariant with respect to the stabilizer of x.

There is a very interesting fact: differences $\Pi_c^+ - \Pi_c^-$ and $\Pi_d^+ - \Pi_d^-$ of projection operators are expressed *only in terms of operators* $W^{\pm}$:

$$\Pi_c^+ - \Pi_c^- = \frac{i}{2\pi}\left(W^+ + W^-\right),$$
$$\Pi_d^+ - \Pi_d^- = \frac{i}{2\pi}\left(W^+ - W^-\right).$$

Subspaces in the right-hand side of (38) are eigenspaces for these differences with eigenvalues $1, -1, 0, 0$ and $0, 0, 1, -1$, respectively.

Finally, (52) and (53) give Cauchy–Szegö kernels corresponding to subspaces $H_c^{\pm}$ and $H_d^{\pm}$:

$$\mathcal{E}_c^{\pm}(\omega, x) = \frac{1}{4\pi^2}\,(1 - [\omega, x])^{-1}\,,\ \ \omega \in \Omega^{\pm},\ x \in \mathcal{X},$$

$$\mathcal{E}_d^{\pm}(y, x) = \frac{1}{4\pi^2}\,([y, x] - 1)^{-1}\,,\ \ \ y \in \mathcal{Y}^{\pm},\ x \in \mathcal{X}.$$

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A Simple Construction of Integrable Whitham Type Hierarchies

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Abstract. A simple construction of Whitham type hierarchies in all genera is suggested. Potentials of these hierarchies are written as integrals of hypergeometric type. Possible generalization for universal moduli space is also briefly discussed.

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1. Introduction

The goal of this paper is to give a simple construction for a wide class of integrable quasi-linear systems of the form

$$\sum_{l=1}^{n}\left(a_{rl}(u_1,\dots,u_n)\frac{\partial u_l}{\partial t}+b_{rl}(u_1,\dots,u_n)\frac{\partial u_l}{\partial x}+c_{rl}(u_1,\dots,u_n)\frac{\partial u_l}{\partial y}\right)=0,\ \ r=1,\dots,m \tag{1}$$

where t, x, y are independent variables and $u_1, \dots, u_n$ are dependent variables. By integrability of a system (1) we mean the existence of the so-called pseudo-potential representation

$$\frac{\partial \psi}{\partial x}=F\left(\frac{\partial \psi}{\partial t},u_1,\dots,u_n\right),\qquad \frac{\partial \psi}{\partial y}=G\left(\frac{\partial \psi}{\partial t},u_1,\dots,u_n\right). \tag{2}$$

In other words, a system (1) is integrable if there exist functions F, G such that the compatibility conditions for (2) are equivalent to (1). Writing the system (2) in parametric form

$$\frac{\partial \psi}{\partial t}=P_1(z,\mathbf{u}),\quad \frac{\partial \psi}{\partial x}=P_2(z,\mathbf{u}),\quad \frac{\partial \psi}{\partial y}=P_3(z,\mathbf{u}),$$

allowing an arbitrary number of independent variables $t_1 = t$, $t_2 = x$, $t_3 = y$, $t_4, \dots, t_N$, and writing compatibility conditions in terms of functions P_i, we obtain the so-called Whitham type hierarchy. An important class of such hierarchies associated with the moduli space of Riemann surfaces of genus g with n punctures (the so-called universal Whitham hierarchy) was constructed and studied in [1, 2]. The universal Whitham hierarchy is important in the theory of Frobenius manifolds [3], matrix models and other areas of mathematics. Note that the set of times in the universal Whitham hierarchy coincides with a set of meromorphic differentials on a Riemann surface (holomorphic outside punctures), and that the potentials $P_i(z)$ are integrals of these differentials.

In papers [4–6] the general theory of quasi-linear systems of the form (1) integrable by hydrodynamic reductions was developed and important classification results were obtained. In particular, in the paper [5] the systems with two equations for two unknowns (i.e., $n = m = 2$) were characterized by a complicated system of non-linear PDEs for coefficients a_{rl}, b_{rl}, c_{rl}. Moreover, it was shown in the same paper that integrability by hydrodynamic reductions (in the case $n = m = 2$) is equivalent to existence of a pseudo-potential representation.

In paper [7] these systems and their pseudo-potentials were constructed explicitly in terms of arbitrary solutions of a linear system of PDEs of hypergeometric type [8] with rational coefficients. Moreover, a generalization of this construction to the case of arbitrary n and $m = n$ was done in the same paper. It was clear that the systems constructed in [7] are associated with $\mathbb{C}P^1 \setminus \{u_1, \dots, u_n, 0, 1, \infty\}$ but constitute a wider class than the universal Whitham hierarchy associated with a rational curve. Further generalization to the case of $n + k$ equations with n unknowns (where $0 \le k < n - 1$) and to the elliptic case was done in papers [9, 10]. Moreover, in was shown in these papers that all constructed systems are also integrable by hydrodynamic reductions. It became clear that similar deformations and generalizations of the universal Whitham hierarchy should exist in all genera. However, constructions of the papers [7, 9, 10] were too complicated for direct generalization to Riemann surfaces of genus larger than one. Indeed, some expressions for derivatives $\frac{\partial P_i}{\partial z}$, $\frac{\partial P_i}{\partial u_j}$ were written down in terms of hypergeometric functions and their derivatives.

Recall that general hypergeometric functions can be constructed and studied in two ways: as solutions of holonomic linear systems of PDEs and/or as periods of some multiple-valued differential forms. In this paper by exploring the second method we have solved explicitly the overdetermined systems for P_i found in [7, 9, 10], and we write down a simple formula for P_i as a single integral of hypergeometric type. This formula can be easily generalized to all genera.

Let us describe the contents of the paper. In Section 2 we recall generalities of Whitham type hierarchies. In Section 3 we construct potentials in terms of hypergeometric type integrals in the rational case, and in Section 4 we give a similar construction in the elliptic case. In Section 5 we generalize these constructions to higher genus. In Section 6 we construct a compatible system of PDEs of hypergeometric type associated with an arbitrary KP tau-function. Some speculations about possible integrable systems associated with universal moduli space containing all the Riemann surfaces of finite genus are made and several directions of future research are pointed out.

2. Whitham type hierarchies

Given a set of independent variables $t_1, \dots, t_N$ called times, a set of dependent variables $u_1, \dots, u_n$ called fields and a set of functions $P_i(z, u_1, \dots, u_n),\ i = 1, \dots, N$ called potentials we define a Whitham type hierarchy as compatibility conditions of the following system of PDEs:

$$\frac{\partial \Psi}{\partial t_i} = P_i(z, u_1, \dots, u_n),\ i = 1, \dots, N. \tag{3}$$

Here $\Psi, u_1, \dots, u_n$ are functions of times $t_1, \dots, t_N$ and z is a parameter. The system (3) is understood as a parametric way of defining $N-1$ relations between partial derivatives $\frac{\partial \Psi}{\partial t_i},\ i = 1, \dots, N$ obtained by excluding z from these equations. Assume that the system (3) is compatible. Compatibility conditions can be written as

$$\begin{aligned}\sum_{l=1}^{n}\Big(\Big(\frac{\partial P_i}{\partial z}\frac{\partial P_j}{\partial u_l} - \frac{\partial P_j}{\partial z}\frac{\partial P_i}{\partial u_l}\Big)\frac{\partial u_l}{\partial t_k} + \Big(\frac{\partial P_j}{\partial z}\frac{\partial P_k}{\partial u_l} - \frac{\partial P_k}{\partial z}\frac{\partial P_j}{\partial u_l}\Big)\frac{\partial u_l}{\partial t_i} \\ + \Big(\frac{\partial P_k}{\partial z}\frac{\partial P_i}{\partial u_l} - \frac{\partial P_i}{\partial z}\frac{\partial P_k}{\partial u_l}\Big)\frac{\partial u_l}{\partial t_j}\Big) = 0\end{aligned} \tag{4}$$

where $i,\ j,\ k = 1, \dots, N$ are pairwise distinct. Let $V_{i,j,k}$ be linear space of functions in z spanned by

$$\frac{\partial P_i}{\partial z}\frac{\partial P_j}{\partial u_l} - \frac{\partial P_j}{\partial z}\frac{\partial P_i}{\partial u_l},\ \frac{\partial P_j}{\partial z}\frac{\partial P_k}{\partial u_l} - \frac{\partial P_k}{\partial z}\frac{\partial P_j}{\partial u_l},\ \frac{\partial P_k}{\partial z}\frac{\partial P_i}{\partial u_l} - \frac{\partial P_i}{\partial z}\frac{\partial P_k}{\partial u_l},\ l = 1, \dots, n.$$

Lemma 1. *Let $V_{i,j,k}$ be finite dimensional and $\dim V_{i,j,k} = m$. Then (4) is equivalent to a hydrodynamic type system of m linearly independent equations of the form*

$$\begin{aligned}\sum_{l=1}^{n}\Big(a_{rl}(u_1, \dots, u_n)\frac{\partial u_l}{\partial t_i} + b_{rl}(u_1, \dots, u_n)\frac{\partial u_l}{\partial t_j} \\ + c_{rl}(u_1, \dots, u_n)\frac{\partial u_l}{\partial t_k}\Big) = 0,\ r = 1, \dots, m.\end{aligned} \tag{5}$$

Proof. Let $\{S_1(z), \dots, S_m(z)\}$ be a basis in $V_{i,j,k}$ and

$$\frac{\partial P_i}{\partial z}\frac{\partial P_j}{\partial u_l} - \frac{\partial P_j}{\partial z}\frac{\partial P_i}{\partial u_l} = \sum_{r=1}^{m} c_{rl}S_r, \quad \frac{\partial P_j}{\partial z}\frac{\partial P_k}{\partial u_l} - \frac{\partial P_k}{\partial z}\frac{\partial P_j}{\partial u_l} = \sum_{r=1}^{m} a_{rl}S_r,$$

$$\frac{\partial P_k}{\partial z}\frac{\partial P_i}{\partial u_l} - \frac{\partial P_i}{\partial z}\frac{\partial P_k}{\partial u_l} = \sum_{r=1}^{m} b_{rl}S_r.$$

Substituting these expressions to (4) and equating to zero coefficients at $S_1, \dots, S_m$ we obtain (5). $\square$

Remark 1. In all known examples of integrable Whitham type hierarchies we have $n \le m \le 2n-1$. Therefore, this inequality can be regarded as a criterion of integrability.

Remark 2. In the theory of integrable systems of hydrodynamic type the system (3) is often referred to as a pseudo-potential representation of the system (5).

3. Genus zero case

Let $u_1, \dots, u_n \in \mathbb{C} \setminus \{0,1\}$ be pairwise distinct. Fix real numbers $s_1, \dots, s_{n+2}$. Define

$$P_\gamma(z, u_1, \dots, u_n) = \int_\gamma \frac{1}{z-t} \frac{(z-u_1)^{s_1} \dots (z-u_n)^{s_n} z^{s_{n+1}} (z-1)^{s_{n+2}}}{(t-u_1)^{s_1} \dots (t-u_n)^{s_n} t^{s_{n+1}} (t-1)^{s_{n+2}}} dt \qquad (6)$$

where γ is a cycle in $\mathbb{C} \setminus \{u_1, \dots, u_n, 0, 1\}$. Note that $u_1, \dots, u_n, 0, 1, \infty$ can be endpoints of γ and we assume that the corresponding s_i are small enough for convergence of our integral.

Proposition 2. *For generic values of $s_1, \dots, s_{n+2}$ the set of functions $P_\gamma(z, u_1, \dots, u_n)$ defines a Whitham type hierarchy with n fields $u_1, \dots, u_n$ and $N = n+1$ times. Compatibility conditions for this potentials are equivalent to a hydrodynamic type system of the form (5) with $m = n$ linearly independent equations.*

Proof. Let I be integrand in (6). Computing $\frac{\partial P_\gamma}{\partial z} = \int_\gamma \frac{\partial I}{\partial z} dt = \int_\gamma \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial t}\right) I dt$ and $\frac{\partial P_\gamma}{\partial u_i} = \int_\gamma \frac{\partial I}{\partial u_i} dt$ we obtain

$$\begin{aligned}
\frac{\partial P_\gamma}{\partial z} &= -\int_\gamma \Big(\sum_{i=1}^n \frac{s_i}{(z-u_i)(t-u_i)} + \frac{s_{n+1}}{zt} + \frac{s_{n+2}}{(z-1)(t-1)} \Big) \\
&\qquad \times \frac{(z-u_1)^{s_1} \dots (z-u_n)^{s_n} z^{s_{n+1}} (z-1)^{s_{n+2}}}{(t-u_1)^{s_1} \dots (t-u_n)^{s_n} t^{s_{n+1}} (t-1)^{s_{n+2}}} dt, \\
\frac{\partial P_\gamma}{\partial u_i} &= \int_\gamma \frac{s_i}{(z-u_i)(t-u_i)} \frac{(z-u_1)^{s_1} \dots (z-u_n)^{s_n} z^{s_{n+1}} (z-1)^{s_{n+2}}}{(t-u_1)^{s_1} \dots (t-u_n)^{s_n} t^{s_{n+1}} (t-1)^{s_{n+2}}} dt.
\end{aligned}$$

These formulas can be written as

$$\begin{aligned}
\frac{\partial P_\gamma}{\partial z} &= \Big(\sum_{i=1}^n \frac{f_{\gamma,i}}{z-u_i} + \frac{f_{\gamma,n+1}}{z} + \frac{f_{\gamma,n+2}}{z-1} \Big) (z-u_1)^{s_1} \\
&\qquad \cdots (z-u_n)^{s_n} z^{s_{n+1}} (z-1)^{s_{n+2}}, \qquad (7) \\
\frac{\partial P_\gamma}{\partial u_i} &= -\frac{f_{\gamma,i}}{z-u_i} (z-u_1)^{s_1} \dots (z-u_n)^{s_n} z^{s_{n+1}} (z-1)^{s_{n+2}},
\end{aligned}$$

where $f_{\gamma,i}$ are independent of z. Note that $f_{\gamma,1}+\cdots+f_{\gamma,n+2}=0$. It is clear from (7) that

$$\frac{\partial P_{\gamma_1}}{\partial z}\frac{\partial P_{\gamma_2}}{\partial u_l}-\frac{\partial P_{\gamma_2}}{\partial z}\frac{\partial P_{\gamma_1}}{\partial u_l}$$
$$=\phi_{\gamma_1,\gamma_2,l}(z)(z-u_1)^{2s_1-1}\cdots(z-u_n)^{2s_n-1}z^{2s_{n+1}-1}(z-1)^{2s_{n+2}-1}$$

where $\phi_{\gamma_1,\gamma_2,l}(z)$ are polynomials in z of degree $n-1$. Therefore, the linear span of these functions is n dimensional and applying Lemma 1 we see that compatibility conditions are equivalent to a hydrodynamic type system of the form (5) with $m=n$ linearly independent equations. It is known that the linear space spanned by P_γ is $n+2$ dimensional for generic values of $s_1,\ldots,s_{n+2}$. If γ is a small circle around z, then P_γ is a constant. Therefore, there are $n+1$ nontrivial times in this hierarchy. □

Remark 3. Let $\omega=\frac{1}{z-t}\frac{(z-u_1)^{s_1}\ldots(z-u_{n+3})^{s_{n+3}}}{(t-u_1)^{s_1}\ldots(t-u_{n+3})^{s_{n+3}}}dt$. If $s_1+\cdots+s_{n+3}=-1$, then ω is invariant with respect to transformations $t\to\frac{at+b}{ct+d}$, $z\to\frac{az+b}{cz+d}$, $u_i\to\frac{au_i+b}{cu_i+d}$. Using these transformations we can send u_{n+1},u_{n+2},u_{n+3} to $0,1,\infty$ and obtain integrand of (6).

Remark 4. More general hierarchy can be defined by

$$P_{\gamma_0,\ldots,\gamma_k}(z,u_1,\ldots,u_n)$$
$$=\frac{\displaystyle\int_{\gamma_0\times\cdots\times\gamma_k}\frac{\prod_{0\le i<j\le k}(t_i-t_j)\cdot(z-u_1)^{s_1}\ldots(z-u_n)^{s_n}z^{s_{n+1}}(z-1)^{s_{n+2}}}{\prod_{i=0}^{k}(z-t_i)(t_i-u_1)^{s_1}\ldots(t_i-u_n)^{s_n}t_i^{s_{n+1}}(t_i-1)^{s_{n+2}}}dt_0\wedge\cdots\wedge dt_k}{\displaystyle\int_{\gamma_1\times\cdots\times\gamma_k}\frac{\prod_{1\le i<j\le k}(t_i-t_j)}{\prod_{i=1}^{k}(t_i-u_1)^{s_1}\ldots(t_i-u_n)^{s_n}t_i^{s_{n+1}}(t_i-1)^{s_{n+2}}}dt_1\wedge\cdots\wedge dt_k}. \tag{8}$$

Here we fix $\gamma_1,\ldots,\gamma_k$ and vary γ_0. There are n fields $u_1,\ldots,u_n$ and $n+1-k$ times in this hierarchy. Compatibility conditions are equivalent to a system of $n+k$ equations of hydrodynamic type.

Remark 5. Yet more general hierarchy can be defined by

$$P_{\gamma_0,\ldots,\gamma_k}(z,\mathbf{u},\mathbf{v})$$
$$=\frac{\displaystyle\int_{\gamma_0\times\cdots\times\gamma_k}\frac{\prod_{0\le i<j\le k}(t_i-t_j)\cdot(z-u_1)^{s_1}\ldots(z-u_n)^{s_n}z^{s_{n+1}}(z-1)^{s_{n+2}}e^{\Omega(z)}}{\prod_{i=0}^{k}(z-t_i)(t_i-u_1)^{s_1}\ldots(t_i-u_n)^{s_n}t_i^{s_{n+1}}(t_i-1)^{s_{n+2}}e^{\Omega(t_i)}}dt_0\wedge\cdots\wedge dt_k}{\displaystyle\int_{\gamma_1\times\cdots\times\gamma_k}\frac{\prod_{1\le i<j\le k}(t_i-t_j)}{\prod_{i=1}^{k}(t_i-u_1)^{s_1}\ldots(t_i-u_n)^{s_n}t_i^{s_{n+1}}(t_i-1)^{s_{n+2}}e^{\Omega(t_i)}}dt_1\wedge\cdots\wedge dt_k} \tag{9}$$

where

$$\Omega(p)=\sum_{i=1}^{n}\sum_{j=1}^{d_i-1}\frac{v_{i,j}}{(p-u_i)^j}+\sum_{j=1}^{d_{n+1}-1}\frac{v_{n+1,j}}{p^j}+\sum_{j=1}^{d_{n+2}-1}\frac{v_{n+2,j}}{(p-1)^j}+\sum_{j=1}^{d_{n+3}-1}v_{n+3,j}p^j.$$

Here we fix $\gamma_1, \ldots, \gamma_k$ and vary γ_0. There are $d_1 + \cdots + d_{n+3}$ fields $u_1, \ldots, u_n, v_{i,j}$ and $d_1 + \cdots + d_{n+3} + 1 - k$ times in this hierarchy. Compatibility conditions are equivalent to a system of $d_1 + \cdots + d_{n+3} + k$ equations of hydrodynamic type. In particular, for $k = 0$ we have

$$P_\gamma(z, u_1, \ldots, u_n) = \int_\gamma \frac{1}{z-t} \frac{(z-u_1)^{s_1} \ldots (z-u_n)^{s_n} z^{s_{n+1}} (z-1)^{s_{n+2}} \exp(\Omega(z))}{(t-u_1)^{s_1} \ldots (t-u_n)^{s_n} t^{s_{n+1}} (t-1)^{s_{n+2}} \exp(\Omega(t))} dt. \quad (10)$$

The numbers $d_1, \ldots, d_{n+3}$ are called multiplicities of $u_1, \ldots, u_n, 0, 1, \infty$ correspondingly. In particular, if all multiplicities are equal to 1, then we return to potentials given by (8), (6).

4. Genus one case

Let $\Gamma = \{l_1 + l_2\tau;\ l_1, l_2 \in \mathbb{Z}\} \subset \mathbb{C}$ be a lattice in $\mathbb{C}$ spanned by 1 and τ where $\operatorname{Im} \tau > 0$. Let $\mathcal{E} = \mathbb{C}/\Gamma$ be the corresponding elliptic curve. Define theta-function $\theta(z, \tau)$ by

$$\theta(z, \tau) = e^{-\pi i z} \sum_{l \in \mathbb{Z}} (-1)^l e^{2\pi i (lz + \frac{l(l-1)}{2}\tau)}.$$

Note that $\theta(z, \tau)$ can be identified with a holomorphic section of a linear bundle on $\mathcal{E}$, the only zero of $\theta(z, \tau)$ modulo Γ is at $z = 0$ (see [11] for details). In the sequel we will omit the second argument of θ as it always will be equal to τ. The notation θ' is used for derivative of θ by the first argument. We will need the following identities:

$$\begin{gathered}
\theta(-z, \tau) = -\theta(z, \tau),\ \theta(z+1) = -\theta(z),\ \theta(z+\tau) = -e^{-2\pi i (z + \frac{\tau}{2})}\theta(z), \\
\frac{\partial \theta}{\partial \tau} = -\frac{i}{4\pi}\theta'' - \frac{\pi i}{4}\theta, \\
\frac{\theta'(z-t+\eta)}{\theta(z-t+\eta)} - \frac{\theta'(\eta)}{\theta(\eta)} + \frac{\theta'(t-u)}{\theta(t-u)} - \frac{\theta'(z-u)}{\theta(z-u)} \\
= -\frac{\theta'(0)\theta(z-t)\theta(z-u+\eta)\theta(t-u-\eta)}{\theta(\eta)\theta(z-t+\eta)\theta(z-u)\theta(t-u)}.
\end{gathered} \quad (11)$$

Let $u_1, \ldots, u_n, 0 \in \mathbb{C}$ be pairwise distinct modulo Γ. Fix real numbers $s_1, \ldots, s_{n+1}$ such that $s_1 + \cdots + s_{n+1} = 0$ and complex numbers a, b. Let $\eta = s_1 u_1 + \cdots + s_n u_n + a$. Define

$$P_\gamma(z, u_1, \ldots, u_n, \tau) = \int_\gamma \frac{\theta'(0)\theta(z-t+\eta)}{\theta(\eta)\theta(z-t)} \frac{\theta(z-u_1)^{s_1} \ldots \theta(z-u_n)^{s_n} \theta(z)^{s_{n+1}}}{\theta(t-u_1)^{s_1} \ldots \theta(t-u_n)^{s_n} \theta(t)^{s_{n+1}}} e^{b(z-t)} dt. \quad (12)$$

where γ is a cycle in $\mathbb{C} \setminus \{u_1, \ldots, u_n, 0\}$. Note that $u_1, \ldots, u_n, 0$ can be endpoints of γ and we assume that the corresponding s_i are small enough for convergence of our integral.

Proposition 3. *For generic values of $s_1, \ldots, s_{n+1}$ the set of functions $P_\gamma(z, u_1, \ldots, u_n, \tau)$ defines a Whitham type hierarchy with $n+1$ fields $u_1, \ldots, u_n, \tau$ and $N = n+1$ times. Compatibility conditions for these potentials are equivalent to a hydrodynamic type system of the form* (5) *with $m = n+1$ linearly independent equations.*

Proof. Let I be integrand in (12). Computing

$$\frac{\partial P_\gamma}{\partial z} = \int_\gamma \frac{\partial I}{\partial z} dt = \int_\gamma \Big(\frac{\partial}{\partial z} + \frac{\partial}{\partial t}\Big) I dt, \ \frac{\partial P_\gamma}{\partial u_i} = \int_\gamma \frac{\partial I}{\partial u_i} dt, \quad \frac{\partial P_\gamma}{\partial \tau} = \int_\gamma \frac{\partial I}{\partial \tau} dt$$

and using (11) we obtain

$$\frac{\partial P_\gamma}{\partial z} = \frac{\theta'(0)^2}{\theta(\eta)^2} \times \int_\gamma \Big(\sum_{i=1}^n \frac{s_i \theta(z-u_i+\eta)\theta(t-u_i-\eta)}{\theta(z-u_i)\theta(t-u_i)} + \frac{s_{n+1}\theta(z+\eta)\theta(t-\eta)}{\theta(z)\theta(t)} \Big) \\ \times \frac{\theta(z-u_1)^{s_1} \ldots \theta(z-u_n)^{s_n} \theta(z)^{s_{n+1}} e^{bz}}{\theta(t-u_1)^{s_1} \ldots \theta(t-u_n)^{s_n} \theta(t)^{s_{n+1}} e^{bt}} dt,$$

$$\frac{\partial P_\gamma}{\partial u_i} = -\int_\gamma \frac{s_i \theta'(0)^2 \theta(z-u_i+\eta)\theta(t-u_i-\eta)}{\theta(\eta)^2 \theta(z-u_i)\theta(t-u_i)} \\ \times \frac{\theta(z-u_1)^{s_1} \ldots \theta(z-u_n)^{s_n} \theta(z)^{s_{n+1}} e^{bz}}{\theta(t-u_1)^{s_1} \ldots \theta(t-u_n)^{s_n} \theta(t)^{s_{n+1}} e^{bt}} dt,$$

$$\frac{\partial P_\gamma}{\partial \tau} = -\frac{\theta'(\eta)}{2\pi i \theta(\eta)} \frac{\partial P_\gamma}{\partial z} + \frac{\theta'(0)^2}{2\pi i \theta(\eta)^2} \\ \times \int_\gamma \Big(\sum_{i=1}^n \frac{s_i \theta'(z-u_i+\eta)\theta(t-u_i-\eta)}{\theta(z-u_i)\theta(t-u_i)} + \frac{s_{n+1}\theta'(z+\eta)\theta(t-\eta)}{\theta(z)\theta(t)} \Big) \\ \times \frac{\theta(z-u_1)^{s_1} \ldots \theta(z-u_n)^{s_n} \theta(z)^{s_{n+1}} e^{bz}}{\theta(t-u_1)^{s_1} \ldots \theta(t-u_n)^{s_n} \theta(t)^{s_{n+1}} e^{bt}} dt.$$

These formulas can be written as

$$\begin{aligned} \frac{\partial P_\gamma}{\partial z} &= \Big(\sum_{i=1}^n \frac{f_{\gamma,i}\theta(z-u_i+\eta)}{\theta(\eta)\theta(z-u_i)} + \frac{f_{\gamma,n+1}\theta(z+\eta)}{\theta(\eta)\theta(z)} \Big) \theta(z-u_1)^{s_1} \\ &\quad \cdots \theta(z-u_n)^{s_n} \theta(z)^{s_{n+1}}, \\ \frac{\partial P_\gamma}{\partial u_i} &= -\frac{f_{\gamma,i}\theta(z-u_i+\eta)}{\theta(\eta)\theta(z-u_i)} \theta(z-u_1)^{s_1} \ldots \theta(z-u_n)^{s_n} \theta(z)^{s_{n+1}}, \\ \frac{\partial P_\gamma}{\partial \tau} &= -\frac{\theta'(\eta)}{2\pi i \theta(\eta)} \frac{\partial P_\gamma}{\partial z} + \Big(\sum_{i=1}^n \frac{f_{\gamma,i}\theta'(z-u_i+\eta)}{2\pi i \theta(\eta)\theta(z-u_i)} + \frac{f_{\gamma,n+1}\theta'(z+\eta)}{2\pi i \theta(\eta)\theta(z)} \Big) \\ &\quad \times \theta(z-u_1)^{s_1} \ldots \theta(z-u_n)^{s_n} \theta(z)^{s_{n+1}} \end{aligned} \qquad (13)$$

where $f_{\gamma,i}$ are independent of z. It is clear from (13) that

$$\frac{\partial P_{\gamma_1}}{\partial z}\frac{\partial P_{\gamma_2}}{\partial u_l} - \frac{\partial P_{\gamma_2}}{\partial z}\frac{\partial P_{\gamma_1}}{\partial u_l} = \phi_{\gamma_1,\gamma_2,l}(z)\theta(z-u_1)^{2s_1} \cdots \theta(z-u_n)^{2s_n}\theta(z)^{2s_{n+1}},\ l=1,\dots,n,$$

$$\frac{\partial P_{\gamma_1}}{\partial z}\frac{\partial P_{\gamma_2}}{\partial \tau} - \frac{\partial P_{\gamma_2}}{\partial z}\frac{\partial P_{\gamma_1}}{\partial \tau} = \phi_{\gamma_1,\gamma_2,n+1}(z)\theta(z-u_1)^{2s_1}\dots\theta(z-u_n)^{2s_n}\theta(z)^{2s_{n+1}}$$

where $\phi_{\gamma_1,\gamma_2,l}(z)$ are meromorphic functions in z with simple poles at $u_1,\dots,u_n,0$ only. Moreover, these functions satisfy quasi-periodicity properties:

$$\phi_{\gamma_1,\gamma_2,l}(z+1) = \phi_{\gamma_1,\gamma_2,l}(z),\ \phi_{\gamma_1,\gamma_2,l}(z+\tau) = e^{-2\pi i\eta}\phi_{\gamma_1,\gamma_2,l}(z),\ l=1,\dots,n+1.$$

Therefore, the linear span of these functions is $n+1$ dimensional and applying Lemma 1 we see that compatibility conditions are equivalent to a hydrodynamic type system of the form (5) with $m=n+1$ linearly independent equations. The linear space spanned by P_γ is $n+2$ dimensional for generic values of $s_1,\dots,s_{n+1}$. If γ is a small circle around z, then P_γ is a constant. Therefore, there are $n+1$ nontrivial times in this hierarchy. □

Remark 6. Let

$$\omega = \frac{\theta'(0)\theta(z-t+\eta)}{\theta(z-t)}\frac{\theta(z-u_1)^{s_1}\dots\theta(z-u_n)^{s_n}\theta(z-u_{n+1})^{s_{n+1}}}{\theta(t-u_1)^{s_1}\dots\theta(t-u_n)^{s_n}\theta(t-u_{n+1})^{s_{n+1}}}e^{b(z-t)}dt.$$

If $s_1+\dots+s_{n+1}=0$, then ω is invariant with respect to simultaneous translations of $z,t,u_1,\dots,u_{n+1}$. Using these translations we can send u_{n+1} to 0 and obtain integrand of (12).

Remark 7. More general hierarchy can be defined by

$$P_{\gamma_0,\dots,\gamma_k}(z,u_1,\dots,u_n,\tau) = \frac{\theta'(0)}{\Delta}\int_{\gamma_0\times\dots\times\gamma_k} \frac{\theta(z-\sum_{i=0}^{k} t_i+\eta)\prod_{0\le i<j\le k}\theta(t_i-t_j)\cdot\theta(z-u_1)^{s_1}\dots\theta(z-u_n)^{s_n}\theta(z)^{s_{n+1}}e^{bz}}{\prod_{i=0}^{k}\theta(z-t_i)\theta(t_i-u_1)^{s_1}\dots\theta(t_i-u_n)^{s_n}\theta(t_i)^{s_{n+1}}e^{bt_i}} \times dt_0\wedge\dots\wedge dt_k, \tag{14}$$

where

$$\Delta = \int_{\gamma_1\times\dots\times\gamma_k}\frac{\theta(\eta-\sum_{i=1}^{k}t_i)\prod_{1\le i<j\le k}\theta(t_i-t_j)}{\prod_{i=1}^{k}\theta(t_i-u_1)^{s_1}\dots\theta(t_i-u_n)^{s_n}\theta(t_i)^{s_{n+1}}e^{bt_i}}dt_1\wedge\dots\wedge dt_k.$$

Here we fix $\gamma_1,\dots,\gamma_k$ and vary γ_0. There are $n+1$ fields $u_1,\dots,u_n,\tau$ and $n+1-k$ times in this hierarchy. Compatibility conditions are equivalent to a system of $n+1+k$ equations of hydrodynamic type.

Remark 8. Yet more general hierarchy can be defined by

$$P_{\gamma_0,\ldots,\gamma_k}(z,\mathbf{u},\mathbf{v},\tau)$$
$$= \frac{\theta'(0)}{\Delta}\int\limits_{\gamma_0\times\cdots\times\gamma_k}\frac{\theta(z-\sum\limits_{i=0}^{k}t_i+\eta)\prod\limits_{0\le i<j\le k}\theta(t_i-t_j)\cdot\theta(z-u_1)^{s_1}\ldots\theta(z-u_n)^{s_n}\theta(z)^{s_{n+1}}e^{bz+\Omega(z)}}{\prod_{i=0}^{k}\theta(z-t_i)\theta(t_i-u_1)^{s_1}\ldots\theta(t_i-u_n)^{s_n}\theta(t_i)^{s_{n+1}}e^{bt_i+\Omega(t_i)}}$$
$$\times\, dt_0\wedge\cdots\wedge dt_k, \tag{15}$$

where

$$\Delta = \int\limits_{\gamma_1\times\cdots\times\gamma_k}\frac{\theta(\eta-\sum\limits_{i=1}^{k}t_i)\prod\limits_{1\le i<j\le k}\theta(t_i-t_j)}{\prod\limits_{i=1}^{k}\theta(t_i-u_1)^{s_1}\ldots\theta(t_i-u_n)^{s_n}\theta(t_i)^{s_{n+1}}e^{bt_i+\Omega(t_i)}}dt_1\wedge\cdots\wedge dt_k,$$

$$\Omega(p) = \sum_{i=1}^{n}\sum_{j=1}^{d_i-1}v_{i,j}\Omega_j(p-u_i) + \sum_{j=1}^{d_{n+1}-1}v_{n+1,j}\Omega_j(p),\ \Omega_j(p) = \frac{\partial^j}{\partial p^j}\log(\theta(p)).$$

Here we fix $\gamma_1,\ldots,\gamma_k$ and vary γ_0. There are $d_1+\cdots+d_{n+1}+1$ fields $u_1,\ldots,u_n,v_{i,j},\tau$ and $d_1+\cdots+d_{n+1}+1-k$ times in this hierarchy. Compatibility conditions are equivalent to a system of $d_1+\cdots+d_{n+1}+1+k$ equations of hydrodynamic type. In particular, for $k=0$ we have

$$P_\gamma(z,u_1,\ldots,u_n,\tau)$$
$$= \int_\gamma \frac{\theta'(0)\theta(z-t+\eta)}{\theta(\eta)\theta(z-t)}\frac{\theta(z-u_1)^{s_1}\ldots\theta(z-u_n)^{s_n}\theta(z)^{s_{n+1}}e^{bz+\Omega(z)}}{\theta(t-u_1)^{s_1}\ldots\theta(t-u_n)^{s_n}\theta(t)^{s_{n+1}}e^{bt+\Omega(t)}}dt. \tag{16}$$

The numbers $d_1,\ldots,d_{n+1}$ are called multiplicities of $u_1,\ldots,u_n,0$ correspondingly. In particular, if all multiplicities are equal to 1, then we return to potentials given by (14), (12).

5. Higher genus case

Let $\mathcal{E}=\mathbb{D}/\Gamma$ be a compact Riemann surface of genus $g>1$, $\mathbb{D}\subset\mathbb{C}$ its universal covering and $\Gamma=\pi_1(\mathcal{E})$. Denote $a_\alpha, b_\alpha,\ \alpha=1,\ldots,g$ a canonical basis in the homology group $H_1(\mathcal{E},\mathbb{Z})$. Let us choose a coordinate in $\mathbb{D}$ and use the same symbols for holomorphic objects on $\mathcal{E}$ and their lifting on $\mathbb{D}$. Let $\omega_\alpha(z)dz$ be the basis of holomorphic 1-forms on $\mathcal{E}$ normalized by $\int_{a_\alpha}\omega_\beta dz=\delta_{\alpha\beta}$. Choose a basepoint z_0 and define the Abel map $q_\alpha(z)=\int_{z_0}^{z}\omega_\alpha(z)dz$. Note that $\omega_\alpha=q'_\alpha$. Denote by $E(x,y)(dx)^{-1/2}(dy)^{-1/2}$ the prime form and by

$$\theta(z_1,\ldots,z_g)=\sum_{\mathbf{m}\in\mathbb{Z}^g}\exp(2\pi i\mathbf{m}\cdot\mathbf{z}+\pi i\mathbf{m}\mathbf{B}\mathbf{m}^t)$$

the Riemann theta-function where $\mathbf{B}=(B_{\alpha\beta})$, $B_{\alpha\beta}=\int_{b_\alpha}\omega_\beta dz$ is the matrix of b-periods. See [11, 12] for details on holomorphic objects on Riemann surfaces.

Here and in the sequel we use bold symbols for the corresponding vectors: $\mathbf{q} = (q_1, \dots, q_g)$, $\mathbf{z} = (z_1, \dots, z_g), \dots$ and $\mathbf{m} \cdot \mathbf{z} = m_1 z_1 + \dots + m_g z_g$. Recall that

$$\begin{aligned} &E(v,u) = -E(u,v),\ E(u,v) = u - v + o((u-v)^2),\\ &E(u,v)E(w,t)\theta(\mathbf{z} + \mathbf{q}(u) + \mathbf{q}(v))\theta(\mathbf{z} + \mathbf{q}(w) + \mathbf{q}(t))\\ &+ E(v,w)E(u,t)\theta(\mathbf{z} + \mathbf{q}(v) + \mathbf{q}(w))\theta(\mathbf{z} + \mathbf{q}(u) + \mathbf{q}(t))\\ &+ E(w,u)E(v,t)\theta(\mathbf{z} + \mathbf{q}(w) + \mathbf{q}(u))\theta(\mathbf{z} + \mathbf{q}(v) + \mathbf{q}(t)) = 0. \end{aligned} \tag{17}$$

The last relation is called Fay identity [13].

Let $u_1, \dots, u_n \in \mathbb{D}$ be pairwise distinct modulo Γ. Fix real numbers $s_1, \dots, s_n$ such that $s_1 + \dots + s_n = 1$ and complex vectors $\mathbf{a}, \mathbf{b} \in \mathbb{C}^g$. Let $\boldsymbol{\eta} = s_1 \mathbf{q}(u_1) + \dots + s_n \mathbf{q}(u_n) + \mathbf{a}$. Define

$$P_\gamma(z, u_1, \dots, u_n) = \int_\gamma \frac{\theta(\mathbf{q}(z) - \mathbf{q}(t) + \boldsymbol{\eta})}{\theta(\boldsymbol{\eta})E(z,t)} \frac{E(z,u_1)^{s_1} \dots E(z,u_n)^{s_n}}{E(t,u_1)^{s_1} \dots E(t,u_n)^{s_n}} e^{\mathbf{b}\cdot(\mathbf{q}(z) - \mathbf{q}(t))} dt \tag{18}$$

where γ is a cycle in $\mathbb{D} \setminus \{u_1, \dots, u_n\}$. Note that $u_1, \dots, u_n$ can be endpoints of γ and we assume that the corresponding s_i are small enough for convergence of our integral.

Remark 9. The function P_γ does not depend on the choice of coordinate in $\mathbb{D}$. Note that P_γ is a function of $n+1$ points of $\mathbb{D}$ (with coordinates $z, u_1, \dots, u_n$) and $3g-3$ moduli of a Riemann surface $\mathcal{E}$.

Proposition 4. *For generic values of $s_1, \dots, s_n$ the set of functions $P_\gamma(z, u_1, \dots, u_n)$ defines a Whitham type hierarchy with $n + 3g - 3$ fields ($u_1, \dots, u_n$ and $3g-3$ moduli of $\mathcal{E}$) and $N = n + 2g - 2$ times. Compatibility conditions for these potentials are equivalent to a hydrodynamic type system of the form* (5) *with $m = n + 3g - 3$ linearly independent equations.*

Let I be integrand in (18). Computing $\frac{\partial P_\gamma}{\partial u_i} = \int_\gamma \frac{\partial I}{\partial u_i} dt$ and using the Fay identity we obtain

$$\begin{aligned} \frac{\partial P_\gamma}{\partial u_i} = \int_\gamma &\frac{s_i \theta(\mathbf{q}(z) - \mathbf{q}(u_i) + \boldsymbol{\eta})\theta(\mathbf{q}(t) - \mathbf{q}(u_i) - \boldsymbol{\eta})}{\theta(\boldsymbol{\eta})^2 E(z,u_i)E(t,u_i)}\\ &\times \frac{E(z,u_1)^{s_1} \dots E(z,u_n)^{s_n} e^{\mathbf{b}\cdot\mathbf{q}(z)}}{E(t,u_1)^{s_1} \dots E(t,u_n)^{s_n} e^{\mathbf{b}\cdot\mathbf{q}(t)}} dt. \end{aligned} \tag{19}$$

Let

$$\frac{\partial P_\gamma}{\partial z} = f_\gamma(z) E(z,u_1)^{s_1 - 1} \dots E(z,u_n)^{s_n - 1} e^{\mathbf{b}\cdot\mathbf{q}(z)}. \tag{20}$$

One can check that $f_\gamma(z)$ is a holomorphic section of a linear bundle of degree $n + 3g - 3$ on $\mathcal{E}$. Moreover,

$$f_\gamma(u_i) = -\int_\gamma \frac{s_i \theta(\mathbf{q}(t) - \mathbf{q}(u_i) - \boldsymbol{\eta})}{\theta(\boldsymbol{\eta})E(t,u_i)} \frac{E(u_i,u_1) \dots \hat{i} \dots E(u_i,u_n)}{E(t,u_1)^{s_1} \dots E(t,u_n)^{s_n} e^{\mathbf{b}\cdot\mathbf{q}(t)}} dt$$

and, therefore, we have

$$\frac{\partial P_\gamma}{\partial u_i} = -\frac{f_\gamma(u_i)\theta(\mathbf{q}(z) - \mathbf{q}(u_i) + \boldsymbol{\eta})}{\theta(\boldsymbol{\eta})E(z, u_i)} \frac{E(z, u_1)^{s_1} \dots E(z, u_n)^{s_n} e^{\mathbf{b}\cdot(\mathbf{q}(z))}}{E(u_i, u_1) \dots \hat{i} \dots E(u_i, u_n)}. \tag{21}$$

It is clear from (20), (21) that

$$\begin{aligned} &\frac{\partial P_{\gamma_1}}{\partial z}\frac{\partial P_{\gamma_2}}{\partial u_l} - \frac{\partial P_{\gamma_2}}{\partial z}\frac{\partial P_{\gamma_1}}{\partial u_l} \\ &\quad = \phi_{\gamma_1,\gamma_2,l}(z)E(z-u_1)^{2s_1-1}\cdots E(z-u_n)^{2s_n-1}, \ l = 1, \dots, n, \end{aligned}$$

where $\phi_{\gamma_1,\gamma_2,l}(z)$ are holomorphic sections of a linear bundle of degree $n+4g-4$ on $\mathcal{E}$. Therefore, the linear span of these functions is $(n+3g-3)$ dimensional and applying Lemma 1 we see that compatibility conditions are equivalent to a hydrodynamic type system of the form (5) with $m = (n+3g-3)$ linearly independent equations. The linear space spanned by P_γ is $(n+2g-1)$ dimensional for generic values of $s_1, \dots, s_n$. If γ is a small circle around z, then P_γ is a constant. Therefore, there are $(n+2g-2)$ nontrivial times in this hierarchy.

Remark 10. More general hierarchy can be defined by

$$\begin{aligned} &P_{\gamma_0,\dots,\gamma_k}(z, u_1, \dots, u_n) \\ &= \frac{\displaystyle\int_{\gamma_0\times\cdots\times\gamma_k} \frac{\theta(\mathbf{q}(z) - \sum_{i=0}^{k}\mathbf{q}(t_i) + \boldsymbol{\eta}) \prod_{0\le i<j\le k} E(t_i,t_j)\cdot E(z,u_1)^{s_1}\dots E(z,u_n)^{s_n} e^{\mathbf{b}\cdot\mathbf{q}(z)}}{\prod_{i=0}^{k} E(z,t_i)E(t_i,u_1)^{s_1}\dots E(t_i,u_n)^{s_n} e^{\mathbf{b}\cdot\mathbf{q}(t_i)}} dt_0\wedge\cdots\wedge dt_k}{\displaystyle\int_{\gamma_1\times\cdots\times\gamma_k} \frac{\theta(\boldsymbol{\eta} - \sum_{i=1}^{k}\mathbf{q}(t_i)) \prod_{1\le i<j\le k} E(t_i,t_j)}{\prod_{i=1}^{k} E(t_i,u_1)^{s_1}\dots E(t_i,u_n)^{s_n} e^{\mathbf{b}\cdot\mathbf{q}(t_i)}} dt_1\wedge\cdots\wedge dt_k}, \end{aligned} \tag{22}$$

where $s_1+\cdots+s_n = k+1$. Here we fix $\gamma_1, \dots, \gamma_k$ and vary γ_0. There are $n+3g-3$ fields and $n+2g-2-k$ times in this hierarchy. Compatibility conditions are equivalent to a system of $n+3g-3+k$ equations of hydrodynamic type.

Remark 11. Yet more general hierarchy can be defined by

$$\begin{aligned} &P_{\gamma_0,\dots,\gamma_k}(z, \mathbf{u}, \mathbf{v}) \\ &= \frac{\displaystyle\int_{\gamma_0\times\cdots\times\gamma_k} \frac{\theta(\mathbf{q}(z) - \sum_{i=0}^{k}\mathbf{q}(t_i) + \boldsymbol{\eta}) \prod_{0\le i<j\le k} E(t_i,t_j)\cdot E(z,u_1)^{s_1}\dots E(z,u_n)^{s_n} e^{\mathbf{b}\cdot\mathbf{q}(z)+\Omega(z)}}{\prod_{i=0}^{k} E(z,t_i)E(t_i,u_1)^{s_1}\dots E(t_i,u_n)^{s_n} e^{\mathbf{b}\cdot\mathbf{q}(t_i)+\Omega(t_i)}} dt_0\wedge\cdots\wedge dt_k}{\displaystyle\int_{\gamma_1\times\cdots\times\gamma_k} \frac{\theta(\boldsymbol{\eta} - \sum_{i=1}^{k}\mathbf{q}(t_i)) \prod_{1\le i<j\le k} E(t_i,t_j)}{\prod_{i=1}^{k} E(t_i,u_1)^{s_1}\dots E(t_i,u_n)^{s_n} e^{\mathbf{b}\cdot\mathbf{q}(t_i)+\Omega(t_i)}} dt_1\wedge\cdots\wedge dt_k} \end{aligned} \tag{23}$$

where $s_1+\cdots+s_n = k+1$,

$$\Omega(p) = \int_{z_0}^{p} \sum_{i=1}^{n}\sum_{j=2}^{d_i} v_{i,j}\zeta_j(t, u_i)dt,$$

$$\zeta_j(t, u) = \frac{1}{(t-u)^j} + O(1), \ \int_{a_\alpha} \zeta_j(t, u)dt = 0, \ \alpha = 1, \dots, g,$$

and $\zeta_j(t,u)$ is holomorphic for $t \neq u$. Here we fix $\gamma_1,\dots,\gamma_k$ and vary γ_0. There are $d_1+\cdots+d_n+3g-3$ fields and $d_1+\cdots+d_n+2g-2-k$ times in this hierarchy. Compatibility conditions are equivalent to a system of $d_1+\cdots+d_n+3g-3+k$ equations of hydrodynamic type. In particular, for $k=0$ we have

$$P_\gamma(z,u_1,\dots,u_n) = \int_\gamma \frac{\theta(\mathbf{q}(z)-\mathbf{q}(t)+\boldsymbol{\eta})}{\theta(\boldsymbol{\eta})E(z,t)} \frac{E(z,u_1)^{s_1}\dots E(z,u_n)^{s_n}e^{\mathbf{b}\cdot\mathbf{q}(z)+\Omega(z)}}{E(t,u_1)^{s_1}\dots E(t,u_n)^{s_n}e^{\mathbf{b}\cdot\mathbf{q}(t)+\Omega(t)}}dt. \quad (24)$$

The numbers $d_1,\dots,d_n$ are called multiplicities of $u_1,\dots,u_n$ correspondingly. In particular, if all multiplicities are equal to 1, then we return to potentials given by (22), (18).

6. Hypergeometric type systems associated with an arbitrary tau-function

Compatibility conditions for (19) and (20) imply that the functions $f_\gamma(z)$ satisfy the following system of PDEs:

$$\frac{\partial f(z)}{\partial u_i} = -\frac{(s_i-1)\frac{\partial E(z,u_i)}{\partial u_i}}{E(z,u_i)}f(z) - \frac{\theta(\mathbf{q}(z)-\mathbf{q}(u_i)+\boldsymbol{\eta})E(z,u_1),,\hat{,i}\dots E(z,u_n)}{\theta(\boldsymbol{\eta})E(u_i,u_1),,\hat{,i}\dots E(u_i,u_n)}f(u_i)\Big(\mathbf{b}\cdot\mathbf{q}'(z)+\sum_{j=1}^n s_j\frac{\frac{\partial E(z,u_j)}{\partial z}}{E(z,u_j)} - \frac{\frac{\partial E(z,u_i)}{\partial z}}{E(z,u_i)} + \frac{\mathbf{q}'(z)\cdot\boldsymbol{\theta}'(\mathbf{q}(z)-\mathbf{q}(u_i)+\boldsymbol{\eta})}{\theta(\mathbf{q}(z)-\mathbf{q}(u_i)+\boldsymbol{\eta})}\Big), i=1,\dots,n \quad (25)$$

where $\mathbf{q}'(z)\cdot\boldsymbol{\theta}'(\boldsymbol{\eta}) = \sum_{j=1}^g q_j'(z)\frac{\partial\theta(\boldsymbol{\eta})}{\partial\eta_j}$. In particular, setting $z=u_j,\ j\neq i$ in (25) and denoting $f_j = \frac{f(u_j)}{E(u_j,u_1)\dots\hat{j}\dots E(u_j,u_n)}$ we obtain the following system:

$$\frac{\partial f_j}{\partial u_i} = -\frac{s_i\frac{\partial E(u_i,u_j)}{\partial u_i}}{E(u_i,u_j)}f_j + \frac{s_j\theta(\mathbf{q}(u_j)-\mathbf{q}(u_i)+\boldsymbol{\eta})}{\theta(\boldsymbol{\eta})E(u_i,u_j)}f_i,\ i\neq j=1,\dots,n. \quad (26)$$

Proposition 5. *Each of the systems* (25), (26) *is compatible by virtue of* (17). *In other words, let* $q_1(z),\dots,q_g(z),E(x,y),\theta(t_1,\dots,t_g)$ *be arbitrary holomorphic functions satisfying* (17). *Then system* (25) *for a single function* $f(z,u_1,\dots,u_n)$ *and system* (26) *for* n *functions* $f_i(u_1,\dots,u_n),\ i=1,\dots,n$ *are both compatible. Recall that* $\boldsymbol{\eta}=s_1\mathbf{q}(u_1)+\cdots+s_n\mathbf{q}(u_n)+\mathbf{a}$.

The proof is a straightforward computation using (17).

Let us set $g=\infty,\ E(x,y)=x-y,\ q_i(z)=\frac{z^i}{i},\ i=1,2,\dots$ and $\theta=\tau$ where τ is an arbitrary KP tau-function [14]. Recall that τ satisfies the following Fay type identity:

$$(a-b)(c-d)\tau(\mathbf{t}+[a]+[b])\tau(\mathbf{t}+[c]+[d])+(b-c)(a-d)\tau(\mathbf{t}+[b]+[c])\tau(\mathbf{t}+[a]+[d]) + (c-a)(b-d)\tau(\mathbf{t}+[c]+[a])\tau(\mathbf{t}+[b]+[d]) = 0$$

where $\mathbf{t} = (t_1, t_2, \dots)$ and $[a] = \left(a, \frac{a^2}{2}, \dots\right)$. The system (26) takes the form

$$\frac{\partial f_j}{\partial u_i} = \frac{s_i}{u_j - u_i} f_j + \frac{s_j \tau([u_j] - [u_i] + \boldsymbol{\eta})}{(u_i - u_j)\tau(\boldsymbol{\eta})} f_i, \; i \neq j = 1, \dots, n, \tag{27}$$

where $\boldsymbol{\eta} = s_1[u_1] + \cdots + s_n[u_n] + \mathbf{a}$. This system is compatible for arbitrary constants $s_1, \dots, s_n$, $a_1, a_2, \dots$ and arbitrary tau-function.

Remark 12. It would be interesting to examine the functions P_γ given by (18), (22), (23) where $g = \infty$, $E(x,y) = x - y$, $q_i(z) = \frac{z^i}{i}$, $i = 1, 2, \dots$ and $\theta = \tau$. For example, (18) takes the form

$$P_\gamma(z, u_1, \dots, u_n) = \int_\gamma \frac{\tau(\boldsymbol{\eta} + [z] - [t])}{(z-t)\tau(\boldsymbol{\eta})} \frac{(z-u_1)^{s_1} \dots (z-u_n)^{s_n}}{(t-u_1)^{s_1} \dots (t-u_n)^{s_n}} e^{\mathbf{b}\cdot([z]-[t])} dt.$$

It particular, one could try to construct a Whitham type hierarchy (with infinitely many fields and times) associated with a universal moduli space containing all the Riemann surfaces of finite genus [15, 16].

Remark 13. It would be interesting to prove that Whitham type hierarchies constructed in this paper are integrable by hydrodynamic reductions for all genera and find corresponding Gibbons–Tsarev type systems [17].

These problems will be addressed in future publications.

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Asymptotic Integration of Linear Third-order Differential Equation

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Abstract. Asymptotic formulas are presented for the solutions of the equation $u''' - (1 + \varphi(t))u = 0$, where function φ is small in a certain sense for large values of the argument. To this end two methods of asymptotic integration are to be used, namely method based on a reduction to nonlinear Poincaré–Liapounoff type equation and method of L-diagonal systems.

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1. Introduction

Consider the equation

$$u''' - (1 + \varphi(t))u = 0, \tag{1}$$

where function φ is small in a certain sense for large values of the argument, which can be treated as perturbation of the equation

$$u''' - u = 0. \tag{2}$$

It is natural to expect that under appropriate assumptions equation (1) has solutions asymptotically equivalent to the ones of unperturbed equation (2) which has a fundamental system of solutions

$$\left\{ e^t,\ e^{\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) t},\ e^{\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) t} \right\}.$$

Usage of two methods of asymptotic integration, namely the method based on a reduction to Poincaré–Liapounoff type equation and the method of L-diagonal systems [1, 2], enables one to obtain similar forms of leading term of the asymptotics. Our goal is to evaluate and compare the estimates for the remainder terms depending on the properties of function φ.

2. Reduction to the *L*-diagonal system

Assume that $\varphi \in L^1(\mathbb{R}_+)$ and consider the first-order system associated with (1)

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}' = \left[\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}}_{A} + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \varphi(t) & 0 & 0 \end{pmatrix}}_{B(t)} \right] \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}, \tag{3}$$

where $z_1 = u,\ z_2 = u',\ z_3 = u''$.

The change of variables $y = Sz$, where the matrix S diagonalizes matrix A reduces the system (3) to the form

$$\begin{aligned} \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}' = & \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} - \frac{\sqrt{3}}{2}i & 0 \\ 0 & 0 & -\frac{1}{2} + \frac{\sqrt{3}}{2}i \end{pmatrix} \right. \\ & \left. + \varphi(t) \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \left(-\frac{1}{6} - \frac{\sqrt{3}}{6}i\right) & \left(-\frac{1}{6} - \frac{\sqrt{3}}{6}i\right) & \left(-\frac{1}{6} - \frac{\sqrt{3}}{6}i\right) \\ \left(-\frac{1}{6} + \frac{\sqrt{3}}{6}i\right) & \left(-\frac{1}{6} + \frac{\sqrt{3}}{6}i\right) & \left(-\frac{1}{6} + \frac{\sqrt{3}}{6}i\right) \end{pmatrix} \right] \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}. \end{aligned} \tag{4}$$

Solving the corresponding system of integral equations by the method of successive approximations we prove the following

Theorem 1. *If $\varphi \in L^1(\mathbb{R}_+)$ then equation* (1) *has the fundamental system of solutions*

$$u_1(t) = e^t \left(1 + \epsilon_1(t)\right), \qquad u_{2,3}(t) = e^{\left(-\frac{1}{2} \mp \frac{\sqrt{3}}{2}i\right)t} \left(1 + \epsilon_{2,3}(t)\right),$$

where the remainder terms admit the estimates

$$\epsilon_i(t) = O\left(\int_t^\infty |\varphi(s)| ds \right), \quad i = 1,\, 2,\, 3.$$

Now, suppose that $\varphi' \in L^1(\mathbb{R}_+)$. Then equation (1) has the fundamental system of solutions with asymptotic behavior

$$\begin{aligned} u_1(t) &\sim \exp\left(\int_{t_0}^t \sqrt[3]{1 + \varphi(\tau)} d\tau \right), \\ u_2(t) &\sim \exp\left(\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i \right) \int_{t_0}^t \sqrt[3]{1 + \varphi(\tau)} d\tau \right), \\ u_3(t) &\sim \exp\left(\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) \int_{t_0}^t \sqrt[3]{1 + \varphi(\tau)} d\tau \right). \end{aligned}$$

To present more accurate formulas we formulate

Theorem 2. *If* $\varphi' \in L^1(\mathbb{R}_+)$ *then equation* (1) *has solutions with the following asymptotics at* $t \to \infty$

$$u_1(t) = \exp\left(\int_{t_0}^{t} \sqrt[3]{1+\varphi(\tau)}d\tau\right)(1+\epsilon_1(t)),$$

$$u_{2,3}(t) = \exp\left(\int_{t_0}^{t} \left(-\frac{1}{2} \mp \frac{\sqrt{3}}{2}i\right) \sqrt[3]{1+\varphi(\tau)}d\tau\right)(1+\epsilon_{2,3}(t)),$$

where

$$\epsilon_1(t) = O\left(\exp\left(-\frac{3}{2}\int_{t_0}^{t} \sqrt[3]{1+\varphi(\tau)}d\tau\right)\right) + O\left(\int_{t}^{\infty} |\varphi'(\tau)|d\tau\right),$$

$$\epsilon_{2,3}(t) = O\left(\int_{t}^{\infty} |\varphi'(s)|ds\right).$$

The proof of this theorem is based on Levinson's theory [3], where the first-order system (3) is reduced to the form

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}' = \left[\sqrt[3]{1+\varphi(t)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2}-\frac{\sqrt{3}}{2}i & 0 \\ 0 & 0 & -\frac{1}{2}+\frac{\sqrt{3}}{2}i \end{pmatrix}\right.$$
$$\left. - \frac{\varphi'(t)}{3(1+\varphi(t))} \begin{pmatrix} 1 & -\frac{1}{2}-\frac{\sqrt{3}}{2}i & -\frac{1}{2}+\frac{\sqrt{3}}{2}i \\ -\frac{1}{2}-\frac{\sqrt{3}}{6}i & \frac{\sqrt{3}}{3}i & \frac{1}{2}-\frac{\sqrt{3}}{6}i \\ -\frac{1}{2}+\frac{\sqrt{3}}{6}i & \frac{1}{2}+\frac{\sqrt{3}}{6}i & -\frac{\sqrt{3}}{3}i \end{pmatrix}\right] \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}$$

by the change of variables $y = Sz$, where matrix S diagonalizes the matrix $A+B(t)$. Denote by $\lambda_i(t)$ the eigenvalues of matrix $A+B(t)$. Each of the first-order equations is converted into an integral equation in the following way

$$y_i(t) = \delta_{ik}\exp\left(\int_0^t \lambda_k(s)ds\right) - \int_t^\infty \left[\exp\left(\int_s^t \lambda_i(\tau)d\tau\right)\left(\sum_{j=1}^{3} r_{ij}(s)y_j(s)\right)\right] ds$$

or

$$y_i(t) = \int_0^t \left[\exp\left(\int_s^t \lambda_i(\tau)d\tau\right)\left(\sum_{j=1}^{3} r_{ij}(s)y_j(s)\right)\right] ds, \quad i = 1,\ 2,\ 3,$$

depending on whether

$$\int_{t_0}^{\infty} \mathrm{Re}\,(\lambda_i(t)-\lambda_j(t))\,dt \le \infty \ \text{ or } \int_{t_0}^{\infty} \mathrm{Re}\,(\lambda_i(t)-\lambda_j(t))\,dt = -\infty.$$

Here δ_{ik} is the Kronecker delta symbol and $r_{ij}(t)$ are elements of the matrix $S'(t)S^{-1}(t)$.

Usage of the method of iterations applied to the integral equations systems gives the desired information about the asymptotic behavior of solutions and remainder terms.

3. Reduction to Poincaré–Liapounoff type equation

Another method of asymptotic integration is based on reducing equation (1) to the Poincaré–Liapounoff type equation

$$v'' + 3v'v + v^3 - (1 + \varphi(t)) = 0$$

where $v = \dfrac{u'}{u}$. After the substitution $v = w + 1$ one obtains

$$w'' + 3w' + 3w = \varphi(t) - 3ww' - w^3 - 3w^2. \tag{5}$$

The asymptotic behavior of solution of this equation is determined by the location of the roots of polynomial $\lambda^2 + 3\lambda + 3$ which is characteristic for the linearized version of (5). Apply now a method of variation of parameters to reduce (5) to the integral equation

$$\begin{aligned} w(t) = {} & \frac{1}{3}\int_0^t 2\sqrt{3}e^{-\frac{3}{2}(t-t_1)} \sin\left(\frac{\sqrt{3}}{2}(t-t_1)\right)\varphi(t_1)dt_1 \\ & - \frac{1}{3}\int_0^t 2\sqrt{3}e^{-\frac{3}{2}(t-t_1)} \sin\left(\frac{\sqrt{3}}{2}(t-t_1)\right) \\ & \times (3w(t_1)w'(t_1) + w^3(t_1) + 3w^2(t_1))dt_1. \end{aligned} \tag{6}$$

We use this construction to get an increasing solution of the equation in question.

Theorem 3. *If $\varphi \in L^2(\mathbb{R}_+)$, and $\varphi \to 0$, as $t \to \infty$, then equation* (1) *has an increasing solution with the following asymptotic behavior, as $t \to \infty$:*

$$u(t) = \exp\left(t + \frac{1}{3}\int_0^t \varphi(t_1)dt_1\right)(1 + \epsilon(t)),$$

where

$$\epsilon(t) = O\left(\int_0^t e^{-\frac{1}{2}(t-t_1)}\varphi(t_1)dt_1 + \int_t^\infty \varphi^2(t_1)dt_1\right).$$

Sketch of the proof. In order to solve the integral equation (6) by using the method of successive approximations let us denote

$$w_0(t) = \frac{1}{3}\int_0^t 2\sqrt{3}e^{-\frac{3}{2}(t-t_1)} \sin\left(\frac{\sqrt{3}}{2}(t-t_1)\right)\varphi(t_1)dt_1,$$

$$\begin{aligned} w_{n+1}(t) = w_0(t) - {} & \frac{1}{3}\int_0^t 2\sqrt{3}e^{-\frac{3}{2}(t-t_1)} \sin\left(\frac{\sqrt{3}}{2}(t-t_1)\right) \\ & \times (3w_n(t_1)w_n'(t_1) + w_n^3(t_1) + 3w_n^2(t_1))dt_1. \end{aligned}$$

First we prove, inductively in n, that successive approximations $w_n(t)$ and their derivatives $w_n'(t)$ are uniformly bounded and the following inequalities are valid

$$|w_n(t)| \le 2\int_0^t e^{-\frac{1}{2}(t-t_1)}|\varphi(t_1)|dt_1,$$
$$|w_n'(t)| \le 4\int_0^t e^{-\frac{1}{2}(t-t_1)}|\varphi(t_1)|dt_1$$

for all $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$.

The solution of equation (6) is then obtained in the form

$$w(t) = w_0(t) + \sum_{n=0}^{\infty}(w_{n+1}(t) - w_n(t)). \tag{7}$$

By using the boundedness of successive approximations and their derivatives we show that

$$|w_{n+1}(t) - w_n(t)| \le \frac{1}{2}C^n\left(\sup|w_1(t) - w_0(t)| + \sup|w_1'(t) - w_0'(t)|\right)$$

with a certain constant $C \in (0,1)$. Due to this fact the series (7) is convergent and its sum $w(t)$ satisfies equation (6). Moreover, the following estimates

$$|w(t)| \le 2\int_0^t e^{-\frac{1}{2}(t-t_1)}|\varphi(t_1)|dt_1$$

and

$$|w'(t)| \le 4\int_0^t e^{-\frac{1}{2}(t-t_1)}|\varphi(t_1)|dt_1$$

hold. Using these inequalities we obtain the asymptotics

$$w(t) = w_0(t) + O\left(\int_0^t e^{-\frac{1}{2}(t-t_1)}\varphi^2(t_1)dt_1\right)$$

which implies

$$\int_0^t w(t_1)dt_1 = \frac{1}{3}\int_0^t \varphi(t_1)dt_1 + O\left(\int_0^t e^{-\frac{1}{2}(t-t_1)}\varphi(t_1)dt_1 + \int_t^{\infty}\varphi^2(t_1)dt_1\right).$$

Substitution of these asymptotics into the formula

$$u(t) = \exp\left(t + \int_0^t w(s)ds\right)$$

gives the desired solution of (1). □

4. Conclusion

Comparison of the asymptotic formulas and estimates for the remainder terms from Theorems 1, 2 and 3, show that they well agree on the common domain of validity and are naturally complementary since the hypotheses are in a general position.

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On New Reduction of Nonlinear Wave Type Equations via Classical Symmetry Method

Joanna Zonenberg and Ivan Tsyfra

Abstract. The paper is devoted to the construction of an ansatz for the first derivatives of an unknown function which reduces a scalar partial differential equation with three independent variables to a system of equations by using the operators of classical point symmetry. The method is applied to nonlinear wave equation with cubic nonlinearity, Liouville equation and Kadomtsev–Petviashvili equation.

Mathematics Subject Classification (2010). 76M60, 35G20.

Keywords. Reduction of nonlinear equations, symmetry, invariant.

1. Introduction

It is well known, that the group analysis is widely used for constructing solutions of nonlinear partial differential equation. In the framework in this approach we use operators of classical point symmetry to construct an ansatz for the unknown function, which reduces partial differential equations to equations with a smaller number of independent variables, in particular to ordinary differential equations. Then, we obtain solutions of partial differential equation by integrating the reduced ordinary differential equation. The method of a reduction of a scalar partial differential equation by using the operators of non-point symmetry is proposed in [4, 5]. In this approach we construct the ansatz for the first derivatives of an unknown function. The validity of the method was illustrated in application to nonlinear evolution and wave type equations with two independent variables. In this presentation we apply the method to nonlinear partial differential equations with three independent variables. As regards the main idea of this approach the presentation is closely related to [1] (see also [2, 3]).

2. Reduction of selected nonlinear wave type equations

We consider the nonlinear wave equation of the form:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = -\lambda u^3 \tag{1}$$

where $u = u(t, x, y)$ and $\lambda \in \mathbb{R}$.

Let us introduce new variables:

$$\begin{cases} \dfrac{\partial u}{\partial t} = v^t(t, x, y, u) \\ \dfrac{\partial u}{\partial x} = v^x(t, x, y, u) \\ \dfrac{\partial u}{\partial y} = v^y(t, x, y, u). \end{cases}$$

We impose on variables v^t, v^x, v^y the compatibility condition:

$$\begin{cases} v^t_x = v^x_t \\ v^t_y = v^y_t \\ v^x_y = v^y_x. \end{cases} \tag{2}$$

Next we use the symmetry operator:

$$X = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u}$$

of (1) to construct the ansatz reducing equation (1) to the system of differential equations by applying the classic method of construction of invariant solutions [2, 3]. Thus it is necessary to solve the invariance surface conditions:

$$\begin{aligned} t\frac{\partial v^t}{\partial t} + x\frac{\partial v^t}{\partial x} + y\frac{\partial v^t}{\partial y} - u\frac{\partial v^t}{\partial u} &= -2v^t, \\ t\frac{\partial v^x}{\partial t} + x\frac{\partial v^x}{\partial x} + y\frac{\partial v^x}{\partial y} - u\frac{\partial v^x}{\partial u} &= -2v^x, \\ t\frac{\partial v^y}{\partial t} + x\frac{\partial v^y}{\partial x} + y\frac{\partial v^y}{\partial y} - u\frac{\partial v^y}{\partial u} &= -2v^t. \end{aligned}$$

By solving the corresponding characteristic equations we obtain six functionally-independent invariants:

$$\begin{aligned} &\omega_1 = tu, \qquad &&\omega_2 = \frac{t}{x}, \qquad &&\omega_3 = \frac{t}{y}, \\ &\omega_4 = t^2 v^t, \qquad &&\omega_5 = x^2 v^x, \qquad &&\omega_6 = y^2 v^y. \end{aligned}$$

By using these invariants we construct the following ansatz:

$$\begin{cases} v^t = \dfrac{1}{t^2}\varphi_1(\omega_1,\omega_2,\omega_3) \\ v^x = \dfrac{1}{x^2}\varphi_2(\omega_1,\omega_2,\omega_3) \\ v^y = \dfrac{1}{y^2}\varphi_3(\omega_1,\omega_2,\omega_3) \end{cases}$$

where $\varphi_1, \varphi_2, \varphi_3$ are unknown functions of independent variables $\omega_1, \omega_2, \omega_3$.

From the compatibility condition (2) we obtain the following system of equations:

$$\begin{cases} \varphi_2\dfrac{\partial\varphi_1}{\partial\omega_1} - \dfrac{\partial\varphi_1}{\partial\omega_2} = \omega_1\dfrac{\partial\varphi_2}{\partial\omega_1} + \varphi_1\dfrac{\partial\varphi_2}{\partial\omega_1} + \omega_2\dfrac{\partial\varphi_2}{\partial\omega_2} + \omega_3\dfrac{\partial\varphi_2}{\partial\omega_3} \\ \varphi_3\dfrac{\partial\varphi_1}{\partial\omega_1} - \dfrac{\partial\varphi_1}{\partial\omega_3} = \omega_1\dfrac{\partial\varphi_3}{\partial\omega_1} + \varphi_1\dfrac{\partial\varphi_3}{\partial\omega_1} + \omega_2\dfrac{\partial\varphi_3}{\partial\omega_2} + \omega_3\dfrac{\partial\varphi_3}{\partial\omega_3} \\ \varphi_3\dfrac{\partial\varphi_2}{\partial\omega_1} - \dfrac{\partial\varphi_2}{\partial\omega_3} = \varphi_2\dfrac{\partial\varphi_3}{\partial\omega_1} - \dfrac{\partial\varphi_3}{\partial\omega_2}. \end{cases} \tag{3}$$

Equation (1) in new variables takes the form: $v^t_t - v^x_x - v^y_y = -\lambda u^3$. Using the ansatz for v^t, v^x, v^y one can obtain the reduced equation:

$$\begin{aligned} -2\varphi_1 + \omega_1\frac{\partial\varphi_1}{\partial\omega_1} + \omega_3\frac{\partial\varphi_1}{\partial\omega_3} + \omega_2\frac{\partial\varphi_1}{\partial\omega_2} + 2\omega_2^3\varphi_2 - \omega_2^4\varphi_2\frac{\partial\varphi_2}{\partial\omega_1} \\ + \omega_2^4\frac{\partial\varphi_2}{\partial\omega_2} + 2\varphi_3\omega_3^3 - \omega_3^4\varphi_3\frac{\partial\varphi_3}{\partial\omega_1} + \omega_3^4\frac{\partial\varphi_3}{\partial\omega_3} = -\lambda\omega_1^3. \end{aligned} \tag{4}$$

Therefore we reduce the equation (1) to the system of equations (3) and (4) with three independent variables $\omega_1, \omega_2, \omega_3$. Note that the independent variables $\omega_1, \omega_2, \omega_3$ depend on four variables t, x, y, u and this is why the reduction procedure essentially appears in this setting.

Next we consider the nonlinear Liouville equation in the three-dimensional case:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = \tilde{\lambda}e^u \tag{5}$$

where $u = u(t,x,y)$ and $\tilde{\lambda}$ is a real constant.

We introduce new variables:

$$\begin{cases} \dfrac{\partial u}{\partial t} = v^t(t,x,y,u) \\ \dfrac{\partial u}{\partial x} = v^x(t,x,y,u) \\ \dfrac{\partial u}{\partial y} = v^y(t,x,y,u) \end{cases}$$

and we assume that the compatibility condition:

$$\begin{cases} v^t_x = v^x_t \\ v^t_y = v^y_t \\ v^x_y = v^y_x \end{cases} \tag{6}$$

is satisfied. We use the operator

$$Q = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 2\frac{\partial}{\partial u}$$

to construct the ansatz reducing equation (5). Thus, it is necessary to solve the invariance surface conditions:

$$t\frac{\partial v^t}{\partial t} + x\frac{\partial v^t}{\partial x} + y\frac{\partial v^t}{\partial y} - 2\frac{\partial v^t}{\partial u} = -v^t,$$
$$t\frac{\partial v^x}{\partial t} + x\frac{\partial v^x}{\partial x} + y\frac{\partial v^x}{\partial y} - 2\frac{\partial v^x}{\partial u} = -v^x,$$
$$t\frac{\partial v^y}{\partial t} + x\frac{\partial v^y}{\partial x} + y\frac{\partial v^y}{\partial y} - 2\frac{\partial v^y}{\partial u} = -v^y.$$

By solving the corresponding characteristic equations we obtain six independent invariants:

$$\omega_1 = \frac{t}{x}, \qquad \omega_2 = \frac{t}{y}, \qquad \omega_3 = u + \ln|t|,$$
$$\omega_4 = tv^t, \qquad \omega_5 = xv^x, \qquad \omega_6 = yv^y.$$

Therefore we can construct the ansatz:

$$\begin{cases} v^t = \frac{1}{t}\varphi_1(\omega_1, \omega_2, \omega_3) \\ v^x = \frac{1}{x}\varphi_2(\omega_1, \omega_2, \omega_3) \\ v^y = \frac{1}{y}\varphi_3(\omega_1, \omega_2, \omega_3). \end{cases}$$

Then we obtain the following system of equations from the compatibility condition (6):

$$\begin{cases} -\frac{\partial\varphi_1}{\partial\omega_1} + \frac{\varphi_2}{\omega_1}\frac{\partial\varphi_1}{\partial\omega_3} = \frac{\partial\varphi_2}{\partial\omega_1} + \frac{\omega_2}{\omega_1}\frac{\partial\varphi_2}{\partial\omega_2} + \frac{2}{\omega_1}\frac{\partial\varphi_2}{\partial\omega_3} + \frac{\varphi_1}{\omega_1}\frac{\partial\varphi_2}{\partial\omega_3} \\ -\frac{\partial\varphi_1}{\partial\omega_2} + \frac{\varphi_3}{\omega_2}\frac{\partial\varphi_1}{\partial\omega_3} = \frac{\partial\varphi_3}{\partial\omega_2} + \frac{\omega_1}{\omega_2}\frac{\partial\varphi_3}{\partial\omega_1} + \frac{2}{\omega_2}\frac{\partial\varphi_3}{\partial\omega_3} + \frac{\varphi_1}{\omega_2}\frac{\partial\varphi_3}{\partial\omega_3} \\ -\frac{\partial\varphi_2}{\partial\omega_2} + \frac{\varphi_3}{\omega_2}\frac{\partial\varphi_2}{\partial\omega_3} = -\frac{\omega_1}{\omega_2}\frac{\partial\varphi_3}{\partial\omega_1} + \frac{\varphi_2}{\omega_2}\frac{\partial\varphi_3}{\partial\omega_3}. \end{cases} \tag{7}$$

Equation (5) in new variables takes the form: $v_t^t - v_x^x - v_y^y = \tilde{\lambda} e^u$. Next we obtain the reduced equation:

$$\begin{aligned} -\varphi_1 + \omega_1 \frac{\partial \varphi_1}{\partial \omega_1} + \omega_2 \frac{\partial \varphi_1}{\partial \omega_2} + 2 \frac{\partial \varphi_1}{\partial \omega_3} + \varphi_1 \frac{\partial \varphi_1}{\partial \omega_3} + \omega_1^2 \varphi_2 + \omega_1^3 \frac{\partial \varphi_2}{\partial \omega_1} \\ - \omega_1^2 \varphi_2 \frac{\partial \varphi_2}{\partial \omega_3} + \omega_2^2 \varphi_3 + \omega_2^3 \frac{\partial \varphi_3}{\partial \omega_2} - \omega_2^2 \varphi_3 \frac{\partial \varphi_3}{\partial \omega_3} = \tilde{\lambda} e^{\omega_3}. \end{aligned} \tag{8}$$

The reduced system of equations is given in this case by (7) and (8).

Let us apply the method to the well-known integrable Kadomtsev–Petviashvili equation:

$$\left(u_t + \frac{3}{2} u u_x + \frac{1}{4} u_{xxx} \right)_x + \frac{3}{4} u_{yy} = 0, \tag{9}$$

where $u_t = \dfrac{\partial u}{\partial t}, u_x = \dfrac{\partial u}{\partial x}, u_{xxx} = \dfrac{\partial^3 u}{\partial x^3}, u_{yy} = \dfrac{\partial^2 u}{\partial y^2}$.

We introduce new variables:

$$\begin{cases} u_t = v^t(t, x, y, u) \\ u_x = v^x(t, x, y, u) \\ u_y = v^y(t, x, y, u) \end{cases}$$

and we assume that the compatibility condition:

$$\begin{cases} v_x^t = v_t^x \\ v_y^t = v_t^y \\ v_y^x = v_x^y \end{cases} \tag{10}$$

is satisfied. Next we use the operator

$$Y = e^t \frac{\partial}{\partial_y} - \frac{2}{3} y e^t \frac{\partial}{\partial_x} - \frac{4}{9} y e^t \frac{\partial}{\partial_u}$$

which belongs to the infinite-dimensional Lie algebra of symmetry of the Kadomtsev–Petviashvili equation, to construct the ansatz reducing this equation. Analysis similar to that in the derivation of reduced systems (7), (8) gives the following invariance surface conditions:

$$\begin{aligned} 9e^t \frac{\partial v^t}{\partial y} - 6ye^t \frac{\partial v^t}{\partial x} - 4ye^t \frac{\partial v^t}{\partial u} &= -(4y - 6yv^x + 9v^y)e^t, \\ 9e^t \frac{\partial v^x}{\partial y} - 6ye^t \frac{\partial v^x}{\partial x} - 4ye^t \frac{\partial v^x}{\partial u} &= 0, \\ 9e^t \frac{\partial v^y}{\partial y} - 6ye^t \frac{\partial v^y}{\partial x} - 4ye^t \frac{\partial v^y}{\partial u} &= (4 - 6v^x)e^t \end{aligned}$$

and six functionally-independent invariants:

$$\begin{aligned} &\omega_1 = t, \quad \omega_2 = v^x, \qquad \omega_3 = 3x + y^2, \quad \omega_4 = 9u + 2y^2, \\ &\omega_5 = 9v^y + (4 - 6v^x)y, \quad \omega_6 = 9v^t + 9v^y y - (-4 + 6v^x)y^2. \end{aligned}$$

Therefore we have the ansatz:

$$\begin{cases} v^x = \varphi_1(\omega_1, \omega_3, \omega_4) \\ v^y = \dfrac{1}{9}\varphi_2(\omega_1, \omega_3, \omega_4) - \dfrac{1}{9}(4 - 6v^x)y \\ v^t = \dfrac{1}{9}\varphi_3(\omega_1, \omega_3, \omega_4) - v^y y + \dfrac{1}{9}(-4 + 6v^x)y^2. \end{cases}$$

We obtain the following system of equations from the compatibility condition (10):

$$\begin{cases} \varphi_2 \dfrac{\partial \varphi_1}{\partial \omega_3} - \dfrac{1}{3}\dfrac{\partial \varphi_2}{\partial \omega_3} - \varphi_1 \dfrac{\partial \varphi_2}{\partial \omega_3} = 0 \\ \dfrac{\partial \varphi_1}{\partial \omega_1} + \varphi_3 \dfrac{\partial \varphi_1}{\partial \omega_3} - \dfrac{1}{3}\dfrac{\partial \varphi_3}{\partial \omega_2} - \varphi_1 \dfrac{\partial \varphi_3}{\partial \omega_3} = 0 \\ \dfrac{1}{9}\dfrac{\partial \varphi_2}{\partial \omega_1} + \dfrac{1}{9}\varphi_3 \dfrac{\partial \varphi_3}{\partial \omega_3} - \dfrac{1}{9}\varphi_2 = 0. \end{cases} \tag{11}$$

In new variables equation (9) takes the form:

$$v_t^x + \frac{3}{2}(v^x)^2 + \frac{3}{2}uv_x^x + \frac{1}{4}v_{xxx}^x + \frac{3}{4}v_y^y = 0.$$

Then we obtain the reduced equation:

$$\begin{aligned} &\frac{\partial \varphi_1}{\partial \omega_1} + \varphi_3 \frac{\partial \varphi_1}{\partial \omega_3} + \frac{1}{2}\left(\frac{\partial \varphi_1}{\partial \omega_2} + 3\frac{\partial \varphi_1}{\partial \omega_3}\right)\omega_3 + \frac{1}{4}\left(27\frac{\partial^3 \varphi_1}{\partial \omega_2^3} + 81\varphi_1 \frac{\partial^3 \varphi_1}{\partial \omega_2^2 \partial \omega_3}\right. \\ &\quad + 243\frac{\partial \varphi_1}{\partial \omega_2}\frac{\partial^2 \varphi_1}{\partial \omega_2 \partial \omega_3} + 1215\varphi_1 \frac{\partial \varphi_1}{\partial \omega_3}\frac{\partial^2 \varphi_1}{\partial \omega_2 \partial \omega_3} + 81\frac{\partial^2 \varphi_1}{\partial \omega_2^2}\frac{\partial \varphi_1}{\partial \omega_3} + \left.729\varphi_1 \frac{\partial \varphi_1}{\partial \omega_2}\frac{\partial^2 \varphi_1}{\partial \omega_3^2}\right) \\ &\quad + \frac{1}{4}\left(243\frac{\partial \varphi_1}{\partial \omega_2}\left(\frac{\partial \varphi_1}{\partial \omega_3}\right)^2 + 729\varphi_1\left(\frac{\partial \varphi_1}{\partial \omega_3}\right)^2 + 2916\varphi_1^2 \frac{\partial \varphi_1}{\partial \omega_3}\frac{\partial^2 \varphi_1}{\partial \omega_3^2} + 243\varphi_1^2 \frac{\partial^3 \varphi_1}{\partial \omega_2 \partial \omega_3^2}\right. \\ &\quad + \left.729\varphi_1^3 \frac{\partial^3 \varphi_1}{\partial \omega_3^3}\right) + \frac{1}{2}\varphi_2 \frac{\partial \varphi_2}{\partial \omega_3} - \frac{1}{3} + \frac{1}{2}\varphi_1 = 0. \end{aligned} \tag{12}$$

The reduced system of equations in this case is given by (11) and (12).

3. Conclusions

We have constructed new reductions of nonlinear wave equations (1), (5) and (9) by using the classical symmetry of these equations via the method proposed in Refs. [4, 5]. It is obvious that we can construct the ansatz for v^t, v^x, v^y where $\varphi_1, \varphi_2, \varphi_3$ depend on two variables ω_1, ω_2 or only one variable ω_1 if we use the multidimensional Lie algebra.These representations can be used for the construction of non-local symmetries, conservation laws and solutions for equations under consideration.

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Part V: General Methods

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C-orbit Function and Image Filtering

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Abstract. We present the first attempt to use the C-orbit functions in image processing. For the image processing we perform a Fourier-like transform of the image. Then we define a convolution on C-orbit functions and we apply the simplest spatial linear filters on several examples. Finally we compare the results with filtering via an ordinary Fourier transformation.

Mathematics Subject Classification (2010). 42B10; 43A75.

Keywords. Image processing, Fourier type transform, orbit functions.

1. Introduction

The development of the theory of orbit functions opens a space for their applications in data processing. In this work we focus on the simplest case, the two-dimensional digital image processing. The most widely used method for image processing is the Fourier analysis (decomposition into exponential series). For the spatial filtering the convolution of functions is used because in the Fourier image it is transferred to the multiplication of functions. Fourier analysis is based on the decomposition of brightness values in each digitized image point along the rows and columns into a Fourier series. This Fourier image is then processed. The inverse discrete Fourier transform gives the modified digital image. In this way we can emphasize some features of the image, remove noise or enhance blur edges. The whole process is described in several papers, for an overview see for example [1, 2]. For the JPEG image compression discrete cosine transforms are used. They are of four types and the introduction of convolution in this case is more complicated because of mixing the discrete cosine and sine transform.

The simplest technique of filtering consists in averaging the image intensity at a point. The value of the intensity in a given pixel is the mean value of the intensities of the 8 neighboring pixels and the pixel itself. In other filters the values of neighboring pixels are multiplied by different relative weights and the pixel is assigned with the mean value of 9 intensities. Other filters take into account higher number of other surrounding pixels, for example 25 pixels including the center. This averaging over neighboring pixels is mathematically expressed as a convolution.

The intensities in 9 or 25 pixels are represented by 3×3 or 5×5 matrices. Averaging over neighboring pixels is mathematically expressed by the convolution of the original intensity matrix with another 3×3 or 5×5 matrix, whose elements are the weights assigned to the corresponding pixel in the area according to the desired filter type. For the processing of the pixel intensities on the boundary in the case of 3×3 matrix we need to extend one line above and below the picture, and the column on the left and the right. In the case of 5×5 we need to add two columns or lines to each side. In this work, we used the C-orbit functions of the Lie algebra A_2, which are the generalized cosine functions. The symmetries of these functions allows us to define them on an equilateral triangle, so-called fundamental domain. In this triangle we introduce discrete grids on which we define discrete C-orbit transform. For the analysis and processing of a square image we split it into two triangles in order to perform the C-orbit transform.

In order to implement a convolution, we use the same procedure as in the discrete cosine transform case. The difference is that the convolution mask is triangular and the calculation time is longer, because unlike the Fourier transform case, C-orbit function is defined as the sum of six exponential members.

This method can be extended to the processing of digital information in a simplex of any dimension, which is the subject of our future research.

The paper is organized as follows. In Section 2 we summarize the needed facts from the theory of orbit functions. Section 3 gives the details of the discretization process. In Section 4 we review the use of convolution in image processing. Section 5 describes the C-orbit convolution and related spatial filters. Finally, the last section contains examples of application of the filters to digital images.

2. Preliminaries

We consider a simple Lie algebra A_2 with the set of simple roots $\{\alpha_1, \alpha_2\}$. According to the notation in [3] we define four lattices in $\mathbb{R}^2$: The root lattice $Q = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$; the co-root lattice $Q^\vee = \mathbb{Z}\alpha_1^\vee + \mathbb{Z}\alpha_2^\vee$, where the co-roots are defined as $\alpha^\vee = 2\alpha/\langle\alpha, \alpha\rangle$; and their duals $P = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and $P^\vee = \mathbb{Z}\omega_1^\vee + \mathbb{Z}\omega_2^\vee$ which are called the weight lattice and co-weight lattice respectively and it holds that $\langle\alpha_i^\vee, \omega_j\rangle = \langle\alpha_i, \omega_j^\vee\rangle = \delta_{ij}$.

Let r_1 and r_2 denote the reflections with respect to the hyperplanes orthogonal to the simple roots. They generate the Weyl group W of the algebra A_2. Let r_0 be the affine reflection, i.e., reflection with respect to the hyperplane orthogonal to the highest root ξ of the root system of A_2 shifted by $\xi/2$. Then, the affine Weyl group W^{aff} is generated by $\{r_0, r_1, r_2\}$. Its fundamental domain is a connected subset of $\mathbb{R}^2$ such that it contains exactly one point of each affine Weyl group orbit. In the similar way we define the dual affine Weyl group corresponding to the root system generated by co-roots, its fundamental domain is denoted $F^\vee$.

The highest root ξ of the root system of A_2 is given by $\xi = \alpha_1 + \alpha_2$, therefore, the fundamental domain F of the corresponding affine Weyl group can be chosen as the convex hull of the points $\{0, \omega_1^\vee, \omega_2^\vee\}$ [3]. Explicitly,

$$F = \{x\omega_1^\vee + y\omega_2^\vee \mid x, y \geq 0 \wedge x + y \leq 1\} .$$

The root system and the fundamental region of A_2 is shown in Figure 1.

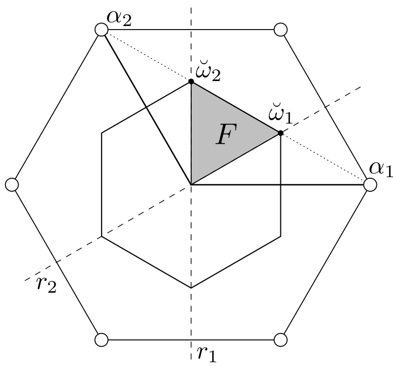

FIGURE 1. Root system of A_2. Dots denote the roots, dashed lines the hyperplanes orthogonal to the simple roots and the gray triangle is the fundamental domain of the corresponding affine Weyl group.

Several families of the Weyl group orbit functions can be defined in the context of the Weyl group of A_2. The family of so-called C-functions can be defined as follows: For every x and $\lambda \in \mathbb{R}^2$ we have

$$\Phi_\lambda(x) = \sum_{w \in W} e^{2\pi i \langle w(\lambda),\, x\rangle}.$$

The C-orbit functions are described in detail in [6]. They are invariant with respect to the affine Weyl group in the following way: For every $w' \in W$ and $w'' \in W^{\mathrm{aff}}$ we have

$$\Phi_{w'\lambda}(x) = \Phi_\lambda(x),$$
$$\Phi_\lambda(w''x) = \Phi_\lambda(x),$$

therefore, we can consider $x \in F$ and we can restrict the choice of the parameter λ to the positive part of the weight lattice, $P^+ = \mathbb{Z}^{\geq 0}\omega_1 + \mathbb{Z}^{\geq 0}\omega_2$. Later we will need the following formula for the product of an orbit function with the complex conjugate of an orbit function with the same label, but a different argument

$$\Phi_\lambda(x)\overline{\Phi_\lambda(y)} = \sum_{w \in W} \Phi_\lambda(x - w(y)). \tag{1}$$

3. Discrete orthogonality and C-transform

The method of discretization of orbit functions was described in detail in the series of papers [3–5], here we summarize it for the case of A_2 only.

Two finite lattice grids depending on an integer parameter M are introduced. The grid of points F_M is defined as $\frac{1}{M}P^\vee/Q^\vee \cap F$ and the grid of parameters as $\Lambda_M = P/MQ \cap MF^\vee$. We consider a space of discrete functions sampled on the points of F_M with a scalar product defined for each pair of functions f, g as

$$\langle f,\, g\rangle_M = \sum_{x \in F_M} \varepsilon(x) f(x) \overline{g(x)}\,. \tag{2}$$

The weight function $\varepsilon(x)$ is given by the order of the Weyl orbit of x, $\varepsilon(x) = \frac{|W|}{|\operatorname{stab}_W(x)|}$. The set of parameters gives us a finite family of orbit functions which are pairwise orthogonal with respect to the scalar product (2).

In the case of the Weyl group of A_2 the two grids are of the following form:

$$F_M = \left\{ \frac{s_1}{M}\omega_1^\vee + \frac{s_2}{M}\omega_2^\vee \,|\, s_0, s_1, s_2 \in \mathbb{Z}^{\geq 0}, s_0 + s_1 + s_2 = M \right\}, \tag{3}$$

$$\Lambda_M = \left\{ t_1\omega_1 + t_2\omega_2 \,|\, t_0, t_1, t_2 \in \mathbb{Z}^{\geq 0}, t_0 + t_1 + t_2 = M \right\}. \tag{4}$$

The orders of the two grids are the same and equal to $1/2(M+1)(M+2)$.

For every $\lambda, \lambda' \in \Lambda_M$ it holds that

$$\langle \Phi_\lambda,\, \Phi_{\lambda'} \rangle = 18 M^2 h_\lambda^\vee \delta_{\lambda\lambda'}, \tag{5}$$

where the coefficient $h_\lambda^\vee$ is the order of the stabilizer of λ. The values of $\varepsilon(x)$ and $h_\lambda^\vee$ are listed in Table 1.

$x \in F_M$	$\varepsilon(x)$	$\lambda \in \Lambda_M$	$h_\lambda^\vee$
$[s_0, s_1, s_2]$	6	$[t_0, t_1, t_2]$	1
$[s_0, s_1, 0]$	3	$[t_0, t_1, 0]$	2
$[s_0, 0, s_2]$	3	$[t_0, 0, t_2]$	2
$[0, s_1, s_2]$	3	$[0, t_1, t_2]$	2
$[0, 0, s_2]$	1	$[0, 0, t_2]$	6
$[0, s_1, 0]$	1	$[0, t_1, 0]$	6
$[s_0, 0, 0]$	1	$[t_0, 0, 0]$	6

TABLE 1. The coefficients $\varepsilon(x)$ and $h_\lambda^\vee$ of A_2. The variables s_i, t_i, $i = 0, 1, 2$, are non negative integers and have the same meaning as in (3) and (4).

The discrete orthogonality allows us to perform a Fourier like transform, called C-orbit transform. We consider a function f sampled on the points of F_M. We can interpolate it by a sum of C-functions

$$I_M(x) = \sum_{\lambda \in \Lambda_M} F_\lambda \Phi_\lambda(x), \tag{6}$$

where we require $f(x) = I_M(x)$ for every $x \in F_M$. Therefore, the coefficients c_λ are equal to

$$F_\lambda = \frac{\langle f, \Phi_\lambda \rangle_M}{\langle \Phi_\lambda, \Phi_\lambda \rangle_M} = \frac{1}{18M^2 h_\lambda^\vee} \sum_{x \in F_M} \varepsilon(x) f(x) \overline{\Phi_\lambda(x)}. \tag{7}$$

4. Convolution

Filters mentioned in the introduction are called linear spatial filters. Their application to a digital image creates a new image using a linear combination of brightness values in the surrounding pixels. The intensities of the digital image in each pixel are defined by the matrix $f(m,n)$. If we want to apply a filter comprising eight neighboring pixels with different weights, we construct the 3×3 weights matrix

$$\begin{pmatrix} a_{-1-1} & a_{-10} & a_{-11} \\ a_{0-1} & a_{00} & a_{01} \\ a_{1-1} & a_{10} & a_{11} \end{pmatrix}.$$

New digital image has the intensity in each pixel given by a matrix $F(m,n)$ and their values are

$$\begin{aligned} F(m,n) = a_{-1-1} f(m-1,n-1) + a_{-10} f(m-1,n) + a_{-11} f(m-1,n+1) \\ + a_{0-1} f(m,n-1) + a_{00} f(m,n) + a_{01} f(m,n+1) \\ + a_{1-1} f(m+1,n-1) + a_{10} f(m+1,n) + a_{11} f(m+1,n+1), \end{aligned}$$

which corresponds to the sum of all the values of the 3×3 matrix we get as a point wise multiplication of the filter 3×3 matrix cut around the filtered pixel. Mathematically, it is a discrete convolution

$$F(m,n) = \sum_{i,j=-1}^{1} f(m+i, n+j) a_{ij}.$$

Similarly, using a convolution matrix we can describe nonlinear Secondary filtering, such as edge detection, etc.

For defining the convolution in the case of the C-orbit convolution we will proceed in a similar way as for the discrete cosine transform DCT II, for two functions f and g it is defined

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y)(g(|x-y|) + g(x+y))dy$$

and for cosine transform F_c the following relation holds [7]

$$F_c(f * g)(x) = (F_c f)(x)(F_c g)(x).$$

5. Orbit convolution

The C-orbit convolution is for every pair of discrete functions f, g sampled on F_M and $u \in F_M$ defined as

$$(f * g)(u) := \sum_{x \in F_M} \varepsilon(x) \sum_{w \in W} f(x) g(u - w(x)). \tag{8}$$

Such a convolution is well defined, the shifts in the convolution kernel g respect the symmetry of the Weyl group pf A_2. We can write the C-orbit convolution theorem. Its proof is straightforward.

Theorem 1. *Let f, g be any functions defined on the points of F_M and $u \in F_M$. Then*

$$(f * g)(u) = \sum_{\lambda \in \Lambda_M} 18 M^2 h_\lambda^\vee F_\lambda G_\lambda \Phi_\lambda(u), \tag{9}$$

where F_λ and G_λ are the C-orbit transforms of f and g given by (6).

6. Spatial image filtering

Spatial image filtering allows us to modify an image to improve its visual properties or to preprocess the image for further operations, i.e., object detection. Image filtering is performed by a convolution process or, using the convolution theorem, by a frequency filtering. Nevertheless, for both approaches an original image and a spatial filter is needed.

To demonstrate a spatial image filtering we apply blurring, sharpening and edge-detecting filter to an image of a sunset, see Figure 2. These filters are easily described by convolution kernels. In a common $\mathbb{R}^2$ image processing one may use the following kernels:

$$h_m = \frac{1}{9}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad h_s = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 0 \end{pmatrix} \quad h_{ed} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Figure 2. Original image.

FIGURE 3. Result of meaning filter, used with Fourier (left) and C-orbit transform (right).

FIGURE 4. Result of sharpening filter, used with Fourier (left) and C-orbit transform (right).

The alternative kernels for the case of C-orbit convolution are:

$$h_m = \frac{1}{3}\begin{pmatrix} 1 & \\ & 1 \\ 1 & \end{pmatrix} \quad h_s = \begin{pmatrix} 0 & \\ & -1 \\ 5 & \end{pmatrix} \quad h_{ed} = \begin{pmatrix} 0 & \\ & -1 \\ 3 & \end{pmatrix}$$

We used these kernels to compare standard Fourier filtering and C-orbit filtering. The result is shown on Figures 3, 4, 5.

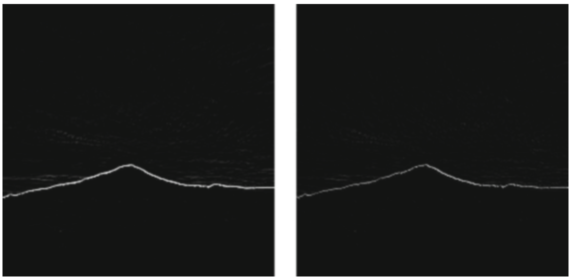

FIGURE 5. Result of edge detection filter, used with Fourier (left) and C-orbit transform (right).

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Conformal Inversion and Maxwell Field Invariants in Four- and Six-dimensional Spacetimes

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Presented in honour of Daniel Sternheimer, on the occasion of his 75th birthday

Abstract. Conformally compactified $(3+1)$-dimensional Minkowski spacetime may be identified with the projective light cone in $(4+2)$-dimensional spacetime. In the latter spacetime the special conformal group acts via rotations and boosts, and conformal inversion acts via reflection in a single coordinate. Hexaspherical coordinates facilitate dimensional reduction of Maxwell electromagnetic field strength tensors to $(3+1)$ from $(4+2)$ dimensions. Here we focus on the operation of conformal inversion in different coordinatizations, and write some useful equations. We then write a conformal invariant and a pseudo-invariant in terms of field strengths; the pseudo-invariant in $(4+2)$ dimensions takes a new form. Our results advance the study of general nonlinear conformal-invariant electrodynamics based on nonlinear constitutive equations.

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1. Introduction

Maxwell's equations in $(3+1)$-dimensional spacetime $M^{(4)}$ (Minkowski space) are not only Poincaré invariant but conformally invariant. But the physical consequences of this symmetry, if any, remain somewhat unclear.

As was observed by Dirac [1], the conformal compactification of $M^{(4)}$ (which we denote $M^{\#}$) can be identified with the projective light cone in a $(4+2)$-

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dimensional spacetime $Y^{(6)}$, in such a way that the special conformal transformations act by rotations and boosts. One may then write a version of Maxwell's equations in $Y^{(6)}$.

Introducing so-called hexaspherical coordinates in the latter space, one obtains a spacetime $Q^{(6)}$. Using this coordinatization one seeks to recover classical electrodynamics in $M^{(4)}$ through a process of "dimensional reduction," which involves restriction to the (projective) light cone and the imposition of various conditions on the Maxwell fields. The result is to gain some insight into additional fields that might, as a consequence, survive in $M^{(4)}$. Many details of these results are described by Nikolov and Petrov [2]. The conventions we adopt here differ in some ways from their development.

Our first goal in this presentation is to consider how conformal inversion acts explicitly in various coordinate systems. This leads to a number of useful equations. Secondly, we introduce conformal invariant (or pseudoinvariant) functionals of the electromagnetic field strength tensor in $(4+2)$-dimensional spacetime. Our ultimate motivation, in the spirit of our earlier work [3–5], is to consider general nonlinear conformal-invariant electrodynamics based on nonlinear constitutive equations. The constitutive equations, in turn, are to be written explicitly in $(4+2)$ dimensions in terms of the conformal-invariant functionals. This allows discussion of both Lagrangian and non-Lagrangian theories. Thus we present here some steps in this overall program.

2. Maxwell's equations and conformal symmetry

2.1. Conformal transformations of Minkowski space

We write $x = (x^\mu) \in M^{(4)}$, with $\mu = 0,1,2,3$. The metric tensor $n_{\mu\nu}$ is $\mathrm{diag}\,[1,-1,-1,-1]$, so that (with the usual summation convention)

$$x_\mu x^\mu = n_{\mu\nu} x^\mu x^\nu = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2,$$

and the light cone $L^{(4)}$ is the submanifold $x_\mu x^\mu = 0$. The conformal group then consists respectively of spacetime translations,

$$x'^{\,\mu} = x^{\,\mu} - b^{\,\mu}\,, \tag{1}$$

spatial rotations and Lorentz boosts, e.g.,

$$x'^{\,0} = \gamma(x^0 - \beta x^1)\,,\ \ x'^{\,1} = \gamma(x^1 - \beta x^0)\,,\ \ -1 < \beta = \frac{v}{c} < 1\,,\ \ \gamma = (1-\beta^2)^{-\frac{1}{2}}\,, \tag{2}$$

and dilations,

$$x'^{\,\mu} = \lambda x^{\,\mu}\,,\ \ \lambda > 0\,, \tag{3}$$

all of which are *causal* in $M^{(4)}$; together with inversion, which breaks causality and acts singularly on the light cone in $M^{(4)}$,

$$x'^{\,\mu} = \frac{x^{\,\mu}}{x_\nu x^\nu}\,. \tag{4}$$

That is, conformal inversion preserves the set of light-like submanifolds (the "light rays"), but not the causal structure. One may write,

$$n_{\mu\nu} dx'^{\mu} dx'^{\nu} = \frac{1}{\Omega(x)^2} n_{\mu\nu} dx^{\mu} dx^{\nu} \,. \tag{5}$$

Following inversion by a translation and inverting again gives us the special conformal transformation,

$$x'^{\mu} = \frac{(x^{\mu} - b^{\mu} x_{\nu} x^{\nu})}{(1 - 2b_{\nu} x^{\nu} + b_{\nu} b^{\nu} x_{\mu} x^{\mu})} \,. \tag{6}$$

These can be continuously connected to the identity in the conformal group; thus special conformal symmetry may be studied with (local) Lie algebraic techniques. However, examining the conformal inversion (4) directly, the main approach taken here, provides valuable insight into the (global) conformal symmetry.

2.2. Conformal symmetry of Maxwell's equations

Under the transformation (4), one has the following symmetry transformations of the electromagnetic potential and the spacetime derivatives:

$$A'_{\mu}(x') = x^2 A_{\mu}(x) - 2x_{\mu}(x^{\alpha} A_{\alpha}(x)) \,, \tag{7}$$

$$\partial'_{\mu} := \frac{\partial}{\partial x'^{\mu}} = x^2 \partial_{\mu} - 2x_{\mu}(x \cdot \partial) \,, \tag{8}$$

where we have here used the abbreviations $x^2 = x_{\mu} x^{\mu}$ and $(x \cdot \partial) = x^{\alpha} \partial_{\alpha}$; with $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$,

$$F'_{\mu\nu}(x') = (x^2)^2 F_{\mu\nu}(x) - 2x^2 x^{\alpha}(x_{\mu} F_{\alpha\nu}(x) + x_{\nu} F_{\mu\alpha}(x)) \,, \tag{9}$$

and with $\Box = \partial^{\mu} \partial_{\mu}$,

$$\Box' = (x^2)^2 \Box - 4x^2 (x \cdot \partial) \,. \tag{10}$$

Additionally, the 4-current j_{μ} transforms by

$$j'_{\mu}(x') = (x^2)^3 j_{\mu}(x) - 2(x^2)^2 x_{\mu}(x^{\alpha} j_{\alpha}(x)) \,. \tag{11}$$

These transformations define a symmetry of the (linear) Maxwell equations,

$$\Box A_{\nu} - \partial_{\nu}(\partial^{\alpha} A_{\alpha}) = j_{\nu} \,; \tag{12}$$

if $A(x)$ and $j(x)$ satisfy (12), then $A'(x')$ and $j'(x')$ satisfy the same equation with $\Box'$ and ∂' in place of $\Box$ and ∂ respectively. Combining this symmetry with that of the Poincaré transformations and dilations, we have the symmetry with respect to the usual conformal group.

Note that (8) and (10) can be obtained by regarding the inversion (4) as if it were a coordinate transformation, and using the corresponding Jacobian matrix. However (7), (9), and (11) are *symmetry* transformations of the fields, not coordinate transformations.

2.3. Conformal-invariant functionals

In $M^{(4)}$ we have the Poincaré-invariant functionals

$$I_1 = \frac{1}{2} F_{\mu\nu}(x) F^{\mu\nu}(x) , \quad I_2 = -\frac{c}{4} F_{\mu\nu}(x) \tilde{F}^{\mu\nu}(x) , \tag{13}$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2}\, \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$, with ϵ the usual totally antisymmetric Levi-Civita symbol. Sometimes the functional I_2 is called a pseudoinvariant, because it changes sign under spatial reflection (parity). These functionals are useful in writing general nonlinear Poincaré-invariant Maxwell systems.

Under conformal inversion, however, I_1 and I_2 are not individually invariant; rather, they transform by,

$$I_1'(x') = \frac{1}{2} F'_{\mu\nu}(x')(F')^{\mu\nu}(x') \quad = (x^2)^4 I_1(x) , \tag{14}$$

$$I_2'(x') = -\frac{c}{4} F'_{\mu\nu}(x') \left(\tilde{F}'\right)^{\mu\nu}(x') = -(x^2)^4 I_2(x) . \tag{15}$$

So the ratio I_2/I_1 is a pseudoinvariant under conformal inversion. This means, however, that it is invariant under the special conformal transformations.

3. The compactification $M^{\#}$ and the conformal group acting in $(4+2)$-dimensional spacetime

3.1. Compactified Minkowski space

We can also describe Minkowski space using light cone coordinates. Choose a particular (spatial) direction in $\mathbf{R}^3$. Such a direction is specified by a unit vector $\hat{u}$, labeled (for example) by an appropriate choice of angles in spherical coordinates. A point $\mathbf{x} \in \mathbf{R}^3$ is then labeled by angles and by the coordinate u, with $-\infty < u < \infty$, and $\mathbf{x} \cdot \mathbf{x} = u^2$.

With respect to the selected direction, introduce the coordinates

$$u^{\pm} = \frac{1}{\sqrt{2}}(x^0 \pm u) . \tag{16}$$

Then $x_\mu x^\mu = 2u^+u^-$, so under conformal inversion, with obvious notation,

$$u'^{+} = 1/2u^{-} , \quad u'^{-} = 1/2u^{+} . \tag{17}$$

Now one can compactify $M^{(4)}$ by formally adjoining to it the set $\mathcal{J}$ of the necessary "points at infinity." These are taken to be the images under inversion of the light cone $L^{(4)}$ (defined by either $u^+ = 0$ or $u^- = 0$), together with the formal limit points of $L^{(4)}$ itself at infinity (which form an invariant submanifold of $\mathcal{J}$ under conformal inversion). Here $\mathcal{J}$ is the well-known "extended light cone at infinity"; see, e.g., [6].

The resulting space $M^{\#} = M^{(4)} \cup \mathcal{J}$ has the topology of $S^3 \times S^1/Z_2$, and conformal inversion acts on $M^{\#}$ in a well-defined manner. There are many different ways to coordinatize $M^{\#}$ and to visualize its structure, which we do not review here.

3.2. The (4 + 2)-dimensional space $Y^{(6)}$ and its projective light cone

One now introduces the (4 + 2)-dimensional spacetime $Y^{(6)}$. For $y \in \mathbf{R}^6$, write $y = (y^m), m = 0, 1, \ldots, 5$, and define the flat metric tensor $\eta_{mn} = \mathrm{diag}[1, -1, -1, -1; -1, 1]$, so that (with summation convention)

$$y_m y^m = \eta_{mn} y^m y^n = (y^0)^2 - (y^1)^2 - (y^2)^2 - (y^3)^2 - (y^4)^2 + (y^5)^2.$$

The light cone $L^{(6)}$ is then specified by the condition $y_m y^m = 0$, or

$$(y^1)^2 + (y^2)^2 + (y^3)^2 + (y^4)^2 = (y^0)^2 + (y^5)^2 . \tag{18}$$

In $Y^{(6)}$, define *projective equivalence* in the usual way, $(y^m) \sim (\lambda y^m)$ for $\lambda \in \mathbf{R}$, $\lambda \neq 0$. The equivalence classes $[y]$ are the rays in $Y^{(6)}$; let $PY^{(6)}$ denote this space of rays. The *projective light cone* $PL^{(6)}$ is likewise the space of rays in $L^{(6)}$. To specify $PL^{(6)}$, one may choose one point from each ray in $L^{(6)}$. Then, referring back to (18), if we consider $(y^1)^2 + (y^2)^2 + (y^3)^2 + (y^4)^2 = (y^0)^2 + (y^5)^2 = 1$, we have $S^3 \times S^1$. But evidently we have here *two* points in each ray; so $PL^{(6)}$ can be identified with (and has the topology of) $S^3 \times S^1/Z_2$.

Furthermore, $PL^{(6)}$ can be identified with $M^{\#}$. When $y^4 + y^5 \neq 0$, the corresponding element of $M^{\#}$ belongs to $M^{(4)}$ (finite Minkowski space), and is given by

$$x^\mu = \frac{y^\mu}{y^4 + y^5} , \quad \mu = 0, 1, 2, 3 , \tag{19}$$

while the "light cone at infinity" corresponds to the submanifold $y^4 + y^5 = 0$ in $PL^{(6)}$.

3.3. The conformal group acting in $Y^{(6)}$

Conformal transformations act in $Y^{(6)}$ via rotations and boosts, so as to leave $PL^{(6)}$ invariant. We may write this in terms of the 15 conformal group generators. Setting $X_{mn} = y_m \partial_n - y_n \partial_m$ $(m < n)$, one has the 6 rotation and boost generators $M_{mn} = X_{mn}$ $(0 \leq m < n \leq 3)$, the 4 translation generators $P_m = X_{m5} - X_{m4}$ $(0 \leq m \leq 3)$, the dilation generator $D = -X_{45}$, and the 4 special conformal generators, $K_m = -X_{m5} - X_{m4}$ $(0 \leq m \leq 3)$.

But of course, from these infinitesimal transformations we can only construct the *special* conformal transformations, which act like (proper) rotations and boosts. Conformal inversion acts in $Y^{(6)}$ by reflection of the y^5 axis, which makes it easy to explore in other coordinate systems too:

$$y'^{\,m} = y^m (m = 0, 1, 2, 3, 4) , \quad y'^5 = -y^5 , \tag{20}$$

or more succinctly, $y'^{\,m} = K^m_n y^n$, where $K^m_n = \mathrm{diag}\,[1, 1, 1, 1, 1, -1]$.

3.4. Maxwell fields and conformal invariants in $Y^{(6)}$

Now one introduces 6-component fields $\mathcal{A}_m$ in $Y^{(6)}$, and writes

$$\mathcal{F}_{mn} = \partial_m \mathcal{A}_n - \partial_n \mathcal{A}_m , \tag{21}$$

so that for any specific choices of k, m, and n,

$$\frac{\partial \mathcal{F}_{mn}}{\partial y^k} + \frac{\partial \mathcal{F}_{nk}}{\partial y^m} + \frac{\partial \mathcal{F}_{km}}{\partial y^n} = 0\,. \tag{22}$$

While this is not really the *most* general possible "electromagnetism" in $(4+2)$-dimensional spacetime, it is the theory most commonly discussed in the linear case. Note that for fields in the space $Y^{(6)}$ we are using the calligraphic font $\mathcal{A}$, $\mathcal{F}$, etc.

To complete Maxwell's equations, we set

$$\frac{\partial \mathcal{G}^{mn}}{\partial y^m} = \mathcal{J}^{\,n}\,, \tag{23}$$

where $\mathcal{J}^{\,n}$ is the 6-current. In the linear theory, $\mathcal{G}$ is proportional to $\mathcal{F}$. For the general nonlinear theory, however, conformal-invariant nonlinear constitutive equations which relate $\mathcal{G}^{mn}$ to $\mathcal{F}_{mn}$ should be written in terms of invariant functionals. Thus the next step is to consider these functionals.

3.5. Conformal invariants for Maxwell theory in $Y^{(6)}$

As we have seen, conformal invariance in $M^{\#}$ means rotational invariance in $Y^{(6)}$. Thus two rotation-invariant functionals of the field strength tensor $\mathcal{F}_{mn}$ in $Y^{(6)}$ can immediately be written (with ϵ now the totally antisymmetric Levi-Civita symbol with six indices):

$$\mathcal{I}_1 = \frac{1}{2}\,\mathcal{F}_{mn}\mathcal{F}^{mn}\,, \quad \mathcal{I}_2 = \frac{1}{2}\,\epsilon^{mnk\ell rs}\mathcal{F}_{mn}\mathcal{F}_{k\ell}\mathcal{F}_{rs}\,. \tag{24}$$

The first rotation invariant functional, perhaps as expected, is analogous to the first invariant in (13) for the $(3+1)$-dimensional case. But the second rotation invariant functional, unlike the second one in (13), is now *trilinear* in the field strengths (due to the presence of six indices rather than four).

Under conformal inversion, we also have the field transformations,

$$\mathcal{A}'_m(y') = K^n_m \mathcal{A}_n(y)\,, \tag{25}$$

and

$$\begin{aligned} \mathcal{F}'_{mn}(y') &= -\mathcal{F}_{mn}(y) \text{ if } m=5 \text{ or } n=5\,,\\ \mathcal{F}'_{mn}(y') &= +\mathcal{F}_{mn}(y) \text{ otherwise}\,. \end{aligned} \tag{26}$$

So $\mathcal{I}_1$ is invariant under conformal inversion, while $\mathcal{I}_2$ is here seen to be a pseudoinvariant.

4. Hexaspherical coordinates and conformal inversion in the space $Q^{(6)}$

4.1. Coordinate transformations

Hexaspherical coordinates, or q-coordinates, are defined conveniently for the eventual process of dimensional reduction. For $q \in \mathbf{R}^6$, write $q = (q^a)$, with the index

$a = 0, 1, 2, 3, +, -$. Then for $y \in Y^{(6)}$ with $y^4 + y^5 \neq 0$, define

$$q^\mu = \frac{y^\mu}{y^4 + y^5} \ (a = \mu = 0, 1, 2, 3); \ \ q^+ = y^4 + y^5; \ \ q^- = \frac{y_m y^m}{(y^4 + y^5)^2} \,. \tag{27}$$

The projective equivalence in $Y^{(6)}$ becomes in $Q^{(6)}$ simply

$$\left(q^0, q^1, q^2, q^3, q^+, q^-\right) \sim \left(q^0, q^1, q^2, q^3, \lambda q^+, q^-\right), \quad \lambda \neq 0. \tag{28}$$

When we take q^- to zero, we have the light cone in $Q^{(6)}$; when we additionally take $q^+ \sim \lambda q^+$, we have the projective light cone and recover Minkowski space.

The inverse coordinate transformation, as well as some later equations, are written more concisely if we introduce the notations

$$(q, q) = (q^0)^2 - \sum_{k=1}^{3} (q^k)^2 \,, \ \text{ and } \ Q_\pm = (q, q) \pm q^- \,. \tag{29}$$

Then

$$y^\mu = q^+ q^\mu \ (m = \mu = 0, 1, 2, 3); \ \ y^4 = q^+ \frac{1 + Q_-}{2}; \ \ y^5 = q^+ \, \frac{1 - Q_-}{2} \,. \tag{30}$$

The Jacobian matrix for this transformation, defined by

$$dy^m = \frac{\partial y^m}{\partial q^a} dq^a = J_a^m\left(q\right) dq^a \,, \tag{31}$$

is given (for rows $m = \mu, 4, 5$; and columns $a = \nu, +, -$) by

$$J_a^m\left(q\right) = \begin{pmatrix} q^+ \delta_\nu^\mu & q^\mu & 0 \\ q^+ n_{\nu\sigma} q^\sigma & \dfrac{1 + Q_-}{2} & -q^+/2 \\ -q^+ n_{\nu\sigma} q^\sigma & \dfrac{1 - Q_-}{2} & q^+/2 \end{pmatrix} ; \tag{32}$$

where $n_{\nu\sigma} = \operatorname{diag}\left[1, -1, -1, -1\right]$. The inverse Jacobian matrix expressed in q-coordinates, i.e., $\bar{J}_m^a\left(q\right) = J_m^{-1,a}\left(y\left(q\right)\right)$, is then given (for rows $a = \nu, +, -$; and columns $m = \mu, 4, 5$) by

$$\bar{J}_m^a\left(q\right) = \begin{pmatrix} \dfrac{1}{q^+} \delta_\mu^\nu & -q^\nu/q^+ & -q^\nu/q^+ \\ 0 & 1 & 1 \\ \dfrac{2 n_{\mu\sigma} q^\sigma}{q^+} & -\dfrac{1 + Q_+}{q^+} & \dfrac{1 - Q_+}{q^+} \end{pmatrix} . \tag{33}$$

In $Q^{(6)}$, the metric tensor (used to raise or lower indices) is no longer flat. In fact,

$$g_{ab}\left(q\right) = J_a^m\left(q\right) \eta_{mn} J_b^n\left(q\right) = \begin{pmatrix} \left(q^+\right)^2 n_{\mu\nu} & 0 & 0 \\ 0 & q^- & \dfrac{q^+}{2} \\ 0 & \dfrac{q^+}{2} & 0 \end{pmatrix} , \tag{34}$$

while (with raised indices),

$$g^{ab}(q) = \begin{pmatrix} \frac{1}{(q^+)^2} n_{\mu\nu} & 0 & 0 \\ 0 & 0 & \frac{2}{q^+} \\ 0 & \frac{2}{q^+} & -\frac{4q^-}{(q^+)^2} \end{pmatrix}. \tag{35}$$

We remark that the coordinate q^+ appears explicitly in $\det[g^{ab}] = 4/(q^+)^{10} = (\det\bar{J})^2$, a fact that is important later.

Our next task is to express in q-coordinates the invariant functionals $\mathcal{I}_1(y)$ and $\mathcal{I}_2(y)$ given by (24), for which we of course need the field strength tensors in q-coordinates. We write the fields $A_a(q)$ and $F_{ab}(q)$ in terms of $\mathcal{A}_m(y)$ and $\mathcal{F}_{mn}(y)$ using the above Jacobian matrices, $A_a(q) = J_a^m(q(y))\mathcal{A}_m(y)$ and $F_{ab}(q) = J_a^m(q(y))\mathcal{F}_{mn}(y)J_b^n(q(y))$. We have the corresponding inverse transformations,

$$\mathcal{A}_m(y) = A_a(q)\bar{J}_m^a(q), \quad \mathcal{F}_{mn}(y) = \bar{J}_m^a(q)F_{ab}(q)\bar{J}_n^b(q). \tag{36}$$

From these equations, it is not hard to demonstrate that $F_{ab}(q) = \partial_a A_b - \partial_b A_a$ (where $\partial_a = \partial/\partial q^a$), using the fact that $\partial_a J_b^n - \partial_b J_a^n = 0$.

In addition, substituting (36) into (24), one may demonstrate explicitly that in $Q^{(6)}$, the invariants (24) take the form,

$$\begin{aligned} I_1(q) &= \frac{1}{2} F_{ab}(q)F^{ab}(q) = \frac{1}{2} g^{ac}g^{bd}F_{ab}(q)F_{cd}(q), \\ I_2(q) &= \frac{1}{(q^+)^5} \epsilon^{abcdeg}F_{ab}(q)F_{cd}(q)F_{eg}(q) \\ &= \frac{1}{2}(\det \bar{J})\, \epsilon^{abcdeg}F_{ab}(q)F_{cd}(q)F_{eg}(q). \end{aligned} \tag{37}$$

Note that ϵ is the Levi-Civita *symbol*. The Levi-Civita *tensor* with raised indices is defined generally as $(1/\sqrt{|g|})\epsilon$, where $g = \det[g_{ab}]$. Here this becomes $(\det \bar{J})\, \epsilon^{abcdeg}$.

4.2. Conformal inversion in $Q^{(6)}$

The conformal inversion transformation contains most of the essential information for a subsequent discussion of nonlinear electrodynamics. From (20), we obtain the formula for conformal inversion in $Q^{(6)}$,

$$q'^{\mu} = \frac{q^\mu}{Q_-}, \quad q'^{+} = q^+Q_-, \quad q'^{-} = \frac{q^-}{Q_-^2}. \tag{38}$$

Recalling that $Q_- = (q,q) - q^-$, we also have

$$Q'_- = \frac{1}{Q_-}. \tag{39}$$

The remaining steps are to express the fields $A'(q')$ and $F'(q')$, transformed under conformal inversion, in terms of $A(q)$ and $F(q)$ respectively, and then to explore the dimensional reduction to Minkowski space with attention to the invariants (37). To do this, we use the conformal inversion of the fields in $Y^{(6)}$ given by (25) and (26), together with the above Jacobian matrices; for example, $A'_a(q') = \mathcal{A}'_m(y(q'))J^m_a(q') = K^n_m\mathcal{A}_n(y(q))J^m_a(q')$. The resulting expressions are rather complicated, so we focus here on components especially relevant to the dimensional reduction.

One finds, for example (with $\mu,\nu,\alpha,\sigma = 0,1,2,3$, and repeated Greek indices summed from 0 to 3),

$$\begin{aligned} A'_\nu(q') = A_\nu(q)\,Q_- - 2q^\alpha A_\alpha(q) n_{\nu\sigma} q^\sigma \\ + \, 2A_+(q)\,q^+ n_{\nu\sigma}q^\sigma - 4A_-(q)\,q^- n_{\nu\sigma}q^\sigma , \end{aligned} \tag{40}$$

while

$$\begin{aligned} F'_{\mu\nu}(q') = Q_-^2\,F_{\mu\nu} - 2Q_-\,q^\alpha\,(q_\mu F_{\alpha\nu} + q_\nu\,F_{\mu\alpha}) \\ + \text{ terms in other components of } F. \end{aligned} \tag{41}$$

5. Remarks on the conformal invariants and dimensional reduction

Note that if $q^- \to 0$, then $Q_- \to (q,q)$, and (38) becomes

$$q'^\mu = \frac{q^\mu}{(q,q)}\,, \quad q'^+ = q^+(q,q)\,, \quad q'^- = 0\,. \tag{42}$$

Thus when we move to the light cone in $Q^{(6)}$, identifying the first four components q^μ ($\mu = 0,1,2,3$) with the point $x = (x^\mu) \in M^{(4)}$ and identifying (q,q) with $x_\mu x^\mu$, we recover the formula (4) for conformal inversion in $M^{(4)}$.

The condition $q^- = 0$ is preserved by conformal inversion, as is the equivalence relation $(q^\mu, q^+, 0) \sim (q^\mu, \lambda q^+, 0)$, $\lambda \neq 0$. However, note that the prescription $q^+ = 1$ for selecting a particular element of each equivalence class is *not* invariant under conformal inversion.

Now it is instructive to compare (41) with the corresponding expression (9) in $M^{(4)}$ for $F'_{\mu\nu}(x')$; the two are formally the same (up to the terms included) when Q_- is taken to $(q,q) = q_\rho q^\rho$. However, $I_1(q) = (1/2)F'_{ab}(q)F'^{\,ab}(q)$ defines an invariant under conformal inversion. In contrast, $I_1(x) = (1/2)F'_{\mu\nu}(x)F'^{\,\mu\nu}(x)$ transforms according to (14) and is not invariant.

The reason for this difference is now clear. The metric tensor g in $Q^{(6)}$, given by (35), is applied twice to raise the indices a and b in the expression for $I_1(q)$. This introduces an additional factor of $1/(q^+)^4$ as compared with the corresponding expression for $I_1(x)$ in $M^{(4)}$. Under conformal inversion, $q'^+ = q^+ Q_-$, which reduces to $q^+\,(q,q)$ when $q^- \to 0$. When we then identify q^μ with the coordinates of Minkowski spacetime, the resulting fourth power of $q_\mu q^\mu$ in the denominator restores the invariance under conformal inversion.

Evidently the dimensional reduction procedure for conformal invariant nonlinear Maxwell theories in $(4+2)$-dimensional spacetime, with compactified Minkowski space identified with the projective light cone, must take account of the fact that setting $q^+ = 1$ (as a device for handling the projective equivalence) is inconsistent with the desired conformal symmetry. This is important if we are to write nonlinear constitutive equations in terms of the $(4+2)$-dimensional invariants.

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Fourier Transforms of E-functions of $O(5)$ and $G(2)$

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Abstract. The discrete Fourier transforms of the six families of E-functions of the groups $O(5)$ and $G(2)$ is summarized. The six types are shown to be generalizations of the Euler formula for the complex exponential function. The fragments of the dual weight lattices, which can be of any density, form the points of the discrete Fourier calculus. Application of the discrete Fourier transforms to interpolation is presented and exemplified on a model function.

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1. Introduction

The standard discrete Fourier analysis of one real variable is known to be a very valuable tool in mathematics, physics and in practical applications [2, 12]. Various approaches can be used to generalize this analysis in higher dimensions. The approach based on Weyl groups has the one-dimensional case as well as its straightforward Cartesian product generalizations as special cases. Four types of the multidimensional generalizations of cosine and sine functions, which can be defined for the root systems with two different lengths of roots, form the core of this approach. Some of these types appeared in various contexts in the literature [1, 4, 7–9, 11]. The standard Euler formula for the complex exponential function is used as a model for the definition of the six types of E-functions using the four types of sines and cosines.

The discrete and continuous Fourier calculus of the six types of E-functions, formulated explicitly for rank two cases $O(5)$ and $G(2)$, is contained in [3]. The discrete Fourier calculus of the one type, which was originally denoted by E-functions and which exists for all root systems, is done in full generality in [5]. This paper focuses on the discrete Fourier calculus and summarizes this calculus for all six types.

An important extension of [3] is considered: the most at hand application to the interpolation of any complex function is formulated and exemplified. The success of the interpolation method indicates possible usefulness of this two-dimensional generalization to other fields where the standard discrete Fourier calculus is widely used.

The paper is organized as follows. In Section 2, the notions and notations which are necessary for the mathematical exposition are summarized. In Section 3, the six types of E-functions are introduced via generalization of Euler's formula. In Section 4, the six types of fragments of the dual weight lattices and the corresponding sets of weights are defined and the discrete orthogonality formulated. In Section 5, the interpolating functions are defined and determined. For two cases, the interpolation formulas are applied to a specific model function.

2. The fundamental domains

Consider the Lie algebra of the compact simply connected simple Lie group of rank two. Its set of simple roots $\Delta = (\alpha_1, \alpha_2)$ forms a basis of the Euclidean vector space $\mathbb{R}^2$ equipped with the standard scalar product $\langle\, ,\, \rangle$. Only two simple algebras of Lie groups $O(5)$ and $G(2)$ which have two different lengths of roots are considered and are denoted standardly as C_2 and G_2. For these algebras the set of simple roots consists of the short simple root α_s and the long simple root α_l. We use the standard ordering of the root systems

$$\Delta(C_2) = (\alpha_1, \alpha_2) = (\alpha_s, \alpha_l)$$
$$\Delta(G_2) = (\alpha_1, \alpha_2) = (\alpha_l, \alpha_s).$$

To unify the notation, we list the following quantities which can be deduced from the entire root system Δ by conventional methods: the Cartan matrix C, the highest root $\xi \equiv -\alpha_0 = m_1\alpha_1 + m_2\alpha_2$, the root lattice $Q = \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_2$, the $\mathbb{Z}$-dual lattice $P^\vee = \mathbb{Z}\omega_1^\vee + \mathbb{Z}\omega_2^\vee$, the dual root lattice $Q^\vee = \mathbb{Z}\alpha_1^\vee + \mathbb{Z}\alpha_2^\vee$, where $\alpha_i^\vee = 2\alpha_i/\langle\alpha_i,\, \alpha_i\rangle$ and the weight lattice $P = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. The set of vectors $\{\alpha_1^\vee, \alpha_2^\vee\}$ also forms a root system called the dual root system $\Delta^\vee$ of G. This dual root system determines the highest dual root $\eta \equiv -\alpha_0^\vee = m_1^\vee\alpha_1^\vee + m_2^\vee\alpha_2^\vee$.

The reflections r_α, $\alpha \in \Delta$ are given as reflections in one-dimensional 'mirrors' orthogonal to the simple roots and intersecting at the origin. Similarly are defined the reflections r_ξ, r_η of the highest root ξ and the highest dual root r_η. The Weyl group W is generated by reflections r_α, $\alpha \in \Delta$. The affine Weyl group W^{aff} is a semidirect product $W^{\mathrm{aff}} = Q^\vee \rtimes W$ and is generated by the reflections r_α and the affine reflection r_0, which is composed of the reflection r_ξ and the shift by $2\xi/\langle\xi,\, \xi\rangle$. The fundamental region $F \subset \mathbb{R}^2$ of W^{aff} consists of precisely one point from each W^{aff}-orbit and can be chosen as the triangle of the form $F = \left\{0, \frac{\omega_1^\vee}{m_1}, \frac{\omega_2^\vee}{m_2}\right\}_\kappa$. The borders of the triangle F which are stabilized by r_s, r_l, i.e., orthogonal to α_s and α_l are denoted by Y_s and Y_l and the borders which are stabilized with respect to the affine reflections r_0, $r_0^\vee$ are denoted by Y_0, $Y_0^\vee$. Similarly, by $Y_s^\vee$, $Y_l^\vee$ and $Y_0^\vee$,

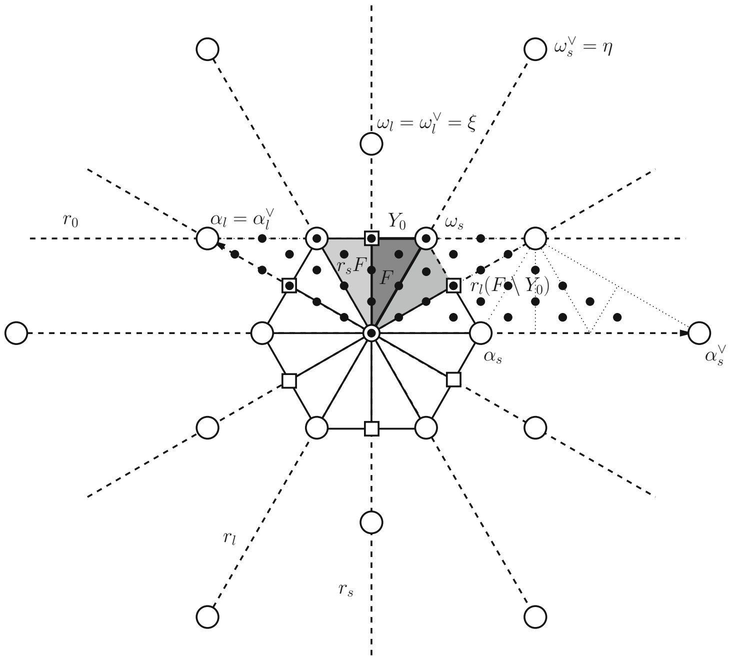


FIGURE 1. The fundamental region F and its reflections r_sF and $r_l(F \setminus Y_0)$ of G_2. The fundamental domain F is depicted as the dark gray triangle and its reflections as lighter gray triangles. The coset representatives of $\frac{1}{6}P^\vee/Q^\vee$ are depicted as 36 black dots. The dashed lines represent 'mirrors' r_0, r_s and r_l.

are denoted the borders of $F^\vee$. Crucial are the following six domains

$$\begin{aligned}
F^{e+} &= F \cup r_sF^\circ, & F^{e-} &= (F \setminus \{(Y_l \cup Y_0) \cap Y_s\}) \cup r_sF^\circ, \\
F^{s+} &= F \cup r_s(F \setminus Y_s), & F^{s-} &= (F \setminus (Y_l \cup Y_0)) \cup r_sF^\circ, \\
F^{l+} &= F \cup r_l(F \setminus (Y_l \cup Y_0)), & F^{l-} &= (F \setminus Y_s) \cup r_lF^\circ.
\end{aligned} \tag{1}$$

The fundamental domain F together with its reflected copies r_sF and $r_l(F \setminus Y_0)$ and the root system of G_2 are depicted in Figure 1. The six dual counterparts of the six domains are the following

$$\begin{aligned}
F^{e+\vee} &= F^\vee \cup r_sF^{\vee\circ}, & F^{e-\vee} &= (F^\vee \setminus \{(Y_s^\vee \cup Y_0^\vee) \cap Y_l^\vee\}) \cup r_sF^{\vee\circ}, \\
F^{s+\vee} &= F^\vee \cup r_s(F^\vee \setminus (Y_s^\vee \cup Y_0^\vee)), & F^{s-\vee} &= (F^\vee \setminus Y_l^\vee) \cup r_sF^{\vee\circ}, \\
F^{l+\vee} &= F^\vee \cup r_l(F^\vee \setminus Y_l^\vee), & F^{l-\vee} &= (F^\vee \setminus (Y_s^\vee \cup Y_0^\vee)) \cup r_lF^{\vee\circ}.
\end{aligned} \tag{2}$$

3. Six types of *E*-functions

Considering a weight $b \in P$ and $a \in \mathbb{R}^2$, the normalized C-function and the S-function are given by

$$\Phi_b(a) = \sum_{w \in W} e^{2\pi i \langle wb,\, a \rangle}, \quad \varphi_b(a) = \sum_{w \in W} (\det w) e^{2\pi i \langle wb,\, a \rangle}.$$

Two 'sign' homomorphisms $\sigma^s, \sigma^l : W \to \{\pm 1\}$ are defined [3, 11] by their values on the generating reflections r_s, r_l of W

$$\sigma^s(r_s) = -1\,, \quad \sigma^s(r_l) = 1\,, \tag{3}$$

$$\sigma^l(r_l) = -1\,, \quad \sigma^l(r_s) = 1\,. \tag{4}$$

These sign homomorphisms σ^s and σ^l determine the S^s-functions and the S^l-functions

$$\varphi^s_b(a) = \sum_{w \in W} \sigma^s(w)\, e^{2\pi i \langle wb,\, a \rangle}, \quad \varphi^l_b(a) = \sum_{w \in W} \sigma^l(w)\, e^{2\pi i \langle wb,\, a \rangle}.$$

The detailed review of C-functions is contained in [8] and the S-functions, which enter the Weyl character formula, are reviewed in [9]. The six types of E-functions are obtained via generalization of the Euler formula $e^{ix} = \cos x + i \sin x$ – see also [10]. For $b \in P$ and $a \in \mathbb{R}^2$ we define these six new families by the relations

$$\Xi^{e+}_b(a) = \frac{1}{2}(\Phi_b(a) + \varphi_b(a)), \qquad \Xi^{e-}_b(a) = \frac{1}{2}(\varphi^l_b(a) + \varphi^s_b(a)),$$

$$\Xi^{s+}_b(a) = \frac{1}{2}(\Phi_b(a) + \varphi^s_b(a)), \qquad \Xi^{s-}_b(a) = \frac{1}{2}(\varphi_b(a) + \varphi^s_b(a)),$$

$$\Xi^{l+}_b(a) = \frac{1}{2}(\Phi_b(a) + \varphi^l_b(a)), \qquad \Xi^{l-}_b(a) = \frac{1}{2}(\varphi_b(a) + \varphi^l_b(a))\,.$$

The family of Ξ^{e+}-functions is called in literature simply E-functions and some of its properties are detailed in [5, 6, 8]. Contour plots of some lower Ξ^{s+}-, Ξ^{l+}-, Ξ^{e-}- and Ξ^{s-}-functions of the group G_2 are depicted in Figures 2, 3, 4 and 5. Note that for the case of the root system A_1, the standard Euler formula is recovered.

4. Discrete orthogonality of *E*-functions

This section summarizes the discrete orthogonality results of six types of E-functions from [3, 5]. Let us consider the values of six types of E-functions on certain finite grids inside the corresponding regions (1). For an arbitrary $M \in \mathbb{N}$, the quotient group $\frac{1}{M}P^\vee/Q^\vee$ is a W-invariant finite group. The six finite grids contain such elements from $\frac{1}{M}P^\vee/Q^\vee$ which have representative points in the

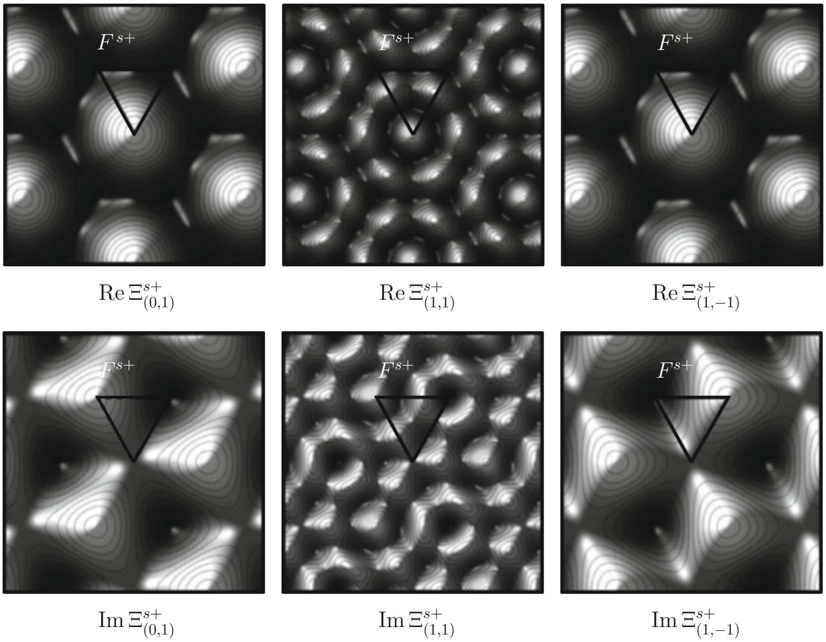


FIGURE 2. The contour plots of Ξ^{s+}-functions of G_2. The black triangle, with the dashed border excluded, forms the fundamental domain F^{s+}.

corresponding regions (1)

$$\begin{aligned} F^{e+}_M &= \frac{1}{M}P^\vee/Q^\vee \cap F^{e+}, & F^{e-}_M &= \frac{1}{M}P^\vee/Q^\vee \cap F^{e-}, \\ F^{s+}_M &= \frac{1}{M}P^\vee/Q^\vee \cap F^{s+}, & F^{s-}_M &= \frac{1}{M}P^\vee/Q^\vee \cap F^{s-}, \\ F^{l+}_M &= \frac{1}{M}P^\vee/Q^\vee \cap F^{l+}, & F^{l-}_M &= \frac{1}{M}P^\vee/Q^\vee \cap F^{l-}. \end{aligned} \tag{5}$$

The group $\frac{1}{6}P^\vee/Q^\vee$, the simple roots and the weights are depicted in Figure 1.

When the variables of the six types of E-functions are restricted on the grids (5), they can be labeled only by the labels from the finite subsets of the weight lattice P. Since the restriction of the E-functions on the grids (5) induces a shifting symmetry of the E-functions by the lattice MQ, the weights can be handled as representing elements of W-invariant finite quotient group P/MQ. The six types of the dual domains (2) induce contain pertinent representative elements of P/MQ for each case

$$\begin{aligned} \Lambda^{e+}_M &= MF^{e+\vee} \cap P/MQ, & \Lambda^{e-}_M &= MF^{e-\vee} \cap P/MQ, \\ \Lambda^{s+}_M &= MF^{s+\vee} \cap P/MQ, & \Lambda^{s-}_M &= MF^{s-\vee} \cap P/MQ, \\ \Lambda^{l+}_M &= MF^{l+\vee} \cap P/MQ, & \Lambda^{l-}_M &= MF^{l-\vee} \cap P/MQ. \end{aligned} \tag{6}$$

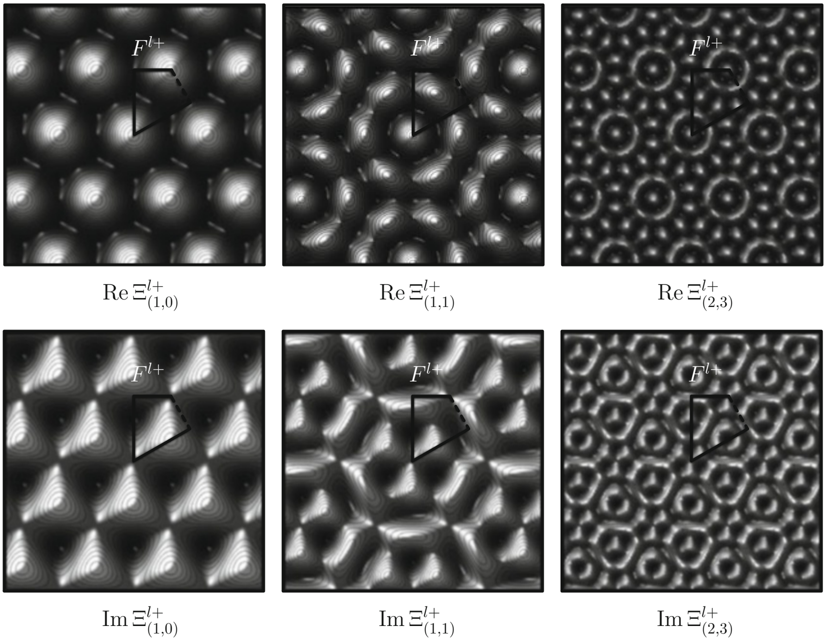


FIGURE 3. The contour plots of Ξ^{l+}-functions of G_2. The black triangle, with the dashed border excluded, forms the fundamental domain F^{l+}.

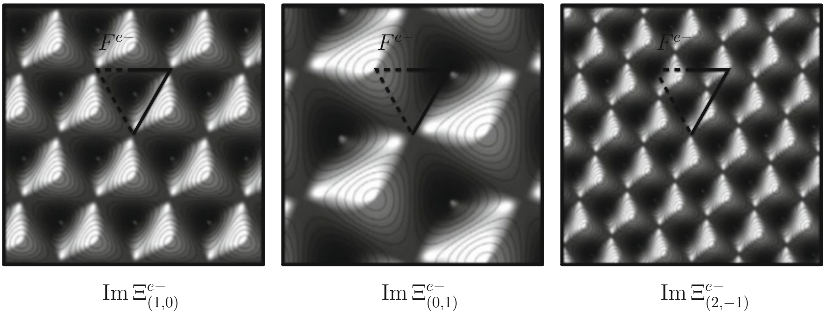
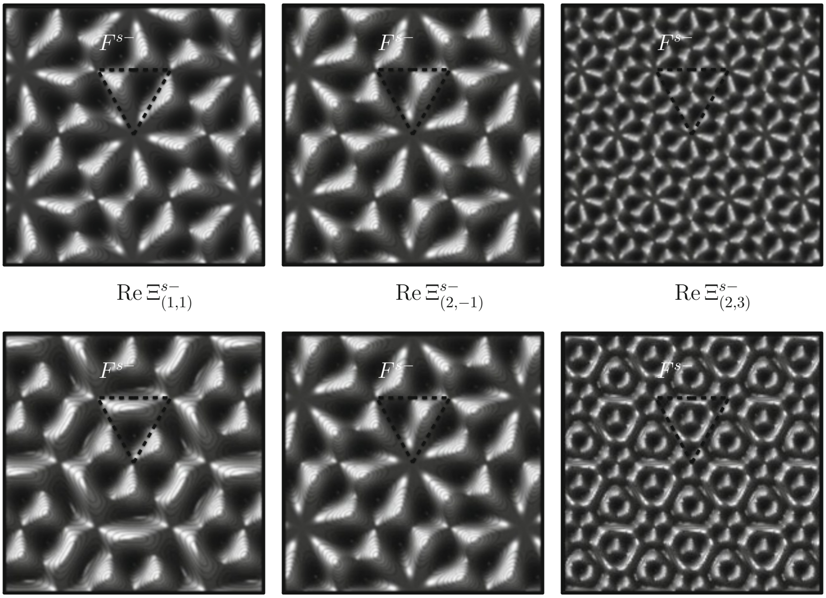


FIGURE 4. The contour plots of Ξ^{e-}-functions of G_2. The black triangle, with the dashed border excluded, forms the fundamental domain F^{e-}. Real parts of these functions are zero.

Analyzing the number of points of (5) and (6) one can derive that the corresponding pairs contain the same number of points

$$\begin{aligned} |\Lambda^{e+}_M| &= |F^{e+}_M|, \quad |\Lambda^{e-}_M| = |F^{e-}_M|, \\ |\Lambda^{s+}_M| &= |F^{s+}_M|, \quad |\Lambda^{s-}_M| = |F^{s-}_M|, \\ |\Lambda^{l+}_M| &= |F^{l+}_M|, \quad |\Lambda^{l-}_M| = |F^{l-}_M|. \end{aligned}$$



$\mathrm{Re}\, \Xi^{s-}_{(1,1)}$ $\mathrm{Re}\, \Xi^{s-}_{(2,-1)}$ $\mathrm{Re}\, \Xi^{s-}_{(2,3)}$

$\mathrm{Im}\, \Xi^{s-}_{(1,1)}$ $\mathrm{Im}\, \Xi^{s-}_{(2,-1)}$ $\mathrm{Im}\, \Xi^{s-}_{(2,3)}$

FIGURE 5. The contour plots of Ξ^{s-}-functions of G_2. The black triangle, with the dashed border excluded, forms the fundamental domain F^{s-}.

The discrete orthogonality of the six types of E-functions has the following form

$$\sum_{x\in F_M^{e+}} \varepsilon^e(x)\Xi_\lambda^{e+}(x)\overline{\Xi_{\lambda'}^{e+}(x)} = kM^2 h_\lambda^{e\vee}\delta_{\lambda\lambda'}, \quad \lambda,\lambda' \in \Lambda_M^{e+}$$
$$\sum_{x\in F_M^{s+}} \varepsilon^s(x)\Xi_\lambda^{s+}(x)\overline{\Xi_{\lambda'}^{s+}(x)} = kM^2 h_\lambda^{s\vee}\delta_{\lambda\lambda'}, \quad \lambda,\lambda' \in \Lambda_M^{s+}$$
$$\sum_{x\in F_M^{l+}} \varepsilon^l(x)\Xi_\lambda^{l+}(x)\overline{\Xi_{\lambda'}^{l+}(x)} = kM^2 h_\lambda^{l\vee}\delta_{\lambda\lambda'}, \quad \lambda,\lambda' \in \Lambda_M^{l+}$$
$$\sum_{x\in F_M^{e-}} \varepsilon^e(x)\Xi_\lambda^{e-}(x)\overline{\Xi_{\lambda'}^{e-}(x)} = kM^2 h_\lambda^{e\vee}\delta_{\lambda\lambda'}, \quad \lambda,\lambda' \in \Lambda_M^{e-}$$
$$\sum_{x\in F_M^{s-}} \varepsilon^s(x)\Xi_\lambda^{s-}(x)\overline{\Xi_{\lambda'}^{s-}(x)} = kM^2 h_\lambda^{s\vee}\delta_{\lambda\lambda'}, \quad \lambda,\lambda' \in \Lambda_M^{s-}$$
$$\sum_{x\in F_M^{l-}} \varepsilon^l(x)\Xi_\lambda^{l-}(x)\overline{\Xi_{\lambda'}^{l-}(x)} = kM^2 \delta_{\lambda\lambda'}, \qquad \lambda,\lambda' \in \Lambda_M^{l-}$$

where k is equal to 8 for C_2 and 6 for G_2 and the three types of coefficients $\varepsilon(x)$, $h_\lambda^\vee$ are listed in Table 2 in [3].

5. Discrete E-transforms and interpolation

For any function sampled on the grids (5) we define continuous interpolating functions with variable $x \in \mathbb{R}^2$

$$
\begin{aligned}
I_M^{e+}(x) &= \sum_{\lambda \in \Lambda_M^{e+}} c_\lambda^{e+} \Xi_\lambda^{e+}(x), \quad & I_M^{e-}(x) &= \sum_{\lambda \in \Lambda_M^{e-}} c_\lambda^{e-} \Xi_\lambda^{e-}(x), \\
I_M^{s+}(x) &= \sum_{\lambda \in \Lambda_M^{s+}} c_\lambda^{s+} \Xi_\lambda^{s+}(x), \quad & I_M^{s-}(x) &= \sum_{\lambda \in \Lambda_M^{s-}} c_\lambda^{s-} \Xi_\lambda^{s-}(x), \\
I_M^{l+}(x) &= \sum_{\lambda \in \Lambda_M^{l+}} c_\lambda^{l+} \Xi_\lambda^{l+}(x), \quad & I_M^{l-}(x) &= \sum_{\lambda \in \Lambda_M^{l-}} c_\lambda^{l-} \Xi_\lambda^{l-}(x).
\end{aligned} \tag{7}
$$

The interpolating functions (7) are defined as finite linear combinations of basis functions with expansion coefficients whose values need to be determined from the condition that (7) coincide with f on the grids (5). The formulas for calculation of the expansion coefficients, which follow from the discrete orthogonality relations and can be also viewed as discrete E-transforms, are of the form

$$
c_\lambda^{e+} = (k\, M^2 h_\lambda^{e\vee})^{-1} \sum_{x \in F_M^{e+}} \varepsilon^e(x) f(x) \overline{\Xi_\lambda^{e+}(x)}
$$

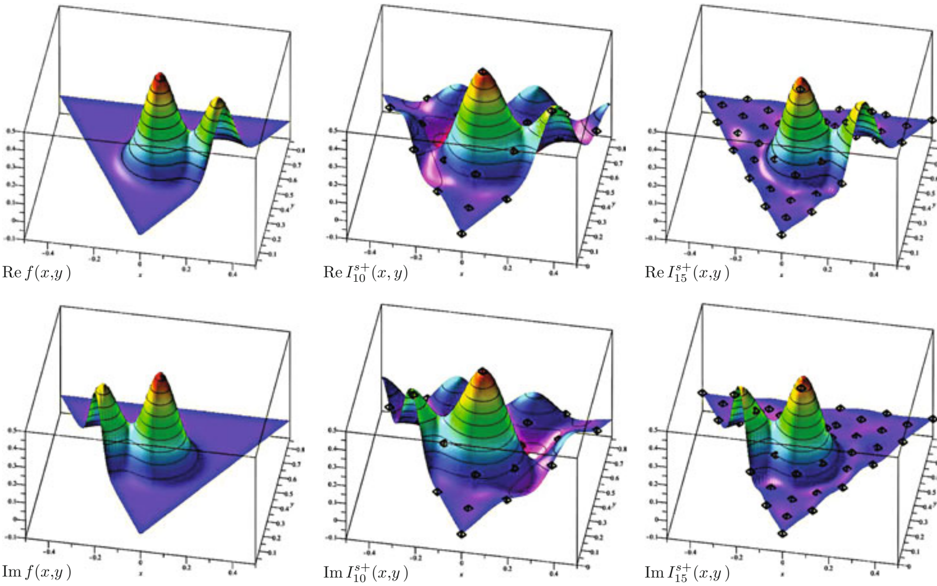


FIGURE 6. The contour plots of interpolation by Ξ^{s+}-functions of G_2. The model functions f is plotted over the domain F^{s+}. The black diamonds represent the discrete function which is the sample of f at the points F_M^{s+} and is interpolated by I_M^{s+} for $M = 10$, 15.

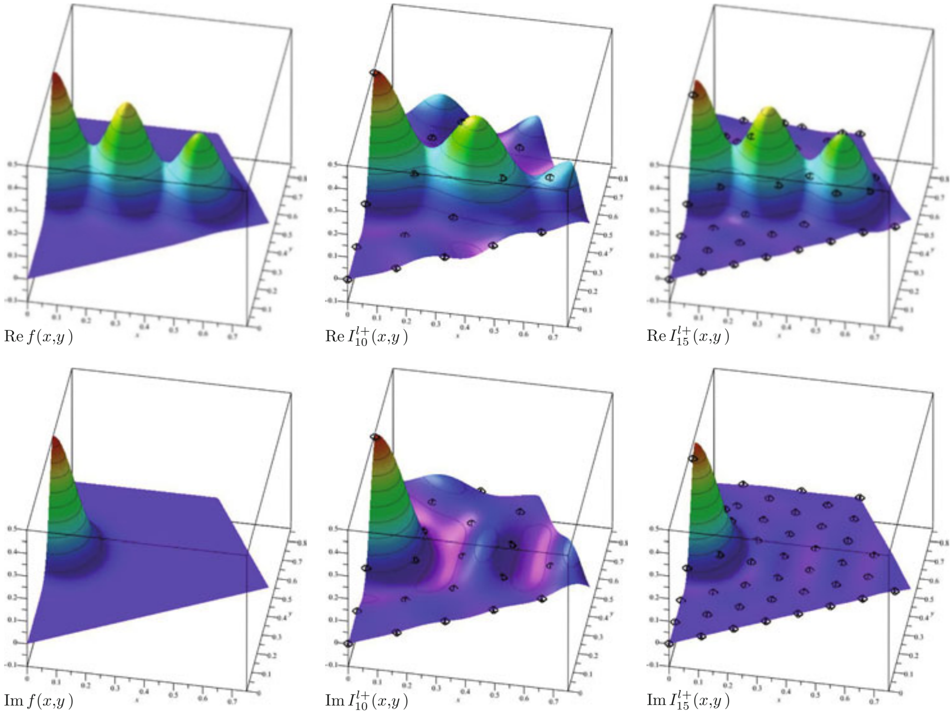


FIGURE 7. The contour plots of interpolation by Ξ^{l+}-functions of G_2. The model functions f is plotted over the domain F^{l+}. The black diamonds represent the discrete function which is the sample of f at the points F^{l+}_M and is interpolated by I^{l+}_M for $M = 10$, 15.

$$c^{s+}_\lambda = (k\, M^2 h^{s\vee}_\lambda)^{-1} \sum_{x\in F^{s+}_M} \varepsilon^s(x) f(x) \overline{\Xi^{s+}_\lambda(x)}$$

$$c^{l+}_\lambda = (k\, M^2 h^{l\vee}_\lambda)^{-1} \sum_{x\in F^{l+}_M} \varepsilon^l(x) f(x) \overline{\Xi^{l+}_\lambda(x)}. \tag{8}$$

The formulas for the remaining three cases are similar.

Consider the following three parameter function of two variables x and y

$$g_{(a,b,c)}(x,y) = ae^{-100((x-b)^2+(y-c)^2)}.$$

The two-variable complex model function f can be written as

$$f = g_{(\frac{1}{2},0,\frac{1}{2})} + g_{(\frac{2}{5},\frac{1}{4},\frac{1}{2})} + g_{(\frac{3}{10},\frac{1}{2},\frac{1}{2})} + i[g_{(\frac{1}{2},0,\frac{1}{2})} + g_{(\frac{2}{5},-\frac{1}{4},\frac{1}{2})} + g_{(\frac{3}{10},-\frac{1}{2},\frac{1}{2})}].$$

This function is sampled on the domains F^{s+}_M and F^{l+}_M of G_2 and the interpolating functions I^{s+}_M and I^{l+}_M are calculated from (8) and depicted in Figures 6 and 7.

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Interpolation of Multidimensional Digital Data Using Weyl Group Orbit Functions

Lenka Háková and Jiří Hrivnák

Abstract. Orbit functions are families of special functions related to the Weyl groups of simple Lie algebras. They are complex functions depending on n variables where n is the rank of the underlying Lie algebra. They possess several remarkable properties, among them a discrete orthogonality when sampled on a lattice fragment of a domain in $\mathbb{R}^n$. This allows applications of orbit functions in processing of digital data. We present a method for an interpolation of discrete functions using the family of so-called S^l-function defined by the Weyl group of the Lie algebra B_3.

Mathematics Subject Classification (2010). Primary 42B10; Secondary 43A75.

Keywords. Orbit functions, Fourier transform, interpolation.

1. Introduction

Several families of multi-variables special functions have their origin in orbits of Weyl groups W of simple Lie algebras. They can be understood as (anti)symmetric (by means of the group W) exponential functions. The number of continuous variables of each family is equal to the rank of the underlying Lie algebra. The symmetric (C-) and antisymmetric (S-) orbit functions are related to the Weyl character formula for irreducible representations of simple Lie algebras. They were fully described in [1] and [2]. Two other families were described in [3]. These so-called S^s- and S^l-functions are defined for simple Lie algebras with the root system of two root lengths. All families of orbit functions possess several remarkable properties. In particular, they are invariant or antiinvariant with respect to the affine Weyl group. They have continuous derivatives of all orders. The most advantageous property for the practical applications is the continuous orthogonality of orbit functions with respect to the integration over the fundamental domain F of the corresponding affine Weyl group.

For a discretization of a bounded region we need, in general, a finite set of points in the region and a family of functions which is pairwise orthogonal when summed over these points. For given digital data (a discrete function f) we can perform discrete Fourier-like analysis. The function f is expanded into n-dimensional finite series of orbit functions. By replacing the discrete variable by a continuous one we obtain interpolations of the digital data. This idea applies equally well to any dimension, to lattices of any symmetry and any density. This method was described in detail in [5, 6].

In this paper we extend the results from [4]. We apply the discretization on the family of S^l-orbit functions of the Lie algebra B_3 on three-dimensional discrete functions to illustrate the interpolation property. The paper is organized as follows. Section 2 summarizes important facts about Weyl groups and orbit functions. In Section 3 we recall the discretization of orbit functions. We conclude with an example of S^l-interpolation.

2. Weyl group orbit functions

2.1. Weyl groups

There are two infinite families of simple Lie algebras and two exceptional ones such that their root systems contain roots of two different lengths. Their Dynkin diagrams are depicted on Figure 1.

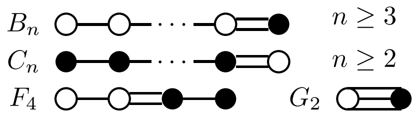

FIGURE 1. Dynkin diagrams of simple Lie algebras with two different lengths of roots in their root system. The meaning of open or full circles and of single or multiple lines is given in Ref. [1].

The set of simple roots (corresponding to the open circles in the diagram) in $\mathbb{R}^n$ is denoted by $\Delta = \{\alpha_1, \dots, \alpha_n\} = \Delta_s \cup \Delta_l$, where s stands for short and l for long roots. The coroots are defined as $\alpha_i^\vee = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}$, the weights ω_i and coweights $\omega_i^\vee$ are orthogonal to the coroots and roots in the sense $\langle \alpha_i, \omega_j^\vee \rangle = \langle \alpha_i^\vee, \omega_j \rangle = \delta_{ij}$. We define four lattices in $\mathbb{R}^n$:

$$Q = \bigoplus_i \mathbb{Z}\alpha_i, \quad Q^\vee = \bigoplus_i \mathbb{Z}\alpha_i^\vee, \quad P = \bigoplus_i \mathbb{Z}\omega_i, \quad P^\vee = \bigoplus_i \mathbb{Z}\omega_i^\vee.$$

The reflections r_i with respect to hyperplanes orthogonal to simple roots and passing through the origin generate a Weyl group W. Its infinite extension by shifts by elements of $Q^\vee$ is called the affine Weyl group, $W^{\mathrm{aff}} = Q^\vee \rtimes W$. Let F denote a fundamental region of W^{aff}.

2.2. Sign homomorphism and orbit functions

The sign homomorphisms are homomorphisms $\sigma : W \to \{\pm 1\}$. For the Weyl group with two lengths of roots there are four possibilities:

$$\mathbf{1}(r_i) = 1, \qquad \alpha_i \in \Delta, \qquad \sigma^e(r_i) = -1, \qquad \alpha_i \in \Delta,$$

$$\sigma^s(r_i) = \begin{cases} 1, & \alpha_i \in \Delta_l, \\ -1, & \alpha_i \in \Delta_s, \end{cases} \qquad \sigma^l(r_i) = \begin{cases} 1, & \alpha_i \in \Delta_s, \\ -1, & \alpha_i \in \Delta_l. \end{cases}$$

Four families of orbit functions, labeled by $\lambda \in P$, are defined as

$$\varphi_\lambda^\sigma(x) = \sum_{w \in W} \sigma(w) e^{2\pi i \langle w\lambda,\, x\rangle}, \quad x \in \mathbb{R}^n, \tag{1}$$

where σ is one of the above homomorphisms. Each family is invariant with respect to the shifts by elements of $Q^\vee$ and (anti-)invariant with respect to W. Therefore, we consider the functions on a subset F^σ of F, we exclude such points of F which are common zeros for the corresponding family of orbit functions.

3. Discrete orthogonality and discrete transform

We fix a Weyl group, a family of orbit functions and an integer M. We define two finite lattice fragments in $\mathbb{R}^n$: grid of points and grid of parameters, with the same order. We consider a space of functions sampled on the grid of points with a discrete scalar product. The grid of parameters specifies the set of orbit functions which are pairwise orthogonal. The method described in [5, 6] defines the grid of points as $F_M^\sigma = \frac{1}{M} P^\vee / Q^\vee \cap F^\sigma$ and the set of weights as $\Lambda_M^\sigma = P/MQ \cap MF^{\sigma\vee}$ ($F^\vee$ is a fundamental domain of the dual affine Weyl group).

The discrete scalar product is defined as

$$\langle f,\, g\rangle_{F_M^\sigma} = \sum_{x \in F_M^\sigma} \varepsilon(x) f(x) \overline{g(x)},$$

where $\varepsilon(x)$ denotes the order of the orbit of the action of W on x. Then for every $\lambda, \lambda' \in \Lambda_M^\sigma$ it holds that

$$\langle \varphi_\lambda^\sigma,\, \varphi_{\lambda'}^\sigma \rangle_{F_M^\sigma} = c|W| M^n h_\lambda^\vee \delta_{\lambda\lambda'},$$

where the following notation is used: $|W|$ denotes the order of the Weyl group, c is the determinant of its Cartan matrix and $h_\lambda^\vee$ is the order of the stabilizer of $\lambda \in P/MQ$ by the action of W.

The discrete transform of a function f sampled on F_M^σ is given by

$$I_M^\sigma(x) = \sum_{\lambda \in \Lambda_M^\sigma} c_\lambda \varphi_\lambda^\sigma(x),$$

where the coefficients c_λ are such that $f(x) = I_M^\sigma(x)$ for $x \in F_M^\sigma$, i.e.,

$$c_\lambda = \frac{1}{c|W| M^n h_\lambda^\vee} \sum_{x \in F_M^\sigma} \varepsilon(x) f(x) \overline{\varphi_\lambda^\sigma(x)}.$$

The functions I^{σ}_M become continuous interpolations of the function f by replacing the discrete variable x by the continuous variable.

4. S^l-functions of B_3

The choice of homomorphism $\sigma = \sigma^l$ gives the so-called S^l-functions, denoted by φ^l_λ. We consider the Weyl group of B_3. The discrete grid $F^l_M = F^{\sigma^l}_M$ and the set of weights $\Lambda^l_M = \Lambda^{\sigma^l}_M$ are given explicitly by

$$F^l_M = \left\{ \frac{u^l_1}{M}\omega_1^\vee + \frac{u^l_2}{M}\omega_2^\vee + \frac{u^l_3}{M}\omega_3^\vee \,|\, u^l_0, u^l_1, u^l_2 \in \mathbb{N}, u^l_3 \in \mathbb{Z}^{\geq 0}, u^l_0 + u^l_1 + 2u^l_2 + 2u^l_3 = M \right\}$$

$$\Lambda^l_M = \left\{ t^l_1\omega_1 + t^l_2\omega_2 + t^l_3\omega_3 \,|\, t^l_0, t^l_3 \in \mathbb{Z}^{\geq 0}, t^l_1, t^l_2 \in \mathbb{N}, t^l_0 + 2t^l_1 + 2t^l_2 + t^l_3 = M \right\}.$$

The discrete orthogonality relations are

$$\langle \varphi^l_\lambda, \varphi^l_{\lambda'} \rangle_{F^l_M} = \sum_{x \in F^l_M} \varepsilon(x)\varphi^l_\lambda(x)\overline{\varphi^l_{\lambda'}(x)} = 96M^3 h^\vee_\lambda \delta_{\lambda\lambda'}, \quad \text{for every } \lambda, \lambda' \in \Lambda_M.$$

The values of $\varepsilon(x)$ and $h^\vee_\lambda$ are listed in Table 1.

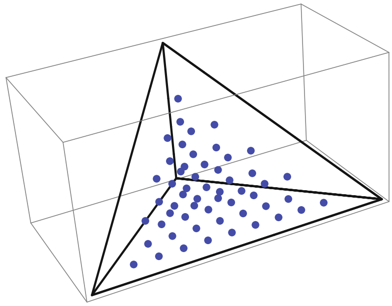

FIGURE 2. The grid F^l_{12} of B_3.

5. Example of S^l-interpolation

We will use the following smooth function $f : \mathbb{R}^3 \to \mathbb{R}$ to define our sample function

$$f_{\alpha,\beta,\gamma,x_0,y_0,z_0}(x, y, z) = \begin{cases} \gamma & \text{if } r < \alpha \\ 0 & \text{if } r > \beta \\ \gamma e \exp\left(\left(\frac{r-\alpha}{\beta-\alpha}\right)^2 - 1\right)^{-1} & \text{otherwise} \end{cases},$$

where $\alpha, \beta, \gamma, x_0, y_0, z_0 \in \mathbb{R}$, $r(x, y, z) = \|(x, y, z) - (x_0, y_0, z_0)\|$.

$x \in F_M^l$	$\varepsilon(x)$
$[u_0^l, u_1^l, u_2^l, u_3^l]$	48
$[u_0^l, u_1^l, u_2^l, 0]$	24
$\lambda \in \Lambda_M^l$	$h_\lambda^\vee$
$[t_0^l, t_1^l, t_2^l, t_3^l]$	1
$[0, t_1^l, t_2^l, t_3^l]$	2
$[t_0^l, t_1^l, t_2^l, 0]$	2
$[0, t_1^l, t_2^l, 0]$	4

TABLE 1. The coefficients $\varepsilon(x)$ and $h_\lambda^\vee$ of B_3. All variables u_i^l, t_i^l, $i = 0, 1, 2, 3$, are assumed to be natural numbers.

We define f^l as a sum of three functions centered at $(1/2, 1/3, 1/15)$, $(2/3, 1/6, 1/15)$ and $(1/3, 1/6, 1/15)$, respectively. The parameters (α, β, γ) are chosen as $(1/20, 1/10, 1)$, $(1/30, 1/15, 3/4)$ and $(1/50, 1/15, 1/2)$. The graph cut at $z = 1/15$ is depicted in Figure 3. This function is defined in the fundamental domain F of B_3. We sample it on F_M^l and we compute the interpolating functions I_M^l:

$$I_M^l = \sum_{\lambda \in \Lambda_M^l} c_\lambda^l \varphi_\lambda^l(x), \quad \text{where } c_\lambda^l = \frac{1}{96 M^3 h_\lambda^\vee} \sum_{x \in F_M^l} \varepsilon(x) f(x) \overline{\varphi_\lambda^l(x)}.$$

By changing the value of the parameter M we can increase the density of the points in F_M^l and compare the corresponding interpolations with the original function. Figure 4 shows the graph cuts ($z = 1/15$) of the interpolating functions I_M^l of B_3 for the values $M = 10$, 20 and 40.

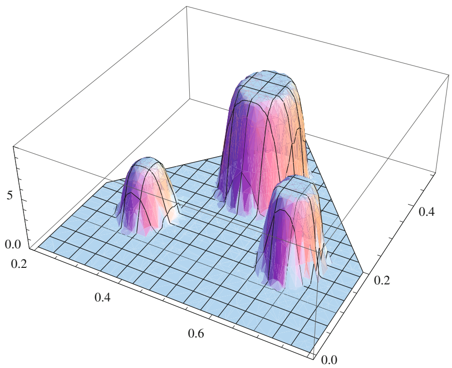

FIGURE 3. The graph cut $\left(z = \frac{1}{15}\right)$ of the function f^l.

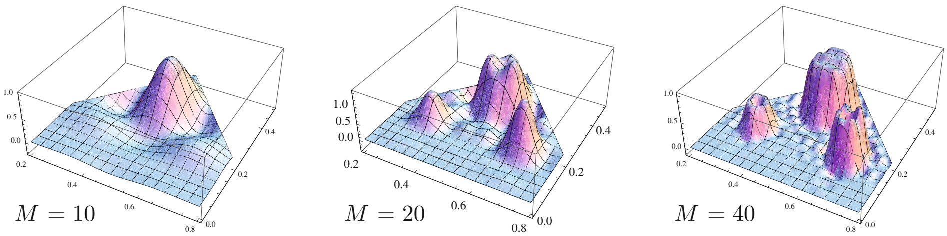

FIGURE 4. The graph cut $\left(z = \frac{1}{15}\right)$ of the f^l-interpolations I^l_M of B_3 for $M = 10$, 20 and 40.

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Semiclassical Spectral Asymptotics for a Magnetic Schrödinger Operator with Non-vanishing Magnetic Field

Bernard Helffer and Yuri A. Kordyukov

Abstract. We consider a magnetic Schrödinger operator H^h on a compact Riemannian manifold, depending on the semiclassical parameter $h > 0$. We assume that there is no electric field. We suppose that the minimal value b_0 of the intensity of the magnetic field b is strictly positive. We give a survey of the results on asymptotic behavior of the eigenvalues of the operator H^h in the semiclassical limit.

Mathematics Subject Classification (2010). 81Q20, 81Q35, 81V99.

Keywords. Magnetic Schrödinger operator, semiclassical approximation, spectrum.

1. Introduction

Let M be a compact oriented manifold of dimension $n \geq 2$ (possibly with boundary). Let g be a Riemannian metric and $\mathbf{B}$ a real-valued closed 2-form on M. Assume that $\mathbf{B}$ is exact and choose a real-valued 1-form $\mathbf{A}$ on M such that $d\mathbf{A} = \mathbf{B}$. Thus, one has a natural mapping

$$u \mapsto ih\, du + \mathbf{A}u$$

from $C_c^\infty(M)$ to the space $\Omega_c^1(M)$ of smooth, compactly supported one-forms on M. The Riemannian metric allows to define scalar products in these spaces and consider the adjoint operator

$$(ih\, d + \mathbf{A})^* : \Omega_c^1(M) \to C_c^\infty(M)\,.$$

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A Schrödinger operator with magnetic potential $\mathbf{A}$ is defined by the formula

$$H^h = (ih\, d + \mathbf{A})^*(ih\, d + \mathbf{A})\,. \tag{1}$$

Here $h > 0$ is a semiclassical parameter. If M has non-empty boundary, we will assume that the operator H^h satisfies the Dirichlet boundary conditions.

From the geometric point of view, the 1-form $\mathbf{A}$ defines a Hermitian connection $\nabla_{\mathbf{A}} = d - i\mathbf{A}$ on the trivial complex line bundle $\mathcal{L}$ over M. The curvature of this connection is $-i\mathbf{B}$. Then the operator H^h is related with the associated covariant (or Bochner) Laplacian

$$H_{\mathbf{A}} = \nabla^*_{\mathbf{A}} \nabla_{\mathbf{A}}$$

by the formula

$$H^h = h^2(d - ih^{-1}\mathbf{A})^*(d - ih^{-1}\mathbf{A}) = h^2 H_{h^{-1}\mathbf{A}}\,.$$

This formula shows, in particular, that the semiclassical limit $h \to 0$ is clearly equivalent to the large magnetic field limit.

We choose local coordinates $\mathbf{x} = (x_1, \dots, x_n)$ on M. We write the 1-form $\mathbf{A}$ in the local coordinates as

$$\mathbf{A} = \sum\nolimits_{j=1}^{n} A_j(\mathbf{x})\, dx_j\;,$$

the matrix of the Riemannian metric g as

$$g(\mathbf{x}) = (g_{j\ell}(\mathbf{x}))_{1\leq j,\ell\leq n}\;,$$

and its inverse as

$$g(\mathbf{x})^{-1} = (g^{j\ell}(\mathbf{x}))_{1\leq j,\ell\leq n}\,.$$

We denote the determinant of g by:

$$|g(\mathbf{x})| = \det(g(\mathbf{x}))\,.$$

Then the magnetic field $\mathbf{B}$ is given by the following formula

$$\mathbf{B} = \sum\nolimits_{j<k} B_{jk}\, dx_j \wedge dx_k\;, \quad B_{jk} = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k}\,.$$

Moreover, the operator H^h has in these coordinates the form

$$H^h = \frac{1}{\sqrt{|g(\mathbf{x})|}} \sum_{1\leq j,\ell\leq n} \left(ih\frac{\partial}{\partial x_j} + A_j(\mathbf{x})\right) \left[\sqrt{|g(\mathbf{x})|} g^{j\ell}(\mathbf{x}) \left(ih\frac{\partial}{\partial x_\ell} + A_\ell(\mathbf{x})\right)\right].$$

In the case when $M = \mathbb{R}^n$ is the flat Euclidean space, the operator H^h takes the form

$$H^h = \sum\nolimits_{1\leq j\leq n} \left(hD_{x_j} - A_j(\mathbf{x})\right)^2, \tag{2}$$

where, as usual, $D_{x_j} = \frac{1}{i}\frac{\partial}{\partial x_j}$, $j = 1, \dots, n$.

When $n = 2$, the magnetic two-form $\mathbf{B}$ is a volume form on M and therefore can be identified with the function $b \in C^\infty(M)$ given by

$$\mathbf{B} = b\, d\mathbf{x}_g\;,$$

where $d\mathbf{x}_g$ denotes the Riemannian volume form M associated with g.

When $n = 3$, the magnetic two-form $\mathbf{B}$ can be identified with a magnetic vector field $\vec{b}$ by the Hodge star-operator. If M is the Euclidean space $\mathbb{R}^3$, we have

$$\vec{b} = (b_1, b_2, b_3) = \operatorname{curl} \mathbf{A} = (B_{23}, -B_{13}, B_{12}) \,, \tag{3}$$

with the usual definition of curl.

We are interested in asymptotic behavior of the spectrum of the operator H^h in the semiclassical limit. This problem was studied in [7, 13, 19, 20, 23, 32–34] (see [8, 14, 36] for surveys including the case of problems with boundary).

After the pioneering works by Kato [30] and his school, the starting reference for the spectral analysis of self-adjoint realizations of the magnetic Schrödinger operator is the paper by Avron–Herbst–Simon [1] where the role of the module of the magnetic field in the three-dimensional case appears for the first time. Further investigations were inspired by R. Montgomery [34], who was asking "Can we hear the locus of the magnetic field" (by analogy with the celebrated question by M. Kac). In [34], this question was studied for the two-dimensional magnetic Schrödinger operator. Motivated by the question of R. Montgomery, the first author and Mohamed in [19] investigated the asymptotic behavior of the low-lying eigenvalues of the Dirichlet realization of the magnetic Schrödinger operator in the case when the magnetic field vanishes. This study was continued more recently in [5, 12, 13, 35] (see also [14]). The case when the magnetic field never vanishes was analyzed in detail for the Dirichlet realization in the two-dimensional case in [20] and more recently in [15, 18, 37]. Moreover, there is a big literature devoted to the spectral analysis of the Neumann realization because of its connection with problems in superconductivity (see [8] and the references therein). Finally, we do not give a complete description of the semi-classical results obtained in the case when an electric potential V is creating the main localization and refer to [23] and [4] for a presentation and references therein.

The purpose of this paper is to give a survey of the results obtained in the case when the magnetic field never vanishes. First, we suppose that M is two-dimensional. Let

$$b_0 = \min_{\mathbf{x} \in M} |b(\mathbf{x})| \,. \tag{4}$$

Note that if M is without boundary then we necessarily have $b_0 = 0$, since

$$\int_M b(\mathbf{x}) d\mathbf{x}_g = \int_M d\mathbf{A} = 0 \,.$$

If we assume that M has a non-empty boundary and the operator H^h satisfies the Dirichlet boundary conditions, it was observed by many authors [31, 34, 40, 41] (as the immediate consequence of the Weitzenböck–Bochner type identity and the positivity of the square of a suitable Dirac operator) that, if U is a domain in M, then, for any $u \in C_c^\infty(U)$, the following estimate holds:

$$\|(ih\, d + \mathbf{A})u\|_U^2 \geq h \int_U b|u|^2 d\mathbf{x}_g \,. \tag{5}$$

In particular, for any $h > 0$,

$$\lambda_0(H^h) \geq hb_0 \,. \tag{6}$$

In the case $M = \mathbb{R}^2$, this estimate follows from the formula

$$hb(x) = -i[hD_{x_1} - A_1, hD_{x_2} - A_2]\,,$$

which implies (after an integration by parts) that

$$h \int b(\mathbf{x})|u(\mathbf{x})|^2 \, d\mathbf{x} \leq \|(hD_{x_1} - A_1)u\|^2 + \|(hD_{x_2} - A_2)u\|^2 \,.$$

Due to this estimate, the function hb can be considered in many spectral problems as an effective electric potential, that is, as a magnetic analog of the electric potential V in a Schrödinger operator $-h^2\Delta + V$.

Any connected component of the minimum set

$$U = \{\mathbf{x} \in M \;:\; b(\mathbf{x}) = b_0\} \tag{7}$$

can be understood as a magnetic well (attached to the given energy hb_0). In particular, an asymptotic description of the spectrum near the bottom strongly depends on the geometry of the magnetic wells and the behavior of b near them.

In higher dimensions, the role of magnetic potential is played by the function $\mathbf{x} \mapsto h \cdot \mathrm{Tr}^+(B(\mathbf{x}))$, which can be defined in the following way. For any $\mathbf{x} \in M$, denote by $B(\mathbf{x})$ the anti-symmetric linear operator on the tangent space $T_\mathbf{x}M$ associated with the 2-form $\mathbf{B}$:

$$g_\mathbf{x}(B(\mathbf{x})u, v) = \mathbf{B}_\mathbf{x}(u, v), \quad u, v \in T_\mathbf{x}M.$$

Recall that the intensity of the magnetic field is defined as

$$\mathrm{Tr}^+(B(\mathbf{x})) = \sum_{\substack{\lambda_j(\mathbf{x})>0 \\ i\lambda_j(\mathbf{x})\in\sigma(B(\mathbf{x}))}} \lambda_j(\mathbf{x}) = \frac{1}{2}\,\mathrm{Tr}([B^*(\mathbf{x}) \cdot B(\mathbf{x})]^{1/2}).$$

In the $(3D)$ case the only positive eigenvalue is $|B(x)|$ and we get

$$\mathrm{Tr}^+(B(\mathbf{x})) = |B(\mathbf{x})|\,.$$

In the general case, we do not have the equivalent of (5) but only the weaker estimate [19]:

$$(h \inf_\mathbf{x} \mathrm{Tr}^+(B(\mathbf{x})) - Ch^{\frac{5}{4}}) \int |u(\mathbf{x})|^2 \, d\mathbf{x} \leq \langle H^h u\,,\, u\rangle\,, \forall u \in C_c^\infty(M)\,. \tag{8}$$

When U is not connected, the spectrum is essentially obtained by analyzing (the union of) the spectra of Dirichlet Laplacians attached to each component. This is true modulo exponentially small errors. This corresponds to the so-called magnetic tunneling. We will not focus on this question (which is widely open) and will emphasize more on the presentation of the known semi-classical results in dimension 2 and 3 which are purely magnetic first at the bottom (Sections 2 and 3 in dimension 2 and Section 4 in dimension 3), secondly in Section 5 for excited states in dimension 2, where we present the newest contributions (Helffer–Kordyukov and Raymond–Vu Ngoc) and will give a few examples in the last section.

2. Discrete wells in dimension 2

In this section, we will discuss the case of discrete wells. We assume that:

$$b_0 > 0\,, \tag{9}$$

and that there exists a unique point x_0, which belongs to the interior of M, $k \in \mathbb{N}$ and $C > 0$ such that for all x in some neighborhood of x_0 the estimates hold:

$$C^{-1}\, d(x, x_0)^2 \leq b(x) - b_0 \leq C\, d(x, x_0)^2\,. \tag{10}$$

We introduce:

$$a = \mathrm{Tr}\left(\frac{1}{2}\,\mathrm{Hess}\; b(x_0)\right)^{1/2}, \quad d = \det\left(\frac{1}{2}\,\mathrm{Hess}\; b(x_0)\right)^{1/2},$$

and denote by $\lambda_0(H^h) \leq \lambda_1(H^h) \leq \lambda_2(H^h) \leq \cdots$ the eigenvalues of the operator H^h in $L^2(M)$.

Theorem 1. *Under current assumptions, for any $j \in \mathbb{N}$, there exists a sequence $(\alpha_{j,\ell})_{\ell\in\mathbb{N}}$ with*

$$\alpha_{j,0} = b_0, \quad \alpha_{j,1} = 0, \quad \alpha_{j,2} = \frac{2d^{1/2}}{b_0} j + \frac{a^2}{2b_0}\,,$$

such that

$$\lambda_j(H^h) \sim h \sum_{\ell=0}^{N} \alpha_{j,\ell} h^{\frac{\ell}{2}}\,. \tag{11}$$

In other words, for any N, there exist $C_{j,N} > 0$ and $h_{j,N} > 0$ such that, for any $h \in (0, h_{j,N}]$,

$$|\lambda_j(H^h) - h \sum_{\ell=0}^{N} \alpha_{j,\ell} h^{\frac{\ell}{2}}| \leq C_{j,N} h^{\frac{N+3}{2}}\,.$$

In particular, we have for the groundstate energy $\lambda_0(H^h)$ a two term asymptotics:

$$\lambda_0(H^h) = hb_0 + h^2 \frac{a^2}{2b_0} + \mathcal{O}(h^{5/2}), \quad h \to 0,$$

and the asymptotics of the splitting between the groundstate energy and the first excited state:

$$\lambda_1(H^h) - \lambda_0(H^h) \sim h^2 \frac{2d^{\frac{1}{2}}}{b_0}\,.$$

This theorem is proved in [15]. A two-terms asymptotics for the ground state energy in the flat case was previously obtained in [20]. Recent improvements by Helffer–Kordyukov [18] and Raymond–Vu Ngoc [37] (see also Section 4) show that no odd powers of $h^{\frac{1}{2}}$ actually occur in the flat case. We believe that this fact also holds in the general case of Riemannian manifold.

The proof of the upper bound is based on a construction of approximate eigenfunctions for the operator H^h. More precisely, we prove in [15] the following accurate upper bound for the eigenvalues of the operator H^h.

Theorem 2. *Under current assumptions, for any j and k in $\mathbb{N}$, there exists a sequence $(\mu_{j,k,\ell})_{\ell\in\mathbb{N}}$ with*

$$\mu_{j,k,0} = (2k+1)b_0, \quad \mu_{j,k,1} = 0\,,$$

and

$$\mu_{j,k,2} = (2j+1)(2k+1)\frac{d^{1/2}}{b_0} + (2k^2+2k+1)\frac{t}{2b_0} + \frac{1}{2}(k^2+k)R(x_0)\,,$$

where R is the scalar curvature, and

$$t = \operatorname{Tr}\left(\frac{1}{2}\operatorname{Hess} b(x_0)\right)\,,$$

and for any N, there exist $\phi^h_{jkN} \in C^\infty(M)$, $C_{jk,N} > 0$ and $h_{jk,N} > 0$ such that

$$(\phi^h_{j_1k_1N}, \phi^h_{j_2k_2N}) = \delta_{j_1j_2}\delta_{k_1k_2} + \mathcal{O}_{j_1,j_2,k_1,k_2}(h)\,, \tag{12}$$

and, for any $h \in (0, h_{jk,N}]$,

$$\|H^h\phi^h_{jkN} - \mu^h_{jkN}\phi^h_{jkN}\| \leq C_{jkN}h^{\frac{N+3}{2}}\|\phi^h_{jkN}\|,$$

where

$$\mu^h_{jkN} = h\sum_{\ell=0}^{N}\mu_{j,k,\ell}h^{\frac{\ell}{2}}\,. \tag{13}$$

Since the operator H^h is self-adjoint, using the Spectral Theorem, we immediately deduce the existence of eigenvalues near the values μ^h_{jkN} .

Corollary 3. *For any j, k and N in $\mathbb{N}$, there exist $C_{jk,N} > 0$ and $h_{jk,N} > 0$ such that, for any $h \in (0, h_{jk,N})$,*

$$\operatorname{dist}(\mu^h_{jkN}, \operatorname{Spec}(H^h)) \leq C_{jk,N}h^{\frac{N+3}{2}}\,.$$

Remark 1. The low-lying eigenvalues of the operator H^h , as $h \to 0$, are obtained by taking $k = 0$ in Theorem 2. Therefore, as an immediate consequence of Theorem 2, we deduce that, for any j and N in $\mathbb{N}$, there exists $h_{j,N} > 0$ such that, for any $h \in (0, h_{j,N}]$, we have

$$\lambda_j(H^h) \leq \mu^h_{j0N} + C_{j0,N}h^{\frac{N+3}{2}}\,.$$

In particular, this implies the upper bound in Theorem 1.

Remark 2. Our interest in the case of arbitrary k in Theorem 2 is motivated, in particular, by its importance for proving the existence of gaps in the spectrum of the operator H^h in the semiclassical limit [11].

Remark 3. The term

$$(2k+1)hb_0 + \frac{1}{2}h^2\left(k^2+k\right)R\,,$$

on the right-hand side of (13) (see also (18) below) has a natural interpretation as the Landau levels. The interpretation depends on whether R is zero, positive or negative and, in all three cases, is given in terms of eigenvalues of the associated magnetic Laplacian with constant magnetic field (Landau operator) on the corresponding simply connected Riemann surface of constant curvature (see [15] for more details).

We also mention the paper [6] by Ferapontov and Veselov, who prove that these three model magnetic Laplacians are integrable in some sense. This observation enables them to give the complete description of the spectra of these operators in the same way as it was done by Schrödinger for the harmonic oscillator.

3. Degenerate wells in dimension 2

In this section, following [16], we will discuss the case when the minimum of the magnetic field is attained on a regular curve γ. We assume that:

- $b_0 > 0$;
- the set $\{x \in M : |b(x)| = b_0\}$ is a smooth curve γ, which is contained in the interior of M;
- there is a constant $C > 0$ such that for all x in some neighborhood of γ the estimates hold:

$$C^{-1} d(x,\gamma)^2 \leq |b(x)| - b_0 \leq C d(x,\gamma)^2 \,. \tag{14}$$

3.1. Asymptotics near the bottom

The main purpose is to give an asymptotics of the groundstate energy $\lambda_0(H^h)$ of the operator H^h. Denote by N the external unit normal vector to γ. Let $\tilde{N}$ denote the natural extension of N to a smooth normalized vector field on M, whose integral curves starting from a point x in a tubular neighborhood of γ are the minimal geodesics to γ. Consider the function β_2 on γ given by

$$\beta_2(x) = \tilde{N}^2 |b(x)| \,, \quad x \in \gamma \,. \tag{15}$$

By (14), it is easy to see that

$$\beta_2(x) > 0 \,, \quad x \in \gamma \,.$$

Theorem 4. *There exists $h_0 > 0$, such that, for any $h \in (0, h_0]$,*

$$\lambda_0(H^h) = hb_0 + h^2 \frac{\mu_0}{4b_0} + \mathcal{O}(h^{17/8}) \,. \tag{16}$$

where

$$\mu_0 := \inf_{x \in \gamma} \beta_2(x) \,. \tag{17}$$

The proof of the upper bound is based on a construction of approximate eigenfunctions for the operator H^h. We denote by R the scalar curvature of the Riemannian manifold (M, g).

Theorem 5. *For any $x \in \gamma$ and for any integer $k \geq 0$, there exist C and $h_0 > 0$, such that, for any $h \in (0, h_0]$, there exists $\Phi_k^h \in C_c^\infty(M), \Phi_k^h \neq 0$, such that*

$$\left\| H^h \Phi_k^h - \lambda^h(k,x)\Phi_k^h \right\| \leq C h^{17/8} \|\Phi_k^h\|\,,$$

where

$$\lambda^h(k,x) = (2k+1)hb_0 + h^2 \left[(2k^2+2k+1)\frac{\beta_2(x)}{4b_0} + \frac{1}{2}\left(k^2+k\right)R(x) \right]. \qquad (18)$$

When $k = 0$, we get:

Corollary 6. *For any $x \in \gamma$, there exist C and $h_0 > 0$, such that, for any $h \in (0, h_0]$, there exists $\Phi_0^h \in C_c^\infty(M), \Phi_0^h \neq 0$, such that*

$$\left\| H^h \Phi_0^h - \lambda^h(x)\Phi_0^h \right\| \leq C h^{17/8} \|\Phi_0^h\|\,,$$

where

$$\lambda^h(x) = hb_0 + h^2\, \frac{\beta_2(x)}{4b_0}\,.$$

3.2. Miniwells

Like in the case of the Schrödinger operator with electric potential (see [22]), one can introduce an internal notion of magnetic well for a fixed closed curve γ in the minimum set of the magnetic field $\mathbf{B}$. Such magnetic wells can be naturally called magnetic miniwells. They are defined by means of the function β_2 on γ given by (15).

Theorem 7. *Assume that there exists a unique minimum point $x_0 \in \gamma$ of the function β_2 on γ, which is nondegenerate:*

$$\mu_2 := \beta_2''(x_0) > 0\,.$$

For any $j \in \mathbb{N}$, there exist C_j and $h_j > 0$, such that for any $h \in (0, h_j)$

$$\lambda_j(H^h) \leq hb_0 + h^2\, \frac{\mu_0}{4b_0} + h^{5/2}\, \frac{(\mu_0\mu_2)^{1/2}}{4b_0^{3/2}}(2j+1) + C_j h^{11/4}\,.$$

Here and below the derivative means the derivative with respect to the natural parameter on γ.

Remark 4. We conjecture that

$$\lambda_0(H^h) = hb_0 + h^2\, \frac{\mu_0}{4b_0} + h^{5/2}\, \frac{(\mu_0\mu_2)^{1/2}}{4b_0^{3/2}} + o(h^{5/2})\,.$$

The proof is based on a construction of approximate eigenfunctions, which can be made near an arbitrary Landau level. For $k \in \mathbb{N}$, consider the function V_k on γ given by (cf. (18))

$$V_k(x) := (2k^2+2k+1)\frac{\beta_2(x)}{4b_0} + \frac{1}{2}\left(k^2+k\right)R(x)\,. \qquad (19)$$

Assume that there exists a unique minimum $x_0 \in \gamma$ of the function V_k on γ, which is nondegenerate, that is satisfying, for all $x \in \gamma$ in some neighborhood of x_0,

$$Cd(x,x_0)^2 \leq V_k(x) - V_k(x_0) \leq C^{-1}d(x,x_0)^2\,. \tag{20}$$

Under these assumptions, one can give the following, more precise construction of approximate eigenvalues of the operator H^h.

Theorem 8. *Under current assumptions, for any $j, k \in \mathbb{N}$, there exist $u^h_{jk} \in C^\infty_c(M)$, $C_{jk} > 0$ and $h_{jk} > 0$ such that*

$$(u^h_{j_1k}, u^h_{j_2k}) = \delta_{j_1j_2} + \mathcal{O}_{j_1,j_2,k}(h)$$

and, for any $h \in (0, h_{jk}]$,

$$\|H^h u^h_{jk} - \mu^h_{jk} u^h_{jk}\| \leq C_{jk}\, h^{11/4} \|u^h_{jk}\|\,,$$

where

$$\mu^h_{jk} = \mu_{j,k,0} h + \mu_{j,k,4} h^2 + \mu_{j,k,6}\, h^{5/2}\,, \tag{21}$$

with

$$\mu_{j,k,0} = (2k+1)b_0, \quad \mu_{j,k,4} = V_k(x_0)\,,$$

and

$$\mu_{j,k,6} = \frac{1}{2b_0} V''_k(x_0)^{1/2} \beta_2(x_0)^{1/2} (2k+1)^{1/2} (2j+1)\,.$$

4. Excited states for discrete wells

If Theorem 1 is satisfactory for the analysis of a finite numbers of eigenvalues at the bottom, it appears to be useful to get an extended description of the bottom of the spectrum including more excited states. Motivated by Karasev's paper [29], it seems to be interesting to produce an effective Hamiltonian whose spectrum will also describe the excited states. In 2013, Helffer–Kordyukov [18] on one side, and Raymond–Vu Ngoc [37] on the other side reanalyzed the problem in the case of discrete wells with two different points of view leading in the two cases to the existence of an effective $(1D)$-Hamiltonian whose spectrum describes the spectrum of our magnetic Schrödinger operator.

4.1. Using a Grushin's problem (after Helffer–Kordyukov [18])

The approach of [18] is based on Grushin's method. This method was initiated in the context of hypoellipticity by V. Grushin [9] and then exploited by J. Sjöstrand alone or with collaborators in many contexts. We refer to [38] for a survey on this method and references or Appendix D in [42]. In spectral theory a variant of this method is known under the name of "Feschbach projection method" or "Schur complement formula" in analytic Fredholm theory.

Let us consider the magnetic Schrödinger operator H^h in the flat Euclidean space $\mathbb{R}^2$:

$$H^h = h^2 D_x^2 + (hD_y + A(x,y))^2\,.$$

The magnetic field $\mathbf{B}$ is given by

$$\mathbf{B} = b\, dx \wedge dy \;\text{ with }\; b(x,y) = \frac{\partial A}{\partial x}(x,y)\,.$$

Let

$$b_0 = \min_{(x,y)\in\mathbb{R}^2} |b(x,y)| > 0\,.$$

We assume that at ∞, we have

$$b_0 < \liminf_{|x|+|y|\to+\infty} |b(x,y)| := b_0 + \eta_0\,.$$

Then one can prove easily [19] that, for any $0 \leq \eta_1 < \eta_0$, there exists $h_1 > 0$ such that

$$\sigma(H^h) \cap [0, h(b_0+\eta_1)) \subset \sigma_d(H^h)\,, \quad \forall h \in (0, h_1].$$

Next, as above, we assume that:

- $b_0 > 0$;
- the set $\{(x,y) \in \mathbb{R}^2 : |b(x,y)| = b_0\}$ is a single point (x_0, y_0);
- (x_0, y_0) is a non-degenerate minimum:

$$\operatorname{Hess} b(x_0, y_0) > 0.$$

We have a diffeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\phi(x,y) = (A(x,y), y), \quad (x,y) \in \mathbb{R}^2\,.$$

We then associate with b a function $\hat{b} \in C^\infty(\mathbb{R}^2)$ by

$$\hat{b} = b \circ \phi^{-1}\,.$$

Theorem 9. *There exist $h_0 > 0$, $\epsilon_0 > 0$, $\gamma_0 \in (0, \eta_0)$, $h \mapsto \gamma_0(h)$ defined for $(0, h_0]$ such that $\gamma_0(h) \to \gamma_0$ as $h \to 0$, and a semiclassical symbol $p_{\rm eff}(y,\eta,h,z)$, which is defined in a neighborhood $\Omega \subset \mathbb{R}^2$ of the set $\{(y,\eta) \in \mathbb{R}^2 : \hat{b}(y,\eta) \leq b_0 + \gamma_0\}$ for $h \in (0, h_0]$ and $z \in \mathbb{C}$ such that $|z| < \gamma_0 + \epsilon_0$, of the form*

$$p_{\rm eff}(y,\eta,h,z) \sim \sum_{j\in\mathbb{N}} p^j_{\rm eff}(y,\eta,z) h^j\,, \tag{22}$$

with

$$p^0_{\rm eff}(y,\eta,z) = \hat{b}(y,\eta) - b_0 - z\,, \tag{23}$$

such that $\lambda_h \in \sigma(H^h) \cap [0, h(b_0+\gamma_0(h)))$, if and only if the associated h-pseudodifferential operator[1] *$p_{\rm eff}(y, hD_y, h, z(h))$ has an approximate 0-eigenfunction $u_h^{qm} \in C^\infty(\mathbb{R})$, i.e.,*

$$p_{\rm eff}(y, hD_y, h, z(h)) u_h^{qm} = \mathcal{O}(h^\infty)\,, \tag{24}$$

with

$$z(h) = \frac{1}{h}(\lambda_h - hb_0) + \mathcal{O}(h^\infty)\,,$$

$|z(h)| < \gamma_0(h)$ for any $h \in (0, h_0]$, and such that the frequency set[2] *of u_h^{qm} is non-empty and contained in Ω .*

---

[1]We use the Weyl semi-classical quantization of the symbol (see for example [26]).

[2]See [42] for a discussion of the frequency set and references therein.

Remark 5. Here (24) makes sense modulo $\mathcal{O}(h^\infty)$ by extending first the symbol $p_{\mathrm{eff}}(y, \eta, h, z)$ outside the neighborhood Ω to a semiclassical symbol in $\mathbb{R}^2$ and defining then the operators $p_{\mathrm{eff}}(y, hD_y, h, z)$ by the Weyl calculus. Using the localization of the frequency set of u_h^{qm}, the left-hand side of (24) does not depend on the extension up to an error which is $\mathcal{O}(h^\infty)$.

Remark 6. By Theorem 9, for any $E \in [b_0, b_0 + \gamma_0)$, the spectrum of the operator H^h (divided by h) is determined near E (say in an interval $(E - Ch^{\frac{1}{2}}, E + Ch^{\frac{1}{2}})$) and modulo $\mathcal{O}(h^{\frac{3}{2}})$ by the spectrum of $\hat{b}(y, hD_y) + hb_1(y, hD_y, E)$, where one can use the Bohr–Sommerfeld rule (see [21] or [24] for a mathematical justification) for determining the energy levels.

Corollary 10. *There exists $\gamma_0 \in (0, \eta_0)$, $h_0 > 0$ and $C > 0$ such that*

$$\lambda_{j+1}(H^h) - \lambda_j(H^h) \geq \frac{1}{C} h^2 \,,\ \forall h \in (0, h_0] \,,$$

for any j such that $\lambda_{j+1}(H^h) < h(b_0 + \gamma_0)$.

4.2. Using a Birkhoff normal form (after Raymond–Vu Ngoc [37])

The proof of Raymond–Vu Ngoc is reminiscent of Ivrii's approach (see his book – old version or new version in progress on his Home Page [28] – and, more accessible but without proofs, the introductory article [27]) and uses a Birkhoff normal form. This approach has the advantage to be semi-global and uses more general symplectomorphisms and their quantizations.

Consider the magnetic Schrödinger operator H^h in $\mathbb{R}^2$ given by (2). Let H be its h-symbol:

$$\begin{aligned} H(x, y, \xi, \eta) &= |\xi - A_1(x, y)|^2 + |\eta - A_2(x, y)|^2, \\ (x, y, \xi, \eta) &\in T^*\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2 \,. \end{aligned} \tag{25}$$

By definition the energy surface Σ_E corresponding to energy E is the set $H^{-1}(E)$. The first result shows the existence of a smooth symplectic diffeomorphism that transforms the initial Hamiltonian into a normal form, up to any order in the distance to the zero energy surface Σ_0. Assume that the magnetic field b does not vanish in an open set $\Omega \subset \mathbb{R}^2$.

Theorem 11 ([37, Theorem 1.1]). *There exists a symplectic diffeomorphism Φ, defined in an open set $\tilde{\Omega} \subset \mathbb{C}_{z_1} \times \mathbb{R}^2_{z_2}$, with values in $T^*\mathbb{R}^2$, which sends the plane $\{z_1 = 0\}$ to Σ_0, and such that*

$$H \circ \Phi = |z_1|^2 f(z_2, |z_1|^2) + \mathcal{O}(|z_1|^\infty),$$

where $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is smooth. Moreover, the map

$$\varphi : \Omega \ni (x, y) \mapsto \Phi^{-1}(x, y, \mathbf{A}(x, y)) \in (\{0\} \times \mathbb{R}^2_{z_2}) \cap \tilde{\Omega}$$

is a local diffeomorphism and

$$f \circ (\varphi(x, y), 0) = |b(x, y)|.$$

The next result gives the quantum counterpart of this theorem. We keep the notation of the previous theorem.

Theorem 12 ([37, Theorem 1.6]). *For h small enough there exists a (semi-classical) Fourier Integral Operator*[3] U_h *such that*

$$U_h^* U_h = I + Z_h, \quad U_h U_h^* = I + Z_h',$$

where Z_h, Z_h' are h-pseudo-differential operators that microlocally vanish in a neighborhood of $\tilde{\Omega} \cap \Sigma_0$, and

$$U_h^* H^h U_h = \mathcal{I}_h F_h + R_h,$$

where:

1. $\mathcal{I}_h := -h^2 \frac{\partial^2}{\partial x_1^2} + x_1^2$.
2. *F_h is a classical h-pseudo-differential operator that commutes with $\mathcal{I}_h$.*
3. *For any Hermite function $h_n(x_1)$ such that $\mathcal{I}_h h_n = h(2n-1)h_n$, the operator $F_h^{(n)}$ acting on $L^2(\mathbb{R}_{x_2})$ by*
$$h_n \otimes F_h^{(n)}(u) = F_h(h_n \otimes u)$$
is a classical h-pseudo-differential operator with principal symbol
$$F^{(n)}(x_2, \xi_2) = b(x, y)\,,$$
where $(0, x_2 + i\xi_2) = \varphi(x, y)$.
4. *Given any h-pseudo-differential operator D_h with principal symbol d_0 such that $d_0(z_1, z_2) = c(z_2)|z_1|^2 + \mathcal{O}(|z_1|^3)$, and any $N \geq 1$, there exist classical pseudo-differential operators $S_{h.N}$ and K_N such that*
$$R_h = S_{h.N}(D_h)^N + K_N + \mathcal{O}(h^\infty)\,,$$
with K_N compactly supported away from a fixed neighborhood of $|z_1| = 0$.
5. *$\mathcal{I}_h F_h = \mathcal{N}_h = \mathcal{H}_h^0 + Q_h$, where $\mathcal{H}_h^0$ is the h-pseudodifferential operator of symbol $H^0(z_1, z_2) = b(\varphi^{-1}(z_2))|z_1|^2$, and the operator Q_h is relatively bounded with respect to $\mathcal{H}_h^0$ with an arbitrarily small relative bound.*

As a consequence, Raymond and Vu Ngoc obtain the following theorem.

Theorem 13 ([37, Theorem 1.5]). *Assume that the magnetic field B is non vanishing on $\mathbb{R}^2$ and confining: there exist constants $\tilde{C}_1 > 0$, $M_0 > 0$ such that*

$$b(q) \geq \tilde{C}_1 \ \text{for}\ |q| \geq M_0\,.$$

Let $\mathcal{H}_h^0 = Op_h^w(H^0)$, where $H^0 = b(\varphi^{-1}(z_2)|z_2|^2$ and $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism. Then there exists a bounded classical pseudo-differential operator Q_h on $\mathbb{R}^2$, such that

- *Q_h commutes with $Op_h^w(|z_1|^2)$;*
- *Q_h is relatively bounded with respect to $\mathcal{H}_h^0$ with an arbitrarily small relative bound;*
- *its Weyl symbol is $O_{z_2}(h^2 + h|z_1|^2 + |z_1|^4)$,*

[3] See [26, 42] for a definition.

so that the following holds. Let $0 < C_1 < \tilde{C}_1$*. Then the spectra of* H^h *and* $\mathcal{N}_h := \mathcal{H}_h^0 + Q_h$ *in* $(-\infty, C_1 h]$ *are discrete. We denote by* $0 < \lambda_1(h) \leq \lambda_2(h) \leq \cdots$ *the eigenvalues of* H^h *and by* $0 < \mu_1(h) \leq \mu_2(h) \leq \cdots$ *the eigenvalues of* $\mathcal{N}_h$*. Then for any* $j \in \mathbb{N}^*$ *such that* $\lambda_j(h) \leq C_1 h$ *and* $\mu_j(h) \leq C_1 h$*, we have*

$$|\lambda_j(h) - \mu_j(h)| = O(h^\infty)\,.$$

Remark 7. Theorem 13 is stronger than Theorem 9 because Theorem 9 gives a description of the spectrum of H^h in the interval $[hb_0, h(b_0 + \gamma_0))$ for some $\gamma_0 \in (0, \eta_0)$, whereas in Theorem 13, $\gamma_0 \in (0, \eta_0)$ is arbitrary. On the other hand, the symbol of the effective Hamiltonian in Theorem 13 seems to be less explicit than in Theorem 9. The other point could be that Theorem 9 allows us to treat an additional term $h^2 V(x, y)$. This will complete the analysis of Helffer–Sjöstrand [25], in the case of the constant magnetic field. The case with an additional term hV could also be interesting.

Remark 8. As communicated to us by F. Faure, there is some hope that the results of [37] can be generalized under a generic assumption to the case of arbitrary even dimension. Some results are also presented in [28, Chapter 13].

5. Discrete wells in dimension 3

In this section, we discuss the three-dimensional case.

5.1. Upper bounds [17]

Consider the magnetic Schrödinger operator H^h in a domain Ω of the flat Euclidean space $\mathbb{R}^3$ (see (2)). As usual, we assume that H^h satisfies the Dirichlet boundary condition. Let $\vec{b} = (b_1, b_2, b_3)$ be the corresponding vector magnetic field (see (3)).

We assume that there exists a constant $C > 0$ such that for $j = 1, 2, 3$ we have

$$|(\nabla b_j)(\mathbf{x})| \leq C(|\vec{b}(\mathbf{x})| + 1), \quad \forall \mathbf{x} \in \Omega\,. \tag{26}$$

Put

$$b_0 = \min\{|\vec{b}(\mathbf{x})| \,:\, \mathbf{x} \in \Omega\}.$$

We assume that there exist a (connected) bounded domain $\Omega_1 \subset\subset \Omega$ and a constant $\epsilon_0 > 0$ such that

$$|\vec{b}(\mathbf{x})| \geq b_0 + \epsilon_0, \quad \mathbf{x} \notin \Omega_1\,. \tag{27}$$

As shown in [19], under conditions (26) and (27), for any ϵ_1 with $0 < \epsilon_1 < \epsilon_0$, there exists $h_1 > 0$ such that, for $h \in (0, h_1]$

$$\sigma(H^h) \cap [0, h(b_0 + \epsilon_1)) \subset \sigma_d(H^h).$$

Denote by $\lambda_0(H^h) \leq \lambda_1(H^h) \leq \lambda_2(H^h) \leq \cdots$ the eigenvalues of the operator H^h contained in $[0, h(b_0 + \epsilon_0))$.

Finally, we assume that:

$$b_0 > 0\,,$$

and that there exists a unique minimum $\mathbf{x}_0 \in \Omega$ such that $|\vec{b}(\mathbf{x}_0)| = b_0$, which is non-degenerate: in some neighborhood of $\mathbf{x}_0$

$$C^{-1}|\mathbf{x}-\mathbf{x}_0|^2 \leq |\vec{b}(\mathbf{x})| - b_0 \leq C|\mathbf{x}-\mathbf{x}_0|^2 \,.$$

We also introduce:

$$d = \det \operatorname{Hess} |\vec{b}|(\mathbf{x}_0)\,, \quad a = \frac{1}{2b_0^2}(\operatorname{Hess} |\vec{b}|\, \vec{b}\cdot\vec{b})(\mathbf{x}_0)\,.$$

Theorem 14. *Under current assumptions, for any $m \in \mathbb{N}$, there exist $C_m > 0$ and $h_m > 0$ such that, for any $h \in (0, h_m]$,*

$$\lambda_m(H^h) \leq hb_0 + h^{3/2}a^{1/2} + h^2 \left[\frac{1}{2b_0}\left(\frac{d}{2a}\right)^{1/2}(2m+1) + \nu\right] + C_m h^{9/4}\,, \quad (28)$$

where ν is some explicit constant[4].

The proof of Theorem 14 is based on a construction of quasimodes.

Theorem 15. *Under current assumptions, for any j, k and m in $\mathbb{N}$, there exist $\phi^h_{j,k,m} \in C^\infty_c(\Omega)$, $C_{j,k,m} > 0$ and $h_{j,k,m} > 0$ such that*

$$(\phi^h_{j_1,k_1,m_1}, \phi^h_{j_2,k_2,m_2}) = \delta_{j_1j_2}\delta_{k_1k_2}\delta_{m_1m_2} + \mathcal{O}_{j_1,k_1,m_1,j_2,k_2,m_2}(h)\,,$$

and, for any $h \in (0, h_{j,k,m}]$,

$$\|H^h\phi^h_{j,k,m} - \mu^h_{j,k,m}\phi^h_{j,k,m}\| \leq C_{j,k,m}\, h^{\frac{9}{4}}\|\phi^h_{j,k,m}\|,$$

where

$$\mu^h_{j,k,m} = \mu_{j,k,m,0}h + \mu_{j,k,m,2}h^{\frac{3}{2}} + \mu_{j,k,m,4}h^2$$

with

$$\mu_{j,k,m,0} = (2k+1)b_0\,, \quad \mu_{j,k,m,2} = (2j+1)(2k+1)^{1/2}a^{1/2}$$

and

$$\mu_{j,k,m,4} = \frac{1}{2b_0}\left(\frac{d}{2a}\right)^{1/2}(2m+1)(2k+1) + \nu(j,k)\,,$$

where $\nu(j,k)$ has the form

$$\nu(j,k) = \nu_{22}(2k+1)^2 + \nu_{11}(2j+1)^2 + \nu_0\,,$$

with some explicit constants ν_0, ν_{11}, ν_{22}.

Remark 9. It is conjectured that

$$\lambda_m(H^h) \geq hb_0 + h^{3/2}a^{1/2} + h^2 \left[\frac{1}{2b_0}\left(\frac{d}{2a}\right)^{1/2}(2m+1) + \nu\right] - C_m h^{9/4}\,.$$

At the moment, we only know from [19]

$$\lambda_m(H^h) \geq hb_0 - Ch^2\,,$$

which is an improvement of the general lower bound (8).

[4] which means that it is given by a rather complicated explicit formula.

5.2. On some statements of V. Ivrii [27, 28]

Here we refer to some results announced in [27] and developed in Chapter 18 in [28]. These results correspond in the (3D)-case to what was discussed in the (2D)-case in the subsection 4.2. Under the assumption that the magnetic field does not vanish, the claim[5] is that (up to conjugation by an h-Fourier integral operator), our Schrödinger operator can microlocally be written in the form:

$$\omega_1(x_1, x_2, hD_{x_2})(h^2 D_{x_3}^2 + x_3^2) + h^2 D_{x_1}^2 \\ + \sum_{2m+n+\ell \geq 3} h^\ell \, a_{mn\ell}(x_1, x_2, hD_{x_2}) \, (h^2 D_{x_3}^2 + x_3^2)^m (hD_{x_1})^n \,,$$

with

$$\omega_1 = |\vec{b}| \circ \psi \,,$$

where ψ is some local unspecified diffeomorphism which plays the role of φ^{-1} in Theorem 11.

Once precisely stated and proved, let us explain what we could expect after. Reducing to the lowest Landau level (the first eigenvalue of $h^2 D_{x_3}^2 + x_3^2$), we obtain that the spectrum of our initial operator near the minimum of $|\vec{b}(x)|$ should be deduced from the spectral analysis in $(-\infty, hb_0 + Ch^{\frac{3}{2}})$ (for some fixed $C > 0$) in the semi-classical limit of the following "formal" pseudo-differential operator:

$$h\omega_1(x_1, x_2, hD_{x_2}) + h^2 D_{x_1}^2 + \sum_{2m+n+\ell \geq 3} h^{m+\ell} \, a_{mn\ell}(x_1, x_2, hD_{x_2})(hD_{x_1})^n \,.$$

If we only look for the principal term (and divide by h), we get as first "effective" operator to analyze for the spectrum now in $(-\infty, b_0 + Ch^{\frac{1}{2}})$:

$$\omega_1(x_1, x_2, hD_{x_2}) + (h^{\frac{1}{2}} D_{x_1})^2 + ha_{020}(x_1, x_2, hD_{x_2}) \,,$$

with the hope to get in this way an approximation modulo $\mathcal{O}(h^{\frac{5}{4}})$. This suggests a semi-classical analysis near the bottom of a pseudodifferential operator of the type met in Born–Oppenheimer theory $p(x_1, x_2, h^{\frac{1}{2}} D_{x_1}, hD_{x_2})$ with two semi-classical parameters (see [36] and references therein for a recent discussion on this subject). This would be coherent with the expansion obtained on the right-hand side of (28) at least modulo $\mathcal{O}(h^{\frac{9}{4}})$.

6. Some remarks and open questions

6.1. Geometry of magnetic fields

Consider the magnetic Schrödinger operator in the flat Euclidean space $\mathbb{R}^n$ (see (2)). Its semiclassical symbol (as defined in (25)) is a smooth function

[5] We have tried to correct many typos of the statement (more specifically the remainder in Formula (25)) in [27]. Note in particular that the sum on the right-hand side is undefined (see however [18] which meets the same problem) and could only be meaningful for some subspace of functions whose energy is for example less than $hb_0 + Ch^{\frac{3}{2}}$.

$H \in C^\infty(\mathbb{R}^{2n})$ whose zero set of H given by

$$\Sigma_0 := H^{-1}(0) = \{(\mathbf{x}, \xi) \in \mathbb{R}^{2n} : \xi_j = A_j(\mathbf{x})\,, j = 1, \dots, n\}\,.$$

Since it is a graph, it is an embedded submanifold of $\mathbb{R}^{2n}$, parameterized by $\mathbf{x} \in \mathbb{R}^n$. It is easy to check that if we denote by $J : \mathbb{R}^n \to \Sigma$ the embedding $J(\mathbf{x}) = (\mathbf{x}, A(\mathbf{x}))$, then, for the canonical symplectic form $\omega = \sum_{j=1}^n d\xi_j \wedge dx_j$ on $\mathbb{R}^{2n}$ we have

$$J^*\omega\,|_\Sigma \cong \mathbf{B}\,.$$

When $n = 2$ and the magnetic field b does not vanish, Σ_0 is symplectic. When $n = 3$, Σ_0 cannot be symplectic. If the magnetic field $\vec{b}$ does not vanish, then $\mathbf{B}$ has a constant rank, and Σ_0 is a presymplectic manifold. When n is arbitrary even, we can hope that, under generic assumptions, Σ_0 is symplectic. This kind of analysis was basic in the seventies for the analysis of the hypoellipticity of operators with multiple characteristics.

Recall that, in Remark 3, we give a geometric interpretation of some terms, entering into the asymptotic formula (13) for approximate eigenvalues of the operator H^h in the two-dimensional case. One can naturally consider similar questions in the three-dimensional case. First, observe that three cases $R = 0$, $R > 0$ and $R < 0$ mentioned in Remark 3 correspond to three cases of two-dimensional model geometries: Euclidean, spherical and hyperbolical, respectively. In the three-dimensional case, the situation is more complicated. There are eight three-dimensional model geometries introduced by Thurston (see, for instance, [39]). The interesting open problem is to construct the magnetic Schrödinger operators with constant magnetic field on each three-dimensional geometric model and compute its spectra. It is also interesting to find examples of integrable magnetic Schrödinger operators on three-dimensional Riemannian manifolds.

6.2. The tunneling effect

Although, as a consequence of magnetic Agmon estimates [19, 23, 36], it is possible to give upper bounds on the tunneling effect (see [42, Section 7.2] or [10] for an introduction) due to the presence of multiconnected magnetic wells, essentially no results are known for lower bounds of this effect analogous to what is proved for the celebrated double well problem for the Schrödinger operator $-h^2\Delta + V$. The only exception is [23], which involves $\sum_j (hD_{x_j} - t(h)A_j)^2 + V$ but this last result is not a "pure magnetic effect" and it is assumed that the magnetic field is small enough ($|t(h)| = \mathcal{O}(h|\log h|)$).

There are however a few models where one can "observe" this effect in particular in domains with corners [2] (numerics with some theoretical interpretation, see also [8] for a presentation of results due to V. Bonnaillie-Noel), the role of the magnetic wells being played by the corners of smallest angle. We describe other toy models, which are closer to the analysis which is presented in this survey:

Example 1. We consider in $\mathbb{R}^2$ the operator:

$$h^2 D_x^2 + (hD_y - a(x))^2 + y^2\,.$$

This model is rather artificial (and not purely magnetic) but by the Fourier transform, it is unitary equivalent to

$$h^2 D_x^2 + (\eta - a(x))^2 + h^2 D_\eta^2 \,,$$

which can be analyzed because it enters in the category of the miniwells problem treated in Helffer–Sjöstrand [22]. We have indeed a well defined in $\mathbb{R}^2_{x,\eta}$ by $\eta = a(x)$ which is unbounded but if we assume a varying curvature $\beta(x) = a'(x)$ (with $\liminf_{|x|\to+\infty} |\beta(x)| > \inf_x |\beta(x)|$) we will have a miniwell localization. A double well phenomenon can be created by assuming $\beta = a'$ even.

Example 2. If we add an electric potential $V(x)$ to the previous example, we get:

$$h^2 D_x^2 + (hD_y - a(x))^2 + y^2 + V(x) \,.$$

For $a(x) = x$, this example was considered by J. Brüning, S.Yu. Dobrokhotov and R.V. Nekrasov in [3].

Here one can measure the explicit effect of the magnetic field by considering

$$h^2 D_x^2 + h^2 D_\eta^2 + (\eta - a(x))^2 + V(x) \,.$$

If V admits as minimum value 0, the wells are defined in $\mathbb{R}^2_{x,\eta}$ by $\eta = a(x)$, $V(x) = 0$ and one can use under suitable assumptions the semi-classical treatment of the double well problem for the Schrödinger operator with electric potential $W(x,\eta) = (\eta - a(x))^2 + V(x)$ (see [10]).

Example 3. One can also imagine that in the case of Sections 2 and 4, we have a magnetic double well, and that a tunneling effect could be measured using the effective (1D)-Hamiltonian introduced in Subsection 4.1 $\hat{b}(x, hD_x)$ (actually a perturbation of it), assuming that b and A are holomorphic with respect to one of the variables. Here we are extremely far to a proof but we could hope for candidates for a formula for the splitting.

Example 4. Similarly, one can hope to measure the tunneling in the case of miniwells, in the situation considered in Subsection 3.2, when $|b|$ admits its minimum along a curve and β_2 has two symmetric miniwells.

Example 5. Finally one can come back to the Montgomery example [34] which was analyzed in [5, 12, 19, 35] and corresponds to the two-dimensional case when the magnetic field vanish to some order on a compact curve. According to a personal communication of V. Bonnaillie-Noël, F. Hérau and N. Raymond, it seems to be reasonable to hope (work in progress) that one could analyze the splitting between the two lowest eigenvalues for the following model in $\mathbb{R}^2$:

$$h^2 D_x^2 + \left(hD_y - \gamma(y) \frac{x^2}{2} \right)^2 ,$$

where γ is a positive even C^∞ function with two non degenerate minima and $\inf \gamma < \liminf \gamma$. By dilation, this problem is unitary equivalent to the analysis of the spectrum of

$$h^{\frac{4}{3}}\left(D_x^2+\left(h^{\frac{1}{3}}D_y-\gamma(y)\frac{x^2}{2}\right)^2\right).$$

After division by $h^{\frac{4}{3}}$, the guess is then that we can understand the tunneling by analyzing the spectrum of the $h^{\frac{2}{3}}$-pseudodifferential operator on $L^2(\mathbb{R})$ whose Weyl symbol is $\gamma(x)^{\frac{2}{3}}E(\gamma(x)^{-\frac{1}{3}}\xi)$, where $E(\alpha)$ is the ground state energy of the Montgomery operator $D_t^2+(\frac{t^2}{2}-\alpha)^2$. This would involve a Born–Oppenheimer analysis like in [36].

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Part VI: Special Talks

Geometric Methods in Physics. XXXII Workshop 2013
Trends in Mathematics, 281–286

Of Ice Cream and Parasites

Environmental talk at the XXXII Workshop on Geometric Methods in Physics

Bogdan Mielnik

Esteemed Colleagues:
In order to honor the ecological reserve of Białowieża (our Workshop place) it might be proper to consider some environment problems which may seem tangential but are not. Apart of the simple need to defend the world forests, it seems also necessary to compare the different time scales of the ecological threats and the proposed means of defense.

In his well-known movie, Al Gore illustrates global warming by showing a little girl who cannot consume her ice cream since it vanishes too fast. We are not indifferent to the little girl's tragedy, but in the present day a lot of children have much worse problems. One of A.G.'s conclusions is that the costs of defending our planet should be shared by the entire world population. If each human being made a modest effort to use less gas, less water, electricity, etc. (or even consume less ice cream?), then the Earth's deterioration could be significantly reduced.

While this might be true, it looks like a proposal to charge the general public with the price of great industry abuses, similar to the recent "austerity" doctrines which demand the world's citizens to alleviate the economic crisis by making personal sacrifices. Even if one accepts this, there is a technical difficulty concerning the time scales.

Despite advances of public consciousness the devastation advances much faster. The trees, the forests, and some species to which we have been accustomed for millenia (e.g., frogs, bees, etc) are vanishing. The destruction of the priceless tropical jungles becomes an explosive process. To try to slow it down through small sacrifices by goodhearted individuals may prove insufficient (more or less like trying to cure a patient with a heart attack by applying homeopathic medicines).

What seems urgent are quick actions and sharp laws implemented with an iron consistency. Yet, the understanding of this necessity is still deficient. One of difficulties is that a lot of meaningful data are too fragmented to be noticed by the wider public. Surprisingly also, some simple facts are rarely visible in enormous piles of the scientific literature, though they can be found in the daily news (again fragmented). The subjects appear and disappear, the discussions are discontinued, or else, they just vanish in the noise of the football games.

A hypothesis arises, that it might be relevant to collect patiently some ordinary press reports to augment the expert visions. While each one of them might be lacking in scientific rigor, their very accumulation can give a striking testimony of our epoch and its unsolved problems… After all, the Egyptian scrolls of papyrus studied by archaeologists are usually far from objective scientific rigor: yet they are of high interest for present day specialists. Today's Egyptologists look desperately to find some still undiscovered writings of an author living, e.g., in the epoch of Tutmosis III (even if not objective!). Antiquity converts ancient trash into scientific treasure. Many of today's specialists would be delighted to go back in time, to visit ancient Egypt, to be personally present in poor cabanas and proud temples (not yet in ruins!) – to talk personally with the authors of the old papyrus scrolls, no matter if they were deep or superficial thinkers. However, impossible, all this is gone…

Yet we have the good (or bad) luck to live in the fascinating XXI century. We can talk with people, read the news, (the serious ones and the trash) – everything that will become the precious remnants for the XXX century archaeologists! (of course, if they exist). What shall they think of us? It is hard to guess… Nor can we guess the possible thoughts of some hypothetical aliens, if by some telepathic mechanism they read our present day news and gossips. Perhaps they ask: "What do these poor creatures truly know about their disaster? Can they survive?…" Without insisting on the special importance of one of these creatures, let me start the collection by reporting an incident in XX century Poland which I still cannot remove from my memory.

Poland 1979. The little town of Brwinów (my family place) was connected with another little settlement Leśna Podkowa by an avenue of about 2 km with a row of splendid oaks more than 100 years old, with great crowns and more than 1.5 m diameter trunks. Along this avenue I was accustomed to walk to the nearby forests first as a schoolboy, then as a student and then as a staff member of the Institute of Theoretical Physics of Warsaw University (IFT UW). However, to my surprise, when repeating my favorite excursion after a short absence in 1979, I saw three big oaks cut, mountains of long branches on the avenue, a team of people in working uniforms with heavy machinery cutting the branches and the crown of the next oak. "Eeey!! What are you doing here?", I shouted. "We are cutting these trees", answered one of the team. "You can pass over here", he added. "Soon, the avenue will be clean". "What does it mean avenue clean?", I asked. "Who authorized you to cut the oaks?" "Oh, just ask the engineer", he told, pointing out an elegantly dressed figure standing nearby. I approached the man. "Pardon me", I asked, "what exactly is going on here?" "But who are you?" he asked. "This does not matter", I replied. "I am from Warsaw Institute of Physics and I simply live here". "Aha… Then check in the town council. I am an agronomic engineer, to your service, and I am in charge of cutting these trees, since they are redundant. Moreover, they have parasites". "How could it be that the hundred year old trees are redundant? Besides, on the trunks you cut, I don't notice any parasites!" "Yes, they have parasites! And you see, there is also another problem. They are an obstacle to a very important project. There will be a new telephone line here, an important investment required by the local citizens and prepared for many years. Now, finally, the telephones will be installed thanks to the joint agreement of the cities of Brwinów and Leśna Podkowa. If you are interested, consult the offices of the town council in Brwinów." "If so", I asked, "then why the telephone line requires cutting the trees? Couldn't it run some 10 or 20 meters aside?" "Impossible, since it would be too expensive", said the official. "This was definitely the cheapest solution". (In fact, the cheapest one: the only price was the destruction of an avenue of more than 100-years old trees! Instead, they have now an outdated technology).

Adios socialism, welcome the market. The situations we observe are not limited to "socialist" regimes. Quite similarly, in 2011 the newspapers in Mexico City revealed that on one of the avenues the old trees were cut for unknown reason. Answering the citizen objections, the executives of the district explained that the decision was taken to create a "cultural corridor". It was never explained why the existence of the trees was an obstacle to the culture; nor was there any report on cultural activity in the "corridor", but the trees no longer exist.

In Poland there is a strict law that the owner of a private garden has no right to cut any of his trees if the trunk diameter is above 30 cm. If he does it, he pays the fine of about 30 000 zł. (approximately 8 000 US dollars). If the tree was damaged by a hurricane, or partly broken and presents a life danger to pedestrians, the owner can cut it, but he must first write a report to the town council and ask permission. The formalities seldom take less than one month. A well-known story is about an owner whose tree was cut and stolen during one night: he lost his tree and he had to pay 30 000 zł... No such difficulties face the town executives, who can "execute" as many trees as they wish. So, how many trees will the executives still execute? In the central square of the ecological city of Białystok the trees were cut just to "open more place for public events". In danger are also the crowns of the trees if they block the visibility of some department store or supermarket. Then it may happen that one morning the tree wakes up without its crown!

Brazil, Brazil... These were just little illustrations, but the mechanisms seem universal. The situation is specially dramatic in Brazil. The scientists in the Ecology Department in Brazil, Manaus (Philip Fearnside and colleagues) for many years have tried to fight the persistently multiplied "development" projects (such as the construction of highways, gasoducts, hydro-electric dams etc.) causing a progressive destruction of the Amazon jungle. It was noticed that each project, even if rejected by the Parliament, was soon returning, slightly reformulated and under a new name to be discussed again and over again (see Philip Fearnside web page)...

PHOTO 1. Krasiński's Park in Warsaw, Poland. The splendid trees cut to open the 'visibility axis', (photo by Stefan Romanik).

In 2002 a group of "specialists", apparently linked to industrial investors, claimed that the construction of a highway through the jungle would bring considerable benefits to the soya industry: the transport of soya from the south of Brazil to the north could save at least 70 millions of US dollars per year, an amount which could be used to help the community of about 18 million poor people in the region. The calculation, though, looks a bit hypocrite. Even if a part of the saving was offered to the poor (which seems doubtful!) each one would obtain less than 4 dollars per year, i.e., about 38 cents per month... Some new arguments on "inevitable character" of the industrial developments were soon formulated. Since the industrial investment cannot be avoided, it is better to sacrifice a little part of the jungle and... use the profit to defend the rest". Hmm! Can you believe this?...

The peripheries of the jungle are frequently invaded and burned by the poor peasants to gain agricultural terrains. The international volunteers who tried to protect the jungle, are frequently murdered. The fast profit industries are also landing inside the forest without the need of highways to cut the trees and sell the wood. In the new presidency of Dilma Rousseff a lot of discussion is dedicated to the laws about selling of lots to the private owners, who are then obliged to preserve 80% of the trees on their terrain (it seems that the new proposal is to reduce it to 50%). Even if 80% is maintained, the law means an enormous destruction of the jungle, the principal lung of our planet. In extreme danger are the mangrove

PHOTO 2. Ixtapan de la Sal, Mexico. This nice central garden no longer exists, (photo by Bogdan Mielnik).

forests, the natural habitat of fauna and flora of incomparable diversity. They are pitilessly cut, split into pieces by tourist centers and shrimp farms, irreversibly poisoning the environments. Do we need so many shrimps?

Little, but... From these global problems let me again jump to the local ones. In an act of incomprehensive irresponsibility the executives of Warsaw administration (the democratic Poland, 2012) ordered the splendid old trees of Krasinskis' park to be cut (see Photo 1). Asked by the angry citizen, who and why permitted this, the Warsaw executive (in charge to protect the monuments of the past), answered: *The tree was removed since its health was not the best and moreover, it was blocking the 'visibility axis'.* As it seems, the mental health of this executive was 'not the best', so he should not occupy any executive position.

In Mexico, the little town Ixtapan de la Sal was proud of an exceptional central square with trees covered by wonderful orange flowers (see Photo 2). They are no longer there. So was there some problem about the *visibility axis* blocked by flowers?

Though quite modest, I hope, these remarks help to complete the fragmented picture. You can observe that certain events are repeated at all latitudes and all social systems. While the general public is under a rigorous bureaucratic control, the higher administration levels are not. The higher the level, the less responsibility!... Perhaps, the only doubt left to the archaeologists of XXX century will be when precisely the last tree disappeared from the surface of the Earth?

In discussions about our Białowieża forest the subject of parasites reappears. Some industry groups (representing mostly the wood commerce) argue that the very old trees have parasites and therefore should be removed for sanitary reasons... Hmm! Some of us are a bit old and we have probably a lot of parasites: hence, we feel some solidarity... However, what is the truth? Yes, the trees have parasites! **We are the parasites!** The humans are the most dangerous parasites devastating the surface of the Earth. Can we evolve from the destructive into benign parasites? As Bob Dylan sings: "The answer, my friend, is blowing in the wind..." Note only, that in recent years those winds are increasingly violent!

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The Hamiltonian?

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Abstract. In classical mechanics the letter H is commonly used to denote the total energy function. This notation was introduced by Lagrange in his *Mécanique Analytique* of 1813, when Hamilton was 8 years of age. We will show that Lagrange most probably used this letter to honor the Dutch scientist Christiaan Huygens. It would thus be better to talk about the Huygensian rather than the Hamiltonian.

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1. Introduction

In the nineteen eighties Jean-Marie Souriau told us (in at least one of his lectures) that the letter H, commonly used in classical mechanics to denote the total energy function which governs the time evolution, was attributed by Lagrange[1] in honor of Huygens[2] in the second edition of his famous book on mechanics *Mécanique Analytique*, which appeared posthumously in 1813. And thus that calling this function the "Hamiltonian" would be historically incorrect, as Hamilton[3] was only 8 years old at that time. When I got to know Souriau a bit better, I asked him where exactly this affirmation could be found. His elusive answer was "somewhere in the *Mécanique Analytique*," but he had forgotten where exactly. Following this indication, Patrick Iglesias and I independently searched the *Mécanique Analytique* and we arrived at the same conclusion: nowhere does Lagrange say explicitly that he uses the letter H to honor Huygens, but there is overwhelming circumstantial evidence that was his intention. Iglesias published his findings in an appendix to his book [1] and here I provide essentially the same arguments, made even stronger by the comparison between the first and second editions of the *Mécanique Analytique*.

[1] Joseph-Louis Lagrange, Turin 1736–Paris 1813.
[2] Christiaan Huygens, Den Haag 1629–Den Haag 1695.
[3] Rowan Hamilton, Dublin 1805–Dublin 1865.

2. The evidence

My circumstantial evidence that Lagrange attributed the letter H in honor of Huygens is based upon a comparison between the first edition of his *Mécanique Analytique* [2] and the second edition [3], as can be found in the fourth edition (which contains notes added posthumously by several specialists to fill some of the gaps in the second edition of 1813 left by Lagrange upon his death). In the first (introductory) chapter of the second part on dynamics (the first part is on statics) [2, pp. 158–189], [3, Vol. I, pp. 207–230], Lagrange shows his admiration for several of his predecessors but especially for the Dutch scientist Christiaan Huygens. Of Huygens he says that "he seemed to be destined to improve and complete most of Galilei's discoveries." He also tells that Huygens solved a problem of mechanics without knowing the exact form of the forces involved, just by applying a general principle. And Lagrange to note that he has no idea how Huygens got this marvellous idea. A few pages later on he mentions that there are four main principles that govern mechanics: *conservation of live forces, conservation of the center of gravity, conservation of moments of rotation* (also called the *principles of areas*) and *the principle of least action.* Of the first he says that it is the principle found by Huygens, and a few lines later he changes the wording by talking of *Huygens' principle.* It thus seems obvious that for Lagrange, the principle of *conservation of live forces* is synonym to *Huygens' principle.*

In the third chapter on dynamics entitled "*General properties of the motion deduced from the preceding formula*" Lagrange deduces (from the preceding formula, as said in the title) the equation [2, p. 207], [3, Vol. I, p. 268],

$$S\left(\frac{dx\,d^2x + dy\,d^2y + dz\,d^2z}{dt^2} + P\,dp + Q\,dq + R\,dr + \cdots\right)m = 0$$

where the P, Q and R denote the accelerating forces and where S denotes the sum/integral over all bodies involved. If, says Lagrange, the quantity $P\,dp + Q\,dq + R\,dr + \cdots$ is integrable, then we can write

$$P\,dp + Q\,dq + R\,dr + \cdots = d\Pi$$

and he then integrates the above equation in the first edition to [2, p. 208]

$$S\left(\frac{dx^2 + dy^2 + dz^2}{2\,dt^2} + \Pi\right)m = F$$

telling us that F is an arbitrary constant, and in the 2nd edition to [3, Vol. I, p. 268]

$$S\left(\frac{dx^2 + dy^2 + dz^2}{2\,dt^2} + \Pi\right)m = H$$

telling us (again) that H is an arbitrary constant. Both editions continue with the remark that this last equation englobes the principle known under the name of *Conservation of live forces.* Comparing the rest of the text, there seems no particular reason to have changed the letter F to the letter H. All other symbols used remain the same. On the other hand, this third chapter is much bigger in the second edition and is subdivided in several sections (which it is not in the first edition). And in the second edition, this formula appears in the section entitled "*Properties concerning live forces.*" If we now remember that for Lagrange the names *Conservation of live forces* and *Huygens principle* are synonym, the reason for this change seems obvious: he wants to honor Huygens.

Now anybody with some knowledge of classical mechanics will recognize in the term

$$\frac{dx^2 + dy^2 + dz^2}{2\,dt^2} m$$

the kinetic energy of a body and in the function Πm the potential energy function whose gradient determines the force exerted on the system. And indeed, in a time-independent system the total energy H, being the sum of these to terms, is conserved. Later on [3, Vol. II, p. 3], in the introduction to Section 7 of Part 2, Lagrange introduces the letter T for the kinetic energy of a system of bodies/particles

$$T = m\frac{dx^2 + dy^2 + dz^2}{2\,dt^2} + m'\frac{dx'^2 + dy'^2 + dz'^2}{2\,dt^2} + \cdots$$

and the letter V for the potential energy of such a system

$$V = m\Pi + m'\Pi' + m''\Pi'' + \cdots$$

He then remarks (once again) that, if the functions T and V are independent of time, then one has the equation

$$T + V = H,$$

H being an arbitrary constant of integration, and that this equation englobes the *principle of live forces*. The use of the letters T and V for these two functions persists up to today, as does the use of the letter H for their sum. As this letter was attributed before Hamilton did any work in mechanics (even though he was precocious), it is historically incorrect to call this function the Hamiltonian. It should be called

the Huygensian!

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